The Structure of C*-Algebras of Product Systems

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Abstract

A prevalent trend in the theory of operator algebras is the study of geometric/topological structures via bounded linear operators on a Hilbert space. The goal is to establish a rigid correspondence between such a structure and a C*-algebra, and use the rich theory of the latter to study the former. This approach has been met with much success in recent years, revealing surprising links with quantum mechanics, graphs, groups, dynamics, subshifts and more. Initially these applications were studied individually; however, the introduction of C*-correspondences and product systems within the past thirty years has presented a unifying framework. Broadly speaking, C*-correspondences and their C*-algebras account for low-rank examples (e.g., directed graphs) and are by now well explored. The more general product systems and their C*-algebras account for higher-rank examples (e.g., higher-rank graphs) and less is known in this context. In turn, there is motivation to analyse the structure of C*-algebras of product systems and interpret the results with respect to the applications that these objects encompass.

The current work falls within the remit of this programme, and focuses on the gauge-invariant ideal structure of C*-algebras associated with the subclass of strong compactly aligned product systems. We parametrise the gauge-invariant ideals of every equivariant quotient of the Toeplitz-Nica-Pimsner algebra (most importantly the Cuntz-Nica-Pimsner algebra) via tuples of ideals of the coefficient algebra. We describe the conditions defining these families via product system operations alone. In the process, we prove a Gauge-Invariant Uniqueness Theorem. We characterise the lattice operations on the parametrising families such that the bijection is a lattice isomorphism. We then interpret the main result in the settings of regular product systems, C*-dynamical systems, higher-rank graphs and product systems on finite frames. We close by examining the case of proper product systems in further detail.

I dedicate this thesis to my mother Rosemary, my father John, my sister Nina, my aunt Anna, my granddad Mario and my uncle Len; for making it all worthwhile.

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Contents

Chapter 1

Introduction

1.1 Background

A prominent feature of the theory of operator algebras is the quantisation procedure by which a geometric/topological object can be studied via bounded linear operators on a Hilbert space. The goal is to associate such an object with a C*-algebra in a rigid way, such that properties of the original structure are reflected by properties of the C*-algebra (and vice versa). In this way, the powerful and well-developed theory of C*-algebras can be brought to bear on the study of other mathematical structures. In recent years, there has been interest in encoding this procedure in a uniform way, i.e., accounting for a multitude of examples via a single framework.

A contemporary tool in this endeavour is that of product systems, whose associated C*-algebras account for a vast array of C*-constructions associated with a unital subsemigroup P of a discrete group G. Structures encompassed by this language include (but are not limited to) C*-dynamical systems, higher-rank graphs and subshifts. A pertinent feature of product systems is their ability to encode transformations that may not be reversible, and as such the associated C*-algebras provide an ample source of examples and counterexamples. In turn, there is motivation to analyse the structure of these C*-algebras, and interpret the results with respect to the applications that the product system construction affords. Much progress has been made in this direction in the case of $P = \mathbb{Z}_+$; however, the situation changes when we consider more general semigroups. There are many open questions even in the case of $P = \mathbb{Z}_+^d$.

The case of $P = \mathbb{Z}_+$ is the case of a single C*-correspondence X, the study of whose C*-algebras was initiated by Pimsner [48]. The quantisation is implemented via a Fock space construction, in which the elements of X are treated as left creation operators. These operators, together with the coefficient algebra of X (suitably viewed as a family of operators itself), give rise to the Toeplitz-Pimsner algebra $\mathcal{T}_X$. Of particular interest is a specific equivariant quotient: the Cuntz-Pimsner algebra $\mathcal{O}_X$. The latter is the minimal C*-algebra that contains an isometric copy of X, and it is this boundary behaviour that allows for the recovery of a wealth of (low-rank) C*-constructions. For example, the

C*-crossed product induced by a single $*$-automorphism and the Cuntz-Krieger algebra associated with a row-finite directed graph are both incarnations of $\mathcal{O}_X$.

In light of the array of applications, C*-algebras associated with C*-correspondences have been explored in detail. Important developments in this direction include the study of ideal structure and simplicity [10], K-theory computation [35] and classification [9], necessary and sufficient conditions for nuclearity and exactness [35], the parametrisation of the KMS-simplex [39] and, central to the current work, the parametrisation of gauge-invariant ideals [36]. Focusing on the latter, the parametrisation is implemented by pairs of ideals of the coefficient algebra satisfying conditions related to the underlying C*-correspondence. If the C*-correspondence is induced by a geometric/topological object, then this description can be translated directly in terms of properties of the inducing object. For example, the gauge-invariant ideals of the Cuntz-Krieger algebra of a row-finite directed graph are in bijection with the hereditary saturated vertex sets of the graph, in accordance with [3].

Moving beyond $\mathbb{Z}_+$, many of the aforementioned results do not have clear extensions to the general case. However, by imposing additional structure on the product system X, progress can be made. One such addition is compact alignment for product systems over quasi-lattices, as pioneered by Fowler [23]. We can also ask that the representations of X preserve compact alignment, leading to the notion of Nica-covariant representations. In this case the associated C*-algebras admit a Wick ordering due to the Nica-covariant relations of the Fock representation, allowing for a tractable analysis via cores. The KMS-simplex of the Fock C*-algebra and particularly KMS-states of finite type have been studied by Afsar, Larsen and Neshveyev [1], unifying multiple works. We can still make sense of compact alignment when extending to product systems over right LCM semigroups, and a thorough study of the associated C*-algebras was provided by Kwaśniewski and Larsen [37, 38]. A key difference compared to the low-rank case is that the Fock C*-algebra is not universal for all representations, in general. However, we do have that the Fock C*-algebra is universal for all Nica-covariant representations when X is compactly aligned over a unital right LCM subsemigroup of an amenable discrete group (in particular $P = \mathbb{Z}_+^d$ resides in this framework). In their recent work, Brix, Carlsen and Sims [7] explore the ideal structure of C*-algebras related to commuting local homeomorphisms, pushing the theory beyond simplicity.

Until recently, the problem of ascertaining the appropriate Cuntz-type object for product systems has been open. Work in this direction commenced with the results of Fowler [23]; however, the proposed object could be trivial if the inducing product system is not injective. Sims and Yeend [56] provided an answer in the case of compactly aligned product systems over quasi-lattices, and showed that this C*-algebra (referred to as the Cuntz-Nica-Pimsner algebra) accounts for numerous examples. Co-universality of the Cuntz-Nica-Pimsner algebra (under an appropriate amenability assumption) was clarified by Carlsen, Larsen, Sims and Vittadello [11]. The appropriate Cuntz-type object for compactly aligned product systems over right LCM semigroups was identified as the C*-envelope of the (nonselfadjoint) tensor algebra (equipped with the natural coaction) by Dor-On, Kakariadis, Katsoulis, Laca and Li [18]. Nuclearity and exactness was addressed by Kakariadis, Katsoulis, Laca and Li [33]. The complete picture was provided in the general case by Sehnem [53, 54] via strong covariance relations, linking the Cuntz-type object with the C*-envelope of the tensor algebra.

The preceding results fall into the broader programme of bringing C*-algebras of product systems into the remit of Elliott's Classification Programme. A key result in this direction for $P = \mathbb{Z}_+$ has been provided by Brown, Tikuisis and Zelenberg [9], wherein a sufficient condition for classifiability of the Cuntz-Pimsner algebra in terms of properties of the C*-correspondence and its coefficient algebra is provided. A corresponding result for the Cuntz-Nica-Pimsner algebra in higher-rank cases has not yet been achieved. Indeed, one of the key advantages of the low-rank case is that the strong covariance relations defining the Cuntz-Pimsner algebra are simple and algebraic in nature, induced by a single ideal of the coefficient algebra introduced by Katsura [34]. In the general case the picture is significantly more complicated, since the strong covariance relations may not adopt the simple algebraic format of the low-rank case. For example, the relations defining the Cuntz-Nica-Pimsner algebra of Sims and Yeend [56] are based on families of compact operators induced by all possible finite subsets of the underlying semigroup.

1.2 Motivation

Henceforth, we restrict to considering a compactly aligned product system X over the semigroup $P = \mathbb{Z}_+^d$ that additionally satisfies the strong compact alignment condition of [17, Definition 2.2]. This condition, introduced by Dor-On and Kakariadis [17], is advantageous because it ensures that the strong covariance relations defining the Cuntz-Nica-Pimsner algebra are simple and algebraic in format and are induced by a family of $2^d - 1$ ideals of the coefficient algebra. This picture is in analogy with the low-rank case, opening a direction for lifting results from this setting. This leads to our principal point of motivation. Katsura's parametrisation of gauge-invariant ideals [36] makes extensive use of the fact that the Cuntz-Pimsner algebra is defined in terms of simple algebraic relations induced by a single ideal of the coefficient algebra. Indeed, the parametrising families can be broken down into a kernel and a covariance ideal related to Katsura's ideal [34]. Appealing to the previously mentioned analogy, it is natural to ask if a similar parametrisation can be established in the case of $P = \mathbb{Z}_+^d$. In the current work we answer this question for the Toeplitz-Nica-Pimsner algebra (i.e., the universal C*-algebra for Nica-covariant representations), thereby resolving [17, Question 9.2].

Next, it should be noted that strong compactly aligned product systems still account for a variety of important examples, e.g., regular product systems, C*-dynamical systems, row-finite higher-rank graphs and product systems on finite frames. In particular, all proper product systems over $\mathbb{Z}_+^d$ (i.e., product systems in which all left actions are by compact operators) are strong compactly aligned. This leads to our second point of motivation. Specifically, we wish to interpret our parametrisation in the context of each of the aforementioned examples, recovering pre-existing results in the process. We highlight three such results here. Firstly, given an automorphic C*-dynamical system $(A, \alpha, \mathbb{Z}_+^d)$, there is a lattice isomorphism between the set of gauge-invariant ideals of the crossed product C*-algebra $A \rtimes_\alpha \mathbb{Z}^d$ and the set of α-invariant ideals of A. Next, given a locally convex row-finite k-graph (Λ, d), there is a lattice isomorphism between the set of gauge-invariant ideals of the graph C*-algebra $\mathrm{C}^*(\Lambda)$ and the set of hereditary saturated vertex sets, as proved by Raeburn, Sims and Yeend [50]. Passing to the more general case of finitely aligned k-graphs, Sims provided a parametrisation implemented by hereditary saturated vertex sets together with satiated path sets [55]. In the current work we will encompass these results uniformly and provide an alternative to [55] in the row-finite case using only vertex sets. Observe also that these results are presented on the level of lattices. To accommodate for this, we will explicitly describe the lattice operations on the parametrising families of our main result that render the bijection a lattice isomorphism. Such a description is new even for $P = \mathbb{Z}_+$.

We also wish to characterise the gauge-invariant ideal structure of every equivariant quotient of the Toeplitz-Nica-Pimsner algebra (i.e., the quotients by gauge-invariant ideals). Our motivation is two-fold. As a base case, consider $\mathcal{T} \otimes \mathcal{T}$ (where $\mathcal{T}$ is the Toeplitz algebra) and associate the vertices of the square (taking $d = 2$) to ideals of $\mathcal{T} \otimes \mathcal{T}$ so that

$$\begin{array}{ccc} (0,1) & \text{——} & (1,1) \\ | & & | \\ (0,0) & \text{——} & (1,0) \end{array} \quad \longrightarrow \quad \begin{array}{ccc} \mathcal{T} \otimes \mathcal{K} & \text{——} & \mathcal{K} \otimes \mathcal{K} \\ | & & | \\ \{0\} & \text{——} & \mathcal{K} \otimes \mathcal{T} \end{array}$$

for the compact operators $\mathcal{K} \subseteq \mathcal{T}$. Then the gauge-invariant ideals of $\mathcal{T} \otimes \mathcal{T}$ can be read off an inclusion-preserving association by considering vertex sets of the square. Specifically, we have that

$$\begin{array}{ccc} \{(0,1),(1,1)\} & \text{——} & \{(1,1)\} \\ | & & | \\ \{(1,0),(0,1),(1,1)\} & \text{——} & \{(1,0),(1,1)\} \end{array} \quad \longrightarrow \quad \begin{array}{ccc} \mathcal{T} \otimes \mathcal{K} & \text{——} & \mathcal{K} \otimes \mathcal{K} \\ | & & | \\ \mathcal{T} \otimes \mathcal{K} + \mathcal{K} \otimes \mathcal{T} & \text{——} & \mathcal{K} \otimes \mathcal{T}. \end{array}$$

A similar decomposition for the boundary ideal $\ker\{\mathcal{NT}_X \to \mathcal{NO}_X\}$, where $\mathcal{NT}_X$ (resp. $\mathcal{NO}_X$) is the Toeplitz-Nica-Pimsner algebra (resp. Cuntz-Nica-Pimsner algebra) of X, is provided in [17] and resembles that of Deaconu [13], further exploited by Fletcher [22]. Such a decomposition has been used successfully for the computation of the K-theory of $\mathcal{NO}_X$ in terms of the coefficient algebra in low-rank cases, e.g., for 2-rank graphs by Evans [20] and for two commuting $*$-automorphisms by Barlak [2]. The general case remains unresolved, and in the current work we aim to shed more light on this construction.

A further motivation comes from the theory of KMS-states, wherein the equivariant quotients of $\mathcal{NT}_X$ (particularly those that may not be injective on X) arise naturally.

Following the seminal work of Exel and Laca [21] and of Laca and Neshveyev [39], there have been many results on KMS-states for C*-algebras induced by finite graphs and dynamics. Finite higher-rank graphs have also undergone study in this direction, with Christensen [12] supplying the complete picture. A parametrisation of the gauge-invariant KMS-states has been obtained by Kakariadis in the presence of finite frames [32]. A key aspect of [12, 32] is the Wold decomposition of a KMS-state into F-finite and F^c-infinite parts, for $F \subseteq \{1, \dots, d\}$. This corresponds to KMS-states annihilating the gauge-invariant ideal generated by the projections along F^c. The construction in [32] uses F to induce a product system whose coefficient algebra arises from the F^c-core. In the current work we are motivated to close the circle with [32]. More specifically, we will provide a full characterisation of each F-quotient of $\mathcal{NT}_X$ as the Cuntz-Nica-Pimsner algebra of an F-induced product system.

The proof of Katsura's parametrisation result [36] also makes use of a Gauge-Invariant Uniqueness Theorem for the Cuntz-Pimsner algebra. This type of result was pioneered by an Huef and Raeburn for Cuntz-Krieger algebras [28], and various generalisations were provided by Doplicher, Pinzari and Zuccante [16], Fowler, Muhly and Raeburn [24], and Fowler and Raeburn [25]. Katsura [36] completed the picture for $P = \mathbb{Z}_+$, providing a Gauge-Invariant Uniqueness Theorem for relative Cuntz-Pimsner algebras. As such, we are motivated to establish an analogous result for $P = \mathbb{Z}_+^d$, both for its independent utility and in order to make progress in generalising Katsura's parametrisation [36]. We note that a Gauge-Invariant Uniqueness Theorem for the Cuntz-Nica-Pimsner algebra of a strong compactly aligned product system is provided in [17], and the proof relies on analysing polynomial equations induced by cores. Due to Nica-covariance and the structure of the cores, the solutions must adhere to invariance and partial ordering. However, simple examples show that different subsets of solutions may induce the same gauge-invariant ideal, raising the question of ascertaining the appropriate compatibility conditions that characterise the maximal solution set. Due to this subtlety, a Gauge-Invariant Uniqueness Theorem for relative Cuntz-Nica-Pimsner algebras has been elusive. In the current work we fill this gap.

1.3 Summary of main results

We parametrise the gauge-invariant ideals of the Toeplitz-Nica-Pimsner algebra $\mathcal{NT}_X$ of an arbitrary strong compactly aligned product system X over $\mathbb{Z}_+^d$, thereby answering [17, Question 9.2]. Our parametrisation is implemented by 2^d-tuples of ideals of the coefficient algebra which satisfy conditions related to X. We characterise these conditions using product system operations alone. As a corollary, we obtain a parametrisation of the gauge-invariant ideals of each relative Cuntz-Nica-Pimsner algebra of X, including the Cuntz-Nica-Pimsner algebra $\mathcal{NO}_X$. In turn, we are able to completely describe the gauge-invariant ideal structure of every equivariant quotient of $\mathcal{NT}_X$. These results are in direct analogy with (and recover) the one-dimensional case [36]. Additionally, we clarify the lattice structure on the parametrising families such that the bijection is rendered a lattice isomorphism (this is new even for $d = 1$). In the process of obtaining the main result, we prove a Gauge-Invariant Uniqueness Theorem for the relative Cuntz-Nica-Pimsner algebras in-between $\mathcal{NT}_X$ and $\mathcal{NO}_X$, recovering the corresponding results of [17, 36].

We apply our results to regular product systems, which is instructive for subsequent examples. In particular, when X is regular and the coefficient algebra is non-zero and simple, we show that the set of gauge-invariant ideals of $\mathcal{NT}_X$ is in bijection with the set of pairwise incomparable subsets of $\{1, \dots, d\}$. In this case we also show that $\mathcal{NO}_X$ does not admit non-trivial proper gauge-invariant ideals. We also apply our results to C*-dynamical systems over $\mathbb{Z}_+^d$. In particular, for injective systems we show that the gauge-invariant ideals of the Cuntz-Nica-Pimsner algebra correspond to ideals of the coefficient algebra that are positively and negatively invariant. In this way we recover the classical C*-crossed product result for automorphic systems. We move on to apply our results to row-finite higher-rank graphs. We show that the parametrisation result of Sims [55] is implemented by vertex sets alone in this context. By requiring the underlying graph to be in addition locally convex, we demonstrate how the parametrisation result of Raeburn, Sims and Yeend [50] is recovered from our own. In the presence of finite frames, we address the decomposition of [32] and show that the F-quotient of $\mathcal{NT}_X$ can be realised as the Cuntz-Nica-Pimsner algebra of an F-induced product system with coefficients in the F^c-core, for all $F \subseteq \{1, \dots, d\}$. We achieve this in two ways, and show that these approaches are equivalent in the sense that they produce the same Cuntz-Nica-Pimsner algebra (up to $*$-isomorphism). Finally, we explore the connection between our work and the recent work of Bilich [4] in the setting of proper product systems over $\mathbb{Z}_+^d$. In particular, we show that the main result of [4] aligns with our own.

1.4 Description of main results

First we remind of the key concepts and results from the low-rank case. Let X be a C*-correspondence over a C*-algebra A. Given an ideal $I \subseteq A$, we define

$$X(I) := [\langle X, IX \rangle] \quad \text{and} \quad X^{-1}(I) := \{a \in A \mid \langle X, aX \rangle \subseteq I\},$$

both of which are ideals of A. Additionally, we define

$$J(I, X) := \{a \in A \mid [\phi_X(a)]_I \in \mathcal{K}([X]_I), aX^{-1}(I) \subseteq I\},$$

which is also an ideal of A. Here we make use of the quotient construction for Hilbert C*-modules, e.g., [24, 36]. We say that I is *positively invariant (for X)* if $X(I) \subseteq I$. Following [36, Definition 5.6, Definition 5.12], we define a *T-pair (of X)* to be a pair $\mathcal{L} = \{\mathcal{L}_\emptyset, \mathcal{L}_{\{1\}}\}$ of ideals of A such that $\mathcal{L}_\emptyset$ is positively invariant for X and $\mathcal{L}_\emptyset \subseteq \mathcal{L}_{\{1\}} \subseteq J(\mathcal{L}_\emptyset, X)$; a

T-pair $\mathcal{L}$ that satisfies $J_X \subseteq \mathcal{L}_{\{1\}}$ (where J_X is Katsura's ideal [34, Definition 2.3]) is called an *O-pair (of X)*. We remind of Katsura's parametrisation of gauge-invariant ideals [36].

