A STUDY OF THE SOLITON SOLUTIONS OF THE BOUSSINESQ AND OTHER NONLINEAR EVOLUTION EQUATIONS OF FLUID MECHANICS

by

Mukheta Bin Isa

Thesis submitted for the degree of Ph.D of the University of Newcastle upon Tyne, 1988.

DEDICATED TO AYAH & IBU, MAMA, JOE, IDA,

YAN AND TAMAR FITRI.

ACKNOWLEDGEMENTS

I am very grateful to my supervisor, Professor N.C.Freeman for his inspiration and unending guidance throughout the period of preparation for this thesis. Thanks are also due to him for his critical reading of the manuscript.

I would also like to thank Dr.R.S.Johnson, Dr.S.Candler and Dr.M.Ito who have contributed in many ways to this work.

Thanks are also due to Miss P.Armstrong and Mrs.H.F.Bliss for their careful typing of the script.

Financial support from the Malaysian government and the Universiti Teknologi Malaysia throughout the period of this work is hereby gratefully acknowledged.

ABSTRACT

After introducing the nonlinear evolution equations of interest: the finite depth fluid (FDF), the Kadomtsev-Petviashvili (KP), the Classical and the ordinary Boussinesq equations, formal asymptotic derivations of the KP and the FDF equations are given for the description of surface and interfacial waves.

The N-soliton solution of the FDF equation is reconstructed as a finite sum of Wronskian type determinants. This solution is then shown to reduce to the solutions of the KdV and the Benjamin - Ono equations under specific limiting conditions. Interactions between two solitons of the FDF equation are studied and their interaction properties are shown to reduce to those of the KdV and the Benjamin - Ono equations. Computer plots of the interactions of two-soliton solutions of the FDF and the Benjamin - Ono equations are given.

Resonance phenomena in solitons are studied with reference to the KP equation. After discussion of the basic concepts of these phenomena, the N-soliton solution is shown to reduce to the Wronskian of N/2 functions (N-even), each of which represents a triad of solitons when the solitons resonate in pairs. Asymptotic behaviour of the interactions between a triad and a soliton and between two triads are examined and the phase shifts of the triads are obtained directly from the Wronskian representation. The interactions are analysed in detail with reference to numerical computations of the full solutions.

After showing that the Classical Boussinesq equations are obtained from Whitham's shallow water wave equations, the basic concept of Hirota's pq=c reduction of the first modified KP hierarchy is outlined. The Classical Boussinesq equations are shown as the pq=0 reduction of the same hierarchy. The solution of the hierarchy is manipulated to incorporate the pq=0 reduction. As a result of these limiting procedures applied to the problem, Wronskian solutions of the Classical Boussinesq equations in terms of rational functions are produced.

Finally the pq=c reduction of the KP hierarchy is applied to the ordinary Boussinesq equation. Using this, the N-soliton solution is expressed as a finite sum of Wronskian type determinants. Analytic verification made for the two-soliton solution shows that a number of Wronskian identities are needed for this purpose. The reason for this behaviour is examined.

CONTENTS

PAGE

CHAPTER 5

THE CLASSICAL BOUSSINESQ EQUATIONS: THE pq=0 REDUCTION

CHAPTER 6

THE ORDINARY BOUSSINESQ EQUATION

CHAPTER 7

CHAPTER 1

INTRODUCTION

Continuing studies in Fluid Mechanics have produced some fascinating nonlinear evolution equations which are worthy of further investigation. Particularly interesting is that many of these equations are shown to exhibit a special kind of steady state wave solutions now known as solitons. This type of wave can be found on the surface of shallow water, on the interface between two fluid layers of different densities, on thermoclines in the tropical oceans and in many other places.

Perhaps the simplest nonlinear evolution equation of this kind is Burgers equation:

$$u_t + uu_x - \delta u_{xx} = 0, \ \delta > 0. \qquad (1.1)$$

This was originally obtained by Burgers (1948) to model a turbulent flow. This equation also describes one dimensional flow of a viscous heat conducting fluid in which δ is a measure of the viscous and thermal diffusion [Cole (1951)].

Equation (1.1) includes both nonlinearity (uu_x) and dissipation (δu_{xx}). The behaviour of the solution of this equation can be explained without actually solving the equation itself. Let us suppose that δ is small. Clearly in the case of $\delta = 0$, it is simply the first order nonlinear equation which shows steepening and breaking of most initial wave profiles at the leading edge. If the nonlinear term uu_x is neglected, then (1.1) is a purely dissipative equation. This means that the initial wave profiles will decay exponentially with time. Now, with both nonlinear and dissipation terms in the equation, initially the waves will steepen. However, the steepening region contains wave components of increasingly large wave numbers and this makes the dissipation effect more dominant than the effect of nonlinearity as the decay is proportional to the square of wave numbers. Therefore, further steepening is counteracted by this decay and thus breaking or multivaluedness of the solution is avoided. Hence, in Burgers equation we have a balance between nonlinearity and dissipation.

One of the solutions of Burgers equation which shows this balance is [Dodd et al (1982)]

$$u = a\delta[1 - \tanh\tfrac{1}{2}(ax - \delta a^2 t)].$$

This can be obtained by using the Cole-Hopf transformation

$$u = -2\delta \frac{\partial}{\partial x}(\log F) \tag{1.2}$$

which converts equation (1.1) to the heat equation

$$F_t = \delta F_{xx}.$$

Another equation which is very familiar in water wave theory is the KdV equation

$$u_t + uu_x + \delta u_{xxx} = 0. \tag{1.3}$$

In contrast to the Burgers equation which is the simplest nonlinear dissipative equation, the KdV equation is the simplest nonlinear dispersive equation due to the presence of the δu_{xxx} term. This equation was first derived by Korteweg and de Vries (1895) for long waves on the surface of shallow water. The equations of this type occur wherever nonlinearity and dispersion come together as, for example, on nonlinear transmission lines [Hirose and Lonngren (1985)], in oscillation of a crystal lattice and in ion-acoustic waves [Scott, Chu and McLaughlin (1973)].

The KdV equation (1.3) was studied numerically by Zabusky and Kruskal (1965) for $\delta = (0.022)^2$ with periodic boundary conditions $u(x+2,t) = u(x,t)$ and a periodic initial condition $u(x,0) = \cos \pi x$. In their study they observed that the initial wave only steepened at early times. They also found that behind the steepening wave, oscillations were developed and grew into separate solitary waves. Furthermore, they found that the amplitudes and velocities of those solitary waves remained fixed and unchanged despite the nonlinear interactions taking place. This particle like property caused them to call the solitary waves "Solitons".

Essentially, the nonlinearity in the KdV equation is balanced by the dispersion. This can be explained as follows. If we neglect the nonlinear term, then with $\delta>0$ we have a purely dispersive equation with negative phase speed proportional to the square of the wave number. Now, as the leading edge steepens, components of increasingly large wave numbers are dispersed and in this way the nonlinear steepening is balanced by the dispersion.

The KdV equation has been studied extensively since the discovery of solitons by Zabusky and Kruskal (1965). An analytic procedure to obtain soliton solutions of this equation was immediately carried out by Gardner et al (1967) by using the inverse scattering transform. Rapid development in soliton theory has been continuing since then and some of this will be examined later.

The form of the KdV equation normally used in employing the inverse scattering transform is

$$u_t + 6uu_x + u_{xxx} = 0 \qquad (1.4)$$

and its solitary wave or single soliton solution is [Satsuma (1979)]

$$u = 2k^2 \operatorname{sech}^2 (kx-4k^3t+\eta) \qquad (1.5)$$

where k is any real constant which characterizes the soliton and η is a phase constant. The expression (1.5) shows that a taller soliton moves faster than the shorter ones.

All the above equations describe surface waves which propagate in one direction only. There is however another equation which also describes waves on the surface of shallow water. This is the Boussinesq equation

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0 \qquad (1.6)$$

which was introduced by Boussinesq (1872) to describe such waves. This equation has also been rederived by Zabusky (1967) for waves in a nonlinear one dimensional lattice.

Both the KdV and the Boussinesq equations are nonlinear dispersive equations. Unlike the KdV equation, the Boussinesq equation describes waves which can propagate in both positive and negative x-directions due to the presence of the second t and x-derivatives in (1.6). If one looks for waves which travel in one direction only from the Boussinesq equation, it is expected that this equation would reduce to the KdV equation. Indeed this can be done for large space and time scales by defining $\xi = \epsilon^{1/2}(x-t)$ and $\tau = \epsilon^{3/2}t$ where ϵ is a small parameter. In this way the Boussinesq equation (1.6) yields the KdV equation in the form

$$2u_{1\tau} + 6u_1u_{1\xi} + u_{1\xi\xi\xi} = 0$$

where

$$u = \epsilon u_1 + \epsilon^2 u_2 + \ldots$$

Solutions of the Boussinesq equation have been obtained by Zakharov (1974) by using the inverse scattering transform, by Hirota (1973b) using a direct method and by Nimmo and Freeman (1983) using the Bäcklund transformations.

Both the KdV and the Boussinesq equations arise from the shallow water wave problem as we have seen in the above discussion: this is one of the common features of these two equations. Another common feature is that both equations are contained in more general shallow water wave equations introduced by Whitham (1974),

$$\begin{aligned} &u_t + \{(1+\alpha u)w\}_x - \frac{\beta}{6} w_{xxx} + O(\alpha\beta,\beta^2) = 0 \\ &w_t + \alpha w w_x + u_x - \frac{\beta}{2} w_{xxt} + O(\alpha\beta,\beta^2) = 0 \qquad (1.7) \\ &w = f_x \end{aligned}$$

where f is the first order term of an expansion for the velocity potential and α,β are small parameters arising from nondimensionalizing the physical variables.

Equations (1.7) can be written in a more convenient form by the substitution $w = v + \frac{1}{2}\beta v_{xx} + O(\alpha\beta,\beta^2)$ as

$$u_t + \{(1+u)v\}_x + \frac{1}{3}v_{xxx} = 0$$
$$v_t + (u + \tfrac{1}{2}v^2)_x = 0 \qquad (1.8)$$

after setting $\alpha = \beta = 1$. Equations (1.8) are referred to as the Classical Boussinesq equations and their solitary wave solutions have been found by Krishnan (1982) and Nakamura and Hirota (1985).

All the equations mentioned in the above were derived for waves which propagate on the surface of shallow water. The propagation of internal waves in deep water has been the subject of study by Phillips (1966), Benjamin (1967), Davis and Acrivos (1967) and many others. Benjamin (1967) found that these solitary waves could be written in the rational form

$$u(x,t) = \frac{2k}{k^2(x-kt+\eta)+1} \qquad (1.9)$$

where k is a real constant and η is a phase constant.

The actual equation for these internal waves was derived by Ono (1975) and written in the integro-differential form

$$u_t + 2uu_x + \frac{1}{\pi}\frac{\partial^2}{\partial x^2}P\int_{-\infty}^{\infty}\frac{u(x',t)}{x'-x}\,dx' = 0. \qquad (1.10)$$

He also showed that the solitary wave solution of (1.10) was given by (1.9). Equation (1.10) is known as the Benjamin - Ono equation. Multisoliton solutions of this equation have been found by Matsuno (1979a).

It can be shown that both the KdV and the Benjamin - Ono equations share a common parent equation: an equation which contains a depth parameter which can be put to zero for the KdV equation and to infinity for the Benjamin - Ono equation. Such an equation is called the finite depth fluid equation

$$u_t + 2uu_x + G[u_{xx}] = 0 \qquad (1.11)$$

where

$$G[f(x)] = \frac{\lambda}{2}\,P\int_{-\infty}^{\infty}[\coth\tfrac{\lambda\pi}{2}(x'-x) - \mathrm{sgn}(x'-x)]\,f(x')dx' \qquad (1.12)$$

where λ is related to the reciprocal of the fluid depth and P denotes the principal value of the integral.

The form of the finite depth fluid equation (1.11) is due to Matsuno (1984). The original equation, however, was introduced by Joseph (1977) by using the representation of the dispersion relation advanced by Whitham (1967). The knowledge of the phase speed c(k) for waves travelling in a dispersive medium can be used to construct the equation governing such waves. The expression inside the square brackets in (1.12) comes from the inverse of the Fourier transform for c(k).

The expression of c(k) used by Joseph to construct equation (1.11) was previously found by Phillips (1966),

$$c(k) = \frac{\lambda}{2}[1 - \frac{\lambda}{2}(\coth(\frac{k}{\lambda}) - \frac{\lambda}{k})]$$

for waves travelling on the thermocline in an ocean.

All the equations we have mentioned so far are in one space dimension. A direct generalization of the KdV equation to two space dimensions has been made by Kadomtsev and Petviashvili (1970). By assuming weak y - coordinate dependence with the dominant wave propagation in the x - direction they have been able to produce the equation

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0 \qquad (1.13)$$

where the coefficients are chosen for later convenience. Although the original derivation of this equation was entirely based on the argument about wavelengths in the x and y - directions, its physical significance has been justified in connection with shallow water waves. A formal derivation of equation (1.13), which is called the Kadomtsev-Petviashvili (KP) equation, for waves propagating at a small angle with x-axis on the surface of shallow water, has been made by Johnson (1980), Freeman (1980) and Thompson (1980).

All the equations we discussed in the above have been shown in the literature to exhibit soliton solutions. The work of searching for soliton solutions has been growing rapidly since the exploitation of soliton properties by Zabusky and Kruskal (1965). The discovery of these type of waves, however, can be traced back as early as 1834, in a report by Scott-Rusell (1844) who called them "Waves of Translation". Many parts of this report can be found quoted in Bullough and Caudrey (1980), Freeman (1980), Calogero and Degasperis (1982), Dodd, Eilbeck, Gibbon and Morris (1982) and in many others.

The first analytical approach to obtain multisoliton solutions of the KdV equation was developed by Gardner et al (1967) and revised in a more mathematical formalism in their later paper of 1974. This method, now referred to as the inverse scattering transform, has received a lot of attention since then. The scope of inverse scattering was widened to include many other equations by Zakharov and Shabat (1974) and Ablowitz et al (1974). Many other equations such as the nonlinear Schrödinger, the modified KdV, the sine and sinh-Gordon, the Kadomtsev-Petviashvili and the Boussinesq equations have been solved by the use of the inverse scattering transform [see Novikov et al (1984)].

Although the inverse scattering transform is widely used, it is an indirect method of solution which seeks to obtain the general solution to the problem. If the soliton solutions themselves are to be studied, a more straightforward approach can be adopted. An alternative technique has been given by Hirota (1971), first for the KdV equation and in a series of later papers for many other nonlinear evolution equations [see Hirota (1980)].

Essentially, Hirota's technique is a direct method. It involves the transformation [similar to the Cole-Hopf transformation (1.2)] of the original equation into a bilinear form. The solution is then obtained from this bilinear equation by using an expansion method. The application of this method to the KdV equation is given in Appendix A since all the equations considered in this thesis will be approached in this way.

The central aim of this thesis will be mainly the study of two aspects of the soliton solutions of the finite depth fluid, the Kadomtsev-Petviashvili, the Classical and ordinary Boussinesq equations. These are the structures of the solutions and the interactions between solitons.

The direct method of Hirota leads to solutions in the form of sums of exponentials [Hirota (1980)] while the inverse scattering transform leads directly to a determinantal form [Lamb (1980)]. Another form of soliton solutions was first given by Satsuma (1979). By considering the inverse scattering scheme of the KdV equation (1.4) he was able to show that its N-soliton solution could be written as

$$u = 2 \frac{\partial^2}{\partial x^2} \log W(\phi_1,\phi_2,\ldots,\phi_N) \tag{1.14}$$

where W is the Wronskian of N functions $\phi_1,\phi_2,\ldots,\ \phi_N$ with

$$\phi_j = \cosh\,(k_j x-4k_j^3 t + \eta_j),\ j=1,2,\ldots,N \tag{1.15}$$

Each function ϕ_j represents a single-soliton solution.

However one can also reduce directly the ordinary determinantal form to a Wronskian form. This has been done by Freeman (1984). The Wronskian representation of the soliton solutions will be used throughout the thesis and the basic derivation of this representation together with its properties is presented in Appendix B.[†]

The use of Wronskian solutions in verifying the multisoliton solutions is relatively simple compared with the other two forms. This is due to the fact that the Wronskian solutions can be made very compact. Furthermore differentiation of a Wronskian determinant is straightforward and does not generate a large number of terms as differentiation of an ordinary determinant would.

Soliton solutions have also been studied in terms of infinite dimensional Lie algebras. In this way, Jimbo and Miwa (1983) found that a particular

---

† See also Adler and Moser (1978) for earlier work on Wronskian solutions.

soliton solution does not belong to a specific equation only, but rather to a whole class of equations which form a hierarchy of increasing complexity. Indeed they have produced seven important hierarchies which can be written in the form of Hirota's bilinear operator. Two of them are the KP hierarchy and the first modified KP hierarchy. These two hierarchies will be of interest in this work and they are listed in Appendix C. One of the advantages of these hierarchies is that they can be tabulated for reference purposes. This means that the solutions of many equations which do not belong directly to any of the hierarchies may be obtained by manipulating the solutions of one of the hierarchies. Such a manipulation leads to a constraint on the solution parameters. This process is called a "reduction" by Hirota (1985). He applied this procedure to the Classical Boussinesq equations (1.8) and obtained the so-called pq = c reduction.

In order to illustrate this procedure, we take the KdV equation (1.4) as an example. By using the transformation

$$u = 2 \frac{\partial^2}{\partial x^2} \log F \tag{1.16}$$

it can be written in the bilinear form [Hirota (1971)]

$$(D_x{}^4 + D_x D_t) F.F = 0 \tag{1.17}$$

where D_x, D_t are the bilinear differential operators defined by

$$D_x{}^m D_t{}^n f.g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x,t) g(x',t') \Big|_{\substack{x'=x \\ t'=t}} \quad . \tag{1.18}$$

The bilinear equation (1.17) does not belong to any of the hierarchies [see Appendix C]. However it can be compared with the first equation in the KP hierarchy,

$$(D_1{}^4 - 4D_1 D_3 + 3D_2{}^2)\ F.F = 0 \tag{1.19}$$

where D_1, D_2 and D_3 refer to the bilinear operators with respect to the variables x_1, x_2 and x_3. The single-soliton solution under this hierarchy is written as

$$F = e^{px_1 + p^2 x_2 + p^3 x_3 + \eta} + e^{qx_1 + q^2 x_2 + q^3 x_3 + \theta} \tag{1.20}$$

where p,q are abritrary solution parameters with $p \neq q$, η and θ are constants.

Now if we define $x = x_1$ and $t = -\frac{1}{4}x_3$ then (1.17) becomes

$$(D_1^4 - 4D_1D_3)\ F.F = 0 \tag{1.21}$$

which is exactly the first two terms of (1.19). Therefore it is easy to see that the solution (1.20) will satisfy (1.21) if

$$3D_2^2 F.F = 0\ .$$

This is equivalent to the condition

$$p^2 - q^2 = 0. \tag{1.22}$$

Due to the relation given by (1.22) this problem has been described as the 2 - reduction by Hirota (1986b). Since $p \neq q$ then (1.22) gives $q = -p$, the solution of the KdV equation can be deduced from (1.20) to give

$$F = e^{\frac{1}{2}(\eta+\theta)+p^2x_2}\cosh(px_1+p^3x_3+\eta_0).$$

It should be noted from this expression that the exponential factor is independent of $x_1(=x)$ and therefore it does not contribute to the final solution u (1.16) and we can simply write

$$F = \cosh\ (px - 4p^3t + \eta_0)$$

in terms of the original variables where $\eta_0 = \frac{1}{2}(\eta-\theta)$. Hence we obtain the same result as (1.15) which gives the final sech^2 solution (1.5) for u.

We note that in the above calculation we have used the transformation $t = -\frac{1}{4}x_3$. However, this is not the only choice that we could use to solve this problem. We could choose, for example, $t = x_3$. This would result in a different reduction problem and a constraint different from (1.22).

Let us now examine this type of reduction problem for the Boussinesq equation (1.6). The bilinear form of this equation is [Hirota (1973b)]

$$(D_x^4 + D_x^2 - D_t^2)F.F = 0 \tag{1.23}$$

upon using the variable transformation (1.16).

We may of course choose $x = x_1$ and $t = \frac{-i}{\sqrt{3}}x_2$ so that (1.23) becomes

$$(D_1^4 + D_1^2 + 3D_2^2)F.F = 0. \tag{1.24}$$

Equation (1.24) can now be separated as

$$(D_1^4 + D_1^2 + 3D_2^2)F.F$$
$$= (D_1^4 - 4D_1D_3 + 3D_2^2)F.F + (4D_1D_3 + D_1^2)F.F \qquad (1.25)$$

Now, in order that (1.20) is to satisfy (1.24) we must have

$$(4D_1D_3 + D_1^2)F.F = 0,$$

since the first expression on the right hand side of (1.25) is automatically zero by (1.19). Now if (1.20) is substituted into the last equation or into equation (1.24) itself, we find

$$p + 4p^3 = q + 4q^3. \qquad (1.26)$$

This would result in the solution which is exactly the same as that obtained by Nimmo and Freeman (1983).

If we now choose $x = x_1$ and $t = x_2$ then (1.23) is

$$(D_1^4 + D_1^2 - D_2^2)F.F = 0, \qquad (1.27)$$

which then separates into

$$(D_1^4 + D_1^2 - D_2^2)F.F$$
$$= (D_1^4 - 4D_1D_3 + 3D_2^2)F.F + (4D_1D_3 + D_1^2 - 4D_2^2)F.F \quad . \qquad (1.28)$$

By similar argument, if we require (1.20) to satisfy (1.27) then

$$(4D_1D_3 + D_1^2 - 4D_2^2)F.F = 0$$

which means that the solution parameters satisfy

$$pq = \frac{1}{4} \quad . \qquad (1.29)$$

We shall see in Chapter 6 that the $pq = \frac{1}{4}$ reduction of the KP hierarchy for the Boussinesq equation generates a new structure of the solution : a sum of Wronskian type determinants.

Interactions between solitons have been studied since the discovery of the solitons by Zabusky and Kruskal (1965). It has become one of the definitions of solitons that with the exception of some phase shifts, solitons preserve their identity after emerging from an interaction with another soliton. However a study of two interacting solitons made by Miles (1977) reveals that under certain conditions a single soliton can be produced after the interaction. He also found that the two incident solitons and the resulting soliton formed a triad of solitons. This phenomenon is typical of soliton interactions which are known as resonances, and are familar in the study of linear waves.

Let us consider this phenomenon for the two-soliton solution of the KP equation (1.13). The two-soliton solution is given by Lamb (1980) as

$$u = 2 \frac{\partial^2}{\partial x^2} \log \Delta \tag{1.30}$$

with

$$\Delta = 1 + e^{\eta_1} + e^{\eta_2} + q_{12} e^{\eta_1+\eta_2} \tag{1.31}$$

where

$$\eta_i = (\ell_i+n_i)x + (\ell_i^2-n_i^2)y - 4(\ell_i^3+n_i^3)t + \log\left(\frac{a_i}{\ell_i+n_i}\right).$$

$$q_{ij} = \frac{(\ell_i-\ell_j)(n_i-n_j)}{(\ell_i+n_j)(\ell_j+n_i)} \;; \quad i,\ j = 1,\ 2;\ i \neq j.$$

Here a_i is a constant, ℓ_i, n_i are the solution parameters obtained upon parameterization of wave number $\underline{k} = (k,m)$ and the frequency ω so that they satisfy the linear dispersion relation of the KP equation [Freeman (1980)]

$$-\omega k + k^4 + 3m^2 = 0$$

with

$$k = \ell+n,\ m = n^2 - \ell^2 \text{ and } \omega = 4(\ell^3+n^3).$$

It is found more convenient for the purpose of this discussion to represent the solution by Δ instead of the actual solution u. It should be noted that a single-soliton solution is represented by only two terms, $\Delta = 1+e^{\eta}$, which is equivalent to the final solution

$$u = \frac{1}{2}(\ell+n)^2 \operatorname{sech}^2 \frac{\eta}{2}.$$

Let us now return to the two-soliton solution (1.31) with $q_{12} = 0$ which can be achieved by choosing either $\ell_1 = \ell_2$ or $n_1 = n_2$. Now (1.31) becomes

$$\Delta = 1 + e^{\eta_1} + e^{\eta_2}. \qquad (1.32)$$

From this expression we see that in the region where $\eta_2 \to -\infty$ and η_1 is fixed we have $\Delta = \Delta_1 = 1+e^{\eta_1}$ which is the first soliton. In the region where $\eta_1 \to -\infty$ and η_2 is fixed we find $\Delta = \Delta_2 = 1 + e^{\eta_2}$: the second soliton. In the region where $\eta_1 \to +\infty$ and $\eta_2 \to +\infty$ with $\eta_1 - \eta_2$ being fixed we have

$$\Delta = \Delta_3 = e^{\eta_2}(1 + e^{\eta_1-\eta_2})$$

which is the third soliton resulting from the interaction between the first and the second solitons. Since the final solution u is expressed as (1.30), the exponential factor in the above equation can therefore be removed and hence

$$\Delta_3 = 1 + e^{\eta_1-\eta_2}. \qquad (1.33)$$

If the wave number of the third soliton is $\underline{k}_3 = (k_3, m_3)$ and its frequency is ω_3, then we see that

$$\underline{k}_3 = \underline{k}_1 - \underline{k}_2$$

and

$$\omega_3 = \omega_1 - \omega_2.$$

This means that the soliton represented by Δ_3 is a resonant soliton produced by the solitons represented by Δ_1 and Δ_2.

Therefore, with the choice $q_{12} = 0$ one can always find that these three solitons come together in the form of a triad. Hence, in a two-resonance soliton solution, the motion of this triad can be dealt with as a single entity.

The interaction will be more complicated if we include more solitons in the solution. The problem with three-soliton interaction of this kind has been considered by Anker and Freeman (1978). A procedure will be developed in Chapter 4 for the interactions of a larger number of solitons when they resonate in pairs.

We have so far explained the basic concern of the thesis. The outline of the remaining chapters is as follows.

We first derive the Kadomtsev-Petviashvili (KP) and the finite depth fluid (FDF) equations in Chapter 2. Our fluid models will be inviscid, incompressible and irrotational so that we deal with only the Laplace's equation for the velocity potential of the fluid.

Chapter 3 looks into the soliton solution of the FDF equation. The N-soliton solution is expressed as a finite sum of Wronskian type determinants. This solution is then shown to reduce to those of the KdV and the Benjamin - Ono equations under certain limiting conditions. An explicit form of the two-soliton solution of the FDF equation is obtained and used to study their interaction. The interaction properties will be shown to reduce to those of the KdV and the Benjamin - Ono equations. The result of the analysis for the FDF and the Benjamin - Ono equations is supported by some computer plots.

In Chapter 4 we discuss resonance phenomena in solitons with reference to the Kadomtsev-Petviashvili equation. After discussion of the basic concepts of this phenomena, the N-soliton solution which resonates in pairs is transformed into the Wronskian of N/2 functions (N-even). Each of these functions represents a triad. We then go on to discuss the interactions between a triad and a soliton and between two triads, with reference to numerical computations of the full solutions.

Soliton solutions of the Classical Boussinesq equations are discussed in Chapter 5. After showing that these equations originate from Whitham's shallow water wave equations we discuss some known solutions and the concept of $pq = c$ reduction. Special attention is then given to the case of $c = 0$ since it is directly related to the Classical Boussinesq equations. A complete theory of the N-soliton solution using the Wronskian technique is discussed and the rational solutions are produced.

In Chapter 6, we discuss the soliton solution of the ordinary Boussinesq equation. This equation is shown as the $pq = \frac{1}{4}$ reduction of the KP hierarchy. This reduction problem leads to a more complex Wronskian representation. The two-soliton solution in this representation is examined and shown analytically to satisfy the equation. The use of REDUCE programs is made in verifying higher order solutions.

Finally we conclude our results and give some suggestions for further study in Chapter 7.

CHAPTER 2

DERIVATIONS OF EQUATIONS

2.1 Derivation of the Kadomtsev-Petviashvili equation

The derivation of the equation for waves on the surfaces of shallow water has been given by Johnson (1980), Freeman (1980) and Thomspon (1980). It will be repeated here as an introduction to a similar result for waves on an interface between two fluids of different densities which leads to the finite depth fluid equation. The scaling procedures for this equation in the form required are not readily available.

For the purpose of this derivation we assume an inviscid, incompressible and irrotational fluid so that we may write the mass conservation equation or the continuity equation of the fluid in the form of Laplace's equation

$$\frac{\partial^2 \phi'}{\partial x'^2} + \frac{\partial^2 \phi'}{\partial y'^2} + \frac{\partial^2 \phi'}{\partial z'^2} = 0. \qquad (2.1.1)$$

Here ϕ' is the velocity potential of the fluid, and (x',y',z') is the chosen coordinate system, with x' and y' the horizontal axes and z' the vertical axis. Primed variables will be used initially to simplify the notation for nondimensional variables later.

We also assume that the bottom of the fluid is horizontal and located at $z' = 0$. Now, if η' is the displacement of the surface from its undisturbed depth h, then the free surface is described by

$$z' = h + \eta'.$$

Let u', v' and w' be the x', y' and z' velocity components of the fluid. Then the vertical velocity of the fluid at the surface can be written as

$$w' = \frac{\partial \eta'}{\partial t'} + u' \frac{\partial \eta'}{\partial x'} + v' \frac{\partial \eta'}{\partial y'} \text{ on } z' = h + \eta'. \qquad (2.1.2)$$

The momentum equation or the pressure condition at the surface is given by Bernoulli's equation

$$\rho \frac{\partial\phi'}{\partial t'} + \frac{\rho}{2}(u'^2+v'^2+w'^2) + p' + \rho gz' = p_0 + pgh$$

on $z' = h + \eta'$. Here p_0 is the atmospheric pressure above the surface, p' the pressure in the fluid at the surface and ρ the density of the fluid. Therefore we must have $p' = p_0$ and hence

$$\frac{\partial\phi'}{\partial t'} + \frac{1}{2}(u'^2 + v'^2 + w'^2) + g\eta' = 0 \text{ on } z' = h + \eta'. \qquad (2.1.3)$$

The final boundary condition is given by the vertical velocity at the bottom

$$w' = 0 \text{ on } z' = 0. \qquad (2.1.4)$$

Our problem here is to solve Laplace's equation (2.1.1) subject to the boundary conditions (2.1.2) - (2.1.4).

We now consider a wave of amplitude a, propogating dominantly in the positive x' direction. If the wavelength in the x' direction is λ and in the y' direction is μ, we then require $\lambda << \mu$. Let us set $\delta = {}^h/_\lambda$. Then for shallow water waves $\delta << 1$ and for deep water waves $\delta >> 1$. Only the case of shallow water waves will be discussed here and so δ small will be the basis of the approximation procedure.

The problem is specified by suitably nondimensionalizing the variables. We define

$$x = \frac{x'}{\lambda}, \quad y = \frac{y'}{\mu}, \; z = \frac{z'}{h}, \; t = \frac{ct'}{\lambda}$$
$$\phi = \frac{\phi'}{\lambda c}, \quad \eta = \frac{\eta'}{h} \qquad (2.1.5)$$

which then implies

$$u = \frac{u'}{c}, \; v = \frac{v'}{\theta c}, \quad w = \frac{\delta w'}{c}$$

where $\theta = \frac{\lambda}{\mu}$ and c is a constant which has the dimension of speed. Indeed c can be chosen upon substituting the above nondimensional variables into the pressure condition (2.1.3). It turns out that the right choice is $c=\sqrt{gh}$.

In terms of these variables the equation of motion and the boundary conditions (2.1.1) - (2.1.4) become

$$\delta^2\left(\frac{\partial^2\phi}{\partial x^2} + \theta^2\frac{\partial^2\phi}{\partial y^2}\right) + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{2.1.7}$$

$$w = \delta^2\left(\frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} + \theta^2 v\frac{\partial\eta}{\partial y}\right) \text{ on } z=1+\eta \tag{2.1.8}$$

$$\delta^2\frac{\partial\phi}{\partial t} + \frac{\delta^2}{2}(u^2+\theta^2v^2) + \frac{w^2}{2} + \delta^2\eta = 0 \text{ on } z=1+\eta \tag{2.1.9}$$

$$w = 0 \text{ on } z = 0. \tag{2.1.10}$$

For small amplitude long waves, we encounter another small parameter $\epsilon = {}^a/_h \ll 1$. We shall make use of the Korteweg-de Vries's approximation $\delta^2 = \epsilon$ which is equivalent to taking $a = h^3/\lambda^2$. In order to achieve a distinguished limit which includes the two dimensionality of the surface waves in the simplest form we set $\theta^2 = \alpha\epsilon$ for some constant α. (For $\alpha = 0$, the analysis would lead to the KdV equation .)

In order to solve the above equations for ϕ and η we first expand them in powers of ϵ,

$$\phi = \epsilon[\phi_0 + \epsilon\phi_1 + \ldots.]$$

$$\eta = \epsilon[N_0 + \epsilon N_1 + \ldots.]$$

and then substitute these into (2.1.7) and (2.1.10) to find

$$\frac{\partial^2\phi_0}{\partial z^2} = 0 \text{ with } \frac{\partial\phi_0}{\partial z} = 0 \text{ on } z = 0$$

$$\frac{\partial^2\phi_1}{\partial z^2} = -\frac{\partial^2\phi_0}{\partial z^2} \text{ with } \frac{\partial\phi_1}{\partial z} = 0 \text{ on } z = 0$$

$$\frac{\partial^2\phi_2}{\partial z^2} = -\frac{\partial^2\phi_1}{\partial x^2} - \alpha\frac{\partial^2\phi_0}{\partial y^2} \text{ with } \frac{\partial\phi_2}{\partial z} = 0 \text{ on } z = 0.$$

The solutions of the above differential equations are then

$$\phi_0 = f_0(x,y,t)$$

$$\phi_1 = f_1(x,y,t) - \frac{z^2}{2}\frac{\partial^2 f_0}{\partial x^2}$$

$$\phi_2 = f_2(x,y,t) - \frac{z^2}{2}\frac{\partial^2 f_1}{\partial x^2} - \frac{\alpha z^2}{2}\frac{\partial^2 f_0}{\partial x^2} + \frac{z^4}{24}\frac{\partial^4 f_0}{\partial x^4}.$$

Inserting these solutions into the velocity condition at the surface (2.1.8) gives to the lowest order

$$\frac{\partial N_0}{\partial t} = - \frac{\partial^2 f_0}{\partial x^2} \tag{2.1.11}$$

and into the pressure condition (2.1.9) gives

$$N_0 = - \frac{\partial f_0}{\partial t} . \tag{2.1.12}$$

Combining the last two equations, we find that N_0 satisfies the linear wave equation

$$\frac{\partial^2 N_0}{\partial t^2} = \frac{\partial^2 N_0}{\partial x^2}.$$

This equation has the solution $N_0 = N_0(x-t,y)$ for waves travelling in the positive x-direction only with speed $\sqrt{gh}$ in the physical coordinate.

The nonlinearity only comes into our calculation in the next order approximation. Carrying out the same calculation as before for the next order we find

$$\frac{\partial^2 N_1}{\partial t^2} - \frac{\partial^2 N_1}{\partial x^2} = G(x-t,y)$$

where G is a function of x-t arising from N_0. Since G itself is a solution of the wave equation, the resonance effects mean that N_1 contains a term like $t\,G(x-t,y)$ which grows much faster than N_0 as $t \to \infty$. Therefore the asymptotic expansion for η is no longer uniformly valid for large t. We then need to seek for a uniformly valid solution in the far field when $t=O(1/\epsilon)$ and $x-t = O(1)$ and write the far field variables

$$\tau = \epsilon t,\ \xi = x-t,\ \phi \to \epsilon\phi,\ u \to \epsilon u,\ v \to \epsilon v \text{ and } w \to \epsilon^2 w.$$

The t and x derivatives are now

$$\frac{\partial}{\partial t} = \epsilon \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \xi}\ ,\ \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}.$$

Now, in terms of the far field variables, equations (2.1.7) - (2.1.10) become

$$\epsilon \frac{\partial^2\phi}{\partial\xi^2} + \alpha\epsilon^2 \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad (2.1.13)$$

$$w = \epsilon \frac{\partial\eta}{\partial\tau} - \frac{\partial\eta}{\partial\xi} + \epsilon u \frac{\partial\eta}{\partial\xi} + \alpha\epsilon^2 v \frac{\partial\eta}{\partial y} \text{ on } z=1+\epsilon\eta \qquad (2.1.14)$$

$$\epsilon\frac{\partial\phi}{\partial\tau} - \frac{\partial\phi}{\partial\xi} + \frac{\epsilon}{2}[u^2+\alpha\epsilon v^2+\epsilon w^2] + \eta = 0 \text{ on } z=1+\epsilon\eta \qquad (2.1.15)$$

$$w = 0 \text{ on } z = 0. \qquad (2.1.16)$$

We now set

$$\phi = \Phi_0 + \epsilon\,\Phi_1 + \epsilon^2\,\Phi_2 + \ldots$$

$$\eta = \eta_0 + \epsilon\,\eta_1 + \epsilon^2\,\eta_2 + \ldots$$

and as before (2.1.13) and (2.1.16) give

$$\Phi_0 = g_0(\xi,\tau,y)$$

$$\Phi_1 = g_1(\xi,\tau,y) - \frac{z^2}{2}\frac{\partial^2 g_0}{\partial\xi^2}$$

$$\Phi_2 = g_2(\xi,\tau,y) - \frac{z^2}{2}\frac{\partial^2 g_1}{\partial\xi^2} - \frac{\alpha z^2}{2}\frac{\partial^2 g_0}{\partial y^2} + \frac{z^4}{24}\frac{\partial^4 g_0}{\partial\xi^4}.$$

From these we can deduce that

$$u = \frac{\partial g_0}{\partial\xi} + \epsilon\left(\frac{\partial g_1}{\partial\xi} - \frac{z^2}{2}\frac{\partial^3 g_0}{\delta\xi^3}\right) + \ldots$$

$$v = \frac{\partial g_0}{\partial y} + \epsilon\left(\frac{\partial g_1}{\partial y} - \frac{z^2}{2}\frac{\partial^3 g_0}{\partial y\partial\xi^2}\right) + \ldots$$

$$w = -z\frac{\partial^2 g_0}{\partial\xi^2} - \epsilon\left(z\frac{\partial^2 g_1}{\partial\xi^2} + \alpha z\frac{\partial^2 g_0}{\partial y^2} - \frac{z^3}{6}\frac{\partial^4 g_0}{\partial\xi^4}\right) + \ldots$$

Using the above results in the velocity condition (2.1.14) we write

$$-(1+\epsilon\eta_0+\ldots)\frac{\partial^2 g_0}{\partial\xi^2} - \epsilon[\frac{\partial^2 g_1}{\partial\xi^2} + \alpha\frac{\partial^2 g_0}{\partial y^2} - \frac{1}{6}\frac{\partial^4 g_0}{\partial\xi^4}]$$
$$= \epsilon\frac{\partial\eta_0}{\delta\tau} - \frac{\partial\eta_0}{\partial\xi} - \epsilon\frac{\partial\eta_1}{\partial\xi} + \epsilon\frac{\partial g_0}{\partial\xi}\frac{\partial\eta_0}{\partial\xi}$$

by keeping only terms of O(1) and O(ϵ).

At each order the last equation gives

$$\frac{\partial^2 g_0}{\partial\xi^2} = \frac{\partial\eta_0}{\partial\xi} \qquad (2.1.17)$$

$$-\frac{\partial^2 g_1}{\partial\xi^2} + \frac{\partial\eta_1}{\partial\xi} = \frac{\partial\eta_0}{\partial\tau} + \eta_0\frac{\partial^2 g_0}{\partial\xi^2} + \frac{\partial g_0}{\partial\xi}\frac{\partial\eta_0}{\partial\xi} - \frac{1}{6}\frac{\partial^4 g_0}{\partial\xi^4} + \alpha\frac{\partial^2 g_0}{\partial y^2}. \qquad (2.1.18)$$

Similarly, from the pressure condition (2.1.15) we can write

$$\epsilon\frac{\partial g_0}{\partial\tau} - \frac{\partial g_0}{\partial\xi} - \epsilon[\frac{\partial g_1}{\partial\xi} - \frac{1}{2}\frac{\partial^3 g_0}{\partial\xi^3}] + \frac{\epsilon}{2}(\frac{\partial g_0}{\partial\xi})^2 + \eta_0 + \epsilon\eta_1 = 0$$

which then gives

$$\frac{\partial g_0}{\partial\xi} = \eta_0 \qquad (2.1.19)$$

$$-\frac{\partial g_1}{\partial\xi} + \eta_1 = -\frac{\partial g_0}{\partial\tau} - \frac{1}{2}\frac{\partial^3 g_0}{\partial\xi^3} - \frac{1}{2}(\frac{\partial g_0}{\partial\xi})^2 . \qquad (2.1.20)$$

It is worth noting here that (2.1.19) is compatible with (2.1.17) showing that we are so far still on the right track. The first order approximation thus relates η_0 and g_0 but does not specify either at this level. An equation for η_0 will thus be obtained at the next order approximation.

Using (2.1.19) into (2.1.18) and (2.1.20) we write

$$-\frac{\partial^2 g_1}{\partial\xi^2} + \frac{\partial\eta_1}{\partial\xi} = \frac{\partial\eta_0}{\partial\tau} + 2\eta_0\frac{\partial\eta_0}{\partial\xi} - \frac{1}{6}\frac{\partial^3\eta_0}{\partial\xi^3} + \alpha\frac{\partial^2 g_0}{\partial y^2}$$

and

$$-\frac{\partial^2 g_1}{\partial\xi^2} + \frac{\partial\eta_1}{\partial\xi} = -\frac{\partial\eta_0}{\partial\tau} - \frac{1}{2}\frac{\partial^3\eta_0}{\partial\xi^3} - \eta_0\frac{\partial\eta_0}{\partial\xi}.$$

When the last equation is subtracted from the first one we find

$$2\,\frac{\partial\eta_0}{\partial\tau} + 3\eta_0\,\frac{\partial\eta_0}{\partial\xi} + \frac{1}{3}\,\frac{\partial^3\eta_0}{\partial\xi^3} + \alpha\,\frac{\partial^2 g_0}{\partial y^2} = 0,$$

and upon differentiating this equation with respect to ξ and making use of (2.1.19) we finally arrive at the desired Kadomtsev - Petviashvili equation

$$\frac{\partial}{\partial\xi}\Big[2\,\frac{\partial\eta_0}{\partial\tau} + 3\eta_0\,\frac{\partial\eta_0}{\partial\xi} + \frac{1}{3}\,\frac{\partial^3\eta_0}{\partial\xi^3}\Big] + \alpha\,\frac{\partial^2\eta_0}{\partial y^2} = 0. \qquad (2.1.21)$$

As mentioned earlier, when $\alpha=0$, this equation reduces to the KdV equation

$$2\,\frac{\partial\eta_0}{\partial\tau} + 3\eta_0\,\frac{\partial\eta_0}{\partial\xi} + \frac{1}{3}\,\frac{\partial^3\eta_0}{\partial\xi^3} = 0.$$

2.2 Derivation of the finite depth fluid equation

Let us now consider one dimensional waves on the interface (y=0) between two fluids of different densities which are confined between two boundaries, one at the top (y=H) and the other at the bottom (y=-h). This model is very similar to the existence of internal waves at a thermocline of sea water if the thickness of the thermocline tends to zero. Such internal waves have been discussed by Phillips (1966), and Joseph (1977). A derivation of an equation describing such waves is given in Kubota et al (1978).

For our model we let the densities of the fluids above and below the interface be R and ρ respectively, with $\rho > R$ for stability. Furthermore we also require the fluids to be inviscid, incompressible and irrotational so that the equations of motion of the two fluids can be written as

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \quad -h < y < 0 \qquad (2.2.1)$$

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0, \quad 0 < y < H \qquad (2.2.2)$$

where Φ and ϕ denote the velocity potentials of the fluids above and below the interface. The schematic diagram for our model is shown in Fig. 2.1.

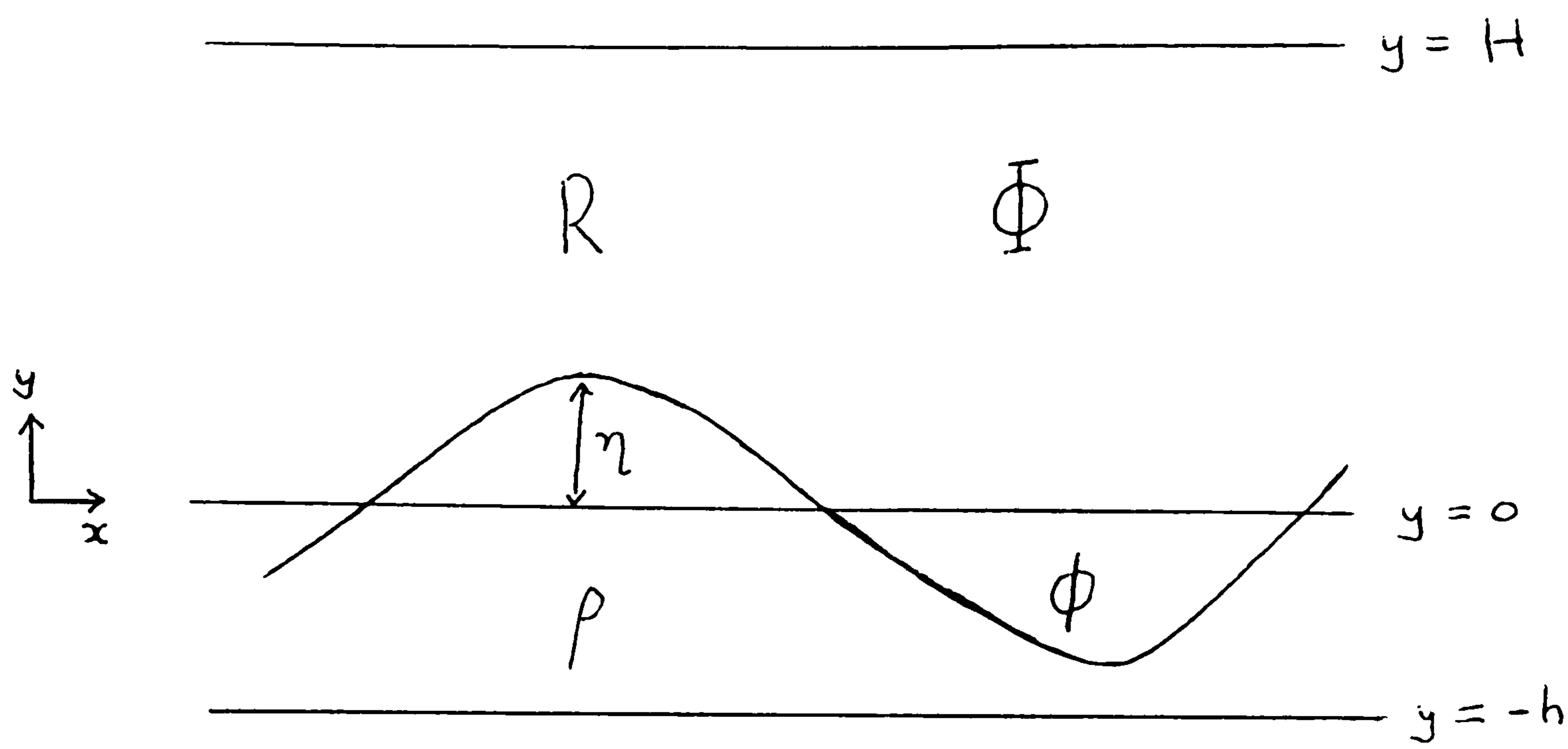


Fig. 2.1 The model for waves at an interface.

The boundary conditions at y=H and y=-h can now be specified;

$$v = \frac{\partial\phi}{\partial y} = 0 \text{ on } y=-h \qquad (2.2.3)$$

$$V = \frac{\partial\Phi}{\partial y} = 0 \text{ on } y=H \qquad (2.2.4)$$

where v, V are the vertical velocities of the lower and the top fluids. Due to some disturbance, the interface is displaced to some distance from the undisturbed position y=0. Let the interface at any time be described by $y=\eta$. We therefore have as before

$$v = \frac{\partial\eta}{\partial t} + u\,\frac{\partial\eta}{\partial x} \text{ on } y = \eta \qquad (2.2.5)$$

$$V = \frac{\partial\eta}{\partial t} + U\,\frac{\partial\eta}{\partial x} \text{ on } y = \eta \qquad (2.2.6)$$

where u, U are the usual notation for the horizontal velocities of the lower and top layer fluids. The final boundary condition is given by the continuity of pressure at the interface, and it can be written as

$$\rho\{\frac{\partial\phi}{\partial t} + \frac{1}{2}(u^2+v^2) + g\eta\} = R\{\frac{\partial\Phi}{\partial t} + \frac{1}{2}(U^2 + V^2) + g\eta\} \text{ on } y=\eta. \quad (2.2.7)$$

We first look at the linear problem by neglecting all nonlinear terms in (2.2.5) - (2.2.7). The solutions of (2.2.1) and (2.2.2) together with (2.2.3) and (2.2.4) are

$$\Phi = Ae^{i(kx-wt)} \cosh k(y-H)$$

$$\phi = Be^{i(kx-wt)} \cosh k(y+h).$$

From (2.2.5) and (2.2.6) we deduce that

$$\frac{\partial\phi}{\partial y} = \frac{\partial\Phi}{\partial y} \text{ on } y = 0$$

whence we find

$$\Phi = Ce^{i(kx-wt)}\frac{\cosh k(y-H)}{\sinh kH}$$

$$\phi = -Ce^{i(kx-wt)}\frac{\cosh k(y+h)}{\sinh kh} \tag{2.2.8}$$

where C is an arbitary constant.

Neglecting the nonlinear term in either (2.2.5) or (2.2.6) and making use of (2.2.8) we have

$$\eta = \frac{Cke^{i(kx-wt)}}{iw} . \tag{2.2.9}$$

The linear counterpart associated with (2.2.7) now reads

$$\rho \frac{\partial\phi}{\partial t} - R\frac{\partial\Phi}{\partial t} = - (\rho-R)g\eta \text{ on } y=0. \tag{2.2.10}$$

Upon using (2.2.8) and (2.2.9) in this equation we find

$$c^2 = (\frac{w}{k})^2 = \frac{g(\rho-R)}{k[\rho\coth kh + R \coth kH]}$$

$$\approx \frac{gh(1 - \frac{R}{\rho})}{1 + \frac{Rkh}{\rho} \coth kH} \text{ for } kh \text{ small.}$$

Hence we find the dispersion relation

$$c \approx \sqrt{gh(1 - \frac{R}{\rho})} \; [1 - \frac{Rkh}{2\rho} \coth kH]$$

$$\approx \sqrt{gh(1 - \frac{R}{\rho})} \; [1 - \frac{Rh}{2\rho H} - \frac{Rkh}{2\rho}(\coth kH - \frac{1}{kH})] \tag{2.2.11}$$

Therefore the linear wave is propagating at the speed c given by (2.2.11).

For the nonlinear problem, we first nondimensionalize the variables as we did earlier for the Kadomtsev - Petviashvili equation. However in this problem we have essentially two fluids and therefore it is reasonable to use two different scalings one in each fluid. We have some freedom in choosing the scaling factors but the following choice turns out to be the correct one to obtain the simplest interaction between the two fluids and in accommodating the linear dispersion relation (2.2.11). We put

$$x_1 = \frac{x}{H} = X_1, \quad y_1 = \frac{y}{h}, \quad v_1 = \frac{v}{C'\delta\epsilon}, \quad \phi_1 = \frac{\phi}{C'\epsilon H}$$
$$t_1 = \frac{C't}{H}, \quad \eta_1 = \frac{\eta}{\epsilon h}, \quad u_1 = \frac{u}{\epsilon C'} \tag{2.2.12}$$

for the fluid in the lower layer, and

$$X_1 = \frac{x}{H}, \quad Y_1 = \frac{y}{H}, \quad V_1 = \frac{V}{C'\delta\epsilon}, \quad U_1 = \frac{U}{C'\delta\epsilon}, \quad \Phi_1 = \frac{\Phi}{C'\epsilon h} \tag{2.2.13}$$

for the fluid in the upper layer.

In the above scalings, ϵ and δ are small parameters and C′ has the speed dimension. For the consistency of the equations, we choose $\delta = {}^{h}/_{H}$ and that δ can be made small as we want h to be much smaller than H. We however still have the freedom in choosing ϵ. The normal choice is $\epsilon = {}^{a}/_{h}$ where a is the wave amplitude.

In terms of the nondimensional variables, equations (2.2.1) - (2.2.7) become

$$\delta^2 \frac{\partial^2\phi_1}{\partial x_1^{\;2}} + \frac{\partial^2\phi_1}{\partial y_1^{\;2}} = 0, \quad -l<y_1<0 \tag{2.2.14}$$

$$\frac{\partial^2\Phi_1}{\partial X_1^{\;2}} + \frac{\partial^2\Phi_1}{\partial Y_1^{\;2}} = 0, \quad 0<Y_1<1 \tag{2.2.15}$$

with

$$v_1 = 0 \text{ on } y_1 = -1, \quad V_1 = 0 \text{ on } Y_1 = 1 \tag{2.2.16}$$

and

$$v_1 = \frac{\partial\eta_1}{\partial t_1} + \epsilon u_1 \frac{\partial\eta_1}{\partial x_1} \text{ on } y_1 = \epsilon\eta_1 \tag{2.2.17}$$

$$V_1 = \frac{\partial\eta_1}{\partial t_1} + \epsilon U_1 \frac{\partial\eta_1}{\partial X_1} \text{ on } Y_1 = \epsilon\delta\eta_1 . \tag{2.2.18}$$

If we substitute the dimensionless variables into the pressure condition (2.2.7) we find that C′ needs to take the following form

$$C' = \sqrt{gh\left(1 - \frac{R}{\rho}\right)}, \quad \frac{R}{\rho}<1 \tag{2.2.19}$$

and with this choice (2.2.7) now reads

$$\frac{\partial\phi_1}{\partial t_1} + \frac{\epsilon}{2}(u_1{}^2+\delta^2 v_1{}^2) + \eta_1 = \frac{R}{\rho}[\delta(\frac{\partial\Phi_1}{\partial t_1}) + \frac{\epsilon}{2}(U_1{}^2+V_1{}^2)]$$

$$\text{on } y_1 = \epsilon\eta_1 \text{ and } Y_1 = \epsilon\delta\eta_1. \qquad (2.2.20)$$

In a different manner from the KP equation, we now choose $\delta = K\epsilon$ for this problem. Then (2.2.14) suggests that ϕ_1 should be expanded in the powers of δ^2. Solving (2.2.14) at each order by using the boundary condition (2.2.16), we find

$$\phi_1 = f(x_1,t_1) + \delta^2[F(x_1,t_1) - (y_1+\frac{y_1{}^2}{2})\frac{\partial^2 f}{\partial x_1{}^2}] + O(\epsilon^4) \cdot \qquad (2.2.21)$$

Since $v = \frac{\partial\phi}{\partial y}$ we find

$$v_1 = \frac{1}{\delta^2}\frac{\partial\phi_1}{\partial y_1} = -(1+y_1)\frac{\partial^2 f}{\partial x_1{}^2} + O(\epsilon^2).$$

Thus to the first order,

$$u_1 = \frac{\partial\phi_1}{\partial x_1} = \frac{\partial f}{\partial x_1}$$

$$\text{on } y_1 = 0.$$

$$v_1 = \frac{\partial\eta_1}{\partial t_1} = -\frac{\partial^2 f}{\partial x_1{}^2}$$

Also, taking only the first order terms from (2.2.20) we obtain

$$\frac{\partial f}{\partial t_1} + \eta_1 = 0 \text{ on } y_1 = 0.$$

Thus we find the linear wave equation

$$\frac{\partial^2\eta_1}{\partial t_1{}^2} = \frac{\partial^2\eta_1}{\partial x_1{}^2} \text{ on } y_1 = 0 \qquad (2.2.22)$$

as one would expect.

For the upper layer, at this order we find

$$\frac{\partial^2\Phi_1}{\partial X_1{}^2} + \frac{\partial^2\Phi_1}{\partial Y_1{}^2} = 0,\ 0<Y_1<1$$

with

$$V_1 = 0 \text{ on } Y_1 = 1,$$

and
$$V_1 = -\frac{\partial^2 f}{\partial x_1^2} \text{ on } Y_1 = 0. \qquad (2.2.23)$$

As in the case of the KP equation, calculation at the next order leads to a nonuniformly valid expansion when $t_1 = O(\epsilon^{-1})$ and $x_1 - t_1 = O(1)$. We then seek for a solution in the far field and set

$$\tau = \epsilon t_1 \text{ and } \xi = x_1 - t_1$$

as before, so that

$$\frac{\partial}{\partial t_1} = -\frac{\partial}{\partial \xi} + \epsilon \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial x_1} = \frac{\partial}{\partial \xi}.$$

We now let

$$\phi_1 = \phi_{11} + \epsilon\phi_{12} + \ldots$$
$$\Phi_1 = \Phi_{11} + \epsilon\Phi_{12} + \ldots$$
$$u_1 = u_{11} + \epsilon u_{12} + \ldots$$
$$v_1 = v_{11} + \epsilon v_{12} + \ldots$$
$$V_1 = V_{11} + \epsilon V_{12} + \ldots$$
$$\eta_1 = \eta_{11} + \epsilon\eta_{12} + \ldots$$

The Laplace's equation for ϕ_1 does not change in the far field and we therefore simply take the solution (2.2.21) but now, in order to suit the above expansion, f is expanded in powers of ϵ,

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \ldots$$

whence

$$\phi_{11} = f_0, \quad \phi_{12} = f_1, \ldots \qquad (2.2.24)$$

Now

$$v_{11} + \epsilon v_{12} = \frac{1}{\partial^2}\frac{\partial\phi_1}{\partial y_1}$$
$$= -(1+y_1)\left[\frac{\partial^2 f_0}{\partial \xi^2} + \epsilon\frac{\partial^2 f_1}{\partial \xi^2}\right] + \ldots .$$

Thus on $y_1 = \epsilon\eta_{11} + \epsilon^2\eta_{12}$ we have

$$v_{11} = -\frac{\partial^2 f_0}{\partial \xi^2}, \quad v_{12} = -\eta_{11}\frac{\partial^2 f_0}{\partial \xi^2} - \frac{\partial^2 f_1}{\partial \xi^2}. \qquad (2.2.25)$$

The velocity condition (2.2.17) on $y_1 = \epsilon\eta_{11} + \epsilon^2\eta_{12}$ now reads

$$v_{11} + \epsilon v_{12} + \dots = - \frac{\partial\eta_{11}}{\partial\xi} + \epsilon \frac{\partial\eta_{11}}{\partial\tau} - \epsilon \frac{\partial\eta_{12}}{\partial\xi} + \epsilon u_{11} \frac{\partial\eta_{11}}{\partial\xi} + \dots$$

which implies

$$v_{11} = - \frac{\partial\eta_{11}}{\partial\xi}$$

$$v_{12} + \frac{\partial\eta_{12}}{\partial\xi} = \frac{\partial\eta_{11}}{\partial\tau} + u_{11} \frac{\partial\eta_{11}}{\partial\xi} . \tag{2.2.26}$$

Substituting (2.2.25) into (2.2.26) and also by making use of the relation

$$u_{11} = \frac{\partial\phi_{11}}{\partial\xi} = \frac{\partial f_0}{\partial\xi}$$

we find

$$\frac{\partial^2 f_0}{\partial\xi^2} = \frac{\partial\eta_{11}}{\partial\xi} \tag{2.2.27}$$

$$- \frac{\partial^2 f_1}{\partial\xi^2} + \frac{\partial\eta_{12}}{\partial\xi} = \frac{\partial\eta_{11}}{\partial\tau} + \frac{\partial f_0}{\partial\xi}\frac{\partial\eta_{11}}{\partial\xi} + \eta_{11} \frac{\partial^2 f_0}{\partial\xi^2} . \tag{2.2.28}$$

The pressure condition (2.2.20) now gives

$$- \frac{\partial\phi_{11}}{\partial\xi} + \epsilon \frac{\partial\phi_{11}}{\partial\tau} - \epsilon \frac{\partial\phi_{12}}{\partial\xi} + \frac{\epsilon}{2} \left(\frac{\partial\phi_{11}}{\partial\xi}\right)^2 + \eta_{11} + \epsilon\eta_{12} = - \frac{RK\epsilon}{\rho} \frac{\partial\Phi_{11}}{\partial\xi}$$

and hence at each order we obtain

$$\frac{\partial\phi_{11}}{\partial\xi} = \eta_{11} \text{ or } \frac{\partial f_0}{\partial\xi} = \eta_{11} \tag{2.2.29}$$

and

$$- \frac{\partial\phi_{12}}{\partial\xi} + \eta_{12} = - \frac{\partial\phi_{11}}{\partial\tau} - \frac{1}{2} \left(\frac{\partial\phi_{11}}{\partial\xi}\right)^2 - \frac{RK}{\rho} \frac{\partial\Phi_{11}}{\partial\xi}$$

or equivalently

$$- \frac{\partial f_1}{\partial\xi} + \eta_{12} = - \frac{\partial f_0}{\partial\tau} - \frac{1}{2} \left(\frac{\partial f_0}{\partial\xi}\right)^2 - \frac{RK}{\rho} \frac{\partial\Phi_{11}}{\partial\xi}. \tag{2.2.30}$$

Again we see here that (2.2.29) is compatible with (2.2.27) and relations between f_0, η_{11} and ϕ_{11} exist but none of them are specified. Now, differentiating (2.2.30) with respect to ξ and making use of (2.2.29) we find

$$- \frac{\partial^2 f_1}{\partial\xi^2} + \frac{\partial\eta_{12}}{\partial\xi} = - \frac{\partial\eta_{11}}{\partial\tau} - \eta_{11} \frac{\partial\eta_{11}}{\partial\xi} - \frac{RK}{\rho} \frac{\partial^2\Phi_{11}}{\partial\xi^2}. \tag{2.2.31}$$

This equation when subtracted from (2.2.28) gives

$$2\,\frac{\partial \eta_{11}}{\partial \tau} + 3\eta_{11}\,\frac{\partial \eta_{11}}{\partial \xi} + \frac{RK}{\rho}\,\frac{\partial^2 \Phi_{11}}{\partial \xi^2} = 0. \qquad (2.2.32)$$

The final task is thus to determine Φ_{11} when evaluated at $Y_1 = 0$. From the Laplace's equation for Φ_1 and the boundary condition (2.2.23) we may obtain Φ_{11} by solving the problem

$$\frac{\partial^2 \Phi_{11}}{\partial \xi^2} + \frac{\partial^2 \Phi_{11}}{\partial {Y_1}^2} = 0 \qquad (2.2.33)$$

with

$$\frac{\partial \Phi_{11}}{\partial Y_1} = 0 \text{ on } Y_1 = 1 \qquad (2.2.34)$$

$$\frac{\partial \Phi_{11}}{\partial Y_1} = -\,\frac{\partial \eta_{11}}{\partial \xi} \text{ on } Y_1 = 0. \qquad (2.2.35)$$

If we define $\mathcal{F}(k,Y_1)$ as the Fourier transform of $\Phi_{11}(\xi,Y_1)$ by

$$\mathcal{F}(k,Y_1) = \int_{-\infty}^{\infty} \Phi_{11}(\xi,Y_1)e^{-ik\xi}d\xi$$

then the inverse is written as

$$\Phi_{11}(\xi,Y_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(k,Y_1)e^{ik\xi}dk.$$

In terms of $\mathcal{F}$, (2.2.33) - (2.2.35) are

$$\frac{\partial^2 \mathcal{F}}{\partial {Y_1}^2} - k^2\mathcal{F} = 0$$

with

$$\frac{\partial \mathcal{F}}{\partial Y_1} = 0 \text{ on } Y_1 = 1$$

$$\frac{\partial \mathcal{F}}{\partial Y_1} = -\int_{-\infty}^{\infty} \frac{\partial \eta_{11}}{\partial \xi}\, e^{-ik\xi}d\xi$$

$$= -\, ik \int_{-\infty}^{\infty} \eta_{11}e^{-ik\xi}d\xi \text{ on } Y_1 = 0$$

as we want η_{11}, $\frac{\partial \eta_{11}}{\partial \xi} \to 0$ as $\xi \to \pm\,\infty$.