Theorem. *([36, Theorem 8.6, Proposition 8.8]) Let X be a C*-correspondence over a C*-algebra A. Then there is a bijection between the set of T-pairs (resp. O-pairs) of X and the set of gauge-invariant ideals of $\mathcal{T}_X$ (resp. $\mathcal{O}_X$). This bijection preserves inclusions and intersections.*

By modifying the definition of T-pairs, we obtain a parametrisation of the gauge-invariant ideals of any relative Cuntz-Pimsner algebra [36, Proposition 11.9]. The parametrisation of the gauge-invariant ideals of $\mathcal{T}_X$ admits the following implementation. Let $\mathfrak{J} \subseteq \mathcal{T}_X$ be a gauge-invariant ideal and let $Q_{\mathfrak{J}} \colon \mathcal{T}_X \to \mathcal{T}_X/\mathfrak{J}$ denote the quotient map. We consider the representation $(Q_{\mathfrak{J}} \circ \overline{\pi}_X, Q_{\mathfrak{J}} \circ \overline{t}_X)$ of X, where $(\overline{\pi}_X, \overline{t}_X)$ is the universal representation of X. We set

$$\mathcal{L}_{\emptyset}^{(Q_{\mathfrak{J}}\circ\overline{\pi}_X, Q_{\mathfrak{J}}\circ\overline{t}_X)} := \ker Q_{\mathfrak{J}} \circ \overline{\pi}_X \quad \text{and} \quad \mathcal{L}_{\{1\}}^{(Q_{\mathfrak{J}}\circ\overline{\pi}_X, Q_{\mathfrak{J}}\circ\overline{t}_X)} := (Q_{\mathfrak{J}} \circ \overline{\pi}_X)^{-1}((Q_{\mathfrak{J}} \circ \overline{\psi}_X)(\mathcal{K}(X))).$$

An application of [36, Proposition 5.11] yields that the pair

$$\mathcal{L}^{\mathfrak{J}} := \{\mathcal{L}_{\emptyset}^{(Q_{\mathfrak{J}}\circ\overline{\pi}_X, Q_{\mathfrak{J}}\circ\overline{t}_X)}, \mathcal{L}_{\{1\}}^{(Q_{\mathfrak{J}}\circ\overline{\pi}_X, Q_{\mathfrak{J}}\circ\overline{t}_X)}\}$$

is a T-pair of X. Next, let $\mathcal{L} = \{\mathcal{L}_{\emptyset}, \mathcal{L}_{\{1\}}\}$ be a T-pair of X. Then $[\mathcal{L}_{\{1\}}]_{\mathcal{L}_{\emptyset}}$ is an ideal of $[A]_{\mathcal{L}_{\emptyset}}$, where $[\cdot]_{\mathcal{L}_{\emptyset}}$ denotes the quotient map. Note that $[\mathcal{L}_{\{1\}}]_{\mathcal{L}_{\emptyset}} \subseteq J_{[X]_{\mathcal{L}_{\emptyset}}}$ by [36, Lemma 5.2]. Taking $(\widetilde{\pi}, \widetilde{t})$ to be the universal $[\mathcal{L}_{\{1\}}]_{\mathcal{L}_{\emptyset}}$-covariant representation of $[X]_{\mathcal{L}_{\emptyset}}$, we may form a representation $(\pi^{\mathcal{L}}, t^{\mathcal{L}})$ of X that generates $\mathcal{O}([\mathcal{L}_{\{1\}}]_{\mathcal{L}_{\emptyset}}, [X]_{\mathcal{L}_{\emptyset}})$ via

$$\pi^{\mathcal{L}}(a) = \widetilde{\pi}([a]_{\mathcal{L}_{\emptyset}}) \quad \text{and} \quad t^{\mathcal{L}}(\xi) = \widetilde{t}([\xi]_{\mathcal{L}_{\emptyset}}) \text{ for all } a \in A, \xi \in X.$$

Universality of $\mathcal{T}_X$ then gives a (unique) canonical $*$-epimorphism

$$\pi^{\mathcal{L}} \times t^{\mathcal{L}} \colon \mathcal{T}_X \to \mathcal{O}([\mathcal{L}_{\{1\}}]_{\mathcal{L}_{\emptyset}}, [X]_{\mathcal{L}_{\emptyset}}).$$

We set

$$\mathfrak{J}^{\mathcal{L}} := \ker \pi^{\mathcal{L}} \times t^{\mathcal{L}},$$

which is a gauge-invariant ideal of $\mathcal{T}_X$. By [36, Proposition 8.8], the maps

$$\mathfrak{J} \mapsto \mathcal{L}^{\mathfrak{J}} \text{ for all gauge-invariant ideals } \mathfrak{J} \text{ of } \mathcal{T}_X,$$
$$\mathcal{L} \mapsto \mathfrak{J}^{\mathcal{L}} \text{ for all T-pairs } \mathcal{L} \text{ of } X,$$

are mutually inverse. Using the preceding proof as a trajectory, our main goal is to extend Katsura's parametrisation result to the setting of strong compactly aligned product systems over $\mathbb{Z}_+^d$, which we now move on to consider.

We fix the following notation when working over $\mathbb{Z}_+^d$. We write $[d] := \{1, \dots, d\}$ for

$d \in \mathbb{N}$ and $\underline{n} = (n_1, \dots, n_d)$ for the elements of $\mathbb{Z}_+^d$. We denote the usual generators of $\mathbb{Z}_+^d$ by $\underline{i}$ for $i \in [d]$. We set $|\underline{n}| := \sum\{n_i \mid i \in [d]\}$ and $\underline{1}_F := \sum\{\underline{i} \mid i \in F\}$ for all $\emptyset \neq F \subseteq [d]$. We define the support of $\underline{n}$ to be the set

$$\operatorname{supp} \underline{n} := \{i \in [d] \mid n_i \neq 0\}.$$

We write $\underline{n} \perp F$ for $F \subseteq [d]$ if $\operatorname{supp} \underline{n} \cap F = \emptyset$.

A *strong compactly aligned product system* $X = \{X_{\underline{n}}\}_{\underline{n} \in \mathbb{Z}_+^d}$ *with coefficients in a C*-algebra* A is a compactly aligned product system that additionally satisfies

$$\mathcal{K}(X_{\underline{n}}) \otimes \mathrm{id}_{X_{\underline{i}}} \subseteq \mathcal{K}(X_{\underline{n}} \otimes_A X_{\underline{i}}) \text{ whenever } \underline{n} \perp \{i\}, \text{ where } i \in [d], \underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}.$$

Henceforth we fix a strong compactly aligned product system X with coefficients in a C*-algebra A. If (π, t) is a Nica-covariant representation of X (acting on a Hilbert space H) and $i \in [d]$, we use an approximate unit $(k_{\underline{i},\lambda})_{\lambda \in \Lambda}$ of $\mathcal{K}(X_{\underline{i}})$ to define the projection $p_{\underline{i}} := \text{w*-}\lim_\lambda \psi_{\underline{i}}(k_{\underline{i},\lambda}) \in \mathcal{B}(H)$, and we set

$$q_\emptyset := I, q_{\underline{i}} := I - p_{\underline{i}}, \text{ and } q_F := \prod_{j \in F}(I - p_{\underline{j}}) \text{ for } \emptyset \neq F \subseteq [d].$$

In Remark 2.5.14 we show that the projections $p_{\underline{i}}$ commute. This fact is not used in [17], and serves to simplify several of the proofs therein. We reserve $(\overline{\pi}_X, \overline{t}_X)$ for the universal Nica-covariant representation of X, which generates the Toeplitz-Nica-Pimsner algebra $\mathcal{NT}_X$. By exploiting a Fock space construction to define a concrete Fock representation, we may view $\mathcal{NT}_X$ as the C*-algebra generated by this representation. Universality follows due to amenability of $\mathbb{Z}^d$.

A 2^d*-tuple* $\mathcal{L} := \{\mathcal{L}_F\}_{F \subseteq [d]}$ *(of* X*)* is a family of 2^d non-empty subsets of A. We define a partial order on such families via $\mathcal{L} \subseteq \mathcal{L}'$ if and only if $\mathcal{L}_F \subseteq \mathcal{L}'_F$ for all $F \subseteq [d]$. Abstracting the construction of relative Cuntz-Pimsner algebras from the $d = 1$ case, we say that a 2^d-tuple $\mathcal{L}$ of X is *relative* if it satisfies

$$\mathcal{L}_F \subseteq \bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in F\} \text{ for all } \emptyset \neq F \subseteq [d].$$

The key property of a relative 2^d-tuple $\mathcal{L}$ is that

$$\pi(a)q_F = \pi(a) + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \in \mathrm{C}^*(\pi, t) \text{ for all } a \in \mathcal{L}_F, F \subseteq [d],$$

though we may not have that $q_F \in \mathrm{C}^*(\pi, t)$. Thus we may form the gauge-invariant ideal

$$\left\langle \overline{\pi}_X(\mathcal{L}_F)\overline{q}_{X,F} \mid F \subseteq [d] \right\rangle \subseteq \mathcal{NT}_X.$$

We write $\mathcal{NO}(\mathcal{L}, X)$ for the corresponding equivariant quotient of $\mathcal{NT}_X$, which is a relative Cuntz-Nica-Pimsner algebra (justifying our choice of nomenclature).

The main result of [17] is that the Cuntz-Nica-Pimsner algebra $\mathcal{NO}_X$ is nothing but $\mathcal{NO}(\mathcal{I}, X)$ for the family $\mathcal{I} := \{\mathcal{I}_F\}_{F \subseteq [d]}$, where

$$\mathcal{I}_F := \bigcap \{X_{\underline{n}}^{-1}(\mathcal{J}_F) \mid \underline{n} \perp F\} \text{ for } \mathcal{J}_F := (\bigcap_{i \in F} \ker \phi_{\underline{i}})^\perp \cap (\bigcap_{i \in [d]} \phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})))$$

for all $\emptyset \neq F \subseteq [d]$ and $\mathcal{I}_\emptyset \equiv \mathcal{J}_\emptyset := \{0\}$. Each $\mathcal{I}_F$ is the largest $F^\perp$-invariant ideal of $\mathcal{J}_F$, and the family $\mathcal{I}$ is partially ordered in the sense that $\mathcal{I}_F \subseteq \mathcal{I}_D$ whenever $F \subseteq D \subseteq [d]$. In order to understand general equivariant quotients of $\mathcal{NT}_X$, we abstract the aforementioned properties to obtain the notions of invariance and partial ordering (respectively) for 2^d-tuples of X.

First we consider the case where $\mathcal{L}$ is a 2^d-tuple of X satisfying $\mathcal{L} \subseteq \mathcal{I}$; we refer to such tuples as *(E)-2^d-tuples*. The "E" stands for "Embedding", since by definition $\mathcal{NO}(\mathcal{L}, X)$ lies in-between $\mathcal{NT}_X$ and $\mathcal{NO}_X$ and thus contains an isometric copy of X. In Propositions 3.2.4 and 3.2.6, we show that we may induce (E)-2^d-tuples $\mathrm{Inv}(\mathcal{L})$ and $\mathrm{PO}(\mathcal{L})$ of X via

$$\mathrm{Inv}(\mathcal{L})_F := \overline{\mathrm{span}}\{X_{\underline{n}}(\mathcal{L}_F) \mid \underline{n} \perp F\} \quad \text{and} \quad \mathrm{PO}(\mathcal{L})_F := \sum \{\langle \mathcal{L}_D \rangle \mid D \subseteq F\}$$

for all $F \subseteq [d]$, such that $\mathcal{L}$ is contained in both $\mathrm{Inv}(\mathcal{L})$ and $\mathrm{PO}(\mathcal{L})$ and

$$\mathcal{NO}(\mathcal{L}, X) = \mathcal{NO}(\mathrm{Inv}(\mathcal{L}), X) = \mathcal{NO}(\mathrm{PO}(\mathcal{L}), X).$$

In particular, we have that $\mathrm{Inv}(\mathcal{L})$ is invariant and $\mathrm{PO}(\mathcal{L})$ is partially ordered, and in fact $\mathrm{PO}(\mathcal{L})$ is invariant if the same is true of $\mathcal{L}$. Hence we may restrict to the case where $\mathcal{L}$ is an invariant, partially ordered (E)-2^d-tuple of ideals by replacing $\mathcal{L}$ with $\mathrm{PO}(\mathrm{Inv}(\mathcal{L}))$. However, it follows from Example 3.1.12 that these properties are not sufficient to provide injectivity of the association $\mathcal{L} \mapsto \mathcal{NO}(\mathcal{L}, X)$. To remedy this, we instead look at the maximal (E)-2^d-tuple $\mathcal{M}$ that induces $\mathcal{NO}(\mathcal{L}, X)$. This family contains $\mathcal{L}$ and all other (E)-2^d-tuples that induce $\mathcal{NO}(\mathcal{L}, X)$ by construction. Existence and uniqueness of $\mathcal{M}$ are provided by Propositions 3.1.9 and 3.2.10. The prototypical example of maximal (E)-2^d-tuples is obtained from injective Nica-covariant representations (π, t) of X that admit a gauge action, by defining

$$\mathcal{L}_\emptyset^{(\pi,t)} := \ker \pi \quad \text{and} \quad \mathcal{L}_F^{(\pi,t)} := \pi^{-1}(B_{(\underline{0}, \underline{1}_F]}^{(\pi,t)}) \text{ for all } \emptyset \neq F \subseteq [d].$$

Here $B_{(\underline{0}, \underline{1}_F]}^{(\pi,t)}$ denotes the $(\underline{0}, \underline{1}_F]$-core of $\mathrm{C}^*(\pi, t)$ and $\mathcal{L}_\emptyset^{(\pi,t)} = \{0\}$ by injectivity. In fact, all maximal (E)-2^d-tuples are of this form. This can be seen via the following theorem, which constitutes a key step towards a Gauge-Invariant Uniqueness Theorem for relative Cuntz-Nica-Pimsner algebras.

Theorem A. *(Theorem 3.2.12) Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be a maximal (E)-2^d-tuple of X and suppose that (π, t) is a Nica-covariant representation of X. Then $\mathcal{NO}(\mathcal{L}, X) \cong \mathrm{C}^*(\pi, t)$ via a (unique)* *canonical $*$-isomorphism if and only if (π, t) admits a gauge action and $\mathcal{L}^{(\pi,t)} = \mathcal{L}$.*

The maximal (E)-2^d-tuples of X parametrise the gauge-invariant ideals $\mathfrak{J} \subseteq \mathcal{NT}_X$ such that $\overline{\pi}_X^{-1}(\mathfrak{J}) = \{0\}$ (see Remark 3.2.11). In order to obtain a genuinely useful parametrisation, we turn our attention to characterising maximality using product system operations alone (without reference to any Nica-covariant representations). To this end, for every $\emptyset \neq F \subseteq [d]$ we define

$$\mathcal{L}_{\mathrm{inv},F} := \bigcap_{\underline{m} \perp F} X_{\underline{m}}^{-1}(\cap_{F \subsetneq D} \mathcal{L}_D) \quad \text{and} \quad \mathcal{L}_{\mathrm{lim},F} := \{a \in A \mid \lim_{\underline{m} \perp F} \|\phi_{\underline{m}}(a) + \mathcal{K}(X_{\underline{m}} \mathcal{L}_F)\| = 0\}.$$

We observe that the definitions of $\mathcal{L}_{\mathrm{inv},F}$ and $\mathcal{L}_{\mathrm{lim},F}$ do not require $\mathcal{L}$ to be an (E)-2^d-tuple, though they do require $\mathcal{L}$ to consist of ideals. When $\mathcal{L}$ is an (E)-2^d-tuple, we define the 2^d-tuple $\mathcal{L}^{(1)}$ by

$$\mathcal{L}_F^{(1)} := \begin{cases} \{0\} & \text{if } F = \emptyset, \\ \mathcal{I}_F \cap \mathcal{L}_{\mathrm{inv},F} \cap \mathcal{L}_{\mathrm{lim},F} & \text{if } \emptyset \neq F \subsetneq [d], \\ \mathcal{L}_{[d]} & \text{if } F = [d]. \end{cases}$$

In Proposition 3.4.5 we show that $\mathcal{L}^{(1)}$ is an (E)-2^d-tuple of ideals that is invariant and partially ordered when $\mathcal{L}$ is so, satisfying $\mathcal{L} \subseteq \mathcal{L}^{(1)}$ and

$$\left\langle \overline{\pi}_X(\mathcal{L}_F)\overline{q}_{X,F} \mid F \subseteq [d] \right\rangle = \left\langle \overline{\pi}_X(\mathcal{L}_F^{(1)})\overline{q}_{X,F} \mid F \subseteq [d] \right\rangle.$$

Maximality is then described in terms of the first iteration.

Theorem B. *(Theorem 3.4.6) Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and suppose that $\mathcal{L}$ is a 2^d-tuple of X. Then $\mathcal{L}$ is a maximal (E)-2^d-tuple of X if and only if $\mathcal{L}$ satisfies the following four conditions:*

(i) $\mathcal{L}$ consists of ideals and $\mathcal{L} \subseteq \mathcal{J}$,

(ii) $\mathcal{L}$ is invariant,

(iii) $\mathcal{L}$ is partially ordered,

(iv) $\mathcal{L}^{(1)} \subseteq \mathcal{L}$.

For $k \in \mathbb{Z}_+$ we write $\mathcal{L}^{(k+1)} := (\mathcal{L}^{(k)})^{(1)}$, where $\mathcal{L}^{(0)} := \mathcal{L}$. When $\mathcal{L}$ is an (E)-2^d-tuple that is invariant, partially ordered and consists of ideals, we obtain that

$$\mathcal{L}^{(k)} \subseteq \mathcal{L}^{(k+1)} \text{ and } \mathcal{NO}(\mathcal{L}, X) = \mathcal{NO}(\mathcal{L}^{(k)}, X) \text{ for all } k \in \mathbb{Z}_+$$

inductively. Thus it is natural to ask if these iterations stabilise, eventually yielding the maximal (E)-2^d-tuple that induces $\mathcal{NO}(\mathcal{L}, X)$. In Theorem 3.4.7 we show that

$$\mathcal{L}^{(d-1)} = \mathcal{L}^{(k)} \text{ for all } k \geq d-1,$$

and thus $\mathcal{L}^{(d-1)}$ is the maximal (E)-2^d-tuple inducing $\mathcal{NO}(\mathcal{L}, X)$. In turn, we obtain an algorithm for computing maximal (E)-2^d-tuples, where the input is an (E)-2^d-tuple $\mathcal{L}$. More precisely, we pass to PO(Inv($\mathcal{L}$)) and then take the $(d-1)$-iteration. Combining Theorems A and B then yields the full form of the Gauge-Invariant Uniqueness Theorem.

Theorem C. *(Theorem 3.4.9) Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be an (E)-2^d-tuple of X and (π, t) be a Nica-covariant representation of X. Then $\mathcal{NO}(\mathcal{L}, X) \cong \mathrm{C}^*(\pi, t)$ via a (unique) canonical $*$-isomorphism if and only if (π, t) admits a gauge action and*

$$\mathcal{L}^{(\pi,t)} = \bigg(\mathrm{PO}\left(\mathrm{Inv}\left(\mathcal{L}\right)\right)\bigg)^{(d-1)}.$$

Next we pass to the parametrisation of all gauge-invariant ideals $\mathfrak{J}$ of $\mathcal{NT}_X$, accounting for the case where $\overline{\pi}_X^{-1}(\mathfrak{J}) \neq \{0\}$. We circumvent this by "deleting the kernel", i.e., by utilising the quotient product system construction to return to the setting of maximal (E)-2^d-tuples. Given an ideal $I \subseteq A$, we say that I is *positively invariant (for X)* if $X_{\underline{n}}(I) \subseteq I$ for all $\underline{n} \in \mathbb{Z}_+^d$. This condition ensures that $[X]_I := \{[X_{\underline{n}}]_I\}_{\underline{n}\in\mathbb{Z}_+^d}$ carries a natural structure as a strong compactly aligned product system with coefficients in $[A]_I$, and as in the low-rank case we use $[\cdot]_I$ to denote the associated quotient maps.

Let $\mathcal{L}$ be a 2^d-tuple of X that consists of ideals and is such that $\mathcal{L}_\emptyset$ is positively invariant for X and satisfies $\mathcal{L}_\emptyset \subseteq \mathcal{L}_F$ for all $F \subseteq [d]$. We say that $\mathcal{L}$ is an *NT-2^d-tuple (of X)* if $[\mathcal{L}]_{\mathcal{L}_\emptyset} := \{[\mathcal{L}_F]_{\mathcal{L}_\emptyset}\}_{F\subseteq[d]}$ is a maximal (E)-2^d-tuple of $[X]_{\mathcal{L}_\emptyset}$. In Section 4.1 we provide a detailed description of the structural properties that render a 2^d-tuple $\mathcal{L}$ an NT-2^d-tuple. In particular, in Proposition 4.1.5 we provide a characterisation of NT-2^d-tuples with no reference to the quotient product system construction when X is proper. Returning to the case of a general strong compactly aligned product system, in Proposition 4.1.12 we show that the NT-2^d-tuples of X are exactly of the form $\mathcal{L}^{(\pi,t)}$ for some Nica-covariant representation of X that admits a gauge action. In Proposition 4.2.1 we show (in particular) that every equivariant quotient of $\mathcal{NT}_X$ can be realised as a relative Cuntz-Nica-Pimsner algebra of a quotient product system. By passing to the quotient and combining with the results for (E)-2^d-tuples, we obtain the parametrisation in its full generality.

Theorem D. *(Proposition 4.2.2, Theorem 4.2.3) Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Then there is a bijection between the set of NT-2^d-tuples of X and the set of gauge-invariant ideals of $\mathcal{NT}_X$ given by*

$$\mathcal{L} \mapsto \mathfrak{J}^{\mathcal{L}} := \ker \pi^{\mathcal{L}} \times t^{\mathcal{L}}, \text{ for } \pi^{\mathcal{L}} \times t^{\mathcal{L}} \colon \mathcal{NT}_X \to \mathcal{NO}([\mathcal{L}]_{\mathcal{L}_\emptyset}, [X]_{\mathcal{L}_\emptyset}),$$
$$\mathfrak{J} \mapsto \mathcal{L}^{\mathfrak{J}} := \mathcal{L}^{(Q_{\mathfrak{J}}\circ\overline{\pi}_X, Q_{\mathfrak{J}}\circ\overline{t}_X)}, \text{ for } Q_{\mathfrak{J}} \colon \mathcal{NT}_X \to \mathcal{NT}_X/\mathfrak{J},$$

where $\pi^{\mathcal{L}} \times t^{\mathcal{L}}$ and $Q_{\mathfrak{J}}$ are canonical $*$-epimorphisms. Moreover, if $\mathcal{L}$ is an NT-2^d-tuple *of X, then we have that*

$$\mathfrak{J}^{\mathcal{L}} = \langle \overline{\pi}_X(a) + \sum_{\underline{0} \neq \underline{n} \leq \underline{1}_F} (-1)^{|\underline{n}|} \overline{\psi}_{X,\underline{n}}(k_{\underline{n}}) \mid F \subseteq [d], a \in \mathcal{L}_F, k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}),$$
$$[\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset} = [k_{\underline{n}}]_{\mathcal{L}_\emptyset} \text{ for all } \underline{0} \neq \underline{n} \leq \underline{1}_F \rangle.$$

We note here that the description of $\mathfrak{J}^{\mathcal{L}}$ is new even in the $d = 1$ case. Moreover, the Nica-covariant representation $(\pi^{\mathcal{L}}, t^{\mathcal{L}})$ is well-defined since the association $X \to [X]_{\mathcal{L}_\emptyset}$ lifts to a canonical $*$-epimorphism $\mathcal{NT}_X \to \mathcal{NT}_{[X]_{\mathcal{L}_\emptyset}}$ (a proof of which is provided in Remark 2.4.5). It is known that the corresponding claim for the Cuntz-Nica-Pimsner algebras is not true in general (even for $d = 1$, see Example 5.3.13). Nevertheless, by using the NT-2^d-tuple machinery, we determine precisely when this holds as part of our applications in Section 5.1. The key requirement is that $\mathcal{I} \subseteq \mathcal{L}$, which implies that $\mathcal{L}_\emptyset$ is both positively and negatively invariant for X (see Definition 5.1.1).

The bijection of Theorem D induces a lattice structure on the NT-2^d-tuples that renders it a lattice isomorphism, where we equip the gauge-invariant ideals of $\mathcal{NT}_X$ with the usual lattice operations. It is then natural to inquire about the join and meet operations in this setting. It is straightforward to check that the bijection preserves inclusions and intersections (by Theorem 4.2.3 and Proposition 4.2.6, respectively). For the join operation we use the iteration process that describes maximality.

Theorem E. *(Proposition 4.2.6, Proposition 4.2.7) Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. We equip the set of NT-2^d-tuples of X with the lattice structure determined by the operations*

$$\mathcal{L}_1 \vee \mathcal{L}_2 := \mathcal{L}^{\mathfrak{J}^{\mathcal{L}_1} + \mathfrak{J}^{\mathcal{L}_2}} \quad \text{and} \quad \mathcal{L}_1 \wedge \mathcal{L}_2 := \mathcal{L}^{\mathfrak{J}^{\mathcal{L}_1} \cap \mathfrak{J}^{\mathcal{L}_2}}.$$

Then we have that

$$(\mathcal{L}_1 \wedge \mathcal{L}_2)_F = \mathcal{L}_{1,F} \cap \mathcal{L}_{2,F} \text{ for all } F \subseteq [d].$$

Additionally, we have that

$$(\mathcal{L}_1 \vee \mathcal{L}_2)_\emptyset = \overline{\pi}_X^{-1}(\mathfrak{J}^{\mathcal{L}_1} + \mathfrak{J}^{\mathcal{L}_2})$$

and that

$$(\mathcal{L}_1 \vee \mathcal{L}_2)_F = [\cdot]^{-1}_{(\mathcal{L}_1 \vee \mathcal{L}_2)_\emptyset} \left[\left((\mathcal{L}_{1,F} + \mathcal{L}_{2,F} + (\mathcal{L}_1 \vee \mathcal{L}_2)_\emptyset) / (\mathcal{L}_1 \vee \mathcal{L}_2)_\emptyset \right)^{(d-1)} \right]$$

for all $\emptyset \neq F \subseteq [d]$.

The parametrisation of gauge-invariant ideals of $\mathcal{NT}_X$ descends naturally to $\mathcal{NO}(\mathcal{K}, X)$ for any relative 2^d-tuple $\mathcal{K}$. The only difference is that the parametrising objects are the NT-2^d-tuples that contain $\mathcal{K}$. The lattice structure on NT-2^d-tuples restricts to this setting and so we obtain another lattice isomorphism (see Proposition 4.2.10 and Theorem 4.2.11).

Combining with Proposition 4.2.1, this elucidates the gauge-invariant ideal structure of every equivariant quotient of $\mathcal{NT}_X$. By choosing $\mathcal{K} = \mathcal{I}$, we obtain the parametrisation of gauge-invariant ideals of $\mathcal{NO}_X$ by what we call *NO-2^d-tuples (of X)*.

Next we pass to applying our results to specific classes of product systems. We begin by studying NO-2^d-tuples in greater depth, which is instructive for subsequent examples. We then deal with regular product systems (i.e., proper product systems such that each left action is injective). In Corollary 5.2.3 we show that the parametrisation of the gauge-invariant ideals of $\mathcal{NO}_X$ is implemented by single ideals of A that are positively and negatively invariant. This relies on the fact that a fixed positively and negatively invariant ideal of A is $\mathcal{L}_\emptyset$ for a specific NO-2^d-tuple $\mathcal{L}$ of X (see Proposition 5.1.6), while by regularity we have that $\mathcal{L}_F = A$ for all $\emptyset \neq F \subseteq [d]$. It follows that $\mathcal{NO}_X$ does not admit non-trivial proper gauge-invariant ideals when A is simple (see Corollary 5.2.4). Passing to $\mathcal{NT}_X$, in Corollary 5.2.7 we show that the gauge-invariant ideals of $\mathcal{NT}_X$ are in bijection with the families of pairwise incomparable subsets of $[d]$, when A is non-zero and simple. A direct application for the Toeplitz algebra $\mathcal{T}^{\otimes d}$ of $\mathbb{Z}_+^d$ produces the parametrisation of its gauge-invariant ideals by vertex sets on the d-hypercube.

The second class of examples that we consider arises from triples $(A, \alpha, \mathbb{Z}_+^d)$, where A is a C*-algebra and $\alpha \colon \mathbb{Z}_+^d \to \operatorname{End}(A)$ is a semigroup action. We refer to such triples as C*-dynamical systems. We write X_α for the associated product system, which will always be proper and so in particular strong compactly aligned. The NT-2^d-tuples are characterised as follows.

Corollary F. *(Corollary 5.3.10) Let $(A, \alpha, \mathbb{Z}_+^d)$ be a C*-dynamical system. Let $\mathcal{K}$ and $\mathcal{L}$ be 2^d-tuples of X_α. Then $\mathcal{L}$ is a $\mathcal{K}$-relative NO-2^d-tuple of X_α if and only if $\mathcal{K} \subseteq \mathcal{L}$ and the following hold:*

(i) $\mathcal{L}$ consists of ideals and $\mathcal{L}_F \cap (\bigcap_{i \in F} \alpha_{\underline{i}}^{-1}(\mathcal{L}_\emptyset)) \subseteq \mathcal{L}_\emptyset$ for all $\emptyset \neq F \subseteq [d]$,

(ii) $\mathcal{L}_F \subseteq \bigcap_{\underline{n} \perp F} \alpha_{\underline{n}}^{-1}(\mathcal{L}_F)$ for all $F \subseteq [d]$,

(iii) $\mathcal{L}$ is partially ordered,

(iv) $\mathcal{L}_{1,F} \cap \mathcal{L}_{2,F} \cap \mathcal{L}_{3,F} \subseteq \mathcal{L}_F$ for all $\emptyset \neq F \subsetneq [d]$, where

- $\mathcal{L}_{1,F} := \bigcap_{\underline{n} \perp F} \alpha_{\underline{n}}^{-1}(\{a \in A \mid a(\bigcap_{i \in F} \alpha_{\underline{i}}^{-1}(\mathcal{L}_\emptyset)) \subseteq \mathcal{L}_\emptyset\})$,
- $\mathcal{L}_{2,F} := \bigcap_{\underline{m} \perp F} \alpha_{\underline{m}}^{-1}(\cap_{F \subsetneq D} \mathcal{L}_D)$,
- $\mathcal{L}_{3,F} := \{a \in A \mid \lim_{\underline{m} \perp F} \|\alpha_{\underline{m}}(a) + [\alpha_{\underline{m}}(A)\mathcal{L}_F \alpha_{\underline{m}}(A)]\| = 0\}$.

If $(A, \alpha, \mathbb{Z}_+^d)$ is injective, then X_α is regular. In turn, we derive a bijection between the set of gauge-invariant ideals of $\mathcal{NO}_{X_\alpha}$ and the set of single ideals I of A such that $\alpha_{\underline{n}}(I) \subseteq I$ and $\alpha_{\underline{n}}^{-1}(I) \subseteq I$ for all $\underline{n} \in \mathbb{Z}_+^d$ (see Corollary 5.3.14). When $(A, \alpha, \mathbb{Z}_+^d)$ is automorphic, we recover the well-known parametrisation of the gauge-invariant ideals of the crossed product $A \rtimes_\alpha \mathbb{Z}^d$ by ideals $I \subseteq A$ satisfying $\alpha_{\underline{n}}(I) = I$ for all $\underline{n} \in \mathbb{Z}_+^d$ (see Corollary 5.3.15).

The third class of examples that we consider relates to finitely aligned higher-rank graphs (Λ, d). We write $X(\Lambda)$ for the associated product system. In keeping with the literature, we write k for the rank and reserve d for the degree map. We use r and s to denote the range and source maps, respectively. We start by considering the class of strong finitely aligned higher-rank graphs, as coined in [17, Definition 7.2]. This class contains row-finite k-graphs as a subclass. The key property is that the strong covariance ideals are characterised in terms of F-tracing vertices [17, Definition 7.5] for all $\emptyset \neq F \subseteq [k]$. A family $H = \{H_F\}_{F \subseteq [k]}$ of vertex sets is called *absorbent (in Λ)* if the following holds for every $\emptyset \neq F \subsetneq [k]$: a vertex $v \in \Lambda^{\underline{0}}$ belongs to H_F whenever it satisfies

(i) v is F-tracing,

(ii) $s(v\Lambda^{\underline{m}}) \subseteq \cap_{F \subsetneq D} H_D$ for all $\underline{m} \perp F$, and

(iii) there exists $\underline{m} \perp F$ such that whenever $\underline{n} \perp F$ and $\underline{n} \geq \underline{m}$, we have that

$$s(v\Lambda^{\underline{n}}) \subseteq H_F \quad \text{and} \quad |v\Lambda^{\underline{n}}| < \infty.$$

By translating our characterisation of NT-2^k-tuples into properties on vertex sets, we obtain the following corollary.

Corollary G. *(Proposition 5.4.19) Let (Λ, d) be a strong finitely aligned k-graph. Let $\mathcal{L}$ be a 2^k-tuple of $X(\Lambda)$ that consists of ideals and let $H_{\mathcal{L}}$ be the corresponding family of sets of vertices of Λ. Then $\mathcal{L}$ is an NT-2^k-tuple of $X(\Lambda)$ if and only if the following four conditions hold:*

(i) for each $\emptyset \neq F \subseteq [k]$, the set $H_{\mathcal{L},F}$ is contained in the union of $H_{\mathcal{L},\emptyset}$ and the set

$$\{v \in H_{\mathcal{L},\emptyset}^c \mid |v\Gamma(\Lambda \setminus H_{\mathcal{L},\emptyset})^{\underline{i}}| < \infty \; \forall i \in [k] \text{ and } v \text{ is not an } F\text{-source in } \Gamma(\Lambda \setminus H_{\mathcal{L},\emptyset})\},$$

(ii) $H_{\mathcal{L}}$ is hereditary in Λ,

(iii) $H_{\mathcal{L}}$ is partially ordered,

(iv) $H_{\mathcal{L}} \setminus H_{\mathcal{L},\emptyset} := \{H_{\mathcal{L},F} \setminus H_{\mathcal{L},\emptyset}\}_{F \subseteq [k]}$ is absorbent in $\Gamma(\Lambda \setminus H_{\mathcal{L},\emptyset})$.

We obtain the following translation for row-finite k-graphs due to Proposition 4.1.5

Corollary H. *(Proposition 5.4.20) Let (Λ, d) be a row-finite k-graph. Let $\mathcal{L}$ be a 2^k-tuple of $X(\Lambda)$ that consists of ideals and let $H_{\mathcal{L}}$ be the corresponding family of sets of vertices of Λ. Then $\mathcal{L}$ is an NT-2^k-tuple of $X(\Lambda)$ if and only if the following four conditions hold:*

(i) for each $\emptyset \neq F \subseteq [k]$, the set $H_{\mathcal{L},F}$ is contained in the union of $H_{\mathcal{L},\emptyset}$ and the set

$$H_F := \{v \in H_{\mathcal{L},\emptyset}^c \mid v \text{ is not an } F\text{-source in } \Gamma := \Gamma(\Lambda \setminus H_{\mathcal{L},\emptyset})\},$$

(ii) $H_{\mathcal{L}}$ *is hereditary in* Λ*,*

(iii) $H_{\mathcal{L}}$ *is partially ordered,*

(iv) $H_{1,F} \cap H_{2,F} \cap H_{3,F} \subseteq H_{\mathcal{L},F}$ *for all* $\emptyset \neq F \subsetneq [k]$*, where*

- $H_{1,F} := \bigcap_{\underline{n} \perp F} \{v \in \Lambda^{\underline{0}} \mid s(v\Lambda^{\underline{n}}) \subseteq H_{\mathcal{L},\emptyset} \cup H_F\}$,
- $H_{2,F} := \bigcap_{\underline{m} \perp F} \{v \in \Lambda^{\underline{0}} \mid s(v\Lambda^{\underline{m}}) \subseteq \cap_{F \subsetneq D} H_{\mathcal{L},D}\}$,
- $H_{3,F}$ *is the set of all* $v \in \Lambda^{\underline{0}}$ *for which there exists* $\underline{m} \perp F$ *such that whenever* $\underline{n} \perp F$ *and* $\underline{n} \geq \underline{m}$*, we have that* $s(v\Lambda^{\underline{n}}) \subseteq H_{\mathcal{L},F}$*.*

If (Λ, d) is locally convex and row-finite, then positive (and negative) invariance of an ideal is equivalent to the related set of vertices being hereditary (and saturated). This follows by Proposition 5.4.25. In this case the NO-2^k-tuples $\mathcal{L}$ are determined solely by $\mathcal{L}_\emptyset$, from which it follows that our results recover the parametrisation of Raeburn, Sims and Yeend [50, Theorem 5.2] (see Corollary 5.4.27).

As a final case study, we restrict our attention to product systems X over $\mathbb{Z}_+^d$ wherein each fibre admits a finite frame (except perhaps for the coefficient algebra A). In connection with the parametrisation of the KMS-states, we exploit a decomposition of X with respect to a fixed $\emptyset \neq F \subseteq [d]$. One direction of this construction was implicit in [32], and here we close the circle. We define

$$B_X^{F^\perp} := \mathrm{C}^*(\overline{\pi}_X(A), \overline{t}_{X,\underline{i}}(X_{\underline{i}}) \mid i \in F^c) \subseteq \mathcal{NT}_X,$$

a collection $Z_X^{F^\perp} := \{X_{\underline{n}}\}_{\underline{n} \perp F}$ and a collection $Y_X^F := \{Y_{X,\underline{n}}^F\}_{\underline{n} \in \mathrm{supp}^{-1}(F)}$ by

$$Y_{X,\underline{0}}^F := B_X^{F^\perp} \quad \text{and} \quad Y_{X,\underline{n}}^F := [\overline{t}_{X,\underline{n}}(X_{\underline{n}}) B_X^{F^\perp}] \subseteq \mathcal{NT}_X \text{ for all } \underline{0} \neq \underline{n} \in \mathrm{supp}^{-1}(F).$$

The collections $Z_X^{F^\perp}$ and Y_X^F become product systems under the structure inherited from X and $\mathcal{NT}_X$, respectively, and satisfy

$$\mathcal{NT}_{Z_X^{F^\perp}} \cong B_X^{F^\perp} \quad \text{and} \quad \mathcal{NT}_{Y_X^F} \cong \mathcal{NT}_X.$$

The final two assertions follow from Proposition 5.5.15 and Theorem 5.5.20, respectively. We are interested in describing the quotient of $\mathcal{NT}_X$ by $\left\langle \overline{\pi}_X(A) \overline{q}_{X,\underline{i}} \mid i \in F \right\rangle$ as a Cuntz-Nica-Pimsner algebra. We provide two equivalent approaches, differing only at which point we wish to delete the kernel.