The solution of the above problem is now

$$\mathcal{F}(k,Y_1) = \frac{i \cosh\ k(1-Y)}{\sinh\ k} \int_{-\infty}^{\infty} \eta_{11}(\xi,\tau)e^{-ik\xi}d\xi$$

and thus we have

$$\Phi_{11}(\xi,0) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \{\coth\ k \int_{-\infty}^{\infty} \eta_{11}(\theta_1,\tau)e^{-ik\theta_1}d\theta_1\}e^{ik\xi}dk. \quad (2.2.36)$$

The integral can be simplified by using the convolution theorem and we need to find the inverse of coth k. However coth z has a pole at z = 0. To remove this pole we consider the function $\coth z - \frac{1}{z}$ and we find that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (\coth\ k - \frac{1}{k})e^{ik\theta_2}dk = \frac{i}{2}[\coth \frac{\pi\theta_2}{2} - \text{sgn}\ \theta_2]. \qquad (2.2.37)$$

Thus we can write

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \{(\coth\ k - \frac{1}{k}) \int_{-\infty}^{\infty} \eta_{11}(\theta_1,\tau)e^{-ik\theta_1}d\theta_1\}e^{ik\xi}dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \{\frac{i}{2}\int_{-\infty}^{\infty} (\coth \frac{\pi\theta_2}{2} - \text{sgn}\ \theta_2)e^{-ik\theta_2}d\theta_2\int_{-\infty}^{\infty} \eta_{11}(\theta_1,\tau)e^{-ik\theta_1}d\theta_1\}e^{ik\xi}dk$$

$$= \frac{i}{2}\int_{-\infty}^{\infty} [\coth \frac{\pi}{2}(\xi-\xi') - \text{sgn}\ (\xi-\xi')]\ \eta_{11}(\xi',\tau)d\xi'. \qquad (2.2.38)$$

Now, by using (2.2.38), (2.2.36) and the relation

$$\frac{\partial^2}{\partial\xi^2}\int_{-\infty}^{\infty} \text{sgn}\ (\xi-\xi')\ \eta_{11}(\xi',\tau)d\xi' = 2\ \frac{\partial\eta_{11}}{\delta\xi}$$

into (2.2.32) we find

$$2\ \frac{\partial\eta_{11}}{\partial\tau} - \frac{RK}{\rho}\ \frac{\partial\eta_{11}}{\partial\xi} + 3\eta_{11}\frac{\partial\eta_{11}}{\delta\xi} - \frac{RK}{2\rho}\ \frac{\partial^2}{\partial\xi^2} \int_{-\infty}^{\infty} [\coth \frac{\pi}{2}(\xi-\xi')-\text{sgn}\ (\xi-\xi')]\eta_{11}(\xi',\tau).d'\xi = 0.$$

This equation can be written as

$$2\ \frac{\partial\eta_{11}}{\partial T} + 3\eta_{11}\ \frac{\partial\eta_{11}}{\partial X} + \frac{RK}{2\rho}\ \frac{\partial^2}{\partial X^2} \int_{-\infty}^{\infty} [\coth \frac{\pi}{2}(X'-X)-\text{sgn}(X'-X)].\eta_{11}(X',T)\ dX' = 0$$

$$(2.2.39)$$

upon putting

$$X = \xi + \frac{RK}{2\rho}\ T$$

$$T = t.$$

Indeed (2.2.39) is the equation we have been looking for. It describes the wave propagation at the interface located at the depth equals unity below the top boundary. It is normally known as the finite depth fluid equation.

The X-coordinate in this equation is a moving coordinate where

$$X = \xi + \frac{RKT}{2\rho}$$

$$= x_1 - t_1 + \frac{RK\epsilon t_1}{2\rho}$$

$$= \frac{1}{H}\{x - (1 - \frac{RK\epsilon}{2\rho})\ C' t\}$$

$$= \frac{1}{H}\{x - \sqrt{gh\ (1 - \frac{R}{\rho})}\ (1 - \frac{Rh}{2\rho H})\ t\ \}.$$

This means that if we observe a stationary wave in the X coordinate, it is actually propagating with a speed of $\sqrt{gh\ (1 - \frac{R}{\rho})}\ (1 - \frac{Rh}{2\rho H})$ with respect to the physical coordinate. This is consistent with the linear wave speed (2.2.11).

CHAPTER 3

THE FINITE DEPTH FLUID EQUATION

3.1 Some preliminaries

We shall consider the finite depth fluid (FDF) equation in the form

$$u_t + 2uu_x + G[u_{xx}] = 0 \qquad (3.1.1)$$

where G is the integro-differential operator defined by

$$G[f(x)] = \frac{\lambda}{2} P\int_{-\infty}^{\infty} \{\coth[\frac{\pi\lambda}{2}(x'-x)]-\mathrm{sgn}(x'-x)\} f(x')dx'. \qquad (3.1.2)$$

The parameter λ^{-1} (>0) represents the distance between the bottom of the fluid and the internal wave layer and P denotes the principal value of the integral. Equation (3.1.1) is due to Matsuno (1979b, 1984). A similar equation to (3.1.1) has been derived by Kubota et al (1978) for internal waves propagating on a pycnocline between two fluids of different densities.

The FDF equation has been studied by a number of researchers. The existence of a steady state solitary wave (single-soliton) solution was obtained and examined analytically by Joseph (1977) and its numerical solution by Kubota et al (1978). Joseph's single soliton solution is written as

$$u = u(x-ct)$$
$$= \frac{\lambda\ \gamma\sin\gamma}{\{\cosh[\lambda\gamma(x-ct)] + \cos\gamma\}} \qquad (3.1.3)$$

where $c = \lambda(1 - \gamma\cot\gamma)$, γ is an arbitary real parameter with $0<\gamma<\pi$. Joseph and Egri (1978) then extended this result to an N-soliton solution. Satsuma et al (1979) gave a Bäcklund transformation, a recursion scheme for conservation laws and an inverse scattering scheme for the equation. However the most suitable work for our purpose is that of Matsuno (1979b, 1984), since the approach used is that of Hirota's direct method.

Most of the previous work shows that equation (3.1.1) and its soliton solution are reducible to the Benjamin-Ono and the KdV equations and to their respective soliton solutions under certain limiting conditions.

Indeed, in the deep water limit $\lambda \to 0$, the equation reduces to the Benjamin-Ono equation

$$u_t + 2uu_x + \frac{1}{\pi} \frac{\partial^2}{\partial x^2} P\int_{-\infty}^{\infty} \frac{u(x',t)}{x'-x} dx' = 0. \qquad (3.1.4)$$

In the shallow water limit $\lambda \to \infty$, and with the variable transformations

$$X = \lambda^{\frac{1}{2}} x, \ T = \lambda^{\frac{1}{2}} t \qquad (3.1.5)$$

it becomes the KdV equation [Matsuno (1979b)]

$$u_T + 2uu_X + \frac{1}{3} u_{XXX} = 0 \qquad (3.1.6)$$

which is the classical equation of soliton theory.

We now give some results due to Matsuno (1984). By defining

$$u = i\frac{\partial}{\partial x}(\log \frac{f^+}{f^-}) \qquad (3.1.7)$$

where f^+ and f^- are defined by

$$f^{\pm} = f(x \mp \frac{i}{\lambda}, t) \qquad (3.1.8)$$

in which the signs are vertically ordered, equation (3.1.1) is then converted into its bilinear form

$$(iD_t + i\lambda D_x - D_x^2) f^+.f^- = 0. \qquad (3.1.9)$$

Assuming that u is real, Matsuno (1984) found the N-soliton solution in the form

$$f = \sum_{\mu=0,1} \exp\{ \sum_{n=1}^{N} \mu_n \lambda \gamma_n \psi_n + \sum_{\ell<m}^{(N)} \mu_\ell \mu_m A_{\ell m}\} \qquad (3.1.10)$$

with
$$\psi_n = x - a_n t - \delta_n$$
$$a_n = \lambda (1 - \gamma_n \cot \gamma_n) \qquad (3.1.11)$$
$$\exp A_{\ell m} = \frac{(a_\ell - a_m)^2 + \lambda^2 (\gamma_\ell - \gamma_m)^2}{(a_\ell - a_m)^2 + \lambda^2 (\gamma_\ell + \gamma_m)^2}$$

where γ_n (satisfying $0<\gamma_n<\pi$) and δ_n, n=1, 2, ..., N, are real constants and $\sum_{\mu=0,1}$ denotes the summation over all possible combinations of $\mu_1 = 0,1$, $\mu_2 = 0,1,\ldots, \mu_N = 0,1$.

We note that the functional form for f is the same as that for the KdV equation [see Appendix A]. In fact (3.1.10) is the generalized N-soliton solution obtained by Hirota (1980).

If we now define θ_n and k_n by

$$\theta_n = k_n\,(x + ik_n t - \delta_n)$$

$$k_n = \frac{1}{2}\,(\lambda\gamma_n + ia_n) \qquad (3.1.12)$$

then expression (3.1.10) is more conveniently written in the form

$$f = \sum_{\mu=0,1} \exp\Big[\sum_{n=1}^{N} \mu_n(\theta_n + \theta_n^*) + \sum_{\ell<m}^{(N)} \mu_\ell\mu_m B_{\ell m}\Big] \qquad (3.1.13)$$

with

$$\exp B_{\ell m} = \frac{(k_\ell-k_m)\,(k_\ell^*-k_m^*)}{(k_\ell+k_m^*)\,(k_\ell^*+k_m)} \qquad (3.1.14)$$

where the asterisks denote complex conjugates.

The sum of exponentials (3.1.13) is precisely the result obtained upon expanding the determinant

$$f = \Big|\delta_{rs} + \Big(\frac{k_r+k_r^*}{k_r+k_s^*}\Big)\, e^{\theta_r+\theta_s^*}\Big|. \qquad (3.1.15)$$

Furthermore, Freeman (1984) has shown that the determinant (3.1.15) may be written in a Wronskian form

$$f = |DV^{-1}|\,W(\phi_1,\phi_2,\ldots\phi_N) \qquad (3.1.16)$$

where

$$D = \{\delta_{rs}(k_r+k_s^*)\, e^{\theta_r} \prod_{p\neq r} (k_p-k_r)\}$$

$$V = \{(-k_s)^{r-1}\}$$

and W is the N X N Wronskian

$$W(\phi_1,\phi_2,\dots\phi_N) = \begin{vmatrix} \phi_1 & \frac{\partial\phi_1}{\partial x} & \cdots & \frac{\partial^{N-1}\phi_1}{\partial x^{N-1}} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \phi_N & \frac{\partial\phi_N}{\partial x} & \cdots & \frac{\partial^{N-1}\phi_N}{\partial x^{N-1}} \end{vmatrix} \tag{3.1.17}$$

with

$$\begin{aligned} \phi_n &= A_n e^{-\theta_n} + B_n e^{\theta_n^*} \\ A_n &= \{(k_n+k_n^*) \prod_{p\neq n}^{N} (k_p-k_n)\}^{-1} \\ B_n &= \{ \prod_{p=1}^{N} (k_p+k_n^*)\}^{-1}, \; n=1,\dots,N. \end{aligned} \tag{3.1.18}$$

We note that the factor $|DV^{-1}|$ in (3.1.16) takes the form

$$|DV^{-1}| = A \exp\left(\sum_{n=1}^{N} k_n x\right)$$

where A is independent of x.

Now, from (3.1.7) and (3.1.16)we find that

$$u = i\frac{\partial}{\partial x} \log\left[\frac{A \exp\left(\sum_{n=1}^{N} k_n(x - \frac{i}{\lambda})\right)W(\phi_1^+,\phi_2^+,\dots,\phi_N^+)}{A \exp\left(\sum_{n=1}^{N} k_n(x + \frac{i}{\lambda})\right)W(\phi_1^-,\phi_2^-,\dots,\phi_N^-)}\right]$$

$$= i \frac{\partial}{\partial x} \log\left[e^{-\frac{i2N}{\lambda}} \frac{W(\phi_1^+,\phi_2^+,\dots,\phi_N^+)}{W(\phi_1^-,\phi_2^-,\dots,\phi_N^-)}\right]$$

$$= i \frac{\partial}{\partial x} \log\left[\frac{W(\phi_1^+,\phi_2^+,\dots,\phi_N^+)}{W(\phi_1^-,\phi_2^-,\dots,\phi_N^-)}\right] \tag{3.1.19}$$

where

$$\phi_n^{\pm} = \phi_n(x \mp \frac{i}{\lambda}, \; t), \; n=1,\dots,N. \tag{3.1.20}$$

From expression (3.1.19) we see that the factor $|DV^{-1}|$ does not contribute to the solution. Therefore we simply write (3.1.16) as

$$f = W(\phi_1,\phi_2,\ldots,\phi_N).^{\dagger} \tag{3.1.21}$$

3.2 The Wronskian solution

We summarize from the previous section that

$$f^+ = W(\phi_1^+,\phi_2^+,\ldots,\phi_N^+)$$

$$(3.2.1)$$

$$f^- = W(\phi_1^-,\phi_2^-,\ldots,\phi_N^-)$$

with $\phi_n^\pm$ defined by (3.1.20) , (3.1.12) and (3.1.18). For later convenience, we shall write $W(\phi_N)$ to mean $W(\phi_1,\phi_2,\ldots,\phi_N)$.

It is found that ϕ_n^+ may be written as a linear combination of ϕ_n^- and $\frac{\partial\phi_n^-}{\partial x}$, and indeed we have

$$\phi_n^+ = \alpha_n\{\frac{\partial\phi_n^-}{\partial x} + \beta_n\phi_n^-\} \tag{3.2.2}$$

where

$$\alpha_n = \frac{e^{-i2k_n^*/\lambda} - e^{i2k_n/\lambda}}{k_n + k_n^*} \tag{3.2.3}$$

$$\beta_n = i\lambda/2, \ n=1,2,\ldots,N \ . \tag{3.2.4}$$

We see from (3.2.4) that β_n is independent of n. This is an important result since it makes further calculation much simpler.

We now assume that

$$f^- = W(\phi_N^-) = (\widehat{N-1}) \tag{3.2.5}$$

where $(\widehat{N-1})$ means that the Wronskian has N-1 consecutive derivatives up to order N-1 [see Appendix B]. We are now looking for a representation of f^+ which is the Wronskian of the functions ϕ_n^+ in terms of Wronskian type determinants of the functions ϕ_n^- .

---

†Expressing f in this way does not imply $|DV^{-1}| = 1$.

From (3.2.1) and (3.2.2) we find

$$\begin{aligned} f^{+} &= W(\phi_n^{+}) \\ &= W(\alpha_n(\frac{\partial\phi_n^{-}}{\partial x} + \frac{i\lambda}{2}\phi_n^{-})) \\ &= \prod_{n=1}^{N} \alpha_n \sum_{r=0}^{N} (\frac{i\lambda}{2})^r (0\ldots\underset{\vee}{r}\ldots N) \end{aligned} \tag{3.2.6}$$

where $(0\ldots\underset{\vee}{r}\ldots N)$ is a Wronskian with consecutive derivatives up to order N but the r-th derivative is absent. In the notation of (3.2.5) this means

$$(0\ldots\underset{\vee}{r}\ldots N) = (\widehat{r-1}, r+1, r+2, \ldots N). \tag{3.2.7}$$

As before the constant $\prod_{n=1}^{N} \alpha_n$ may be removed from f^{+} in (3.2.6) since it gives no contribution to the final solution u. We thus find f^{+} as the sum of N+1 Wronskian type determinants

$$f^{+} = \sum_{r=0}^{N} (\frac{i\lambda}{2})^r (0\ldots\underset{\vee}{r}\ldots N). \tag{3.2.8}$$

We should remark here that all the functions which define f^{-} and f^{+} are now ϕ_n^{-} and that we are not dealing with ϕ_n^{+} at all. We should also note from the definition of ϕ_n^{-} that

$$\frac{\partial}{\partial t}\phi_n^{-} = i\frac{\partial^2\phi_n^{-}}{\partial x^2} \tag{3.2.9}$$

which will be used in calculating the t-derivatives of f^{+} and f^{-}.

We shall now prove that f^{-} and f^{+}, defined by (3.2.5) and (3.2.8), satisfy the bilinear equation (3.1.9).

The derivatives of f^{+} and f^{-} can easily be calculated by shifting the appropriate columns [see Appendix B]. For f^{+} we find

$$f_x^{+} = \sum_{r=0}^{N} (\frac{i\lambda}{2})^r [(0\ldots\underset{\vee}{r}\ldots N-1, N+1) + (0\ldots \underset{\vee}{r-1}\ldots N)]$$

$$f^{+}_{2x} = \sum_{r=0}^{N} \left(\frac{i\lambda}{2}\right)^{r} [(0\ldots \underset{\vee}{r-2}\ldots N) + 2(0\ldots \underset{\vee}{r-1}\ldots N-1, N+1)$$
$$+(0\ldots \underset{\vee}{r}\ldots N-1,N+2) + (0\ldots \underset{\vee}{r}\ldots N-2, N, N+1)] \,, \tag{3.2.10}$$
$$f^{+}_{t} = i \sum_{r=0}^{N} \left(\frac{i\lambda}{2}\right)^{r} [(0\ldots \underset{\vee}{r-2}\ldots N) - (0\ldots \underset{\vee}{r}\ldots N-1,N+2)$$
$$+(0\ldots \underset{\vee}{r}\ldots N-2,N,N+1)] \,.$$

Similarly for f^{-} we find

$$f^{-}_{x} = (\widehat{N-2},N)$$
$$f^{-}_{2x} = (\widehat{N-2},N+1) + (\widehat{N-3},N-1,N) \tag{3.2.11}$$
$$f^{-}_{t} = i[(\widehat{N-3},N-1,N) - (\widehat{N-2},N+1)].$$

We note here that in calculating f^{+}_{t} and f^{-}_{t} we have made use of relation (3.2.9). Using (3.2.10) and (3.2.11) we now calculate

$$iD_{t} f^{+}.f^{-} = i(f^{+}_{t}f^{-} - f^{+}f^{-}_{t})$$
$$= \sum_{r=0}^{N} \left(\frac{i\lambda}{2}\right)^{r} \{(0\ldots \underset{\vee}{r}\ldots N)[(\widehat{N-3}, N-1, N) - (\widehat{N-2}, N+1)]$$
$$- [(0\ldots \underset{\vee}{r-2}\ldots N) - (0\ldots \underset{\vee}{r}\ldots N-1, N+2)$$
$$+ (0\ldots \underset{\vee}{r}\ldots N-2,N,N+1)]\ (\widehat{N-1})\} \tag{3.2.12}$$

$$i\lambda D_{x} f^{+}.f^{-} = i\lambda(f^{+}_{x}f^{-} - f^{+}f^{-}_{x})$$
$$= 2 \sum_{r=0}^{N} \left(\frac{i\lambda}{2}\right)^{r+1} \{[(0\ldots \underset{\vee}{r}\ldots N-1,N+1) + (0\ldots \underset{\vee}{r-1}\ldots N)]\ (\widehat{N-1})$$
$$- (0\ldots \underset{\vee}{r}\ldots N)\ (\widehat{N-2},N)\} \tag{3.2.13}$$

and

$$- D^{2}_{x} f^{+}.f^{-} = - f^{+}_{2x}f^{-} + 2f^{+}_{x}f^{-}_{x} - f^{+}f^{-}_{2x}$$
$$= \sum_{r=0}^{N} \left(\frac{i\lambda}{2}\right)^{r} \{[-(0\ldots \underset{\vee}{r-2}\ldots N) - 2(0\ldots \underset{\vee}{r-1}\ldots N-1,N+1)$$
$$- (0\ldots \underset{\vee}{r}\ldots N-1,N+2) - (0\ldots \underset{\vee}{r}\ldots N-2,N,N+1)](\widehat{N-1})$$
$$+ 2[(0\ldots \underset{\vee}{r}\ldots N-1,N+1) + (0\ldots \underset{\vee}{r-1}\ldots N)]\ (\widehat{N-2},N)$$
$$- (0\ldots \underset{\vee}{r}\ldots N)[(\widehat{N-2},N+1) + (\widehat{N-3},N-1,N)]\}. \tag{3.2.14}$$

If we sum up all the expressions (3.2.12), (3.2.13) and (3.2.14) we find

$$(iD_t+i\lambda D_x-D_x^2)f^+.f^- = \sum_{r=0}^{N}\left(\frac{i\lambda}{2}\right)^r G(r)$$

where

$$\begin{aligned} G(r) = -\,2\{&(0\ldots\underset{\vee}{r}\ldots N-2,N,N+1)\ (\widehat{N-1}) \\ &-(0\ldots\underset{\vee}{r}\ldots N-1,N+1)\ (\widehat{N-2,N}) \\ &+\ (0\ldots\underset{\vee}{r}\ldots N)\ (\widehat{N-2,N+1})\}. \end{aligned}$$

The above expression is the Laplace expansion of a determinant [Appendix B]

$$G(r) = (-1)^{r+1}\begin{vmatrix} 0\ldots\underset{\vee}{r}\ldots N-2 & . & r\ N-1\ N\ N+1 \\ . & o\ldots\underset{\vee}{r}\ldots N-2 & r\ N-1\ N\ N+1 \end{vmatrix}.$$

The determinant $G(r)$ can now easily be shown to be zero by elementary methods. Thus we have shown that the Wronskian solutions f^+ and f^- satisfy the FDF equation.

3.3 Reductions in the KdV and the Benjamin-Ono limits

Let us first reduce the N-soliton solution of the FDF equation in the limit for the KdV equation. Under this limit, beside $\lambda \to \infty$ and the transformation (3.1.5), we also require $\gamma_n \to 0$ with $\lambda^{1/2}\gamma_n$ kept fixed [Matsuno (1984)].

From (3.2.5) and (3.2.8) as $\lambda \to \infty$ we have

$$\frac{f^+}{f^-} = \left(\frac{i\lambda}{2}\right)^N \left\{1 + \left(\frac{2}{i\lambda}\right) \frac{(\widehat{N-2},N)}{(\widehat{N-1})} + O(\lambda^{-2})\right\}$$

where we have used the fact that

$$(0 \ldots \underset{v}{N-1},N) = (\widehat{N-2},N)$$

and

$$(0 \ldots \underset{v}{N}) = (\widehat{N-1}).$$

The factor $\left(\frac{i\lambda}{2}\right)^N$ can be removed from the expression since it does not contribute to the final solution u. Now in the limit $\lambda \to \infty$ we have

$$\log\left(\frac{f^+}{f^-}\right) = \log\left\{1 + \left(\frac{2}{i\lambda}\right) \frac{(\widehat{N-2},N)}{(\widehat{N-1})} + O(\lambda^{-2})\right\}$$

$$= \left(\frac{2}{i\lambda}\right) \frac{(\widehat{N-2},N)}{(\widehat{N-1})} + O(\lambda^{-2}). \tag{3.3.1}$$

We note here that

$$(\widehat{N-2},N) = \frac{\partial}{\partial x} (\widehat{N-1}).$$

By using this relation, (3.3.1) and the transformation $X = \lambda^{1/2}x$, the solution (3.1.7) now becomes

$$u = 2 \frac{\partial^2}{\partial X^2}[\log(\widehat{N-1})]. \tag{3.3.2}$$

This is the familiar N-soliton solution in the Wronskian form for the KdV equation [Freeman and Nimmo (1983)].

The remaining task is to find the appropriate functions ϕ_n^- which define the Wronskian $(\hat{N-1})$. Now from (3.1.11)

$$a_n = \lambda(1 - \gamma_n \cot \gamma_n)$$

$$= \lambda[1 - \frac{\gamma_n(1 - \frac{\gamma_n^2}{2} + \ldots)}{(\gamma_n - \frac{\gamma_n^3}{6} + \ldots)}]$$

$$= \frac{\lambda\gamma_n^2}{3} \text{ as } \gamma_n \ll 1 \qquad (3.3.3)$$

and from (3.1.12)

$$k_n = \frac{\lambda\gamma_n}{2}(1 + \frac{i\gamma_n}{3}). \qquad (3.3.4)$$

The function ϕ_n^- is given from (3.1.18) and (3.1.12) as

$$\phi_n^- = A_n e^{-k_n(x + \frac{i}{\lambda} + ik_n t - \delta_n)} + B_n e^{k_n^*(x + \frac{i}{\lambda} - ik_n^* t - \delta_n)}. \qquad (3.3.5)$$

By using (3.1.5) and (3.3.4) we find as $\lambda \to \infty$

$$-k_n(x + \frac{i}{\lambda} + ik_n t - \delta_n) = -\frac{P_n}{2}(X - \frac{P_n^2}{3} T - d_n') - \frac{i\lambda^{1/2}P_n^2 T}{4} \qquad (3.3.6)$$

and

$$k_n^*(x + \frac{i}{\lambda} - ik_n^* t - \delta_n) = \frac{P_n}{2}(X - \frac{P_n^2}{2} T - d_n') - \frac{i\lambda^{1/2}P_n^2 T}{4} \qquad (3.3.7)$$

where $P_n = \lambda^{1/2}\gamma_n$ and $d'_n = \lambda^{1/2}\delta_n$ are kept fixed.

The coefficients A_n and B_n can also be calculated by making use of (3.3.4) in the limit $\lambda \to \infty$, $\gamma_n \ll 1$. From (3.1.18) we have

$$A_n = \{(k_n + k_n^*) \prod_{\ell \neq n}^{N} (k_\ell - k_n)\}^{-1}$$

$$= \{\lambda\gamma_n \prod_{\ell \neq n}^{N} \frac{\lambda}{2}(\gamma_\ell - \gamma_n)\}^{-1}$$

$$= 2^{N-1}\lambda^{-N/2} P_n^{-1} \{\prod_{\ell \neq n}^{N} (P_\ell - P_n)\}^{-1} \qquad (3.3.8)$$

and

$$B_n = \{ \prod_{\ell=1}^{N} (k_\ell + k_n^*)\}^{-1}$$

$$= \{\lambda\gamma_n \prod_{\ell\neq n}^{N} \frac{\lambda}{2}(\gamma_\ell+\gamma_n)\}^{-1}$$

$$= 2^{N-1}\lambda^{-N/2}P_n^{-1}\{\prod_{\ell\neq n}^{N}(P_\ell+P_n)\}^{-1} . \qquad (3.3.9)$$

Inserting (3.3.6) - (3.3.9) into (3.3.5) we find

$$\phi_n^- = 2^{N-1}\lambda^{-N/2}P_n^{-1}e^{-i\lambda^{1/2}P_n^2T/4}\{[\prod_{\ell\neq n}^{N}(P_\ell-P_n)]^{-1}\exp(-\frac{\xi'_n}{2})$$
$$+ [\prod_{\ell\neq n}^{N}(P_\ell+P_n)]^{-1}\exp(\frac{\xi'_n}{2})\}$$

where $\xi'_n = P_n(X - \frac{P_n^2T}{3} - d'_n)$.

However the common factor $2^{N-1}\lambda^{-N/2}P_n^{-1}e^{-i\lambda^{1/2}P_n^2T/4}$ does not give any contribution to the final solution u and therefore it may be removed from ϕ_n^-. After rearranging the terms and removing a common factor we finally arrive at the same result as obtained by Satsuma (1979)

$$\phi_n^- = \cosh\,(\frac{\xi_n}{2}) \qquad (3.3.10)$$

where $\xi_n = P_n(X - \frac{P_n^2}{3}T - d_n)$

with $d_n = \frac{1}{2}[2d_n' + \log \prod_{\ell\neq n}^{N}(P_\ell+P_n) - \log \prod_{\ell\neq n}^{N}(P_\ell-P_n)]$.

We have thus recovered the N-soliton solution of the KdV equation which is given by (3.3.2) and (3.3.10). Indeed it is not difficult to show that this solution satisfies the bilinear form of the KdV equation

$$D_X(D_T + \frac{1}{3}D_X^3)\ F.F = 0 \qquad (3.3.11)$$

where $F = (\widehat{N-1})$ with functions ϕ_n^- defined by (3.3.10). However one needs to be very careful in doing this because,as noted by Freeman and Nimmo (1983), a Wronskian identity is needed for this purpose.

We next consider the N-soliton solution in the Benjamin-Ono limit. In this limit it is required that $\lambda \to 0$ and that the parameter γ_n satisfies [Matsuno (1984)]

$$\gamma_n = \pi(1 - \frac{\lambda}{V_n}) \tag{3.3.12}$$

where V_n will become the solution parameter of the nth soliton.

As $\lambda \to 0$, we see immediately from (3.2.8) that f^+ is dominated by the term with r=0, hence

$$f^+ = (\underset{\vee}{0}\ldots N) = (1,2,\ldots,N).$$

From now on we shall denote any Wronskian in which its first column is the first derivative by the symbol ~ and therefore f^+ can be written as

$$f^+ = (\tilde{N}). \tag{3.3.13}$$

Because there is no independent variable transformation from the FDF equation to the Benjamin-Ono equation, the bilinear form of the Benjamin-Ono equation is obtained directly from (3.1.9) in the limit $\lambda \to 0$,

$$(iD_t - D_x^2)f^+.f^- = 0. \tag{3.3.14}$$

In order to obtain the form of the function ϕ_n^-, we first calculate a_n and k_n. From (3.1.11) and (3.3.12) we have

$$\begin{aligned} a_n &= \lambda(1 - \gamma_n \cot \gamma_n) \\ &= \lambda\{1 + \pi(1 - \frac{\lambda}{V_n}) \cos \frac{\pi\lambda}{V_n}/\sin (\frac{\pi\lambda}{V_n})\} \\ &= V_n\{1 - \frac{\pi^2\lambda^2}{2V_n^2}+\ldots\}/\{1 - \frac{\pi^2\lambda^2}{6V_n^2}+\ldots\} \\ &= V_n\{1 - \frac{\pi^2\lambda^2}{3V_n^2}\}+ 0(\lambda^3). \end{aligned} \tag{3.3.15}$$

Now from (3.1.12), (3.3.12) and (3.3.15) we obtain

$$k_n = \frac{\lambda\pi}{2} - \frac{\lambda^2\pi}{2V_n} + i(\frac{V_n}{2} - \frac{\lambda^2\pi^2}{6V_n}) + 0(\lambda^3). \tag{3.3.16}$$

Using the above expression we find

$$\begin{aligned} &-k_n(x + \frac{i}{\lambda} + ik_n t - \delta_n) \\ &= \frac{V_n}{2\lambda} - \frac{\pi^2\lambda}{6V_n} - \frac{i\pi}{2}(1 - \frac{\lambda}{V_n}) - \frac{\pi\lambda}{2} \zeta_n - \frac{iV_n}{2} \eta_n + 0(\lambda^2) \end{aligned} \tag{3.3.17}$$

and

$$k_n^*(x + \frac{i}{\lambda} - ik_n^* t - \delta_n)$$
$$= \frac{V_n}{2\lambda} - \frac{\pi^2\lambda}{6V_n} + \frac{i\pi}{2}(1 - \frac{\lambda}{V_n}) + \frac{\pi\lambda}{2}\zeta_n - \frac{iV_n}{2}\eta_n + O(\lambda^2) \qquad (3.3.18)$$

where

$$\zeta_n = x - V_n t - \delta_n$$
$$(3.3.19)$$
$$\eta_n = x - \frac{V_n}{2}t - \delta_n.$$

For the coefficients A_n, B_n we find

$$A_n = \{(k_n+k_n^*) \prod_{p\neq n}^{N} (k_p-k_n)\}^{-1}$$
$$= \{(\pi\lambda+O(\lambda^2)] \prod_{p\neq n}^{N} \frac{i}{2}(V_p-V_n)\}^{-1}$$
$$= \{\pi\lambda \prod_{p\neq n}^{N} \frac{i}{2}(V_p-V_n)\}^{-1} \qquad (3.3.20)$$

and

$$B_n = \{\prod_{p=1}^{N} (k_p+k_n^*)\}^{-1}$$
$$= \{(k_n+k_n^*) \prod_{p\neq n}^{N} (k_p+k_n^*)\}^{-1}$$
$$= \{[\pi\lambda + O(\lambda^2)] \prod_{p\neq n}^{N} [\lambda\pi+\frac{i}{2}(V_p-V_n) + O(\lambda^2)]\}^{-1}$$
$$= \{\pi\lambda \prod_{p\neq n}^{N} \frac{i}{2}(V_p-V_n)\}^{-1} \{1 + i2\pi\lambda\alpha_n\}, \qquad (3.3.21)$$

where

$$\alpha_n = \sum_{p\neq n}^{N} \frac{1}{V_p-V_n}. \qquad (3.3.22)$$

Combining the result (3.3.17) - (3.3.22) for ϕ_n^- we find that $\{\lambda\pi \prod_{p\neq n}^{N} \frac{i}{2} (V_p-V_n)\}^{-1} \exp \{\frac{V_n}{2\lambda} - \frac{\pi^2\lambda}{6V_n}\}$ is a common factor which may be removed. We are then left with

$$\phi_n^- = e^{-\frac{i\pi}{2}(1-\frac{\lambda}{V_n}) - \frac{\pi\lambda}{2}\zeta_n - \frac{iV_n}{2}\eta_n}$$

$$+ (1+i2\pi\lambda\alpha_n)\, e^{\frac{i\pi}{2}(1-\frac{\lambda}{V_n}) + \frac{\pi\lambda}{2}\zeta_n - \frac{iV_n}{2}\eta_n}$$

$$= i\{-e^{\frac{\pi\lambda}{2}(\frac{i}{V_n}-\zeta_n)} + (1+i2\pi\lambda\alpha_n)e^{-\frac{\pi\lambda}{2}(\frac{i}{V_n}-\zeta_n)}\}e^{-\frac{iV_n}{2}\eta_n} \quad . \qquad (3.3.23)$$

Since $\lambda \to 0$, all the exponentials inside the curly brackets in (3.3.23) can be expanded in their power series to give

$$\phi_n^- = \{\zeta_n - \frac{i}{V_n} + i2\alpha_n\}e^{-\frac{iV_n}{2}\eta_n}. \qquad (3.3.24)$$

We note here that the factor $\exp(-\frac{iV_n}{2}\eta_n)$ was not removed from (3.3.24) because ϕ_n^- would not define a non-zero NxN Wronskian without this term.

We now show that $f^+ = (\tilde{N})$ and $f^- = (\widehat{N-1})$, with functions ϕ_n^- given by (3.3.24), satisfy the bilinear form of the Benjamin-Ono equation (3.3.14). We note here that it is not difficult to see from (3.3.24) that

$$\frac{\partial}{\partial t}\phi_n^- = -i\,\frac{\partial^2}{\partial x^2}\phi_n^- \qquad (3.3.25)$$

which is needed to calculate f_t^+ and f_t^-.

The derivatives of f^+ and f^- are obtained as before

$f_x^+ = (\widetilde{N-1}, N+1)$

$f_{2x}^+ = (\widetilde{N-1}, N+2) + (\widetilde{N-2}, N, N+1)$

$f_t^+ = -i\{(\widetilde{N-1}, N+2) - (\widetilde{N-2}, N, N+1)\}$

$f_x^- = (\widehat{N-2}, N)$

$f_{2x}^- = (\widehat{N-2}, N+1) + (\widehat{N-3}, N-1, N)$

$f_t^- = -i\{(\widehat{N-2}, N+1) - (\widehat{N-3}, N-1, N)\}$.

Substitution of f^+, f^- and the related derivatives into the bilinear equation (3.3.14) yields

$$(iD_t - D_x^2)f^+.f^-$$
$$= -2\{(\widetilde{N-2}, N, N+1)\ (\widehat{N-1}) - (\widetilde{N-1}, N+1)\ (\widehat{N-2}, N) + (\widetilde{N})\ (\widehat{N-2}\ N+1)\}.$$

Again this is the Laplace expansion of a 2N × 2N determinant which can be written as

$$(-1)^{N+1}\ 2\begin{vmatrix} \widehat{N-2} & \cdot & N-1 & N & N+1 \\ \cdot & \widetilde{N-2} & N-1 & N & N+1 \end{vmatrix}$$

which can be shown to be zero.

As we have seen in the above we did not use any identity in proving the N-soliton solution of the Benjamin-Ono equation. This is in contrast with the KdV equation in which an identity is needed for the same purpose. For the Benjamin-Ono equation if we put x_1=x and x_2=-it then its bilinear equation becomes

$$(D_1^{\ 2}+D_2^{\ 2})f^-.f^+ = 0. \qquad (3.3.26)$$

We note that the order of f^- and f^+ in (3.3.26) is now opposite to the original equation (3.3.14) due to the property $D_2f^+.f^- = -D_2f^-.f^+$. Now (3.3.26) belongs to the first modified KP hierarchy (Appendix C). Indeed it can be shown that all the equations under this hierarchy are satisfied by the Wronskian solutions [c.f. Chapter 5]

$$f^- = (\widehat{N-1}),\ f^+ = (\widetilde{N})$$

with the defining functions ϕ_n^- satisfying

$$\frac{\partial}{\partial x_k}\phi_n^- = \frac{\partial^k}{\partial x_1^{\ k}}\phi_n^-,$$

and therefore no identities are needed.

3.4 The two-soliton solutions

Two-soliton solutions of any nonlinear evolution equation are the simplest ones that one can use to explain soliton interactions. In this section we first deduce the explicit form of the two-soliton solution for the FDF equation and reduce it to the two-soliton solutions of the KdV and the Benjamin-Ono equations under their respective limits. Interactions between the two solitons will also be discussed.

From equations (3.2.5) and (3.2.8), for N=2 we have

$$f^- = (0,1) \tag{3.4.1}$$

$$f^+ = (1,2) + \frac{i\lambda}{2}(0,2) + \left(\frac{i\lambda}{2}\right)^2(0,1) \tag{3.4.2}$$

with functions

$$\phi_n^- = A_n e^{-k_n(x + \frac{i}{\lambda} + ik_n t - \delta_n)} + B_n e^{k_n^*(x + \frac{i}{\lambda} - ik_n^* t - \delta_n)} \tag{3.4.3}$$

where

$$\begin{aligned} A_n &= \{(k_n+k_n^*)(k_p-k_n)\}^{-1} \\ B_n &= \{(k_n+k_n^*)(k_p+k_n^*)\}^{-1} \end{aligned} \tag{3.4.4}$$

with p, n = 1,2 and p ≠ n.

We first calculate all the determinants in (3.4.1) and (3.4.2) in order to find f^- and f^+ and then substitute k_n from (3.1.12). It emerges from our calculations that there are some factors in f^- and f^+ which can be cancelled in f^+/f^-. The actual calculation is straightforward but very laborious and hence only some necessary results are given here.

We eventually find

$$\frac{f^+}{f^-} = \frac{1+e^{\xi_1-i\gamma_1}+e^{\xi_2-i\gamma_2}+C_{12}e^{\xi_1+\xi_2-i(\gamma_1+\gamma_2)}}{1+e^{\xi_1+i\gamma_1}+e^{\xi_2+i\gamma_2}+C_{12}e^{\xi_1+\xi_2+i(\gamma_1+\gamma_2)}} \tag{3.4.5}$$

after removing a constant factor as it does not contribute to the final solution. In (3.4.5) we have defined

$$C_{12} = \frac{(\gamma_1-\gamma_2)^2+(b_1-b_2)^2}{(\gamma_1+\gamma_2)^2+(b_1-b_2)^2} \qquad (3.4.6)$$

and

$$\xi_n = \lambda\gamma_n(x - \lambda b_n t - \delta_n) \qquad (3.4.7)$$

where

$$b_n = 1-\gamma_n \cot\gamma_n . \qquad (3.4.8)$$

In (3.4.5) we see that the numerator is the complex conjugate of the denominator and hence it can be written as

$$\frac{f^+}{f^-} = e^{-i2\Theta} \qquad (3.4.9)$$

where

$$\Theta = \tan^{-1}\left\{\frac{e^{\xi_1}\sin\gamma_1 + e^{\xi_2}\sin\gamma_2 + C_{12}e^{\xi_1+\xi_2}\sin(\gamma_1+\gamma_2)}{1 + e^{\xi_1}\cos\gamma_1 + e^{\xi_2}\cos\gamma_2 + C_{12}e^{\xi_1+\xi_2}\cos(\gamma_1+\gamma_2)}\right\}. \qquad (3.4.10)$$

From (3.4.9) we now have

$$u = i\frac{\partial}{\partial x}\log\frac{f^+}{f^-}$$
$$= 2\frac{\partial\Theta}{\partial x} \qquad (3.4.11)$$

and therefore u is real.

Carrying out the actual differentiation in (3.4.11) we find the explicit form of the two-soliton solution of the FDF equation as

$$u = 2\lambda\{e^{\xi_1}\gamma_1 \sin\gamma_1 + e^{\xi_2}\gamma_2 \sin\gamma_2 + e^{\xi_1+\xi_2}[(\gamma_1-\gamma_2)\sin(\gamma_1-\gamma_2)$$

$$+C_{12}(\gamma_1+\gamma_2)\sin(\gamma_1+\gamma_2)] + e^{2\xi_1+\xi_2}C_{12}\gamma_2 \sin\gamma_2$$

$$+e^{\xi_1+2\xi_2}C_{12}\gamma_1 \sin\gamma_1\}/\{1+2e^{\xi_1}\cos\gamma_1 + 2e^{\xi_2}\cos\gamma_2$$

$$+e^{2\xi_1}+e^{2\xi_2}+2e^{\xi_1+\xi_2}[\cos(\gamma_1-\gamma_2) + C_{12}\cos(\gamma_1+\gamma_2)]$$

$$+2e^{2\xi_1+\xi_2}C_{12}\cos\gamma_2+2e^{\xi_1+2\xi_2}C_{12}\cos\gamma_1 + C_{12}^2e^{2(\xi_1+\xi_2)}\}. \qquad (3.4.12)$$

In order to check this result with Joseph's single-soliton solution we put $\gamma_2 = \xi_2 = 0$ which implies $C_{12} = 1$ in (3.4.12) to give

$$u = \frac{2\lambda e^{\xi_1}\gamma_1 \sin\gamma_1}{1+e^{2\xi_1}+2e^{\xi_1}\cos\gamma_1} = \frac{\lambda\gamma_1 \sin\gamma_1}{\cosh\xi_1 + \cos\gamma_1}$$

which is precisely identical to (3.1.3).

In the KdV limit, with b_n and k_n given by (3.3.3) and (3.3.4) respectively we find

$$\xi_n = \lambda\gamma_n(x - \lambda b_n t - \delta_n)$$

$$= \lambda\gamma_n(\lambda^{-\frac{1}{2}}X - \lambda^{\frac{1}{2}} \frac{\gamma_n^2}{3} T - \delta_n)$$

$$= \lambda^{\frac{1}{2}}\gamma_n(X - \frac{\lambda\gamma_n^2}{3} T - \lambda^{\frac{1}{2}}\delta_n)$$

$$= P_n(X - \frac{P_n^2 T^3}{3} - d'_n) \qquad (3.4.13)$$

where P_n, d'_n, X and T are as defined before.

We also have

$$\sin \gamma_n = \sin (\frac{P_n}{\lambda^{\frac{1}{2}}}) = \frac{P_n}{\lambda^{\frac{1}{2}}} + 0\ [(\frac{1}{\lambda^{\frac{1}{2}}})^3]$$

$$\cos \gamma_n = \cos (\frac{P_n}{\lambda^{\frac{1}{2}}}) = 1 + 0\ [(\frac{1}{\lambda^{\frac{1}{2}}})^2]$$

$$(3.4.14)$$

$$\sin (\gamma_1+\gamma_2) = \sin (\frac{P_1+P_2}{\lambda^{\frac{1}{2}}})$$

$$= \frac{P_1+P_2}{\lambda^{\frac{1}{2}}} + 0[(\frac{1}{\lambda^{\frac{1}{2}}})^3]$$

$$\cos (\gamma_1+\gamma_2) = \cos (\frac{P_1+P_2}{\lambda^{\frac{1}{2}}})$$

$$= 1 + 0\ [(\frac{1}{\lambda^{\frac{1}{2}}})^2]$$

and

$$C_{12} = \frac{(\gamma_1-\gamma_2)^2 + (b_1-b_2)^2}{(\gamma_1+\gamma_2)^2 + (b_1-b_2)^2}$$

$$= \frac{\frac{1}{\lambda}(P_1-P_2)^2 + (P_1{}^2-P_2{}^2)^2/9\lambda^3}{\frac{1}{\lambda}(P_1+P_2)^2 + (P_1{}^2-P_2{}^2)^2/9\lambda^3}$$

$$= \left(\frac{P_1-P_2}{P_1+P_2}\right)^2 \text{ as } \lambda \to \infty. \qquad (3.4.15)$$

We find that it is much simpler to work with Θ (3.4.10) rather than working with the final solution u (3.4.12). Substituting (3.4.14) into (3.4.10) we find

$$\Theta = \tan^{-1}\left\{\frac{\lambda^{-1/2}[P_1 e^{\xi_1}+P_2 e^{\xi_2}+C_{12}(P_1+P_2)e^{\xi_1+\xi_2}]}{1+e^{\xi_1}+e^{\xi_2}+C_{12}e^{\xi_1+\xi_2}}\right\}.$$

In the above expression, apart from the factor $\lambda^{-1/2}$, the numerator is the X-derivative of the denominator, or we can write

$$\Theta = \tan^{-1}\left\{\frac{\lambda^{-1/2}F_X}{F}\right\} \qquad (3.4.16)$$

where

$$F = 1+e^{\xi_1}+e^{\xi_2}+C_{12}e^{\xi_1+\xi_2}. \qquad (3.4.17)$$

Now we have

$$u = 2\frac{\partial\Theta}{\partial x}$$

$$= 2\lambda^{1/2}\frac{\partial\Theta}{\partial X}$$

$$= 2\frac{\partial^2}{\partial X^2}\log F \text{ as } \lambda \to \infty. \qquad (3.4.18)$$

By carrying out the differentiation in (3.4.18) we eventually find the two-soliton solution to the KdV equation (3.1.6)

$$u = \frac{(P_1+P_2)^2}{2} \frac{\{P_1^2 s_1^2 (P_1-P_2t_2)^2+P_2^2 s_2^2 (P_2-P_1t_1)^2\}}{\{(P_1+P_2)^2-P_1P_2(1+t_1)(1+t_2)\}^2} \tag{3.4.19}$$

where $s_n = \text{sech}\,(\frac{\xi_n}{2})$

$t_n = \tanh\,(\frac{\xi_n}{2})$, $n = 1, 2$.

Now, in the Benjamin-Ono limit ($\lambda \to 0$), from (3.3.12) and (3.3.15) we find

$$\begin{aligned} \xi_n &= \lambda\gamma_n(x - \lambda b_n t - \delta_n) \\ &= \lambda\pi(x - V_n t - \delta_n) - \frac{\pi\lambda^2}{V_n}(x - V_n t - \delta_n) + O(\lambda^3) \\ &= \lambda\pi\zeta_n - \frac{\pi\lambda^2}{V_n}\zeta_n + O(\lambda^3) \end{aligned} \tag{3.4.20}$$

and

$$C_{12} = 1 - \frac{4\pi^2\lambda^2}{(V_1-V_2)^2} + O(\lambda^3) \tag{3.4.21}$$

where $\zeta_n = x - V_n t - \delta_n$.

In this limit we also have

$$\begin{aligned} \sin\gamma_n &= \sin\pi(1 - \frac{\lambda}{V_n}) = \frac{\pi\lambda}{V_n} + O(\lambda^3) \\ \sin(\gamma_1+\gamma_2) &= \sin(2\pi-\pi\lambda(\frac{1}{V_1} + \frac{1}{V_2})) \\ &= -\pi\lambda(\frac{1}{V_1} + \frac{1}{V_2}) + O(\lambda^3) \\ \cos\gamma_n &= \cos\pi(1 - \frac{\lambda}{V_n}) = -1 + \frac{\pi^2\lambda^2}{2V_n^2} + O(\lambda^4) \end{aligned} \tag{3.4.22}$$

$$\begin{aligned} \cos(\gamma_1 + \gamma_2) &= \cos(2\pi - \pi\lambda(\frac{1}{V_1} + \frac{1}{V_2})) \\ &= 1 - \frac{\pi^2\lambda^2}{2}(\frac{1}{V_1} + \frac{1}{V_2})^2 + O(\lambda^4). \end{aligned}$$

If we write (3.4.10) as

$$\Theta = \tan^{-1}(\frac{\theta_N}{\theta_D}), \tag{3.4.23}$$

then by using (3.4.20) - (3.4.22) we find

$$\theta_N = e^{\xi_1} \sin \gamma_1 + e^{\xi_2} \sin \gamma_2 + C_{12}\, e^{\xi_1+\xi_2} \sin(\gamma_1+\gamma_2)$$

$$= \frac{\lambda\pi}{V_1V_2}\left[V_2 e^{\lambda\pi\zeta_1 - \frac{\pi\lambda^2}{V_1}\zeta_1} + V_1 e^{\lambda\pi\zeta_2 - \frac{\pi\lambda^2}{V_2}\zeta_2} - \left(1 - \frac{4\pi^2\lambda^2}{(V_1-V_2)^2}\right)(V_1+V_2)e^{\lambda\pi(\zeta_1+\zeta_2)-\pi\lambda^2(\frac{\zeta_1}{V_1}+\frac{\zeta_2}{V_2})}\right].$$

Expanding the last expression in its power series we find

$$\theta_N = -\frac{\lambda^2\pi^2}{V_1V_2}(V_1\zeta_1 + V_2\zeta_2) + 0(\lambda^3). \qquad (3.4.24)$$

Similarly for θ_D we have

$$\theta_D = 1 + e^{\zeta_1} \cos \gamma_1 + e^{\zeta_2} \cos \gamma_2 + C_{12}e^{\zeta_1+\zeta_2}\cos(\gamma_1 + \gamma_2)$$

$$= 1 + \left(\frac{\pi^2\lambda^2}{2V_1^2} - 1\right)e^{\lambda\pi\zeta_1 - \frac{\pi\lambda^2}{V_1}\zeta_1} + \left(\frac{\pi^2\lambda^2}{2V_2^2} - 1\right)e^{\lambda\pi\zeta_2 - \frac{\pi\lambda^2}{V_2}\zeta_2}$$

$$+\left(1 - \frac{4\pi^2\lambda^2}{(V_1-V_2)^2}\right)\left(1 - \frac{\pi^2\lambda^2}{2}\left(\frac{1}{V_1} + \frac{1}{V_2}\right)^2\right)e^{\lambda\pi(\zeta_1+\zeta_2)-\pi\lambda^2(\frac{\zeta_1}{V_1}+\frac{\zeta_2}{V_2})}$$

$$= -\frac{\lambda^2\pi^2}{V_1V_2}\left[\left(\frac{V_1+V_2}{V_1-V_2}\right)^2 - V_1V_2\zeta_1\zeta_2\right] + 0(\lambda^3) \qquad (3.4.25)$$

where we have also used the power series expansions to obtain the last expression. Thus as $\lambda \to 0$ we find

$$\Theta = \tan^{-1}\left\{\frac{V_1\zeta_1+V_2\zeta_2}{V_{12}-V_1V_2\zeta_1\zeta_2}\right\} \qquad (3.4.26)$$

where

$$V_{12} = \left(\frac{V_1+V_2}{V_1-V_2}\right)^2. \qquad (3.4.27)$$

The explicit form for the two-soliton solution in the Benjamin-Ono limit is then obtained from (3.4.11) by using (3.4.26) to give

$$u = 2\left[\frac{V_{12}(V_1+V_2)+V_1V_2(V_1\zeta_1^{\;2}+V_2\zeta_2^{\;2})}{(V_1\zeta_1+V_2\zeta_2)^2+(V_{12}-V_1V_2\zeta_1\zeta_2)^2}\right]. \qquad (3.4.28)$$

This result is identical to the one obtained by Matsuno (1979a).

We note that the single-soliton solution of the Benjamin-Ono equation can be obtained from (3.4.28) by putting $V_2 = 0$ and $V_{12} = 1$ to give

$$u = \frac{2V_1}{1+V_1^{\;2}\zeta_1^{\;2}}.$$

We have so far shown that the two-soliton solutions of the KdV and the Benjamin-Ono equations can be deduced directly from the two-soliton solution of the FDF equation.

We now return to the two-soliton solution of the FDF equation and look for the interaction between two solitons. In the analysis that follows we assume $\gamma_1 > \gamma_2$. Although it is not very straightforward from (3.4.8), it can be shown that this assumption implies $b_1 > b_2$ for $0 < \gamma_2 < \gamma_1 < \pi$. The procedure to follow in this analysis is to look at the solution at time $t \to -\infty$, long before the two solitons interact, and at time $t \to +\infty$, long after they have collided. We shall also look at the form of the solution in the middle of the interaction. The results will then be deduced in the Benjamin-Ono and the KdV limits. We find that it is much simpler to work with Θ (3.4.10) rather than the full solution (3.4.12).

Let $t \to -\infty$ and ξ_1 fixed. Then we have

$$\xi_2 \approx \lambda\gamma_2(b_1-b_2)t \to -\infty.$$

Thus, in this region the solution (3.4.10) is dominated by

$$\Theta \approx \tan^{-1}\left[\frac{e^{\xi_1}\sin\gamma_1}{1+e^{\xi_1}\cos\gamma_1}\right]$$

which corresponds to the first soliton u_1,

$$u_1 = \frac{\lambda\gamma_1 \sin\gamma_1}{\cosh \xi_1 + \cos \gamma_1}. \tag{3.4.29}$$

The situation is quite different when we look for the soliton in the region where ξ_2 is fixed as $t \to -\infty$. In this case we find

$$\xi_1 \approx \lambda\gamma_1(b_2-b_1)t \to +\infty$$

and this implies

$$\Theta \approx \tan^{-1}\left[\frac{\sin \gamma_1 + C_{12}e^{\xi_2}\sin(\gamma_1+\gamma_2)}{\cos \gamma_1 + C_{12}e^{\xi_2}\cos(\gamma_1+\gamma_2)}\right]$$

which corresponds to the second soliton u_2,

$$u_2 = \frac{\lambda\gamma_2 \sin \gamma_2}{\cosh(\xi_2 + \log C_{12}) + \cos \gamma_2}. \tag{3.4.30}$$

Thus as $t \to -\infty$ we have two well separated solitons, u_1 centred at $\xi_1 = 0$ and u_2 centred at $\xi_2 = -\log C_{12}$, with u_1 placed far behind u_2.

By carrying out the calculation for $t \to +\infty$ in a similar manner we find in the region where ξ_1 is fixed the soliton u_1,

$$u_1 = \frac{\lambda\gamma_1 \sin \gamma_1}{\cosh(\xi_1 + \log C_{12}) + \cos \gamma_1}, \tag{3.4.31}$$

and in the region where ξ_2 is fixed we find only soliton u_2,

$$u_2 = \frac{\lambda\gamma_2 \sin \gamma_2}{\cosh \xi_2 + \cos \gamma_2}. \tag{3.4.32}$$

Again as $t \to +\infty$ we have two well separated solitons, u_1 now centred at $\xi_1 = -\log C_{12}$ and u_2 at $\xi_2 = 0$, with u_1 placed far in front of u_2.

From (3.4.29) - (3.4.32) we see that both of the solitons have been shifted by a distance $|\log C_{12}|$ after the interaction. A schematic diagram for the interaction discussed above is shown in Fig 3.1. In this figure we show that soliton 1 which was placed behind soliton 2 (their phase lines are drawn as continuous lines) undergoes an interaction in the circled region. After the interaction soliton 1 overtakes soliton 2 with their phases being shifted by $\log C_{12}$. The dotted lines are the phase lines if there had been no interaction or if an interaction with $\log C_{12} = 0$ had taken place.

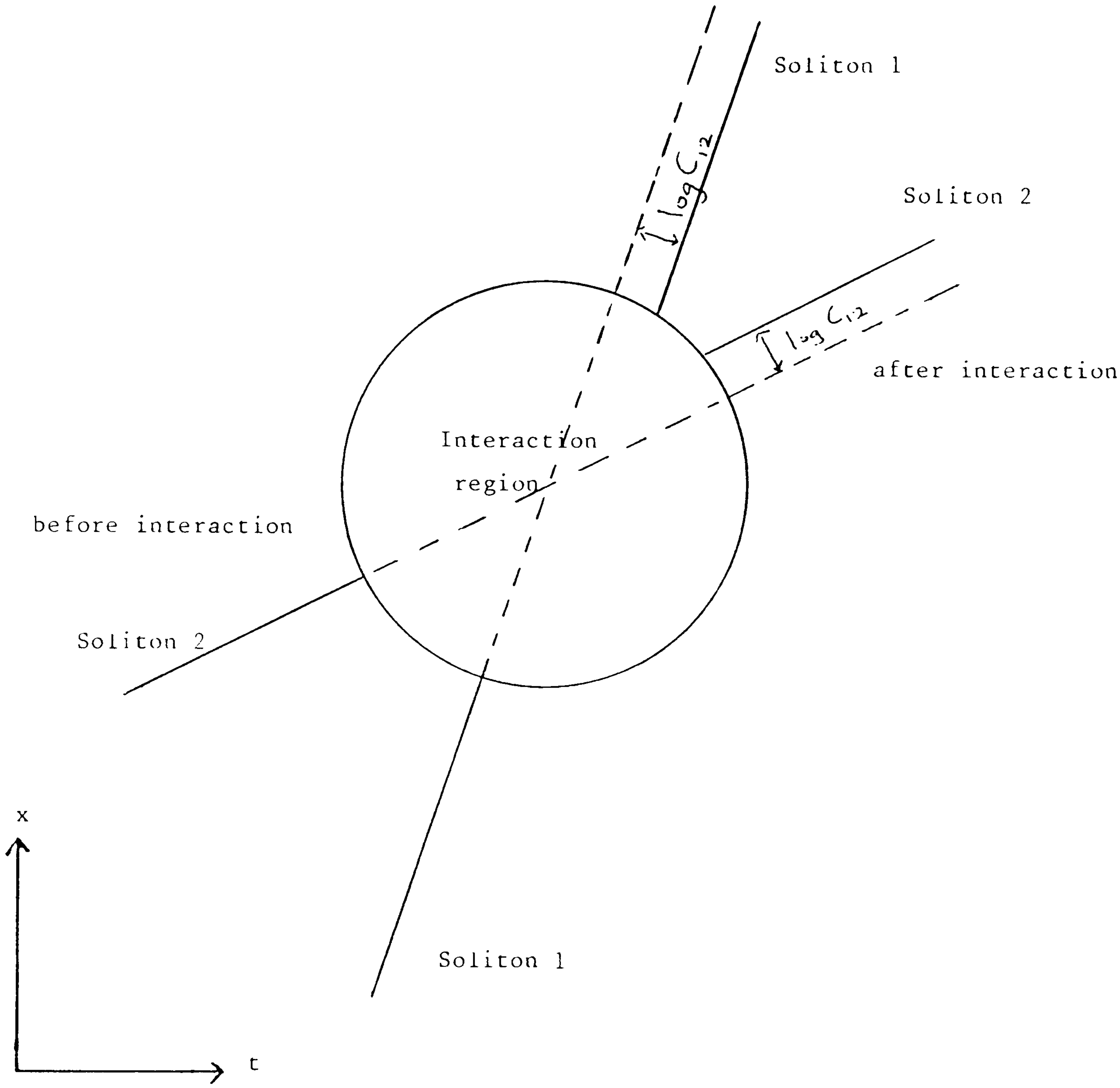


Fig 3.1 Schematic diagram showing the phase shifts of the solitons after the interaction.

In the Benjamin-Ono limit since $C_{12} \to 1$ as $\lambda \to 0$ [c.f.(3.4.21)] then the phase shift, $\log C_{12} = 0$. Thus unlike most of the soliton interactions, Benjamin-Ono solitons interact together without producing any phase shift.

For the KdV solitons the phase shift is

$$\log C_{12} = 2 \log \left[\frac{P_1 - P_2}{P_1 + P_2}\right], \quad P_1 > P_2$$

after the interaction [c.f.(3.4.15)].

The form of the solution in the middle of the interaction can be examined by making the following variable transformations

$$x = x' + h, \quad t = t' + k \tag{3.4.33}$$

such that

$$h - \lambda b_1 k - \delta_1 = - \frac{\log C_{12}}{2\lambda\gamma_1}$$

and (3.4.34)

$$h - \lambda b_2 k - \delta_2 = - \frac{\log C_{12}}{2\lambda\gamma_2}.$$

Substituting (3.4.33) and (3.4.34) into (3.4.29), (3.4.30), (3.4.31) and (3.4.32) we find that as $t \to -\infty$,

$$u \approx u_1 = \frac{\lambda\gamma_1 \quad \sin \gamma_1}{\cosh[\lambda\gamma_1(x - \lambda b_1 t) - \frac{\log C_{12}}{2}] + \cos \gamma_1} \tag{3.4.35}$$

in the region where ξ_1 is fixed, and

$$u \approx u_2 = \frac{\lambda\gamma_2 \quad \sin \gamma_2}{\cosh[\lambda\gamma(x - \lambda b_2 t) + \frac{\log C_{12}}{2}] + \cos \gamma_2} \tag{3.4.36}$$

in the region where ξ_2 is fixed.

Also we find as $t \to +\infty$

$$u \approx u_1 = \frac{\lambda\gamma_1 \quad \sin \gamma_1}{\cosh[\lambda\gamma_1(x - \lambda b_1 t) + \frac{\log C_{12}}{2}] + \cos \gamma_1} \tag{3.4.37}$$

in the region where ξ_1 is fixed, and

$$u \approx u_2 = \frac{\lambda\gamma_2 \sin\gamma_2}{\cosh[\lambda\gamma_2(x-\lambda b_2 t) - \frac{\log C_{12}}{2}] + \cos\gamma_2} \qquad (3.4.38)$$

in the region where ξ_2 is fixed. In all expressions (3.4.35) - (3.4.38) we have dropped the primes for convenience.

We note that the solitons (3.4.35), (3.4.36) and (3.4.37), (3.4.38) are symmetric with respect to the new x and t-axes. This symmetric property means that the centre of the interaction is at x=0, t=0.

If the full solution (3.4.12) is written in the new variables we find

$$u = \frac{2\lambda u_N}{u_D} \qquad (3.4.39)$$

where

$$u_N = 2C_{12}^{1/2}\gamma_2 \sin\gamma_2 \cosh\xi_1 + 2C_{12}^{1/2}\gamma_1 \sin\gamma_1 \cosh\xi_2$$
$$+(\gamma_1-\gamma_2)\sin(\gamma_1-\gamma_2) + C_{12}(\gamma_1+\gamma_2)\sin(\gamma_1+\gamma_2) \qquad (3.4.40)$$

and

$$u_D = 2C_{12}^{1/2}\cos\gamma_2 \cosh\xi_1 + 2C_{12}^{1/2}\cos\gamma_1 \cosh\xi_2 + \cosh(\xi_1-\xi_2)$$
$$+C_{12}\cosh(\xi_1+\xi_2) + \cos(\gamma_1-\gamma_2) + C_{12}\cos(\gamma_1+\gamma_2), \qquad (3.4.41)$$

where in the new variables x,t ξ_n is redefined as

$$\xi_n = \lambda\gamma_n(x - \lambda b_n t). \qquad (3.4.42)$$

We now determine the number of peaks in the middle of the interaction (x=0, t=0). To do this we expand all the hyperbolic functions in (3.4.39) in their power series at t=0 about x = 0 to find

$$u(x,0) \approx 2\lambda\{\frac{Q_1 + Q_2\lambda^2x^2}{R_1 + R_2\lambda^2x^2}\}$$

$$\approx \frac{2\lambda}{R_1}\{Q_1+Q_2\lambda^2x^2\}\{1 - \frac{\lambda^2R_2}{R_1}x^2\}$$

$$\approx 2\lambda\{\frac{Q_1}{R_1} + \frac{\lambda^2(Q_2R_1-Q_1R_2)x^2}{R_1^2}\}, \qquad (3.4.43)$$

where

$$Q_1 = 2C_{12}^{1/2}(\gamma_1 \sin \gamma_1+\gamma_2 \sin \gamma_2) + (\gamma_1-\gamma_2) \sin (\gamma_1-\gamma_2) + C_{12}(\gamma_1+\gamma_2) \sin (\gamma_1+\gamma_2)$$

$$Q_2 = C_{12}^{1/2}\gamma_1\gamma_2(\gamma_2 \sin \gamma_1+\gamma_1 \sin \gamma_2)$$

(3.4.44)

$$R_1 = 2C_{12}^{1/2}(\cos \gamma_1+\cos \gamma_2) + \cos(\gamma_1-\gamma_2) + C_{12} + 1 + C_{12} \cos (\gamma_1+\gamma_2)$$

$$R_2 = C_{12}^{1/2}(\gamma_1^2 \cos \gamma_2+\gamma_2^2 \cos \gamma_1) + \frac{(\gamma_1-\gamma_2)^2}{2} + \frac{C_{12}(\gamma_1+\gamma_2)^2}{2}.$$

From the quadratic expression (3.4.43) we can see immediately that u will have a maximum (corresponding to a single peak) at x=0 when

$$Q_2R_1 - Q_1R_2 < 0 \qquad (3.4.45)$$

and a minimum (corresponding to double peaks) at x=0 when

$$Q_2R_1 - Q_1R_2 > 0 \ . \qquad (3.4.46)$$

Since expression $Q_2R_1-Q_1R_2$ is very complicated and does not give a linear relation between γ_1 and γ_2, Table 3.1 is then produced to show the possibilities of having both types of the peaks discussed above.

Figs 3.2a, b, c, d and e are produced to show the interactions at various time instants between the two-soliton solution of the FDF equation with $\lambda=1$, $\gamma_1=1.5$ and $\gamma_2=1.0$. From Table 3.1 the value of $Q_2R_1 - Q_1R_2$ is positive and thus we have two peaks in the middle of interaction, t=0. All the plots are drawn by using the symmetric solution (3.4.39). As is shown in these figures while the taller soltion is trying to overtake the shorter one (t = - 4.0), it transfers some of its 'mass' to the shorter soliton [as shown in Fig 3.2b] until both of them have the same mass [Fig 3.2c]. Figures 3.2d and 3.2e are the reversals of those of Figs 3.2a and 3.2b.

γ_1	γ_2	C_{12}	$Q_2R_1-Q_1R_2$
1.10	1.00	0.00311	0.21882
1.20	1.00	0.01153	0.45311
1.30	1.00	0.02407	0.63119
1.40	1.00	0.03977	0.85157
1.50	1.00	0.05785	0.83721
1.60	1.00	0.07766	0.74058
1.70	1.00	0.09867	0.35911
1.80	1.00	0.12043	-0.29936
1.90	1.00	0.14262	-1.25146
2.00	1.00	0.16493	-2.53396
2.10	1.00	0.18715	-4.09995
2.20	1.00	0.20911	-5.91642
2.30	1.00	0.23067	-7.91257
2.40	1.00	0.25173	-9.98947
2.50	1.00	0.27223	-12.02078
2.60	1.00	0.29210	-13.85455
2.70	1.00	0.31132	-15.31618
2.80	1.00	0.32988	-16.21217
2.90	1.00	0.34775	-16.33502
3.00	1.00	0.36496	-15.45882
3.10	1.00	0.38150	-13.39532
1.60	1.50	0.00169	0.60479
1.70	1.50	0.00640	1.25429
1.80	1.50	0.01365	1.89876
1.90	1.50	0.02303	2.48467
2.00	1.50	0.03415	2.95891
2.10	1.50	0.04667	3.27301
2.20	1.50	0.06030	3.38717
2.30	1.50	0.07478	3.27367
2.40	1.50	0.08991	2.91967
2.50	1.50	0.10547	2.32925
2.60	1.50	0.12133	1.52458
2.70	1.50	0.13734	0.54623
2.80	1.50	0.15339	-0.54724
2.90	1.50	0.16940	-1.63062
3.00	1.50	0.18528	-2.75401
3.10	1.50	0.20097	-3.69536
2.10	2.00	0.00107	0.99351
2.20	2.00	0.00410	1.99707
2.30	2.00	0.00884	2.95965
2.40	2.00	0.01505	3.83270
2.50	2.00	0.02253	4.57297
2.60	2.00	0.03108	5.14500
2.70	2.00	0.04053	5.52305
2.80	2.00	0.05072	5.69226
2.90	2.00	0.06151	5.64930
3.00	2.00	0.07279	5.40239
3.10	2.00	0.08444	4.97069
2.60	2.50	0.00073	0.99708
2.70	2.50	0.00282	1.90469
2.80	2.50	0.00612	2.69115
2.90	2.50	0.01048	3.33327
3.00	2.50	0.01579	3.81687
3.10	2.50	0.02192	4.13692
3.10	3.00	0.00053	0.24244

Table 3.1 Showing possibilities of having single and double peaks in the middle of interactions between two FDF solitons.