Corollary I. *(Corollary 5.5.26) Let* X *be a product system over* $\mathbb{Z}_+^d$ *with coefficients in a* C^**-algebra* A*, wherein* $X_{\underline{i}}$ *admits a finite frame for all* $i \in [d]$*, and fix* $\emptyset \neq F \subseteq [d]$*. On the one hand, define the positively invariant ideal*

$$I_{Y_X^F} := \ker\{Y_{X,\underline{0}}^F \to \mathcal{NT}_{Y_X^F} / \left\langle \overline{\pi}_{Y_X^F}(Y_{X,\underline{0}}^F) \overline{q}_{Y_X^F,\underline{i}} \mid i \in F \right\rangle\}$$

for the product system Y_X^F *related to* X *and* F*. On the other hand, define the positively invariant ideal*

$$I_X^F := \ker\{A \to \mathcal{NT}_X / \left\langle \overline{\pi}_X(A)\overline{q}_{X,\underline{i}} \mid i \in F \right\rangle\}$$

for X*, and consider the product system* $Y^F_{[X]_{I_X^F}}$ *related to* $[X]_{I_X^F}$ *and* F*. Then there are canonical* $*$*-isomorphisms*

$$\mathcal{NO}_{[Y_X^F]_{I_{Y_X^F}}} \cong \mathcal{NT}_X / \left\langle \overline{\pi}_X(A)\overline{q}_{X,\underline{i}} \mid i \in F \right\rangle \cong \mathcal{NO}_{Y^F_{[X]_{I_X^F}}}.$$

If in addition $X_{\underline{i}}$ *is injective for all* $i \in F$*, then* Y_X^F *is regular,* $I_{Y_X^F} = \{0\}$ *and* $I_X^F = \{0\}$*.*

In turn, we arrive at another avenue for obtaining the results of [32]. The F^c-equivariant KMS-states of $\mathcal{NT}_X$ that annihilate $\left\langle \overline{\pi}_X(A)\overline{q}_{X,\underline{i}} \mid i \in F \right\rangle$ can be obtained from tracial states of A annihilating I_X^F by first inducing a KMS-state of finite type on the Toeplitz-Nica-Pimsner algebra of $Z^{F\perp}_{[X]_{I_X^F}}$ (by using the Fock space construction) and then extending it to a KMS-state on the Cuntz-Nica-Pimsner algebra of $Y^F_{[X]_{I_X^F}}$ (by using a direct limit argument on the fixed point algebra).

Finally, we explore the connection between our main result and that of [4] in the case of a proper product system X. Adopting the nomenclature of [4], we say that a 2^d-tuple $\mathcal{L}$ of X is a *T-family* if it consists of ideals and satisfies

$$\mathcal{L}_F = X_{\underline{i}}^{-1}(\mathcal{L}_F) \cap \mathcal{L}_{F\cup\{i\}} \text{ for all } F \subsetneq [d], i \in [d] \setminus F.$$

We say that a T-family $\mathcal{L}$ is an *O-family* if $\mathcal{I} \subseteq \mathcal{L}$. The main result of [4] asserts that the gauge-invariant ideals of $\mathcal{NT}_X$ (resp. $\mathcal{NO}_X$) are in order-preserving bijection with the T-families (resp. O-families) of X. We show that this aligns with our parametrisation by demonstrating that the T-families (resp. O-families) of X are exactly the NT-2^d-tuples (resp. NO-2^d-tuples) of X. We show that the passage from NT-2^d-tuples to T-families can be achieved directly (i.e., using the definitions alone). A direct passage from T-families to NT-2^d-tuples is less clear, so instead we take a detour via Nica-covariant representations and exploit results of both the current work and [4]. The corresponding result for O-families and NO-2^d-tuples follows as an immediate consequence.

We close this section by noting that the problem of parametrising the gauge-invariant ideals in the case of a general product system necessitates a different approach, as there is no direct connection to ideals of the coefficient algebra alone. Indeed, this is the case even for the reduced strong covariant algebra [18, 53, 54].

1.5 Contents of chapters

In Chapter 2 we provide a detailed exposition on the aspects of C*-correspondence and product system theory that we will need. Upon collecting the requisite results concerning C*-correspondences, we present Katsura's parametrisation of gauge-invariant ideals [36].

We then move on to consider product systems, commencing at full generality before moving on to compactly aligned product systems over right LCM semigroups and finally strong compactly aligned product systems over $\mathbb{Z}_+^d$. We pay particular attention to the quotient product system construction and unitary equivalence of product systems, both of which are used frequently in the sequel. We also present the main results of [17] and elucidate some of the key points that are used in subsequent chapters. Most importantly, we show that injectivity on the fixed point algebra reduces to checking just on the $[\underline{0}, \underline{1}_{[d]}]$-core, a trick that was used implicitly in [17]. We also study the IXI construction and show an association between $\mathcal{NT}_{IXI}$ (resp. $\mathcal{NO}_{IXI}$) and $\mathcal{NT}_X$ (resp. $\mathcal{NO}_X$) that is exploited in Section 5.4. Full proofs are provided throughout.

In Chapter 3 we focus on 2^d-tuples of a fixed strong compactly aligned product system X and the gauge-invariant ideals that they induce. We give a step-by-step analysis of relativity, invariance and partial ordering, and we illustrate the maximality condition required for our parametrisation. We then proceed to consider the "injective" case, centring on (E)-2^d-tuples and maximal (E)-2^d-tuples. Upon clarifying the role of the latter in the parametrisation of gauge-invariant ideals of $\mathcal{NT}_X$, we turn our attention to capturing maximality using product system operations alone, without reference to Nica-covariant representations. We end the chapter by providing an algorithm for computing maximal (E)-2^d-tuples and a Gauge-Invariant Uniqueness Theorem for the equivariant quotients of $\mathcal{NT}_X$ in-between $\mathcal{NT}_X$ and $\mathcal{NO}_X$.

In Chapter 4 we address the "non-injective" case. We demonstrate the interaction between the quotient product system construction and Nica-covariant representations of X. We exploit this interaction to pass back to the "injective" case, thereby providing a full parametrisation of the gauge-invariant ideals of $\mathcal{NT}_X$ by NT-2^d-tuples. We also demonstrate how the parametrisation descends to the relative Cuntz-Nica-Pimsner algebras. Moreover, we study the induced lattice structure on the parametrising families and give an explicit description of the join and meet operations.

In Chapter 5 we give applications of our results and connections with the literature. First we study positive and negative invariance of ideals and how it relates to the machinery developed in prior chapters. We then move on to consider regular product systems, accounting for the case where the coefficient algebra is simple. We proceed to interpret our parametrisation in the settings of C*-dynamical systems and higher-rank graphs, via the structural data that characterises these objects. In turn, we show that our parametrisation recovers various results from the literature. We close this chapter by examining product systems on finite frames, in connection with [32].

In Chapter 6 we focus on proper product systems, clarifying the connection between our main result and that of [4]. More precisely, we show that the NT-2^d-tuples of a proper product system X are exactly the T-families of X in the sense of [4]. The passage from NT-2^d-tuples to T-families is accomplished directly (i.e., using the definitions alone). The passage from T-families to NT-2^d-tuples is achieved by considering Nica-covariant representations and utilising results of both the current work and [4]. We provide an avenue via which a T-family could be proven to be an NT-2^d-tuple directly. We show that this approach is successful in the particular context of row-finite higher-rank graphs.

In Appendix A we provide a proof of the Hewitt-Cohen Factorisation Theorem, which is used regularly in the main chapters. We also give a proof that $\mathcal{NO}_X$ satisfies a well-known co-universal property in the specific case of a strong compactly aligned product system X. We proceed via the theory of nonselfadjoint operator algebras, and clarify the connection between $\mathcal{NO}_X$ and the C*-envelope of the tensor algebra of X in the process.

Chapter 2

C*-correspondences and product systems

2.1 Notation

By a lattice we will always mean a distributive lattice with operations $\vee$ and $\wedge$. We write $\mathbb{Z}_+$ for the nonnegative integers $\{0, 1, \dots\}$ and $\mathbb{N}$ for the positive integers $\{1, 2, \dots\}$. We denote the unit circle in the complex plane by $\mathbb{T}$. If A, B and C are sets and $f \colon A \times B \to C$ is a map, then we set

$$f(A, B) := \{f(a, b) \mid a \in A, b \in B\};$$

for example, if H is a Hilbert space, then $\langle H, H \rangle := \{\langle \xi, \eta \rangle \mid \xi, \eta \in H\}$. If V is a normed vector space and $S \subseteq V$ is a subset, then $[S]$ denotes the norm-closed linear span of S inside V. If we only wish to take the linear span then this will always be clearly stated. We recall the Hewitt-Cohen Factorisation Theorem, e.g., [52, Proposition 2.33].

Theorem 2.1.1 (Hewitt-Cohen Factorisation Theorem)**.** *Let A be a C*-algebra, X be a Banach space and $\pi \colon A \to \mathcal{B}(X)$ be a bounded homomorphism. Then $[\pi(A)X] = \pi(A)X$.*

Proof. See Appendix A.1. □

All ideals of C*-algebras are taken to be two-sided and norm-closed. If A is a C*-algebra and $S \subseteq A$ is a subset, then $\langle S \rangle$ denotes the ideal of A generated by S. If $I \subseteq A$ is an ideal, then we set $I^\perp := \{a \in A \mid aI = \{0\}\}$.

If $A = \mathrm{C}^*(a_i \mid i \in \mathbb{I})$ and $B = \mathrm{C}^*(b_i \mid i \in \mathbb{I})$ are C*-algebras, then a map $\Phi \colon A \to B$ is called *canonical* if it preserves generators of the same index, i.e., $\Phi(a_i) = b_i$ for all $i \in \mathbb{I}$.

2.2 C*-correspondences

We assume familiarity with the elementary theory of right Hilbert C*-modules. The reader is addressed to [40, 43] for an excellent introduction to the subject. We will briefly outline the fundamentals of the theory of C*-correspondences. We also recount Katsura's parametrisation of gauge-invariant ideals [36].

Let A be a C*-algebra and X be a right Hilbert A-module. We write $\mathcal{L}(X)$ for the C*-algebra of adjointable operators on X, and $\mathcal{K}(X)$ for the ideal of (generalised) compact operators on X. Recall that $\mathcal{K}(X)$ is densely spanned by the rank-one operators $\Theta^X_{\xi,\eta} \colon \zeta \mapsto \xi \langle \eta, \zeta \rangle$, for $\xi, \eta, \zeta \in X$. When the right Hilbert C*-module X is clear from the context, we will write $\Theta_{\xi,\eta}$ instead of $\Theta^X_{\xi,\eta}$. We remind of the following useful fact.

Lemma 2.2.1. *Let X be a right Hilbert module over a C*-algebra A and let $(k_\lambda)_{\lambda\in\Lambda}$ be an approximate unit of $\mathcal{K}(X)$. Then we have that*

$$\|\cdot\| \text{-}\lim_\lambda k_\lambda \xi = \xi \text{ for all } \xi \in X.$$

Proof. First recall that $X[\langle X, X\rangle]$ is dense in X, e.g., [40, p. 5]. Thus it suffices to prove the claim for $\xi \in X[\langle X, X\rangle]$, as an $\varepsilon/3$ argument[1] then yields the result in full generality. By analogous reasoning, it suffices to prove the claim for $\xi \in X \langle X, X\rangle$.

Accordingly, fix $\xi, \eta, \zeta \in X$ and $\lambda \in \Lambda$. We obtain that

$$\|k_\lambda(\xi \langle \eta, \zeta\rangle) - \xi \langle \eta, \zeta\rangle \| = \|(k_\lambda \Theta_{\xi,\eta} - \Theta_{\xi,\eta})(\zeta)\| \leq \|k_\lambda \Theta_{\xi,\eta} - \Theta_{\xi,\eta}\| \cdot \|\zeta\|,$$

from which it follows that $\|\cdot\|\text{-}\lim_\lambda k_\lambda(\xi\langle \eta,\zeta\rangle) = \xi \langle \eta, \zeta\rangle$ since $\|\cdot\|\text{-}\lim_\lambda k_\lambda \Theta_{\xi,\eta} = \Theta_{\xi,\eta}$. This finishes the proof. □

A *C*-correspondence* X over a C*-algebra A is a right Hilbert A-module equipped with a left action implemented by a $*$-homomorphism $\phi_X \colon A \to \mathcal{L}(X)$. When the left action is clear from the context, we will abbreviate $\phi_X(a)\xi$ as $a\xi$, for $a \in A$ and $\xi \in X$. We say that X is *non-degenerate* if $[\phi_X(A)X] = X$. If ϕ_X is injective, then we say that X is *injective*. If X is injective and $\phi_X(A) \subseteq \mathcal{K}(X)$, then we say that X is *regular*.

Any C*-algebra A can be viewed as a non-degenerate C*-correspondence over itself, with right (resp. left) action given by right (resp. left) multiplication in A, and A-valued inner product given by $\langle a, b\rangle = a^*b$ for all $a, b \in A$. Then $A \cong \mathcal{K}(A)$ by the left action ϕ_A, and thus A is non-degenerate by an application of an approximate unit.

Let X and Y be C*-correspondences over a C*-algebra A. We call an A-bimodule linear map $u \colon X \to Y$ a *unitary* if it is a surjection that preserves the A-valued inner product. If such a unitary exists, then it is adjointable, and we say that X and Y are *unitarily equivalent* (symb. $X \cong Y$).

We write $X \otimes_A Y$ for the A-balanced tensor product. Given $S \in \mathcal{L}(X)$, there exists an operator $S \otimes \mathrm{id}_Y \in \mathcal{L}(X \otimes_A Y)$ defined on simple tensors by $x \otimes y \mapsto (Sx) \otimes y$ for all $x \in X$ and $y \in Y$, e.g., [40, p. 42]. The assignment $S \mapsto S \otimes \mathrm{id}_Y$ constitutes a unital $*$-homomorphism from $\mathcal{L}(X)$ to $\mathcal{L}(X \otimes_A Y)$. In this way we can define a left action $\phi_{X\otimes_A Y}$ on $X \otimes_A Y$ by $\phi_{X\otimes_A Y}(a) = \phi_X(a) \otimes \mathrm{id}_Y$ for all $a \in A$, thereby endowing $X \otimes_A Y$ with the

[1] See the proof of Lemma A.1.1 for full details on arguments of this type.

structure of a C*-correspondence over A. The A-balanced tensor product is associative. Moreover, the right action of X yields a unitary $X \otimes_A A \to X$ determined by $\xi \otimes a \mapsto \xi a$ for all $\xi \in X$ and $a \in A$. The left action of X yields a unitary $A \otimes_A X \to [\phi_X(A)X]$ determined by $a \otimes \xi \mapsto \phi_X(a)\xi$ for all $a \in A$ and $\xi \in X$.

A *(Toeplitz) representation* (π, t) of the C*-correspondence X on $\mathcal{B}(H)$ is a pair of a $*$-homomorphism $\pi\colon A \to \mathcal{B}(H)$ and a linear map $t\colon X \to \mathcal{B}(H)$ that preserves the left action and inner product of X. Then (π, t) automatically preserves the right action of X. Every representation (π, t) on $\mathcal{B}(H)$ induces a $*$-homomorphism $\psi\colon \mathcal{K}(X) \to \mathcal{B}(H)$.

Proposition 2.2.2. *[8, Proposition 4.6.3] Let X be a C*-correspondence over a C*-algebra A and let (π, t) be a representation of X on $\mathcal{B}(H)$ for some Hilbert space H. Then there exists a unique $*$-homomorphism $\psi\colon \mathcal{K}(X) \to \mathcal{B}(H)$ such that*

$$\psi(\Theta_{\xi,\eta}) = t(\xi)t(\eta)^* \text{ for all } \xi, \eta \in X.$$

Proof. We begin by defining a map

$$\psi\colon \operatorname{span}\{\Theta_{\xi,\eta} \mid \xi, \eta \in X\} \to \mathcal{B}(H); \sum_{j=1}^{n} \Theta_{\xi_j,\eta_j} \mapsto \sum_{j=1}^{n} t(\xi_j)t(\eta_j)^*,$$

for all $\xi_j, \eta_j \in X, j \in \{1, \dots, n\}$ and $n \in \mathbb{N}$. Note that ψ is linear by construction. Fix $\xi_j, \eta_j \in X$ for all $j \in \{1, \dots, n\}$, where $n \in \mathbb{N}$. To see that ψ is well-defined and bounded, it suffices to show that

$$\|\sum_{j=1}^{n} t(\xi_j)t(\eta_j)^*\|_{\mathcal{B}(H)} \le \|\sum_{j=1}^{n} \Theta_{\xi_j,\eta_j}\|_{\mathcal{K}(X)}.$$

To this end, first we prove that

$$\|\sum_{j=1}^{n} \Theta_{\xi_j,\eta_j}\|_{\mathcal{K}(X)} = \|(\langle \xi_i, \xi_j \rangle)_{ij}^{\frac{1}{2}} (\langle \eta_i, \eta_j \rangle)_{ij}^{\frac{1}{2}}\|_{M_n(A)}, \tag{2.1}$$

noting that $(\langle \xi_i, \xi_j \rangle)_{ij}, (\langle \eta_i, \eta_j \rangle)_{ij} \in M_n(A)_+$ by [40, Lemma 4.2].

It will be useful to associate each $\xi \in X$ with an operator $\tau_0(\xi)$ defined by

$$\tau_0(\xi)\colon A \to X; a \mapsto \xi a \text{ for all } a \in A.$$

Observe that $\tau_0(\xi) \in \mathcal{L}(A, X)$, with adjoint $\tau_0(\xi)^*$ defined by

$$\tau_0(\xi)^*\colon X \to A; \eta \mapsto \langle \xi, \eta \rangle \text{ for all } \eta \in X.$$

It is routine to check that

$$\tau_0(\xi)^*\tau_0(\eta) = \phi_A(\langle \xi, \eta \rangle) \quad \text{and} \quad \tau_0(\xi)\tau_0(\eta)^* = \Theta_{\xi,\eta} \text{ for all } \xi, \eta \in X.$$

Next we define the operator $\tau_0(\underline{\xi}) := [\tau_0(\xi_1)\dots\tau_0(\xi_n)] \in \mathcal{L}(A^n, X)$, where A^n is the usual Hilbert A-module direct sum, e.g., [40, p. 5]. More precisely, we have that

$$\tau_0(\underline{\xi})(a_1,\dots,a_n) = \sum_{j=1}^n \tau_0(\xi_j)a_j = \sum_{j=1}^n \xi_j a_j \text{ for all } (a_1,\dots,a_n) \in A^n.$$

The adjoint $\tau_0(\underline{\xi})^*$ is defined by

$$\tau_0(\underline{\xi})^*\xi = (\tau_0(\xi_1)^*\xi,\dots,\tau_0(\xi_n)^*\xi) = (\langle \xi_1, \xi\rangle,\dots,\langle \xi_n,\xi\rangle) \text{ for all } \xi \in X.$$

Observe that $\tau_0(\underline{\xi})^*\tau_0(\underline{\xi}) \in \mathcal{L}(A^n)_+$ by [40, Lemma 4.1]. Note also that $\tau_0(\underline{\xi})^*\tau_0(\underline{\xi})$ is associated with the matrix $(\phi_A(\langle \xi_i, \xi_j\rangle))_{ij}$ under the usual identification $\mathcal{L}(A^n) \cong M_n(\mathcal{L}(A))$. This matrix is nothing but $\phi_A^{(n)}((\langle \xi_i,\xi_j\rangle)_{ij})$, where $\phi_A^{(n)} \colon M_n(A) \to M_n(\mathcal{L}(A))$ is the n-th ampliation of ϕ_A (see Appendix A.2).