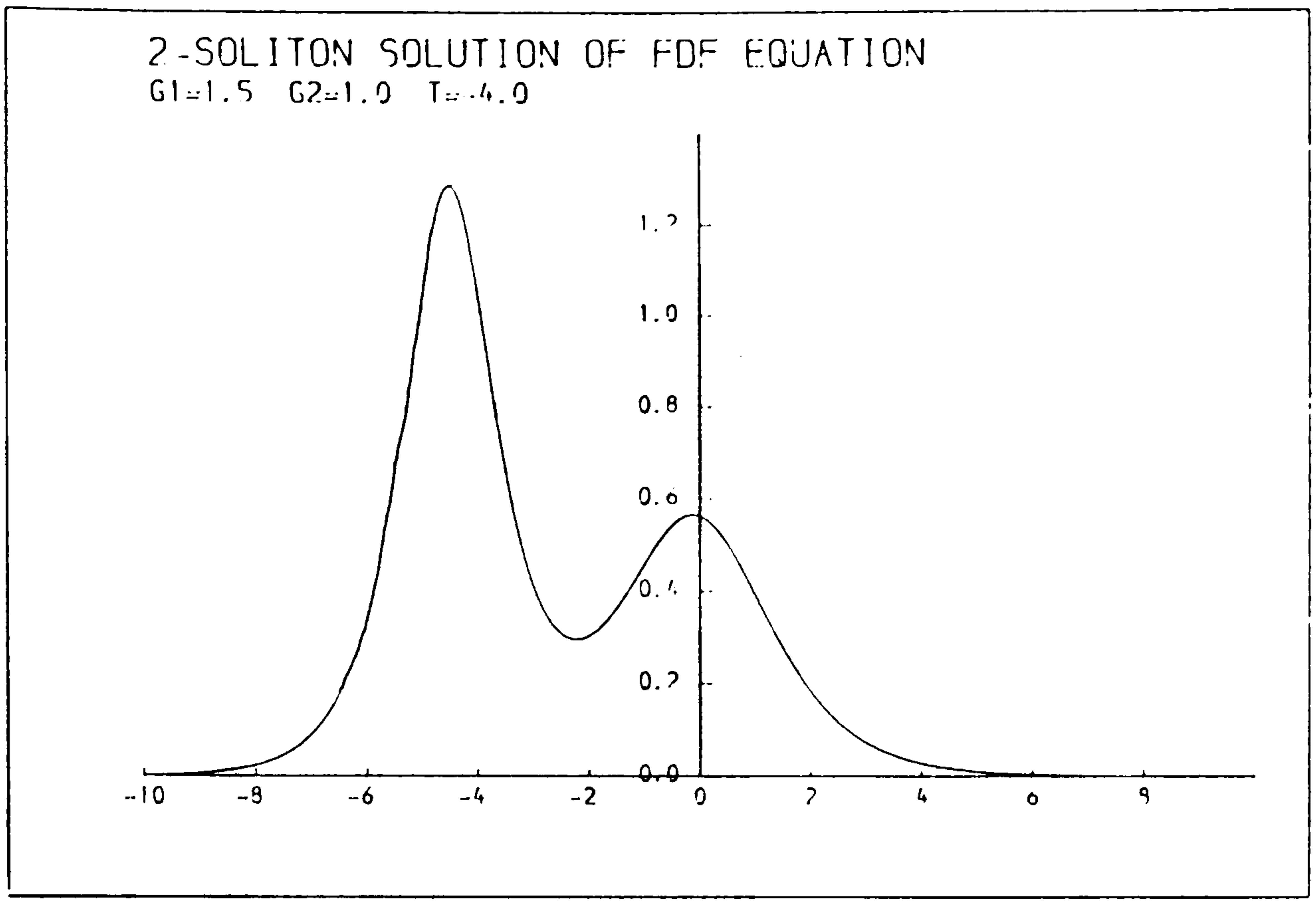


Fig. 3.2a

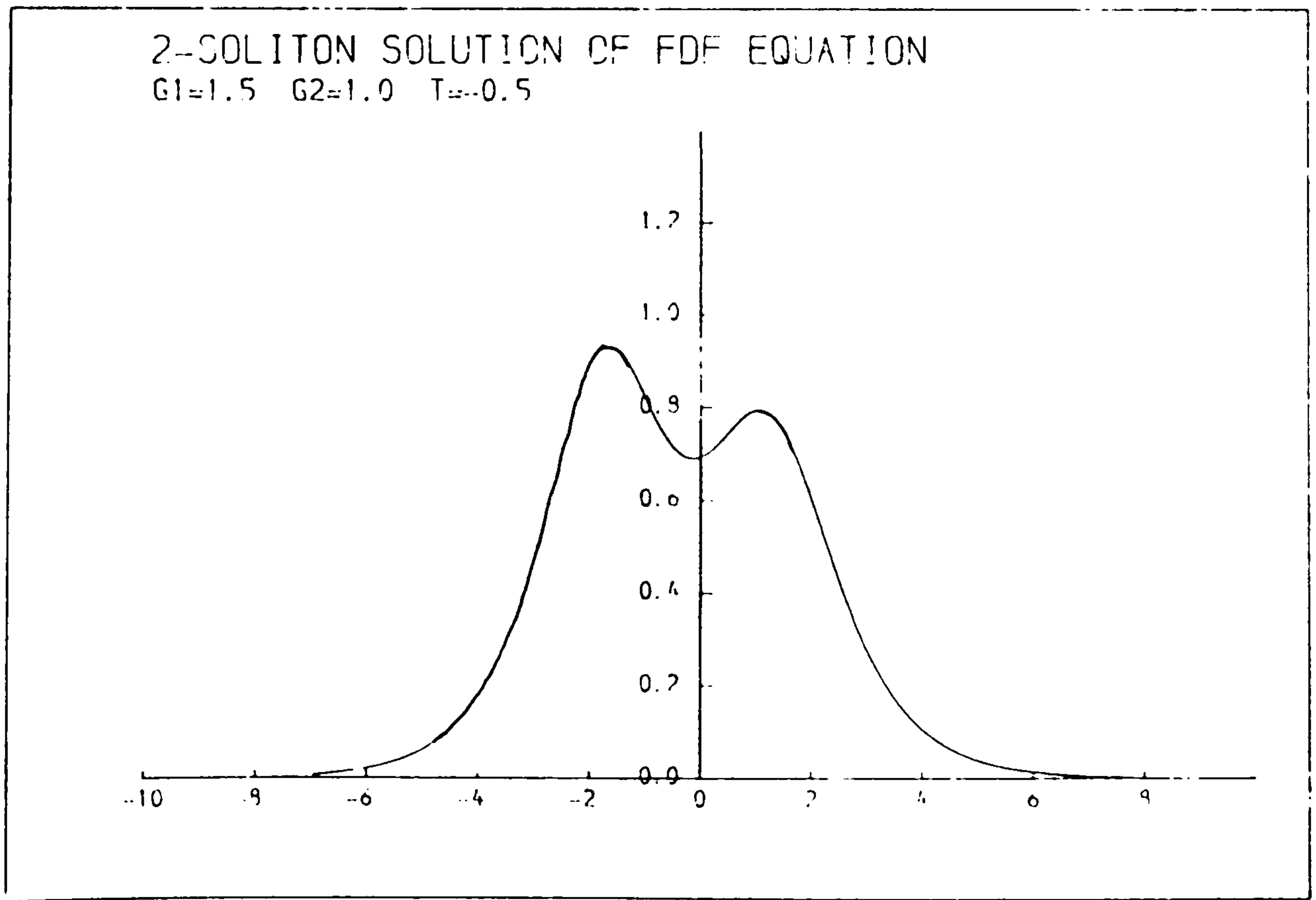


Fig. 3.2b

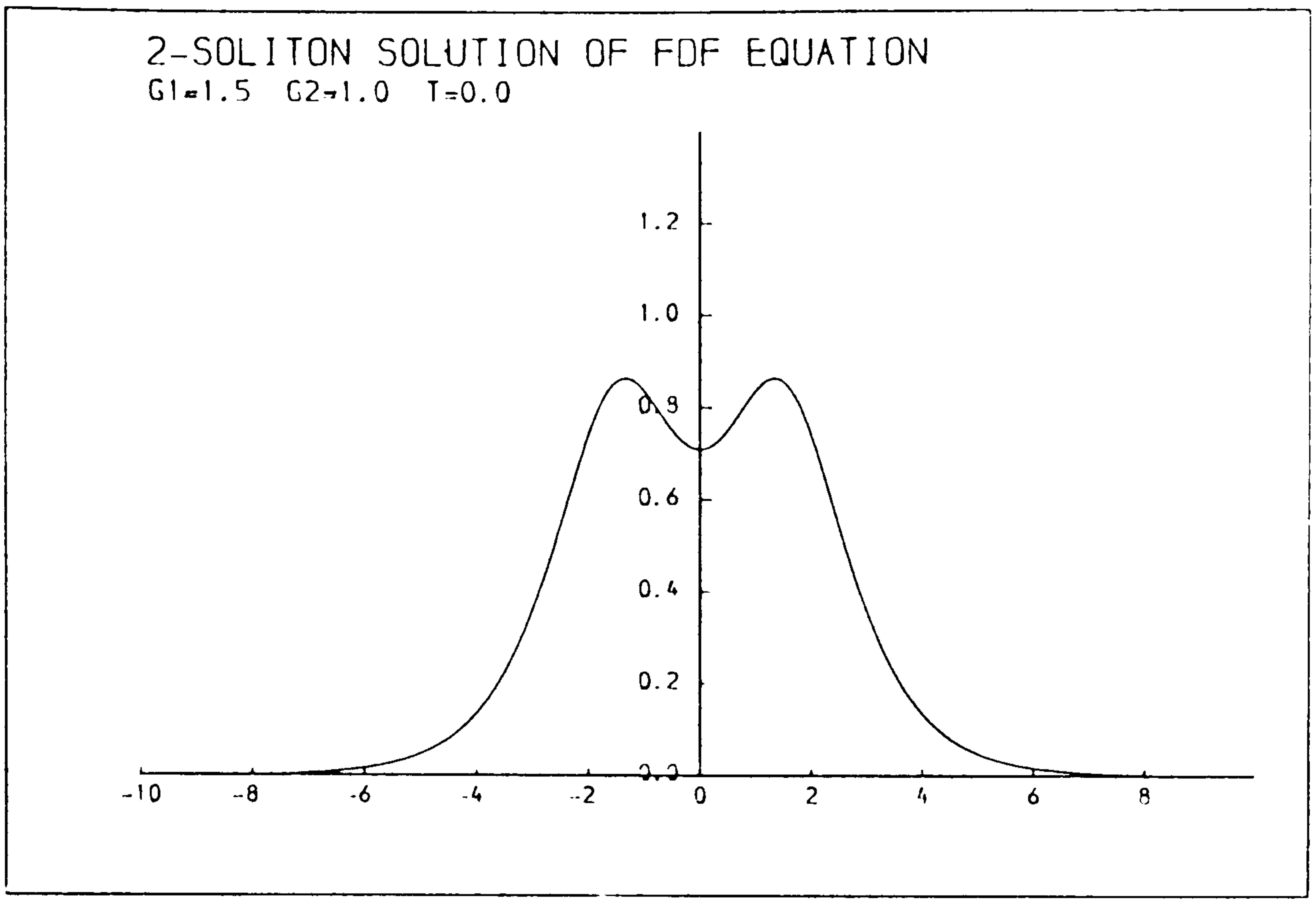


Fig. 3.2c

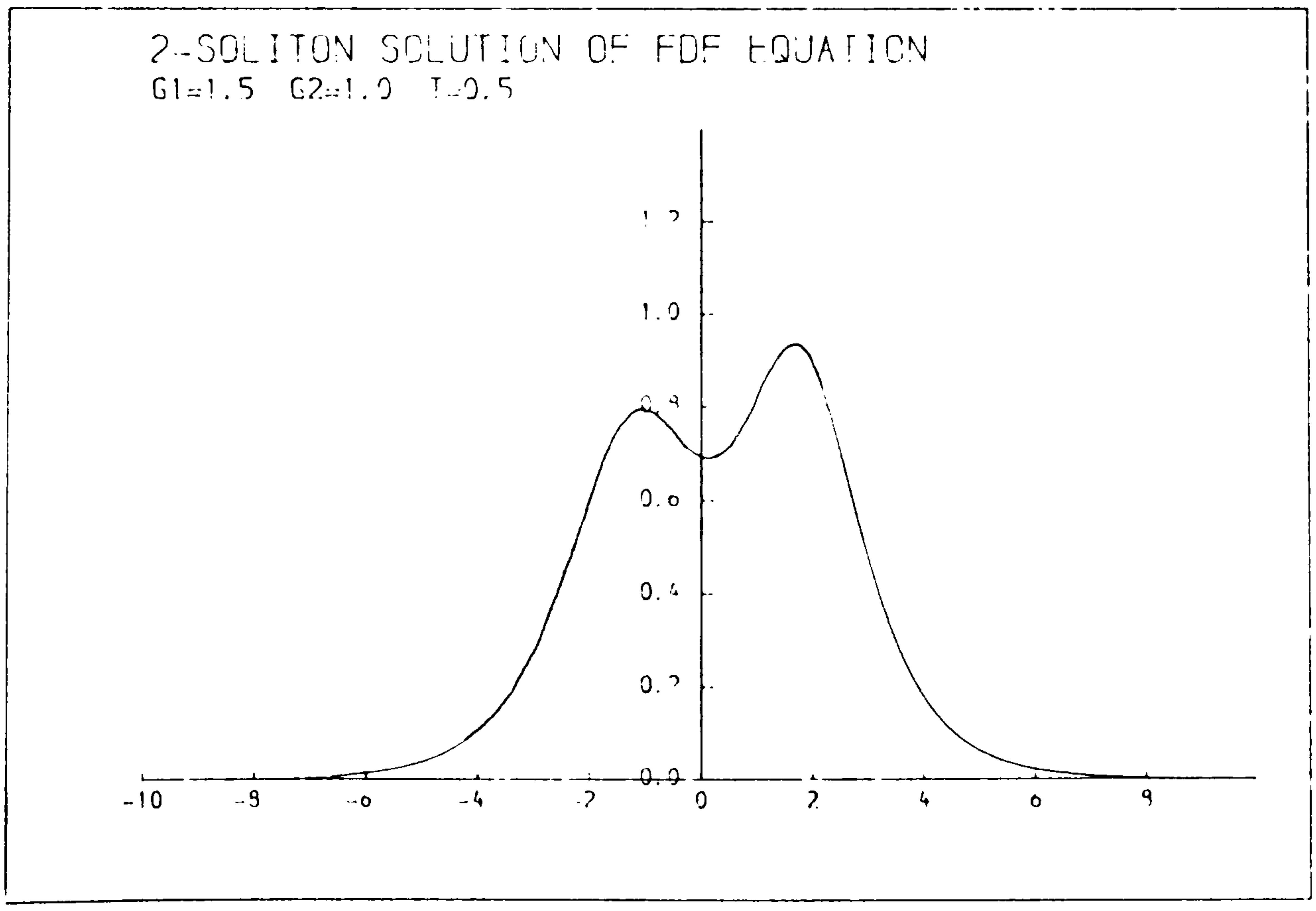


Fig. 3.2d

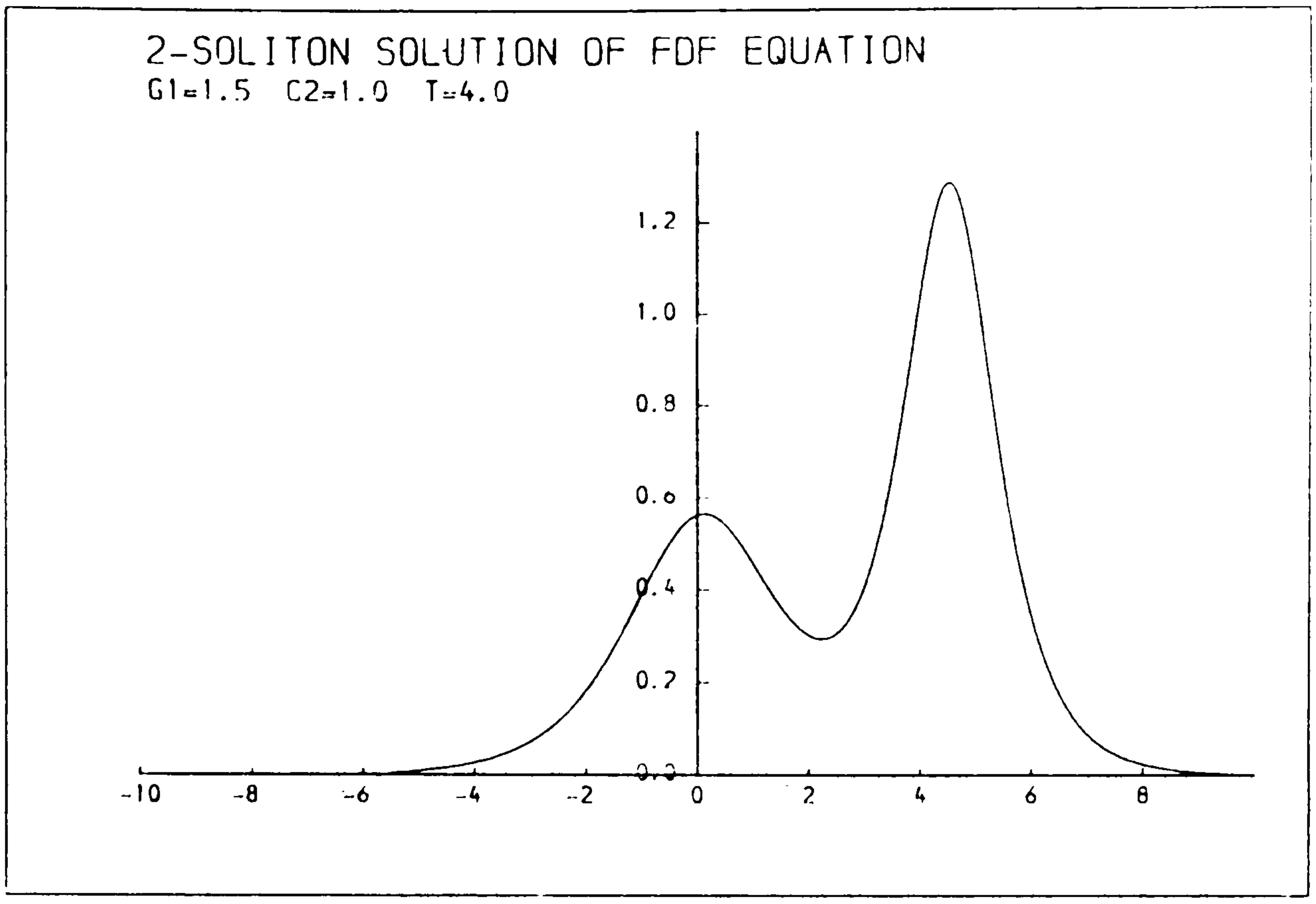


Fig. 3.2e

Fig. 3.2 Showing interactions at time instants (a) t = - 4.0 (b) t = - 0.5 (c) t = 0 (d) t = 0.5 (e) t = 4.0 between two FDF solitons with λ = 1.0, γ_1 = 1.5 and γ_2 = 1.0. This corresponds to the case of double peaks in the mid-interaction since $Q_2R_1 - Q_1R_2 \approx 0.88721$. Note the symmetry between figures 3.3a and 3.3e and also between figures 3.3b and 3.3d.

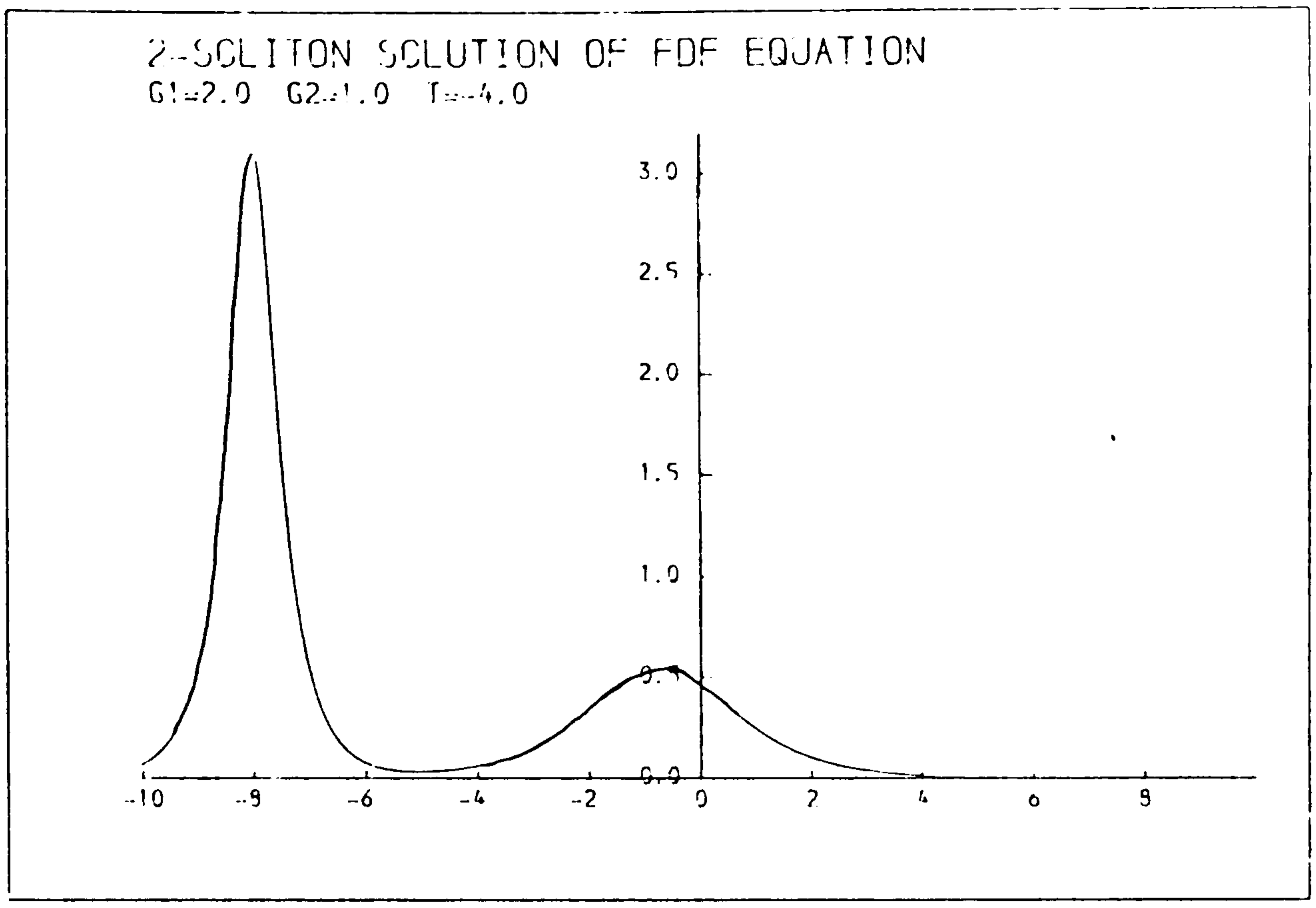


Fig. 3.3a

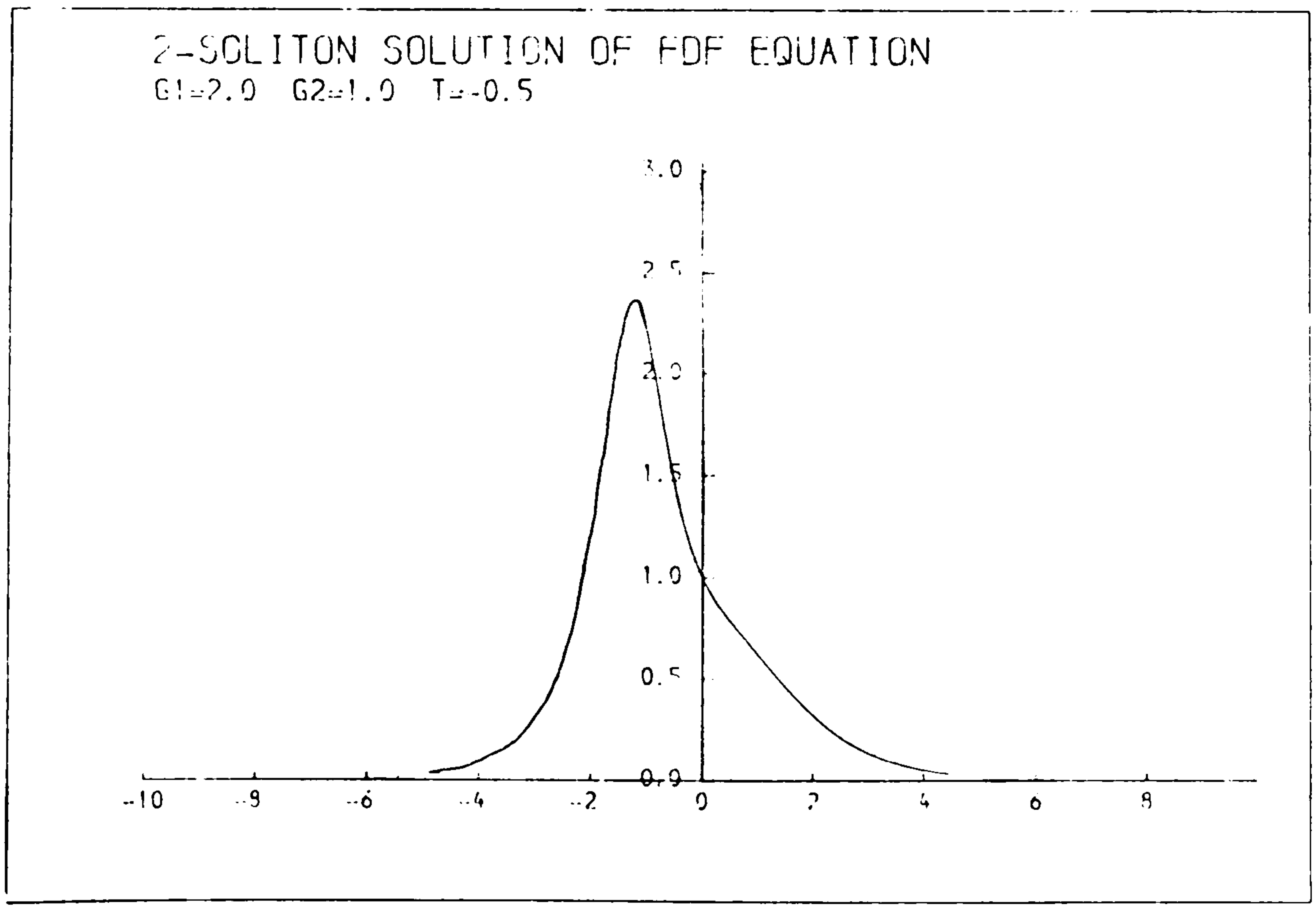


Fig. 3.3b

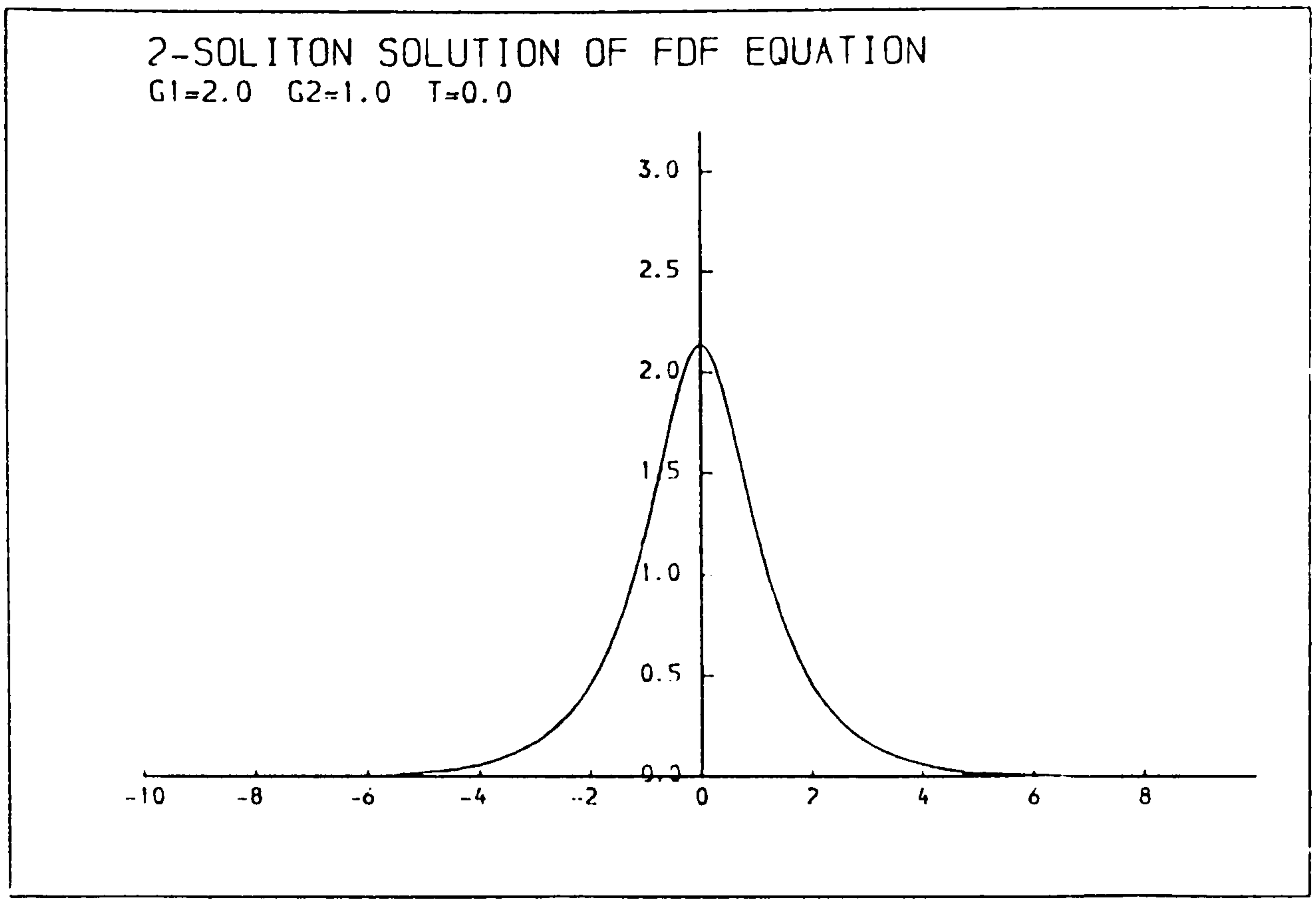


Fig. 3.3c

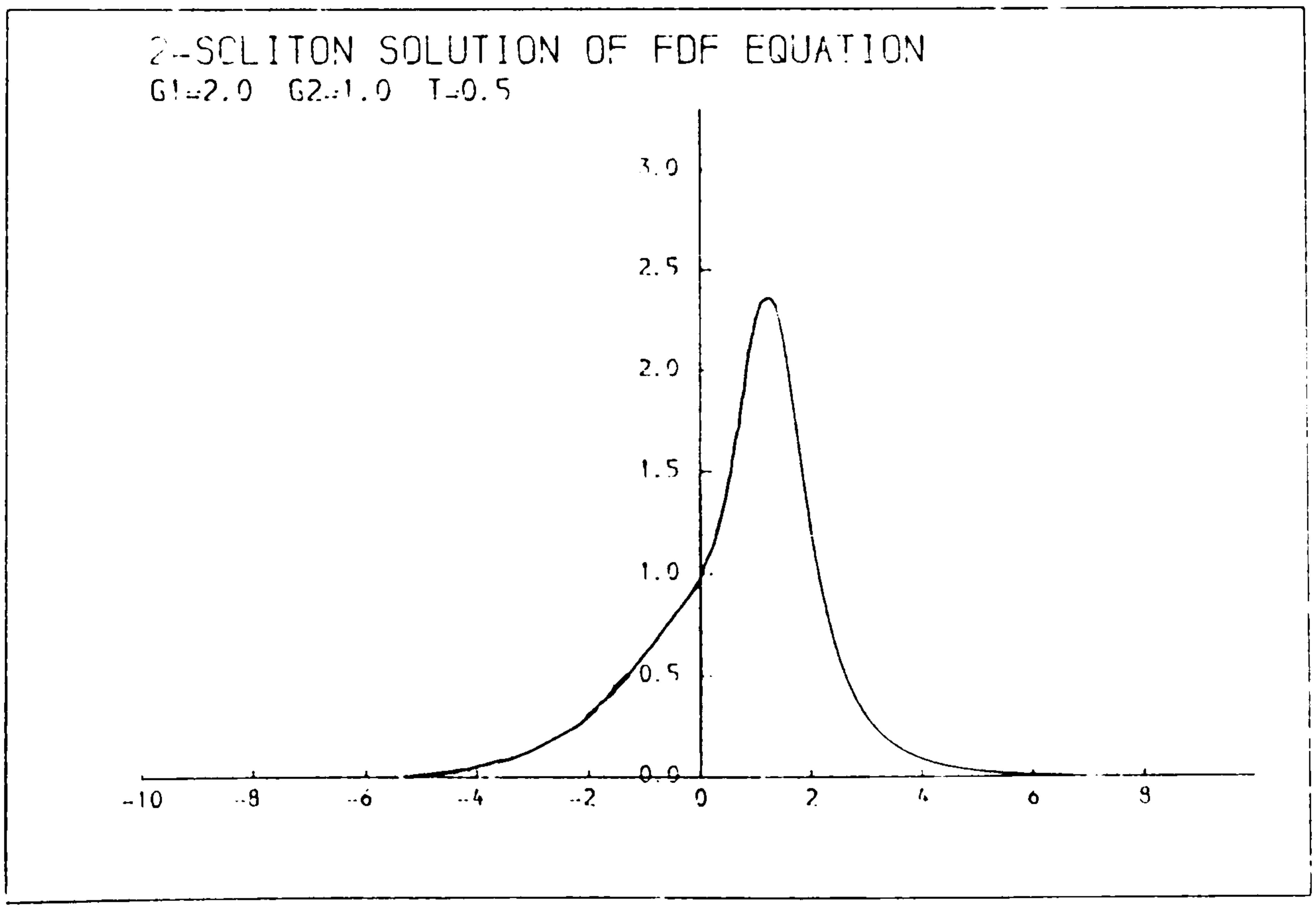


Fig. 3.3d

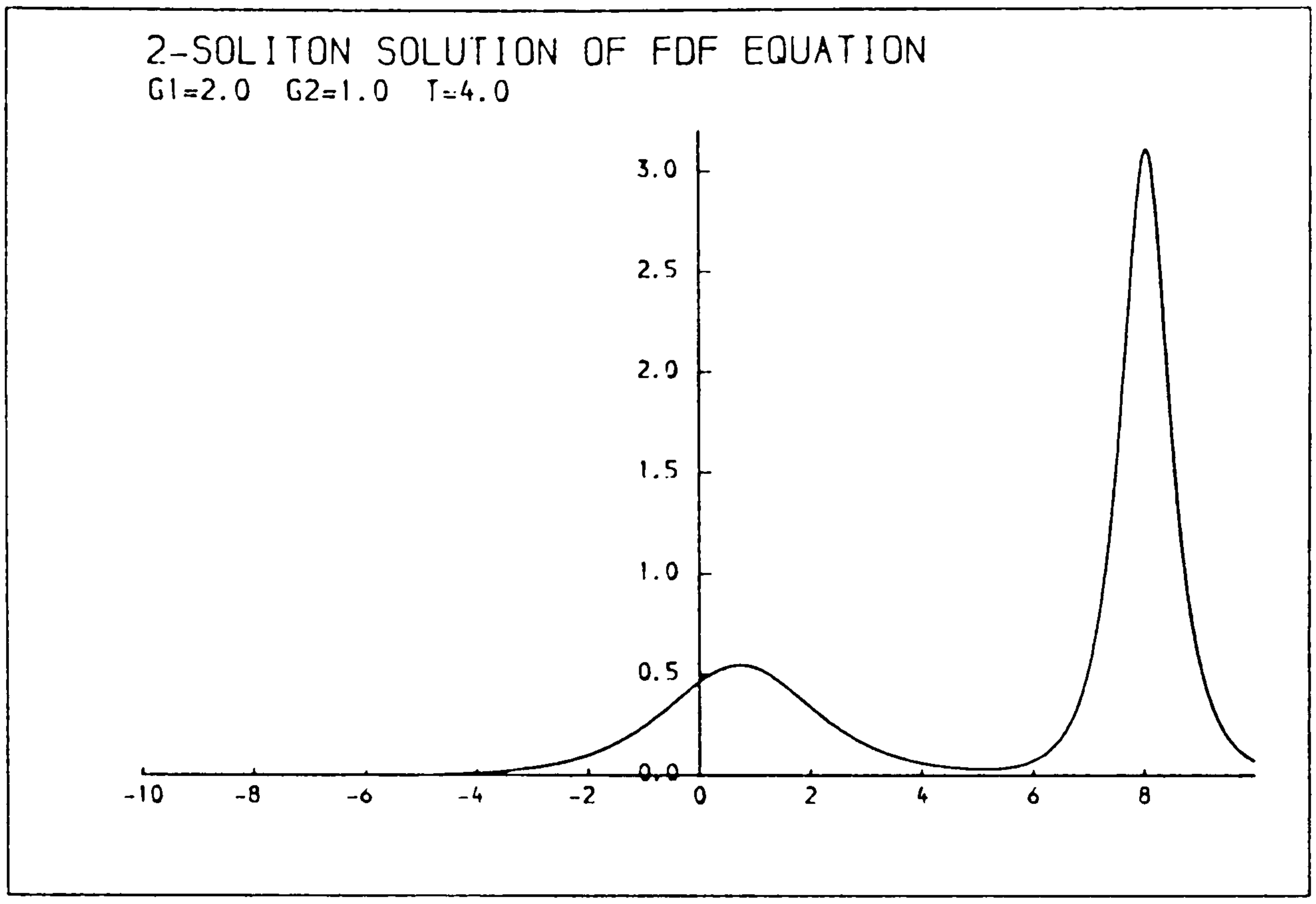


Fig. 3.3e

Fig. 3.3 Showing interactions at time instants (a) $t = -4.0$ (b) $t = -0.5$ (c) $t = 0$ (d) $t = 0.5$ (e) $t = 4.0$ between two FDF solitons with $\lambda = 1.0, \gamma_1 = 2.0$ and $\gamma_2 = 1.0$ This corresponds to the case of a single peak in the middle of the interaction since $Q_2R_1 - Q_1R_2 \approx -2.53396$. Note the symmetry between figures 3.3a and 3.3b and also between figures 3.3b and 3.3d.

For $\lambda=1$, $\gamma_1=2.0$ and $\gamma_2=1.0$, from Table 3.1, $Q_2R_1 - Q_1R_2$ is negative and thus gives only a single peak in the middle of the interaction. In this case the taller soliton, while catching up the shorter one, tries to 'combine' its mass with the second soliton [Figs 3.3a, b] before both of the solitons become one [Fig 3.3c].

After time t=0 they split again until the taller soliton leads the way. These interactions are shown in Figs 3.3d, e.

For the KdV solitons ($\lambda\to\infty$) the transformation $x = \lambda^{-1/2}X$ does not alter the peak conditions (3.4.45) and (3.4.46). In this limit we have

$$Q_1 = 2C_{12}^{1/2}(\gamma_1 \sin \gamma_1 + \gamma_2 \sin \gamma_2) + (\gamma_1-\gamma_2) \sin (\gamma_1-\gamma_2) + C_{12}(\gamma_1+\gamma_2) \sin (\gamma_1+\gamma_2)$$

$$= 2\frac{(P_1-P_2)}{P_1+P_2}\left(\frac{P_1^2}{\lambda} + \frac{P_2^2}{\lambda}\right) + \frac{(P_1-P_2)^2}{\lambda} + \left[\frac{P_1-P_2}{P_1+P_2}\right]^2 \frac{(P_1+P_2)^2}{\lambda}$$

$$= \frac{4P_1^2(P_1-P_2)}{\lambda(P_1+P_2)}$$

$$Q_2 = C_{12}^{1/2}\gamma_1\gamma_2(\gamma_2 \sin \gamma_1 + \gamma_1 \sin \gamma_2)$$

$$= \left(\frac{P_1-P_2}{P_1+P_2}\right)\frac{P_1P_2}{\lambda}\left(\frac{P_1P_2}{\lambda} + \frac{P_1P_2}{\lambda}\right)$$

$$= \frac{2(P_1-P_2)P_1^2P_2^2}{\lambda^2(P_1+P_2)}$$

$$R_1 = 2C_{12}^{1/2}(\cos\gamma_1 + \cos\gamma_2) + \cos(\gamma_1-\gamma_2) + C_{12} + 1$$
$$+C_{12}\cos(\gamma_1+\gamma_2)$$
$$= 2\left(\frac{P_1-P_2}{P_1+P_2}\right)(2) + 1 + \left[\frac{P_1-P_2}{P_1+P_2}\right]^2 + 1 + \left[\frac{P_1-P_2}{P_1+P_2}\right]^2$$
$$= \frac{8P_1^{\,2}}{(P_1+P_2)^2}$$

$$R_2 = C_{12}^{1/2}(\gamma_1^{\,2}\cos\gamma_2 + \gamma_2^{\,2}\cos\gamma_1) + \frac{(\gamma_1-\gamma_2)^2}{2} + C_{12}\frac{(\gamma_1+\gamma_2)^2}{2}$$

$$= \left(\frac{P_1-P_2}{P_1+P_2}\right)\left(\frac{P_1^{\,2}}{\lambda} + \frac{P_2^{\,2}}{\lambda}\right) + \frac{(P_1-P_2)^2}{2\lambda} + \left[\frac{P_1-P_2}{P_1+P_2}\right]^2\frac{(P_1+P_2)^2}{2\lambda}$$

$$= \frac{2(P_1-P_2)P_1^{\,2}}{\lambda(P_1+P_2)}.$$

Using the above expressions for Q_1, Q_2, R_1 and R_2 we then find

$$Q_2R_1 - Q_1R_2 = \frac{8P_1^{\,2}}{(P_1+P_2)^2}\cdot\frac{2(P_1-P_2)P_1^{\,2}P_2^{\,2}}{\lambda^2(P_1+P_2)}$$

$$- \frac{4P_1^2(P_1-P_2)}{\lambda(P_1+P_2)}\cdot\frac{2(P_1-P_2)P_1^2}{\lambda(P_1+P_2)}$$

$$= \frac{8P_1^{\,4}(P_1-P_2)}{\lambda^2(P_1+P_2)^3}\{3P_2^{\,2} - P_1^{\,2}\};\ P_1>P_2.$$

Hence from (3.4.45) and (3.4.46) the peak conditions for the KdV solitons are

$$3P_2^{\,2} > P_1^{\,2} \text{ for the double peaks} \tag{3.4.47}$$

and

$$3P_2^{\,2} < P_1^{\,2} \text{ for a single peak .} \tag{3.4.48}$$

We note that these results can be obtained directly from the KdV solution itself and this has been treated elsewhere [Williams (1974)].

In the Benjamin-Ono limit we must be careful in working with the limiting process involved. Indeed, calculations for Q_1 and R_1 show that $O(\lambda^4)$ is the lowest order term. Therefore $O(\lambda^4)$ terms must be included in the expressions for C_{12} and b_n, otherwise a number of dominant terms will be lost. In this limit we find from (3.4.8) and (3.4.6)

$$b_n = \frac{1}{\lambda}(V_n - \frac{\pi^2\lambda^2}{3V_n} + \frac{\pi^2\lambda^3}{3V_n^2} - \frac{\pi^4\lambda^4}{45V_n^4})$$

$$C_{12} = 1 - \frac{4\pi^2\lambda^2}{(V_1-V_2)^2} + \frac{4\pi^2\lambda^3(V_1+V_2)}{V_1V_2(V_1-V_2)^2} - \frac{4\pi^2\lambda^4}{V_1V_2(V_1-V_2)^2}$$

$$+ \frac{8\pi^4\lambda^4}{3V_1V_2(V_1-V_2)^2} + \frac{16\pi^4\lambda^4}{(V_1-V_2)^4} + O(\lambda^5).$$

This gives

$$C_{12}^{1/2} = 1 - \frac{2\pi^2\lambda^2}{(V_1-V_2)^2} + \frac{2\pi^2\lambda^3(V_1+V_2)}{V_1V_2(V_1-V_2)^2} - \frac{2\pi^2\lambda^4}{V_1V_2(V_1-V_2)^2} + \frac{4\pi^4\lambda^4}{3V_1V_2(V_1-V_2)^2}$$

$$+ \frac{6\pi^4\lambda^4}{(V_1-V_2)^4} + O(\lambda^5)$$

upon using the Binomial expansion. Using the above expressions we then find

$$Q_1 = \lambda^4\pi^4\{\frac{4}{(V_1-V_2)^2}\frac{(V_1+V_2)}{V_1V_2} + \frac{1}{V_1^2V_2} + \frac{1}{V_1V_2^2}\}$$

$$= \frac{\lambda^4\pi^4}{(V_1-V_2)^2V_1^2V_2^2}(V_1+V_2)^3$$

$$Q_2 = \lambda^2\pi^4(\frac{1}{V_1} + \frac{1}{V_2})$$

$$= \frac{\lambda^2\pi^4}{(V_1-V_2)^2V_1^2V_2^2}\{V_1V_2(V_1+V_2)(V_1-V_2)^2\}$$

$$R_1 = \lambda^4\pi^4\{\frac{8}{(V_1-V_2)^4} + \frac{4}{(V_1-V_2)^2V_1V_2} + \frac{1}{2V_1^2V_2^2}\}$$

$$= \frac{\lambda^4\pi^4}{2(V_1-V_2)^4V_1^2V_2^2} (V_1+V_2)^4$$

and

$$R_2 = \lambda^2\pi^4\{-\frac{4}{(V_1-V_2)^2} + \frac{1}{2V_1^2} + \frac{1}{2V_2^2}\}$$

$$= \frac{\lambda^2\pi^4}{2(V_1-V_2)^4V_1^2V_2^2} (V_1+V_2)^2 \{(V_1-V_2)^4 - 2V_1V_2(V_1-V_2)^2\}.$$

Thus we eventually arrive at

$$Q_2R_1 - Q_1R_2 = \frac{\lambda^6\pi^8}{2(V_1-V_2)^6V_1^4V_2^4} (V_1+V_2)^5 (V_1-V_2)^2 \{3V_1V_2 - (V_1-V_2)^2\}.$$

From this expression the peak conditions are

$$3V_1V_2 - (V_1-V_2)^2 > 0 \text{ for double peaks} \qquad (3.4.49)$$

and

$$3V_1V_2 - (V_1-V_2)^2 < 0 \text{ for a single peak.} \qquad (3.4.50)$$

We have thus obtained all the conditions for double and single peaks in the middle of the interaction for the Benjamin-Ono equation.

We note here that the symmetric two-soliton solution in the Benjamin-Ono limit is exactly the form (3.4.28) except that in the symmetric coordinates x,t ζ_n is now defined as

$$\zeta_n = x - V_n t, \; n=1, 2.$$

In Figs. 3.4a, b, c, d, e we show the consecutive interactions at various time instants between two solitons which satisfy condition (3.4.49). The double peaks are shown in Fig. 3.4c.

For the condition (3.4.50), the plots for the interactions are given in Figs. 3.5a, b, c, d, e in which the single peak is shown in Fig. 3.5c.

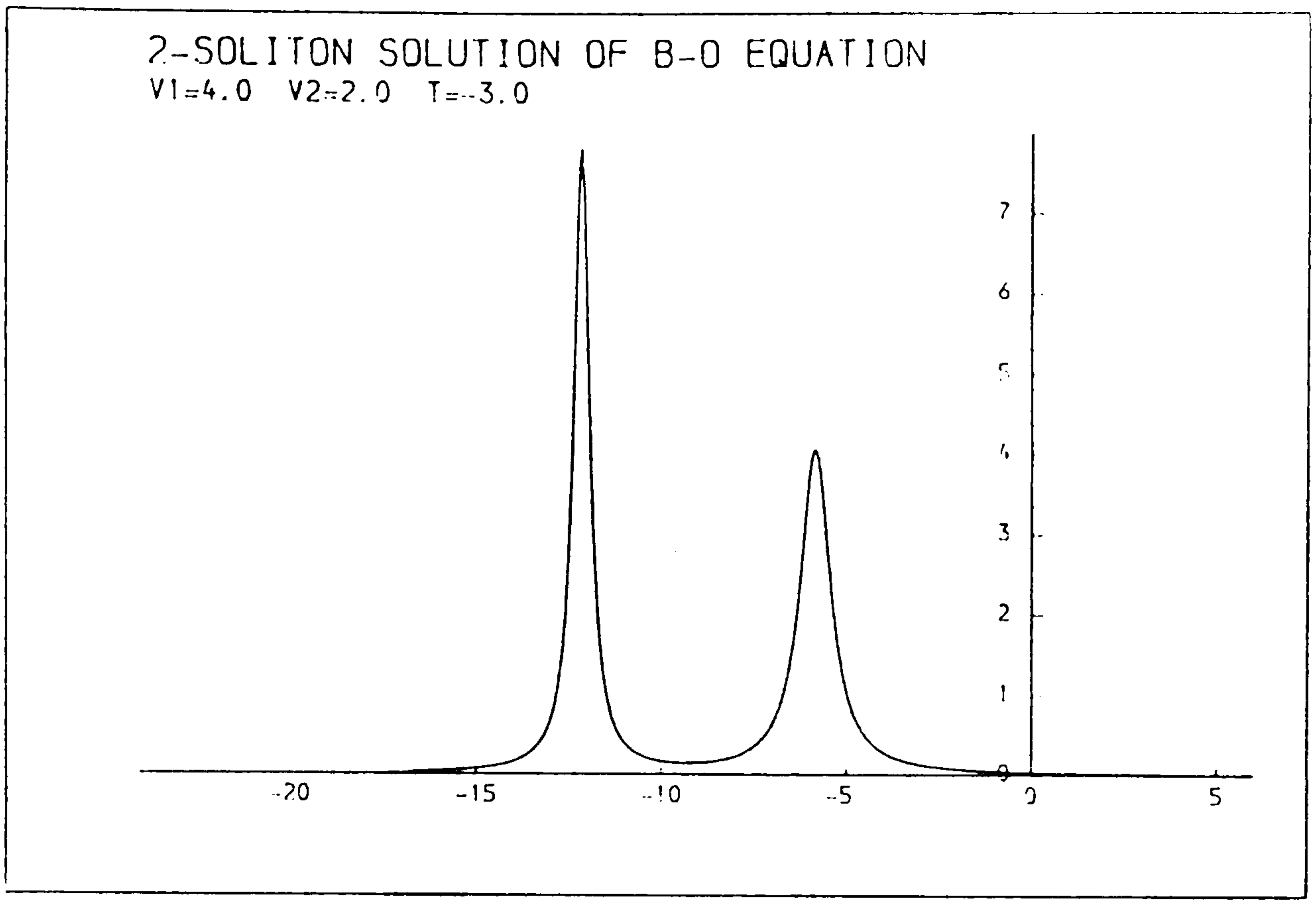


Fig. 3.4a

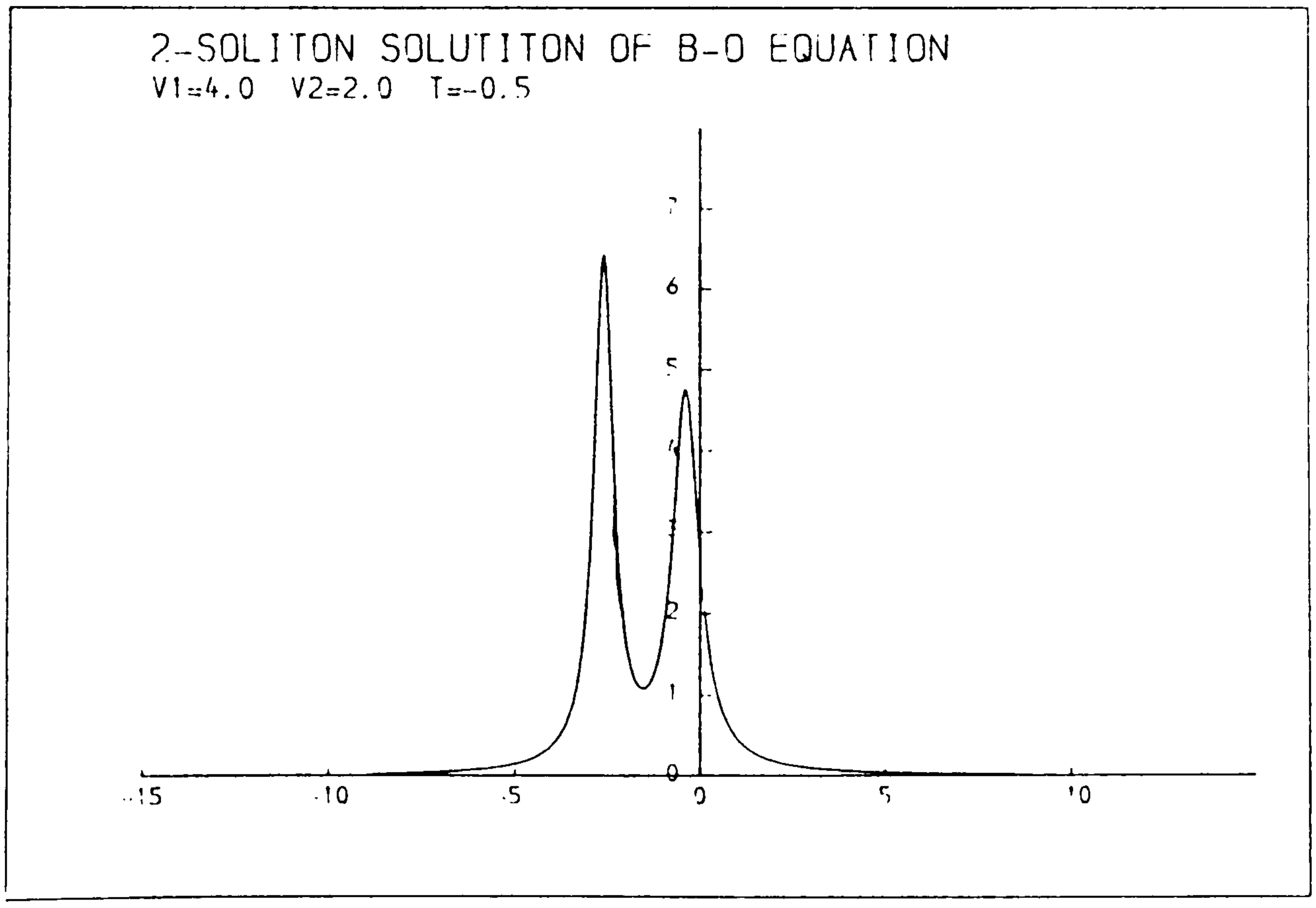


Fig. 3.4b

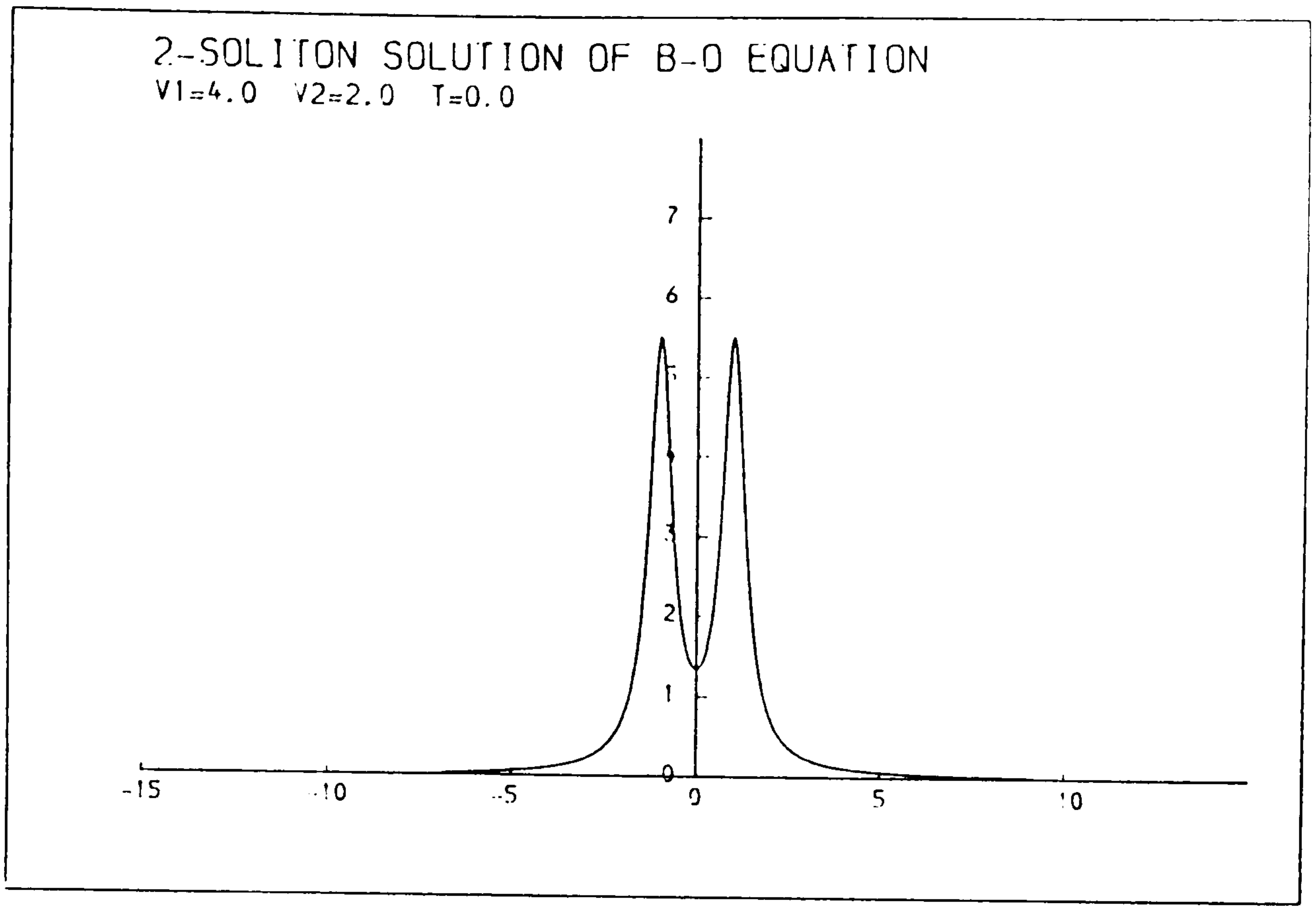


Fig. 3.4c

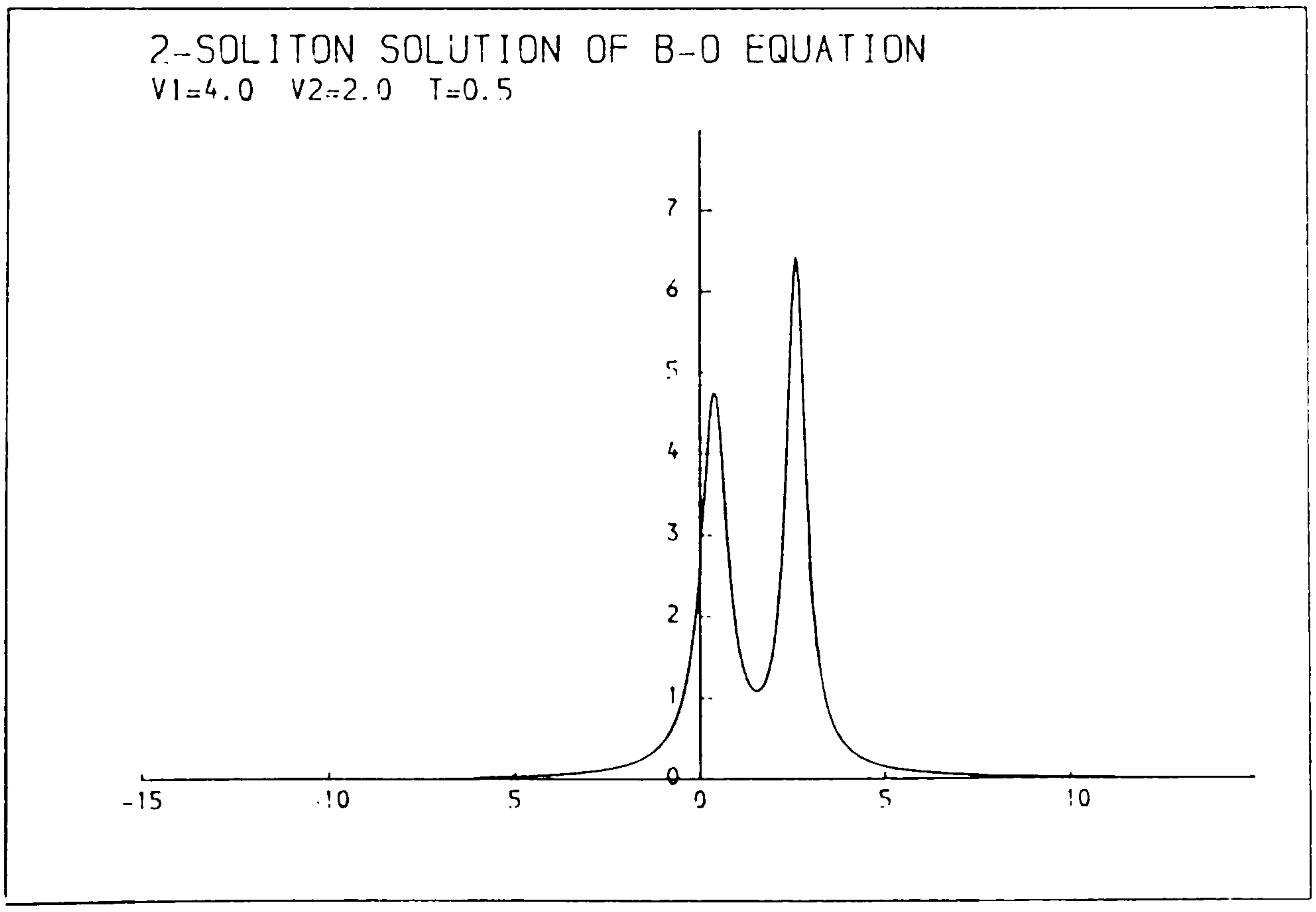


Fig. 3.4d

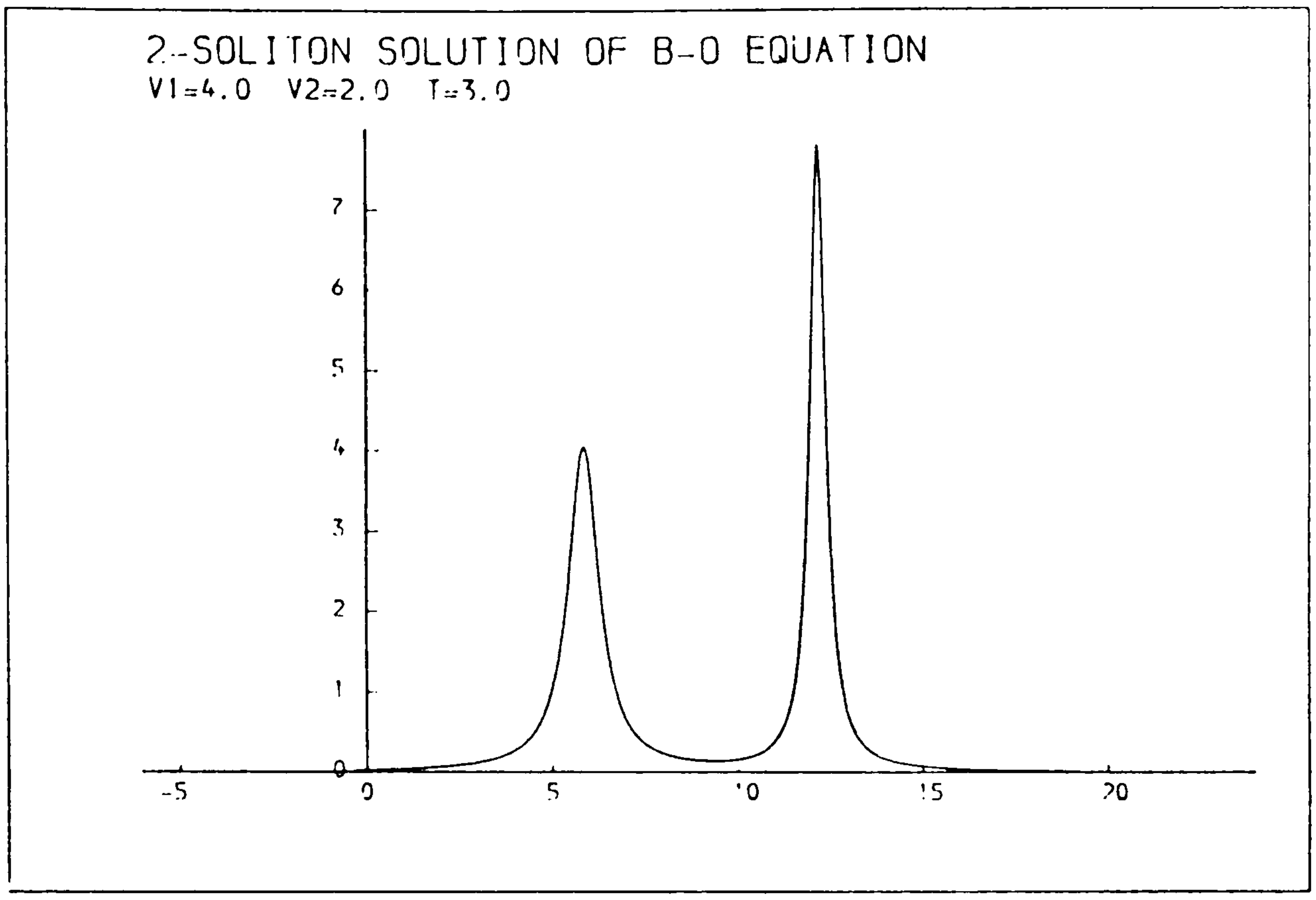


Fig. 3.4e

Fig. 3.4 Showing interactions at time instants (a) $t = -3.0$ (b) $t = = 0.5$ (c) $t = 0$ (d) $t = 0.5$ (e) $t = 3.0$ between two B-O solitons with $V_1 = 4.0$ and $V_2 = 2.0$. There are two peaks in the middle of the interaction ($t = 0$) as $3V_1V_2 - (V_1 - V_2)^2 = 20.0$. Note the symmetry between Fig. 3.4a and Fig. 3.4e and also between Fig. 3.4b and Fig. 3.4d.