We also define the operator $\tau_0(\underline{\eta}) := [\tau_0(\eta_1)\dots\tau_0(\eta_n)] \in \mathcal{L}(A^n, X)$. We obtain that

$$\sum_{j=1}^n \Theta_{\xi_j,\eta_j} = \tau_0(\underline{\xi})\tau_0(\underline{\eta})^*.$$

Next, given any C*-algebra B and elements $a, b \in B$, repeated applications of the C*-identity yield that

$$\|ab^*\|_B^2 = \|ba^*ab^*\|_B = \|(a^*a)^{\frac{1}{2}}b^*\|_B^2 = \|(a^*a)^{\frac{1}{2}}b^*b(a^*a)^{\frac{1}{2}}\|_B = \|(a^*a)^{\frac{1}{2}}(b^*b)^{\frac{1}{2}}\|_B^2. \quad (2.2)$$

Therefore, identifying $\mathcal{L}(A^n)$ and $\mathcal{L}(X)$ isometrically within the matrix C*-algebra

$$\begin{pmatrix} \mathcal{L}(A^n) & \mathcal{L}(X, A^n) \\ \mathcal{L}(A^n, X) & \mathcal{L}(X) \end{pmatrix},$$

we deduce that

$$\|\sum_{j=1}^n \Theta_{\xi_j,\eta_j}\|_{\mathcal{K}(X)} = \|\tau_0(\underline{\xi})\tau_0(\underline{\eta})^*\|_{\mathcal{K}(X)} = \|(\tau_0(\underline{\xi})^*\tau_0(\underline{\xi}))^{\frac{1}{2}}(\tau_0(\underline{\eta})^*\tau_0(\underline{\eta}))^{\frac{1}{2}}\|_{\mathcal{L}(A^n)}.$$

Here we also use that $*$-homomorphisms preserve square roots. By the preceding remarks, we have that

$$\begin{aligned}\|(\tau_0(\underline{\xi})^*\tau_0(\underline{\xi}))^{\frac{1}{2}}(\tau_0(\underline{\eta})^*\tau_0(\underline{\eta}))^{\frac{1}{2}}\|_{\mathcal{L}(A^n)} &= \|\phi_A^{(n)}((\langle \xi_i,\xi_j\rangle)_{ij}^{\frac{1}{2}}(\langle \eta_i,\eta_j\rangle)_{ij}^{\frac{1}{2}})\|_{M_n(\mathcal{L}(A))} \\ &= \|(\langle \xi_i,\xi_j\rangle)_{ij}^{\frac{1}{2}}(\langle \eta_i,\eta_j\rangle)_{ij}^{\frac{1}{2}}\|_{M_n(A)},\end{aligned}$$

using that $\phi_A^{(n)}$ is injective and thus isometric in the final equality. We conclude that (2.1) holds, as claimed.

Returning to the proof, we define an operator $\sigma(\underline{\xi}) \in \mathcal{B}(H^n, H)$ by

$$\sigma(\underline{\xi})(h_1, \ldots, h_n) = \sum_{j=1}^{n} t(\xi_j)h_j \text{ for all } (h_1, \ldots, h_n) \in H^n,$$

where H^n is the usual Hilbert space direct sum. Analogously, we define an operator $\sigma(\underline{\eta}) \in \mathcal{B}(H^n, H)$. The operator $\sigma(\underline{\xi})^* \in \mathcal{B}(H, H^n)$ is determined by

$$\sigma(\underline{\xi})^* h = (t(\xi_1)^* h, \ldots, t(\xi_n)^* h) \text{ for all } h \in H.$$

We obtain that

$$\sum_{j=1}^{n} t(\xi_j)t(\eta_j)^* = \sigma(\underline{\xi})\sigma(\underline{\eta})^*.$$

Arguing as in the proof of (2.1), we have that

$$\begin{aligned} \|\sigma(\underline{\xi})\sigma(\underline{\eta})^*\|_{\mathcal{B}(H)} &= \|(\sigma(\underline{\xi})^*\sigma(\underline{\xi}))^{\frac{1}{2}}(\sigma(\underline{\eta})^*\sigma(\underline{\eta}))^{\frac{1}{2}}\|_{\mathcal{B}(H^n)} \\ &= \|(t(\xi_i)^*t(\xi_j))_{ij}^{\frac{1}{2}}(t(\eta_i)^*t(\eta_j))_{ij}^{\frac{1}{2}}\|_{M_n(\mathcal{B}(H))} \\ &= \|(\pi(\langle \xi_i, \xi_j \rangle))_{ij}^{\frac{1}{2}}(\pi(\langle \eta_i, \eta_j \rangle))_{ij}^{\frac{1}{2}}\|_{M_n(\mathcal{B}(H))}, \end{aligned}$$

using the canonical identification $\mathcal{B}(H^n) \cong M_n(\mathcal{B}(H))$ in the second equality. Consider the ampliation $\pi^{(n)} \colon M_n(A) \to M_n(\mathcal{B}(H))$, which is a $*$-homomorphism and therefore contractive. We obtain that

$$\begin{aligned} \|(\pi(\langle \xi_i, \xi_j \rangle))_{ij}^{\frac{1}{2}}(\pi(\langle \eta_i, \eta_j \rangle))_{ij}^{\frac{1}{2}}\|_{M_n(\mathcal{B}(H))} &= \|\pi^{(n)}((\langle \xi_i, \xi_j \rangle)_{ij}^{\frac{1}{2}}(\langle \eta_i, \eta_j \rangle)_{ij}^{\frac{1}{2}})\|_{M_n(\mathcal{B}(H))} \\ &\leq \|(\langle \xi_i, \xi_j \rangle)_{ij}^{\frac{1}{2}}(\langle \eta_i, \eta_j \rangle)_{ij}^{\frac{1}{2}}\|_{M_n(A)} = \|\sum_{j=1}^{n} \Theta_{\xi_j, \eta_j}\|_{\mathcal{K}(X)}, \end{aligned}$$

using (2.1) in the final equality. Thus $\| \sum_{j=1}^{n} t(\xi_j)t(\eta_j)^* \|_{\mathcal{B}(H)} \leq \| \sum_{j=1}^{n} \Theta_{\xi_j, \eta_j} \|_{\mathcal{K}(X)}$ and we deduce that ψ is a well-defined bounded linear map. In turn, the map ψ extends to a bounded linear map $\psi \colon \mathcal{K}(X) \to \mathcal{B}(H)$. It is routine to verify that ψ is a $*$-homomorphism by first checking on rank-one compacts and then invoking linearity and continuity. Since ψ is determined by its action on rank-one compacts, uniqueness immediately follows. This completes the proof. □

We say that (π, t) is *injective* if π is injective; then both t and ψ are isometric. We provide a short proof to this effect.

Corollary 2.2.3. *Let X be a C*-correspondence over a C*-algebra A and let (π, t) be an injective representation of X. Then the maps t and ψ are isometric.*

Proof. Fixing $\xi \in X$, we obtain that

$$\|t(\xi)\|^2 = \|t(\xi)^*t(\xi)\| = \|\pi(\langle \xi, \xi \rangle)\| = \| \langle \xi, \xi \rangle \| = \|\xi\|^2,$$

using the C*-identity in the first equality and the fact that π is isometric in the third equality. Hence t is isometric, as required.

The fact that ψ is isometric can be seen by tweaking the proof of Proposition 2.2.2 to account for the additional information that (π, t) is injective. Using the nomenclature of the latter, the ampliation $\pi^{(n)}$ is injective since π is injective. Being a $*$-homomorphism between C*-algebras, the map $\pi^{(n)}$ is therefore isometric. Thus we obtain that

$$\|\pi^{(n)}((\langle \xi_i, \xi_j \rangle)_{ij}^{\frac{1}{2}}(\langle \eta_i, \eta_j \rangle)_{ij}^{\frac{1}{2}})\|_{M_n(\mathcal{B}(H))} = \|(\langle \xi_i, \xi_j \rangle)_{ij}^{\frac{1}{2}}(\langle \eta_i, \eta_j \rangle)_{ij}^{\frac{1}{2}}\|_{M_n(A)}$$

and therefore

$$\|\psi(\sum_{j=1}^{n} \Theta_{\xi_j, \eta_j})\|_{\mathcal{B}(H)} = \|\sum_{j=1}^{n} \Theta_{\xi_j, \eta_j}\|_{\mathcal{K}(X)}.$$

It follows that $\|\psi(k)\|_{\mathcal{B}(H)} = \|k\|_{\mathcal{K}(X)}$ for all $k \in \mathcal{K}(X)$, completing the proof. □

We write $\mathrm{C}^*(\pi, t)$ for the C*-algebra generated by $\pi(A)$ and $t(X)$. We say that (π, t) *admits a gauge action* γ if there exists a family $\{\gamma_z\}_{z \in \mathbb{T}}$ of $*$-endomorphisms of $\mathrm{C}^*(\pi, t)$ such that

$$\gamma_z(\pi(a)) = \pi(a) \text{ for all } a \in A \text{ and } \gamma_z(t(\xi)) = zt(\xi) \text{ for all } \xi \in X,$$

for each $z \in \mathbb{T}$. When such a gauge action γ exists, it is necessarily unique. We also have that each γ_z is a $*$-automorphism, the family $\{\gamma_z\}_{z \in \mathbb{T}}$ is point-norm continuous, and we obtain a group homomorphism

$$\gamma \colon \mathbb{T} \to \mathrm{Aut}(\mathrm{C}^*(\pi, t)); z \mapsto \gamma_z \text{ for all } z \in \mathbb{T}.$$

These claims are recovered in the $d = 1$ case of Proposition 2.5.3 to come, and so we defer their proofs until this point. An ideal $\mathfrak{J} \subseteq \mathrm{C}^*(\pi, t)$ is called *gauge-invariant* or *equivariant* if $\gamma_z(\mathfrak{J}) \subseteq \mathfrak{J}$ for all $z \in \mathbb{T}$ (and thus $\gamma_z(\mathfrak{J}) = \mathfrak{J}$ for all $z \in \mathbb{T}$).

The *Toeplitz-Pimsner algebra* $\mathcal{T}_X$ is the universal C*-algebra with respect to the representations of X. Let J be a subset of A satisfying $J \subseteq \phi_X^{-1}(\mathcal{K}(X))$. The *$J$-relative Cuntz-Pimsner algebra* $\mathcal{O}(J, X)$ is the universal C*-algebra with respect to the *J-covariant* representations of X; that is, the representations (π, t) of X satisfying $\pi(a) = \psi(\phi_X(a))$ for all $a \in J$. When $J = \{0\}$, we have that $\mathcal{O}(J, X) = \mathcal{T}_X$. For the ideal

$$J_X := (\ker \phi_X)^\perp \cap \phi_X^{-1}(\mathcal{K}(X)) \subseteq A,$$

we obtain that $\mathcal{O}(J_X, X)$ is the *Cuntz-Pimsner algebra* $\mathcal{O}_X$ [34].

Remark 2.2.4. Traditionally the relative Cuntz-Pimsner algebras are defined with respect to *ideals* of A rather than just subsets. The two versions are equivalent since

$$\mathcal{O}(J, X) = \mathcal{O}(\langle J \rangle, X) \text{ for all } J \subseteq \phi_X^{-1}(\mathcal{K}(X)). \tag{2.3}$$

Indeed, suppose that $J \subseteq \phi_X^{-1}(\mathcal{K}(X))$ and that (π, t) is a J-covariant representation of X. For $a \in J$ and $b, c \in A$, we have that

$$\pi(bac) = \pi(b)\pi(a)\pi(c) = \pi(b)\psi(\phi_X(a))\pi(c) = \psi(\phi_X(bac)), \tag{2.4}$$

using that

$$\begin{aligned}\pi(b)\psi(\Theta_{\xi,\eta})\pi(c) &= \pi(b)t(\xi)t(\eta)^*\pi(c) = t(\phi_X(b)\xi)t(\phi_X(c)^*\eta)^* \\ &= \psi(\Theta_{\phi_X(b)\xi,\phi_X(c)^*\eta}) = \psi(\phi_X(b)\Theta_{\xi,\eta}\phi_X(c))\end{aligned}$$

for all $\xi, \eta \in X$, using [40, p. 9, (1.6)] in the final equality. It then follows that $\pi(b)\psi(k)\pi(c) = \psi(\phi_X(b)k\phi_X(c))$ for all $k \in \mathcal{K}(X)$. Applying for $k = \phi_X(a)$ yields (2.4). Since π, ψ and ϕ_X are continuous and (in particular) linear, it follows that (π, t) is $\langle J \rangle$-covariant. We then obtain (2.3), since any $\langle J \rangle$-covariant representation is J-covariant.

The significance of Katsura's ideal J_X is encapsulated by the following proposition, which follows from [34, Lemma 2.2]. We include a direct proof for convenience.

Proposition 2.2.5. *[34] Let X be a C*-correspondence over a C*-algebra A. Then the ideal J_X is the largest ideal of A to which the restriction of ϕ_X is injective with image contained in $\mathcal{K}(X)$.*

Proof. First we check that $\phi_X|_{J_X}$ is injective and that $\phi_X(J_X) \subseteq \mathcal{K}(X)$. The latter is immediate by definition of J_X. For the former, it suffices to show that $\ker \phi_X|_{J_X} = \{0\}$. Indeed, we have that

$$\ker \phi_X|_{J_X} = \ker \phi_X \cap J_X = \ker \phi_X \cap (\ker \phi_X)^\perp \cap \phi_X^{-1}(\mathcal{K}(X)) = \{0\},$$

using that $\ker \phi_X \cap (\ker \phi_X)^\perp = \{0\}$ in the final equality.

Thus we have that J_X is an ideal of A to which the restriction of ϕ_X is injective with image contained in $\mathcal{K}(X)$. To see that J_X is the largest such ideal, fix another ideal I with this property. It suffices to show that $I \subseteq J_X$. Fixing $a \in I$, we have that $\phi_X(a) \in \mathcal{K}(X)$ by assumption, and hence $a \in \phi_X^{-1}(\mathcal{K}(X))$. It remains to verify that $a \in (\ker \phi_X)^\perp$. To this end, fix $b \in \ker \phi_X$. Since I is an ideal, we have that $ab \in I$. Likewise, since $\ker \phi_X$ is an ideal, we have that $ab \in \ker \phi_X$. Hence $ab \in \ker \phi_X \cap I = \ker \phi_X|_I = \{0\}$, where the final equality holds since $\phi_X|_I$ is injective. This shows that $a \in (\ker \phi_X)^\perp$ and so in total $a \in J_X$. Hence $I \subseteq J_X$, finishing the proof. □

One of the main tools in the theory is the Gauge-Invariant Uniqueness Theorem, obtained in its full generality by Katsura [36]. An alternative proof can be found in [30], and Frei [26] extended this method to include all relative Cuntz-Pimsner algebras, in connection with [36].

Theorem 2.2.6 ($\mathbb{Z}_+$-GIUT)**.** *[36, Corollary 11.8] Let X be a C*-correspondence over a C*-algebra A, let $J \subseteq A$ be an ideal satisfying $J \subseteq J_X$ and let (π, t) be a representation of*

X. Then $\mathcal{O}(J,X) \cong \mathrm{C}^*(\pi,t)$ via a (unique) canonical $*$-isomorphism if and only if (π,t) is injective, admits a gauge action and satisfies $\pi^{-1}(\psi(\mathcal{K}(X))) = J$.

Let X be a right Hilbert module over a C*-algebra A and let $I \subseteq A$ be an ideal. Then the set XI is a closed linear subspace of X that is invariant under the right action of A. In particular, we have that $[XI] = XI$. Thus XI is itself a right Hilbert A-module under the structure inherited from X. We include a full proof for completeness.

Proposition 2.2.7. *[24, p. 576], [36, Corollary 1.4] Let X be a right Hilbert module over a C*-algebra A and let $I \subseteq A$ be an ideal. Then XI is a closed linear subspace of X that is also a right A-submodule of X. Thus XI is itself a Hilbert A-module.*

Proof. First note that for any $\xi \in X$, we have that $\xi \in XI$ if and only if $\langle \eta, \xi \rangle \in I$ for all $\eta \in X$. To see this, first assume that $\xi \in XI$ and fix $\eta \in X$. By assumption we may write $\xi = \zeta a$ for some $\zeta \in X$ and $a \in I$. Hence we obtain that

$$\langle \eta, \xi \rangle = \langle \eta, \zeta a \rangle = \langle \eta, \zeta \rangle a \in I,$$

as required. Now assume that $\langle \eta, \xi \rangle \in I$ for all $\eta \in X$. In particular, we have that $\langle \xi, \xi \rangle \in I$. Consequently, we deduce that $\langle \xi, \xi \rangle^{\frac{1}{4}} \in I$. An application of [40, Lemma 4.4] then gives that $\xi = \zeta \langle \xi, \xi \rangle^{\frac{1}{4}}$ for some $\zeta \in X$. Hence $\xi \in XI$, establishing the equivalence.

In turn, we have that

$$XI = \{\xi \in X \mid \langle \eta, \xi \rangle \in I \text{ for all } \eta \in X\}.$$

The fact that XI is a linear subspace of X that is also a right A-submodule now follows from the A-valued inner product axioms. Likewise, the fact that XI is closed in X follows by using that the A-valued inner product satisfies a Cauchy-Schwarz inequality, together with the fact that I is closed in A. This finishes the proof. □

We may also view XI as a right Hilbert I-module. We will identify $\mathcal{K}(XI)$ as an ideal of $\mathcal{K}(X)$ in the following natural way:

$$\mathcal{K}(XI) = \overline{\operatorname{span}}\{\Theta^X_{\xi,\eta} \mid \xi, \eta \in XI\} \subseteq \mathcal{K}(X).$$

The veracity of this identification follows from [24, Lemma 2.6], or by tweaking the proof of Lemma 2.6.1 to come. When X is in addition a C*-correspondence over A, we may equip XI with a C*-correspondence structure via the left action

$$\phi_{XI} \colon A \to \mathcal{L}(XI); \phi_{XI}(a) = \phi_X(a)|_{XI} \text{ for all } a \in A.$$

By restricting ϕ_{XI} to I, we may also view XI as a C*-correspondence over I.

Following [36], and in order to ease notation, we will use the symbol $[\cdot]_I$ to denote the quotient maps associated with a right Hilbert A-module X and an ideal $I \subseteq A$. For example, we use it for both the quotient map $A \to A/I \equiv [A]_I$ and the quotient map $X \to X/XI \equiv [X]_I$. Next we proceed in steps to endow $[X]_I$ with the structure of a right Hilbert $[A]_I$-module.