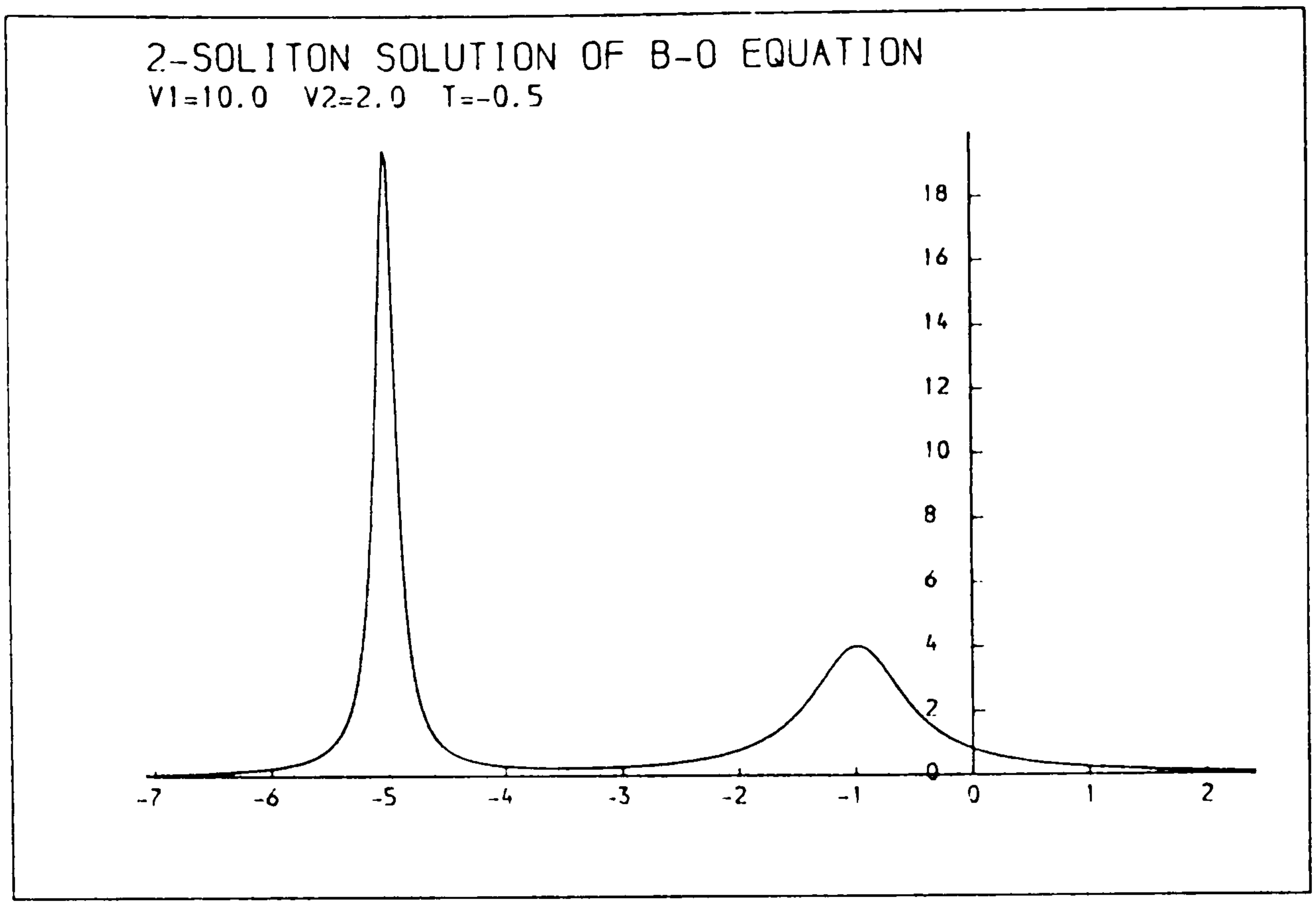


Fig. 3.5a

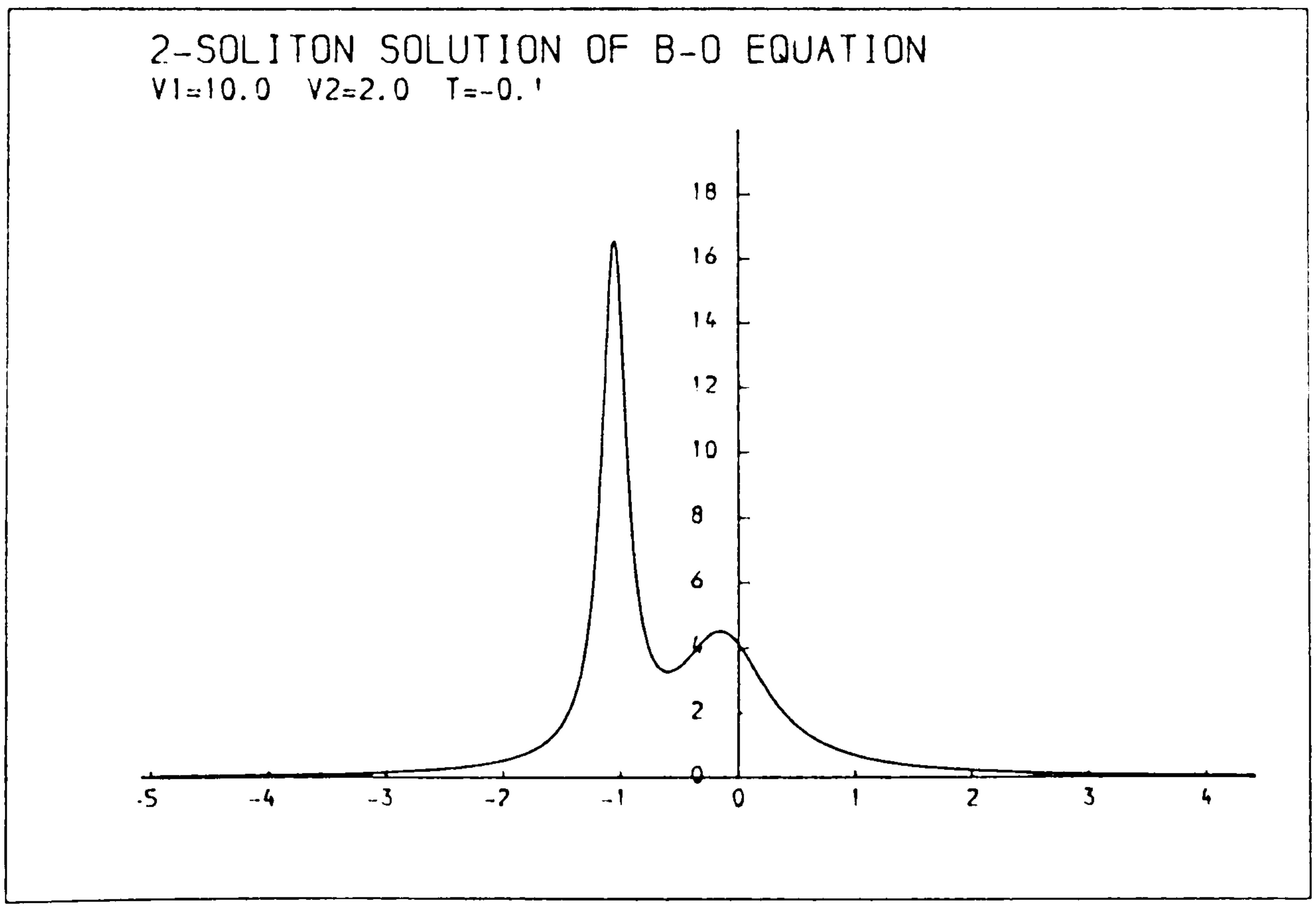


Fig. 3.5b

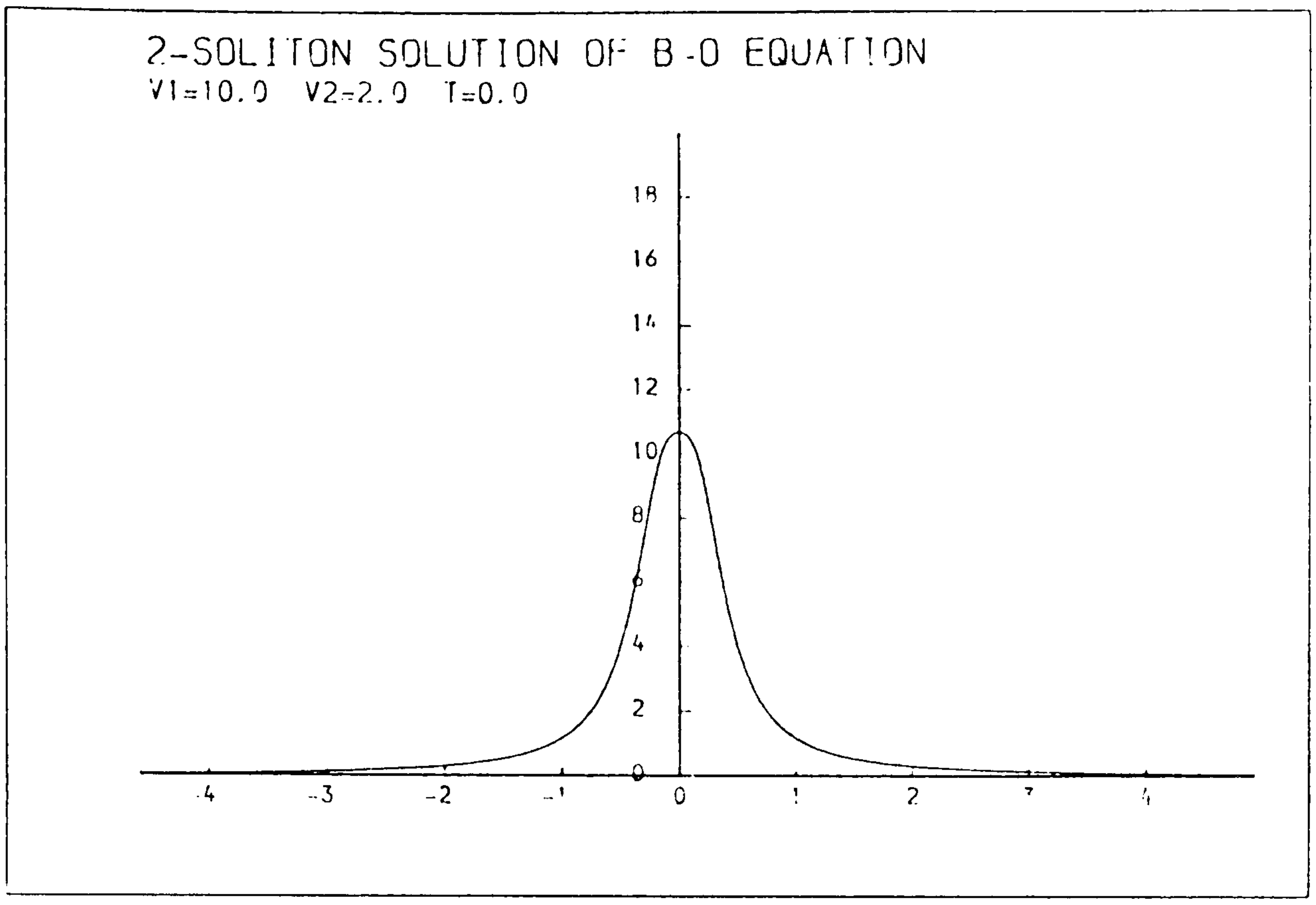


Fig. 3.5c

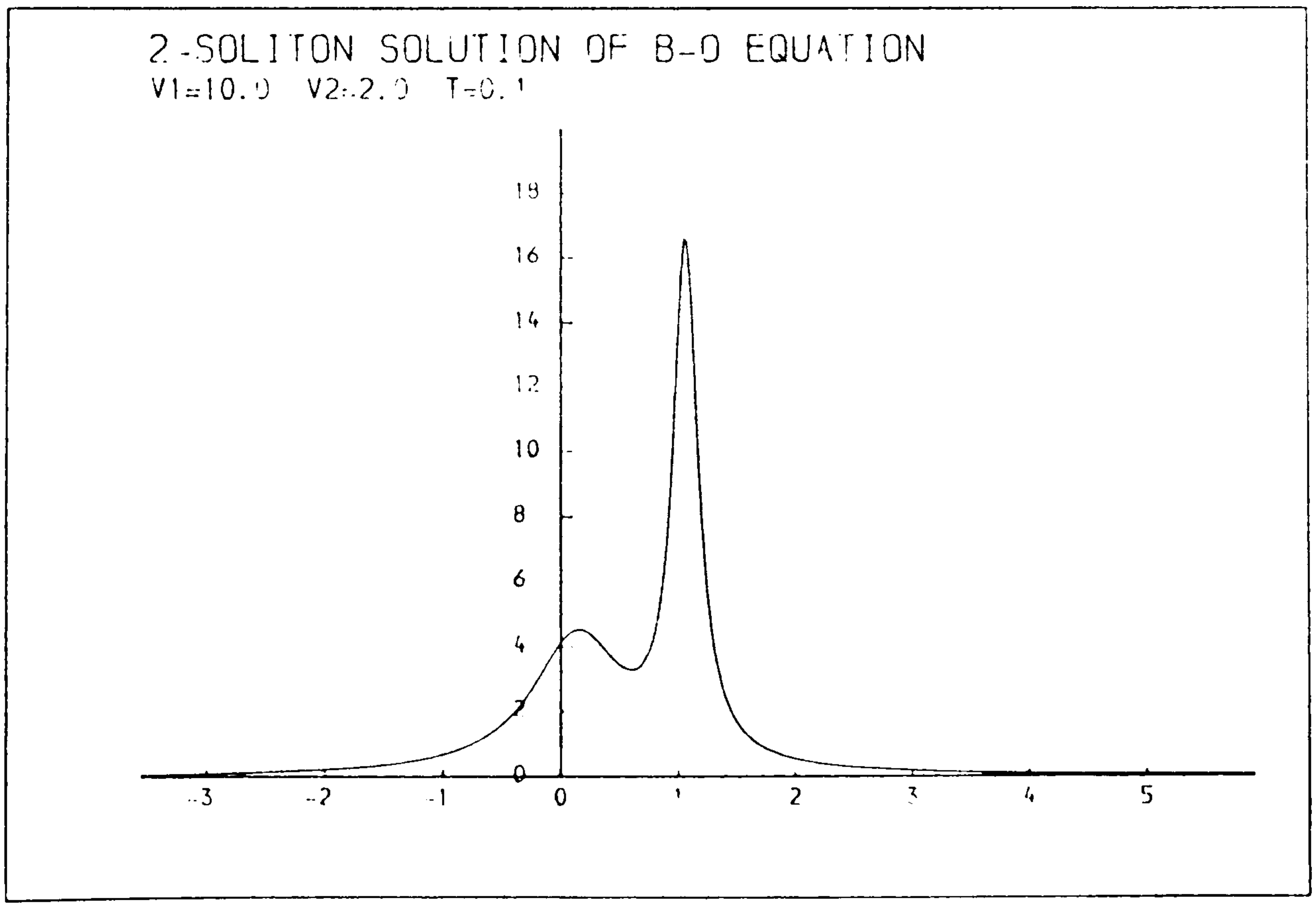


Fig. 3.5d

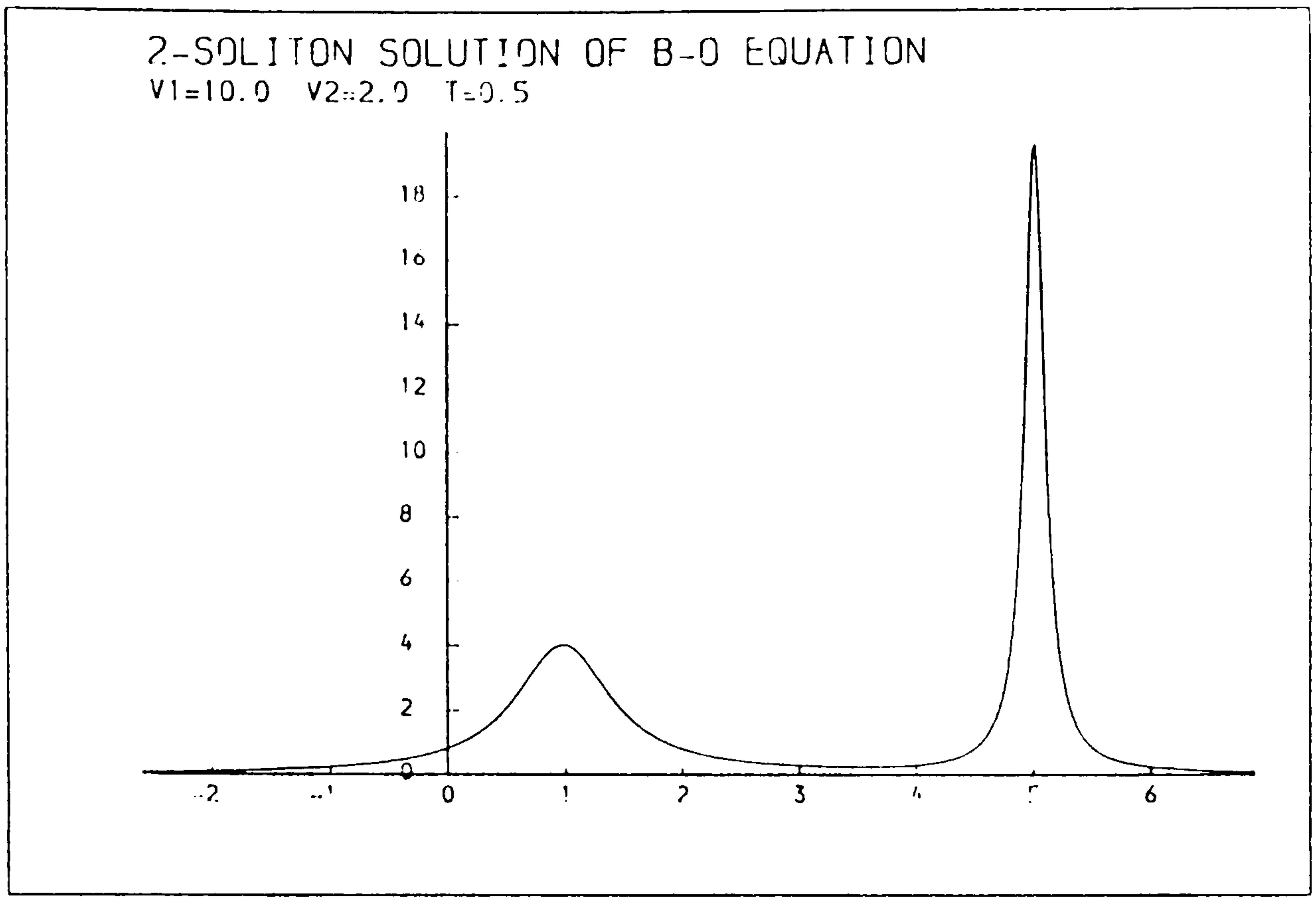


Fig. 3.5e

Fig. 3.5 Showing interactions at time instants (a) t = -0.5 (b) t = -0.1 (c) t = 0 (d) t = 0.1 (e) t = 0.5 between two Benjamin-Ono solitons characterised by $V_1 = 10.0$ and $V_2 = 2.0$. This corresponds to a single peak in the middle of the interaction since $3V_1V_2 - (V_1 - V_2)^2 = -4.0$. Note the symmetry between figures 3.5a and 3.5e and also between figures 3.5b and 3.5d.

CHAPTER 4

RESONANCE PHENOMENA WITH REFERENCE TO THE KADOMTSEV - PETVIASHVILI EQUATION

4.1 Notes on previous work

Perhaps the first observation on resonance phenomena in solitons was made by Scott-Russell when he wrote "the magnitude of the reflected wave diminishes as the angle of incidence diminishes, until at length, when the angle of the ridge of the wave is within 15^o or 20^o of being perpendicular to the plane, reflexion ceases, the size of the wave near the point of incidence and its velocity rapidly increases, and it moves forward rapidly with a high crest at right angles to the resisting surface". [Scott-Russell (1844)].

However, the study of this phenomenon in soliton interactions was only carried out much later by Miles (1977). In his study of two interacting solitons in terms of shallow water wave theory in two space dimensions, he showed that when the angle of intersection of the two solitons is between certain critical angles, the two-soliton solution (with sech^2 profile) becomes singular (with cosech^2 profile) through the interaction, while at the critical angles the two incident solitons interact strongly to produce a third soliton known as a resonant soliton.

This phenomenon has been considered as a breakdown of the Zakharov-Shabat theory of integrable systems with more than one space dimension by Newell and Redekopp (1977). Essentially this breakdown corresponds to the resonance condition

$$\omega(\underline{k}_1-\underline{k}_2) = \omega(\underline{k}_1) - \omega(\underline{k}_2)$$

where $\underline{k}_i$ and ω are the wavelength vectors and frequencies of the phases of the two solitons. This means that the third soliton with frequency $\omega(\underline{k}_1-\underline{k}_2)$

together with the incident solitons with frequencies $w(\underline{k}_1)$ and $w(\underline{k}_2)$ form a triad of solitons. Therefore in a two-resonance-soliton interaction the motion of this triad can be dealt with as a single entity.

The motion of such triads has been used by Anker and Freeman (1978) to describe the interaction of a three-soliton solution of the Kadomtsev -Petviashvili equation. The schematic development of the interaction with time has been obtained by them and shown to approximate closely to computer calculations of the analytic solution.

More recently Ohkuma and Wadati (1983) have described analytically resonant interactions of two and three-soliton solutions of the Kadomtsev-Petviashvili equation. By considering the asymptotic behaviour of the solutions they showed that there are two types of resonances, plus and minus resonances. They also found that there are three arms of solitons stretching to infinity for the two-resonance-soliton interaction and five arms for the three-resonance-soliton interaction. However, the motion of the triads has not been considered in their work.

In this chapter we shall study resonance phenomena between solitons with reference to the Kadomtsev-Petviashvili equation. In section 4.2 resonances in two-soliton solutions are discussed. This concept is then taken as the basis in explaining interactions of a larger number of solitons. In section 4.3, the N-soliton solution of the Kadomtsev-Petviashvili equation in the form of an ordinary determinant is reduced to a Wronskian form when the solitons are assumed to resonate in pairs. The rest of the chapter will discuss the interactions between a triad and a soliton and between two triads.

4.2 Resonant interactions in two-soliton solutions

The Kadomtsev-Petviashvili (KP) equation is generally written in the form

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0. \qquad (4.2.1)$$

The N-soliton solution of this equation has been found by many authors, including Zakharov and Shabat (1974), Satsuma (1976) and Lamb (1980). The N-soliton solution due to Lamb (1980) and parameterized by Freeman (1984) is expressed in terms of an N x N determinant Δ as

$$u = 2\frac{\partial^2}{\partial x^2}(\log \Delta) \qquad (4.2.2)$$

with

$$\Delta = \left| \delta_{ij} + \frac{a_i}{\ell_i + n_j} e^{\theta_{ij}} \right| \qquad (4.2.3)$$

where a_i, ℓ_i, n_i are arbitrary real constants and

$$\theta_{ij} = (\ell_i + n_j)x - (\ell_i^2 - n_j^2)y - 4(\ell_i^3 + n_j^3)t, \quad i,j=1,2,\ldots N. \qquad (4.2.4)$$

The single-soliton solution is given by two terms

$$\Delta = 1 + e^{\eta} \qquad (4.2.5)$$

which gives

$$u = ½(\ell+n)^2 \text{sech}^2 \frac{\eta}{2} \qquad (4.2.6)$$

with $\eta = (\ell+n)x - (\ell^2-n^2)y - 4(\ell^3+n^3)t + \log(\frac{a}{\ell+n})$ as we have seen in the Introduction.

The sech^2 profile given by (4.2.6) is a skewed soliton localized in some neighbourhood of line $\eta=0$ and it extends its length to infinity along this line.

As we mentioned earlier in the Introduction, all the information about a particular soliton solution given by (4.2.6) may also be obtained from its Δ expression (4.2.5). However the form of Δ is more convenient to use for our purpose. Therefore we shall be using Δ instead of the actual solution u in describing soliton interactions throughout this chapter.

For the two-soliton solution, we find from (4.2.3)

$$\Delta = 1 + e^{\eta_1} + e^{\eta_2} + q_{12}e^{\eta_1+\eta_2} \qquad (4.2.7)$$

(1) (2) (3) (4)

where

$$\eta_i = (\ell_i+n_i)x - (\ell_i^2-n_i^2)y - 4(\ell_i^3+n_i^3)t + \log\left(\frac{a_i}{\ell_i+n_i}\right) \qquad (4.2.8)$$

and

$$q_{ij} = \frac{(\ell_i-\ell_j)(n_i-n_j)}{(\ell_i+n_j)(\ell_j+n_i)}, \quad i\neq j=1,2. \qquad (4.2.9)$$

We note that all the terms in (4.2.7) have been numbered (1), (2), (3) and (4) respectively in order to facilitate our description later. The procedure of numbering the terms was introduced by Anker and Freeman (1978) to describe a three-soliton interaction.

As seen from (4.2.5), a soliton is represented by two terms of Δ. Any single term from (4.2.7) will give zero contribution to the final solution u due to (4.2.2). Therefore, a procedure can be chosen so that we look for two dominant terms of Δ in order to locate which soliton is present in a region. For example in the region where $\eta_2 \to -\infty$ and η_1 is fixed, Δ is dominated by

$$\Delta = 1 + e^{\eta_1}$$

(1) (2)

which can be recognized as the soliton characterized by the phase η_1 centered at $\eta_1 = 0$. Continuing the asymptotic process for all possible regions which provide two dominant terms we find

(i) $\Delta = 1 + e^{\eta_1}$ as $\eta_2 \to -\infty$, η_1 fixed

(1) (2)

(ii) $\Delta = 1 + e^{\eta_2}$ as $\eta_1 \to -\infty$, η_2 fixed

(1) (3)

(iii) $\Delta = e^{\eta_1}(1 + q_{12}e^{\eta_2})$ as $\eta_1 \to +\infty$, η_2 fixed

(2) (4)

(iv) $\Delta = e^{\eta_2}(1 + q_{12}e^{\eta_1})$ as $\eta_2 \to +\infty$, η_1 fixed

(3) (4)

(v) $\Delta = e^{\eta_2}(1 + e^{\eta_1-\eta_2})$ as $\eta_1 \to +\infty$, $\eta_2 \to +\infty$

(3) (2)

with $\eta_1-\eta_2$ fixed and q_{12} sufficiently small.

All the exponential factors in (iii), (iv) and (v) may be removed for the actual solution (4.2.2). We note that the number underneath every term in the above expressions is obtained from the number which corresponds to its original term in (4.2.7). Each soliton will then be denoted by these numbers. For example in (i) we have soliton (12) with phase η_1. We note that from (iv) and (i) soliton (34) has the same phase as soliton (12) but it is shifted by $\delta_{12} = \log(q_{12})$. Also soliton (24) has the same phase as soliton (13) but it is shifted by δ_{12}. The lines of the phases of all the above solitons in (i) - (v) can be drawn as in Fig. 4.1.

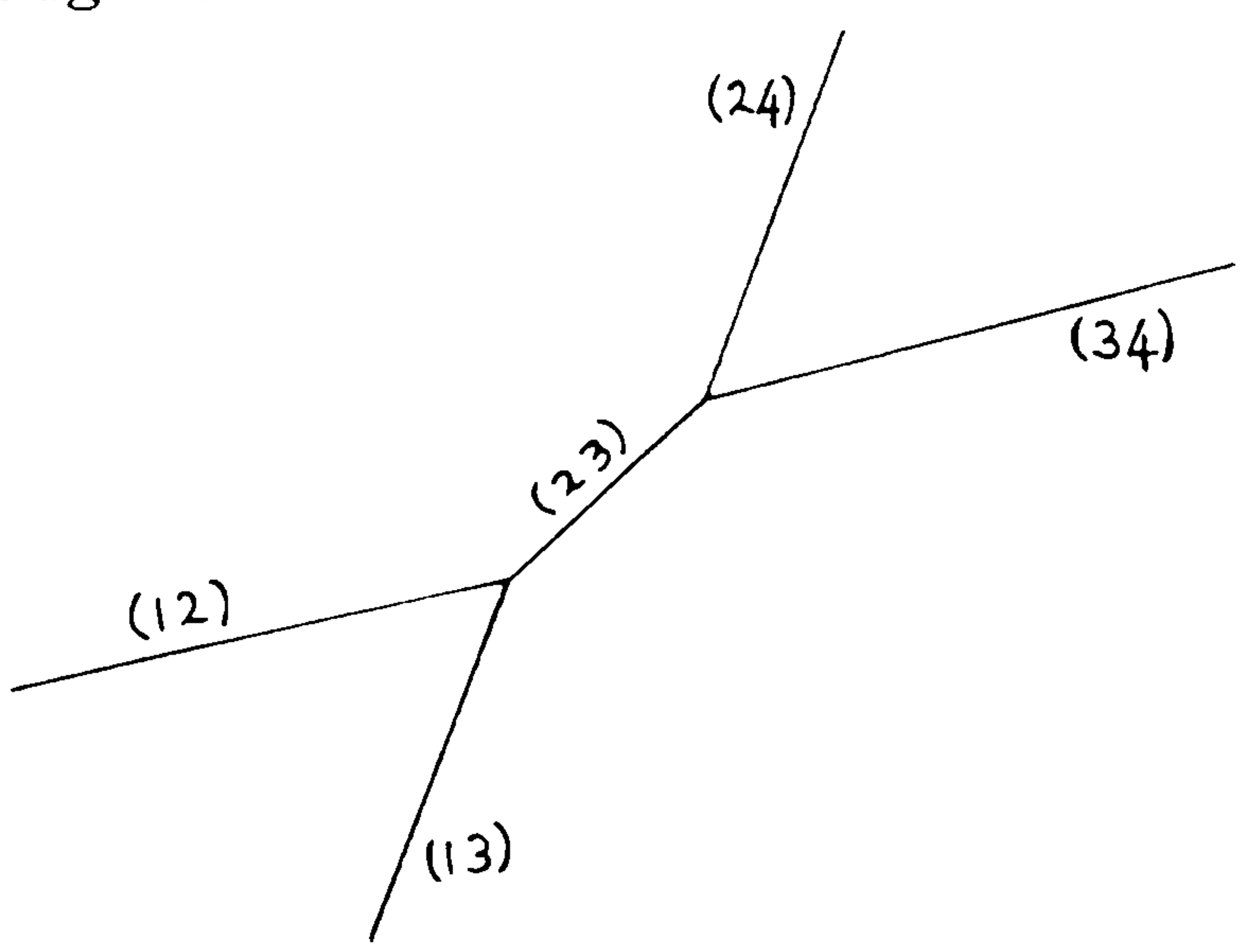


Fig.4.1 A near-resonant interaction ($|\delta_{12}|$=finite).

Now, solitons (12) and (13) can be viewed as the two incident solitons before the interaction, (23) is the interaction region while (34) and (24) are the post-interaction solitons. Solitons (34) and (24) are in fact the same solitons as (12) and (13) respectively but they are shifted by $|\delta_{12}|$.

Therefore the actual configuration depends on the value of $|\delta_{12}|$, whether it is finite, zero or infinite. Fig.4.1 corresponds to the case when $|\delta_{12}|$ is finite so that (23) has a finite arm. This situation can be achieved by choosing $\ell_1-\ell_2 = O(1)$ and $n_1 \approx n_2$. Fig.4.1 shall be referred to as a near-resonant interaction.

In the case of $|\delta_{12}| = 0$, both of the post-interaction solitons (34) and (24) are not shifted at all from the pre-interaction solitons (12) and (13). This means that the arm of (23) is zero and thus (12) and (34) lie on the same line $\eta_1 = 0$ and (13) and (24) lie on the same line $\eta_2 = 0$. This interaction is sketched in Fig. 4.2.

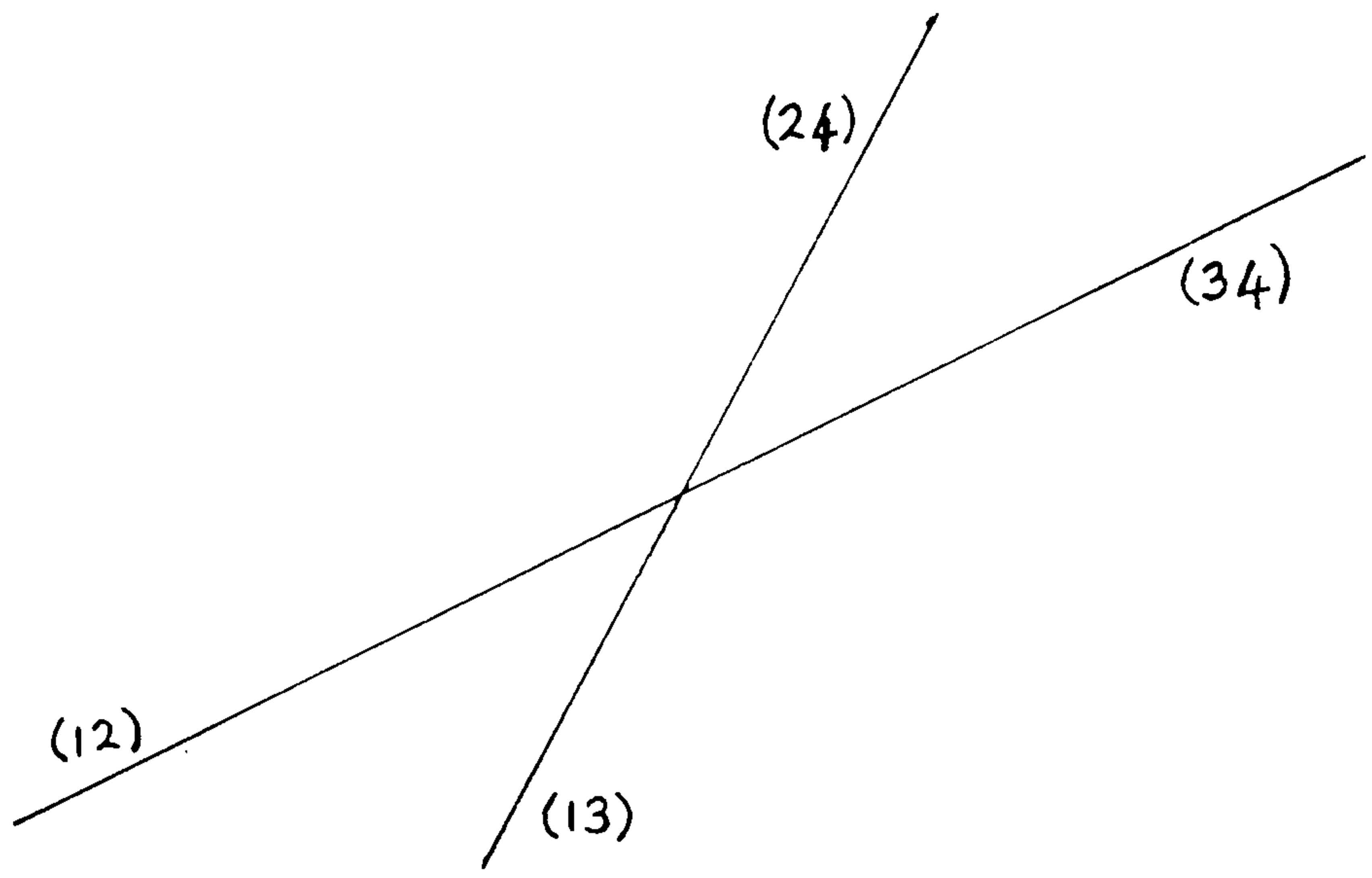


Fig. 4.2 A non-resonant interaction ($|\delta_{12}|=0$).

We shall refer to Fig. 4.2 as a non-resonant interaction since all solitons emerge form the interaction without being shifted at all. If we choose $\ell_1 - \ell_2 = O(1)$ and $n_1 - n_2 = O(1)$ then $|\delta_{12}|$ is small. Therefore we have a situation which is very close to Fig. 4.2. Indeed this choice of parameters will produce a small phase shift which is difficult to see in practice and therefore we shall use this condition in order to mean or to produce a non-resonant interaction. In this limit we have from (4.2.7)

$$\Delta = (1 + e^{\eta_1})(1 + e^{\eta_2})$$

which corresponds to the superposition of the two solitons, (12) and (13).

Now as $|\delta_{12}| \to \infty$ the arm of (23) stretches to infinity and hence we have a triad of solitons as observed by Miles (1977). This is a pure-resonant interaction. It can be achieved by taking $q_{12} = 0$ or $q_{12} \to \infty$. For $q_{12} = 0$ we have

$$\ell_1 = \ell_2 \text{ or } n_1 = n_2 \tag{4.2.10}$$

and for $q_{12} \to \infty$ we have

$$\ell_1 = - n_2 \text{ or } \ell_2 = - n_1. \tag{4.2.11}$$

Therefore there are two types of resonance phenomena. The one which corresponds to condition (4.2.10) is termed as a "minus resonance" and to condition (4.2.11) as a "plus resonance" by Ohkuma and Wadati (1983). We shall only consider resonance phenomena of type (4.2.10) and choose the resonance conditions $n_1 = n_2$, $\ell_1 - \ell_2 = O(1)$ for the purpose of the work in this chapter. A schematic diagram for this behaviour can be drawn as in Fig. 4.3.

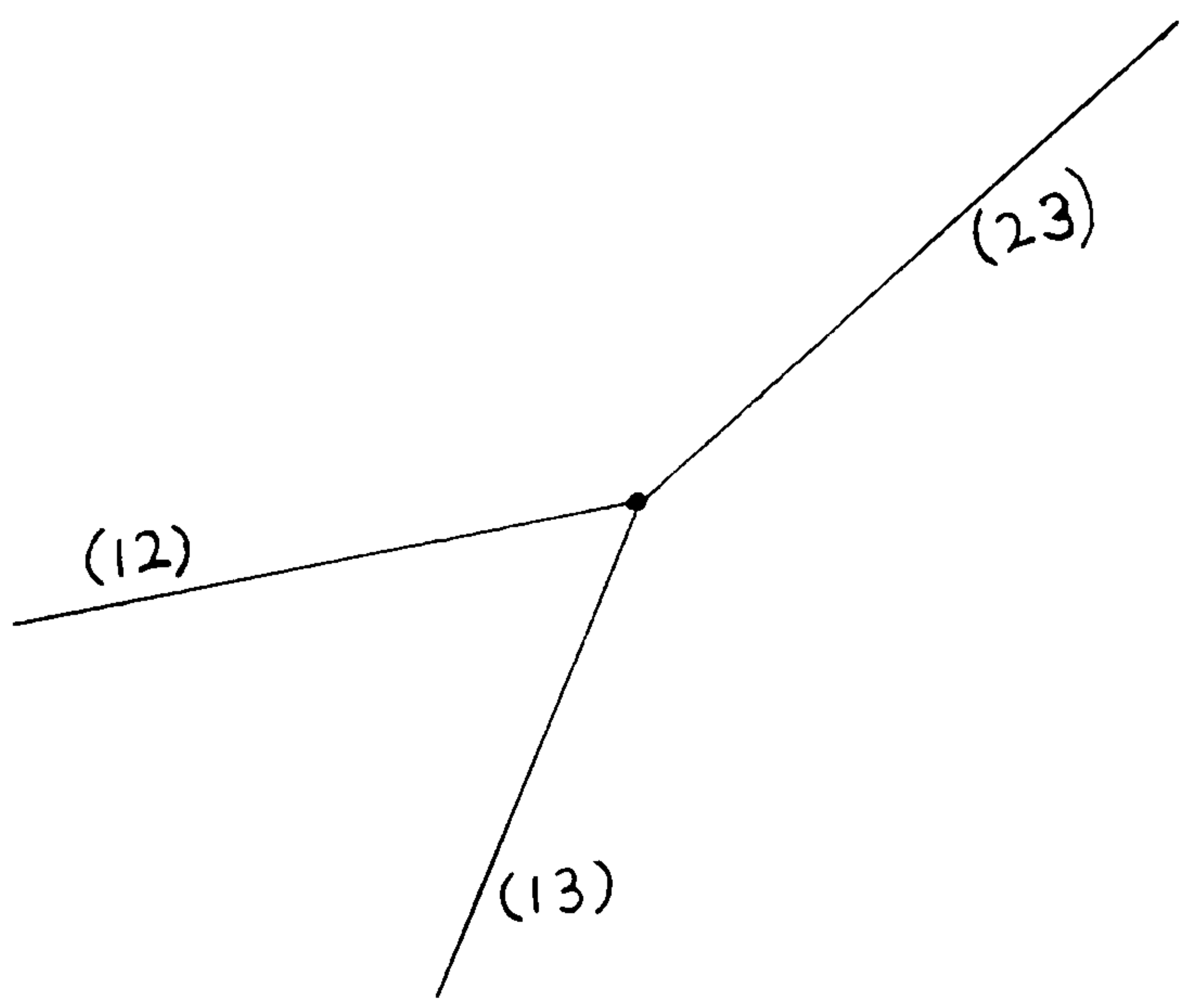


Fig. 4.3 <u>A pure-resonant interaction ($|\delta_{12}| = \infty$).</u>

If the expression for q_{12} is examined we find

$$q_{12} = \frac{(\ell_1-\ell_2)(n_1-n_2)}{(\ell_1+n_2)(\ell_2+n_1)}$$

$$= - \frac{D(k_1-k_2,\ m_1-m_2,\ w_1-w_2)}{D(k_1+k_2,\ m_1+m_2,\ w_1+w_2)} \qquad (4.2.12)$$

where D is defined as

$$D(k,\ m,\ w) = k^4 - 4kw + 3m^2. \qquad (4.2.13)$$

We note that

$$D(k,\ m,\ w) = 0 \qquad (4.2.14)$$

is the dispersion relation of the KP equation and that

$$k = \ell + n,\ m = n^2 - \ell^2,\ w = \ell^3 + n^3. \qquad (4.2.15)$$

The resonance condition (4.2.10) is therefore equivalent to

$$D(k_1-k_2,\ m_1-m_2,\ w_1-w_2) = 0. \qquad (4.2.16)$$

If the phase of the resonant soliton in Fig. 4.3 is η_3, then

$$\begin{aligned}\eta_3 &= (\ell_3+n_3)x - (\ell_3^2-n_3^2)y - 4(\ell_3^3+n_3^3)t + \delta_3 \\ &= (\ell_1-\ell_2)x - (\ell_1^2-\ell_2^2)y - 4(\ell_1^3-\ell_2^3) + \delta_1-\delta_2 \qquad (4.2.17)\end{aligned}$$

where $\delta_i = \log \frac{a_i}{\ell_i+n_i}$. Therefore we find that $\ell_3 = \ell_1$ and $n_3 = -\ell_2$ upon using the condition $n_1 = n_2$.

Since ℓ_i may take both positive and negative values, the amplitude of the resonant soliton $\frac{1}{2}(\ell_1-\ell_2)^2$ may be less or greater than the aplitudes of the incident solitons (12) and (13). We note that the term "minus resonance" refers to $\ell_1-\ell_2$ which appears in (4.2.17).

The three types of interactions discussed in the above can be used as the basis in explaining more complicated interactions. In such interactions we shall find combinations of all or part of near-resonant, non-resonant and pure-resonant interactions. In other words the detail of an interaction will be a combination of figures (4.1), (4.2) and (4.3).

We note from Fig. 4.1, that the interaction between solitons (12) and (13) produces the intermediate soliton (23) and the shifted solitons (34) and (24). Soliton (23) is therefore obtained simply by eliminating number "1" from (12) and (13). Soliton (23) then splits into (34) and (24) where the common number "4" has been introduced. In fact "4" has not been used so far.

This means that we can always construct an interaction between two solitons directly from the Δ expression (4.2.7). In the two-soliton interaction (4.2.7), if we start with an interaction between (12) and (13), the shifted solitons must be produced from the remaining term "4". However in a more complicated interaction we shall have many more terms left and the choice must be the one that produces two solitons which differ from (12) and (13) by some phase shift.

All the above rules will be observed in describing the interactions between a triad and a soliton and between two triads further on in the study.

4.3 <u>The reduction of determinant Δ into a Wronskian.</u>

The determinantal form of Δ (4.2.3) has already been transformed into a Wronskian form for the N-soliton solution by Freeman (1984). An advantage of the Wronskian $|\phi_1, \phi_2, \ldots, \phi_N|$ as constructed from individual soliton solutions $\phi_1, \phi_2, \ldots, \phi_N$ is that the behaviour of the solution before and after the interaction can be deduced directly from the Wronskian itself.

We now consider the case of N even and assume the resonance condition

$$\begin{aligned} &n_{K+i} = n_i, \qquad i = 1, 2, \ldots, K \\ &n_i \neq n_j, \qquad i \neq j,\ i, j = 1, 2, \ldots K \end{aligned} \tag{4.3.1}$$

with K = N/2.

Condition (4.3.1) implies that we have K pairs of resonating solitons in the N-soliton solution. As we have seen earlier two resonating solitons interact together to produce a triad of solitons; we therefore now have K triads. Thus the solution now represents an interaction between K triads. We shall show that, with condition (4.3.1), the determinantal form of Δ reduces to a Wronskian of K functions, each of them representing a triad.

The determinant Δ (4.2.3) can be written as

$$\Delta = |E_n^{-1} + E_\ell A|\,|E_n| \tag{4.3.2}$$

where

$$E_n = \{\delta_{ij}\, e^{n_i x + n_i^2 y - 4n_i^3 t}\}$$

$$E_\ell = \{\delta_{ij}\, e^{\ell_i x - \ell_i^2 y - 4\ell_i^3 t}\} \tag{4.3.3}$$

$$A = \{\frac{a_i}{\ell_i + n_j}\} \quad i,\ j=1,\ 2, \ldots ,N.$$

Due to the fact that the final solution u is given by (4.2.2) and that the factor $|E_n|$ is exponentially linear in x, it can be removed and thus (4.3.2) is simply written as

$$\Delta = |E_n^{-1} + E_\ell A|. \tag{4.3.4}$$

A close inspection of matrix $E_\ell A$ shows that it contains K pairs of identical columns $C_{K+j} = C_j$, j=1, 2,...,K, due to the resonance condition (4.3.1). Therefore the determinant (4.3.4) can be simplified by using the row and column operations. This procedure can be summarized by introducing a partitioned matrix J of size K × K, which is defined as

$$J = \left[\begin{array}{c|c} I & I \\ \hline 0 & I \end{array}\right] \tag{4.3.5}$$

where I is the identity matrix of size K × K, and 0 is the zero matrix.

This matrix has the property

$$(i)\quad |J| = |J^{-1}| = 1$$
$$(ii)\quad J E_n^{-1} J^{-1} = E_n^{-1} \tag{4.3.6}$$

where the inverse, J^{-1} can be found as

$$J^{-1} = \left[\begin{array}{c|c} I & -I \\ \hline 0 & I \end{array}\right] . \tag{4.3.7}$$

Since J is a constant matrix, multiplying (4.3.4) by $|J|$ or $|J^{-1}|$ does not change the final solution u. We thus have

$$\begin{aligned} \Delta &= |J E_n^{-1} J^{-1} + J E_\ell A J^{-1}| \\ &= |E_n^{-1} + J E_\ell A J^{-1}| \\ &= \left|\begin{array}{c|c} F_{(K)} & 0_{(K)} \\ \hline G_{(K)} & E_{n(K)} \end{array}\right| \\ &= |F_{(K)}|\ |E_{n(K)}| \end{aligned} \tag{4.3.8}$$

where the subscript (K) denotes the size of the matrix, $0_{(K)}$ the zero matrix, $G_{(K)}$ is the matrix

$$G_{(K)ij} = \frac{a_{K+i}}{\ell_{K+i} + n_j}\, e^{(\ell_{K+i}x)}$$

and the matrix $F_{(K)}$ is given by

$$F_{(K)ij} = \delta_{ij} e^{(-n_i x)} + \frac{a_i}{\ell_i + n_j} e^{(\ell_i x)} + \frac{a_{K+i}}{\ell_{K+i} + n_j} e^{(\ell_{K+i} x)} \tag{4.3.9}$$

where we have shortened the arguments of all the exponentials as

$$e^{(-n_i x)} \equiv e^{-(n_i x + n_i^2 y - 4 n_i^3 t)}$$

$$e^{(\ell_i x)} \equiv e^{\ell_i x - \ell_i^2 y - 4 \ell_i^3 t}$$

for convenience.

We note that from (4.3.8), $|E_{n(K)}|$ is an exponential factor, whose argument is linear in x and hence may be removed for the usual reason. Thus

$$\Delta = |F_{(K)}|. \tag{4.3.10}$$

We note further that Δ now is a K × K, or $\frac{N}{2} \times \frac{N}{2}$, determinant. We now demonstrate that this determinant can be transformed into a Wronskian.

It is found convenient to work with the transposed form of matrix $F_{(K)}$. Now

$$F^T_{(K)ij} = \delta_{ij} e^{(-n_j x)} + \frac{a_j}{\ell_j + n_i} e^{(\ell_j x)} + \frac{a_{K+j}}{\ell_{K+j} + n_i} e^{(\ell_{K+j} x)} \tag{4.3.11}$$

where T stands for the transposed matrix.

From (4.3.10) and (4.3.11) we can write

$$\Delta = |E_n^{-1} + M_1 A_1 E_{\ell 1} + M_2 A_2 E_{\ell 2}| \tag{4.3.12}$$

where we have suppressed the subscript (K) since all the matrices are now of size K × K. In (4.3.12) we have defined

$$E_{\ell 1} = \{\delta_{ij} e^{(\ell_j x)}\}$$
$$E_{\ell 2} = \{\delta_{ij} e^{(\ell_{K+j} x)}\}$$
$$A_1 = \{\delta_{ij} a_j\}$$
$$A_2 = \{\delta_{ij} a_{K+j}\}$$
$$M_1 = \{\frac{1}{\ell_j + n_i}\}$$
$$M_2 = \{\frac{1}{\ell_{K+j} + n_i}\}.$$

We also need to use the Van der Monde matrices V, W_1 and W_2 defined by

$$V = \{(-n_j)^{i-1}\}$$
$$W_1 = \{(-1)^{j-1}(\ell_j)^{i-1}\}$$
$$W_2 = \{(-1)^{j-1}(\ell_{K+j})^{i-1}\}$$

and the diagonal matrices P, Q_1 and Q_2 as

$$P = \{\delta_{ij} \prod_{p\neq i}^{K} (n_p - n_i)\}$$

$$Q_1 = \{\delta_{ij} (-1)^{i-1} \prod_{p=1}^{K} (n_p + \ell_i)\}$$

$$Q_2 = \{\delta_{ij} (-1)^{i-1} \prod_{p=1}^{K} (n_p + \ell_{K+i})\}.$$

It has been shown by Freeman (1984) that

$$V^{-1}W_r = P^{-1}M_r Q_r, \quad r=1,\ 2.$$

Therefore

$$M_r = PV^{-1}W_r Q_r^{-1}, \quad r=1,\ 2. \tag{4.3.13}$$

Introducing (4.3.13) into (4.3.12) we then have

$$\Delta = |E_n^{-1} + PV^{-1}W_1Q_1^{-1}A_1E_{\ell 1} + PV^{-1}W_2Q_2^{-1}A_2E_{\ell 2}|$$

$$= |PV^{-1}|\,|VP^{-1}E_n^{-1} + W_1Q_1^{-1}A_1E_{\ell 1} + W_2Q_2^{-1}A_2E_{\ell 2}|.$$

Since $|PV^{-1}|$ is only a constant factor in the above expression it can be removed, leaving

$$\Delta = |VD_n + W_1D_{\ell 1} + W_2D_{\ell 2}| \tag{4.3.14}$$

where

$$D_n = P^{-1}E_n^{-1}$$

(4.3.15)

$$D_{\ell r} = Q_r^{-1}A_rE_{\ell r}, \quad r=1,\ 2.$$

It can be checked that (4.3.14) is the $\frac{N}{2}$ - Wronskian of the functions

$$\Phi_j = e^{-\theta_j} + e^{\psi_{1j}} + e^{\psi_{2j}} \tag{4.3.16}$$

where

$$\theta_j = n_jx + n_j^2y - 4n_j^3t + \log\{\prod_{p\neq j}^{K} (n_p - n_j)\}$$

$$\psi_{1j} = \ell_j x - \ell_j^2 y - 4\ell_j^3 t + \log\left\{\frac{a_j}{\prod_{p=1}^{K}(n_p+\ell_j)}\right\} \tag{4.3.17}$$

$$\psi_{2j} = \ell_{K+j} x - \ell^2_{K+j} y - 4\ell_{K+j}^3 t + \log\left\{\frac{a_{K+j}}{\prod_{p=1}^{K}(n_p+\ell_{K+j})}\right\},$$

$j = 1, 2, \ldots K.$

For $N = 2$, and making $n_1=n_2$, the above gives

$$\Phi_1 = e^{(-n_1 x)} + e^{(\ell_1 x) + \log(\frac{a_1}{\ell_1+n_1})} + e^{(\ell_2 x) + \log(\frac{a_2}{\ell_2+n_1})}$$
$$= e^{(-n_1 x)}\{1 + e^{\eta_1} + e^{\eta_2}\} \tag{4.3.18}$$

which can be recognized as a triad with parameters ℓ_1, ℓ_2, n_1.

Each of the functions Φ_1, $\Phi_2, \ldots \Phi_{N/2}$ of the Wronskian (4.3.14) can be arranged in the form (4.3.18) which represents a triad. Thus the Wronskian solution (4.3.14) represents an interaction between N/2 triads, each of which can be treated as a single moving entity.

We now give the result for N odd ($N \geq 3$). In this case we choose $K=(N-1)/2$ and use the same assumption that $n_{K+i} = \eta_i$. This means that we have K pairs of solitons in the resonant state and a single-soliton. By using a similar procedure to the one used for N even, the determinant Δ is transformed into a Wronskian of K+1 functions Φ_1, $\Phi_2, \ldots, \Phi_{K+1}$ where Φ_1, $\Phi_2, \ldots, \Phi_K$ are defined by (4.3.16) and (4.3.17), while Φ_{K+1} is defined by

$$\Phi_{K+1} = e^{-\theta_N} + e^{\psi_N} \tag{4.3.19}$$

where

$$\theta_N = n_N x + n_N^2 y - 4n_N^3 t + \log \{ \prod_{p=1}^{K} (n_p - n_N) \}$$

$$\psi_N = \ell_N x - \ell_N^2 y - 4\ell_N^3 t + \log \left\{ \frac{a_N}{\prod_{p=1}^{K,N} (n_p + \ell_N)} \right\}. \qquad (4.3.20)$$

The function Φ_{K+1} represents a single-soliton while each of all other functions $\Phi_1, \Phi_2, \ldots, \Phi_K$ represents a triad. Hence we have an interaction between $K = (N-1)/2$ triads and a single-soliton for N odd.

4.4 <u>Interactions between a triad and a soliton</u>

If we put $n_1 = n_2$ leaving n_3 unpaired in the three-soliton solution, we find from the previous section the associated Wronskian Δ as

$$\Delta = \begin{vmatrix} e^{(-n_1 x)} + e^{(\ell_1 x)} + e^{(\ell_2 x)} & e^{(-n_3 x)} + e^{(\ell_3 x)} \\ -n_1 e^{(-n_1 x)} + \ell_1 e^{(\ell_1 x)} + \ell_2 e^{(\ell_2 x)} & -n_3 e^{(-n_3 x)} + \ell_3 e^{(\ell_3 x)} \end{vmatrix}$$

which is recognized as an interaction between a triad with parameters ℓ_1, ℓ_2, n_1 and a soliton with parameters ℓ_3, n_3.

In the above Wronskian, the term $e^{-(n_1+n_3)x}$ can be factored out and removed from it, leaving

$$\Delta = \begin{vmatrix} 1+e^{(\ell_1+n_1)x} + e^{(\ell_2+n_1)x} & 1 + e^{(\ell_3+n_3)x} \\ -n_1+\ell_1 e^{(\ell_1+n_1)x} + \ell_2 e^{(\ell_2+n_1)x} & -n_3 + \ell_3 e^{(\ell_3+n_3)x} \end{vmatrix} \qquad (4.4.1)$$

where all the arguments of the exponentials have been shortened appropriately, for convenience, as before. We note that

$$e^{(\ell_i+n_i)x} \equiv e^{(\ell_i+n_i)x - (\ell_i^2-n_i^2)y - 4(\ell_i^3+n_i^3)t + \log \frac{a_i(n_3-n_i)}{(n_1+\ell_i)(n_3+\ell_i)}}$$

for i = 1, 2 and

$$e^{(\ell_3+n_3)x} \equiv e^{(\ell_3+n_3)x - (\ell_3^2-n_3^2)y - 4(\ell_3^3+n_3^3)t + \log \frac{a_3(n_1-n_3)}{(n_1+\ell_3)(n_3+\ell_3)}}.$$

We first write

$$(\ell_i+n_i)x - (\ell_i^2-n_i^2)y - 4(\ell_i^3+n_i^3)t$$
$$= P_i[x - Q_iy - (P_i^2+3Q_i^2)t]$$

where

$$P_i = \ell_i+n_i, \quad Q_i = \ell_i-n_i.$$

We now assume $\ell_1<\ell_2<n_1<n_3<\ell_3$, $\ell_i-\ell_j = O(1)$ for $i\neq j$, $n_1\approx n_3$ and $P_i>0$ both for convenience and also in order to incorporate the numerical computations presented in Section 4.6.

In the region of the maximum of the triad, we have

$$x - Q_1y - (P_1^2+3Q_1^2)t \approx 0$$
$$x - Q_2y - (P_2^2+3Q_2^2)t \approx 0$$

which can be solved to give

$$x = \{\frac{Q_2(P_1^2+3Q_1^2) - Q_1(P_2^2+3Q_2^2)}{Q_2-Q_1}\}t$$

$$y = \{\frac{(P_1^2+3Q_1^2) - (P_2^2+3Q_2^2)}{Q_2-Q_1}\}t.$$

In this region we then find

$$P_3[x - Q_3y - (P_3^2 + 3Q_3^2)t]$$
$$= \frac{P_3}{Q_3-Q_1}[- (Q_3-Q_2)(P_1^2+3Q_1^2) - (Q_2-Q_1)(P_3^2+3Q_3^2)$$
$$+ (Q_3-Q_1)(P_2^2+3Q_2^2)]t. \qquad (4.4.2)$$

We now show that the sign of the coefficient of t in (4.4.2) is negative under present assumptions:

$$-(Q_3-Q_2)(P_1^2+3Q_1^2) - (Q_2-Q_1)(P_3^2+3Q_3^2)$$
$$+(Q_3-Q_1)(P_2^2+3Q_2^2)$$

$$=(Q_3-Q_2)[P_3^2+3Q_3^3-P_1^2-3Q_1^2]$$
$$+(Q_3-Q_1)[P_2^2+3Q_2^2-P_3^2-3Q_3^2]$$

$$=(Q_3-Q_2)[(P_3-P_1)(P_3+P_1) + 3(Q_3-Q_1)(Q_3+Q_1)]$$
$$+(Q_3-Q_1)[(P_2-P_3)(P_2+P_3) + 3(Q_2-Q_3)(Q_2+Q_3)]$$

$$\approx 4(Q_3-Q_2)(Q_3-Q_1)(Q_1-Q_2).$$

The last expression is negative since $Q_1 - Q_2 < 0$. Note that in the above we have used the relation $P_3 - P_1 \approx Q_3 - Q_1$ since $n_1 \approx n_3$.

Therefore

$$P_3[x - Q_3y - (P_3^2+3Q_3^2)t] \to +\infty \text{ as } t \to -\infty \qquad (4.4.3)$$
$$\to -\infty \text{ as } t \to +\infty.$$

Now as $t \to -\infty$, we have from (4.4.1)

$$\Delta = \begin{vmatrix} 1 + e^{(\ell_1+n_1)x} + e^{(\ell_2+n_1)x} & 1 \\ -n_1 + \ell_1 e^{(\ell_1+n_1)x} + \ell_2 e^{(\ell_2+n_1)x} & \ell_3 \end{vmatrix}.$$

This gives

$$\Delta = 1 + e^{\eta_1+\delta_{13}} + e^{\eta_2+\delta_{23}} \text{ as } t \to -\infty, \qquad (4.4.4)$$

where we have used the usual notation

$$\eta_i = (\ell_i+n_i)x - (\ell_i^2-n_i^2)y - 4(\ell_i^3+n_i^3)t + \log \frac{a_i}{\ell_i+n_i}$$

$$\delta_{ij} = \log \left\{\frac{(\ell_i-\ell_j)(n_i-n_j)}{(\ell_i+n_j)(\ell_j+n_i)}\right\}.$$

We note that (4.4.4) is the triad with parameters ℓ_1, ℓ_2, n_1 which individual solitons are centred on $\eta_1 + \delta_{13} = 0$, $\eta_2 + \delta_{23} = 0$ and $\eta_1 - \eta_2 + \delta_{13} - \delta_{23} = 0$.

As $t \to +\infty$, we find from (4.4.3) and (4.4.1) that

$$\Delta = \begin{vmatrix} 1 + e^{(\ell_1+n_1)x} + e^{(\ell_2+n_1)x} & 1 \\ -n_1 + \ell_1 e^{(\ell_1+n_1)x} + \ell_2 e^{(\ell_2+n_1)x} & -n_3 \end{vmatrix}$$

which yields

$$\Delta = 1 + e^{\eta_1} + e^{\eta_2} \text{ as } t \to +\infty. \tag{4.4.5}$$

This is the same triad as (4.4.4) but the individual solitons are now centred on $\eta_1 = 0$, $\eta_2 = 0$ and $\eta_1 - \eta_2 = 0$.

From (4.4.4) and (4.4.5) we notice the phase shift of the triad. Before the interaction (4.4.4), the triad which can be represented by the point of intersection between the soliton with parameters ℓ_1, n_1 centred on $\eta_1 + \delta_{13} = 0$ and the soliton with parameters ℓ_2, n_1 centred on $\eta_2 + \delta_{23} = 0$, has been displaced after the interaction (4.4.5) by the intersection between the solitons which are centred on $\eta_1 = 0$ and $\eta_2 = 0$.

Similar analysis can also be carried out in order to locate the positions of the solitons with parameters ℓ_3, n_3 before and after the interaction. This will subsequently be confirmed by computer calculations of the full solution.

We now describe the detail of the interaction between a triad and a soliton. In order to explain the interaction we choose the notations of a doublet (ij) to represent a soliton, a triplet (ijk) to represent a pure-resonant interaction (triad) and a quadruplet (ijkℓ) to represent a near-resonant or non-resonant interaction. The basis for the interaction between two solitons outlined in Section 4.2 will also be observed.

Expanding the determinant Δ in (4.4.1) we find

$$\Delta = \underset{(1)}{1} + \underset{(2)}{e^{\eta_1}} + \underset{(3)}{e^{\eta_2}} + \underset{(4)}{e^{\eta_3}} + \underset{(5)}{q_{23}e^{\eta_2+\eta_3}} + \underset{(6)}{q_{13}e^{\eta_1+\eta_3}}. \tag{4.4.6}$$

The triad given by (4.4.4) can be recognized as triad (456) from expression (4.4.6) since

$$\Delta_{456} = e^{\eta_3}[\ 1 + e^{\eta_1+\delta_{13}} + e^{\eta_2+\delta_{23}}].$$

(4) (6) (5)

This triad will interact with a soliton with parameters ℓ_3, n_3 which is recognized from (4.4.6) as (14), (26) and (35). For the values of ℓ_i's and n_i's chosen in the examples presented in section 4.6, soliton (26) will first interact with the triad.

The associated Δ for soliton (26) is

$$\Delta = e^{\eta_1}(1 + e^{\eta_3+\delta_{13}}) \tag{4.4.7}$$

(2) (6)

which is centred on $\eta_3 + \delta_{13} = 0$.

Now soliton (26) interacts with soliton (46) from triad (456). Each of the post-interaction solitons must contain (4) and (2) respectively, and a common term which can be obtained from the unused terms in (4.4.6). Such a term must be (1), and therefore the post-interaction solitons are (14) and (12). Examining the Δ for the interaction (1246) we find from (4.4.6)

$$\Delta_{1246} = 1 + e^{\eta_1} + e^{\eta_3} + q_{13}e^{\eta_1+\eta_3}.$$

(1) (2) (4) (6)

Since $n_1 \approx n_3$, then $|\delta_{13}| = |\log(q_{13})|$ is much bigger than zero and this implies that (1246) is a near-resonant interaction with intermediate soliton (24). The interaction at this stage can be sketched schematically as in Fig. 4.4a

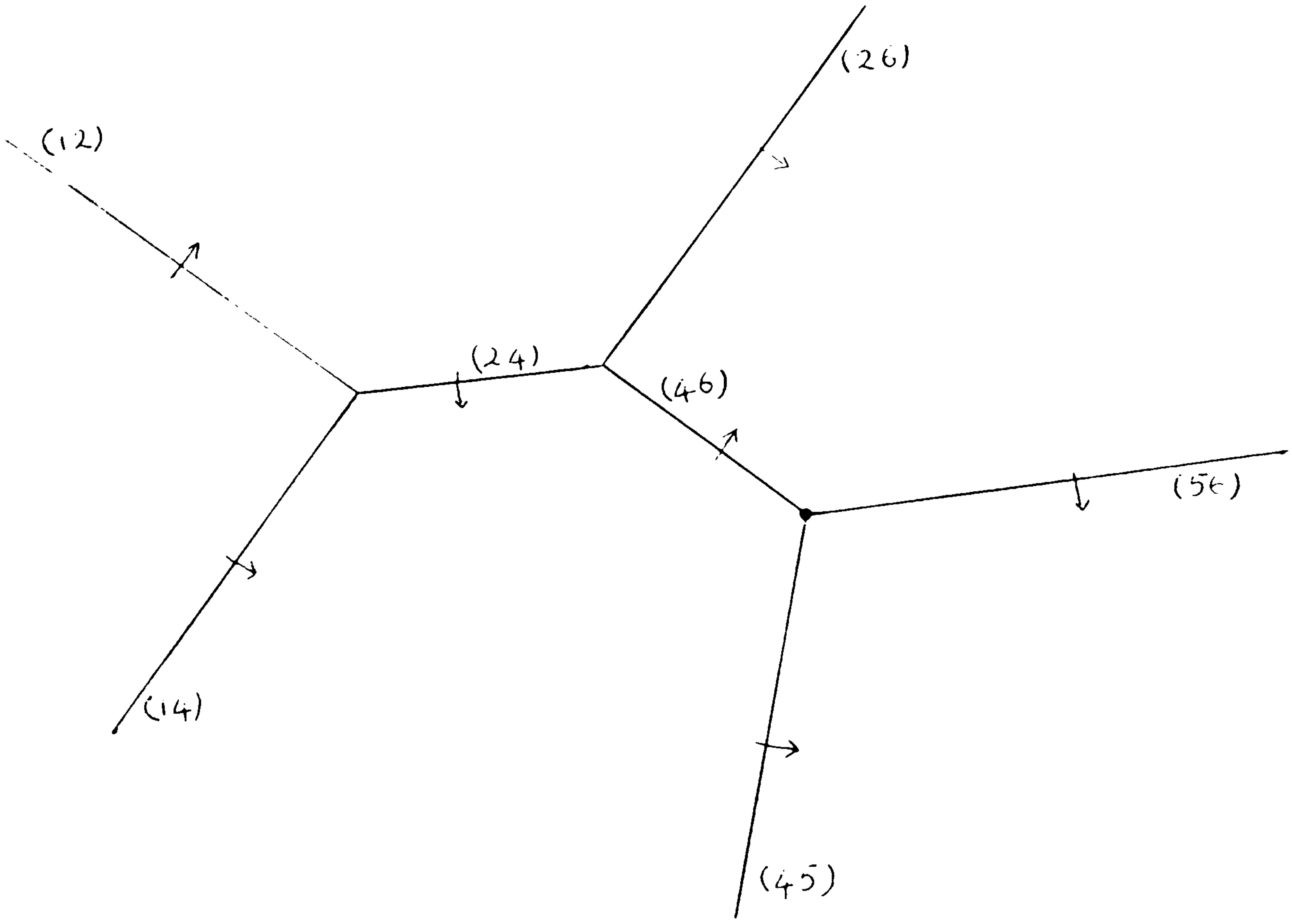


Fig. 4.4a <u>The earlier interaction between (456) and (26).</u>

In Fig. 4.4a the arrows show the directions of propagation of the individual solitons. The next stage of the interaction can be easily deduced from Fig. 4.4a as we notice that solitons (12), (14), (26) and (46) do not interact because they have performed a complete interaction at that stage.

As time goes on soliton (26) will sweep over triad (456). While doing so, during the interaction the length of (46) will decrease to nothing and thus (26) interacts with (56). The expected shifted solitons of (26) and (56) are, from (4.4.6), (35) and (23) respectively. It can be shown that this interaction is non-resonant by examining the associated Δ (by non-resonant we mean a near-resonant interaction with very small phase shift). We have from (4.4.6)

$$\Delta_{2356} = \underset{(2)}{e^{\eta_1}} + \underset{(3)}{e^{\eta_2}} + \underset{(5)}{q_{23}e^{\eta_2+\eta_3}} + \underset{(6)}{q_{13}e^{\eta_1+\eta_3}}$$

$$= e^{\eta_2}[\underset{(3)}{1} + \underset{(2)}{e^{\eta_1-\eta_2}} + \underset{(5)}{e^{\eta_3+\delta_{23}}} + \underset{(6)}{\frac{q_{13}}{q_{23}} e^{(\eta_1-\eta_2) + \eta_3+\delta_{23}}}].$$

Now

$$\frac{q_{13}}{q_{23}} = \frac{(\ell_1-\ell_3)(n_1-n_3)}{(\ell_1+n_3)(\ell_3+n_1)} \cdot \frac{(\ell_2+n_3)(\ell_3+n_1)}{(\ell_2-\ell_3)(n_1-n_3)}$$

$$= \frac{(\ell_1-\ell_3)(\ell_2+n_3)}{(\ell_1+n_3)(\ell_2-\ell_3)}.$$

Since we made the assumption that $\ell_1-\ell_3 = O(1)$, $\ell_2-\ell_3 = O(1)$ and $n_1 = n_2 \approx n_3$ then the above ratio must be of $O(1)$, and hence the interaction is non-resonant.

The interaction continues with soliton (35) interacting with soliton (45), which can be shown to produce a near-resonant interaction (1345) with shifted solitons (14) and (13), and an intermediate soliton (34). Soliton (13) then interacts with soliton (12), which combines with (23) to form triad (123). The interaction is now complete (Fig. 4.4b).

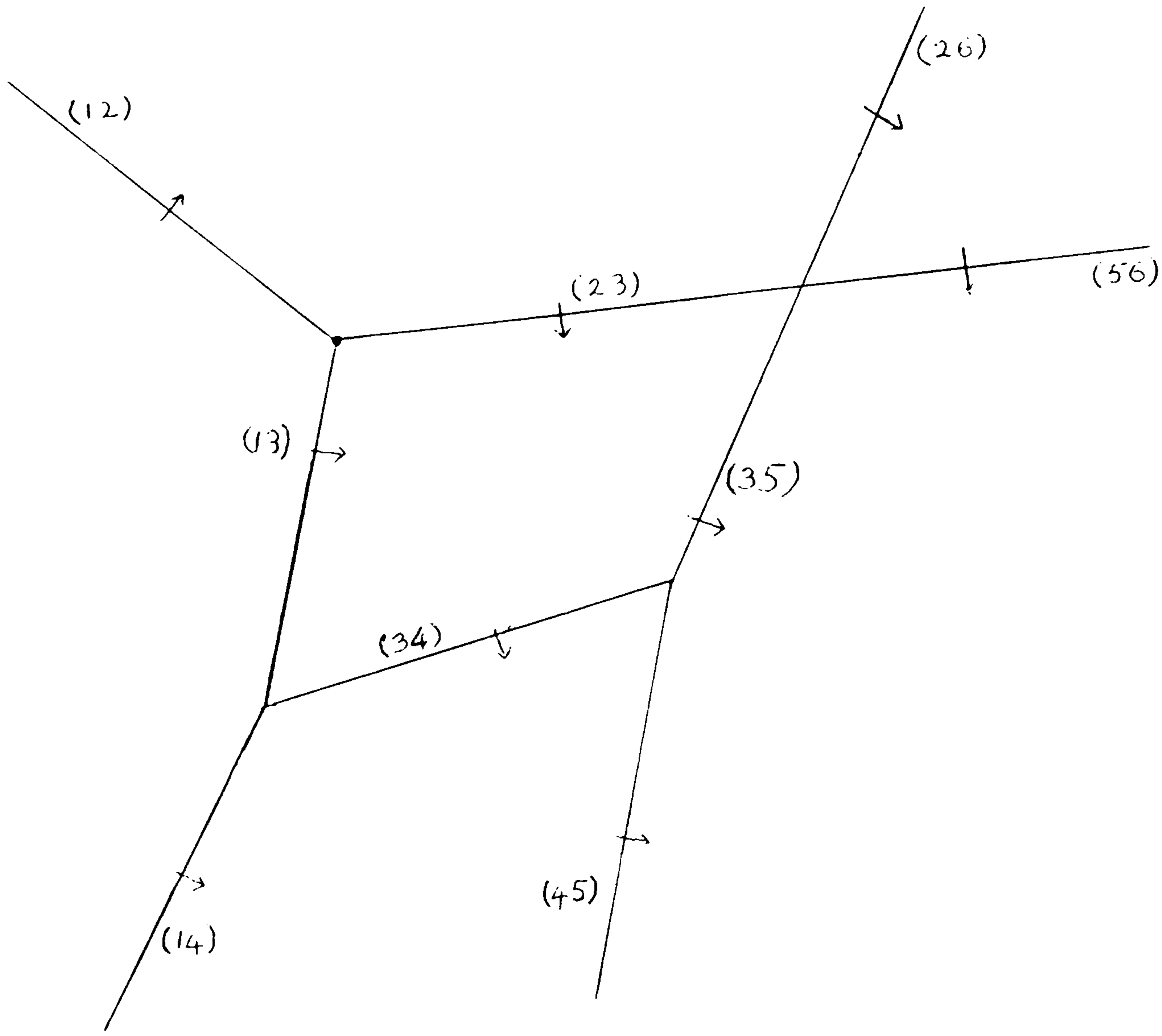


Fig. 4.4b <u>The final interaction between (123) and (14).</u>

The structure of the configuration in Fig. 4.4b remains as it is as time goes on with the quadrangle formed by solitons (13), (23), (35) and (34) getting bigger and bigger.

Essentially the earlier interaction between (456) and (26) in Fig. 4.4a has now become an interaction between (123) and (14) in Fig.(4.4b). Examining the triad (123) from (4.4.6) we have

$$\Delta_{123} = 1 + e^{\eta_1} + e^{\eta_2}$$

(1) (2) (3)

which agrees with (4.4.5).

We may equally well have a different configuration from Fig. 4.4a or Fig. 4.4b. Depending on the gradients of (26) and (56), from Fig. 4.4a we see that it is possible for them to intersect. In fact, we have found from the numerical calculations that this is the case when $\ell_1 < n_1 \approx n_3 < \ell_2 < \ell_3$. In this case beside interacting with (46), soliton (26) also interacts with (56) to produce a non-resonant interaction (2356). Therefore an alternative configuration could be as in Fig. 4.5a.

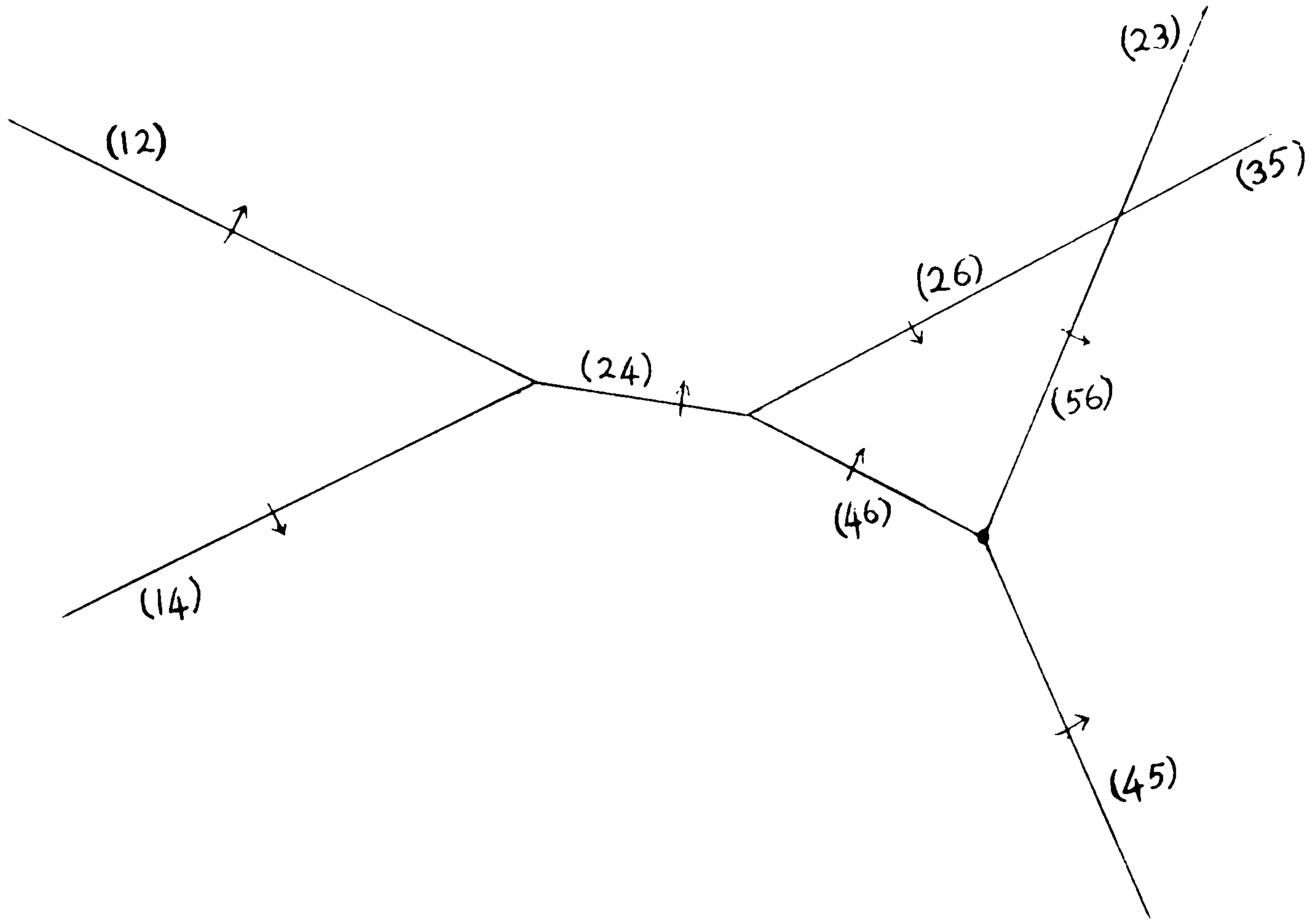


Fig. 4.5a <u>An alternative earlier interaction between (456) and (26)</u>

In the next interaction, soliton (35) interacts with soliton (45) to produce a near-resonant interaction (1345), as in Fig. 4.4b and solitons (12) and (23) interact to produce triad (123) as before; therefore we have a configuration such as Fig. 4.5b.

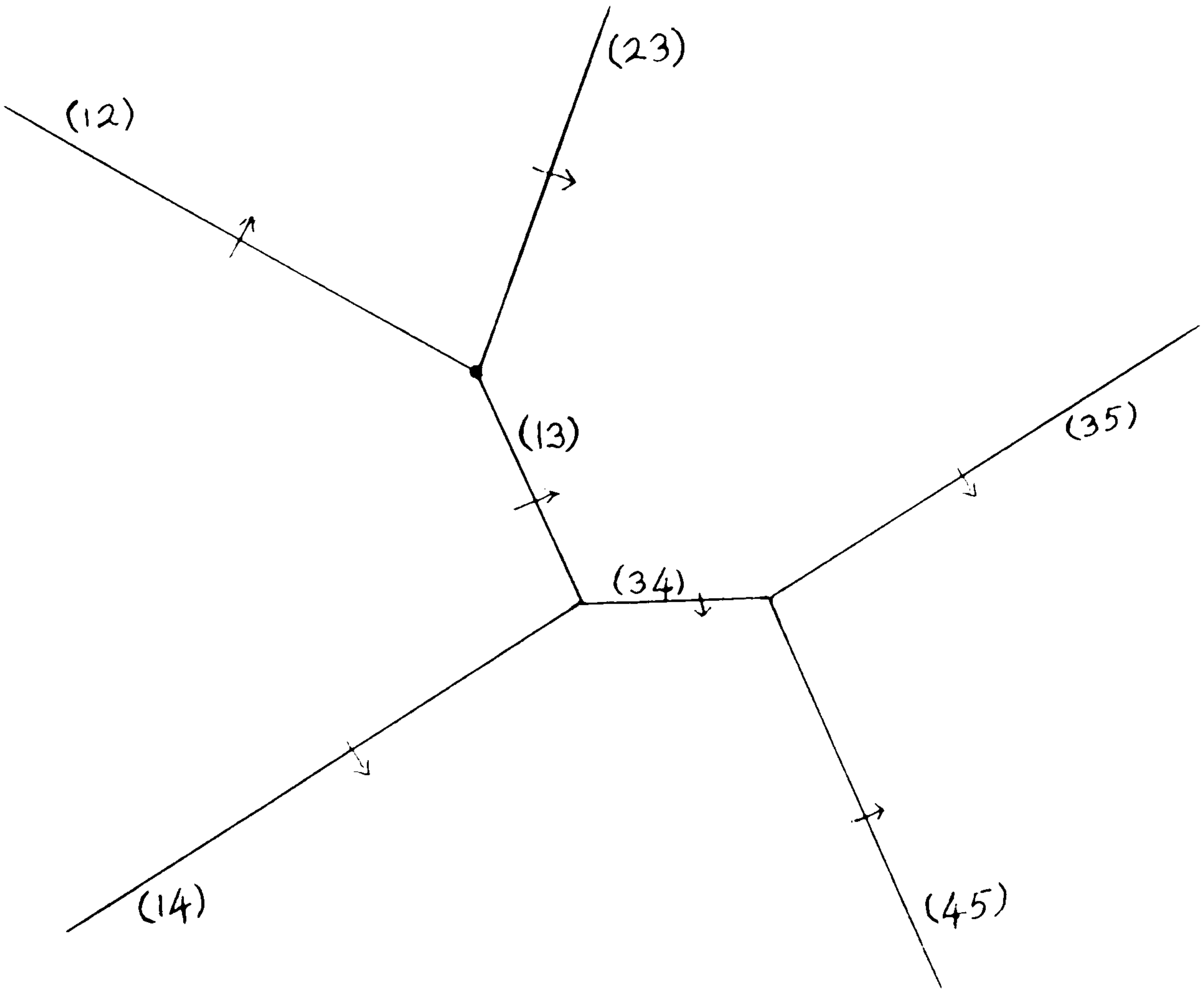


Fig. 4.5b An alternative final interaction between (123) and (14).

At this stage triad (123) and the near-resonant section (1345) are moving on their own way with soliton (13) getting longer and longer; thus no other interaction will take place.

So far we have discussed the interactions between a triad and a soliton under the assumption that $n_1 \approx n_3$. In this case we have found that the triad has experienced significant phase shift after the interaction.

Let us now consider the case when $n_1 - n_3 = 0(1)$. In this case, the interaction between a triad with parameters ℓ_1, ℓ_2, n_1 and a soliton with parameters ℓ_3, n_3 does not produce significant phase shift since the interactions between the individual solitons are all non-resonant.

Without loss of generality let us start the interaction between triad (123) and (26) with (123) behind (26). In this case we can no longer take $\ell_1 < \ell_2 < \ell_3$ as before. The first interaction is simply shown as in Fig. 4.6a.

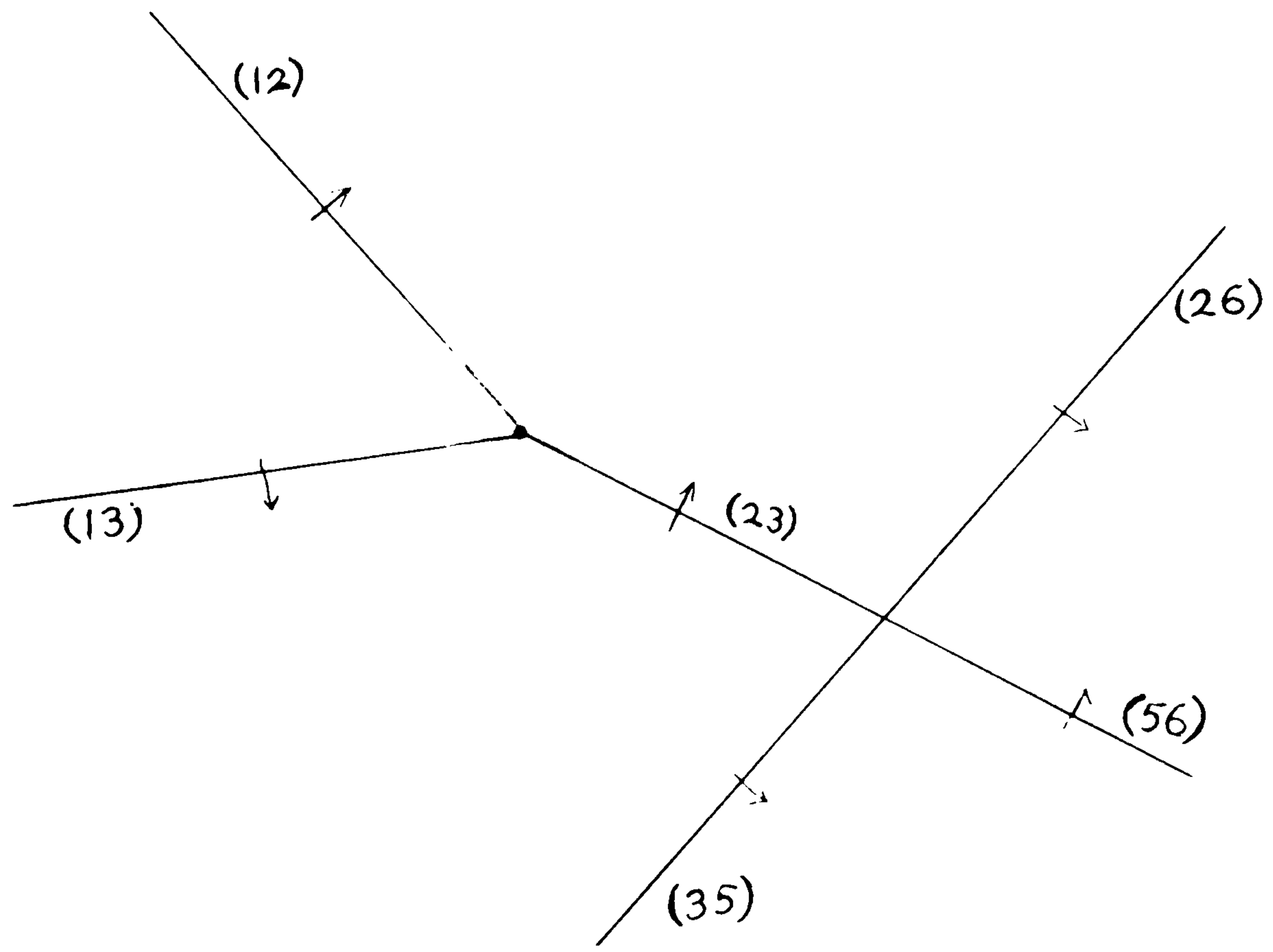


Fig. 4.6a A non-resonant interaction between (123) and (26) at earlier times

As time goes on triad (123) will simply pass soliton (26) with very small phase shifts in the final triad and the final soliton (Fig. 4.6b).

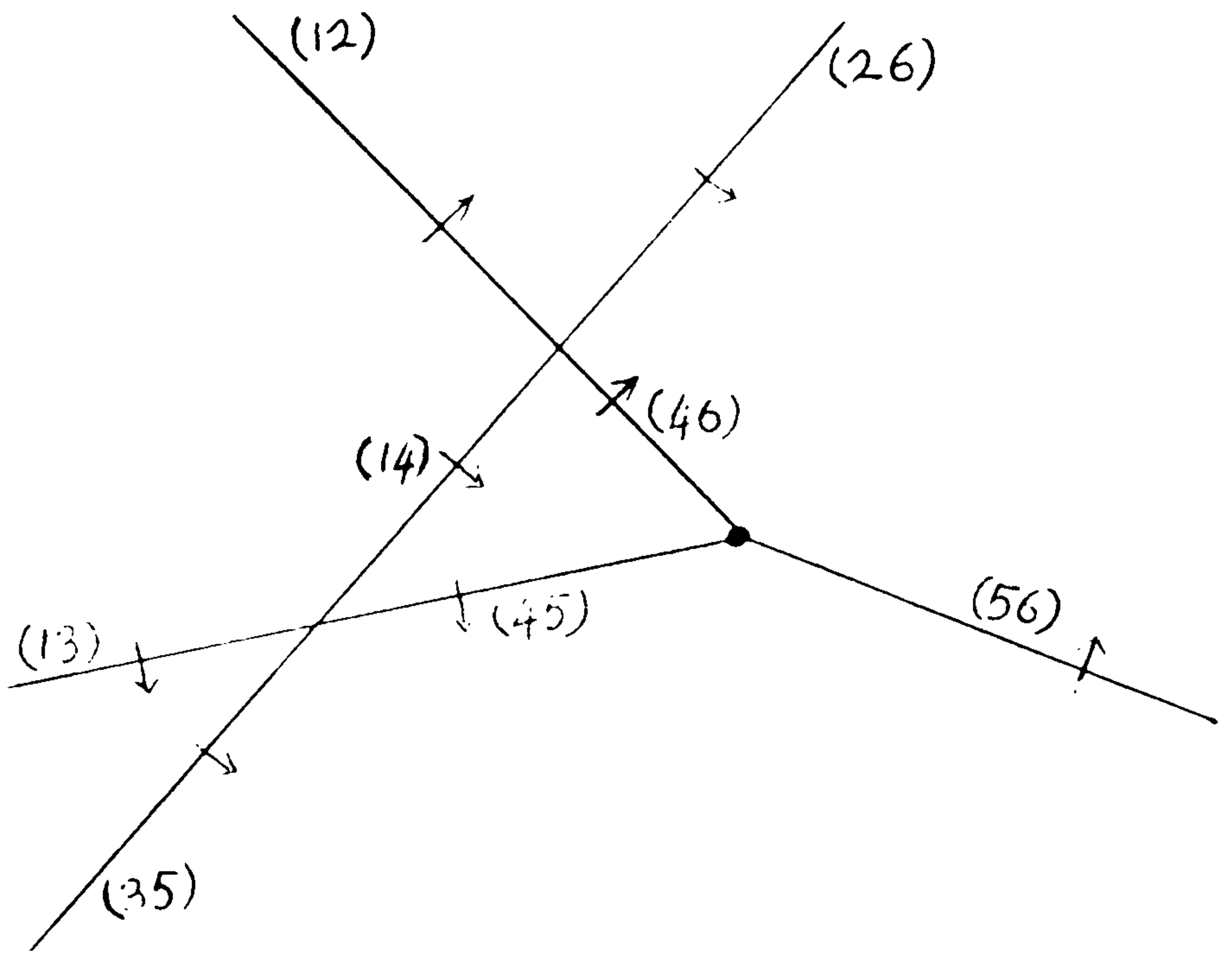


Fig. 4.6b The non-resonant interaction at later times

The configuration in Fig. 4.6b remains unchanged but as we increase time, the triangle formed by solitons (14), (45) and (46) becomes bigger.

We have thus seen that in the interaction between a triad with parameters ℓ_1, ℓ_2, n_1 and a soliton with parameters ℓ_3, n_3, significant phase shift is obtained in the final triad when $n_1 \approx n_3$, while the phase shift is very small when $n_1 - n_3 = O(1)$.

4.5 <u>Interactions between two triads</u>

If we put $n_1 = n_3$ and $n_2 = n_4$ into the four-soliton solution (4.2.3), the analysis in Section 4.3 will end up with a Wronskian (4.3.14) of the functions given by (4.3.15). We thus have

$$\Delta = \begin{vmatrix} e^{(-n_1x)} + e^{(\ell_1x)} + e^{(\ell_3x)} & e^{(-n_2x)} + e^{(\ell_2x)} + e^{(\ell_4x)} \\ -n_1e^{(-n_1x)} + \ell_1e^{(\ell_1x)} + \ell_3e^{(\ell_3x)} & -n_2e^{(-n_2x)} + \ell_2e^{(\ell_2x)} + \ell_4e^{(\ell_4x)} \end{vmatrix},$$

and after removing a factor, Δ becomes

$$\Delta = \begin{vmatrix} 1 + e^{(\ell_1+n_1)x} + e^{(\ell_3+n_3)x} & 1 + e^{(\ell_2+n_2)x} + e^{(\ell_4+n_2)x} \\ -n_1 + \ell_1e^{(\ell_1+n_1)x} + \ell_3e^{(\ell_3+n_3)x} & -n_2 + \ell_2e^{(\ell_2+n_2)x} + \ell_4e^{(\ell_4+n_2)x} \end{vmatrix} \tag{4.5.1}$$

where the arguments of all the exponentials have been shortened as before. Note that

$$e^{(\ell_i+n_i)} \equiv e^{(\ell_i+n_i)x - (\ell_i^2-n_i^2)y - 4(\ell_i^3+n_i^3)t + \log\{\frac{a_i(n_p-n_i)}{(n_1+\ell_i)(n_2+\ell_i)}\}}$$

for $p = 1, 2$ and $n_p \neq n_i$, $i = 1, 2, 3, 4$.

We assume for convenience that $\ell_1 < \ell_3 < n_1 < n_2 < \ell_2 < \ell_4$, $n_1 \approx n_2$ and $P_i > 0$ for $i = 1, 2, 3, 4$ where P_i is defined in Section 4.4.

Now, in the region of the maximum of the triad with parameters ℓ_1, ℓ_3, n_1 we have

$$x - Q_1 y - (P_1^2+3Q_1^2)t \approx 0$$
$$x - Q_3 y - (P_3^2+3Q_3^3)t \approx 0.$$

which gives

$$x = \{\frac{Q_3(P_1^2+3Q_1^2) - Q_1(P_3^2+3Q_3^2)}{Q_3 - Q_1}\}t$$

$$y = \{\frac{(P_1^2+3Q_1^2) - (P_3^2+3Q_3^2)}{Q_3-Q_1}\}t.$$

Hence

$$\begin{aligned} &P_2(x - Q_2 y - (P_2^2+3Q_2^2)t) \\ &= \frac{P_2}{Q_3-Q_1}\{-(Q_3-Q_1)(P_2^2+3Q_2^2) - (Q_2-Q_3)(P_1^2+3Q_1^2) \\ &\qquad + (Q_2-Q_1)(P_3^2+3Q_3^2)\}t, \end{aligned} \quad (4.5.2)$$

$$\begin{aligned} &P_4(x - Q_2 y - (P_2^2+3Q_2^2)t) \\ &= \frac{P_4}{Q_3-Q_1}\{-(Q_3-Q_1)(P_4^2+3Q_4^2) - (Q_4-Q_3)(P_1^2+3Q_1^2) \\ &\qquad + (Q_4-Q_1)(P_3^2+3Q_3^2)\}t. \end{aligned} \quad (4.5.3)$$

It can be shown, as we did in Section 4.4, that the coefficients of t in (4.5.2) and (4.5.3) are both negative, and furthermore that

$$\begin{aligned} &(Q_3-Q_1)(P_4^2+3Q_4^2)+(Q_4-Q_3)(P_1^2+3Q_1^2)-(Q_4-Q_1)(P_3^2+3Q_3^2) \\ &> (Q_3-Q_1)(P_2^2+3Q_2^2)+(Q_2-Q_3)(P_1^2+3Q_1^2)-(Q_2-Q_1)(P_3^2+3Q_3^2). \end{aligned} \quad (4.5.4)$$

Therefore as $t \to -\infty$ both (4.5.2) and (4.5.3) tend to $+\infty$, with

$$P_4(x-Q_2y-(P_2^2+3Q_2^2)t) > P_2(x-Q_2y-(P_2^2+3Q_2^2)t). \quad (4.5.5)$$

Hence as $t \to -\infty$, (4.5.1) becomes

$$\Delta = \begin{vmatrix} 1 + e^{(\ell_1+n_1)x} + e^{(\ell_3+n_1)x} & 1 \\ -n_1 + \ell_1 e^{(\ell_1+n_1)x} + \ell_3 e^{(\ell_3+n_1)x} & \ell_4 \end{vmatrix}$$

which is equivalent to

$$\Delta = 1 + e^{\eta_1+\delta_{14}} + e^{\eta_3+\delta_{34}} \text{ as } t \to -\infty, \tag{4.5.6}$$

where η_i, δ_{ij} are defined as before.

The expression given by (4.5.6) represents the triad with parameters ℓ_1, ℓ_3, n_1 with its individual solitons centred on $\eta_1 + \delta_{14} = 0$, $\eta_3 + \delta_{34} = 0$ and $\eta_1 - \eta_3 + \delta_{14} - \delta_{34} = 0$.

Now as $t \to +\infty$, both (4.5.2) and (4.5.3) tend to $-\infty$ and thus (4.5.1) becomes

$$\Delta = \begin{vmatrix} 1 + e^{(\ell_1+n_1)x} + e^{(\ell_3+n_1)x} & 1 \\ -n_1+\ell_1 e^{(\ell_1+n_1)x} + e^{(\ell_3+n_1)x} & -n_1 \end{vmatrix}$$

which yields

$$\Delta = 1 + e^{\eta_1} + e^{\eta_3} \text{ as } t \to +\infty. \tag{4.5.7}$$

This is the same triad as (4.5.6), but the individual solitons are now centred on $\eta_1 = 0$, $\eta_2 = 0$ and $\eta_1 - \eta_2 = 0$.

The expressions given by (4.5.6) and (4.5.7) show that the triad has been shifted after the interaction. The triad which was originally represented by the intersection of lines $\eta_1 + \delta_{14} = 0$ and $\eta_3 + \delta_{34} = 0$ before the interaction (4.5.6) has then become the intersection between $\eta_1 = 0$ and $\eta_3 = 0$ after the interaction (4.5.7).

We shall now explain the detail of the interaction under the above assumptions. Expanding (4.5.1) we find

$$\begin{aligned}\Delta = \underset{(1)}{1} + \underset{(2)}{e^{\eta_1}} + \underset{(3)}{e^{\eta_2}} + \underset{(4)}{e^{\eta_3}} + \underset{(5)}{e^{\eta_4}} + \underset{(6)}{q_{12}e^{\eta_1+\eta_2}} + \underset{(7)}{q_{14}e^{\eta_1+\eta_4}} \\ + \underset{(8)}{q_{23}e^{\eta_2+\eta_3}} + \underset{(9)}{q_{34}e^{\eta_3+\eta_4}}\ .\end{aligned} \tag{4.5.8}$$

We have already specified the triad with phase ℓ_1, ℓ_3, n_1 before the interaction by (4.5.6). This is also recognized as triad (579) from (4.5.8) since

$$\Delta_{579} = e^{\eta_4}[1 + e^{\eta_1+\delta_{14}} + e^{\eta_3+\delta_{34}}]. \qquad (4.5.9)$$
$$\quad (5) \quad (7) \qquad (9)$$

This triad will then interact with a triad with parameters ℓ_2, ℓ_4, n_2. Such triads are recognized from (4.5.8) as (135), (267) and (489). For the values of ℓ's and n's used in the numerical computation in Section (4.6), it can be shown that (267) will first interact with (579). The corresponding Δ for triad (267) is obtained from (4.5.8) as

$$\Delta_{267} = e^{\eta_1}[1 + e^{\eta_2+\delta_{12}} + e^{\eta_4+\delta_{14}}] . \qquad (4.5.10)$$
$$\quad (2) \quad (6) \qquad (7)$$

Now soliton (27) from triad (267) interacts with soliton (57) from triad (579). Since $n_1 \approx n_2$, this is a near-resonant interaction producing the intermediate soliton (25). The post-interaction solitons must respectively contain the numbers (2) and (5) and another common number taken from the unused terms in (4.5.8). The only choice is soliton (12) and soliton (15).

If (67) and (79) are allowed to intersect then the interaction between these solitons will be non-resonant. The post-interaction solitons are (68) and (89). The corresponding Δ for this interaction is

$$\Delta_{6789} = q_{12}e^{\eta_1+\eta_2} + q_{14}e^{\eta_1+\eta_4} + q_{23}e^{\eta_2+\eta_3} + q_{34}e^{\eta_3+\eta_4}$$
$$(6) \qquad\qquad (7) \qquad\qquad (8) \qquad\qquad (9)$$

$$= q_{34}e^{\eta_3+\eta_4}\{1 + e^{\eta_1-\eta_3+\delta_{14}-\delta_{34}} + e^{\eta_2-\eta_4+\delta_{23}-\delta_{34}} + e^{\eta_1-\eta_3+\eta_2-\eta_4+\delta_{12}-\delta_{34}}\}$$
$$(9) \quad (7) \qquad\qquad (8) \qquad\qquad (6)$$

$$= q_{34}e^{\eta_3+\eta_4}\{1 + e^{\eta_1-\eta_3+\delta_{14}-\delta_{34}} + e^{\eta_2-\eta_4+\delta_{23}-\delta_{34}}$$
$$(9) \quad (7) \qquad\qquad (8)$$

$$+ \frac{q_{12}\, q_{34}}{q_{23}\, q_{14}} e^{\eta_1-\eta_3+\delta_{14}-\delta_{34}+\eta_2-\eta_4+\delta_{23}-\delta_{34}}\}.$$

(6)

Now we have

$$\frac{q_{12}\, q_{34}}{q_{23}\, q_{14}} = \frac{(\ell_1-\ell_2)(\ell_3-\ell_4)}{(\ell_3-\ell_2)(\ell_1-\ell_4)} .$$

This is a O(1) quantity and therefore implies small phase shift, and hence (6789) is a non-resonant interaction.

The schematic configuration of the whole interaction can now be drawn as in Fig. 4.7a.

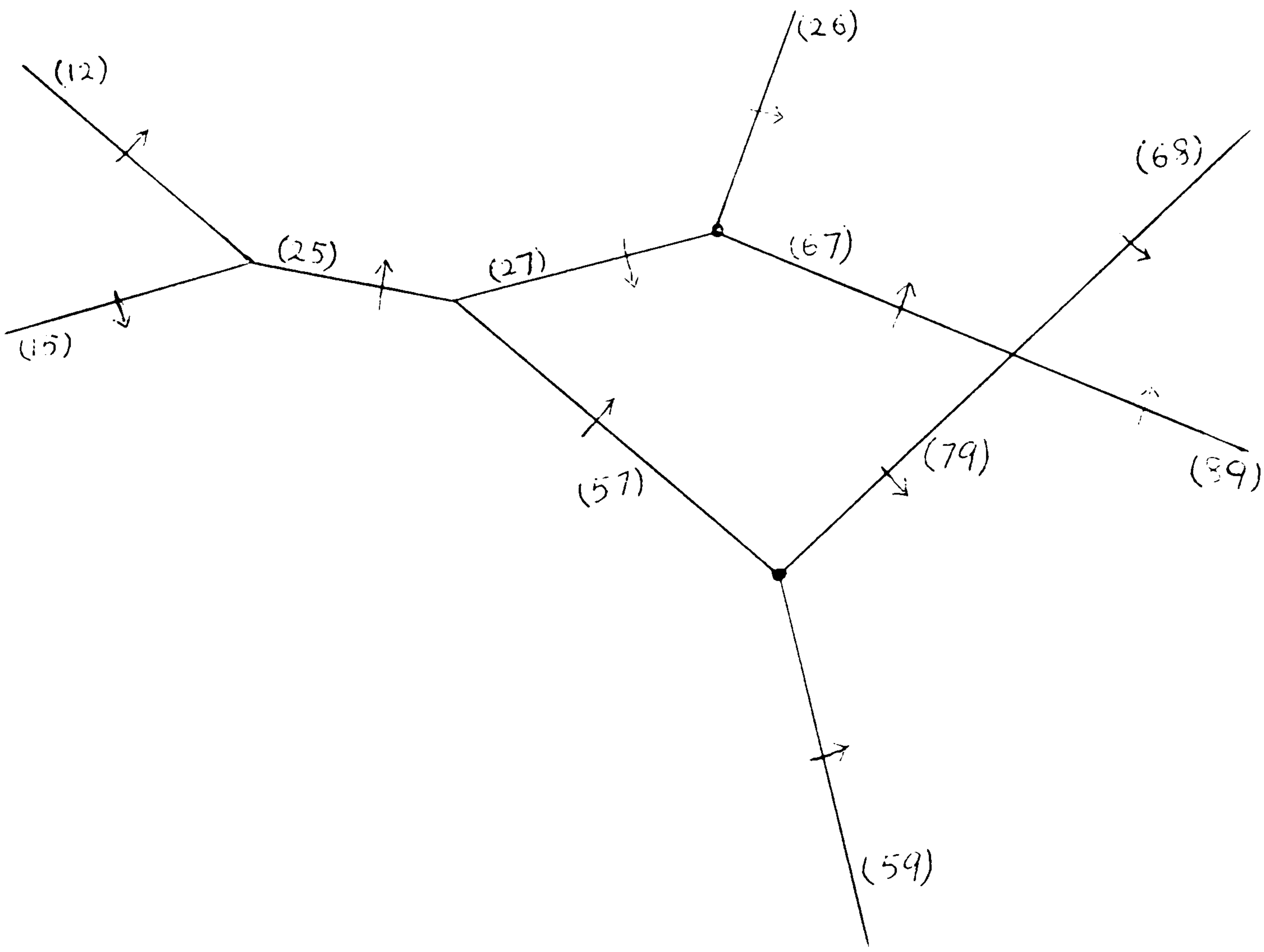


Fig. 4.7a <u>The interaction between triads (267) and (579) at earlier times.</u>

In Fig. 4.7a, we have put the faster triad (267) "behind" the slower triad (579).

As we increase the time, soliton (26) from triad (267) and soliton (6S) intersect to produce a non-resonant interaction (2468), and soliton (24) produced from this interaction in turn intersects with (12) to produce triad (124). Meanwhile solitons (89) and (59) interact to produce another non-resonant interaction (3589). Soliton (38) from (3589) in turn interacts with soliton (48) from (2468) to produce a near-resonant interaction with intermediate soliton (34) and the shifted solitons (14) and (13). Solitons (13) and (15) interact to produce triad (135). The interaction at this stage is now complete (Fig. 4.7b).

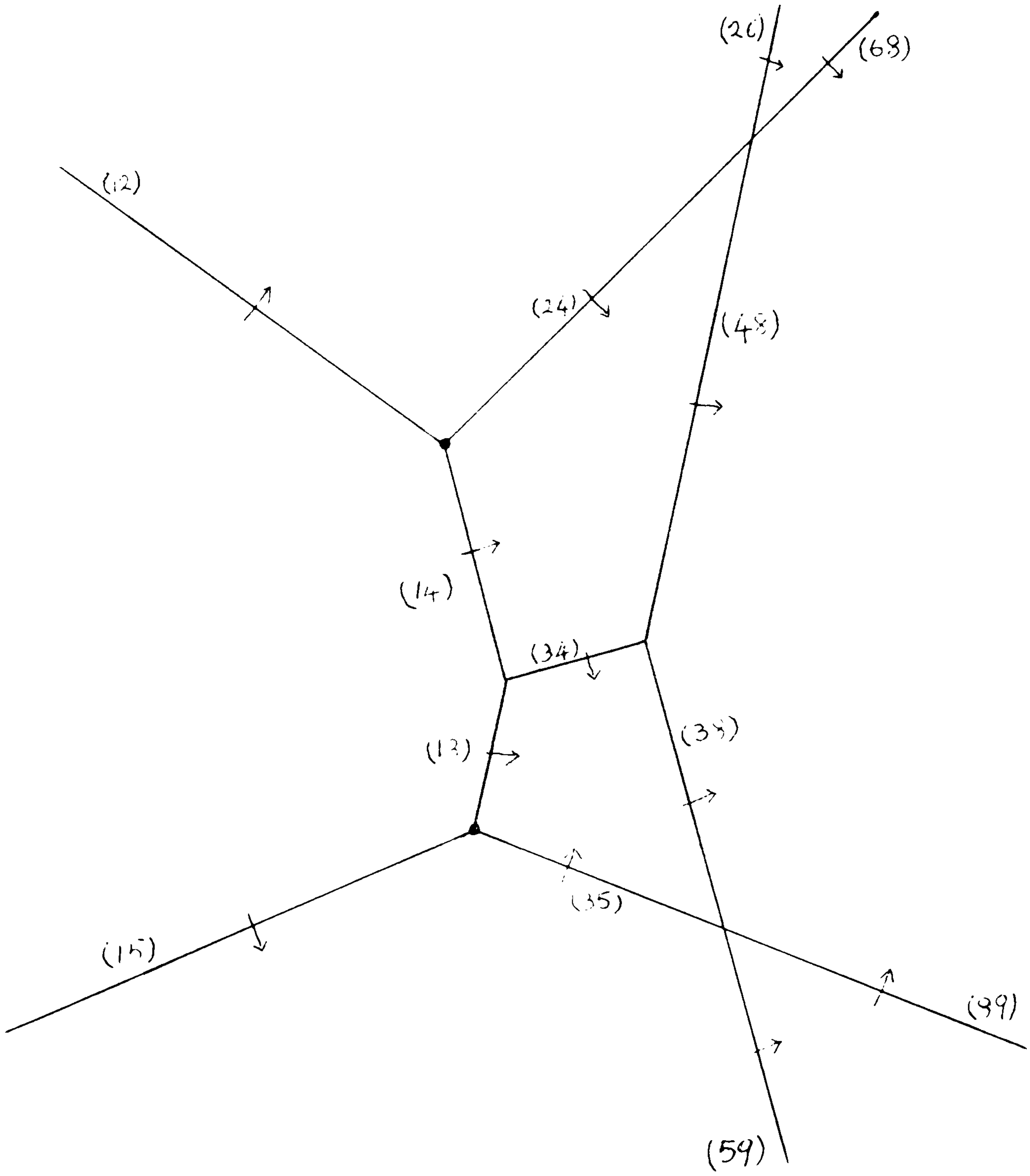


Fig. 4.7b <u>The final interaction between triads (124) and (135).</u>

In Fig. 4.7b, triad (135), originally triad (267), has moved "in front" of triad (124), originally (579). Examining the corresponding Δ's we find from (4.5.8)

$$\Delta_{135} = 1 + e^{\eta_2} + e^{\eta_4}$$

and

$$\Delta_{124} = 1 + e^{\eta_1} + e^{\eta_3}.$$

The last expression is exactly (4.5.7).

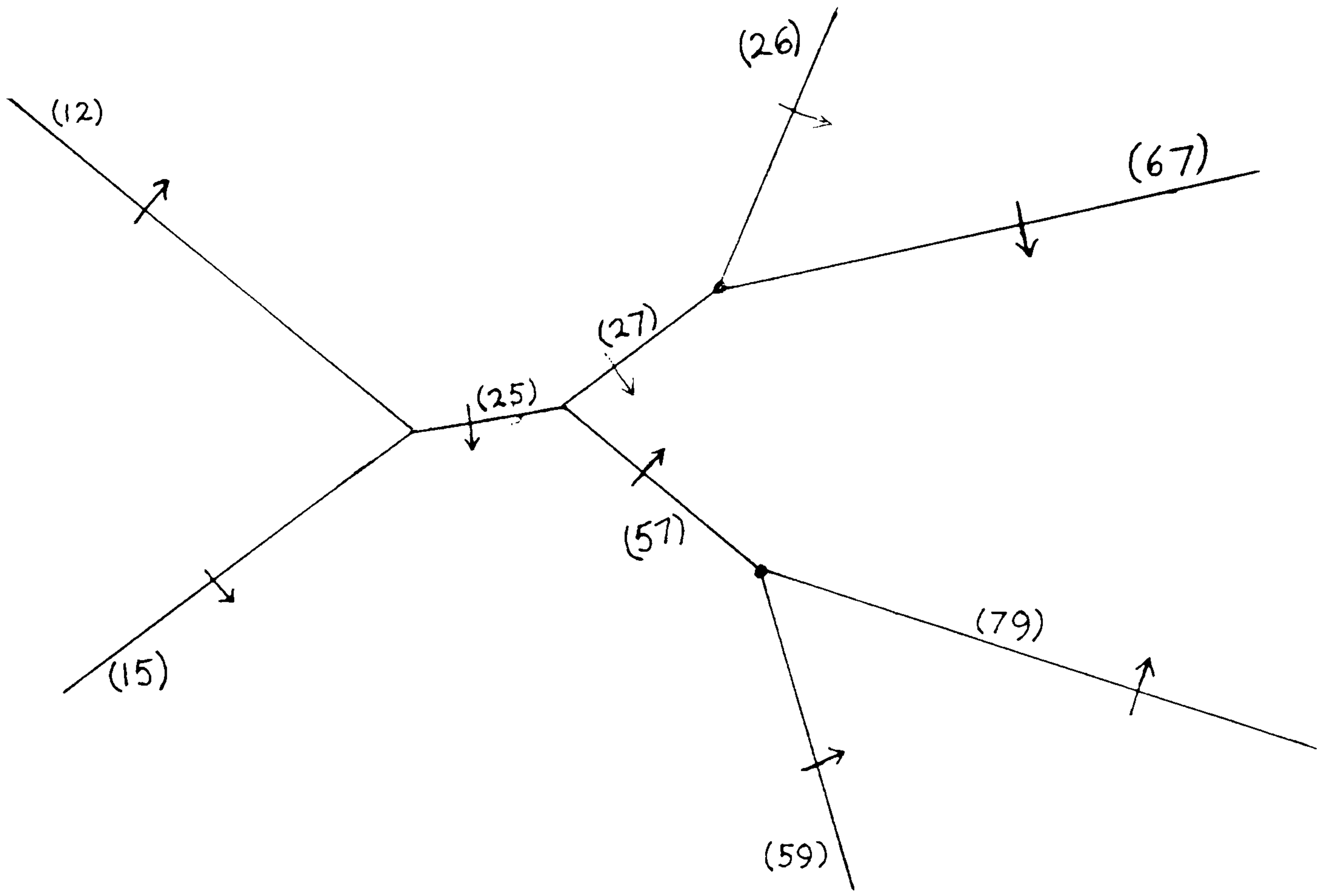


Fig. 4.8a <u>An alternative interaction between triads (267) and (579) at earlier times.</u>

We note here that depending on the slopes of (67) and (79), these two solitons might not intersect. Therefore an alternative to Fig. 4.7a is given by Fig. 4.8a

However solitons (67) and (79) interact at last to produce a non-resonant interaction (6789). Solitons (68), just produced, interacts with soliton (26) to produce also a non-resonant interaction (2468), while soliton (89) interacts with soliton (59), also producing a non-resonant interaction (3589).

The rest of the interactions are the same as in Fig. 4.7b. The full interaction is sketched in Fig. 4.8b.

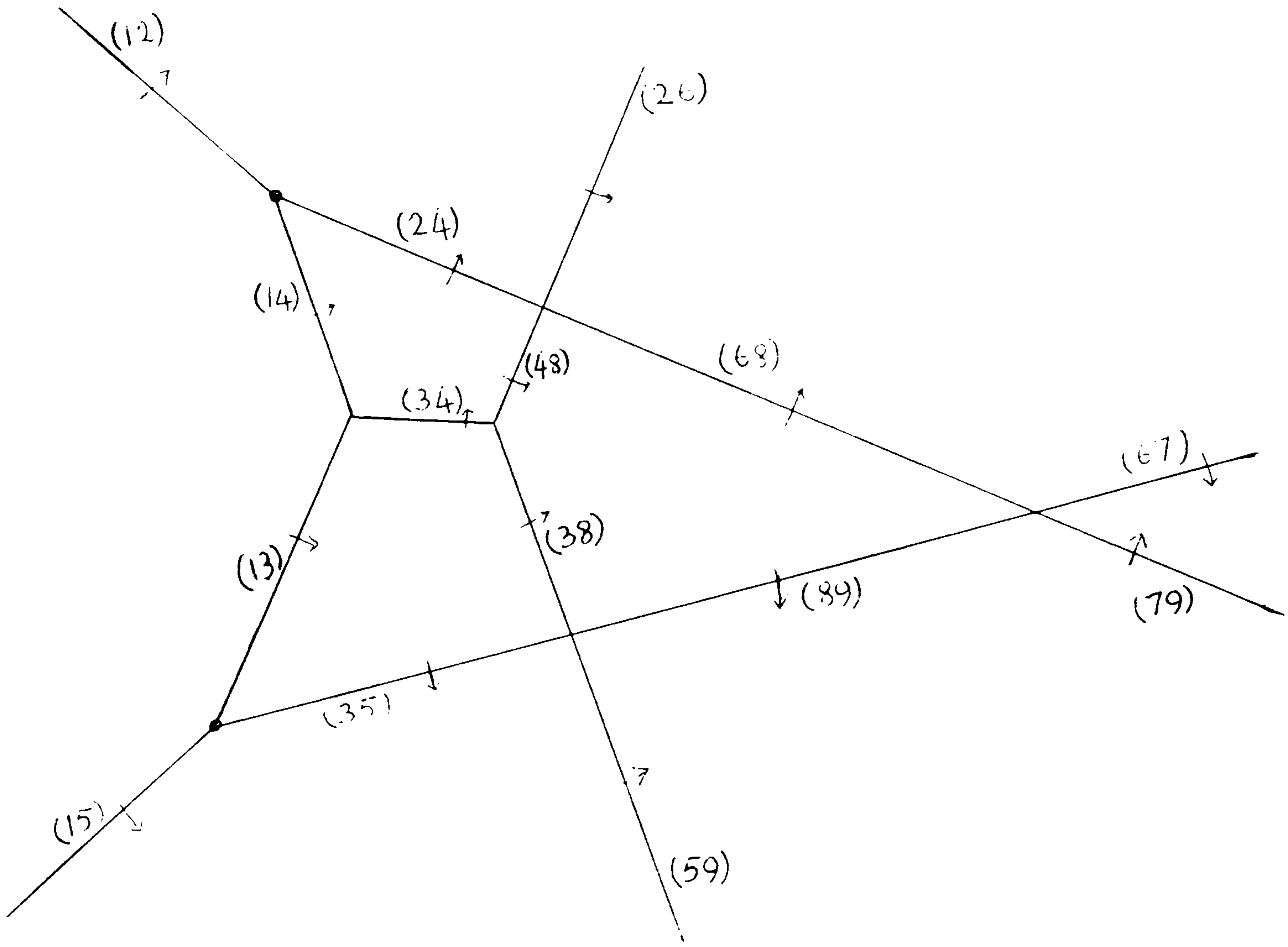


Fig. 4.8b <u>The alternative interaction between triads (124) and (135) at later times.</u>

We have thus seen in the above that upon choosing $n_1 \approx n_2$ both triads experience significant phase shifts after the interaction.

For the case $n_1 - n_2 = O(1)$, all δ_{ij}'s are very close to zero and therefore the phase shifts of all triads are very small. Thus the faster triad simply sweeps over the slower triad and all the intersections between individual solitons produce only non-resonant interactions.

4.6 Numerical computations

Some numerical computations have been carried out for the interactions between a triad and a soliton and between two triads. The expressions for Δ (4.4.6) and (4.5.8) are first calculated and then the full analytic solution (4.2.2) is used to obtain the actual solution.

We follow the method previously used by Anker and Freeman (1978) in plotting the solution graphs by using symbols. The main procedure is to draw the lines on which the amplitudes of all individual solitons are centred. This is done by locating all the possible local maxima in a run. These local maxima are the amplitudes of some solitons. In order to make the result easier to interpret we have chosen the values of parameters ℓ_i's and n_i's in such a way that the amplitudes of all possible solitons arising from them are different. To each of these amplitudes a different symbol is allocated. These symbols are then plotted in the region of their existence in the x-y coordinates.

For the interaction between a triad and a soliton, three different sets of ℓ_i's and n_i's are used and they are listed in Table 4.1a. The slopes of the phase lines are given in Table 4.1b while the amplitudes and the symbols used are given in Table 4.1c. In all the calculations we take $a_i = 1$ for simplicity.

For the values of ℓ_i's and n_i's in Set 1, we find that expression

$$-(Q_3-Q_2)(P_1^2+3Q_1^2) - (Q_2-Q_1)(P_3^2+3Q_3^2) + (Q_3-Q_1)(P_2^2+3Q_2^2) \qquad (4.6.1)$$

is negative. Thus as $t \to -\infty$ we have triad (4.4.4) while as $t \to +\infty$ we have triad (4.4.5)

	Set 1	Set 2	Set 3
ℓ_1	1.0	-1.0	-2.01
ℓ_2	1.5	2.5	3.99
ℓ_3	2.5	4.0	1.99
$n_1 = n_2$	2.0	2.0	3.01
n_3	$2.0 + 10^{-6}$	$2.0 + 10^{-6}$	1.41
δ_{13}	-16.013	-13.998	0.758
δ_{23}	-16.572	-16.708	-2.133

Table 4.1a Data for Figures 4.9, 4.10 and 4.11

	Set 1	Set 2	Set 3
$\eta_1 = C$	-1.0	-0.333	-0.199
$\eta_2 = C$	-2.0	2.0	1.02
$\eta_3 = C$	2.0	0.5	1.72
$\eta_1 - \eta_2 = C$	0.4	0.667	0.50
$\eta_1 - \eta_3 = C$	0.286	0.333	0.788
$\eta_2 - \eta_3 = C$	0.25	0.154	0.176

Table 4.1b Slopes of phase lines for Figures 4.9, 4.10 and 4.11

		Amplitude		
Soliton	Symbol	Set 1	Set 2	Set 3
(12),(46)	1	4.5	0.5	0.5
(13),(45)	2	6.125	10.125	24.5
(14),(26),(35)	3	10.125	18.0	5.78
(23),(56)	4	0.125	6.125	18.0
(24)	5	1.125	12.5	-
(34)	A	0.5	1.125	-

Table 4.1c Showing symbols and amplitudes used for Figures 4.9, 4.10 and 4.11

For the values of ℓ_i's and n_i's in Set 1, we have plotted the interactions at several time instants but only those at T = -1.5, T = -0.5 and T = 2.0 are shown as they give different configurations. These are illustrated in Figures 4.9a, b and c.

At time T = -1.5 (Fig. 4.9a), soliton (26) interacts with soliton (46) from triad (456) to produce a near-resonant interaction (1246). The net configuration is exactly as in Fig. 4.4a. We note that at this stage solitons (26) and (56) do not interact due to their slopes.

At time T = -0.5 (Fig. 4.9b), soliton (26) is about to interact with soliton (56) as the length of (46) is shrinking. We note that at this stage the length of (24) remains unchanged, indicating that all the solitons in the near-resonant interaction (1246) do not interact among themselves.

The change is very clear at T = 2.0 (Fig. 4.9c). In this figure (26) and (56) interact to produce a non-resonant interaction (2356); soliton (35), just produced in (2356), interacts with (45) to form a near-resonant interaction (1345); and soliton (13), produced from this interaction, interacts with (12) to produce triad (123). The whole configuration now agrees with Fig. 4.4b. If we increase the time further, the configuration remains the same but the length of (13) will be greater.

For the values of ℓ_i's and n_i's listed in Set 2 of Table 4.1a, we find that expression (4.6.1) is still negative. Again, in this case as $t \to -\infty$ we have the triad given by (4.4.4) and as $t \to +\infty$ it is given by (4.4.5). The situation here is exactly the same as the one for Set 1 except that solitons (26) and (56) intersect. The interactions between the triad and the soliton arising from Set 2 are presented in Figures 4.10 a, b, c and d.

At time T = -1.25 (Fig. 4.10a) as well as interacting with soliton (46), soliton (26) also interacts with soliton (56) to produce a non-resonant interaction (2356). Other interactions are similar to Fig. 4.9a.

At time T = -0.5 (Fig. 4.10b), the configuration is the same as Fig. 4.10b except that the triangle formed by solitons (26), (46) and (56) is now becoming smaller.

A change can be seen at time T = 1.0 (Fig. 4.10c) when soliton (23) interacts with soliton (12) to produce triad (123), and solitons (35) and (45) interact to give a near-resonant soliton (1345). We note that the scale for Fig. 4.10c has been enlarged so that we can see soliton (34).

Fig. 4.10d is for the interaction at T = 1.5. Its configuration is similar to the one in Fig. 4.10c, except that the length of soliton (13) is greater than before. Therefore, if we increase the time further, the final triad (123) and soliton (14) will be very far apart.

For the values of ℓ_i's and n_i's listed in Set 3 of Table 4.1a, the interactions are plotted in Figures 4.11a,b. For these values of ℓ_i's and n_i's the situation is different from those of the other two sets. It can be found that expression (4.6.1) is now positive, implying the associated Δ from (4.4.1), as $t \to -\infty$, is

$$\Delta = \begin{vmatrix} 1 + e^{(\ell_1+n_1)x} + e^{(\ell_2+n_1)x} & 1 \\ -n_1 + \ell_1 e^{(\ell_1+n_1)x} + \ell_2 e^{(\ell_2+n_1)x} & -n_3 \end{vmatrix} ,$$

which is equivalent to

$$\Delta = 1 + e^{\eta_1} + e^{\eta_2} \text{ as } t \to -\infty . \qquad (4.6.2)$$

Similarly, we shall have

$$\Delta = 1 + e^{\eta_1+\delta_{13}} + e^{\eta_2+\delta_{23}} \text{ as } t \to +\infty . \qquad (4.6.3)$$

However, since now $n_1 - n_3 = O(1)$, we then have δ_{13} and δ_{23} very small. Therefore the final triad (4.6.3) is not very much shifted from the original triad (4.6.2). Indeed all the interactions in this case produce small phase shifts, and they are illustrated in Figures 4.11a,b.

At time T = -2.0 (Fig. 4.11a), triad (123) is placed behind soliton (26). Interaction between (26) and (23) produces a non-resonant interaction (2356).