Lemma 2.2.8. *Let X be a right Hilbert module over a C*-algebra A and let $I \subseteq A$ be an ideal. Then $[X]_I$ carries the structure of a linear space and a right $[A]_I$-module, where the module multiplication is implemented by*

$$[\xi]_I[a]_I = [\xi a]_I \textit{ for all } \xi \in X, a \in A.$$

The scalar multiplication is compatible with the module multiplication in the sense that

$$\lambda([\xi]_I[a]_I) = (\lambda[\xi]_I)[a]_I = [\xi]_I(\lambda[a]_I) \textit{ for all } \lambda \in \mathbb{C}, \xi \in X, a \in A.$$

Proof. We equip $[X]_I$ with the usual quotient vector space structure, recalling that XI is in particular a linear subspace of X by Proposition 2.2.7. To see that the module multiplication is well-defined, take $\xi, \eta \in X$ and $a, b \in A$ and assume that $[\xi]_I = [\eta]_I$ and $[a]_I = [b]_I$. Then $\xi = \eta + \zeta$ for some $\zeta \in XI$ and $a = b + c$ for some $c \in I$. We have that

$$[\xi]_I[a]_I = [\xi a]_I = [(\eta + \zeta)(b + c)]_I = [\eta b + \eta c + \zeta b + \zeta c]_I = [\eta b]_I + [\eta c]_I + [\zeta b]_I + [\zeta c]_I.$$

Note that $\eta c, \zeta b$ and ζc belong to XI by definition. Hence we obtain that

$$[\xi]_I[a]_I = [\eta b]_I = [\eta]_I[b]_I,$$

as required. It is routine to check that this operation obeys the module axioms using the corresponding axioms for the module multiplication of X. It is similarly straightforward to verify that the compatibility condition for $[X]_I$ holds by using the compatibility condition for X. This finishes the proof. □

Lemma 2.2.9. *Let X be a right Hilbert module over a C*-algebra A and let $I \subseteq A$ be an ideal. Then $[X]_I$ carries a canonical structure as an inner-product $[A]_I$-module, where the $[A]_I$-valued inner product is defined by*

$$\langle [\xi]_I, [\eta]_I \rangle = [\langle \xi, \eta \rangle]_I \textit{ for all } \xi, \eta \in X.$$

Proof. By Lemma 2.2.8, it suffices to show that the stated map constitutes a well-defined $[A]_I$-valued inner product on $[X]_I$. Accordingly, take $\xi, \xi', \eta, \eta' \in X$ and suppose that $[\xi]_I = [\xi']_I$ and $[\eta]_I = [\eta']_I$. Then there exist $\zeta, \zeta' \in XI$ such that $\xi = \xi' + \zeta$ and $\eta = \eta' + \zeta'$. We have that

$$\langle [\xi]_I, [\eta]_I \rangle = [\langle \xi, \eta \rangle]_I = [\langle \xi' + \zeta, \eta' + \zeta' \rangle]_I = [\langle \xi', \eta' \rangle]_I + [\langle \xi', \zeta' \rangle]_I + [\langle \zeta, \eta' \rangle]_I + [\langle \zeta, \zeta' \rangle]_I.$$

Write $\zeta = \mu a$ and $\zeta' = \mu' b$ for some $\mu, \mu' \in X$ and $a, b \in I$. Notice that

$$\langle \xi', \zeta' \rangle = \langle \xi', \mu' b \rangle = \langle \xi', \mu' \rangle \, b \in I.$$

A similar computation shows that $\langle \zeta, \zeta' \rangle \in I$. Analogously, we have that

$$\langle \zeta, \eta' \rangle = \langle \mu a, \eta' \rangle = a^* \langle \mu, \eta' \rangle \in I.$$

In total, we obtain that

$$\langle [\xi]_I, [\eta]_I \rangle = [\langle \xi', \eta' \rangle]_I = \langle [\xi']_I, [\eta']_I \rangle,$$

showing that the map of the statement is well-defined.

It is routine to check that the inner product axioms are satisfied using the corresponding axioms for X. For positive-definiteness, suppose that $\langle [\xi]_I, [\xi]_I \rangle = 0$ for some $\xi \in X$. Then by definition we have that $[\langle \xi, \xi \rangle]_I = 0$, and hence $\langle \xi, \xi \rangle \in I$. An application of [36, Proposition 1.3] gives that $\xi \in XI$ and hence $[\xi]_I = 0$, finishing the proof. □

Proposition 2.2.10. *Let X be a right Hilbert module over a C*-algebra A and let $I \subseteq A$ be an ideal. Then the quotient norm on $[X]_I$ coincides with the inner product norm guaranteed by Lemma 2.2.9. In particular, the space $[X]_I$ carries a canonical structure as a right Hilbert $[A]_I$-module.*

Proof. Let $\| \cdot \|_{\text{quot}}$ and $\| \cdot \|_{\text{inn}}$ denote the quotient and inner product norms on $[X]_I$, respectively. Fixing $\xi \in X$, we have that

$$\begin{aligned} \|[\xi]_I\|_{\text{quot}}^2 &= \inf\{\|\xi + \zeta\|_X^2 \mid \zeta \in XI\} \\ &= \inf\{\| \langle \xi + \zeta, \xi + \zeta \rangle \|_A \mid \zeta \in XI\} \\ &= \inf\{\| \langle \xi, \xi \rangle + \langle \xi, \zeta \rangle + \langle \zeta, \xi \rangle + \langle \zeta, \zeta \rangle \|_A \mid \zeta \in XI\}. \end{aligned}$$

Similarly, we have that

$$\|[\xi]_I\|_{\text{inn}}^2 = \| \langle [\xi]_I, [\xi]_I \rangle \|_{[A]_I} = \|[\langle \xi, \xi \rangle]_I\|_{[A]_I} = \inf\{\| \langle \xi, \xi \rangle + a\|_A \mid a \in I\}.$$

Take $\zeta \in XI$, so that $\zeta = \eta a$ for some $\eta \in X$ and $a \in I$. We obtain that

$$\langle \xi, \xi \rangle + \langle \xi, \zeta \rangle + \langle \zeta, \xi \rangle + \langle \zeta, \zeta \rangle = \langle \xi, \xi \rangle + \langle \xi, \eta \rangle \, a + a^* \langle \eta, \xi \rangle + a^* \langle \eta, \eta \rangle \, a.$$

Notice that the final three summands belong to I, so that

$$\|[\xi]_I\|_{\text{inn}}^2 \leq \| \langle \xi, \xi \rangle + \langle \xi, \zeta \rangle + \langle \zeta, \xi \rangle + \langle \zeta, \zeta \rangle \|_A$$

by definition. It follows that $\|[\xi]_I\|_{\text{inn}} \leq \|[\xi]_I\|_{\text{quot}}$.

To see that $\|[\xi]_I\|_{\text{quot}} \leq \|[\xi]_I\|_{\text{inn}}$, fix an approximate unit $(u_\lambda)_{\lambda \in \Lambda}$ of I. An application of [44, Theorem 3.1.3] yields that

$$\|[a]_I\|_{[A]_I} = \lim_\lambda \|(1-u_\lambda)a\|_A = \lim_\lambda \|a(1-u_\lambda)\|_A \text{ for all } a \in A.$$

In turn, we obtain that

$$\begin{aligned}\|[\langle \xi, \xi\rangle]_I\|_{[A]_I} &= \|[\langle \xi, \xi\rangle^{\frac{1}{2}}]_I\|^2_{[A]_I} \\ &= \lim_\lambda \| \langle \xi, \xi\rangle^{\frac{1}{2}} (1-u_\lambda)\|_A^2 \\ &= \lim_\lambda \|(1-u_\lambda) \langle \xi, \xi\rangle (1-u_\lambda)\|_A,\end{aligned}$$

using the C*-identity in the first and third equalities. Next, observe that

$$\begin{aligned}\|(1-u_\lambda) \langle \xi, \xi\rangle (1-u_\lambda)\|_A &= \| \langle \xi, \xi\rangle - \langle \xi, \xi\rangle u_\lambda - u_\lambda \langle \xi, \xi\rangle + u_\lambda \langle \xi, \xi\rangle u_\lambda\|_A \\ &= \| \langle \xi, \xi\rangle + \langle \xi, \xi(-u_\lambda)\rangle + \langle \xi(-u_\lambda), \xi\rangle + \langle \xi(-u_\lambda), \xi(-u_\lambda)\rangle \|_A\end{aligned}$$

for all $\lambda \in \Lambda$. Since $(\xi(-u_\lambda))_{\lambda\in\Lambda} \subseteq XI$, we have that

$$\|[\xi]_I\|^2_{\text{quot}} \leq \| \langle \xi, \xi\rangle + \langle \xi, \xi(-u_\lambda)\rangle + \langle \xi(-u_\lambda), \xi\rangle + \langle \xi(-u_\lambda), \xi(-u_\lambda)\rangle \|_A \text{ for all } \lambda \in \Lambda$$

by definition, from which it follows that $\|[\xi]_I\|_{\text{quot}} \leq \|[\xi]_I\|_{\text{inn}}$. Hence the quotient and inner product norms on $[X]_I$ coincide, as required.

For the final claim, note that $[X]_I$ is complete with respect to $\|\cdot\|_{\text{quot}}$ by Proposition 2.2.7. Thus $[X]_I$ is complete with respect to $\|\cdot\|_{\text{inn}}$ by the first claim. In total, we have that $[X]_I$ is a right Hilbert $[A]_I$-module, completing the proof. □

We may define a $*$-homomorphism $[\cdot]_I \colon \mathcal{L}(X) \to \mathcal{L}([X]_I)$ by

$$[S]_I[\xi]_I = [S\xi]_I \text{ for all } S \in \mathcal{L}(X), \xi \in X.$$

We include [36, Lemma 1.6] in its entirety, as we will be making frequent reference to it.

Lemma 2.2.11. *[36, Lemma 1.6] Let X be a right Hilbert module over a C*-algebra A and let $I \subseteq A$ be an ideal. Then for all $\xi, \eta \in X$, we have that $[\Theta^X_{\xi,\eta}]_I = \Theta^{[X]_I}_{[\xi]_I,[\eta]_I}$. The restriction of the map $[\cdot]_I \colon \mathcal{L}(X) \to \mathcal{L}([X]_I)$ to $\mathcal{K}(X)$ is a surjection onto $\mathcal{K}([X]_I)$ with kernel $\mathcal{K}(XI)$.*

Proof. Fix $\xi, \eta, \zeta \in X$. We have that

$$[\Theta^X_{\xi,\eta}]_I[\zeta]_I = [\Theta^X_{\xi,\eta}(\zeta)]_I = [\xi \langle \eta, \zeta\rangle]_I = [\xi]_I[\langle \eta, \zeta\rangle]_I = [\xi]_I \langle [\eta]_I, [\zeta]_I\rangle = \Theta^{[X]_I}_{[\xi]_I,[\eta]_I}([\zeta]_I),$$

from which it follows that $[\Theta^X_{\xi,\eta}]_I = \Theta^{[X]_I}_{[\xi]_I,[\eta]_I}$, as required. Consequently, we obtain that $[\mathcal{K}(X)]_I \subseteq \mathcal{K}([X]_I)$ since $[\cdot]_I$ is in particular linear and continuous. Next, because $[\cdot]_I|_{\mathcal{K}(X)}$ is a $*$-homomorphism between C*-algebras, its image is a C*-subalgebra of $\mathcal{K}([X]_I)$.

Combining this with the fact that the generators of $\mathcal{K}([X]_I)$ are contained in the image of $[\cdot]_I|_{\mathcal{K}(X)}$, we deduce that $[\cdot]_I|_{\mathcal{K}(X)}$ is a surjection onto $\mathcal{K}([X]_I)$.

Finally, we show that $\ker[\cdot]_I|_{\mathcal{K}(X)} = \mathcal{K}(XI)$. To this end, take $\xi, \eta \in X$ and $a, b \in I$. By the first claim, we obtain that

$$[\Theta^X_{\xi a, \eta b}]_I = \Theta^{[X]_I}_{[\xi a]_I, [\eta b]_I} = \Theta^{[X]_I}_{[\xi]_I[a]_I, [\eta]_I[b]_I} = 0,$$

using that $[a]_I = [b]_I = 0$ in the last equality. Thus $\ker[\cdot]_I|_{\mathcal{K}(X)}$ contains the generators of $\mathcal{K}(XI)$. Since $\ker[\cdot]_I|_{\mathcal{K}(X)}$ is in particular a closed linear subspace of $\mathcal{K}(X)$, it follows that $\mathcal{K}(XI) \subseteq \ker[\cdot]_I|_{\mathcal{K}(X)}$.

Finally, take $k \in \mathcal{K}(X)$ and suppose that $[k]_I = 0$. Fix $\xi \in X$ and note that

$$[k]_I[\xi]_I = [k\xi]_I = 0,$$

so $k\xi \in XI$. In turn, we have that $\langle k\xi, k\xi \rangle \in I$. It follows that $a := \langle k\xi, k\xi \rangle^{\frac{1}{4}}$ is a positive element of I and thus $\sqrt{a} \in I$. An application of [40, Lemma 4.4] yields an element $\xi' \in X$ such that $k\xi = \xi' a$. Fixing $\eta \in X$, we obtain that

$$k\Theta^X_{\xi,\eta} = \Theta^X_{k\xi,\eta} = \Theta^X_{\xi' a,\eta} = \Theta^X_{\xi'\sqrt{a}, \eta\sqrt{a}} \in \mathcal{K}(XI),$$

using [40, p. 9, (1.6)] in the first equality. Since $\xi, \eta \in X$ were arbitrarily chosen, it follows that $k \cdot \mathcal{K}(X) \subseteq \mathcal{K}(XI)$, as $\mathcal{K}(XI)$ is in particular a closed linear subspace of $\mathcal{K}(X)$. By using an approximate unit of $\mathcal{K}(X)$, we therefore deduce that $k \in \mathcal{K}(XI)$. We conclude that $\ker[\cdot]_I|_{\mathcal{K}(X)} = \mathcal{K}(XI)$, finishing the proof. □

In total, given an ideal $I \subseteq A$, we obtain the surjective maps

$$A \to A/I \text{ with kernel } I,$$
$$X \to X/XI \text{ with kernel } XI,$$
$$\mathcal{K}(X) \to \mathcal{K}(X/XI) \text{ with kernel } \mathcal{K}(XI),$$

as well as the map $\mathcal{L}(X) \to \mathcal{L}(X/XI)$ (which may *not* be surjective), all of which will be denoted by the same symbol $[\cdot]_I$. Lemma 2.2.11 implies that if $k \in \mathcal{K}(X)$, then

$$k \in \mathcal{K}(XI) \iff \langle X, kX \rangle \subseteq I. \tag{2.5}$$

Consequently, we may write

$$\mathcal{K}(XI) = \overline{\operatorname{span}}\{\Theta^X_{\xi a,\eta} \mid \xi, \eta \in X, a \in I\} \subseteq \mathcal{K}(X).$$

The forward inclusion follows from the observation that $\Theta^X_{\xi a, \eta b} = \Theta^X_{\xi ab^*, \eta}$ for all $\xi, \eta \in X$ and $a, b \in I$, and the reverse inclusion follows by applying (2.5). Lemma 2.2.11 provides a straightforward way of seeing $\mathcal{K}(XI)$ as an ideal in $\mathcal{K}(X)$, and in turn as an ideal in

$\mathcal{L}(X)$. Hence we may consider the quotient C*-algebra $\mathcal{L}(X)/\mathcal{K}(XI)$.

We recall [40, Lemma 4.6], slightly rewritten to match our setting.

Lemma 2.2.12. *[40, Lemma 4.6] Let X and Y be C*-correspondences over a C*-algebra A. For $x \in X$, the equation $\Theta_x(y) = x \otimes y$ ($y \in Y$) defines an element $\Theta_x \in \mathcal{L}(Y, X \otimes_A Y)$ which satisfies*

$$\|\Theta_x\| = \|\phi_Y(\langle x, x\rangle^{1/2})\| \leq \|x\| \quad \textit{and} \quad \Theta_x^*(x' \otimes y) = \phi_Y(\langle x, x'\rangle)y \ (x' \in X, y \in Y).$$

Proof. To see that Θ_x is adjointable, we begin by defining (in an abuse of notation) a map Θ_x^* by

$$\Theta_x^* \colon X \times Y \to Y; (x', y) \mapsto \phi_Y(\langle x, x'\rangle)y \text{ for all } x' \in X, y \in Y.$$

Observe that Θ_x^* is bilinear and A-balanced in the sense that

$$\Theta_x^*(x'a, y) - \Theta_x^*(x', \phi_Y(a)y) = 0 \text{ for all } x' \in X, y \in Y, a \in A.$$

Thus Θ_x^* induces a unique linear map

$$\Theta_x^* \colon X \odot_A Y \to Y; x' \otimes y \mapsto \phi_Y(\langle x, x'\rangle)y \text{ for all } x' \in X, y \in Y.$$

Recall that $X \odot_A Y$ is an inner-product A-module (e.g., [40, Proposition 4.5]), and the completion with respect to the induced norm is $X \otimes_A Y$. Accordingly, in order to extend Θ_x^* to $X \otimes_A Y$, it suffices to show that Θ_x^* is bounded. To this end, fix $x_1, \dots, x_n \in X$ and $y_1, \dots, y_n \in Y$, where $n \in \mathbb{N}$. We have that

$$\begin{aligned}
\|\Theta_x^*(\sum_{j=1}^n x_j \otimes y_j)\|^2 &= \|\langle \sum_{j=1}^n \phi_Y(\langle x, x_j\rangle)y_j, \sum_{j=1}^n \phi_Y(\langle x, x_j\rangle)y_j\rangle\| \\
&= \| \sum_{j,k=1}^n \langle y_j, \phi_Y(\langle x_j, x\rangle \langle x, x_k\rangle)y_k\rangle \| \\
&= \| \sum_{j,k=1}^n \big\langle y_j, \phi_Y(\big\langle x_j, \Theta_{x,x}^X(x_k)\big\rangle)y_k\big\rangle \| \\
&= \| \sum_{j,k=1}^n \langle y_j, \phi_Y(\langle (\Theta_{x,x}^X)^{\frac{1}{2}}(x_j), (\Theta_{x,x}^X)^{\frac{1}{2}}(x_k)\rangle)y_k\rangle\| \\
&= \|\langle ((\Theta_{x,x}^X)^{\frac{1}{2}} \otimes \mathrm{id}_Y)(\sum_{j=1}^n x_j \otimes y_j), ((\Theta_{x,x}^X)^{\frac{1}{2}} \otimes \mathrm{id}_Y)(\sum_{j=1}^n x_j \otimes y_j)\rangle\| \\
&= \|((\Theta_{x,x}^X)^{\frac{1}{2}} \otimes \mathrm{id}_Y)(\sum_{j=1}^n x_j \otimes y_j)\|^2,
\end{aligned}$$

using that $\Theta_{x,x}^X$ is positive (e.g., [40, Lemma 4.1]) in the fourth equality. It follows that $\|\Theta_x^*\| \leq \|(\Theta_{x,x}^X)^{\frac{1}{2}} \otimes \mathrm{id}_Y\|$ and hence Θ_x^* is bounded. Consequently, we obtain a bounded linear map

$$\Theta_x^* \colon X \otimes_A Y \to Y; x' \otimes y \mapsto \phi_Y(\langle x, x' \rangle) y \text{ for all } x' \in X, y \in Y,$$

and it is routine to check that Θ_x^* is the adjoint of Θ_x. Thus $\Theta_x \in \mathcal{L}(Y, X \otimes_A Y)$.

Finally, we check that $\|\Theta_x\| = \|\phi_Y(\langle x, x \rangle^{1/2})\| \leq \|x\|$. Fixing $y \in Y$, we have that

$$\|\Theta_x(y)\|^2 = \| \langle y, \phi_Y(\langle x, x \rangle) y \rangle \| = \|\langle \phi_Y(\langle x, x \rangle^{\frac{1}{2}}) y, \phi_Y(\langle x, x \rangle^{\frac{1}{2}}) y \rangle\| = \|\phi_Y(\langle x, x \rangle^{\frac{1}{2}}) y\|^2,$$

from which it follows that $\|\Theta_x\| = \|\phi_Y(\langle x, x \rangle^{\frac{1}{2}})\|$. For the last inequality, note that

$$\| \langle x, x \rangle^{\frac{1}{2}} \|^2 = \| \langle x, x \rangle \| = \|x\|^2$$

by the C*-identity. Thus we obtain that

$$\|\phi_Y(\langle x, x \rangle^{\frac{1}{2}})\| \leq \| \langle x, x \rangle^{\frac{1}{2}} \| = \|x\|,$$

finishing the proof. □

We apply Lemma 2.2.12 to obtain the following lemma and corollary.

Lemma 2.2.13. *Let X and Y be C*-correspondences over a C*-algebra A. Let $I \subseteq A$ be an ideal and suppose that $a \in A$ satisfies $\phi_Y(a) \in \mathcal{K}(YI)$. Then*

$$\Theta^X_{\xi a, \eta} \otimes \mathrm{id}_Y \in \mathcal{K}((X \otimes_A Y)I) \text{ for all } \xi, \eta \in X.$$

Proof. Let $\Theta_\xi, \Theta_\eta \in \mathcal{L}(Y, X \otimes_A Y)$ be defined as in Lemma 2.2.12. For each $x \in X$ and $y \in Y$, we directly verify that

$$\begin{aligned}(\Theta^X_{\xi a, \eta} \otimes \mathrm{id}_Y)(x \otimes y) &= (\xi a \langle \eta, x \rangle) \otimes y = \xi \otimes (\phi_Y(a)\phi_Y(\langle \eta, x \rangle) y) \\ &= \Theta_\xi(\phi_Y(a)\phi_Y(\langle \eta, x \rangle) y) = (\Theta_\xi \phi_Y(a) \Theta_\eta^*)(x \otimes y).\end{aligned}$$

Since the maps involved are adjointable and therefore linear and bounded, we deduce that

$$\Theta^X_{\xi a, \eta} \otimes \mathrm{id}_Y = \Theta_\xi \phi_Y(a) \Theta_\eta^* \in \mathcal{K}(X \otimes_A Y),$$

using that $\phi_Y(a) \in \mathcal{K}(YI) \subseteq \mathcal{K}(Y)$ together with [40, p. 9, (1.6)] to establish the membership to $\mathcal{K}(X \otimes_A Y)$. Hence, for $\zeta, \zeta' \in X \otimes_A Y$, we have that

$$\left\langle \zeta, (\Theta^X_{\xi a, \eta} \otimes \mathrm{id}_Y)(\zeta') \right\rangle = \left\langle \Theta_\xi^* \zeta, \phi_Y(a) \Theta_\eta^* \zeta' \right\rangle \in I,$$

using (2.5) applied to Y and $k = \phi_Y(a)$ to establish the membership to I. Another application of (2.5) to $X \otimes_A Y$ and $k = \Theta^X_{\xi a, \eta} \otimes \mathrm{id}_Y$ finishes the proof. □

Corollary 2.2.14. *Let X and Y be C*-correspondences over a C*-algebra A. Let $I \subseteq A$ be an ideal and suppose that $\phi_Y(I) \subseteq \mathcal{K}(YI)$. Then*

$$k \otimes \mathrm{id}_Y \in \mathcal{K}((X \otimes_A Y)I) \text{ for all } k \in \mathcal{K}(XI).$$

Proof. First observe that $\Theta^X_{\xi a,\eta} \otimes \mathrm{id}_Y \in \mathcal{K}((X \otimes_A Y)I)$ for all $\xi, \eta \in X$ and $a \in I$ by Lemma 2.2.13. Recall that the assignment $S \mapsto S \otimes \mathrm{id}_Y$ constitutes a $*$-homomorphism from $\mathcal{L}(X)$ to $\mathcal{L}(X \otimes_A Y)$. In particular, this mapping is linear and continuous. Also note that $\mathcal{K}((X \otimes_A Y)I)$ is an ideal in $\mathcal{L}(X \otimes_A Y)$ and is hence in particular a closed linear subspace. These facts, together with the observation that $\mathcal{K}(XI)$ is densely spanned by elements of the form $\Theta^X_{\xi a,\eta}$ for $\xi, \eta \in X$ and $a \in I$, imply that $k \otimes \mathrm{id}_Y \in \mathcal{K}((X \otimes_A Y)I)$ for all $k \in \mathcal{K}(XI)$, as required. □

If X is a C*-correspondence over A, then we need to make an additional imposition on I in order for $[X]_I$ to carry a canonical structure as a C*-correspondence over $[A]_I$. More specifically, we say that I is *positively invariant (for X)* if it satisfies

$$X(I) := [\langle X, IX \rangle] \subseteq I.$$

Notice that the space $X(I)$ is an ideal of A (in fact, this is true even when I is replaced by any subset of A).

Proposition 2.2.15. *Let X be a C*-correspondence over a C*-algebra A and let $I \subseteq A$ be an ideal that is positively invariant for X. Then the map*

$$\phi_{[X]_I} \colon [A]_I \to \mathcal{L}([X]_I); [a]_I \mapsto [\phi_X(a)]_I \text{ for all } a \in A$$

is a $*$-homomorphism and so $[X]_I$ carries the structure of a C-correspondence over $[A]_I$.*

Proof. First we check that $\phi_{[X]_I}$ is well-defined. To this end, take $a, b \in A$ and suppose that $[a]_I = [b]_I$. Then $a = b + c$ for some $c \in I$. In turn, we obtain that

$$\phi_{[X]_I}([a]_I) = [\phi_X(a)]_I = [\phi_X(b + c)]_I = [\phi_X(b) + \phi_X(c)]_I = [\phi_X(b)]_I + [\phi_X(c)]_I.$$

Positive invariance of I gives that $\langle X, cX \rangle \subseteq I$ and therefore $cX \subseteq XI$ by [36, Proposition 1.3]. Fixing $\xi \in X$, we therefore deduce that

$$[\phi_X(c)]_I[\xi]_I = [\phi_X(c)\xi]_I = 0$$

and hence $[\phi_X(c)]_I = 0$. In turn, we obtain that $\phi_{[X]_I}([a]_I) = \phi_{[X]_I}([b]_I)$, showing that $\phi_{[X]_I}$ is well-defined. It is routine to check that $\phi_{[X]_I}$ is a $*$-homomorphism, using that ϕ_X and $[\cdot]_I \colon \mathcal{L}(X) \to \mathcal{L}([X]_I)$ are $*$-homomorphisms. Combining this with Proposition 2.2.10 completes the proof. □

To ease notation, we will denote $\phi_{[X]_I}$ by $[\phi_X]_I$.

Lemma 2.2.16. *Let X be a C*-correspondence over a C*-algebra A and let $I \subseteq A$ be an ideal. Then the following are equivalent:*

(i) I is positively invariant for X;

(ii) $IX \subseteq XI$;

(iii) $IXI = IX$.

Proof. [(i) $\Rightarrow$ (ii)]: Fix $a \in I$ and $\xi \in X$. An application of [40, Lemma 4.4] yields an element $\eta \in X$ such that

$$\phi_X(a)\xi = \eta \left\langle \phi_X(a)\xi, \phi_X(a)\xi \right\rangle^{\frac{1}{4}}.$$

Notice that $\langle \phi_X(a)\xi, \phi_X(a)\xi \rangle = \langle \xi, \phi_X(a^*a)\xi \rangle \in I$ using positive invariance of I, and hence $\phi_X(a)\xi \in XI$. This shows that $IX \subseteq XI$, as required.

[(ii) $\Rightarrow$ (iii)]: Trivially $IXI \subseteq IX$. Next, note that $I = [II] = II$. Indeed, the first equality follows from the existence of approximate units in I and the second follows by applying the Hewitt-Cohen Factorisation Theorem (see Theorem 2.1.1). From this we deduce that

$$IX = IIX \subseteq IXI,$$

using the assumption in the final inclusion. Thus we obtain the required equality.

[(iii) $\Rightarrow$ (i)]: A direct computation yields that $[\langle X, IX \rangle] = [\langle X, IXI \rangle] = [\langle X, IX \rangle I] \subseteq I$ and thus I is positively invariant, finishing the proof. □

Corollary 2.2.17. *Let X be a C*-correspondence over a C*-algebra A and let $I \subseteq A$ be an ideal that is positively invariant for X. Then we have that*

$$\phi_X(I)\mathcal{K}(X)\phi_X(I) \subseteq \overline{\operatorname{span}}\{\Theta^X_{\xi,\eta} \mid \xi, \eta \in IXI\}.$$

Proof. Fix $a, b \in I$ and $\xi, \eta \in X$. Since $\overline{\operatorname{span}}\{\Theta^X_{\xi',\eta'} \mid \xi', \eta' \in IXI\}$ is a closed linear subspace of $\mathcal{K}(X)$, it suffices to show that

$$\phi_X(a)\Theta^X_{\xi,\eta}\phi_X(b) \in \overline{\operatorname{span}}\{\Theta^X_{\xi',\eta'} \mid \xi', \eta' \in IXI\}.$$

To this end, first note that

$$\phi_X(a)\Theta^X_{\xi,\eta}\phi_X(b) = \Theta^X_{a\xi,b^*\eta}$$

by [40, p.9, (1.6)]. We have that $a\xi$ and $b^*\eta$ belong to $IX = IXI$ by Lemma 2.2.16, which applies due to positive invariance of I. Hence we obtain that

$$\phi_X(a)\Theta^X_{\xi,\eta}\phi_X(b) \in \overline{\operatorname{span}}\{\Theta^X_{\xi',\eta'} \mid \xi', \eta' \in IXI\},$$

as required. □

The following lemma will be useful in Chapter 4.

Lemma 2.2.18. *Let X be a C*-correspondence over a C*-algebra A such that ϕ_X acts by compact operators, and let $I, J \subseteq A$ be ideals. If I is positively invariant for X and $I \subseteq J$, then*

$$\|[\phi_X]_I([a]_I) + \mathcal{K}([X]_I[J]_I)\| = \|\phi_X(a) + \mathcal{K}(XJ)\| \text{ for all } a \in A.$$

Proof. Fix $a \in A$. For notational convenience, we set

$$M := \|\phi_X(a) + \mathcal{K}(XJ)\| = \inf\{\|\phi_X(a) + k\| \mid k \in \mathcal{K}(XJ)\}.$$

Likewise, we set

$$N := \|[\phi_X]_I([a]_I) + \mathcal{K}([X]_I[J]_I)\| = \inf\{\|[\phi_X]_I([a]_I) + \dot{k}\| \mid \dot{k} \in \mathcal{K}([X]_I[J]_I)\}.$$

First we show that $M \geq N$. To this end, it suffices to show that

$$N \leq \|\phi_X(a) + k\| \text{ for all } k \in \mathcal{K}(XJ).$$

Accordingly, fix $k \in \mathcal{K}(XJ)$. Note that

$$\mathcal{K}([X]_I[J]_I) = \mathcal{K}([XJ]_I) = [\mathcal{K}(XJ)]_I, \tag{2.6}$$

using Lemma 2.2.11 in the final equality. Thus $[k]_I \in \mathcal{K}([X]_I[J]_I)$ and so we have that

$$N \leq \|[\phi_X]_I([a]_I) + [k]_I\| = \|[\phi_X(a) + k]_I\| \leq \|\phi_X(a) + k\|,$$

as required.

To prove that $M \leq N$, it suffices to show that

$$M \leq \|[\phi_X]_I([a]_I) + \dot{k}\| \text{ for all } \dot{k} \in \mathcal{K}([X]_I[J]_I).$$

Accordingly, fix $\dot{k} \in \mathcal{K}([X]_I[J]_I)$ and note that $\dot{k} = [k]_I$ for some $k \in \mathcal{K}(XJ)$ by (2.6). By Lemma 2.2.11, we obtain a $*$-isomorphism

$$\Phi \colon \mathcal{K}(X)/\mathcal{K}(XI) \to \mathcal{K}([X]_I); k' + \mathcal{K}(XI) \mapsto [k']_I \text{ for all } k' \in \mathcal{K}(X).$$

In turn, we deduce that

$$\|[\phi_X]_I([a]_I) + \dot{k}\| = \|[\phi_X]_I([a]_I) + [k]_I\| = \|[\phi_X(a) + k]_I\| = \|\Phi((\phi_X(a) + k) + \mathcal{K}(XI))\|,$$

using the assumption that $\phi_X(A) \subseteq \mathcal{K}(X)$ in the final equality. Simplifying further, we have that

$$\begin{aligned}\|\Phi((\phi_X(a)+k)+\mathcal{K}(XI))\| &= \|(\phi_X(a)+k)+\mathcal{K}(XI)\| \\ &= \inf\{\|\phi_X(a)+k+k'\| \mid k' \in \mathcal{K}(XI)\}.\end{aligned}$$

Thus it suffices to show that

$$M \leq \|\phi_X(a)+k+k'\| \text{ for all } k' \in \mathcal{K}(XI).$$

However, this is immediate since $I \subseteq J$ and so $\mathcal{K}(XI) \subseteq \mathcal{K}(XJ)$. In total, we have that $M = N$, finishing the proof. □

Moreover, we define two ideals of A that are related to I and X, namely

$$X^{-1}(I) := \{a \in A \mid \langle X, aX\rangle \subseteq I\},$$

and

$$J(I,X) := \{a \in A \mid [\phi_X(a)]_I \in \mathcal{K}([X]_I), aX^{-1}(I) \subseteq I\}.$$

Note that $A^{-1}(I) = I$. The use of the ideal $J(I,X)$ is pivotal in the work of Katsura [36] for accounting for $*$-representations of $\mathcal{T}_X$ that may not be injective on X. When I is positively invariant and $J \subseteq A$ is an ideal satisfying $I \subseteq J$, the following lemma illustrates the relationship between $X^{-1}(J)$ and $[X]_I^{-1}([J]_I)$.

Lemma 2.2.19. *Let X be a C*-correspondence over a C*-algebra A and let $I, J \subseteq A$ be ideals. If I is positively invariant for X and $I \subseteq J$, then*

$$X^{-1}(J) = [\cdot]_I^{-1}([X]_I^{-1}([J]_I)).$$

Proof. The forward inclusion is immediate by definition of the C*-correspondence operations of $[X]_I$. For the reverse inclusion, fix $a \in A$ such that $[a]_I \in [X]_I^{-1}([J]_I)$. Then we have that

$$[\langle X, aX\rangle]_I = \langle [X]_I, [aX]_I\rangle = \langle [X]_I, [a]_I[X]_I\rangle \subseteq [J]_I.$$

In turn, we deduce that $\langle X, aX\rangle \subseteq J + I$, and the fact that $I \subseteq J$ then yields that $\langle X, aX\rangle \subseteq J$. Hence $a \in X^{-1}(J)$, completing the proof. □

Following [36, Definition 5.6, Definition 5.12], we define a *T-pair* of X to be a pair $\mathcal{L} = \{\mathcal{L}_\emptyset, \mathcal{L}_{\{1\}}\}$ of ideals of A such that $\mathcal{L}_\emptyset$ is positively invariant for X and

$$\mathcal{L}_\emptyset \subseteq \mathcal{L}_{\{1\}} \subseteq J(\mathcal{L}_\emptyset, X).$$

A T-pair $\mathcal{L}$ that satisfies $J_X \subseteq \mathcal{L}_{\{1\}}$ is called an *O-pair*. Our choice of notation will be clarified in the sequel. Proposition 8.8 of [36] is fundamental to the current work.

Theorem 2.2.20. *[36, Theorem 8.6, Proposition 8.8] Let X be a C*-correspondence over a C*-algebra A. Then there is a bijection between the set of T-pairs (resp. O-pairs) of X and the set of gauge-invariant ideals of $\mathcal{T}_X$ (resp. $\mathcal{O}_X$). This bijection preserves inclusions and intersections.*

The bijection of Theorem 2.2.20 restricts appropriately to a parametrisation of the gauge-invariant ideals of any relative Cuntz-Pimsner algebra [36, Proposition 11.9]. The parametrisation of the gauge-invariant ideals of $\mathcal{T}_X$ can be implemented as follows.

Firstly, if $\mathfrak{J} \subseteq \mathcal{T}_X$ is a gauge-invariant ideal, then we consider the representation $(Q_{\mathfrak{J}} \circ \overline{\pi}_X, Q_{\mathfrak{J}} \circ \overline{t}_X)$, where $Q_{\mathfrak{J}} \colon \mathcal{T}_X \to \mathcal{T}_X/\mathfrak{J}$ is the quotient map and $(\overline{\pi}_X, \overline{t}_X)$ is the universal representation of X. We define

$$\mathcal{L}_{\emptyset}^{(Q_{\mathfrak{J}}\circ\overline{\pi}_X, Q_{\mathfrak{J}}\circ\overline{t}_X)} := \ker Q_{\mathfrak{J}} \circ \overline{\pi}_X \quad \text{and} \quad \mathcal{L}_{\{1\}}^{(Q_{\mathfrak{J}}\circ\overline{\pi}_X, Q_{\mathfrak{J}}\circ\overline{t}_X)} := (Q_{\mathfrak{J}} \circ \overline{\pi}_X)^{-1}((Q_{\mathfrak{J}} \circ \overline{\psi}_X)(\mathcal{K}(X))).$$

It follows that the pair

$$\mathcal{L}^{\mathfrak{J}} := \{\mathcal{L}_{\emptyset}^{(Q_{\mathfrak{J}}\circ\overline{\pi}_X, Q_{\mathfrak{J}}\circ\overline{t}_X)}, \mathcal{L}_{\{1\}}^{(Q_{\mathfrak{J}}\circ\overline{\pi}_X, Q_{\mathfrak{J}}\circ\overline{t}_X)}\}$$

is a T-pair of X [36, Proposition 5.11].

Next, let $\mathcal{L} = \{\mathcal{L}_{\emptyset}, \mathcal{L}_{\{1\}}\}$ be a T-pair of X. Then $[\mathcal{L}_{\{1\}}]_{\mathcal{L}_{\emptyset}}$ is an ideal of $[A]_{\mathcal{L}_{\emptyset}}$. Since $\mathcal{L}_{\emptyset}$ is positively invariant and $\mathcal{L}_{\{1\}} \subseteq J(\mathcal{L}_{\emptyset}, X)$, an application of [36, Lemma 5.2] gives that

$$[\mathcal{L}_{\{1\}}]_{\mathcal{L}_{\emptyset}} \subseteq J_{[X]_{\mathcal{L}_{\emptyset}}}.$$

Let $(\widetilde{\pi}, \widetilde{t})$ be the universal $[\mathcal{L}_{\{1\}}]_{\mathcal{L}_{\emptyset}}$-covariant representation of $[X]_{\mathcal{L}_{\emptyset}}$. Then we may form a representation $(\pi^{\mathcal{L}}, t^{\mathcal{L}})$ of X which generates $\mathcal{O}([\mathcal{L}_{\{1\}}]_{\mathcal{L}_{\emptyset}}, [X]_{\mathcal{L}_{\emptyset}})$ via

$$\pi^{\mathcal{L}}(a) = \widetilde{\pi}([a]_{\mathcal{L}_{\emptyset}}) \text{ and } t^{\mathcal{L}}(\xi) = \widetilde{t}([\xi]_{\mathcal{L}_{\emptyset}}) \text{ for all } a \in A, \xi \in X.$$

The universal property of $\mathcal{T}_X$ then guarantees a (unique) canonical $*$-epimorphism

$$\pi^{\mathcal{L}} \times t^{\mathcal{L}} \colon \mathcal{T}_X \to \mathcal{O}([\mathcal{L}_{\{1\}}]_{\mathcal{L}_{\emptyset}}, [X]_{\mathcal{L}_{\emptyset}}).$$

We define

$$\mathfrak{J}^{\mathcal{L}} := \ker \pi^{\mathcal{L}} \times t^{\mathcal{L}}$$

and observe that $\mathfrak{J}^{\mathcal{L}}$ is a gauge-invariant ideal of $\mathcal{T}_X$.