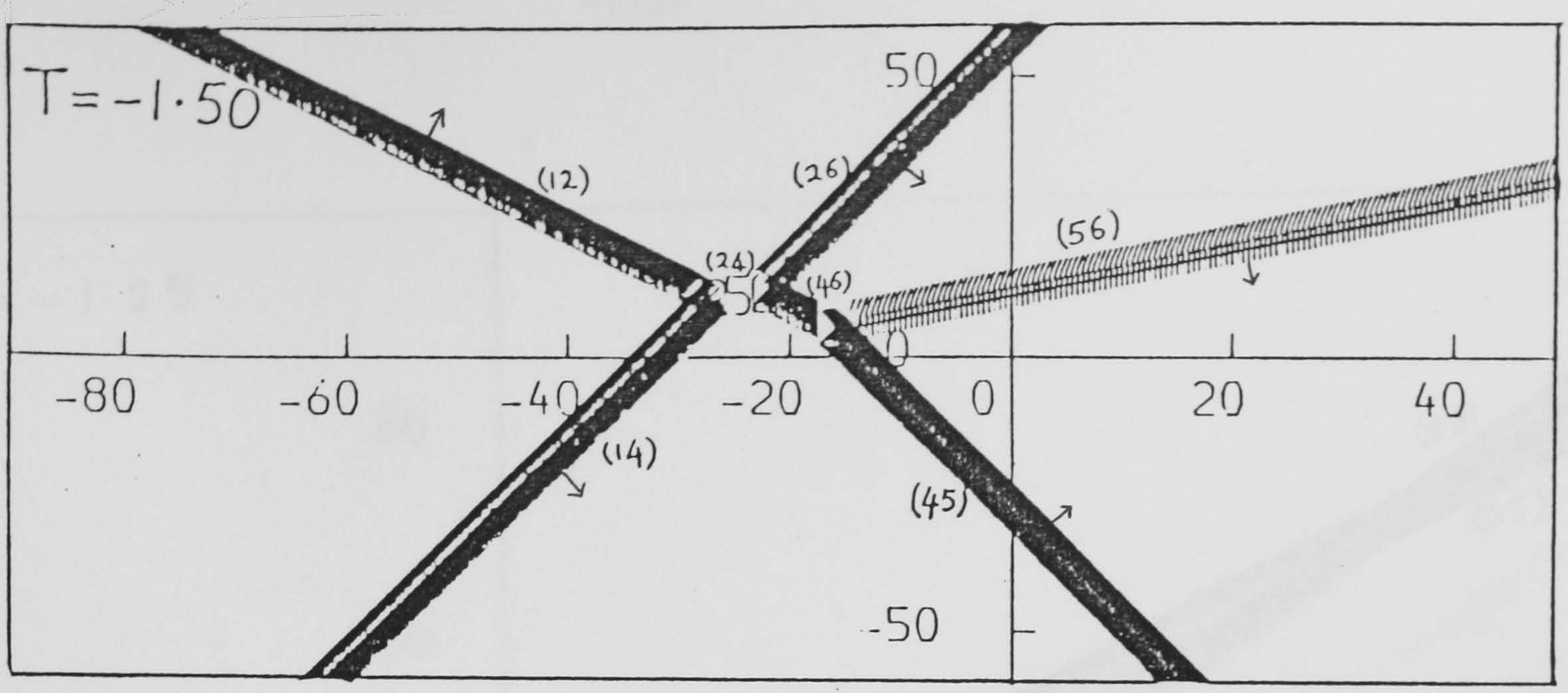


Fig. 4.9a

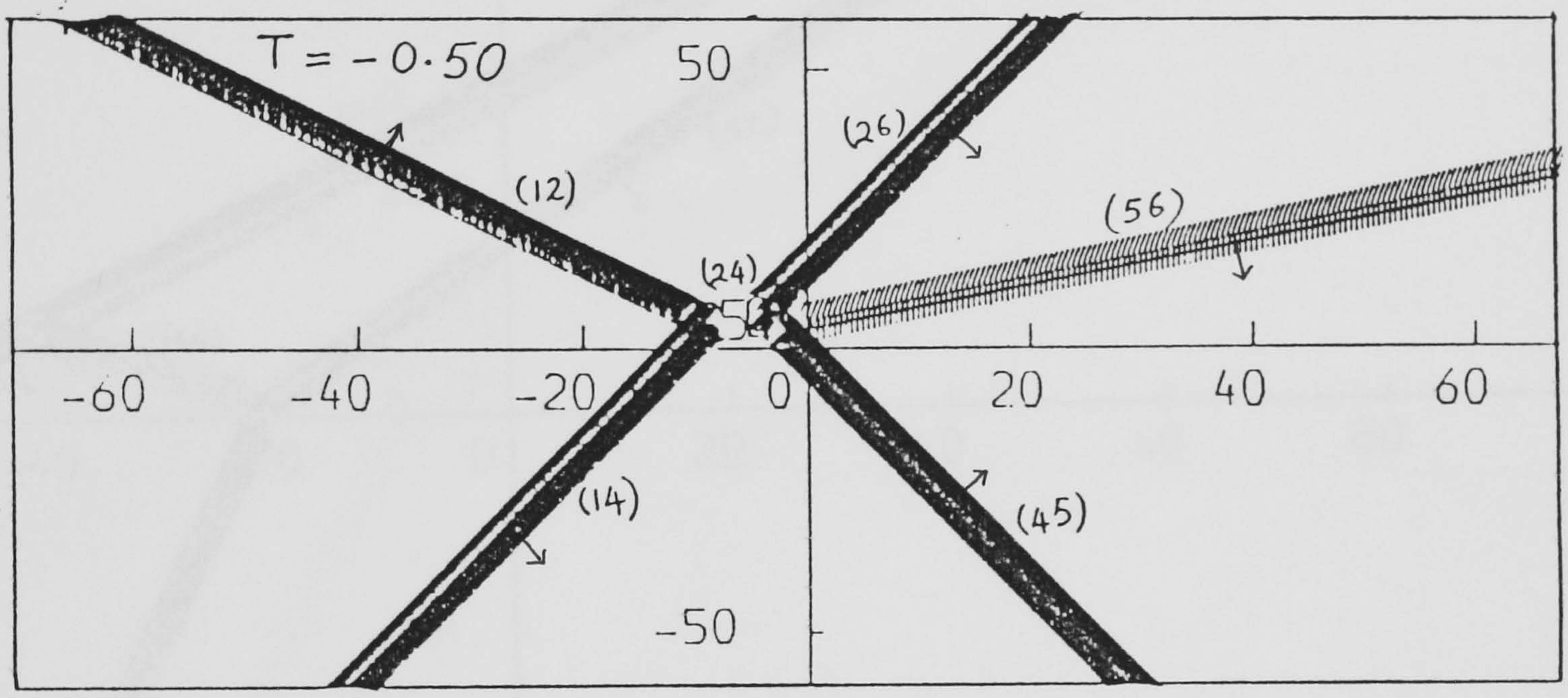


Fig. 4.9b

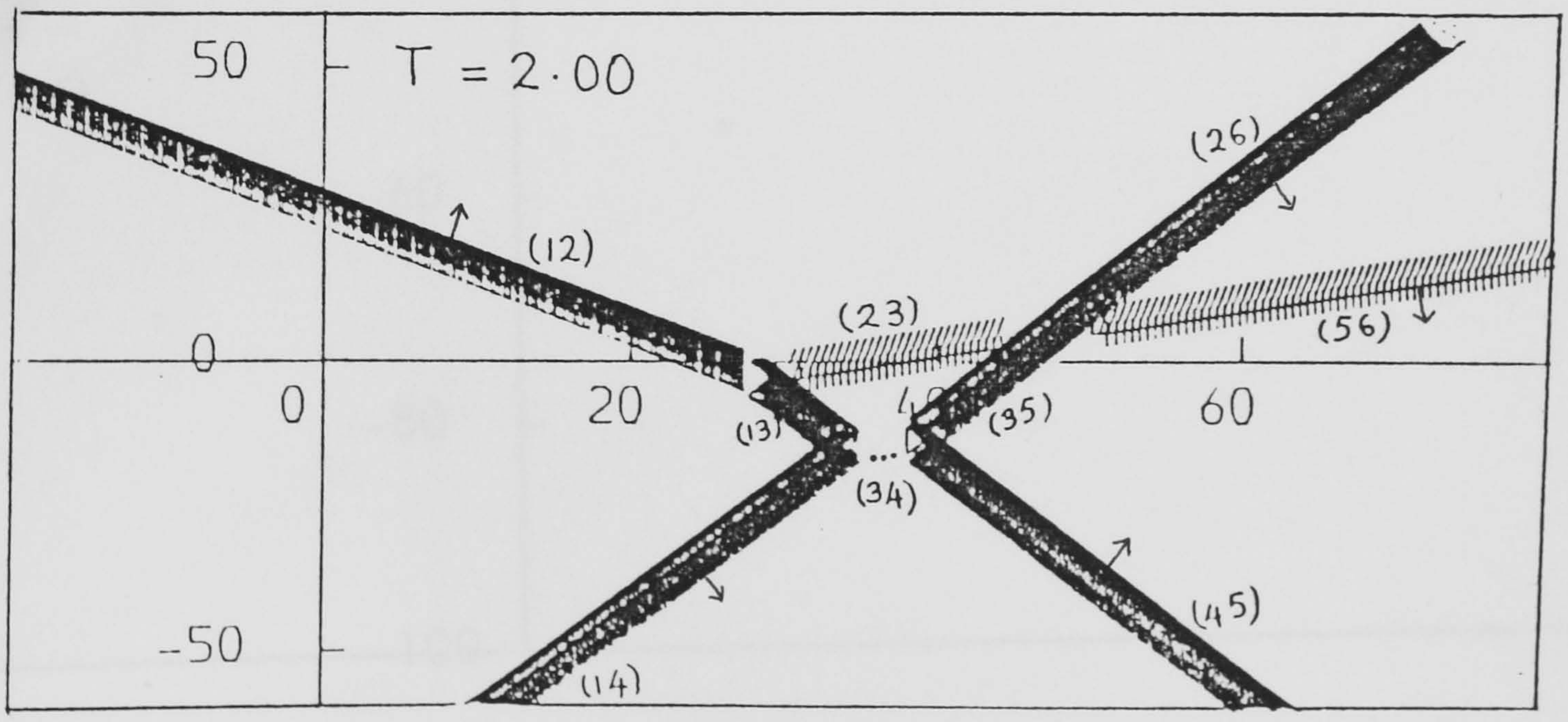
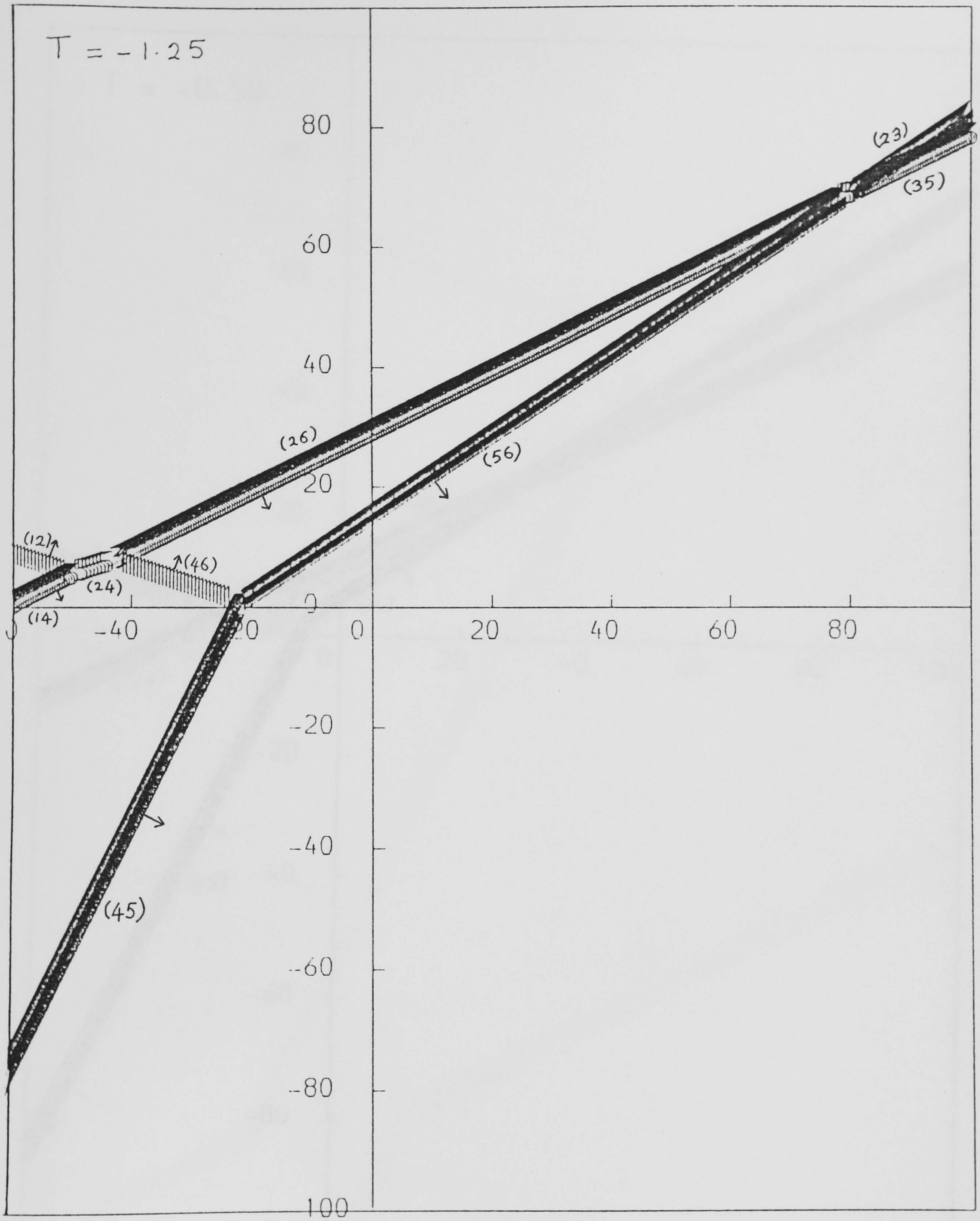


Fig. 4.9c

Fig. 4.9. Interactions between triad with parameters $\ell_1 = 1.0$, $\ell_2 = 1.5$ $n_1 = 2.0$ and soliton with parameters $\ell_3 = 2.5$, $n_3 = 2.0 + 10^{-6}$ at time instants (a) T = -1.5 (b) T = -0.5 (c) T = 2.0



T = -1·25
80
60
40
20
0
-20
-40
-60
-80
-100
-40
-20
0
20
40
60
80
(23)
(35)
(26)
(56)
(12)
(46)
(24)
(14)
(45)

Fig. 4.10a

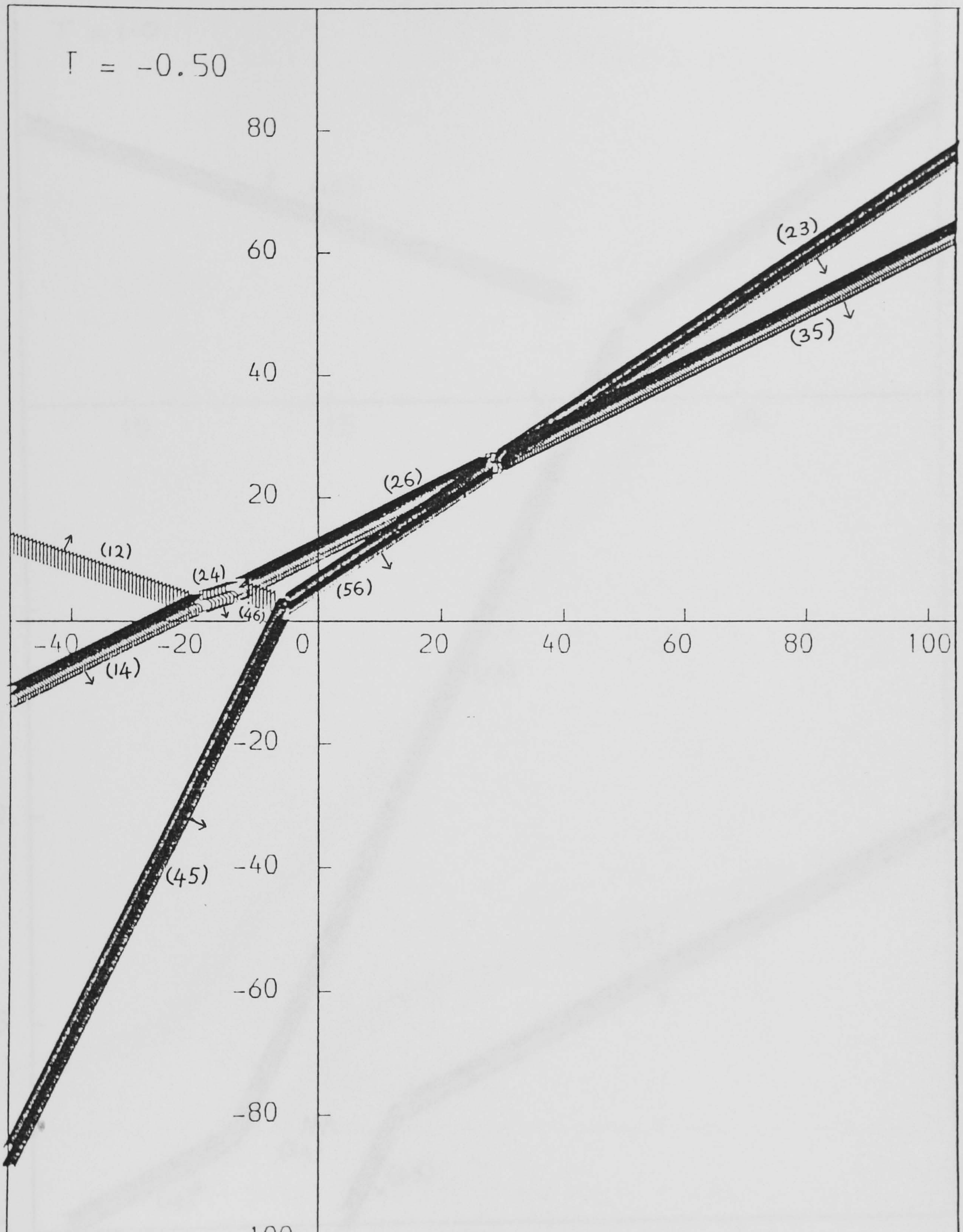

Γ = -0.50
80
60
40
20
-20
-40
-60
-80
100
-40
-20
0
20
40
60
80
100
(23)
(35)
(26)
(12)
(24)
(46)
(56)
(14)
(45)

Fig. 4.10b

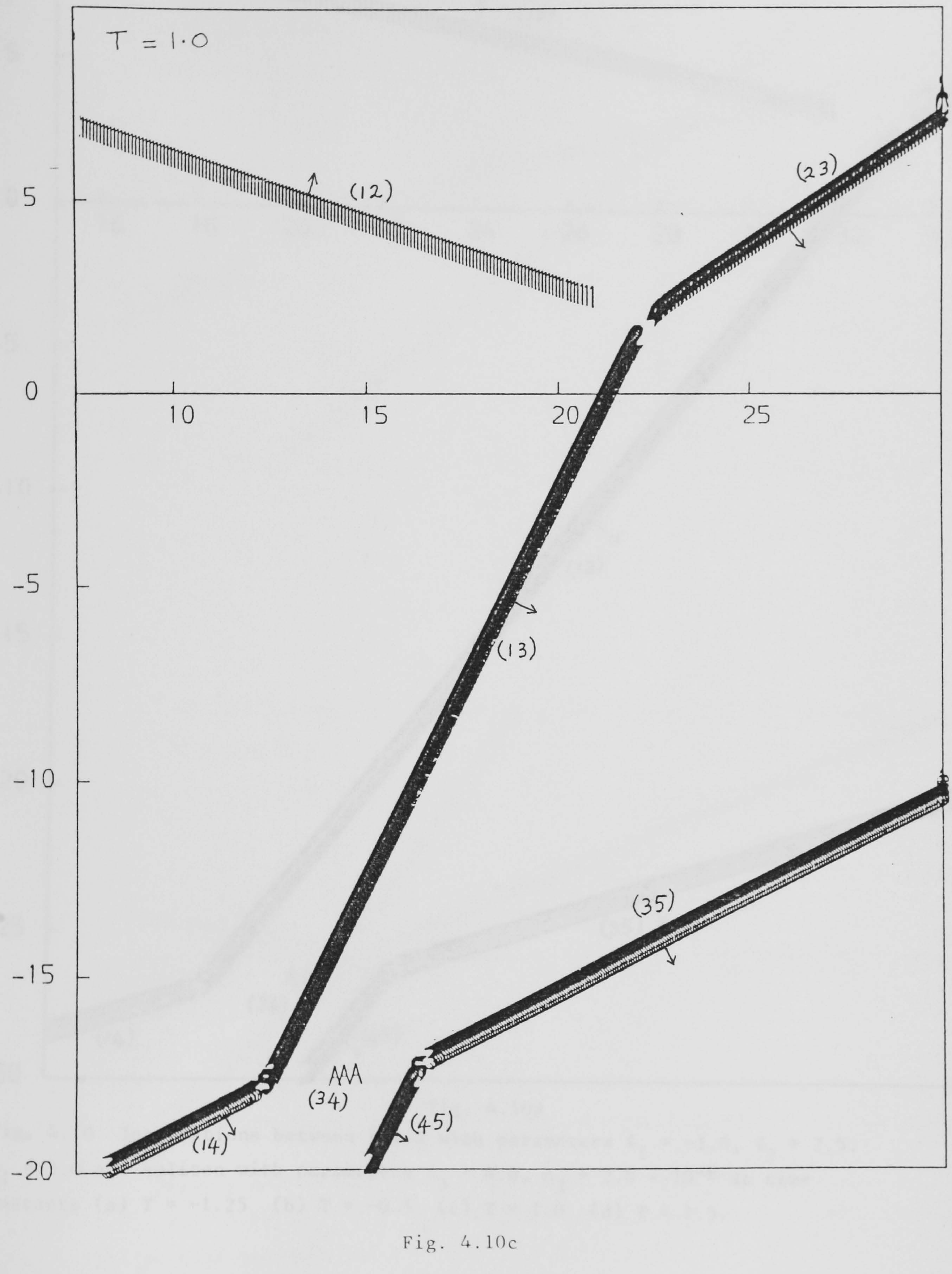


Fig. 4.10c

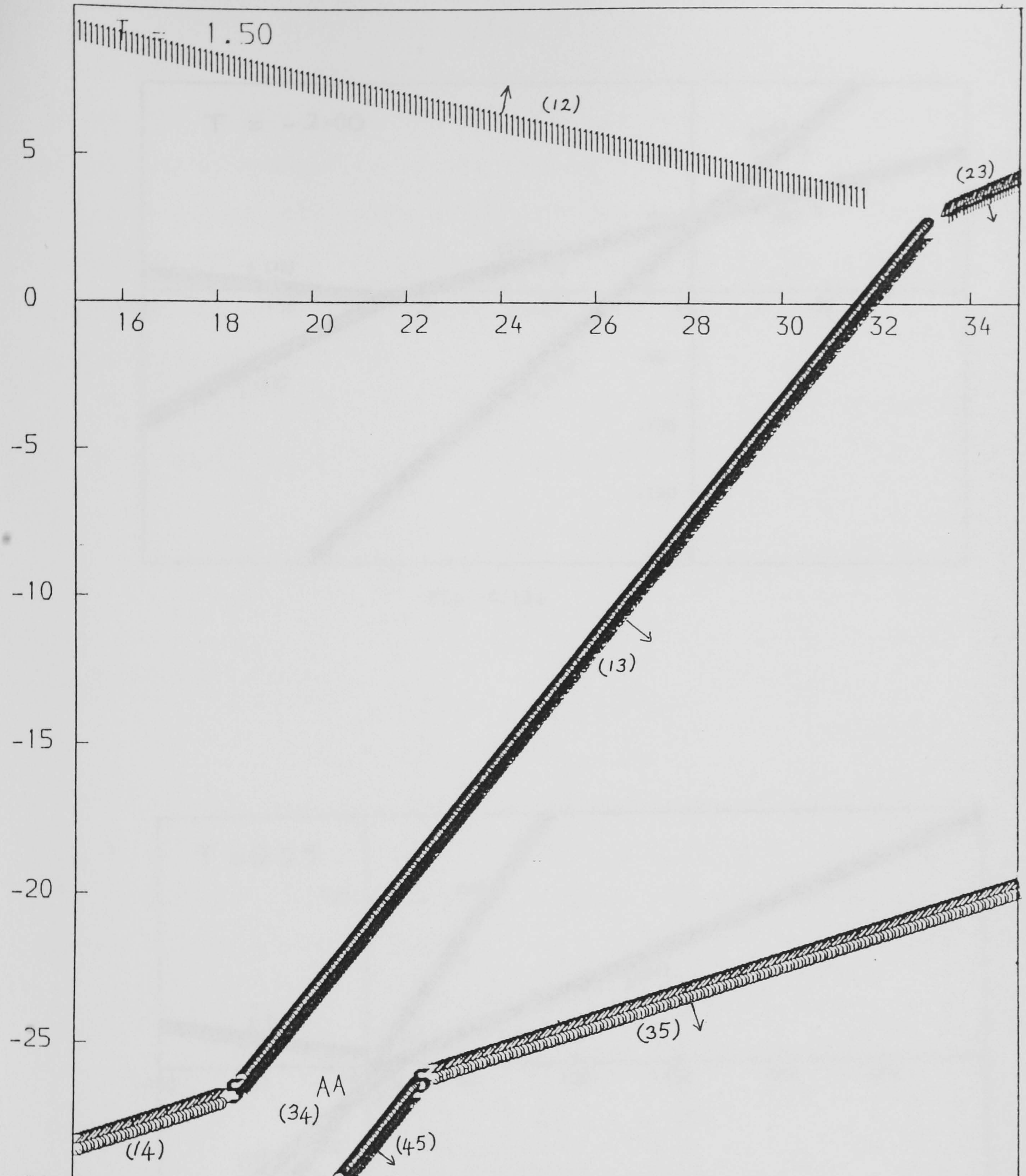


Fig. 4.10d

Fig. 4.10 Interactions between triad with parameters $\ell_1 = -1.0$, $\ell_2 = 2.5$. $n_1 = 2.0$ and soliton with parameters $\ell_3 = 4.0$, $n_3 = 2.0 + 10^{-6}$ at time instants (a) T = -1.25 (b) T = -0.5 (c) T = 1.0 (d) T = 1.5.

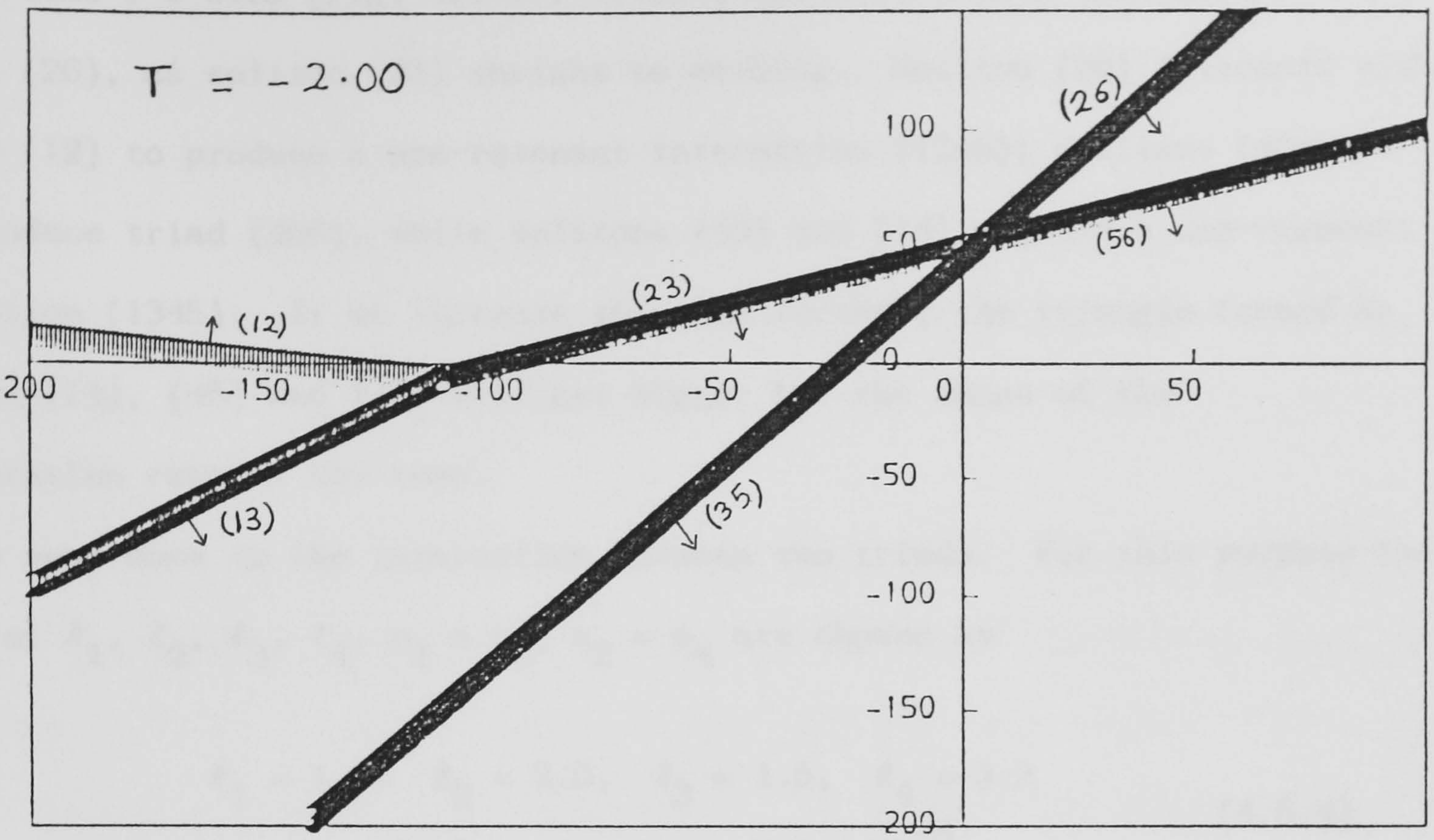


Fig. 4.11a

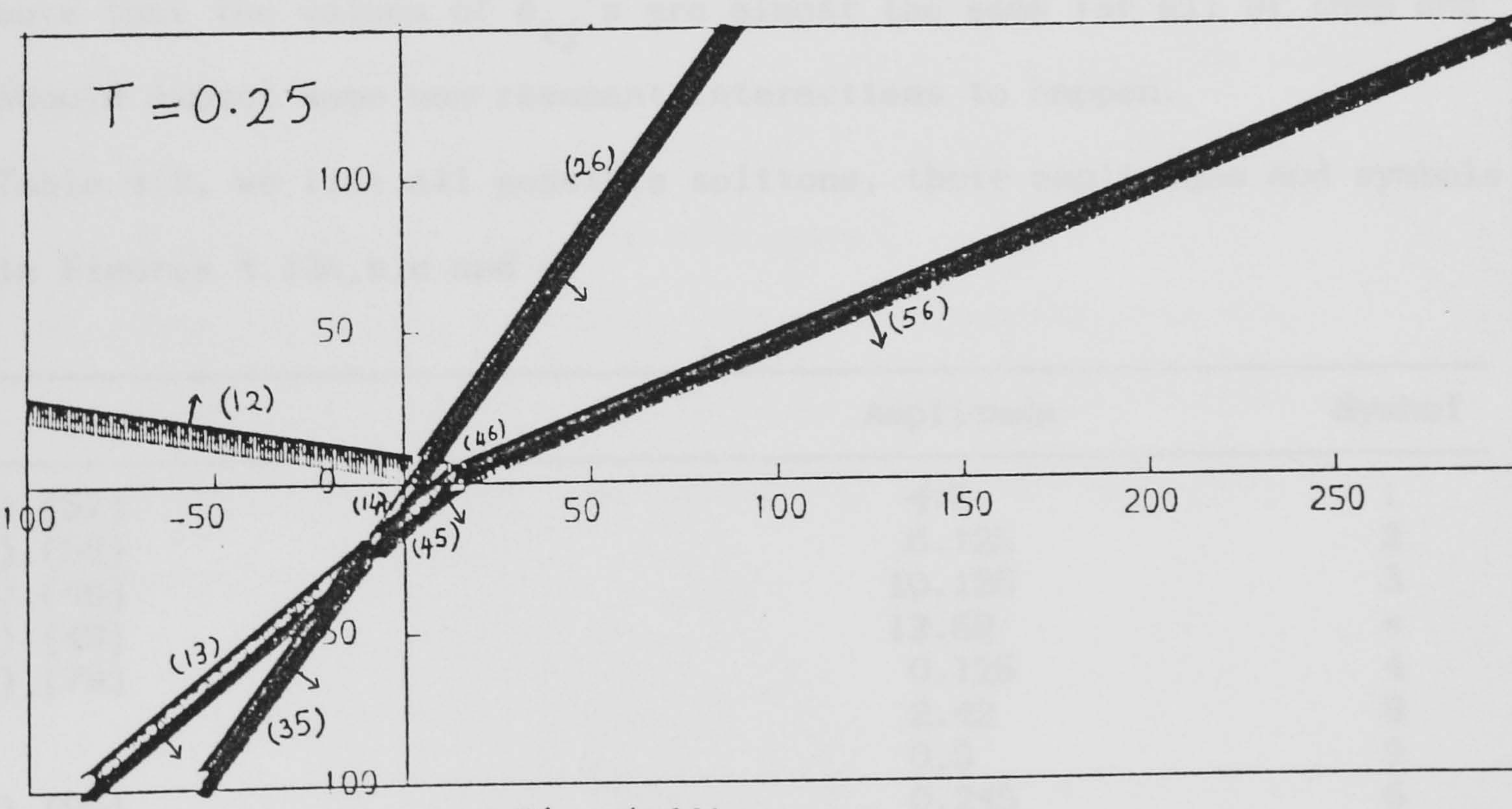


Fig. 4.11b

Fig. 4.11 Showing nonresonant interactions between triad with parameters $\ell_1 = -2.01$, $\ell_2 = 3.99$, $n_1 = 3.01$ and soliton with parameters $\ell_3 = 1.99$, $n_3 = 1.41$ at time instants (a) T = -2.0 (b) T = 0.25

At time T = 0.25 (Fig. 4.11b), triad (123) has already overtaken soliton (26), as soliton (23) shrinks to nothing. Soliton (26) interacts with soliton (12) to produce a non-resonant interaction (1246); solitons (46) and (56) produce triad (456), while solitons (45) and (14) produce a non-resonant interaction (1345). If we increase the time further, the triangle formed by solitons (14), (45) and (46) will get bigger but the shape of the configuration remains the same.

We next move to the interaction between two triads. For this purpose the values of ℓ_1, ℓ_2, ℓ_3, ℓ_4, $n_1 = n_3$, $n_2 = n_4$ are chosen as

$$\ell_1 = 1.0, \quad \ell_2 = 2.5, \quad \ell_3 = 1.5, \quad \ell_4 = 3.2$$
$$n_1 = n_3 = 2.0 \ , \quad n_2 = n_4 = 2.0 + 10^{-6} \quad . \tag{4.6.4}$$

The values for δ_{ij}'s are

$$\delta_{12} = -16.013 \ ; \qquad \delta_{14} = -15.8$$
$$\delta_{23} = -16.6 \quad ; \qquad \delta_{34} = -16.2. \tag{4.6.5}$$

We note that the values of δ_{ij}'s are almost the same for all of them and thus we should expect some non-resonant interactions to happen.

In Table 4.2, we list all possible solitons, their amplitudes and symbols for use in Figures 4.12a,b,c and d.

Soliton	Amplitude	Symbol
(12),(36),(57)	4.5	1
(14),(38),(58)	6.125	2
(13),(26),(48)	10.125	3
(15),(27),(49)	13.52	*
(24),(68),(79)	0.125	4
(25)	2.42	8
(34)	0.5	9
(35),(67),(89)	0.245	6

Table 4.2 Amplitudes and symbols of the solitons for Figures 4.12a,b,c and d

We note that this choice of ℓ_i's and n_i's satisfies all the assumptions used in the analysis in Section 4.5.

From (4.6.4) we can see that the slope of soliton (67) [which corresponds to line $\eta_2 - \eta_4 = 0$] is less than the slope of soliton (79) [corresponding to line $\eta_1 - \eta_3 = 0$]. Thus according to Fig. 4.7a they intersect.

At time T = -1.0 (Fig. 4.12a) triad (267) is placed above triad (579). Solitons (67) and (79) interact to produce a non-resonant interaction (6789), while solitons (27) and (57) intersect to produce a near-resonant interaction (1247).

At time T = -0.5 (Fig. 4.12b), the quadrangle formed by (27), (57), (79) and (67) is indeed getting smaller. Note that the scale has been enlarged in Fig. 4.12b in order to observe that (67) and (89) lie on a straight line.

A significant change is recorded at T = 1.0 (Fig. 4.12c) after the triangle has disappeared. In Fig. 4.12c we note that triad (135), which differs from the original triad (267) by some phase shift, is now below triad (124), which was originally triad (579). This means that the triad with parameters ℓ_2, ℓ_4,n_2 has overtaken the triad with parameters ℓ_1, ℓ_3, n_1.

The configuration at T = 2.0 (Fig. 4.12d) is the same as the one for Fig. 4.12c, except that both of the triangles are much bigger than they were before, and thus the two triads (135) and (124) are much further apart. This means that no further interaction will take place.

All the plots presented in Figures 4.9-4.12 should therefore be sufficient to observe the interactions between triads. Interactions between three or more triads will be the same as for the two, but with more complicated configurations.

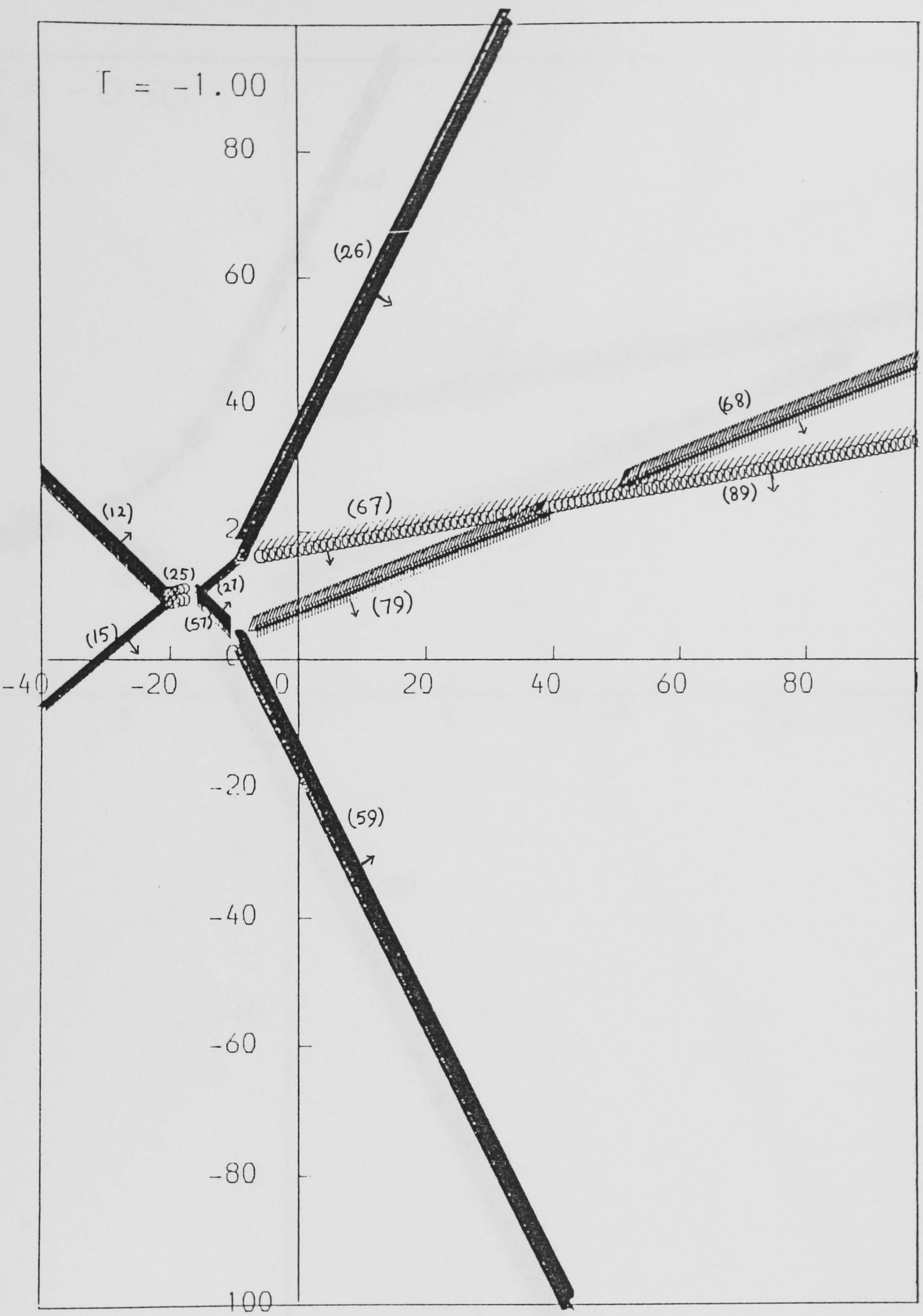
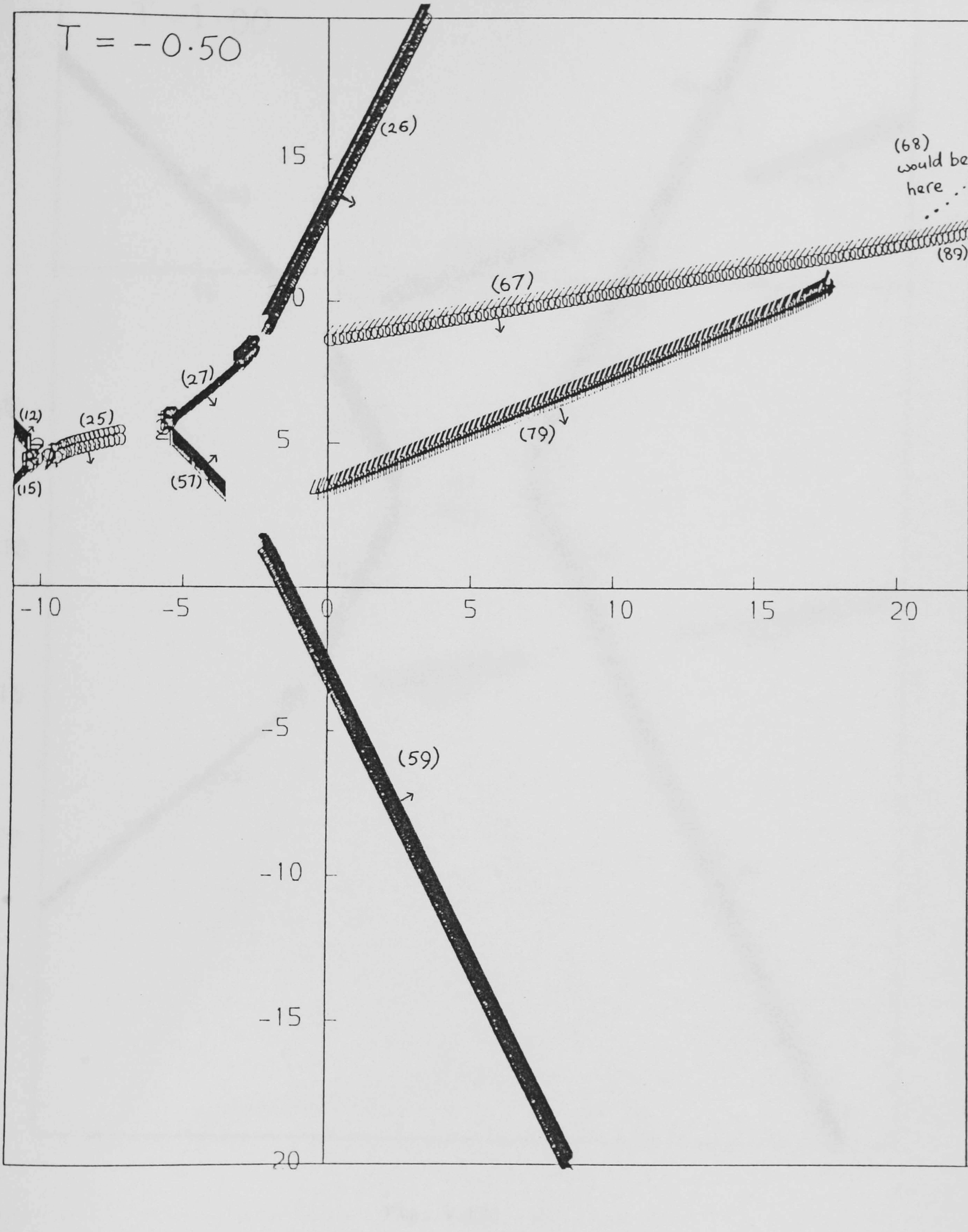
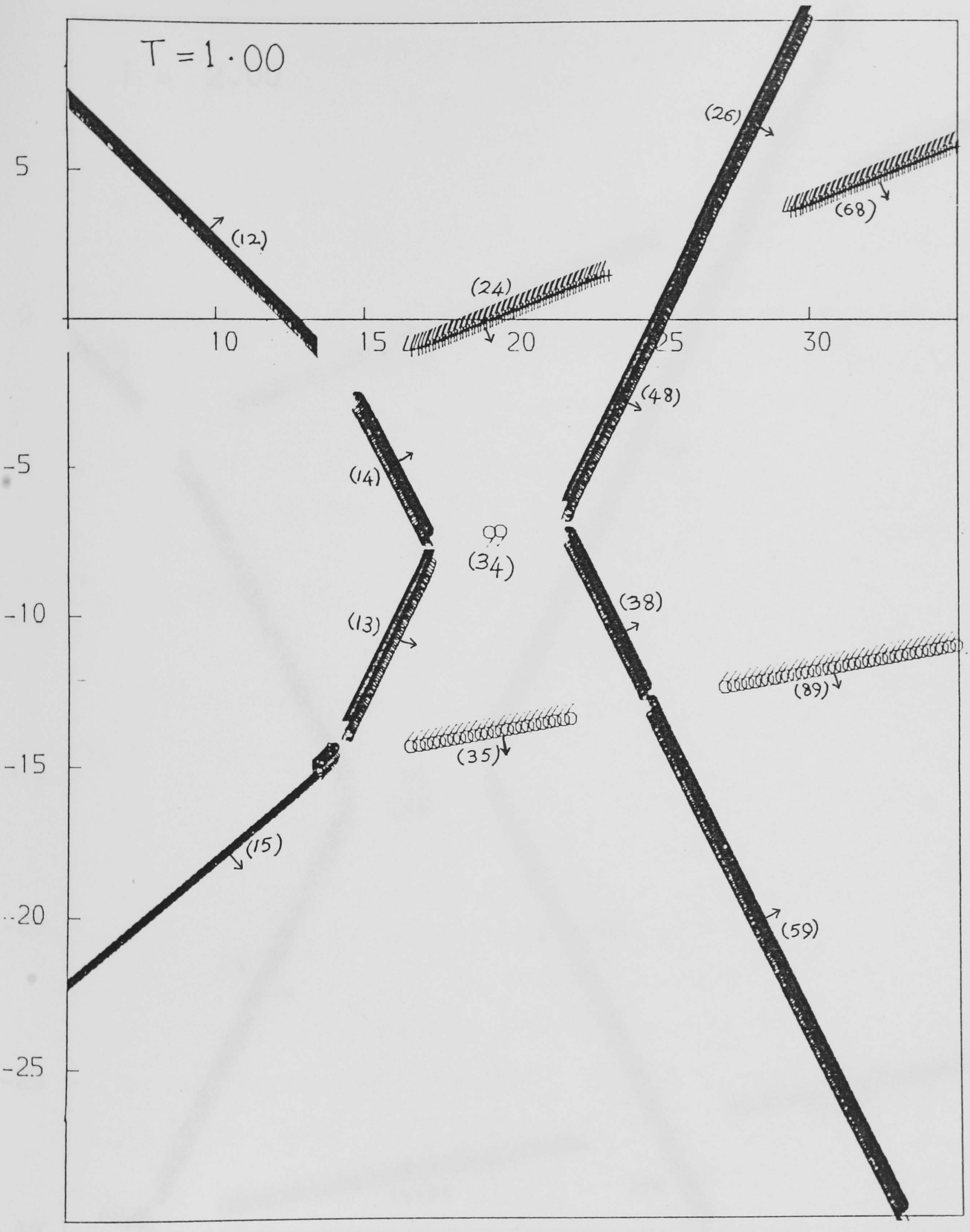

Γ = -1.00
80
60
40
20
-20
-40
-60
-80
100
-40
-20
0
20
40
60
80
(26)
(12)
(25)
(27)
(57)
(15)
(67)
(68)
(89)
(79)
(59)

Fig. 4.12a



T = -0.50
(26)
(68) would be here
(89)
(67)
(27)
(12)
(25)
(15)
(57)
(79)
(59)
15
10
5
-5
-10
-15
-20
-10
-5
0
5
10
15
20

Fig. 4.12b



T = 1.00
5
0
-5
-10
-15
-20
-25
10
15
20
25
30
(12)
(24)
(26)
(68)
(48)
(14)
(34)
(13)
(38)
(89)
(35)
(15)
(59)

Fig. 4.12c

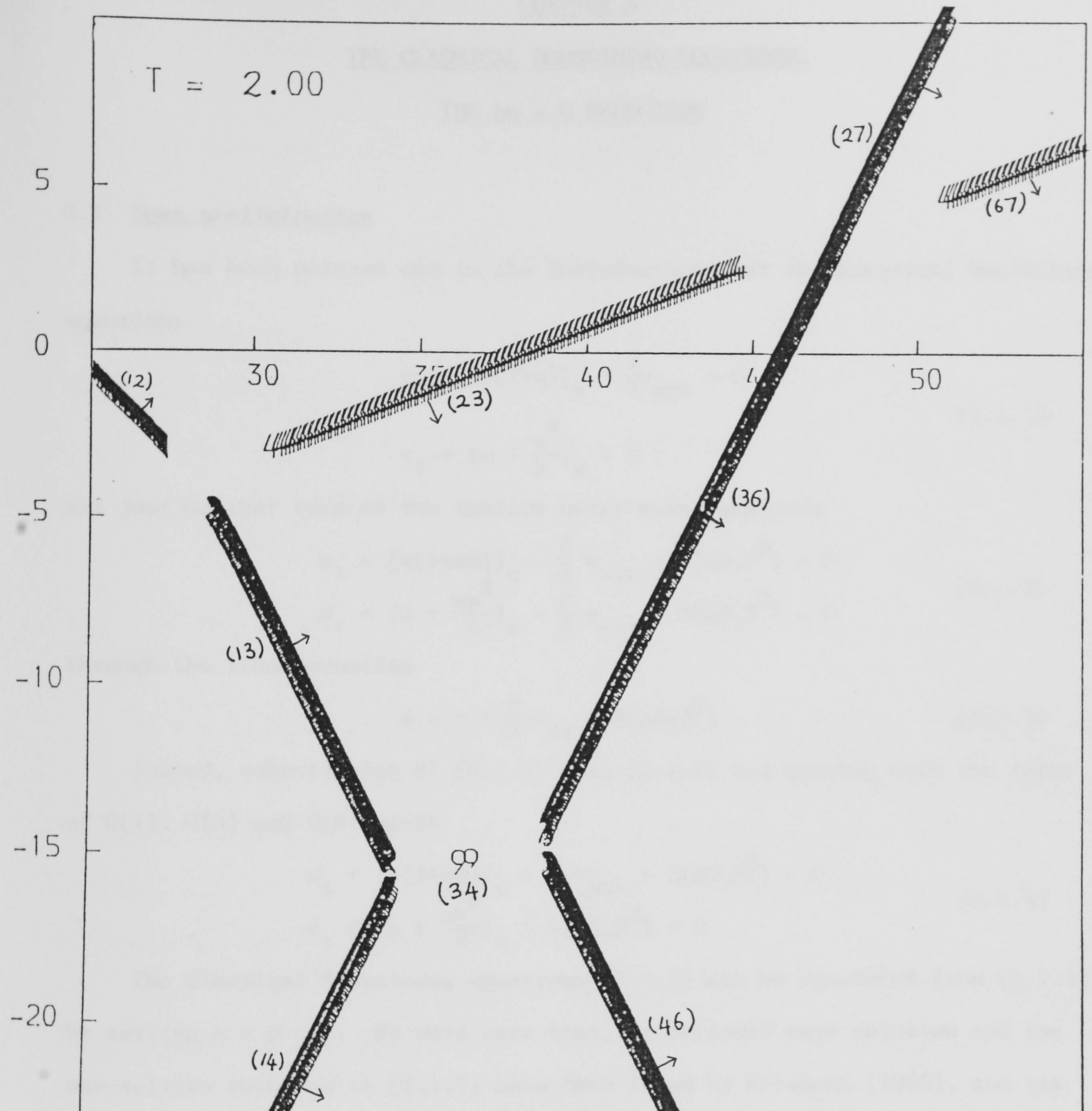


Fig. 4.12d

Fig. 4.12 Interactions between triad with parameters $\ell_1 = 1.0$, $\ell_3 = 1.5$, $n_1 = 2.0$ and triad with parameters $\ell_2 = 2.5$, $\ell_4 = 3.2$, $n_2 = 2.0 + 10^{-6}$ at time instants (a) T = -1.0 (b) T = -0.5 (c) T = 1.0 (d) T = 2.0.

CHAPTER 5

THE CLASSICAL BOUSSINESQ EQUATIONS: THE pq = 0 REDUCTION

5.1 Some preliminaries

It has been pointed out in the Introduction that the Classical Boussinesq equations

$$\begin{aligned} & u_t + \{v(1+u)\}_x + \tfrac{1}{3}v_{xxx} = 0 \\ & v_t + \{u + \tfrac{v^2}{2}\}_x = 0 \end{aligned} \qquad (5.1.1)$$

are just another form of the shallow water wave equations

$$\begin{aligned} & u_t + \{w(1+\alpha u)\}_x - \frac{\beta}{6} w_{xxx} + O(\alpha\beta,\beta^2) = 0 \\ & w_t + \{u + \frac{\alpha w^2}{2}\}_x - \frac{\beta}{2} w_{xxt} + O(\alpha\beta,\beta^2) = 0 \end{aligned} \qquad (5.1.2)$$

through the transformation

$$w = v + \frac{\beta}{2} v_{xx} + O(\alpha\beta,\beta^2) \ . \qquad (5.1.3)$$

Indeed, substitution of (5.1.3) into (5.1.2) and keeping only the terms of $O(1)$, $O(\alpha)$ and $O(\beta)$ gives

$$\begin{aligned} & u_t + \{v(1+\alpha u)\}_x + \frac{\beta}{3} v_{xxx} + O(\alpha\beta,\beta^2) = 0 \\ & v_t + \{u + \frac{\alpha v^2}{2}\}_x + O(\alpha\beta,\beta^2) = 0 \end{aligned} \ . \qquad (5.1.4)$$

The Classical Boussinesq equations (5.1.1) can be recovered from (5.1.4) by setting $\alpha = \beta = 1$. We note here that the periodic wave solution and the one-soliton solution to (5.1.1) have been found by Krishnan (1982), and its multisoliton solutions by Nakamura and Hirota (1985) and Hirota (1985,1986a).

Following Nakamura and Hirota (1985), equations (5.1.1) are transformed into the bilinear forms

$$\begin{aligned} & (iD_t + D_x^2)\tau\cdot\tau' = 0 \\ & (iD_xD_t + D_x^3)\tau\cdot\tau' = 0 \end{aligned} \qquad (5.1.5)$$

by using the transformations

$$u = -1 - 2 \frac{\partial^2}{\partial x^2}(\log \tau\tau')$$
$$v = -2i \frac{\partial}{\partial x}(\log \frac{\tau'}{\tau}) \quad . \tag{5.1.6}$$

They have also found the N+1-soliton solution by solving (5.1.5) in the form

$$\tau' = W(H_{2N}(z), H_{2N-1}(z), \ldots, H_N(z))$$
$$\tau = \tau'^{*} \tag{5.1.7}$$

where W is the Wronskian of N+1 functions $H_N(z)$, $H_{N+1}(z)$, ..., $H_{2N}(z)$ and where H is the Hermite polynomial,

$$H_0(z) = 1,\ H_1(z) = z,\ H_2(z) = z^2 - 1$$
$$H_3(z) = z^3 - 3z\ , \ \ldots \tag{5.1.8}$$
$$H_n(z) = (-1)^n e^{z^2/2} \frac{d^n}{dz^n} e^{-z^2/2} \quad ,$$

and z is defined by

$$z = x/\sqrt{2it}\ . \tag{5.1.9}$$

Hirota (1985) has considered what is termed as the "pq = c" reduction in relation with the Classical Boussinesq equations. Because our work in this chapter will be dealing with this kind of reduction problem, we shall now explain very carefully the meaning of the pq = c reduction. Let us first explain this in the sense used by Hirota.

We start with the first modified KP hierarchy introduced by Jimbo and Miwa (1983) [see Appendix C]. The first two equations of this hierarchy are

$$(D_1^2+D_2)\tau\cdot\tau' = 0 \tag{5.1.10}$$

$$(D_1^3-4D_3-3D_1D_2)\tau\cdot\tau' = 0 \tag{5.1.11}$$

where D_i denotes the bilinear differential operator with respect to the independent variable x_i. The N-soliton solution of all equations under this hierarchy is described by them and also by Hirota (1985) as

$$\tau = \sum_{\mu=0,1} \exp[\sum_{i=1}^{N} \mu_i(\eta_i+\varphi_i) + \sum_{i>j}^{(N)} \mu_i\mu_j\gamma_{ij}] \tag{5.1.12}$$

$$\tau' = \sum_{\mu=0,1} \exp[\sum_{i=1}^{N} \mu_i(\eta_i+\varphi_i') + \sum_{i>j}^{(N)} \mu_i\mu_j\gamma_{ij}] \tag{5.1.13}$$

where

$$\eta_i = \sum_{n=0}^{\infty} (p_i^n - q_i^n) x_n$$

$$\exp \gamma_{ij} = \frac{(p_i-p_j)(q_i-q_j)}{(p_i-q_j)(q_i-p_j)}$$

$$\exp \varphi_i = q_i$$

$$\exp \varphi_i' = p_i \ ,$$

and all the Σ notations have been defined in equation (3.1.10). Here p_i and q_i are the solution parameters which characterize the ith soliton and they are arbitrary for the solutions of all the equations under this hierarchy.

Hirota (1985) showed that with τ and τ' defined by (5.1.12) and (5.1.13) respectively, the choice of

$$p_i q_i = c \text{ for all } i \tag{5.1.14}$$

requires τ and τ' to satisfy

$$(D_1^3+3cD_1-D_3)\tau\cdot\tau' = 0 \ . \tag{5.1.15}$$

Now let us look at a nonlinear evolution equation which has the bilinear form

$$(D_1^3+4cD_1+D_1D_2)\tau\cdot\tau' = 0 \ . \tag{5.1.16}$$

By using (5.1.16), equation (5.1.11) can be separated in the following way

$$\begin{aligned}
&(D_1^3-4D_3-3D_1D_2)\tau\cdot\tau' \\
&\quad = (D_1^3+4cD_1+D_1D_2)\tau\cdot\tau' - 4(D_3+D_1D_2+cD_1)\tau\cdot\tau' \\
&\quad = (D_1^3+4cD_1+D_1D_2)\tau\cdot\tau' - 4[D_3 + \tfrac{1}{3}(D_1^3-4D_3) + cD_1]\tau\cdot\tau' \\
&\quad = (D_1^3+4cD_1+D_1D_2)\tau\cdot\tau' - \tfrac{4}{3}(D_1^3+3cD_1-D_3)\tau\cdot\tau' \ .
\end{aligned} \tag{5.1.17}$$

This means that τ and τ' which satisfy (5.1.11) and (5.1.15), must also satisfy (5.1.16). In other words (5.1.16) is satisfied by τ and τ', given by (5.1.12) and (5.1.13), provided that $p_i q_i = c$ for all i. Therefore (5.1.16) is said to be the pq = c reduction of (5.1.11).

The pq = c reduction can also be explained in the following way. Let us define τ and τ' by

$$\left.\begin{aligned}\tau &= P\exp\left(\sum_{n=1}^{3} p^n x_n\right) + Q\exp\left(\sum_{n=1}^{3} q^n x_n\right)\\ \tau' &= p\,P\exp\left(\sum_{n=1}^{3} p^n x_n\right) + q\,Q\exp\left(\sum_{n=1}^{3} q^n x_n\right)\end{aligned}\right\} \qquad (5.1.18)$$

where all other variables x_n, $n \geq 4$ have been put to zero as they do not actually appear in the equations.

τ and τ' defined by (5.1.18) can be shown to satisfy the first two equations of the first modified KP hierarchy (5.1.10) and (5.1.11) for any P, Q, p and q. If they are substituted into (5.1.15) or (5.1.16) we find that for $p \neq q$, p and q are related by $pq = c$. Now since equations (5.1.15) and (5.1.16) are 'reduced' from the second equation of the first modified KP hierarchy through (5.1.17), we say that both equations (5.1.15) and (5.1.16) are the $pq = c$ reductions of the first modified KP hierarchy.

5.2 The first modified KP hierarchy

In this section we shall claim that all the equations of the first modified KP hierarchy have the n-soliton solutions in the Wronskian form

$$\begin{aligned}\tau &= (\widehat{n-1})\\ \tau' &= (\tilde{n})\end{aligned} \qquad (5.2.1)$$

of the functions

$$\phi_i = P_i \exp\left[\sum_{r=1}^{m} (-p_i)^r x_r\right] + Q_i \exp\left[\sum_{r=1}^{m} (q_i)^r x_r\right], \quad i=1,2,\dots,n\,. \qquad (5.2.2)$$

The Wronskian notations have been defined in Sections 3.2, 3.3 and also in Appendix B, m is the associated number of the independent variables which are present in the equation under consideration. We note that we have put $-p_i$ instead of just p_i in (5.2.2) in order to relate this analysis to earlier work on the Wronskian method. We also note that the Wronskians in (5.2.1) are defined for the derivatives of x_1.

From the definition of ϕ_i (5.2.2) we see that

$$\frac{\partial \phi_i}{\partial x_r} = \frac{\partial^r \phi_i}{\partial x_1^r}\,. \qquad (5.2.3)$$

Relation (5.2.3) will be used in calculating the derivatives of τ and τ' with respect to x_r.

For the single-soliton solution, τ and τ' assume the form (5.1.18) and it is not difficult to show that this pair satisfies the first two equations of the modified KP hierarchy. We now show that the n-soliton solution (5.2.1) satisfies equations (5.1.10) and (5.1.11).

If (5.1.10) is expanded we find

$$(D_1^2+D_2)\tau\cdot\tau' = (\tau_{2x_1}+\tau_{x_2})\tau' + \tau(\tau'_{2x_1}-\tau'_{x_2}) - 2\tau_{x_1}\tau'_{x_1} \quad . \qquad (5.2.4)$$

The derivatives of τ and τ' are found by shifting the appropriate columns as usual,

$$\begin{aligned}
\tau_{x_1} &= (\widehat{n-2},n) \\
\tau_{2x_1} &= (\widehat{n-2},n+1) + (\widehat{n-3},n-1,n) \\
\tau_{x_2} &= (\widehat{n-2},n+1) - (\widehat{n-3},n-1,n) \\
\tau'_{x_1} &= (\widetilde{n-1},n+1) \\
\tau'_{2x_1} &= (\widetilde{n-1},n+2) + (\widetilde{n-2},n,n+1) \\
\tau'_{x_2} &= (\widetilde{n-1},n+2) - (\widetilde{n-2},n,n+1) \ .
\end{aligned} \qquad (5.2.5)$$

Substituting τ and τ' and all their derivatives (5.2.5) into the right hand side of (5.2.4) we find

$$\begin{aligned}(D_1^2+D_2)\tau\cdot\tau' = 2\{&(\widehat{n-1})(\widetilde{n-2},n,n+1) - (\widehat{n-2},n)(\widetilde{n-1},n+1) \\ &+ (\widehat{n-2},n+1)(\widetilde{n})\}.\end{aligned} \qquad (5.2.6)$$

The expression on the right of the above equation can be written as the Laplace expansion of a determinant

$$2(-1)^n \begin{vmatrix} \widehat{n-2} & \cdot & n-1 & n & n+1 \\ & & & & \\ \cdot & \widetilde{n-2} & n-1 & n & n+1 \end{vmatrix}$$

which can be shown to be zero, by using row and column operations. We have therefore shown that the Wronskian solutions τ and τ' defined by (5.2.1) satisfy the first equation of the first modified KP hierarchy.

For the second equation of the first modified KP hierarchy we have

$$(D_1^3-4D_3-3D_1D_2)\tau\cdot\tau' = (\tau_{3x_1}-4\tau_{x_3} - 3\tau_{x_1x_2})\tau'$$
$$- \tau(\tau'_{3x_1}-4\tau'_{x_3}+3\tau'_{x_1x_2}) + 3(\tau_{x_2}-\tau_{2x_1})\tau'_{x_1}$$
$$+ 3\tau_{x_1}(\tau'_{x_2}+\tau'_{2x_1}) \ . \qquad (5.2.7)$$

Beside the derivatives in (5.2.5), some extra derivatives are also needed and we find

$$\begin{aligned}
\tau_{3x_1} &= (\widehat{n-2},n+2) + 2(\widehat{n-3},n-1,n+1) + (\widehat{n-4},n-2,n-1,n) \\
\tau_{x_3} &= (\widehat{n-2},n+2) - (\widehat{n-3},n-1,n+1) + (\widehat{n-4},n-2,n-1,n) \\
\tau_{x_1x_2} &= (\widehat{n-2},n+2) - (\widehat{n-4},n-2,n-1,n) \qquad (5.2.8)\\
\tau'_{3x_1} &= (\widetilde{n-1},n+3) + 2(\widetilde{n-2},n,n+2) + (\widetilde{n-3},n-1,n,n+1) \\
\tau'_{x_3} &= (\widetilde{n-1},n+3) - (\widetilde{n-2},n,n+2) + (\widetilde{n-3},n-1,n,n+1) \\
\tau'_{x_1x_2} &= (\widetilde{n-1},n+3) - (\widetilde{n-3},n-1,n,n+1) \ .
\end{aligned}$$

Now substituting τ, τ' and all their derivatives (5.2.5) and (5.2.8) into the right-hand side of (5.2.7) we find

$$\begin{aligned}
&(D_1^3-4D_3-3D_1D_2)\tau\cdot\tau' \\
&= 6[(\widehat{n-1})(\widetilde{n-3},n-1,n,n+1) - (\widehat{n-3},n-1,n)(\widetilde{n-1},n+1) \\
&+ (\widehat{n-3},n-1,n+1)(\widetilde{n})] - 6[(\widehat{n-1})(\widetilde{n-2},n,n+2) \\
&- (\widehat{n-2},n)(\widetilde{n-1},n+2) + (\widehat{n-2},n+2)(\widetilde{n})] \\
&= (-1)^{n+1}6\left[\begin{vmatrix} \widehat{n-3} & n-1 & \cdot & \cdot & n-2 & n & n+1 \\ \cdot & \cdot & \widetilde{n-3} & n-1 & n-2 & n & n+1 \end{vmatrix}\right. \\
&\left. + \begin{vmatrix} \widehat{n-2} & \cdot & n-1 & n & n+2 \\ \cdot & \widetilde{n-2} & n-1 & n & n+2 \end{vmatrix}\right] \ . \qquad (5.2.9)
\end{aligned}$$

All the determinants in the last expression can be shown to be zero as usual and thus is verified the solution of the second equation of the first modified KP hierarchy.

Indeed, we can always show that the Wronskian solutions defined by (5.2.1) satisfy the rest of the equations of the hierarchy, but these two examples are sufficient for the purpose of the work in this chapter.

5.3 The pq = −c reduction

We are interested in the equations

$$(D_1^2+D_2)\tau\cdot\tau' = 0 \qquad (5.3.1)$$

$$(D_1^3+4cD_1+D_1D_2)\tau\cdot\tau' = 0 \qquad (5.3.2)$$

because they are related directly to the bilinear forms of the Classical Boussinesq equations (5.1.5) by some independent variable transformations. As we have seen earlier, equation (5.3.1) is the first equation of the first modified KP hierarchy, while (5.3.2) is a reduced equation of the second equation of the hierarchy, or as we rewrite from (5.1.17)

$$(D_1^3-4D_3-3D_1D_2)\tau\cdot\tau'$$
$$= (D_1^3+4cD_1+D_1D_2)\tau\cdot\tau' - \tfrac{4}{3}(D_1^3+3cD_1-D_3)\tau\cdot\tau' . \qquad (5.3.3)$$

If we put n = 1 into (5.2.1) and substitute the resulting τ and τ' into (5.3.2) we shall find

$$(pq+c)(p+q)^2 = 0 .$$

We thus have pq = −c for q ≠ −p. The same result would also be obtained if we applied similar treatment to equation

$$(D_1^3+3cD_1-D_3)\tau\cdot\tau' = 0 .$$

Therefore by virtue of (5.2.2) we say that (5.3.2) is the pq = −c reduction of the first modified KP hierarchy. We have already shown that relation pq = −c satisfies (5.3.2) for the single-soliton solutions. We shall now prove this is also true for the case of the n-soliton solutions (5.2.1).

With the relation $p_iq_i = -c$, the function ϕ_i (5.2.2) can be written as

$$\phi_i = P_i \exp[\sum_{r=1}^{2} (-p_i)^r x_r] + Q_i \exp[\sum_{r=1}^{2} (-\frac{c}{p_i})^r x_r] \qquad (5.3.4)$$

where we have put $x_r = 0$, $r \geq 3$ for convenience. From the form of this function, we may deduce immediately that

$$\rho_i \phi_i^{(k)} = \phi_i^{(k+1)} + c\,\phi_i^{(k-1)} \qquad (5.3.5)$$

where

$$\rho_i = -(p_i + \frac{c}{p_i}) , \qquad (5.3.6)$$

and $\phi_i^{(k)}$ denotes the kth derivative of ϕ_i with respect to x_1.

We note that relation (5.3.5) is an identity for the single-soliton solution. An identity for the n-soliton solution may also be obtained from this relation by using the determinant property

$$\sum_{k=1}^{N} \alpha_k \left| a_1, a_2, \dots, a_N \right| = \sum_{j=1}^{N} \left| a_1, a_2, \dots, \alpha a_j, a_{j+1}, \dots, a_N \right| \qquad (5.3.7)$$

where $a_1, a_2, \dots, a_N$ are the columns of an N×N determinant $|a_{ij}|$ and αa_j denotes the jth column with

$$\alpha a_j = \begin{bmatrix} \alpha_1 a_{1j} \\ \alpha_2 a_{2j} \\ \vdots \\ \alpha_N a_{Nj} \end{bmatrix} .$$

If we apply (5.3.7), for example, to the Wronskian $(\widehat{N-1})$ of the functions ϕ_i which satisfy relation (5.3.5), we find

$$\sum_{i=1}^{N} \rho_i (\widehat{N-1}) = (\widehat{N-2}, N) + c(-1, \widetilde{N-1}) .$$

Now, expanding (5.3.2) we find

$$\begin{aligned}(D_1^3 + 4cD_1 + D_1D_2)\tau \cdot \tau' &= (\tau_{3x_1} + \tau_{x_1x_2} + 4c\tau_{x_1})\tau' \\ &+ \tau(-\tau'_{3x_1} + \tau'_{x_1x_2} - 4c\tau'_{x_1}) \\ &- (3\tau_{2x_1} + \tau_{x_2})\tau'_{x_1} + \tau_{x_1}(3\tau'_{2x_1} - \tau'_{x_2}) . \end{aligned} \qquad (5.3.8)$$

Subsituting τ, τ' (5.2.1) and their derivatives which have already been calculated before [(5.2.5) and (5.2.8)] into the right-hand side of (5.3.8), we find

$$\begin{aligned}&(D_1^3 + D_1D_2 + 4cD_1)\tau \cdot \tau' \\ &= [2(\widehat{n-2}, n+2) + 2(\widehat{n-3}, n-1, n+1) + 4c(\widehat{n-2}, n)](\tilde{n}) \\ &-(\widehat{n-1})[2(\widetilde{n-2}, n, n+2) + 2(\widetilde{n-3}, n-1, n, n+1) + 4c(\widetilde{n-1}, n+1)] \\ &- [4(\widehat{n-2}, n+1) + 2(\widehat{n-3}, n-1, n)](\widetilde{n-1}, n+1) \\ &+ (\widehat{n-2}, n)[4(\widetilde{n-2}, n, n+1) + 2(\widetilde{n-1}, n+2)] . \end{aligned} \qquad (5.3.9)$$

The expression (5.3.9) can be simplified into a convenient form by substituting $(\widehat{n-3},n-1,n+1)(\widetilde{n})$ and $(\widehat{n-2},n)(\widetilde{n-1},n+2)$ by an expression which can be found from (5.2.9). Thus

$$
\begin{aligned}
&(D_1^3+D_1D_2+4cD_1)\tau\cdot\tau' \\
&\quad = -4\{(\widehat{n-1})(\widetilde{n-3},n-1,n,n+1) - (\widehat{n-2},n)(\widetilde{n-2},n,n+1) \\
&\quad + [(\widehat{n-2},n+1) + c(\widehat{n-1})](\widetilde{n-1},n+1) \\
&\quad - [(\widehat{n-2},n+2) + c(\widehat{n-2},n)](\widetilde{n})\}\ . \qquad (5.3.10)
\end{aligned}
$$

Making use of the property (5.3.5) we see that

$$
\begin{aligned}
(\widehat{n-2},n+1) + c(\widehat{n-1}) &= (\widehat{n-2},(n+1) + c(n-1)) \\
&= (\widehat{n-2},\rho(n))
\end{aligned}
$$

and (5.3.11)

$$
\begin{aligned}
(\widehat{n-2},n+2) + c(\widehat{n-2},n) &= (\widehat{n-2},(n+2) + c(n)) \\
&= (\widehat{n-2},\rho(n+1))
\end{aligned}
$$

where $\rho(k)$ is a column given by

$$
\rho(k) = \begin{bmatrix} \rho_1\phi_1^{(k)} \\ \rho_2\phi_2^{(k)} \\ \vdots \\ \rho_n\phi_n^{(k)} \end{bmatrix} . \qquad (5.3.12)
$$

Thus, by using (5.3.11), the expression on the right of (5.3.10) is written as

$$
\begin{aligned}
&-4\{(\widehat{n-1})(\widetilde{n-3},n-1,n,n+1) - (\widehat{n-2},n)(\widetilde{n-2},n,n+1) \\
&\quad + (\widehat{n-2},\rho(n))(\widetilde{n-1},n+1) - (\widehat{n-2},\rho(n+1))(\widetilde{n})\}\ .
\end{aligned}
$$

The above expression can now be written in the form of the Laplace expansion of a determinant

$$
4(-1)^n \begin{vmatrix} \widehat{n-2} & \cdot & n-1 & n & \rho(n) & \rho(n+1) \\ \cdot & \widetilde{n-3} & n-2 & n-1 & n & n+1 \end{vmatrix} .
$$

Multiplying each of the last n rows of this determinant by ρ_i gives

$$
\frac{4(-1)^n}{\prod_{i=1}^{n} \rho_i} \begin{vmatrix} \widehat{n-2} & \cdot & n-1 & n & \rho(n) & \rho(n+1) \\ \cdot & \rho(\widetilde{n-3}) & \rho(n-2) & \rho(n-1) & \rho(n) & \rho(n+1) \end{vmatrix} .
$$

Subtracting the bottom n rows from the top n rows and using relation

(5.3.5) yields

$$4(-1)^n \begin{vmatrix} \widehat{n-2} & -(\check{n-2}) & -c(\widehat{n-4}) & -c(n-3) & -c(n-2) & \cdot & \cdot \\ \cdot & & (\widetilde{n-3}) & n-2 & n-1 & n & n+1 \end{vmatrix}$$

where $(\check{n-2})$ denotes columns with derivatives 2,3, ..., n-2.

By adding the individual columns, the last determinant becomes

$$4(-1)^n \begin{vmatrix} \widehat{n-2} & \cdot \\ \cdot & (\widetilde{n+1}) \end{vmatrix} , \tag{5.3.13}$$

which is obviously zero, since it contains n+1 zero columns in the first n rows. This means that we have shown τ and τ', defined by (5.2.1), with functions ϕ_i's defined by (5.3.4), satisfy equation (5.3.2).

We note here that similar proof can also be applied to equation (5.1.15).

5.4 <u>The pq = 0 reduction</u>

If we now choose $P_i = (-p_i)^{-r}$ and $Q_i = (q_i)^{-r}$ in equation (5.3.4) then we have

$$\phi_i = \frac{e^{\xi_i}}{(-p_i)^r} + \frac{e^{\eta_i}}{(q_i)^r} \tag{5.4.1}$$

where

$$\xi_i = \sum_{m=1}^{2} (-p_i)^m x_m ,$$

$$(5.4.2)$$

$$\eta_i = \sum_{m=1}^{2} (q_i)^m x_m .$$

Introducing P_i and Q_i in this way has the effect of shifting all the derivatives in τ and τ' r places down. We may now write

$$\tau = (\tilde{r}^-, \widehat{n-r-1})$$

$$(5.4.3)$$

$$\tau' = (\widetilde{r-1}^-, \widehat{n-r})$$

where $\tilde{k}^-$ denotes all the negative derivatives from k, k-1, ..., 1. The functions ϕ_i which define the Wronskians in (5.4.3) are not of the form

(5.4.1) any more but

$$\phi_i = e^{\xi_i} + e^{\eta_i} . \qquad (5.4.4)$$

We now look at the first r columns of τ and the first r-1 columns of τ'. The element of the first column of τ, for example, takes the form

$$\frac{e^{\xi_i}}{(-p_i)^r} + \frac{e^{\eta_i}}{(q_i)^r} .$$

Now, if we let $q_i \to 0$ then the dominant behaviour of this expression comes from the inverse powers of q_i and the exponential term e^{η_i} is unity. Indeed we have

$$\frac{e^{\xi_i}}{(-p_i)^r} + \frac{e^{\eta_i}}{(q_i)^r} \to \frac{1}{q_i^r} \quad \text{as } q_i \to 0 .$$

Therefore in this limit we have

$$\begin{aligned} \tau &= ([(\tilde{\tfrac{1}{q}})^r], n-\hat{r}-1) \\ \tau' &= ([(\tilde{\tfrac{1}{q}})^{r-1}], n-\hat{r}) \end{aligned} \qquad (5.4.5)$$

where $[(\tilde{\frac{1}{q}})^r]$ denotes the columns $((\frac{1}{q_i})^r, (\frac{1}{q_i})^{r-1}, \ldots \frac{1}{q_i})$.

However, it is more convenient to express (5.4.5) in terms of p, since in the limit $q_i \to 0$ [i.e. the pq = 0 reduction] the function ϕ_i is

$$\phi_i = 1 + e^{\xi_i} = 1 + e^{-p_i x_1 + p_i^2 x_2} . \qquad (5.4.6)$$

By using the fact that pq = -c we then write (5.4.5) as

$$\begin{aligned} \tau &= ([(-\tilde{p})^r], n-\hat{r}-1) \\ \tau' &= ([(-\tilde{p})^{r-1}], n-\hat{r}) . \end{aligned} \qquad (5.4.7)$$

A certain symmetry will be necessary if the physical variables u and v [equation (5.1.1)] are to be real. This will be discussed in Section 5.5. For the present it will be sufficient to say that we require τ and τ' to be complex conjugates as far as possible. This may be achieved by taking n = 2N and r = N. Thus (5.4.7) becomes

$$\tau = ([(-\tilde{p})^N], \widehat{N-1})$$

$$\tau' = ([(-\tilde{p})^{N-1}], \hat{N}) \ . \qquad (5.4.8)$$

If we make a further choice that $p_{i+N} = -p_i^*$ then

$$\phi_{i+N} = 1 + e^{\xi_{i+N}} = 1 + e^{-\xi_i^*} \ . \qquad (5.4.9)$$

We thus have

$$\tau' = \begin{vmatrix} (-p_1)^{N-1} & \cdots & (-p_1) & 1+e^{\xi_1} & (-p_1)e^{\xi_1} & \cdots & (-p_1)^N e^{\xi_1} \\ \vdots & & & & & & \\ (-p_N)^{N-1} & & (-p_N) & 1+e^{\xi_N} & (-p_N)e^{\xi_N} & \cdots & (-p_N)^N e^{\xi_N} \\ (p_1^*)^{N-1} & & (p_1^*) & 1+e^{-\xi_1^*} & (p_1^*)e^{-\xi_1^*} & \cdots & (p_1^*)^N e^{-\xi_1^*} \\ \vdots & & & & & & \\ (p_N^*)^{N-1} & & (p_N^*) & 1+e^{-\xi_N^*} & (p_N^*)e^{-\xi_N^*} & \cdots & (p_N^*)^N e^{-\xi_N} \end{vmatrix} \ .$$

Multiplying each of the first N rows by $e^{-\xi_i}$ and the last N rows by $e^{\xi_i^*}$ we find

$$\tau' = \begin{vmatrix} (-p_1)^{N-1}e^{-\xi_1} & \cdots & (-p_1)e^{-\xi_1} & 1+e^{-\xi_1} & (-p_1) & \cdots & (-p_1)^N \\ \vdots & & & & & & \\ (-p_N)^{N-1}e^{-\xi_N} & & (-p_1)e^{-\xi_N} & 1+e^{-\xi_N} & (-p_N) & \cdots & (-p_N)^N \\ (p_1^*)^{N-1}e^{\xi_1^*} & & (p_1^*)e^{\xi_1^*} & 1+e^{\xi_1^*} & (p_1^*) & \cdots & (p_N^*)^N \\ \vdots & & & & & & \\ (p_N^*)^{N-1}e^{\xi_N^*} & & (p_N^*)e^{\xi_N^*} & 1+e^{\xi_N^*} & (p_N^*) & \cdots & (p_N^*)^N \end{vmatrix} \prod_{i=1}^{N} e^{(\xi_i - \xi_i^*)} \ .$$

This form can be simplified further by exchanging the columns between the first and the last, the second first and the second last and so on. The rows of the resulting determinants are also exchanged in this way. The sign of p_i and p_i^* may also be changed by multiplying the relevant columns by a power of -1. The resulting determinant is now

$$\tau' = \begin{vmatrix} (-p_1^*)^N & \cdots & (-p_1^*) & 1+e^{\xi_1^*} & (-p_1^*)e^{\xi_1^*} & \cdots & (-p_1^*)^{N-1}e^{\xi_1^*} \\ \vdots \\ (-p_N^*)^N & \cdots & (-p_N^*) & 1+e^{\xi_N^*} & (-p_N^*)e^{\xi_N^*} & \cdots & (-p_N^*)^{N-1}e^{\xi_N^*} \\ (p_1)^N & \cdots & (p_1) & 1+e^{-\xi_1} & (p_1)e^{-\xi_1} & \cdots & (p_1)^{N-1}e^{-\xi_1} \\ \vdots \\ (p_N)^N & \cdots & (p_N) & 1+e^{-\xi_N} & (p_N)e^{-\xi_N} & \cdots & (p_N)^{N-1}e^{-\xi_N} \end{vmatrix} (-1)^N \prod_{i=1}^N e^{\xi_i-\xi_i^*} .$$

We thus have

$$\tau' = (-1)^N \prod_{i=1}^N e^{\xi_i-\xi_i^*} \tau^* , \tag{5.4.10}$$

which means that apart from an exponential factor, τ' is just the complex conjugate of τ.

We now show that τ and τ' defined by (5.4.8) satisfy the equation for the $pq = 0$ reduction [i.e., equation (5.3.2) with $c = 0$],

$$(D_1^3+D_1D_2)\tau\cdot\tau' = 0 . \tag{5.4.11}$$

Expanding (5.4.11) as before we find

$$\begin{aligned} &(D_1^3+D_1D_2)\tau\cdot\tau' \\ &\quad = (\tau_{3x_1}+\tau_{x_1x_2})\tau' - \tau(\tau'_{3x_1}-\tau'_{x_1x_2}) \\ &\quad -(3\tau_{2x_1}+\tau_{x_2})\tau'_{x_1} + \tau_{x_1}(3\tau'_{2x_1}-\tau'_{x_2}) . \end{aligned} \tag{5.4.12}$$

Since τ and τ' defined by (5.4.8) are the special case of the more general solution (5.2.1), one may conclude that the substitution of τ,τ' from (5.4.8) and their derivatives into (5.4.12) would eventually give the determinant

$$\begin{vmatrix} [(-\tilde{p})^N] & \widehat{N-2} & & \cdot & \\ & & & & \\ & \cdot & & [(-\tilde{p})^{N-1}] & \widehat{N+1} \end{vmatrix} \tag{5.4.13}$$

by comparison with the general result (5.3.13).

We shall show from first principles that (5.4.13) is true. The derivatives of τ are

$$\tau_{x_1} = ([(-\tilde{p})^N],\hat{N-2},N)$$
$$\tau_{2x_1} = ([(-\tilde{p})^N],\hat{N-2},N+1) + ([(-\tilde{p})^N],\hat{N-3},N-1,N)$$
$$\tau_{3x_1} = ([(-\tilde{p})^N],\hat{N-2},N+2) + 2([(-\tilde{p})^N],\hat{N-3},N-1,N+1)$$
$$+ ([(-\tilde{p})^N],\hat{N-4},N-2,N-1,N) \qquad (5.4.14)$$
$$\tau_{x_2} = ([(-\tilde{p})^N],\hat{N-2},N+1) - ([(-\tilde{p})^N],\hat{N-3},N-1,N)$$
$$\tau_{x_1x_2} = ([(-\tilde{p})^N],\hat{N-2},N+2) - ([(-\tilde{p})^N],\hat{N-4},N-2,N-1,N) \ .$$

The derivatives of τ' are very similar to those of τ, except that the constant columns $[(-\tilde{p})^N]$ are changed to $[(-\tilde{p})^{N-1}]$ for τ', and for the rest of the terms we simply change N to N+1. For example we have

$$\tau'_{x_1} = ([(-\tilde{p})^{N-1}],\hat{N-1},N+1) \ .$$

Using the above we find from (5.4.12)

$$(D_1^3+D_1D_2)\tau\cdot\tau'$$
$$= -4\{([(-\tilde{p})^N],\hat{N-1})([(-\tilde{p})^{N-1}],\hat{N-3},N-1,N,N+1)$$
$$-([(-\tilde{p})^N],\hat{N-2},N)([(-\tilde{p})^{N-1}],\hat{N-2},N,N+1)$$
$$+([(-\tilde{p})^N],\hat{N-2},N+1)([(-\tilde{p})^{N-1}],\hat{N-1},N+1)$$
$$-([(-\tilde{p})^N],\hat{N-2},N+2)([(-\tilde{p})^{N-1}],\hat{N}\}$$
$$= (-1)^N 4 \begin{vmatrix} [(-\tilde{p})^N] & \hat{N-2} & \cdot & & \cdot & N-1 & N & N+1 & N+2 \\ & & & & & & & & \\ \cdot & & \cdot & [(-\tilde{p})^{N-1}] & \hat{N-3} & N-2 & N-1 & N & N+1 \end{vmatrix} \ .$$

(5.4.15)

The 4N × 4N determinant in (5.4.15) can be shown to be zero by realizing an identity relation which is equivalent to (5.3.5). In the present problem we simply have

$$\phi_i = 1 + e^{-p_ix_1+p_i^2x_2} \qquad (5.4.16)$$

which gives

$$(-p_i)\ \phi_i^{(k)} = \phi_i^{(k+1)} \quad \text{for } k \neq 0 \qquad (5.4.17)$$

and

$$(-p_i)\,\phi_i = -p_i + \phi_i^{(1)} .$$

If we multiply the last 2N rows of the determinant (5.4.15) by $-p_i$ we find

$$(D_1^3+D_1D_2)\tau\cdot\tau'$$
$$= \frac{(-1)^N 4}{\prod_{i=1}^{2N}(-p_i)} \begin{vmatrix} [(-\tilde{p})^N] & N-2 & \cdot & \cdot & N-1 & N & N+1 & N+2 \\ \cdot & \cdot & [(-\check{p})^N] & (\tilde{N-2})' & N-1 & N & N+1 & N+2 \end{vmatrix}$$

(5.4.18)

where $([(-\check{p})^N]$ indicates the columns with powers $(-p_i)^N$, $(-p_i)^{N-1}$, ... $(-p_i)^2$, and $'$ indicates the first column is $-p_i(1+e^{\xi_i})$. Now, subtracting the second 2N rows from the first 2N rows of the determinant (5.4.18), and adding the appropriate columns we find

$$\frac{(-1)^N 4}{\prod_{i=1}^{2N}(-p_i)} \begin{vmatrix} [(-\tilde{p})^N] & \hat{N-2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & [(-\check{p})^N] & 1' & \check{N-2} & N-1 & N & N+1 & N+2 \end{vmatrix} .$$

(5.4.19)

This determinant is obviously zero, because there are 2N+1 zero columns in the first 2N rows.

We note that (5.4.19) is simply

$$(-1)^N 4 \begin{vmatrix} [(-\tilde{p})^N] & \hat{N-2} & \cdot & \cdot \\ \cdot & \cdot & [(-\tilde{p})^{N-1}] & \hat{N+1} \end{vmatrix} ,$$

the form we mentioned earlier (5.4.13).

Thus, the functions τ and τ' given by equations (5.4.8) satisfy

$$(D_1^2+D_2)\tau\cdot\tau' = 0$$

(5.4.20)

$$(D_1^3+D_1D_2)\tau\cdot\tau' = 0$$

in which the first equation is from the first modified KP hierarchy, while the second is the equation for the pq = 0 reduction of the hierarchy.

5.5 The Classical Boussinesq equations

The Classical Boussinesq equations (5.1.5) are recovered from (5.4.20) by choosing $x_1 = x$, $x_2 = -it$ to give

$$(iD_t + D_x^2)\tau \cdot \tau' = 0$$
$$\tag{5.5.1}$$
$$(iD_x D_t + D_x^3)\tau \cdot \tau' = 0 .$$

The N-soliton solutions are thus given by [equation (5.4.8)]

$$\tau = ([(-\tilde{p})^N], \hat{N-1})$$
$$\tag{5.5.2}$$
$$\tau' = ([(-\tilde{p})^{N-1}], \hat{N})$$

with $p_{i+N} = -p_i^*$ and the single-soliton solutions $\phi_i = 1+e^{\xi_i}$, $\phi_{i+N} = 1+e^{-\xi_i^*}$, $i = 1,2, \ldots, N$, where

$$\xi_k = -p_k x - ip_k^2 t . \tag{5.5.3}$$

The physical variables u,v, which satisfy equations (5.1.1) are related to τ and τ^* by the transformation [equation (5.1.6)]

$$u = -1 - 2\frac{\partial^2}{\partial x^2}(\log \tau\tau')$$
$$\tag{5.5.4}$$
$$v = -2i\frac{\partial}{\partial x}(\log \frac{\tau'}{\tau}) .$$

Recalling from the previous section, τ and τ' are complex conjugates except for an exponential factor,

$$\tau' = [(-1)^N \exp \sum_{k=1}^{N} (\xi_k - \xi_k^*)]\tau^* . \tag{5.5.5}$$

The exponential factor does not contribute to u and thus we have

$$u = -1 - \frac{\partial^2}{\partial x^2}(\log \tau\tau^*)$$

which is obviously real.

However, the exponential factor gives an added constant for v,

$$v = -2i\frac{\partial}{\partial x}(\log \frac{\tau^*}{\tau}) + A . \tag{5.5.6}$$

The constant A can readily be removed by a Galilean transformation, and thus v is real.

5.6 The rational solutions

It has been mentioned in Section 5.1 that Nakamura and Hirota (1985) obtained the solutions of the Classical Boussinesq equations in the Wronskian form of Hermite polynomials.

We shall now show that these solutions can also be produced from the pq = 0 reduction problem discussed earlier.

We start with writing [Abramowitz and Stegun (1965)]

$$e^{\xi} = e^{-px_1+p^2x_2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n\left[\frac{ix_1}{2x_2^{½}}\right](ipx_2^{½})^n \tag{5.6.1}$$

where H_n is an Hermite polynomial and we are interested in the solutions for $p_1, p_2, \ldots, p_{2N} \to 0$.

Consider first

$$\tau' = ([(-\tilde{p})^N]^{-1}, \hat{N}). \tag{5.6.2}$$

We have from (5.6.1)

$$\begin{aligned}
\frac{\partial^r \phi_k}{\partial x_1^r} &= \frac{\partial^r}{\partial x_1^r}(1+e^{\xi_k}) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^r H_n(\eta)}{\partial x_1^r} (ip_k x_2^{½})^n \quad \text{with } \eta = \frac{ix_1}{2x_2^{½}} \\
&= \sum_{n=r}^{\infty} \frac{1}{n!} \frac{\partial^r H_n(\eta)}{\partial \eta^r} \left[\frac{i}{2x_2^{½}}\right]^r (ip_k x_2^{½})^n \\
&= \sum_{n=r}^{\infty} \frac{1}{(n-r)!} H_{n-r}(\eta) \left[\frac{i}{2x_2^{½}}\right]^r (ip_k x_2^{½})^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} H_n(\eta) \left[\frac{i}{2x_2^{½}}\right]^r (ip_k x_2^{½})^{n+r} ,
\end{aligned}$$

where we have used the relation

$$\frac{\partial^r}{\partial s^r} H_n(s) = \frac{n!}{(n-r)!} H_{n-r}(s) . \tag{5.6.3}$$

We note that in the expression for $\dfrac{\partial^r \phi_k}{\partial x_1^r}$, the term $\left[\dfrac{i}{2x_2^{1/2}}\right]^r$ can be factored out of the summation and is also independent of x_1. Therefore it does not contribute to the solutions u,v and thus we simply write

$$\frac{\partial^r \phi_k}{\partial x_1^r} = \sum_{n=0}^{\infty} \frac{p_k^{n+r} z^{n+r}}{n!} H_n(\eta) \tag{5.6.4}$$

where $z = ix_2^{1/2}$.

In these terms, τ' now takes the form

$$\tau' = \Big|(-p_i)^{N-1} \ldots (-p_i),\ 1 + \sum_{n=0}^{\infty} \frac{p_i^n z^n}{n!} H_n(\eta),\ \sum_{n=0}^{\infty} \frac{p_i^{n+1} z^{n+1}}{n!} H_n(\eta)\ ,$$
$$\ldots,\ \sum_{n=0}^{\infty} \frac{p_i^{n+N} z^{n+N}}{n!} H_n(\eta)\Big|\ .$$

Since the first N-1 columns are simply $(-p_i)^j$, $j = 1,2, \ldots, N-1$, multiples of these can be subtracted reiteratively from the last N columns until all the terms containing $(-p_i)$, $(-p_i)^2$, ..., $(-p_i)^{N-1}$ disappear from the summation.

Now, the first N columns remain unchanged and the N+rth column takes the form

$$\sum_{n=N-r}^{\infty} \frac{p_i^{n+r} z^{n+r}}{n!} H_n(\eta)$$
$$= \sum_{n=0}^{\infty} \frac{p_i^{N+n} z^{N+n}}{(N+n-r)!} H_{N+n-r}(\eta)\ , \quad r = 1,2, \ldots, N\ . \tag{5.6.5}$$

All the elements of τ' can be written in the form $A_j(p_i)$, $i,j = 1,2, \ldots, 2N$. For $p_1, p_2, \ldots, p_{2N} \to 0$, writing the elements of τ' in this way we have

$$\tau' = \lim_{\substack{p_i \to 0 \\ i=1,\ldots,2N}} \left| \frac{\partial^{i-1} A_j(p_i)}{\partial p_i^{i-1}} \right| \tag{5.6.6}$$

as the largest term in an expansion in $p_1, \ldots, p_{2N}$ $(p_i \neq p_j, i \neq j)$.

Introducing the limit $p_1, p_2, \ldots, p_{2N} \to 0$ by using (5.6.6) we find

$$\tau' = \left|\begin{array}{c|c} \tau'_{11} & \cdot \\ \hline \cdot & \tau'_{22} \end{array}\right|$$

where

$$\tau'_{11} = \left|\begin{array}{ccccc} \cdot & & & & 1+H_0 \\ & & & -(1) & \\ & & -(2!) & & \\ & \cdot^{\cdot^{\cdot}} & & & \\ -(N-1)! & & & & \cdot \end{array}\right|$$

and

$$\tau'_{22} = \lim_{\substack{p_i\to 0\\ i=1,\ldots,2N}} \left| \frac{\partial^{N+i-1}}{\partial p_{N+i}^{N+i-1}} \sum_{n=0}^{\infty} \frac{z^{N+n} p_{N+i}^{N+n}}{(N+n-j)!} H_{N+n-j}(\eta) \right|$$

$$= \left| \frac{z^{N+i-1} H_{N+i-j-1}(\eta)(N+i-1)!}{(N+i-j-1)!} \right|$$

in which the contribution only comes from the term with $n = i-1$. Since τ'_{11} is independent of x_1, we just have $\tau' = \tau'_{22}$. However, in τ'_{22} the term $z^{N+i-1}(N+i-1)!$ can always be factored out, leaving

$$\tau' = \left| \frac{H_{N+i-j-1}(\eta)}{(N+i-j-1)!} \right| . \tag{5.6.7}$$

The form of τ' in (5.6.7) can be rearranged so that the Nth row becomes the first, the N-1th becomes the second and so on; we thus write (with $i \to N-i+1$)

$$\tau' = \left| \frac{H_{2N-i-j}(\eta)}{(2N-i-j)!} \right| . \tag{5.6.8}$$

Written in this form, τ' is a Wronskian in both rows and columns. The matrix is thus symmetric.

We note that, if in (5.6.8) N is replaced by N+1, we find the N+1-soliton solution

$$\tau' = \left| \frac{H_{2N-i-j+2}(\eta)}{(2N-i-j+2)!} \right|$$

the form obtained by Nakamura and Hirota (1985).

We now consider

$$\tau = ([(-\tilde{p})^N],\ \hat{N-1}) \ . \tag{5.6.9}$$

By using (5.6.4) we can write

$$\tau = \left|(-p_i)^N, (-p_i)^{N-1}, \ldots, (-p_i),\ 1 + \sum_{n=0}^{\infty} \frac{p_i^n z^n H_n(\eta)}{n!},\ \sum_{n=0}^{\infty} \frac{p_i^{n+1} z^{n+1}}{n!} H_n(\eta)\ , \ldots,\ \sum_{n=0}^{\infty} \frac{p_i^{n+N-1} z^{n+N-1}}{n!} H_n(\eta)\right| .$$

Reiterative subtraction of the multiples of the first N columns from the last N-1 columns yields for the N+rth column, r = 2 ,..., N

$$\sum_{n=N-r+1}^{\infty} \frac{p_i^{n+r} z^{n+r}}{n!} H_n(\eta)$$

$$= \sum_{n=0}^{\infty} \frac{z^{n+N+1} p_i^{n+N+1}}{(n+N-r+1)!} H_{n+N-r+1}(\eta) \ . \tag{5.6.10}$$

If (5.6.6) is applied to τ it will give zero, so we must consider the next term of the expansion and take instead

$$\tau = \lim_{\substack{p_i \to 0 \\ i=1,2,..,2N}} \left| \frac{\partial^i A_j(p_i)}{\partial p_i^i} \right| . \tag{5.6.11}$$

Using this we find

$$\tau = \left| \begin{array}{c|c} \tau_{11} & \cdot \\ \hline \cdot & \tau_{22} \end{array} \right|$$

where

$$\tau_{11} = \left| \begin{array}{cccc} \cdot & & & -1 \\ & & -2 & \\ & & -(3)! & \\ & \cdot^{\cdot^{\cdot}} & & \cdot \\ -(N!) & & & \end{array} \right|$$

and

$$\tau_{22} = \lim_{\substack{p_i \to 0 \\ i=1,2,\dots .2N}} \left| \frac{\partial^{N+i}}{\partial p_i^{N+i}} \sum_{n=0}^{\infty} \frac{z^{N+n} p_i^{N+n}}{(n+N-j+1)!} H_{N+n-j+1}(\eta) \right|$$

$$= \left| \frac{z^{N+i} H_{N+i-j+1}(\eta)}{(N+i-j+1)!} (N+i)! \right| ,$$

where the non-zero contributions only come from the terms with $n = i$. Again, all the coefficients $z^{N+i}(N+i)!$ can be factored out of the determinant and thus

$$\tau = \tau_{22} = \left| \frac{H_{N+i-j+1}(\eta)}{(N+i-j+1)!} \right| .$$

Changing the rows as before, $i \to N-i+1$, we find the Wronskian form

$$\tau = \left| \frac{H_{2N-i-j+2}(\eta)}{(2N-i-j+2)!} \right| . \qquad (5.6.12)$$

We have thus found the rational solutions of the Classical Boussinesq equations by considering the $pq = 0$ reduction problem using the Wronskian technique.

CHAPTER 6

THE ORDINARY BOUSSINESQ EQUATION

6.1 Derivation of the ordinary Boussinesq equation from the shallow water wave equations

The ordinary Boussinesq equation is normally written in the form [Hirota (1973b), Nimmo and Freeman (1983)]

$$u_{tt} - u_{xx} - 6(u^2)_{xx} - u_{xxxx} = 0 \ . \tag{6.1.1}$$

This equation was introduced by Boussinesq (1872) to describe the propagation of long waves in shallow water. The equation has also been derived by Zabusky (1967) to describe similar waves in a nonlinear one-dimensional lattice.

We shall now derive the ordinary Boussinesq equation from the shallow water wave equations

$$\begin{aligned} &\eta_t + \{(1+\alpha\eta)w\}_x - \frac{\beta}{6} w_{xxx} + O(\alpha\beta,\beta^2) = 0 \\ &w_t + \alpha w w_x + \eta_x - \frac{\beta}{2} w_{xxt} + O(\alpha\beta,\beta^2) = 0 \ . \end{aligned} \tag{6.1.2}$$

We note that these equations have been introduced in Chapter 5 [c.f. equations (5.1.2)] but here we choose to write η in place of u as we want to reserve u to denote the amplitude of the waves described by (6.1.1).

We start with the variable transformations

$$\begin{aligned} U &= \eta(X,T) - \alpha\eta^2 \\ X &= x + \alpha \int_{-\infty}^{x} \eta(x,t)dx \\ T &= t \end{aligned} \tag{6.1.3}$$

which have been introduced by Johnson (1983). In the following calculation all terms other than O(1), $O(\alpha)$ and $O(\beta)$ will be omitted.

The x and t derivatives can be calculated as

$$\frac{\partial}{\partial x} = (1+\alpha\eta)\frac{\partial}{\partial X} \tag{6.1.4}$$

$$\begin{aligned}\frac{\partial}{\partial t} &= \frac{\partial}{\partial T} + \frac{\partial X}{\partial t}\frac{\partial}{\partial X} \\ &= \frac{\partial}{\partial T} + \alpha \int_{-\infty}^{x} \eta_t(x,t)dx \frac{\partial}{\partial X} \\ &= \frac{\partial}{\partial T} - \alpha w \frac{\partial}{\partial X}, \text{ [from (6.1.2)]} .\end{aligned} \tag{6.1.5}$$

Using (6.1.4) and (6.1.5) into the first of (6.1.2) and keeping only the required terms we find

$$\eta_T + 2\alpha\eta w_X + w_X - \frac{\beta}{6} w_{XXX} = 0 \tag{6.1.6}$$

and thus

$$\eta_{TT} + 2\alpha(\eta w_X)_T + w_{XT} - \frac{\beta}{6} w_{XXXT} = 0 \tag{6.1.7}$$

upon differentiating (6.1.6) with respect to T.

The second of the equations (6.1.2) now becomes

$$w_T + \eta_X + \frac{\alpha}{2}(\eta^2)_X - \frac{\beta}{2} w_{XXT} = 0 . \tag{6.1.8}$$

The derivatives of w in (6.1.7) can be obtained from (6.1.6) and (6.1.8) as

$$w_X = -\eta_T + \alpha 2\eta\eta_T - \frac{\beta}{6}\eta_{XXT} + O(\alpha\beta,\alpha^2,\beta^2) \tag{6.1.9}$$

$$w_{XT} = -\eta_{XX} - \frac{\alpha}{2}(\eta^2)_{XX} + \frac{\beta}{2} w_{XXXT} + O(\alpha\beta,\alpha^2,\beta^2) . \tag{6.1.10}$$

From the last equation we also have

$$w_{XXXT} = -\eta_{XXXX} + O(\alpha,\beta) . \tag{6.1.11}$$

Inserting (6.1.9)-(6.1.11) into (6.1.7) we find

$$\eta_{TT} - 2\alpha(\eta\eta_T)_T - \eta_{XX} - \frac{\alpha}{2}(\eta^2)_{XX} - \frac{\beta}{3}\eta_{XXXX} = 0 . \tag{6.1.12}$$

It can be realized from (6.1.3) that

$$\eta = U + \alpha U^2 + O(\alpha^2)$$

$$\eta_{TT} - \alpha(\eta^2)_{TT} = U_{TT} + O(\alpha^2)$$

$$\eta_{XX} = U_{XX} + \alpha(U^2)_{XX} + O(\alpha^2)$$

$$(\eta^2)_{XX} = U^2_{XX} + O(\alpha)$$

$$\eta_{XXXX} = U_{XXXX} + O(\alpha) .$$

Using these relations, (6.1.12) can be written as

$$U_{TT} - U_{XX} - \frac{3\alpha}{2}(U^2)_{XX} - \frac{\beta}{3}U_{XXXX} = 0 . \tag{6.1.13}$$

The ordinary Boussinesq equation (6.1.1) can be recovered from (6.1.13) by the linear transformation $U \to au$, $X \to bx$ and $T \to ct$ for some a, b and c.

We note here that if we took into account the $O(\alpha^2)$ terms and make the transformation

$$U = \eta - \alpha\eta^2 + \alpha^2\eta^3$$

the corresponding equation to (6.1.13) would be

$$U_{TT} - U_{XX} - \frac{3}{2}\alpha(U^2)_{XX} - 2\alpha^2(U^3)_{XX} - \frac{\beta}{3}U_{XXXX} = 0$$

which could be considered as the modified Boussinesq equation.

6.2 Some previous results

The inverse scattering scheme for the ordinary Boussinesq equation (6.1.1) has been developed by Zakharov (1974). The solvability of this equation using the inverse scattering transform has also been pointed out by Kaup (1980) and Caudrey (1980).

The direct method has also been applied to this equation by Hirota (1973b). By using the variable transformation

$$u = \frac{\partial^2}{\partial x^2}(\log f) \tag{6.2.1}$$

equation (6.1.1) is transformed into the bilinear form

$$(D_t^2 - D_x^2 - D_x^4) f \cdot f = 0 . \tag{6.2.2}$$

One interesting fact that one should note with regard to Hirota's formulation is that the bilinear equation (6.2.2) is directly related to the linear differential operator of the ordinary Boussinesq equation

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^4}{\partial x^4} = 0$$

which is normally used to deduce the dispersion relation of the equation,

$$w^2 - k^2 - k^4 = 0 .$$

Therefore

$$w = \pm\, k(1+k^2)^{1/2} ,$$

where the plus sign is for the wave which travels to the right and the minus sign for the wave which travels to the left of the x-axis.

The N-soliton solution found by Hirota is written in the form

$$f(x,t) = \sum_{\mu=0,1} \exp\left[\sum_{i<j}^{(N)} A_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \mu_i\eta_i\right]$$

$$\eta_i = k_i x - \epsilon_i\Omega_i t - \eta_i^0$$

$$\epsilon_i = +1 \quad \text{or} \quad -1$$

$$\Omega_i = k_i(1+k_i^2)^{1/2}$$

$$\exp A_{ij} = \frac{(\epsilon_i\Omega_i-\epsilon_j\Omega_j)^2 - (k_i-k_j)^2 - (k_i-k_j)^4}{(\epsilon_i\Omega_i+\epsilon_j\Omega_j)^2 - (k_i+k_j)^2 - (k_i+k_j)^4} ,$$

where the notation for the summations has already been explained in Chapter 3. We note that the above functional form of f is the same as that for the Korteweg-de Vries equation [Hirota (1980)].

Nimmo and Freeman (1983) have also found the N-soliton solution of the ordinary Boussinesq equation in the form of a Wronskian of N functions. Because our work here is closely related to theirs, we shall now examine their result.

The Bäcklund transformations which relate between two soliton solutions f' and f of the equation are

$$(D_t+aD_x^2)f'\cdot f = 0 \qquad (6.2.3a)$$

$$(aD_xD_t+D_x+D_x^3)f'\cdot f = \lambda f'f , \qquad (6.2.3b)$$

where $a^2 = -3$. We note that for $\lambda = 0$ and a suitable value for a, equations (6.2.3) are the bilinear form of the Classical Boussinesq equations discussed in Chapter 5. Thus as we have seen in Chapter 5, for $\lambda = 0$, f' and f differ only in their phases, not in the number of solitons.

By taking $f' = 1$, which corresponds to the trivial solution, $u = 0$, they found from equations (6.2.3a,b) the single-soliton solution

$$f = A\exp(-\ell x+a\ell^2 t) + B\exp(nx+an^2 t) \qquad (6.2.4)$$

where A and B are real constants and ℓ,n the soliton parameters, and they are related by

$$\lambda = -(\ell+4\ell^3) = n + 4n^3 . \qquad (6.2.5)$$

The N-soliton solution of the ordinary Boussinesq equation is written as the Wronskian

$$f = (\widehat{N-1}) \tag{6.2.6}$$

of the functions

$$\phi_i = A_i \exp(-\ell_i x + a\ell_i^2 t) + B_i \exp(n_i x + an_i^2 t)$$
$$i = 1,2,\ldots,N \;. \tag{6.2.7}$$

We note here that in order to show that (6.2.7) satisfies (6.2.2) one needs an identity which can be obtained by making use of relation (6.2.5). By using this relation we see that

$$(\ell_i + 4\ell_i^3)\phi_i^{(r)} = \phi_i^{(r+1)} + 4\phi_i^{(r+3)} \tag{6.2.8}$$

where the superscript (r) denotes the r-th derivative with respect to x.

From the fundamental properties of determinants (5.3.7), relation (6.2.8) can be used to obtain

$$\begin{aligned}\sum_{i=1}^{N} (\ell_i + 4\ell_i^3)(\widehat{N-1}) &= (\widehat{N-2},N) + 4(\widehat{N-2},N+2)\\ &\quad - 4(\widehat{N-3},N-1,N+1) + 4(\widehat{N-4},N-2,N-1,N)\end{aligned} \tag{6.2.9}$$

$$\begin{aligned}\sum_{i=1}^{N} (\ell_i + 4\ell_i^3)(\widehat{N-2},N) &= (\widehat{N-2},N+1) + (\widehat{N-3},N-1,N)\\ &\quad + 4(\widehat{N-2},N+3) - 4(\widehat{N-3},N,N+1)\\ &\quad + 4(\widehat{N-5},N-3,N-2,N-1,N) \;.\end{aligned} \tag{6.2.10}$$

Substitution of (6.2.6) into (6.2.2) yields an expression which contains $(\widehat{N-2},N)^2$. This term can be removed by using (6.2.9) and (6.2.10) to give

$$\begin{aligned}(\widehat{N-2},N)^2 &= (\widehat{N-1})[(\widehat{N-2},N+1) + (\widehat{N-3},N-1,N)\\ &\quad + 4(\widehat{N-2},N+3) - 4(\widehat{N-3},N,N+1)\\ &\quad + 4(\widehat{N-5},N-3,N-2,N-1,N)]\\ &\quad - (\widehat{N-2},N)[4(\widehat{N-2},N+2) - 4(\widehat{N-3},N-1,N+1)\\ &\quad + 4(\widehat{N-4},N-2,N-1,N)] \;.\end{aligned} \tag{6.2.11}$$

Using this identity one eventually finds

$$(D_t^2-D_x^2-D_x^4)(\widehat{N-1})\cdot(\widehat{N-1})$$

$$= 12[(\widehat{N-1})(\widehat{N-3},N,N+1) - (\widehat{N-2},N)(\widehat{N-3},N-1,N+1) + (\widehat{N-2},N+1)(\widehat{N-3},N-1,N)]$$

$$= 6\begin{vmatrix} \widehat{N-3} & \cdot & N-2 & N-1 & N & N+1 \\ \cdot & \widehat{N-3} & N-2 & N-1 & N & N+1 \end{vmatrix} .$$

Since this determinant is obviously zero, $(\widehat{N-1})$ is thus verified as the solution of (6.2.2).

We shall now go on to show that the Bäcklund transformations (6.2.3a,b) are satisfied by two sets of identical solitons which differ in their phases for $\lambda = 0$ and satisfied by N and N+1-soliton solutions when $\lambda = -(\ell_{N+1}+4\ell_{N+1}^3)$.

For $\lambda = 0$ we take two phase-difference N-soliton solutions

$$f' = (\widehat{N-1})$$
$$f = (\tilde{N}) \qquad (6.2.12)$$

where $(\widehat{N-1})$ and $(\tilde{N})$ are the Wronskians of the functions ϕ_i and $\frac{\partial}{\partial x}\phi_i$, $i = 1,2, \ldots, N$ respectively. Essentially the order of the derivative in each column of $(\tilde{N})$ is one degree higher than the order of the same column in $(\widehat{N-1})$.

Substituting (6.2.12) and their derivatives into (6.2.3a) we find

$$(D_t+aD_x^2)f'\cdot f = 2a\{(\widehat{N-1})(\widetilde{N-2},N,N+1) - (\widehat{N-2},N)(\widetilde{N-1},N+1) + (\widehat{N-2},N+1)(\tilde{N})\}$$

$$= (-1)^N 2a\begin{vmatrix} \widehat{N-2} & \cdot & N-1 & N & N+1 \\ \cdot & \widetilde{N-2} & N-1 & N & N+1 \end{vmatrix} .$$

This determinant is zero and thus we have verified the solution of (6.2.3a).

Now substituting (6.2.12) and their derivatives into (6.2.3b) yields (with $\lambda = 0$)

$$(aD_xD_t+D_x+D_x^3)f'\cdot f = \{-2(\widehat{N-2},N+2) + 2(\widehat{N-3},N-1,N+1) + (\widehat{N-2},N) + 4(\widehat{N-4},N-2,N-1,N)\}(\tilde{N})$$
$$+ (\widehat{N-1})\{-4(\widetilde{N-1},N+3) + 2(\widetilde{N-3},N-1,N,N+1) - (\widetilde{N-1},N+1) - 2(\widetilde{N-2},N,N+2)\}$$
$$- 6(\widehat{N-3},N-1,N)(\widetilde{N-1},N+1) + 6(\widehat{N-2},N)(\widetilde{N-1},N+2). \qquad (6.2.13)$$

The expression on the right side of equation (6.2.13) looks quite complicated and it is not obviously zero. However, it can be simplified by using the identity

$$(\widehat{N-1}) \sum_{i=1}^{N} (\ell_i+4\ell_i^3)(\widetilde{N}) = \{ \sum_{i=1}^{N} (\ell_i+4\ell_i^3)(\widehat{N-1})\}(\widetilde{N})$$

which gives

$$\begin{aligned} &\{(\widehat{N-2},N) + 4(\widehat{N-4},N-2,N-1,N)\}(\widetilde{N}) \\ &\quad = (\widehat{N-1})\{(\widetilde{N-1},N+1) + 4(\widetilde{N-1},N+3) - 4(\widetilde{N-2},N,N+2) \\ &\quad + 4(\widetilde{N-3},N-1,N,N+1)\} \\ &\quad - \{4(\widehat{N-3},N-1,N+1) - 4(\widehat{N-2},N+2)\}(\widetilde{N}) \ . \end{aligned} \tag{6.2.14}$$

Using (6.2.14), we find (6.2.13) becomes

$$\begin{aligned} &(aD_xD_t+D_x+D_x^3)f'\cdot f \\ &= 6\{(\widehat{N-1})(\widetilde{N-3},N-1,N,N+1) - (\widehat{N-3},N-1,N)(\widetilde{N-1},N+1) \\ &\quad + (\widehat{N-3},N-1,N+1)(\widetilde{N})\} \\ &\quad - 6\{(\widehat{N-1})(\widetilde{N-2},N,N+2) - (\widehat{N-2},N)(\widetilde{N-1},N+2) + (\widehat{N-2},N+2)(\widetilde{N})\} \\ &= (-1)^{N+1}6\begin{vmatrix} \widehat{N-3} & N-1 & \cdot & \cdot & N-2 & N & N+1 \\ \cdot & \cdot & \widetilde{N-3} & N-1 & N-2 & N & N+1 \end{vmatrix} \\ &\quad + (-1)^{N+1}6\begin{vmatrix} \widehat{N-2} & \cdot & N-1 & N & N+2 \\ \cdot & \widetilde{N-2} & N-1 & N & N+2 \end{vmatrix} \ . \end{aligned}$$

Both of the determinants are obviously zero and therefore we have verified the solution of (6.2.3b).

One important point that one should note from the above verification is that we do not require any identity for equation (6.2.3a) while we do need one for equation (6.2.3b). If we put $x_1 = x$ and $x_2 = at$ equations (6.2.3a) and (6.2.3b) respectively become

$$(D_1^2+D_2)f'\cdot f = 0 \tag{6.2.15}$$

$$(-3D_1D_2+D_1+D_1^3)f'\cdot f = 0 \ . \tag{6.2.16}$$

We thus realize that the first equation is an equation of the first modified KP hierarchy, while the second one is not. Therefore for any value of ℓ_i and n_i the Wronskians $f' = (\widehat{N-1})$ and $f = (\widetilde{N})$ of the functions ϕ_i defined by (6.2.7) will automatically satisfy (6.2.3a), while these will only satisfy (6.2.3b) if ℓ_i and η_i satisfy the identity relation (6.2.5). This can be explained if we recall from Appendix C that the second equation of the first modified KP hierarchy is

$$(D_1^3-4D_3-3D_1D_2)f'\cdot f = 0 . \qquad (6.2.17)$$

This equation can be separated as

$$\begin{aligned}(D_1^3-4D_3-3D_1D_2)f'\cdot f \\ = (-3D_1D_2+D_1+D_1^3)f'\cdot f - (D_1+4D_3)f'\cdot f .\end{aligned} \qquad (6.2.18)$$

In terms of variables x_1 and x_2 the function ϕ_i (6.2.7) becomes

$$\phi_i = A_i \exp(-\ell_i x_1+\ell_i^2 x_2) + B_i(n_i x_1+n_i^2 x_2) . \qquad (6.2.19)$$

It has been proved in Chapter 5 that the Wronskians $f' = (\widehat{N-1})$ and $f = (\tilde{N})$ of the functions ϕ_i, $i = 1,2,\dots,N$, defined by (6.2.19), satisfy equation (6.2.17) for any value of ℓ_i and η_i. Thus in order that f' and f satisfy (6.2.16) we require from (6.2.18)

$$(D_1+4D_3)f'\cdot f = 0 . \qquad (6.2.20)$$

Substituting (6.2.19) into (6.2.20) yields immediately

$$\ell_i + \ell_i^3 = -(n_i+4n_i^3) , \qquad i = 1,2,\dots,N ,$$

the relation which has been used to produce identity (6.2.14).

We now define f' and f as

$$\begin{aligned} f' &= (\widehat{N-1}) \\ f &= (\hat{N}) \end{aligned} \qquad (6.2.21)$$

where $(\widehat{N-1})$ is the Wronskian of N functions ϕ_i, $i = 1,2,\dots,N$ and $(\hat{N})$ the Wronskian of N+1 functions ϕ_i, $i = 1,2,\dots,N,N+1$ with ϕ_i defined by equation (6.2.7). We shall show that (6.2.21) satisfies the Bäcklund transformations (6.2.3a,b) for $\lambda = -(\ell_{N+1}+4\ell_{N+1}^3)$. We note immediately that λ is also given by

$$\lambda = \rho_N - \rho_{N+1} \qquad (6.2.22)$$

where

$$\rho_K = \sum_{i=1}^{K} (\ell_i+4\ell_i^3) , \qquad K = N,N+1 . \qquad (6.2.23)$$

Verification of the solution (6.2.21) for equation (6.2.3a) is straightforward as was the case of $\lambda = 0$. We thus verify only equation (6.2.3b). We first write equation (6.2.3b) as

$$(aD_xD_t+D_x+D_x^3)f'\cdot f - \lambda f'\cdot f = 0 \ . \qquad (6.2.24)$$

Substitution of f′ and f given by (6.2.21) and their derivatives into equation (6.2.24) yields

$$\begin{aligned}&\{-2(\widehat{N-2},N+2) + 2(\widehat{N-3},N-1,N+1) + (\widehat{N-2},N)\\ &\quad + 4(\widehat{N-4},N-2,N-1,N)\}(\hat{N})\\ &\quad + (\widehat{N-1})\{-4(\widehat{N-1},N+3) + 2(\widehat{N-3},N-1,N,N+1)\\ &\qquad - (\widehat{N-1},N+1) - 2(\widehat{N-2},N,N+2)\}\\ &\qquad - 6(\widehat{N-3},N-1,N)(\widehat{N-1},N+1) + 6(\widehat{N-2},N)(\widehat{N-1},N+2) - \lambda(\widehat{N-1})(\hat{N}).\end{aligned} \qquad (6.2.25)$$

In order to work out the actual expression for $\lambda(\widehat{N-1})(\hat{N})$ we should note that the scalar operator ρ_N must act upon $(\widehat{N-1})$ and ρ_{N+1} upon $(\hat{N})$ and their derivatives. We thus have

$$\begin{aligned}\lambda(\widehat{N-1})(\hat{N}) &= [\rho_N(\widehat{N-1})](\hat{N}) - (\widehat{N-1})[\rho_{N+1}(\hat{N})]\\ &= [(\widehat{N-2},N) + 4(\widehat{N-2},N+2) - 4(\widehat{N-3},N-1,N+1)\\ &\quad + 4(\widehat{N-4},N-2,N-1,N)](\hat{N})\\ &\quad - (\widehat{N-1})[(\widehat{N-1},N+1) + 4(\widehat{N-1},N+3) - 4(\widehat{N-2},N,N+2)\\ &\quad + 4(\widehat{N-3},N-1,N,N+1)] \ .\end{aligned} \qquad (6.2.26)$$

Using (6.2.26), expression (6.2.25) becomes

$$\begin{aligned}&6\{(\widehat{N-1})(\widehat{N-3},N-1,N,N+1) - (\widehat{N-3},N-1,N)(\widehat{N-1},N+1)\\ &\quad + (\widehat{N-3},N-1,N+1)(\hat{N})\}\\ &\quad - 6\{(\widehat{N-1})(\widehat{N-2},N,N+2) - (\widehat{N-2},N)(\widehat{N-1},N+2)\\ &\quad + (\widehat{N-2},N+2)(\hat{N})\} \ .\end{aligned}$$

The above expression is the Laplace expansion of two (2N+1) × (2N+1) determinants

$$6(-1)^N \begin{vmatrix} \widehat{N-3} & N-1 & \cdot & \cdot & N-2 & N & N+1 \\ \cdot & \cdot & \widehat{N-3} & N-1 & N-2 & N & N+1 \end{vmatrix}$$
$$+ 6(-1)^N \begin{vmatrix} \widehat{N-2} & \cdot & N-1 & N & N+2 \\ \cdot & \widehat{N-2} & N-1 & N & N+2 \end{vmatrix} \qquad (6.2.27)$$

where the upper matrices contain N rows while the lower ones contain N+1 rows. Both of these determinants can be shown to be zero, thus verifying the solutions of (6.2.3b).

6.3 The $pq = \frac{1}{4}$ reduction of the KP hierarchy

If we write $x_1 = x$, $x_2 = t$ then the bilinear form of the Boussinesq equation (6.2.2) becomes

$$(D_1^4+D_1^2-D_2^2)f\cdot f = 0 . \tag{6.3.1}$$

In terms of x_1 and x_2 the single-soliton solution (6.2.4) is simply

$$\begin{aligned} f &= A\exp(-\ell x_1+a\ell^2x_2) + B\exp(nx_1+an^2x_2) \\ &= A\exp(-\ell x_1+a\ell^2x_2)\{1+\frac{B}{A}\exp[(\ell+n)x_1+a(n^2-\ell^2)x_2]\} . \end{aligned} \tag{6.3.2}$$

Since the exponential factor $A\exp(-\ell x_1+a\ell^2x_2)$ has the argument which is linear in x_1, it gives only zero contribution to the final solution due to [c.f. (6.2.1)]

$$u = \frac{\partial^2}{\partial x_1^2}(\log f) . \tag{6.3.3}$$

Equation (6.3.1) is to be compared with the first equation of the KP hierarchy

$$(D_1^4-4D_1D_3+3D_2^2)f\cdot f = 0 \tag{6.3.4}$$

which has the single-soliton solution in the form

$$\begin{aligned} f &= A\exp(qx_1+q^2x_2+q^3x_3) + B\exp(px_1+p^2x_2+p^3x_3) \\ &= A\exp(qx_1+q^2x_2+q^3x_3)\{1+\frac{B}{A}\exp[(p-q)x_1+(p^2-q^2)x_2 \\ &\quad + (p^3-q^3)x_3]\} , \end{aligned} \tag{6.3.5}$$

for any p,q as long as $p \neq q$.

As we have seen in the Introduction, equation (6.3.4) can be separated as

$$\begin{aligned} &(D_1^4-4D_1D_3+3D_2^2)f\cdot f \\ &= (D_1^4+D_1^2-D_2^2)f\cdot f - (D_1^2+4D_1D_3-4D_2^2)f\cdot f . \end{aligned} \tag{6.3.6}$$

Now for (6.3.5) to satisfy (6.3.1) we also require that it satisfies

$$(D_1^2+4D_1D_3-4D_2^2)f\cdot f = 0 . \tag{6.3.7}$$

Using (6.3.5) we find the dispersion relation of (6.3.7) as

$$(p-q)^2 + 4(p-q)(p^3-q^3) - 4(p^2-q^2)^2 = 0 \ . \qquad (6.3.8)$$

This in turn simplifies to

$$pq = \frac{1}{4} \ . \qquad (6.3.9)$$

We note that relation (6.3.9) can also be found by substituting (6.3.5) into (6.3.1).

Thus the ordinary Boussinesq equation can be viewed as the $pq = \frac{1}{4}$ reduction of the KP hierarchy. The $pq = \frac{1}{4}$ is the type of the $pq = c$ reduction which normally occurs in the first modified KP hierarchy. Normally in the KP hierarchy one has the problems of n-reduction. For example, putting $x_1 = x$ and $x_2 = at$ would result in the ordinary Boussinesq equation in the $p + 4p^3 = q + 4p^3$ reduction problem [Hirota (1986b)].

In the remaining sections of this chapter we shall be dealing with a new representation of the soliton solution of the ordinary Boussinesq equation as a result of the $pq = \frac{1}{4}$ reduction.

6.4 New representation of the solution

From the previous section, we have expressed the soliton solution of the ordinary Boussinesq equation in two forms. One is with parameters ℓ, n which are related by

$$\ell + 4\ell^3 = -(n+4n^3) \ ,$$

and the other one is with parameters p,q related by

$$pq = \frac{1}{4} \ .$$

Using equations (6.3.2) and (6.3.5), and neglecting the factors, we may write (with $q = \frac{1}{4p}$)

$$\ell + n = p - \frac{1}{4p}$$

$$(6.4.1)$$

$$a(n^2-\ell^2) = p^2 - \frac{1}{(4p)^2} \ ,$$

where we shall take $a = i\sqrt{3}$.

Solving equations (6.4.1) simultaneously for ℓ,n we find

$$\ell = \frac{1}{\sqrt{3}}(pz - \frac{1}{4pz})$$

$$n = \frac{1}{\sqrt{3}}(\frac{p}{z} - \frac{z}{4p}) \tag{6.4.2}$$

where

$$z = e^{i\pi/6} . \tag{6.4.3}$$

A number of identities in z may be produced from (6.4.3). Two of them are

$$\frac{1}{\sqrt{3}}(z + \frac{1}{z}) = 1 \tag{6.4.4a}$$

$$z^4 = z^2 - 1 . \tag{6.4.4b}$$

Now, using (6.4.2)-(6.4.4) we may write

$$\begin{aligned} A_i e^{\theta_{1i}} + B_i e^{\theta_{2i}} &= A_i \exp(-\ell_i x_1 + a\ell_i^2 x_2) + B_i \exp(n_i x_1 + a n_i^2 x_2) \\ &= A_i \exp\{-\frac{1}{\sqrt{3}}(p_i z - \frac{1}{4p_i z})x_1 + \frac{a}{3}(p_i^2 z_i^2 + \frac{1}{(4p_i)^2 z^2} - \frac{1}{2})x_2\} \\ &+ B_i \exp\{\frac{1}{\sqrt{3}}(\frac{p_i}{z} - \frac{z}{4p_i})x_1 + \frac{a}{3}(\frac{p_i^2}{z^2} + \frac{z^2}{(4p_i)^2} - \frac{1}{2})x_2\} \\ &= \exp\{-\frac{z}{\sqrt{3}}(p_i + \frac{1}{4p_i})x_1 + \frac{a}{3}[(p_i^2 + \frac{1}{(4p_i)^2})z^2 - \frac{1}{2}]x_2\} \\ &\cdot \{B_i \exp(p_i x_1 + p_i^2 x_2) + A_i \exp(\frac{x_1}{4p_i} + \frac{x_2}{(4p_i)^2})\} \\ &= E_i\{B_i e^{\theta'_{1i}} + A_i e^{\theta'_{2i}}\} \text{ (say)} \end{aligned} \tag{6.4.5}$$

where

$$E_i = \exp\{-\frac{z}{\sqrt{3}}(p_i + \frac{1}{4p_i})x_1 + \frac{a}{3}[(p_i^2 + \frac{1}{(4p_i)^2})z^2 - \frac{1}{2}]x_2\} . \tag{6.4.6}$$

For simplicity we write (6.4.5) as

$$\phi_i = E_i \, \Phi_i \tag{6.4.7}$$

where

$$\phi_i = A_i \exp[-\ell_i x_1 + a\ell_i^2 x_2] \quad + \quad B_i \exp[n_i x_1 + a n_i^2 x_2] \tag{6.4.8}$$

and

$$\Phi_i = A_i \exp\left[\frac{x_1}{4p_i} + \frac{x_2}{(4p_i)^2}\right] + B_i \exp[p_i x_1 + p_i^2 x_2]. \qquad (6.4.9)$$

The x_1-derivatives of ϕ_i can now be deduced as

$$\frac{\partial^n}{\partial x_1^n}\phi_i = E_i\{\frac{1}{z\sqrt{3}}\partial_+ - \frac{z}{4\sqrt{3}}\partial_-\}^n \Phi_i \qquad (6.4.10)$$

where $\partial_+ \equiv \frac{\partial}{\partial x_1}$ and ∂_- denotes the integral operator with respect to x_1.

For the purpose of the following analysis we shall denote [ ] to mean a Wronskian of functions ϕ_i (with parameters ℓ_i, n_i) and ( ) to mean a Wronskian of functions Φ_i (with parameters $p_i, q_i = \frac{1}{4p_i}$). Using (6.4.10) for the single-soliton solution we find

$$[0] = E(0)$$

$$[1] = \frac{E}{z\sqrt{3}}\{(1) - \frac{z^2}{4}(-1)\}$$

$$[2] = \frac{E}{(z\sqrt{3})^2}\{(2) - \frac{z^2}{2}(0) + \frac{z^4}{4^2}(-2)\}$$

$$[3] = \frac{E}{(z\sqrt{3})^3}\{(3) - 3\frac{z^2}{4}(1) + 3\frac{z^4}{4^2}(-1) - \frac{z^8}{4^3}(-3)\}$$

or, in general, we have

$$[N] = \frac{E}{(z\sqrt{3})^N}\sum_{r=0}^{N}\frac{N!}{(N-r)!r!}\{-\frac{z^2}{4}\}^r(N-2r) \;,\; N = 1,2, \ldots . \qquad (6.4.11)$$

We can now proceed to reconstruct the soliton solutions which are originally in the form of a Wronskian of ϕ_i into a form of a finite sum of Wronskian type determinants of Φ_i (with parameters p_i, q_i). From (6.4.11) we have the two-soliton solution

$$[0,1] = \frac{E_1E_2}{z\sqrt{3}}((0),(1) - \frac{z^2}{4}(-1))$$

$$= \frac{E_1E_2}{z\sqrt{3}}\{(0,1) + \frac{z^2}{4}(-1,0)\} . \qquad (6.4.12)$$

For the three-soliton solution we find

$$[0,1,2] = \frac{E_1E_2E_3}{(z\sqrt{3})^3}((0),(1) - \frac{z^2}{4}(-1),(2) - \frac{z^2}{2}(0) + \frac{z^4}{4^2}(-2))$$

$$= \frac{E_1E_2E_3}{(z\sqrt{3})^3}\{(0,1,2) + \frac{z^2}{4}(-1,0,2) + \frac{z^4}{4^2}(-2,0,1) + \frac{z^6}{4^3}(-2,-1,0)\} . \quad (6.4.13)$$

By using the similar procedure we can also produce the four-soliton solution

$$[0,1,2,3] = \frac{E_1E_2E_3E_4}{(z\sqrt{3})^4}\{(0,1,2,3) + \frac{z^2}{4}(-1,0,2,3) + \frac{z^4}{4^2}(-2,0,1,3) + \frac{z^6}{4^3}[(-3,0,1,2) + (-2,-1,0,3)] + \frac{z^8}{4^4}(-3,-1,0,2) + \frac{z^{10}}{4^5}(-3,-2,0,1) + \frac{z^{12}}{4^6}(-3,-2,-1,0)\} . \quad (6.4.14)$$

We note that all the factors outside { } in the above solutions can be removed as they do not contribute to the final solution u of the ordinary Boussinesq equation.

The structure of the new representation can be generalized to the N-soliton solution as

$$[\hat{N-1}] = \sum_{i=0}^{K} (\frac{z^2}{4})^{K-i} W_{L_i} \quad (6.4.15)$$

where

$$K = \frac{N(N-1)}{2}$$

$$L_i = 2i - K, \quad i=0,1,2,\ldots,K . \quad (6.4.16)$$

Also here W_{L_i} is the summation of all possible Wronskian type determinants $(c_{i0},c_{i1},c_{i2},\ldots,c_{iN-1})$ with one of the c_{in}'s being the column (0) and other c_{in}'s being any column ($\pm n$) chosen from n = 1, 2, ..., N-1, without repeating, arranged in the order of $c_{i0} < c_{i1} < c_{i2} < \ldots < c_{iN-1}$ and satisfying

$$\sum_{n=0}^{N-1} c_{in} = L_i . \quad (6.4.17)$$

For example, for the two-soliton solution, we have $K = 1$ and thus $L_0 = -1$, $L_1 = 1$ from (6.4.16). To construct W_{L_0} we choose c_{00}, c_{01} from (0), (-1) and (1), such that $c_{00} + c_{01} = -1$. Thus we have $W_{L_0} = (-1,0)$. Similarly we can find $W_{L_1} = (0,1)$.

In the four-soliton solution we check for the z^6 term. Here $K = 6$ and the z^6 term corresponds to $i = 3$ which gives $L_3 = 0$. Therefore for W_0 we can only choose the columns -2, -1, 0, 3 and -3, 0, 1, 2 as they sum to zero. Thus

$$W_0 = (-2,-1,0,3) + (-3,0,1,2) \ .$$

Although the solution (6.4.15) gives rise to a polynomial in z of order $N(N-1)$, it can always be grouped into two terms only as the result of identity (6.4.4b): the z^0 and z^2 terms. It is therefore reasonable, in verifying the solution to look at the coefficients of z^0 and z^2. However, one should notice that the coefficients of z^0 and z^2 in a solution are themselves not the solutions.

6.5 The two-soliton solution

Before proceeding to verify that the two-soliton solution in its new representation (6.4.12) satisfies the ordinary Boussinesq equation we first give some relations which will be used repeatedly for this purpose. We rewrite the function

$$\Phi_i = B_i \exp(p_i x_1 + p_i^2 x_2) + A_i \exp\left(\frac{x_1}{4p_i} + \frac{x_2}{(4p_i)^2}\right) \ . \qquad (6.5.1)$$

From (6.5.1) we see immediately that

$$\frac{\partial \Phi_i}{\partial x_2} = \frac{\partial^2 \Phi_i}{\partial x_1^2} \qquad (6.5.2)$$

which is the property of the KP hierarchy. This relation will be used as usual to calculate the derivatives of the Wronskians with respect to x_2.

Another relation which can be deduced immediately from (6.5.1) is

$$\alpha_{n_i}\,\Phi_i^{(r)} = \Phi_i^{(r+n)} + \frac{1}{4^n}\,\Phi_i^{(r-n)} \tag{6.5.3}$$

with

$$\alpha_{n_i} = p_i^n + \frac{1}{(4p_i)^n} \tag{6.5.4}$$

where $\Phi_i^{(k)}$ is the k-th derivative of Φ_i with respect to x_1.

Relation (6.5.3) gives rise to a number of Wronskian identities as we have seen in Section 6.2. However, in application of (6.5.3), beside shifting up a column by n degrees, we also shift that column n degrees down. Thus we should expect to obtain many terms in our expressions. For example,

$$\alpha_1(0,2) = (0,3) + (1,2) + \tfrac{1}{4}(0,1) + \tfrac{1}{4}(-1,2)$$

in which we have used the notation

$$\alpha_n = \sum_{i=1}^{N} \alpha_{n_i}\,, \qquad n = 1,2,\ \ldots\ . \tag{6.5.5}$$

We now show that the two-soliton solution

$$f = (0,1) + \frac{z^2}{4}(-1,0) \tag{6.5.6}$$

satisfies the bilinear form of the ordinary Boussinesq equation

$$(D_1^4+D_1^2-D_2^2)f\cdot f = 0\ . \tag{6.5.7}$$

Expanding (6.5.7) we find

$$\begin{aligned}(D_1^4+D_1^2-D_2^2)&f\cdot f\\ &= 2(ff_{4x_1}-4f_{x_1}f_{3x_1}+3f_{2x_1}^2+ff_{2x_1}-f_{x_1}^2-ff_{2x_2}+f_{x_2}^2)\ .\end{aligned} \tag{6.5.8}$$

The required derivatives of f can be calculated as

$$\begin{aligned}
f_{x_1} &= (0,2) + \frac{z^2}{4}(-1,1)\\
f_{2x_1} &= (0,3) + (1,2) + \frac{z^2}{4}\{(-1,2) + (0,1)\}\\
f_{3x_1} &= (0,4) + 2(1,3) + \frac{z^2}{4}\{(-1,3) + 2(0,2)\}\\
f_{4x_1} &= (0,5) + 3(1,4) + 2(2,3) + \frac{z^2}{4}\{(-1,4)+3(0,3)+2(1,2)\}\\
f_{x_2} &= (0,3) - (1,2) + \frac{z^2}{4}\{(-1,2) - (0,1)\}\\
f_{2x_2} &= (0,5) + 2(2,3) - (1,4) + \frac{z^4}{4}\{(-1,4) + 2(1,2) - (0,3)\}\ .
\end{aligned}$$

It can be seen from the above derivatives that all the Wronskian type determinants in the coefficients of $\frac{z^2}{4}$ can be found by shifting down by one every column of the appropriate determinants in z^0. This is simply due to the form of f given by (6.5.6).

Now, substituting f and all the above derivatives into the right-hand side of the equation (6.5.8) we find

$$(D_1^4+D_1^2-D_2^2)f\cdot f = 2\{c_o + \frac{z^2}{4}c_2\} \tag{6.5.9}$$

where

$$\begin{aligned} c_0 = & (0,1)\{4(1,4) + (0,3) + (1,2)\} \\ & + 4\{(0,3)^2 + (1,2)^2 + (0,3)(1,2)\} \\ & - (0,2)^2 - 4(0,2)\{(0,4) + 2(1,3)\} \\ & - \frac{(-1,0)}{4^2}\{4(0,3) + (-1,2) + (0,1)\} \\ & - \frac{1}{4}\{(-1,2)^2 + (0,1)^2 + (-1,2)(0,1)\} \\ & + \frac{(-1,1)^2}{4} + \frac{(-1,1)}{4}\{(-1,3) + 2(0,2)\} \end{aligned} \tag{6.5.10}$$

and

$$\begin{aligned} c_2 = & (0,1)\{ 4(0,3) + (-1,2) + (0,1)\} \\ & + 4(0,3)\{2(-1,2) + (0,1)\} + 4(1,2)\{2(0,1) + (-1,2)\} \\ & - 2(0,2)(-1,1) - 4(0,2)\{(-1,3) + 2(0,2)\} \\ & - 4(-1,1)\{(0,4) + 2(1,3)\} + \frac{(-1,0)}{4}\{4(0,3) + (-1,2) + (0,1)\} \\ & + (-1,2)^2 + (0,1)^2 + (-1,2)(0,1) - (-1,1)^2 \\ & - 4(-1,1)\{(-1,3) + 2(0,2)\} . \end{aligned} \tag{6.5.11}$$

We shall show that both coefficients c_0 and c_2 are zero. However, the expressions for c_0 and c_2 as seen in (6.5.10) and (6.5.11), seem very complicated and of course they are not obviously zero. Furthermore application of a single identity does not reduce them into a more manageable form. However, we have managed to use a number of identities for this purpose.

For c_0, the identities used can be written as follows

$$\{4(0,3) + 4(1,2) - (0,1)\}\alpha_1(0,2) = (0,2)\alpha_1\{4(0,3) + 4(1,2) - (0,1)\}$$
$$(-1,1)\alpha_1\{\frac{(-1,2)}{4} + (1,2)\} = \{\frac{(-1,2)}{4} + (1,2)\}\alpha_1(-1,1)$$
$$4(0,1)\alpha_1(1,3) = 4(1,3)\alpha_1(0,1) \qquad (6.5.12)$$
$$(1,2)(\alpha_1^2-\alpha_2)(-1,0) = (-1,0)(\alpha_1^2-\alpha_2)(1,2).$$

Essentially identities (6.5.12) remove all the squared terms and terms with a common column such as (0,1)(1,4). These types of terms are not wanted for our purpose because they do not appear in the Laplace expansion of a determinant.

By using all the identities in (6.5.12) in (6.5.10) we eventually find that

$$\begin{aligned} c_0 = {} & (-1,0)(2,3) - (-1,2)(0,3) + (-1,3)(0,2) \\ & - (0,1)(2,3) + (0,2)(1,3) - (0,3)(1,2) \\ & + \frac{1}{4}\{(-1,0)(1,2) - (-1,1)(0,2) + (-1,2)(0,1)\} \\ & - \frac{1}{4^2}\{(-2,-1)(1,2) - (-2,1)(-1,2) + (-2,2)(-1,1)\} . \end{aligned}$$

The above expression is exactly the form of the Laplace expansion of some determinants and it can be written as

$$\begin{aligned} c_0 = {} & \begin{vmatrix} -1 & 0 & 2 & 3 \\ \cdot & 0 & 2 & 3 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 2 & 3 \\ \cdot & 1 & 2 & 3 \end{vmatrix} \\ & + \frac{1}{4}\begin{vmatrix} -1 & 0 & 1 & 2 \\ \cdot & 0 & 1 & 2 \end{vmatrix} - \frac{1}{4^2}\begin{vmatrix} -2 & -1 & 1 & 2 \\ \cdot & -1 & 1 & 2 \end{vmatrix} \end{aligned} \qquad (6.5.13)$$

and each of the determinants is obviously zero and thus $c_0 = 0$.

For c_2, the identities used are

$$(0,1)\alpha_1\{8(0,2) - 4(-1,3)\} = \{8(0,2) - 4(-1,3)\}\alpha_1(0,1)$$
$$\{(-1,2) - 4(0,3)\}\alpha_1(-1,1) = (-1,1)\alpha_1\{(-1,2) - 4(0,3)\} \qquad (6.5.14)$$
$$4(-1,0)\alpha_1(1,3) = 4(1,3)\alpha_1(-1,0) .$$

These identities will then give

$$c_2 = (-2,-1)(1,2) - (-2,1)(-1,2) + (-2,2)(-1,1)$$
$$- 4\{(-1,0)(2,3) - (-1,2)(0,3) + (-1,3)(0,2)\}$$
$$+ (-2,0)(1,3) - (-2,1)(0,3) + (-2,3)(0,1)$$
$$+ 4\{(-1,2) + \frac{1}{4^2}(-1,-2)\}(1,2)$$
$$- 4(-1,3)(0,2) + 4\{(-1,4) + \frac{1}{4^2}(-1,0)\}(0,1) .$$

Again, c_2 is seen as a Laplace expansion of some determinants; indeed we can write

$$c_2 = \begin{vmatrix} -2 & -1 & 1 & 2 \\ \cdot & -1 & 1 & 2 \end{vmatrix} - 4\begin{vmatrix} -1 & 0 & 2 & 3 \\ \cdot & 0 & 2 & 3 \end{vmatrix}$$

$$+ \begin{vmatrix} -2 & 0 & 1 & 3 \\ \cdot & 0 & 1 & 3 \end{vmatrix} + 4\begin{vmatrix} -1 & \alpha_2(0) & \alpha_2(1) & \alpha_2(2) \\ \cdot & 0 & 1 & 2 \end{vmatrix} \qquad (6.5.15)$$

where $\alpha_2(n)$ denotes that the column takes the form $\alpha_{2_i} \phi_i^{(n)}$.

Now, except for the last determinant, all other determinants in (6.5.15) are clearly zero.

For the last determinant, multiplying the last two rows each by α_{2_1} and α_{2_2} respectively we have

$$\begin{vmatrix} -1 & \alpha_2(0) & \alpha_2(1) & \alpha_2(2) \\ \cdot & 0 & 1 & 2 \end{vmatrix}$$

$$= \frac{1}{\prod_{i=1}^{2} \alpha_{2_i}} \begin{vmatrix} -1 & \alpha_2(0) & \alpha_2(1) & \alpha_2(2) \\ \cdot & \alpha_2(0) & \alpha_2(1) & \alpha_2(2) \end{vmatrix}$$

and thus it is now zero by the method of subtraction. We have therefore established that $c_2 = 0$ and hence verified the two-soliton solution (6.5.6) of the ordinary Boussinesq equation.

The use of multiple identities such as (6.5.12) and (6.5.14) in verifying Wronskian solutions is something which has never occurred before. For example, we take the KdV equation in its bilinear form

$$(D_1^4 - 4D_1D_3)f \cdot f = 0 . \qquad (6.5.16)$$

As we have seen in the Introduction, (6.5.16) is a reduction of the KP hierarchy since it can be separated as

$$(D_1^4-4D_1D_3)f\cdot f = (D_1^4-4D_1D_3+3D_2^2)f\cdot f - 3D_2^2 f\cdot f \ . \qquad (6.5.17)$$

Therefore, if we take the N-soliton solution in the Wronskian form $f = (\widehat{N-1})$ of functions

$$\phi_i = A_i e^{q_i x_1 + q_i^2 x_2 + q_i^3 x_3} + B_i e^{p_i x_1 + p_i^2 x_2 + p_i^3 x_3} ,$$

we then require

$$3D_2^2 f\cdot f = 0 \ . \qquad (6.5.18)$$

In other words, (6.5.16) is satisfied by f if it also satisfies (6.5.18). Therefore (6.5.18) is the identity needed in verifying the solution of (6.5.16). Substituting $f = (\widehat{N-1})$ and its derivatives into (6.5.18) we find the single identity

$$\begin{aligned}
&(\widehat{N-2},N+1)^2 + (\widehat{N-3},N-1,N)^2 \\
&\quad = (\widehat{N-1})\{(\widehat{N-2},N+3) + 2(\widehat{N-3},N,N+1) - (\widehat{N-3},N-1,N+2) \\
&\quad - (\widehat{N-4},N-2,N-1,N+1) + (\widehat{N-5},N-3,N-2,N-1,N)\} \\
&\quad + 2(\widehat{N-2},N+1)(\widehat{N-3},N-1,N) \ . \qquad (6.5.19)
\end{aligned}$$

If the N-soliton solution is substituted into the KdV equation (6.5.16) one should find an expression with some undesired terms like $(\widehat{N-1},N+1)^2$, $(\widehat{N-3},N-1,N)^2$ and $(\widehat{N-1})(\widehat{N-3},N-1,N+2)$, which do not appear in the Laplace expansion of a determinant. Such terms are removed at once by applying (6.5.19). Therefore the final expression that one should get for the KdV equation (6.1.16) is exactly the same expression that one would obtain from the KP equation

$$(D_1^4-4D_1D_3+3D_2^2)f\cdot f = 0 ,$$

since the last term in equation (6.5.17) which is $-3D_2^2 f\cdot f$ has actually been put to zero.

The situation is quite different for the $pq = \frac{1}{4}$ reduction problem in relation to the ordinary Boussinesq equation. In order to illustrate the difference here we first define

$$\mathcal{D}_B = D_1^4 + D_1^2 - D_2^2$$
$$\mathcal{D}_{KP} = D_1^4 - 4D_1D_3 + 3D_2^2 \qquad (6.5.20)$$
$$\mathcal{D}_L = 4D_1D_3 + D_1^2 - 4D_2^2 \ .$$

Thus, the reduction problem (6.3.6) becomes

$$\mathcal{D}_B f\cdot f = \mathcal{D}_{KP} f\cdot f + \mathcal{D}_L f\cdot f \ . \qquad (6.5.21)$$

If the two-soliton solution (6.5.6) is substituted into (6.5.21) we find

$$\begin{aligned}
&\mathcal{D}_B\{(0,1) + \tfrac{z^2}{4}(-1,0)\}\cdot\{(0,1) + \tfrac{z^2}{4}(-1,0)\} \\
&= \mathcal{D}_{KP}\{(0,1)\cdot(0,1) + \tfrac{2z^2}{4}(-1,0)\cdot(0,1) + \tfrac{z^4}{4^2}(-1,0)\cdot(-1,0) \\
&+ \mathcal{D}_L\{(0,1)\cdot(0,1) + \tfrac{2z^2}{4}(-1,0)\cdot(0,1) + \tfrac{z^4}{4^2}(-1,0)\cdot(-1,0)\} \\
&= \mathcal{D}_{KP}(0,1)\cdot(0,1) + \tfrac{z^4}{4^2}\mathcal{D}_{KP}(-1,0)\cdot(-1,0) \\
&+ \mathcal{D}_L(0,1)\cdot(0,1) - \tfrac{1}{4^2}\mathcal{D}_L(-1,0)\cdot(-1,0) \\
&+ \tfrac{z^2}{4}\{2\mathcal{D}_{KP}(-1,0)\cdot(0,1) + 2\mathcal{D}_L(-1,0)\cdot(0,1) - \tfrac{1}{4}\mathcal{D}_L(-1,0)\cdot(-1,0)\} \ .
\end{aligned} \qquad (6.5.22)$$

However, $\mathcal{D}_{KP}(0,1)\cdot(0,1)$ and $\mathcal{D}_{KP}(-1,0)\cdot(-1,0)$ are both zero, since (0,1) and (-1,0) are the solutions of the KP hierarchy. Thus from (6.5.22) we then have

$$\begin{aligned}
&\mathcal{D}_L(0,1)\cdot(0,1) - \tfrac{1}{4^2}\mathcal{D}_L(-1,0)\cdot(-1,0) \\
&+ \tfrac{z^2}{4}\{2\mathcal{D}_{KP}(-1,0)\cdot(0,1) + 2\mathcal{D}_L(-1,0)\cdot(0,1) - \tfrac{1}{4}\mathcal{D}_L(-1,0)\cdot(-1,0)\} = 0 \ .
\end{aligned} \qquad (6.5.23)$$

By virtue of the reduction problem of the KdV equation (6.5.17), equation (6.5.23) should give the required identity. However, we found that

$$\mathcal{D}_L(0,1)\cdot(0,1) - \tfrac{1}{4^2}\mathcal{D}_L(-1,0)\cdot(-1,0) = 2c_0 \qquad (6.5.24)$$

and

$$2D_{KP}(-1,0)\cdot(0,1) + 2\mathcal{D}_L(-1,0)\cdot(0,1) - \tfrac{1}{4}\mathcal{D}_L(-1,0)\cdot(-1,0) = 2c_2 \qquad (6.5.25)$$

where c_0 and c_2 are given by (6.5.10) and (6.5.11) respectively. This means that both (6.5.9) and the expected identity (6.5.23) give the same expression. This is the reason why the identities (6.5.12) and (6.5.14) have been constructed independently of (6.5.23) by using relation (6.5.3).

Analytic verification of solution (6.4.15) for a higher number of solitons in the way we did for the two-soliton solution is very complicated. However, simple computer programs may be used for this purpose. Furthermore, it is now possible to use REDUCE programs to solve many problems in solitons. Applications of a REDUCE program to verify the four-soliton solution of the ordinary Boussinesq equation will be discussed in Section 6.7.

6.6 The Bäcklund transformations

The Bäcklund transformations (6.2.3a,b) of the Boussinesq equation (6.2.2) can be rewritten with $x = x_1$ and $t = x_2$ as

$$(D_2+aD_1^2)f'\cdot f = 0 \qquad (6.6.1a)$$

$$(aD_1D_2+D_1+D_1^3)f'\cdot f = 0 \qquad (6.6.1b)$$

where we have put $\lambda = 0$. Choosing $\lambda = 0$ means that we require f' and f to differ only in their phase and not in the number of solitons. We shall show that equations (6.6.1a,b) are satisfied by single-soliton solutions [c.f. (6.4.11)]

$$f' = (0)$$

$$\qquad (6.6.2)$$

$$f = (1) - \frac{z^2}{4}(-1).$$

We first realise that

$$(D_2+D_1^2)(0)\cdot(1) = (D_2+D_1^2)(-1)\cdot(0) = 0 \qquad (6.6.3)$$

due to the property of the first modified KP hierarchy, and then we show that

$$(D_1^3+D_1D_2+D_1)(0)\cdot(1) = (D_1^3+D_1D_2+D_1)(-1)\cdot(0) = 0 . \qquad (6.6.4)$$

We have

$$(D_1^3+D_1D_2+D_1)(0)\cdot(1)$$
$$= (3)(1) - 3(2)(2) + 3(1)(3) - (0)(4)$$
$$+ (3)(1) - (2)(2) - (1)(3) + (0)(4)$$
$$+ (1)(1) - (0)(2)$$
$$= 4(3)(1) - 4(2)(2) + (1)(1) - (0)(2) \; . \qquad (6.6.5)$$

From relation (6.5.3) we find

$$\alpha_1(1) = (2) + \tfrac{1}{4}(0)$$
$$\alpha_1(2) = (3) + \tfrac{1}{4}(1) \; .$$

These relations then yield

$$4(2)(2) = 4(3)(1) + (1)(1) - (0)(2) \; .$$

Using the above identity in (6.6.5) we find that

$$(D_1^3+D_1D_2+D_1)(0)\cdot(1) = 0 \; .$$

The proof of

$$(D_1^3+D_1D_2+D_1)(-1)\cdot(0) = 0$$

is very similar to the above and thus we leave it to the reader. Now, substitution of (6.6.2) into (6.6.1a) yields

$$(D_2+aD_1^2)(0)\cdot\{(1) - \tfrac{z^2}{4}(-1)\}$$
$$= [D_2+(2z^2-1)D_1^2](0)\cdot\{(1) - \tfrac{z^2}{4}(-1)\}$$
$$= (D_2+D_1^2)(0)\cdot(1) + 2(z^2-1)D_1^2(0)\cdot(1)$$
$$- \tfrac{z^2}{4}(D_2-D_1^2)(0)\cdot(-1) - \tfrac{z^4}{2}D_1^2(0)\cdot(-1)$$
$$= (D_2+D_1^2)(0)\cdot(1) + \tfrac{z^2}{4}(D_2+D_1^2)(-1)\cdot(0)$$
$$+ 2(z^2-1)\{D_1^2(0)\cdot(1) - \tfrac{1}{4}D_1^2(-1)\cdot(0)\} \qquad (6.6.6)$$

where we have used the relations $a = 2z^2 - 1$, $D_2(0)\cdot(-1) = -D_2(-1)\cdot(0)$ and $z^4 = z^2 - 1$.

The first two terms in (6.6.6) are zero by (6.6.3). Now

$$D_1^2(0)\cdot(1) - \tfrac{1}{4}D_1^2(-1)\cdot(0)$$

can be shown to be zero by making use of relation (6.5.3), and thus verifying (6.6.3a).

For equation (6.6.3b) we can write

$$(aD_1D_2+D_1+D_1^3)(0)\cdot\{(1) - \frac{z^2}{4}(-1)\}$$

$$= \{(2z^2-1)D_1D_2 + D_1 + D_1^3\}(0)\cdot\{(1) - \frac{z^2}{4}(-1)\}$$

$$= (D_1^3+D_1D_2+D_1)(0)\cdot(1) + 2(z^2-1)D_1D_2(0)\cdot(1)$$

$$- \frac{z^2}{4}(D_1^3-D_1D_2+D_1)(0)\cdot(-1) - 2\frac{z^4}{4}D_1D_2(-1)\cdot(0)$$

$$= (D_1^3+D_1D_2+D_1)(0)\cdot(1) + \frac{z^2}{4}(D_1^3+D_1D_2+D_1)(-1)\cdot(0)$$

$$+ 2(z^2-1)\{D_1D_2(0)\cdot(1) - \tfrac{1}{4}D_1D_2(-1)\cdot(0)\} . \qquad (6.6.7)$$

Again the first two terms in (6.6.7) are zero by (6.6.4).

By using relation (6.5.3) we can also show that

$$D_1D_2(0)\cdot(1) - \tfrac{1}{4}D_1D_2(-1)\cdot(0) = 0$$

and thus we have verified (6.6.3b).

We have therefore seen how the soliton solutions (6.4.15) of the ordinary Boussinesq equation satisfy the equation and its Bäcklund transformations.

6.7 REDUCE programs

REDUCE is a special algebraic programming system produced by Hearn (1984). The capabilities of this sytem include:

(1) Expansion and ordering of polynomials and rational functions;

(2) Substitution and pattern matching in a wide variety of forms;

(3) Automatic and user-controlled simplification of expressions;

(4) Calculations with symbolic matrices;

(5) Arbitrary precision integer and real arithmetic;

(6) Facilities for defining new functions and extending program syntax;

(7) Analytic differentiation and integration;

(8) Factorisation of polynomials;

(9) Dirac matrix calculations of interest to high energy physicists.

Since the above capabilities cover calculations with matrices, facilities for defining new functions, expansion and ordering of functions, REDUCE programs are very suitable for solving soliton problems. Indeed, in the preparation of this chapter, extensive use of such programs has been made. Calculations in Section 6.5 were very laborious in some stages and a REDUCE program was used to keep calculations on the correct path.

Furthermore, a program package using REDUCE has been developed by Ito (1987). This package, called DOP (Differential Operator Package) includes Hirota's derivatives, Wronskian manipulation and evaluation. It is very suitable to use in soliton problems. Subroutines for Hirota's bilinear operators and Wronskians are built in the package. Indeed, we have used this package to verify the three- and four-soliton solutions of the ordinary Boussinesq equation in the new representation (6.4.15). The program used for the four-soliton solution is listed below.

```
% 4-SOLITON SOLUTION FOR BOUSSINESQ EQUATION
WRONSKIAN W;
F:=W(0,1,2,3)+W(-1,0,2,3)*Z**2/4+W(-2,0,1,3)*Z**4/16
  +(W(-3,0,1,2)+W(-2,-1,0,3))*Z**6/64 + W(-3,-1,0,2)*Z**8/256
  + W(-3,-2,0,1)*Z**10/1024 + W(-3,-2,-1,0)*Z**12/4096$

  G:=4096*F$
% THE BOUSSINESQ EQUATION
FOR I:=1:8  DO U(I):=PART(G,I)$
FOR I:=1:8  DO FOR J:=1:8  DO
Y(I,J):=D(U(I),U(J),X1,4)+D(U(I),U(J),X1,2)-D(U(I),U(J),X2,2)$


 P:= FOR I:=1:8 SUM FOR J:=1:8 SUM Y(I,J)$

LET Z**4=Z**2-1$
COEFF(NUM(P),Z,C)$
CC0:=EVALW(C0)$
CC2:=EVALW(C2)$
% PQ = 1/4 REDUCTION
FOR ALL N LET FI#(1,N) = A1*P1**N + B1/(4*P1)**N;
FOR ALL N LET FI#(2,N) = A2*P2**N + B2/(4*P2)**N;
FOR ALL N LET FI#(3,N) = A3*P3**N + B3/(4*P3)**N;
FOR ALL N LET FI#(4,N) = A4*P4**N + B4/(4*P4)**N;
CC0;
CC2;
END;
```

In the above program, we first write the four-soliton solution from (6.4.14) as a polynomial in z. Each term is numbered from 1 to 8 in order to avoid stack overflow when the program is run. The D operator used in the program defines the bilinear operator as

$$D(U,V,X_N,M) = D_N^M \, U \cdot V \ .$$

The expression for the Boussinesq equation is represented by P. Expression P is then separated into two parts by requiring $z^4 = z^2 - 1$ and the coefficient statement COEFF(Num(P),z,C). The actual coefficients of z^0 and z^2: CC0 and CC2 are then calculated after declaring the functions which define the Wronskian with $p_i q_i = \frac{1}{4}$. The value of CC0 and CC2 are both zero and thus verify the four-soliton solution.

CHAPTER 7

CONCLUSION

After introducing the nonlinear evolution equations of fluid mechanics which have been the interest of this thesis, we first derive the Kadomtsev-Petviashvili (KP) equation in dimensionless variables by using a formal scaling procedure. Such a derivation has been carried out previously by a number of authors. It was repeated in this work as an introduction to the derivation of a similar equation, the finite depth fluid (FDF) equation. A suitable scaling procedure has been found for the derivation of the FDF equation which, in our model, described a wave propagation on the interface between two fluid layers of different densities. An advantage of using this scaling procedure is that the continuity of density at the interface did not play any role in the derivation. This should be compared to the derivation of the similar equation by Kubota et al (1978) in terms of physical variables, in which the continuity of density has been taken into account.

The main theme of this thesis has been the study of the soliton solutions of the FDF, the KP, the Classical and the ordinary Boussinesq equations. The soliton solutions to all these equations have already been found using Hirota's direct approach. This is the appropriate method to use when the soliton solutions themselves are to be studied.

The result of Matsuno (1984) was employed to deduce the N-soliton solution of the FDF equation in the form of a finite sum of Wronskian type determinants. The new structure of this equation has also been shown to fit the equation without making use of any algebraic identity. This means that the FDF equation has a similar property to that of the KP equation and not to that of the KdV equation, which requires an identity in the verification of its Wronskian solution [Freeman (1984)]. Since the verification of the solution of the FDF equation does not require any identity, and by virtue of the hierarchies of equations produced by Jimbo and Miwa (1983), this suggests a different hierarchy of equations, of which the FDF equation is one. Such a hierarchy, with the general bilinear equation of the form

$$P(D)f^{+}\cdot f^{-} = 0$$

where f^{-} is a single Wronskian and f^{+} is a finite sum of Wronskian type determinants of the same functions as those for f^{-}, however, has not yet been found. This then requires a further study.

The N-soliton solution of the FDF equation in the new representation has also been shown to reduce to those of the KdV and the Benjamin-Ono equations under specific limiting conditions. Also shown was that the interaction properties of the solution of the FDF equation reduced to those of the KdV and the Benjamin-Ono equations. Indeed, this is in contradiction to the result of Chen and Lee (1979) who concluded that the solution of the FDF equation reduced only to the solution of the KdV equation, but not to the solution of the Benjamin-Ono equation. We note here, to our advantage, that Chen and Lee's conclusion was rejected by Henyey (1980).

The use of Wronskians to represent multisoliton solutions has been made extensively throughout the thesis. The advantage of the Wronskian $|\phi_1,\phi_2,\ldots,\phi_N|$ as being constructed directly from individual soliton solutions $\phi_1,\phi_2,\ldots,\phi_N$ has enabled us to reconstruct the N-soliton solution of the KP equation which resonates in pairs. From the knowledge that two resonating solitons interact together to produce a triad of solitons, we have been able to express the N-soliton solution originally in the form of an ordinary determinant as the Wronskian $|\Phi_1,\Phi_2,\ldots,\Phi_{N/2}|$ for N-even and the Wronskian $|\Phi_1,\Phi_2,\ldots,\Phi_{(N-1)/2},\phi_N|$ for N-odd where Φ_i are the individual triads. Therefore we have been able to study in detail the interactions between triads and solitons. Specifically, we chose to consider interactions between N/2 triads and between (N-1)/2 triads and a soliton.

The asymptotic behaviour of the triads or the soliton can be examined from the Wronskian itself. The phase shifts of the triads and soliton after the interaction can be readily computed.

The detail of the interactions between a triad and a soliton and between two triads has been examined both analytically and numerically. It is interesting to note that the interaction between two triads have been found similar to the interaction between two solitons in one space dimension in the way that the faster triad will overtake the slower triad and they both emerge from the interaction without changing their shape but with only some phase shifts.

The $pq = c$ reduction of the first modified KP hierarchy presented in Chapter 5 was first given by Hirota (1985) for the Classical Boussinesq equations. By use of the Wronskian technique, a complete theory of the N-soliton solutions has been given, including the rational solutions. This therefore gives a good example of how the hierarchies can be used to give solutions of 'reduced' equations of practical importance.

A similar technique applied to the ordinary Boussinesq equation itself, leads to a different separation of the first equation of the KP hierarchy from that used by Nimmo and Freeman (1983) and, earlier, by Hirota and Satsuma (1977). Such a separation, however, leads to a more complex Wronskian representation of the solution. Indeed we have found the N-soliton solution of the ordinary Boussinesq equation as a finite sum of Wronskian type determinants. The solution can be viewed as the polynomial in a parameter z where $z = e^{i\pi/6}$. Since z satisfies the identity relation $z^4 = z^2 - 1$, the polynomial can always be reduced into two terms: z^0 and z^2 terms. It is shown that the two-soliton solution in the new representation satisfies the equation in a similar way to that originally proposed by Nimmo and Freeman (1983). This suggests that the N-soliton solution can also be verified in a similar manner. This requires a further investigation.

APPENDIX A

THE DIRECT METHOD OF HIROTA

Hirota (1971) has developed a direct method to obtain multisoliton solutions of the KdV equation. This method has been successfully applied to a number of nonlinear evolution equations including the KP, the Classical and ordinary Boussinesq and the finite depth fluid equations.

The method involves the substitution of the original dependent variable in the form G/F and requires the nonlinear evolution equation to be written in the form of a bilinear equation. The nonbilinear part is equated to zero so that the relation between F and G is found.

In order to illustrate this method we consider the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{A.1}$$

with boundary conditions $u, u_x, u_{xx} \to 0$ as $x \to \pm\infty$.

Equation (A.1) is first written in terms of w, where $u = w_x$. After one integration it becomes

$$w_t + 3w_x^2 + w_{xxx} = 0 \ . \tag{A.2}$$

Substituting $w = G/F$ into (A.2) gives

$$\begin{aligned}(G_tF-GF_t)/F^2 &+ 3(G_xF-GF_x)^2/F^4 \\ &+ (G_{xxx}F-3G_{xx}F_x-3G_xF_{xx}-GF_{xxx})/F^2 \\ &\quad + 6(FG_xF_x^2+FGF_xF_{xx}-GF_x^3)/F^4 = 0 \ .\end{aligned} \tag{A.3}$$

We notice that the resulting equation (A.3) looks more complicated then the original equation (A.2) or (A.1). However, the terms can be rearranged as follows:

$$\begin{aligned}&[G_tF-GF_t+G_{xxx}F-3G_{xx}F_x+3G_xF_{xx}-GF_{xxx}]/F^2 \\ &\quad + 3(G_xF-GF_x)[G_xF-GF_x-2(FF_{xx}-F_x^2)]/F^4 = 0 \ .\end{aligned}$$

This equation can be decoupled into

$$G_tF - GF_t + G_{xxx}F - 3G_{xx}F_x + 3G_xF_{xx} - GF_{xxx} = 0 \tag{A.4}$$

$$G_xF - GF_x - 2(FF_{xx}-F_x^2) = 0 . \tag{A.5}$$

From (A.5) we can deduce the relation between G and F,

$$\left(\frac{G}{F}\right)_x = 2\left(\frac{F_x}{F}\right)_x$$

which means

$$u = 2\frac{\partial^2}{\partial x^2}(\log F) \tag{A.6}$$

and

$$G = F_x . \tag{A.7}$$

Using (A.7) into (A.4), it can be written in the form

$$(D_t+D_x^3)F_x\cdot F = 0 \tag{A.8}$$

where the bilinear operators D_t and D_x^3 have been introduced in (1.18). However (A.8) is more conveniently written in the form

$$D_x(D_t+D_x^3)F\cdot F = 0 \tag{A.9}$$

where we have made use of the relations

$$2D_tF_x\cdot F = D_xD_tF\cdot F$$

and

$$2D_x^{m-1}F_x\cdot F = D_x^mF\cdot F \text{ , m even .}$$

Essentially what we have been doing in the above is to transform the KdV equation (A.1) into the bilinear form (A.9) by using the Cole-Hopf-like transformation (A.6).

The solution to the bilinear equation (A.9) is obtained by expanding F as a power series in a small parameter ϵ

$$F = 1 + \epsilon F_1 + \epsilon^2F_2 + \dots . \tag{A.10}$$

Substituting (A.10) into (A.9) and collecting terms of the same powers in ϵ we find

$$2\frac{\partial}{\partial x}\left[\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right]F_1 = 0 \tag{A.11}$$

$$2\frac{\partial}{\partial x}\left[\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right]F_2 = -D_x(D_t+D_x^3)F_1\cdot F_1 \tag{A.12}$$

$$2\frac{\partial}{\partial x}\left[\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right]F_x = -D_x(D_t+D_x^3)(F_2\cdot F_1+F_1\cdot F_2)\ \ldots \tag{A.13}$$

and so on.

The single-soliton solution is obtained by taking

$$F_1 = a\ \exp(kx - k^3t)\ ,$$

and

$$F_2 = F_3 = \ldots = 0\ .$$

Therefore

$$F = 1 + \exp(kx- k^3t+\delta)$$

where the parameter ϵ has been absorbed into δ. Using (A.6) the solution is

$$u = \frac{1}{2}k^2\ \text{sech}^2\ \frac{1}{2}(kx-k^3t+\delta)\ .$$

For the two-soliton solution we choose $F_1 = a_1 \exp \eta_1 + a_2 \exp \eta_2$ where

$$\eta_i = k_ix - k_i^3t\ .$$

It is interesting to note that with this choice we find from (A.12)

$$F_2 = \left[\frac{k_1-k_2}{k_1+k_2}\right]^2 \exp(\eta_1+\eta_2)$$

and that we can always choose $F_3 = F_4 = \ldots = 0$ to terminate the series (A.10). This means that we obtain an exact solution. Therefore for the two-soliton solution we have

$$F = 1 + e^{\theta_1} + e^{\theta_2} + \left[\frac{k_1-k_2}{k_1+k_2}\right]^2 e^{\theta_1+\theta_2}$$

where $\theta_i = \eta_i + \delta_i$.

Similarly, the N-soliton solution is obtained by taking

$$F_1 = \sum_{i=1}^{N} a_i\ \exp(\eta_i)\ ,$$

and $F_{N+1} = F_{N+2} = \ldots = 0$. The form of F for the N-soliton solution is expressed as a sum of exponentials [Hirota (1980)]

$$F = \sum_{\mu=0,1} \exp\left[\sum_{i>j}^{(N)} A_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \mu_i \theta_i \right] \qquad (A.14)$$

where

$$\exp(A_{ij}) = - \frac{(k_i-k_j)[k_i^3-k_j^3-(k_i-k_j)^3]}{(k_i+k_j)[k_i^3+k_j^3-(k_i+k_j)^3]} ,$$

$\sum_{\mu=0,1}$ is the summation over all possible combinations of $\mu_1 = 0,1$, $\mu_2 = 0,1$, ..., $\mu_N = 0,1$ and $\sum_{i>j}^{(N)}$ the summation over all possible pairs, with $i > j$ chosen from N elements.

APPENDIX B

THE WRONSKIAN SOLUTIONS

The N-soliton solutions of the KdV and the modified KdV equations have been expressed by Satsuma (1979) in the form of the Wronskians of some functions by considering the inverse scattering schemes for the two equations. The structure of the solutions in this form can also be deduced directly from the determinantal form of the solutions obtained from the inverse scattering transform. This has been shown by Freeman (1984) and the outline of the proof is given below.

For this purpose we take the KP equation in the form

$$(u_t+6uu_x+u_{xxx})_x + 3u_{yy} = 0 . \tag{B.1}$$

The N-soliton solution to this equation, obtained from the inverse scattering transform, can be written as [Lamb (1980)]

$$u = 2\frac{\partial^2}{\partial x^2}(\log F) \tag{B.2}$$

with

$$F = \left| \delta_{ij} + \frac{a_i}{\ell_i+n_j} \exp(\theta_i+\psi_j) \right| \tag{B.3}$$

where

$$\theta_i = \ell_i x - \ell_i^2 y - 4\ell_i^3 t$$

$$\psi_j = n_i x + n_i^2 y - 4n_i^3 t$$

with ℓ_i, n_i, a_i real constants, $i = 1, 2, \ldots, N$.

If matrices M, A, D_1 and D_2 are defined as

$$M = \left[\frac{1}{\ell_i+n_j}\right] , \qquad A = \left[\delta_{ij}a_i\right]$$

$$D_1 = \left[\delta_{ij} \exp \theta_i\right] , \; D_2 = \left[\delta_{ij} \exp \psi_i\right]$$

then (B.3) can be written as

$$F = |I + AD_1MD_2| . \tag{B.4}$$

The determinantal form of F in (B.4) is transformed into a Wronskian form by making use of the following matrices

$$P = \left[\delta_{ij} \prod_{p\neq j}^{N} (\ell_p - \ell_j)\right]$$

$$Q = \left[\delta_{ij}(-1)^{i-1} \prod_{p=1}^{N} (\ell_p + n_i)\right]$$

$$V = \left[(-\ell_j)^{i-1}\right]$$

$$W = \left[(-1)^{j-1} n_j^{i-1}\right]$$

where V and W are the Van der Monde-type matrices.

The matrix M is related to the above matrices by

$$V^{-1}W = P^{-1}MQ$$

where V^{-1} has been obtained by making use of the theory of symmetric functions.

From this relation, M is expressed as

$$M = PV^{-1}WQ^{-1} \ .$$

By using the last relation in (B.4) we find

$$\begin{aligned} F &= |I + AD_1PV^{-1}WQ^{-1}D_2| \\ &= |AD_1PV^{-1}|\,|VP^{-1}D_1^{-1}A^{-1}+WQ^{-1}D_2| \ . \end{aligned}$$

The factor $|AD_1PV^{-1}|$ in the above expression takes the form $C \exp\left(\sum_{i=1}^{N} \ell_i x\right)$ where C is a function of t only. By virtue of the final solution u expressed by (B.2) this factor may therefore be ignored and thus

$$F = |VP^{-1}D_1^{-1}A^{-1}+WQ^{-1}D_2| \ . \qquad \text{(B.5)}$$

A close look at the determinant (B.5) shows that

$$F_{(i+1)j} = \frac{\partial}{\partial x} F_{ij}$$

where $F = |\{F_{ij}\}|$. Therefore (B.5) is the Wronskian

$$F = \begin{vmatrix} \Phi_1 & \Phi_2 & \cdots\cdots & \Phi_N \\ \dfrac{\partial\Phi_1}{\partial x} & \dfrac{\partial\Phi_2}{\partial x} & & \dfrac{\partial\Phi_N}{\partial x} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial^{N-1}\Phi_1}{\partial x^{N-1}} & \dfrac{\partial^{N-1}\Phi_2}{\partial x^{N-1}} & \cdots\cdots & \dfrac{\partial^{N-1}\Phi_N}{\partial x^{N-1}} \end{vmatrix} \qquad \text{(B.6)}$$

with

$$\Phi_j = e^{-(\ell_j x-\ell_j^2 y-4\ell_j^3 t+\delta_j)} + e^{n_j x+n_j^2 y-4n_j^3 t-\gamma_j} \qquad (B.7)$$

where

$$\delta_j = \log\{a_j \prod_{p\neq j}^{N} (\ell_p-\ell_j)\}$$

and

$$\gamma_j = \log\{\prod_{p=1}^{N} (\ell_p+n_j)\} .$$

The Wronskian (B.6) is more conveniently written in its transposed form in which the columns represent the orders of differentiation of increasing degree and thus it can be denoted by

$$F = (0,1,2,\ldots,N-1) . \qquad (B.8)$$

The notation in (B.8) can be made more compact still by writing

$$F = (\widehat{N-1}) \qquad (B.9)$$

where ^ indicates the Wronskian which begins with order zero and ends up with order N-1.

In verifying soliton solutions we need to differentiate the function F. Differentiation of a Wronskian is much simpler than differentiation of an ordinary determinant since differentiation of a column may produce two identical columns and hence gives no contribution. For example, we have

$$\begin{aligned} F_x = &\left|\frac{\partial\Phi_i}{\partial x}, \frac{\partial\Phi_i}{\partial x}, \frac{\partial^2\Phi_i}{\partial x^2}, \cdots \frac{\partial^{N-1}\Phi_i}{\partial x^{N-1}}\right| \\ &+\left|\Phi_i, \frac{\partial^2\Phi_i}{\partial x^2}, \frac{\partial^2\Phi_i}{\partial x^2}, \cdots \frac{\partial^{N-1}\Phi_i}{\partial x^{N-1}}\right| \\ &\vdots \\ &+ \\ &\vdots \\ &+\left|\Phi_i, \frac{\partial\Phi_i}{\partial x}, \cdots \frac{\partial^{N-3}\Phi_i}{\partial x^{N-3}}, \frac{\partial^{N-1}\Phi_i}{\partial x^{N-1}}, \frac{\partial^{N-1}\Phi_i}{\partial x^{N-1}}\right| \\ &+\left|\Phi_i, \frac{\partial\Phi_i}{\partial x}, \cdots \frac{\partial^{N-2}\Phi_i}{\partial x^{N-2}}, \frac{\partial^{N}\Phi_i}{\partial x^{N}}\right| . \end{aligned}$$

This means that the first N-1 determinants are zero since they contain two identical columns. Thus in the notation of (B.9) we simply have

$$F_x = (\widehat{N-2},N) .$$

What we are doing in here is simply shifting up the order of every column by one and collecting the ones which do not have two identical columns. Using this we can find

$$F_{2x} = (\widehat{N-2},N+1) + (\widehat{N-3},N-1,N)$$

$$F_{3x} = (\widehat{N-2},N+2) + 2(\widehat{N-3},N-1,N-1) + (\widehat{N-4},N-2,N-1,N+1)$$

and so on.

Differentiation of F with respect to other variables such as t and y for the KP equation follows the same pattern. Normally, differentiation with respect to other variables is directly related to the differentiation with respect to x. Indeed, this can be obtained from the functions which define the Wronskians. For the KP equation we find from (B.7) that

$$\frac{\partial \Phi_i}{\partial t} = -4 \frac{\partial^3 \Phi_i}{\partial x^3} , \quad \frac{\partial \Phi_i}{\partial y} = \frac{\partial^2 \Phi_i}{\partial x^2} .$$

Therefore to differentiate F with respect to t we first multiply the columns by -4 and shift up the order of that column by three. In this way we find

$$\begin{aligned} F_t &= -4\{(\widehat{N-2},N+2) + (\widehat{N-3},N+1,N-2)\} \\ &= -4\{(\widehat{N-2},N+2) - (\widehat{N-3},N-1,N+1)\} . \end{aligned}$$

Similarly, we may also obtain

$$\begin{aligned} F_y &= (\widehat{N-2},N+1) + (\widehat{N-3},N,N-1) \\ &= (\widehat{N-2},N+1) - (\widehat{N-3},N-1,N) . \end{aligned}$$

It should be noted that the number of determinants obtained in all the above derivatives does not depend on the size of the Wronskian F. It depends only on the order of the derivatives.

Now, substituting the solution (B.2) into the KP equation (B.1), we find the bilinear equation

$$F_{xt}F - F_xF_t + FF_{4x} - 4F_xF_{3x} + 3F_{xx}^2 + 3(FF_{2y}-F_y^2) = 0 \quad . \qquad (B.10)$$

In order to verify the Wronskian solution (B.9) for the KP equation (B.1) we need to show that (B.9) satisfies (B.10). Substituting (B.9) and all its necessary derivatives into (B.10) we find the expression on the left side of (B.10) gives

$$6\{(\widehat{N-1})(\widehat{N-3},N,N+1) - (\widehat{N-2},N)(\widehat{N-3},N-1,N+1) + (\widehat{N-2},N+1)(\widehat{N-3},N-1,N)\} \ , \qquad (B.11)$$

which must be shown to be zero. The appropriate method to do this is to make use of the Laplace expansion of a determinant which can be described as follows [Aitken (1954)].

Let us choose the first m rows $r_1, r_2, \ldots, r_m$ of a determinant A of size N × N. From these m rows and from all possible combinations of m different columns we can form n minors A_i, $i = 1,2, \ldots, n$ where

$$n = \frac{N!}{(N-m)!m!} \ .$$

Associated with minor A_i is its complementary minor or cofactor A_i' formed from the remaining N-m rows and N-m columns. The determinant A is then written as

$$A = \sum_{i=1}^{n} (-1)^{S_i} A_i A_i' \qquad (B.12)$$

where

$$S_i = r_1 + r_2 + \ldots + r_m + c_{i1} + c_{i2} + \ldots + c_{im}$$

with $c_{i1}, c_{i2}, \ldots, c_{im}$ the original column numbers used to form the minor A_i.

As a special case we consider a determinant A of size 2N × 2N in the form

$$\begin{vmatrix} D & \cdot & a & b & c & d \\ \cdot & D & a & b & c & d \end{vmatrix} \qquad (B.13)$$

where D is an N × (N-2) matrix, the dot $\cdot$ is the zero matrix, and a,b,c,d the column matrices. By choosing m = N, most of the minors give zero contribution due to the zero matrices in the first and second N rows and we are left with only six non-zero minors. Thus, the Laplace expansion (B.12) yields

$$A = |Dab||Dad| - |Dac||Dbd| + |Dad||Dbc|$$

$$+ |Dbc||Dad| - |Dbd||Dac| + |Dcd||Dab|$$

$$= 2\{|Dab||Dcd| - |Dac||Dbd| + |Dad||Dbc|\} \ . \qquad (B.14)$$

If we put $D = (\widehat{N-1})$, $a = (N-2)$, $b = (N-1)$, $c = (N)$ and $d = (N+1)$ in (B.14) we can see that (B.11) is

$$3\begin{vmatrix} \widehat{N-1} & \cdot & N-2 & N-1 & N & N+1 \\ \cdot & \widehat{N-1} & N-2 & N-1 & N & N+1 \end{vmatrix}$$

which can be shown to be zero by elementary row and column subtraction method. Hence we have verified that the Wronskian solution (B.9) satisfies the KP equation.

APPENDIX C

The Hierarchies

P(D)

degree 4	$D_1^4 - 4D_1D_3 + 3D_2^2$
degree 5	$(D_1^3 + 2D_3)D_2 - 3D_1D_4$
degree 6	$D_1^6 - 20D_1^3D_3 - 80D_3^2 + 144D_1D_5 - 45D_1^2D_2^2$
	$D_1^6 + 4D_1^3D_3 - 32D_3^2 - 9D_1^2D_2^2 + 36D_2D_4$
degree 7	$(D_1^5 + 10D_1^2D_3 + 24D_5)D_2 + 5D_1^3D_4 - 40D_1D_6$
	$D_1D_2^3 + (D_1^3 + 2D_3)D_4 - 4D_1D_6$
	$D_1^2D_3D_2 + D_3D_4 - 2D_1D_6$
degree 8	$D_1^8 + 14D_1^5D_3 + 84D_1^3D_5 - 504D_3D_5 - 120D_1D_7 - 105D_1^2D_2D_4 + 210D_4^2 + 420D_2D_6$
	$-2D_1^2D_3^2 + 4D_1^3D_5 + 4D_3D_5 - 12D_1D_7 + D_1^4D_2^2 - 9D_4^2 + 14D_2D_6$
	$-6D_1^2D_3^2 + 4D_1^3D_5 - 4D_3D_5 + 12D_1D_7 + D_2^4 - 6D_1^2D_2D_4 - 3D_4^2 + 2D_2D_6$
	$D_1^5D_3 - 16D_3D_5 - 5D_1D_3D_2^2 + 20D_2D_6$
	$2D_1^2D_3^2 + 4D_3D_5 - 12D_1D_7 + 2D_1D_3D_2^2 + 3D_1^2D_2D_4 + 3D_4^2 - 2D_2D_6$
degree 9	$(D_1^7 + 35D_1D_3^2 - 21D_1^2D_5 + 90D_7)D_2 + 105D_4D_5 + (35D_1^3 - 140D_3)D_6 - 105D_1D_8$
	$D_1^3D_2^3 + 9D_1D_2^2D_4 + 6D_1^2D_3D_4 + (4D_1^3 + 16D_3)D_6 - 36D_1D_8$
	$(3D_1^4D_3 + 12D_1^2D_5 + 48D_7)D_2 + (-21D_1^2D_3 - 36D_5)D_4 - 24D_1D_2^2D_4 + (2D_1^3 - 8D_3)D_6 + 24D_1D_8$
	$-3D_1^2D_3D_4 + D_3D_2^3 + 2D_1^3D_6$
	$(3D_1D_3^2 + 3D_1^2D_5 - 6D_7)D_2 + 3D_1D_4D_2^2 + (6D_1^2D_3 + 9D_5)D_4 + (-D_1^3 + 4D_3)D_6 - 21D_1D_8$
	$(D_1^5 + 20D_1^2D_3 + 22D_5)D_4 + 15D_1D_2^2D_4 + 2D_3D_6 - 60D_1D_8$

C.1 The KP hierarchy: P(D)F.F = 0 after Jimbo and Miwa (1983). The N-soliton solutions of the equations under this hierarchy take the Wronskian form F = $(\widehat{N-1})$ of the functions

$$\phi_i = a_i \exp\left(\sum_{n=1}^{\infty} p_i^n x_n\right) + b_i \exp\left(\sum_{n=1}^{\infty} q_i^n x_n\right),$$

for i = 1, 2, ..., N with p_i, q_i arbitrary solution parameters.

P(D)

degree 2	$D_1^2 + D_2$
degree 3	$D_1^3 - 4D_3 - 3D_1D_2$
degree 4	$D_1^4 + 8D_1D_3 + 3D_1^2D_2 - 6D_2^2$
	$-D_1^2D_2 + D_2^2 + 2D_4$
degree 5	$D_1^5 - 16D_5 + 5D_1D_2^2 - 10D_1D_4$
	$(D_1^3 - 4D_3)D_2 + 3D_1D_2^2 + 6D_1D_4$
	$D_1^5 - 4D_1^2D_3 + 3D_1^3D_2 + 6D_1D_2^2$
degree 6	$D_1^6 - 20D_1^3D_3 - 80D_3^2 + 144D_1D_5 + (-15D_1^4 + 60D_1D_3)D_2$
	$(-D_1^4 - 8D_1D_3)D_2 - 3D_1^2D_2^2 + 6D_1^2D_4$
	$-D_1^6 + 16D_1^3D_3 + 3D_1^4D_2 + 12D_2^3$
	$D_1^3D_3 + 2D_3^2 - 3D_1D_2D_3 + 6D_6$
	$3D_1^6 + 192D_1D_5 + (-35D_1^4 - 160D_1D_3)D_2 - 90D_1^2D_2^2 + 180D_1^2D_4 - 120D_2D_4$
degree 7	$11D_1^7 - 70D_1^4D_3 - 336D_1^2D_5 + 560D_1D_3^2 - 480D_7 + (-7D_1^5 + 490D_1^2D_3 - 168D_5)D_2 + (210D_1^3 + 420D_3)D_2^2$
	$3D_1^7 + 112D_1^2D_5 - 640D_7 + (14D_1^5 - 224D_5)D_2 + 35D_1^3D_2^2 - 70D_1^3D_4 - 70D_1D_2^3 + 140D_1D_2D_4$
	$5D_1^4D_3 - 120D_1^2D_5 + 40D_1D_3^2 + 240D_7 + (D_1^5 + 35D_1^2D_3 + 84D_5)D_2 + (-5D_1^3 - 10D_3)D_2^2 + 30D_1D_2^3$
	$(-3D_1^5 + 48D_5)D_2 + (-5D_1^3 + 20D_3)D_2^2 + (10D_1^3 - 40D_3)D_4$
	$D_1^7 - 280D_1D_3^2 + 294D_1^2D_5 - 120D_7 + (7D_1^5 - 70D_1^2D_3 - 42D_5)D_2 + (-35D_1^3 - 70D_3)D_2^2 - 105D_1D_2^3$
	$3D_1^4D_3 - 56D_1^2D_5 + 24D_1D_3^2 + 80D_7 + (17D_1^2D_3 + 28D_5)D_2 + 2D_3D_2^2 + 2D_1^3D_4 + 14D_1D_2^3 + 8D_1D_2D_4$
	$-D_1^2D_5 + 4D_7 + (D_1^2D_3 - D_5)D_2 + D_3D_2^2 + D_1^3D_4 + D_1D_2D_4 + 4D_1D_6$

C.2 The First Modified KP hierarchy: P(D)F.F' = 0 after Jimbo and Miwa (1983). The N-soliton solutions of the equations under this hierarchy take the Wronskian form F = $(\widehat{N-1})$, F' = $(\tilde{N})$. The functionswhich define these Wronskians are the same as those for the KP hierarchy.

REFERENCES

Ablowitz, M.J., Kaup, D.J., Newell, A.C. and Segur, H. 1974. The inverse scattering transform - Fourier series for nonlinear problems. Stud.Appl.Math. 53, 249-315.

Abramowitz, M. and Stegun, I.A. 1965. Handbook of Mathematical Functions. New York: Dover.

Aitken, A.C. 1954. Determinants and Matrices. Edinburgh, London: Oliver and Boyd.

Anker, D. and Freeman, N.C. 1978. Interpretation of the three-soliton interactions in terms of resonant triads. J.Fluid Mech. 87, 17-31.

Benjamin, T.B. 1967. Internal waves of permanent form in fluids of great depth. J.Fluid Mech. 29, 559-591.

Boussinesq, J. 1872. Theorie des ondes et des remous qui se propagent le long d'un canal horizontal, en communiquant an liquide contenu dan ce canal des vitesses sensiblement pareilles de la surface au fond. J.Math.Pures Appl. 7, 55-108.

Bullough, R.K. and Caudrey, P.J. 1980. The soliton and its history. In: Bullough, R.K. and Caudrey, P.J. (eds.), Solitons: Topics in Current Physics 17. Berlin: Springer.

Burgers, J.M. 1948. A mathematical model illustrating the theory of turbulence. Adv.Appl.Mech. 1, 171-199.

Calogero, F. and Degasperis, A. 1982. Spectral Transforms and Solitons I. Amsterdam: North Holland.

Caudrey, P.J. 1980. The inverse problem for the third order equation $u_{xxx} + q(x)u_x + r(x)u = -i\zeta^3 u$. Phys.Lett. 79A, 264-268.

Adler, M. and Moser, J. 1978. On a class of polynomials connected with the Korteweg-de Vries equation. Comm. Math. Phys. 61, 1 - 30.

Chen, H.H. and Lee, Y.C. 1979. Internal-wave solitons of fluids with finite depth. Phys.Rev.Lett. 43, 264-266.

Cole, J.D. 1951. On a quasilinear parabolic equation occurring in aerodynamics. Q.Appl.Math. 9, 225-236.

Davis, R.E. and Acrivos, A. 1967. Solitary internal waves in deep water. J.Fluid Mech. 29, 593-607.

Dodd, R.K., Eilbeck, J.C., Gibbon, J.Q. and Morris, H.C. 1982. Solitons and Nonlinear Wave Equations. London, New York: Academic Press.

Freeman, N.C. 1980. Soliton interactions in two dimensions. Adv.Appl.Mech. 20, 1-37.

Freeman, N.C. 1984. Soliton solutions of non-linear evolution equations. IMA J.Appl.Math. 32, 125-145.

Freeman, N.C. and Johnson, R.S. 1970. Shallow water waves on shear flow. J.Fluid Mech. 42, 401-409.

Freeman, N.C., Horrocks, G. and Wilkinson, P. 1981. Backlund transformation applied to the cylindrical Korteweg-de Vries equation. Phys.Lett. 81A, 305-309.

Freeman, N.C. and Nimmo, J.J.C. 1983. Soliton solutions of the Korteweg-de Vries and the Kadomtsev-Petviashvili equations: the Wronskian technique. Proc.R.Soc.London A389, 319-329.

Gardner, C.S, Greene, J.M., Kruskal, M.D. and Miura, R.M. 1967. Method for solving the Korteweg-de Vries equation. Phys.Rev.Lett. 19, 1095-1097.

Gardner, C.S., Greene, J.M., Kruskal, M.D. and Miura, R.M. 1974. Korteweg-de Vries equation and generalizations VI. Methods of exact solution. Comm.Pure Appl.Math. 27, 197-233.

Hearn, A.C. 1984. REDUCE User's Manual - Version 3.1. Santa Monica: The Rand Corporation.

Hirose, A. and Lonngren, K.E. 1985. Introduction to Wave Phenomena. New York: John Wiley.

Henyey, S.F. 1980. Finite-depth and infinite-depth internal wave solitons. Phys.Rev.A 21, 1054-1056.

Hirota, R. 1971. Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons. Phys.Rev.Lett. 27, 1192-1194.

Hirota, R. 1973a. Exact envelope-soliton solutions of a nonlinear wave equation. J.Math.Phys. 14, 805-809.

Hirota, R. 1973b. Exact N-soliton solutions of the wave equation of long waves in shallow water and in nonlinear lattices. J.Math.Phys. 14, 810-814.

Hirota, R. 1980. Direct methods in soliton theory. In: Bullough, R.K. and Caudrey, P.J. (eds.). Solitons: Tropics in Current Physics 17. Berlin: Springer.

Hirota, R. 1985. Classical Boussinesq equation is a reduction of the modified KP hierarchy. J.Phys.Soc.Japan 54, 2409-2415.

Hirota, R. 1986a. Solutions of the Classical Boussinesq equations: the Wronskian technique. J.Phys.Soc.Japan 55, 2137-2150.

Hirota, R. 1986b. Reduction of soliton equations in bilinear form. Physica D 18, 161-170.

Hirota, R. and Satsuma, J. 1977. Nonlinear evolution equations generated from the Backlund transformation for the Boussinesq equation. Prog.Theor.Phys. 57, 797-807.

Hopf, E. 1950. The partial differential equation $u_t + uu_x = \mu u_{xx}$. Comm.Pure Appl.Math. 3, 201-230.

Ito, M. 1987. Manual for REDUCE Programs - Version 1.2. To be published.

Jimbo, M. and Miwa, T. 1983. Solitons and infinite dimensional Lie algebras. Publ.RIMS. Kyoto University 19, 943-1001.

Johnson, R.S. 1980. Water waves and Korteweg-de Vries equations. J.Fluid Mech. 97, 701-719.

Johnson, R.S. 1983. On the phase-shifts due to the interaction of a large and a small solitary wave. Phys.Rev.Lett. 94A, 7-11.

Joseph, R.I. 1977. Solitary waves in a finite depth fluid. J.Phys.A: Math.Gen. 10, L225-L227.

Joseph, R.I. and Egri, R. 1978. Multisoliton solutions in a finite depth fluid. J.Phys.A: Math.Gen. 11, L97-L102.

Kadomtsev, B.B. and Petviashvili, V.I. 1970. On the stability of solitary waves in weakly dispersive media. Sov.Phys.Dokl. 15, 539-541.

Kako, F. and Yajima, N. 1980. Interactions of ion-acoustic solitons in two-dimensional space. J.Phys.Soc.Japan 49, 2063-2071.

Kaup, D.J. 1980. On the inverse scattering problem for cubic eigenvalue problems of the class $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$. Stud.Appl.Math. 62, 189-216.

Kawamoto, S. 1984. Linearization of the Classical Boussinesq and related equations. J.Phys.Soc.Japan 53, 2922-2929.

Korteweg, E.V. and de-Vries, G. 1895. On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves. Phil.Mag. 39, 422-443.

Krishnan, E.V. 1982. An exact solution of the Classical Boussinesq equation. J.Phys.Soc.Japan 51, 2391-2392.

Kubota, T., Ko, D.R.S. and Dobbs, L.D. 1978. Weakly-nonlinear, long internal gravity waves in stratified fluids of finite depth. J.Hydronautics 12, 157-165.

Lamb, G.L. Jr. 1980. Elements of Soliton Theory. New York, Chichester, Brisbane, Toronto: John Wiley.

Matsuno, Y. 1979a. Exact multisoliton solution of the Benjamin-Ono equation. J.Phys.A: Math.Gen.12, 619-621.

Matsuno, Y. 1979b. Exact multisoliton solution for nonlinear waves in a stratified fluid of finite depth. Phys.Lett. 74A, 233-235.

Matsuno, Y. 1984. The Bilinear Transformation Method. Orlando, San Diego, New York, London, Montreal, Sydney, Tokyo: Academic Press.

Miles, J.W. 1977. Resonantly interacting solitary waves. J.Fluid Mech. 79, 171-179.

Nakamura, A. and Hirota, R. 1985. A new example of explode-decay solitary waves in one-dimension. J.Phys.Soc.Japan 54, 491-499.

Newell, A.C. and Redekopp, L.G. 1977. Breakdown of Zakharov-Shabat theory and soliton creation. Phys.Rev.Lett. 38, 377-380.

Nimmo, J.J.C. and Freeman, N.C. 1983. A method of obtaining the N-soliton solution of the Boussinesq equation in terms of a Wronskian. Phys.Lett. 95A, 4-6.

Novikov, S., Manakov, S.V., Pitaevskii and Zakharov, V.E. 1984. Theory of Solitons. New York, London: Consultants Bureau.

Ohkuma, K. and Wadati, M. 1983. The Kadomtsev-Petviashvili equation: The trace method and the soliton resonances. J.Phys.Soc.Japan 52, 749-760.

Ono, H. 1975. Algebraic solitary waves in stratified fluids. J.Phys.Soc.Japan 39, 1082-1091.

Phillips, O.M. 1966. The Dynamics of the Upper Ocean. Cambridge: Cambridge University Press.

Satsuma, J. 1976. N-soliton solution of the two-dimensional Korteweg-de Vries equation. J.Phys.Soc.Japan 40, 286-290.

Satsuma, J. 1979. A Wronskian representation of N-soliton solutions of nonlinear evolution equations. J.Phys.Soc.Japan 46, 359-360.

Satsuma, J., Ablowitz, M.J. and Kodama, Y. 1979. On an internal wave equation describing a stratified fluid with finite depth. Phys.Lett. 73A, 283-286.

Scott-Russell, J. 1844. Report on waves. In: Report of the 14th Meeting of the British Association for the Advancement of Science. London: John Murray.

Scott, A.C., Chu, F.Y.F. and McLaughlin, D. 1973. The soliton: a new concept in applied science. Proc.IEEE 61, 1443-1483.

Thompson, S. 1980. Ph.D. thesis. University of Newcastle upon Tyne.

Williams, J.R. 1974. M.Sc. dissertation. University of Newcastle upon Tyne.

Whitham, G.B. 1967. Variational methods and applications to water waves. Proc.R.Soc.London A299, 6-25.

Whitham, G.B. 1974. Linear and Nonlinear Waves. New York, London, Sydney, Toronto: John Wiley.

Zabusky, N.J. 1967. A synergetic approach to problems of nonlinear dispersive wave propagation and interaction. In: Ames, W. (ed.). Nonlinear Partial Differential Equations. New York: Academic Press.

Zabusky, N.J. and Kruskal, M.D. 1965. Interactions of 'solitons' in a collisionless plasma and the recurrence of initial states. Phys.Rev.Lett. 15, 240-243.

Zakharov, V.E. 1974. On stochastization of one-dimensional chains of nonlinear oscillators. Sov.Phys.JETP 38, 108-110.

Zakharov, V.E. and Shabat, A.B. 1974. Scheme for integrating the nonlinear equations of mathematical physics by the method of inverse scattering problem I. Funct.Anal.Appl. 8, 226-235.