By [36, Proposition 8.8], the maps

$$\mathfrak{J} \mapsto \mathcal{L}^{\mathfrak{J}} \text{ for all gauge-invariant ideals } \mathfrak{J} \text{ of } \mathcal{T}_X,$$
$$\mathcal{L} \mapsto \mathfrak{J}^{\mathcal{L}} \text{ for all T-pairs } \mathcal{L} \text{ of } X,$$

are mutually inverse. The notation that we have used here will be revisited in the sequel.

2.3 Product systems

Let P be a subsemigroup of a discrete group G that contains the identity e of G (i.e., P is *unital*). A *product system X over P with coefficients in a C*-algebra A* is a family $\{X_p\}_{p\in P}$ of C*-correspondences over A together with multiplication maps $u_{p,q}\colon X_p \otimes_A X_q \to X_{pq}$ for all $p, q \in P$, such that:

(i) $X_e = A$, viewing A as a C*-correspondence over itself in the usual way;

(ii) if $p = e$, then $u_{e,q}\colon A \otimes_A X_q \to [\phi_q(A)X_q]$ is the unitary implementing the left action of A on X_q;

(iii) if $q = e$, then $u_{p,e}\colon X_p \otimes_A A \to X_p$ is the unitary implementing the right action of A on X_p;

(iv) if $p, q \in P \setminus \{e\}$, then $u_{p,q}\colon X_p \otimes_A X_q \to X_{pq}$ is a unitary;

(v) the multiplication maps are associative in the sense that

$$u_{pq,r}(u_{p,q} \otimes \mathrm{id}_{X_r}) = u_{p,qr}(\mathrm{id}_{X_p} \otimes u_{q,r}) \text{ for all } p, q, r \in P.$$

Note that we use ϕ_p to denote the left action ϕ_{X_p} of X_p for each $p \in P$. We refer to the C*-correspondences X_p as the *fibres* of X. We do *not* assume that the fibres are non-degenerate. If X_p is injective (resp. regular) for all $p \in P$, then we say that X is *injective* (resp. *regular*). For brevity, we will write $u_{p,q}(\xi_p \otimes \xi_q)$ as $\xi_p\xi_q$ for all $\xi_p \in X_p, \xi_q \in X_q$ and $p, q \in P$, with the understanding that ξ_p and ξ_q are allowed to differ when $p = q$. Axioms (i) and (ii) imply that the unitary $u_{e,e}\colon A \otimes_A A \to A$ is simply multiplication in A. Axioms (ii) and (v) imply that

$$\phi_{pq}(a)(\xi_p\xi_q) = (\phi_p(a)\xi_p)\xi_q \text{ for all } \xi_p \in X_p, \xi_q \in X_q, p, q \in P.$$

Note that the maps involved in axiom (v) are linear and bounded, and are therefore determined by their respective actions on simple tensors.

For $p \in P \setminus \{e\}$ and $q \in P$, we use the product system structure of X to define a $*$-homomorphism $\iota_p^{pq}\colon \mathcal{L}(X_p) \to \mathcal{L}(X_{pq})$ by

$$\iota_p^{pq}(S) = u_{p,q}(S \otimes \mathrm{id}_{X_q})u_{p,q}^* \text{ for all } S \in \mathcal{L}(X_p).$$

In turn, we obtain that

$$\iota_p^{pq}(S)(\xi_p\xi_q) = (S\xi_p)\xi_q \text{ for all } \xi_p \in X_p \text{ and } \xi_q \in X_q.$$

This formula completely describes $\iota_p^{pq}(S)$, since any bounded linear operator on X_{pq} is determined by its action on the subset $\{u_{p,q}(\xi_p \otimes \xi_q) \mid \xi_p \in X_p, \xi_q \in X_q\}$. Indeed, we have that $X_p \otimes_A X_q \cong X_{pq}$ via the multiplication map $u_{p,q}$, and therefore

$$X_{pq} = \overline{\operatorname{span}}\{u_{p,q}(\xi_p \otimes \xi_q) \mid \xi_p \in X_p, \xi_q \in X_q\}.$$

We will make use of this observation frequently. We also define a $*$-homomorphism $\iota_e^q \colon \mathcal{K}(A) \to \mathcal{L}(X_q)$ by $\iota_e^q(\phi_e(a)) = \phi_q(a)$ for all $a \in A$. Moreover, we have that

$$\iota_p^p = \mathrm{id}_{\mathcal{L}(X_p)} \text{ for all } p \in P \setminus \{e\} \quad \text{and} \quad \iota_e^e = \mathrm{id}_{\mathcal{K}(A)}.$$

The theory of product systems includes that of C*-correspondences in the sense that every C*-correspondence X over a C*-algebra A can be viewed as the product system $\{X_n\}_{n\in\mathbb{Z}_+}$ with

$$X_0 := A \quad \text{and} \quad X_n := X^{\otimes n} \text{ for all } n \in \mathbb{N},$$

and multiplication maps $u_{n,m}$ for $n, m \neq 0$ given by the natural inclusions.

The notion of isomorphism for product systems is given by unitary equivalence.

Definition 2.3.1. Let P be a unital subsemigroup of a discrete group G and let A and B be C*-algebras. Let X and Y be product systems over P with coefficients in A and B, respectively. Denote the multiplication maps of X by $\{u_{p,q}^X\}_{p,q\in P}$ and the multiplication maps of Y by $\{u_{p,q}^Y\}_{p,q\in P}$. We say that X and Y are *unitarily equivalent* (symb. $X \cong Y$) if there exist surjective linear maps $W_p \colon X_p \to Y_p$ for all $p \in P$ with the following properties:

(i) $W_e \colon A \to B$ is a $*$-isomorphism;

(ii) $\langle W_p(\xi_p), W_p(\xi_p')\rangle = W_e(\langle \xi_p, \xi_p'\rangle)$ for all $\xi_p, \xi_p' \in X_p$ and $p \in P \setminus \{e\}$;

(iii) $\phi_{Y_p}(W_e(a))W_p(\xi_p) = W_p(\phi_{X_p}(a)\xi_p)$ for all $a \in A, \xi_p \in X_p$ and $p \in P \setminus \{e\}$;

(iv) $W_p(\xi_p)W_e(a) = W_p(\xi_p a)$ for all $a \in A, \xi_p \in X_p$ and $p \in P \setminus \{e\}$;

(v) $u_{p,q}^Y \circ (W_p \otimes W_q) = W_{pq} \circ u_{p,q}^X$ for all $p, q \in P$.

In this case, we say that $\{W_p\}_{p\in P}$ implements a unitary equivalence between X and Y.

Remark 2.3.2. Many structural properties of product systems are preserved under unitary equivalence. Let $\{W_p\}_{p\in P}$ implement a unitary equivalence between product systems X and Y. Item (i) guarantees that items (ii)-(iv) hold when $p = e$. Items (i) and (ii) ensure that W_p is an isometry and therefore injective for all $p \in P$. Hence W_p is bijective for all $p \in P$. Consequently, the collection $\{W_p^{-1}\}_{p\in P}$ defines a unitary equivalence between Y and X. In turn, we deduce that unitary equivalence is an equivalence relation and therefore items (i)-(v) have duals obtained by reversing the arrows.

For all $p, q \in P$, the map $W_p \otimes W_q \in \mathcal{B}(X_p \otimes_A X_q, Y_p \otimes_B Y_q)$ is defined on simple tensors via

$$(W_p \otimes W_q)(\xi_p \otimes \xi_q) = W_p(\xi_p) \otimes W_q(\xi_q) \text{ for all } \xi_p \in X_p, \xi_q \in X_q,$$

from which it follows that $(W_p \otimes W_q)^{-1} = W_p^{-1} \otimes W_q^{-1}$.

We also have that

$$W_p \mathcal{K}(X_p) W_p^{-1} = \mathcal{K}(Y_p). \tag{2.7}$$

To see this, take $x_p, x_p' \in X_p$. Then we have that $x_p = W_p^{-1}(y_p)$ and $x_p' = W_p^{-1}(y_p')$ for some $y_p, y_p' \in Y_p$. Fixing $y_p'' \in Y_p$, we obtain that

$$\begin{aligned}
(W_p \Theta^{X_p}_{x_p,x_p'} W_p^{-1})(y_p'') &= W_p(\Theta^{X_p}_{W_p^{-1}(y_p),W_p^{-1}(y_p')}(W_p^{-1}(y_p''))) \\
&= W_p(W_p^{-1}(y_p)\left\langle W_p^{-1}(y_p'), W_p^{-1}(y_p'')\right\rangle) \\
&= W_p(W_p^{-1}(y_p) W_e^{-1}(\left\langle y_p', y_p''\right\rangle)) \\
&= W_p(W_p^{-1}(y_p\left\langle y_p', y_p''\right\rangle)) = \Theta^{Y_p}_{y_p,y_p'}(y_p''),
\end{aligned}$$

using the dual of item (ii) in the third line and the dual of item (iv) in the fourth. Hence $W_p \Theta^{X_p}_{x_p,x_p'} W_p^{-1} = \Theta^{Y_p}_{y_p,y_p'}$ and we conclude that $W_p \Theta^{X_p}_{x_p,x_p'} W_p^{-1} \in \mathcal{K}(Y_p)$ for all $x_p, x_p' \in X_p$. Using linearity and continuity of W_p and W_p^{-1}, together with the fact that $\mathcal{K}(Y_p)$ is in particular a closed linear subspace of $\mathcal{B}(Y_p)$, it follows that $W_p \mathcal{K}(X_p) W_p^{-1} \subseteq \mathcal{K}(Y_p)$. Consequently, the mapping

$$\Phi\colon \mathcal{K}(X_p) \to \mathcal{K}(Y_p); k_p \mapsto W_p k_p W_p^{-1} \text{ for all } k_p \in \mathcal{K}(X_p)$$

is well-defined. It is routine to check that Φ is linear, and the calculation

$$\Phi(k_p k_p') = W_p k_p k_p' W_p^{-1} = W_p k_p W_p^{-1} W_p k_p' W_p^{-1} = (W_p k_p W_p^{-1})(W_p k_p' W_p^{-1}) = \Phi(k_p)\Phi(k_p'),$$

where $k_p, k_p' \in \mathcal{K}(X_p)$, demonstrates that Φ is an algebra homomorphism. Fixing $y_p, y_p' \in Y_p$ and $k_p \in \mathcal{K}(X_p)$, we obtain that

$$\begin{aligned}
\left\langle (W_p k_p W_p^{-1})(y_p), y_p'\right\rangle &= \left\langle W_p(k_p(W_p^{-1}(y_p))), W_p(W_p^{-1}(y_p'))\right\rangle \\
&= W_e(\left\langle k_p(W_p^{-1}(y_p)), W_p^{-1}(y_p')\right\rangle) \\
&= W_e(\left\langle W_p^{-1}(y_p), k_p^*(W_p^{-1}(y_p'))\right\rangle) \\
&= W_e(\left\langle W_p^{-1}(y_p), W_p^{-1}(W_p(k_p^*(W_p^{-1}(y_p'))))\right\rangle) \\
&= W_e(W_e^{-1}(\left\langle y_p, (W_p k_p^* W_p^{-1})(y_p')\right\rangle)) = \left\langle y_p, (W_p k_p^* W_p^{-1})(y_p')\right\rangle,
\end{aligned}$$

using item (ii) in the second line and its dual in the fifth line. Hence $\Phi(k_p^*) = \Phi(k_p)^*$ for all $k_p \in \mathcal{K}(X_p)$, and thus Φ is a $*$-homomorphism. In turn, we have that $\mathrm{Im}(\Phi) = W_p \mathcal{K}(X_p) W_p^{-1}$ is a C*-subalgebra of $\mathcal{K}(Y_p)$. Thus, to show that $\mathcal{K}(Y_p) \subseteq W_p \mathcal{K}(X_p) W_p^{-1}$, it suffices to show that each rank-one operator in $\mathcal{K}(Y_p)$ belongs to $W_p \mathcal{K}(X_p) W_p^{-1}$. To this end, take $y_p, y_p' \in Y_p$. By the preceding arguments, we obtain that

$$\Theta^{Y_p}_{y_p,y_p'} = W_p \Theta^{X_p}_{W_p^{-1}(y_p),W_p^{-1}(y_p')} W_p^{-1} \in W_p \mathcal{K}(X_p) W_p^{-1}, \tag{2.8}$$

as required. In total, we have that $W_p\mathcal{K}(X_p)W_p^{-1} = \mathcal{K}(Y_p)$, showing that (2.7) holds.

Fixing $p \in P$ and an ideal $I \subseteq A$, we have that

$$W_p\mathcal{K}(X_pI)W_p^{-1} = \mathcal{K}(Y_pW_e(I)). \tag{2.9}$$

The proof follows the same trajectory as that of (2.7), and so is omitted.

Fixing $a \in A$, we have that

$$W_p\phi_{X_p}(a)W_p^{-1} = \phi_{Y_p}(W_e(a)). \tag{2.10}$$

To see this, fix $y_p \in Y_p$. We deduce that

$$\begin{aligned}(W_p\phi_{X_p}(a)W_p^{-1})(y_p) &= W_p(\phi_{X_p}(W_e^{-1}(W_e(a)))W_p^{-1}(y_p)) \\ &= W_p(W_p^{-1}(\phi_{Y_p}(W_e(a))y_p)) \\ &= \phi_{Y_p}(W_e(a))y_p,\end{aligned}$$

using the dual of item (iii) in the second line. Hence $W_p\phi_{X_p}(a)W_p^{-1} = \phi_{Y_p}(W_e(a))$, as required.

Finally, let $\{\iota_p^{pq}\}_{p,q\in P}$ denote the connecting $*$-homomorphisms of X and let $\{j_p^{pq}\}_{p,q\in P}$ denote the connecting $*$-homomorphisms of Y. For $p \in P \setminus \{e\}$ and $q \in P$, we have that

$$\iota_p^{pq}(\Theta^{X_p}_{x_p,x_p'}) = W_{pq}^{-1}j_p^{pq}(\Theta^{Y_p}_{W_p(x_p),W_p(x_p')})W_{pq} \text{ for all } x_p, x_p' \in X_p. \tag{2.11}$$

To see this, take $x_p, x_p', x_p'' \in X_p$ and $x_q \in X_q$. We define

$$y_p := W_p(x_p), y_p' := W_p(x_p'), y_p'' := W_p(x_p'') \in Y_p \quad \text{and} \quad y_q := W_q(x_q) \in Y_q.$$

We obtain that

$$\begin{aligned}\iota_p^{pq}(\Theta^{X_p}_{x_p,x_p'})(x_p''x_q) &= (W_p^{-1}(y_p)\left\langle W_p^{-1}(y_p'), W_p^{-1}(y_p'')\right\rangle)W_q^{-1}(y_q) \\ &= W_p^{-1}(y_p\left\langle y_p', y_p''\right\rangle)W_q^{-1}(y_q) \\ &= (u^X_{p,q} \circ (W_p^{-1} \otimes W_q^{-1}))(\Theta^{Y_p}_{y_p,y_p'}(y_p'') \otimes y_q) \\ &= (W_{pq}^{-1} \circ u^Y_{p,q})(\Theta^{Y_p}_{y_p,y_p'}(y_p'') \otimes y_q) \\ &= W_{pq}^{-1}j_p^{pq}(\Theta^{Y_p}_{y_p,y_p'})(y_p''y_q) \\ &= W_{pq}^{-1}j_p^{pq}(\Theta^{Y_p}_{y_p,y_p'})((u^Y_{p,q} \circ (W_p \otimes W_q))(x_p'' \otimes x_q)) \\ &= W_{pq}^{-1}j_p^{pq}(\Theta^{Y_p}_{y_p,y_p'})W_{pq}(x_p''x_q),\end{aligned}$$

using the duals of items (ii) and (iv) in the second line, the dual of item (v) in the fourth line and item (v) in the last line. By linearity and continuity of the maps involved, we deduce that (2.11) holds.

A *(Toeplitz) representation* (π, t) of X on $\mathcal{B}(H)$ consists of a family $\{(\pi, t_p)\}_{p \in P}$, where (π, t_p) is a representation of X_p on $\mathcal{B}(H)$ for all $p \in P$, $t_e = \pi$ and

$$t_p(\xi_p)t_q(\xi_q) = t_{pq}(\xi_p \xi_q) \text{ for all } \xi_p \in X_p, \xi_q \in X_q, p, q \in P.$$

We write ψ_p for the induced $*$-homomorphism $\mathcal{K}(X_p) \to \mathcal{B}(H)$ for all $p \in P$. We say that (π, t) is *injective* if π is injective; in this case t_p and ψ_p are isometric for all $p \in P$. We denote the C*-algebra generated by $\pi(A)$ and every $t_p(X_p)$ by $\mathrm{C}^*(\pi, t)$. We write $\mathcal{T}_X$ for the universal C*-algebra with respect to the Toeplitz representations of X, and refer to it as the *Toeplitz algebra (of X)*. The following proposition is well known.

Proposition 2.3.3. *Let P be a unital subsemigroup of a discrete group G. Let X be a product system over P with coefficients in a C*-algebra A. Then there exists a C*-algebra $\mathcal{T}_X$ and a representation (π_X, t_X) of X on $\mathcal{T}_X$ such that:*

(i) $\mathcal{T}_X = \mathrm{C}^*(\pi_X, t_X)$*;*

(ii) if (π, t) is a representation of X, then there exists a (unique) canonical $*$-epimorphism $\pi \times t \colon \mathcal{T}_X \to \mathrm{C}^*(\pi, t)$, i.e., $(\pi \times t)(t_{X,p}(\xi_p)) = t_p(\xi_p)$ for all $\xi_p \in X_p$ and $p \in P$.

The pair $(\mathcal{T}_X, (\pi_X, t_X))$ is unique up to canonical $*$-isomorphism.

Unitary equivalence induces a bijection between representations in the following sense.

Proposition 2.3.4. *Let P be a unital subsemigroup of a discrete group G and let A and B be C*-algebras. Let X and Y be product systems over P with coefficients in A and B, respectively. Suppose that X and Y are unitarily equivalent by a collection $\{W_p \colon X_p \to Y_p\}_{p \in P}$. Then there is a bijection between the set $\mathcal{S}_X$ of Toeplitz representations of X and the set $\mathcal{S}_Y$ of Toeplitz representations of Y given by*

$$\{(\pi, t_p)\}_{p \in P} \mapsto \{(\pi \circ W_e^{-1}, t_p \circ W_p^{-1})\}_{p \in P} \textit{ for all } \{(\pi, t_p)\}_{p \in P} \in \mathcal{S}_X.$$

Proof. We denote the map of the statement by Φ and the multiplication maps of X (resp. Y) by $\{u^X_{p,q}\}_{p,q \in P}$ (resp. $\{u^Y_{p,q}\}_{p,q \in P}$). First we check that Φ is well-defined. Fix a representation (π, t) of X on some $\mathcal{B}(H)$. Then $\pi \circ W_e^{-1} \colon B \to \mathcal{B}(H)$ is a composition of $*$-homomorphisms and is hence itself a $*$-homomorphism. Analogously, the map $t_p \circ W_p^{-1} \colon Y_p \to \mathcal{B}(H)$ is linear for all $p \in P \setminus \{e\}$.

Fix $p \in P, b \in B$ and $y_p \in Y_p$. We have that

$$\pi(W_e^{-1}(b))t_p(W_p^{-1}(y_p)) = t_p(\phi_{X_p}(W_e^{-1}(b))W_p^{-1}(y_p)) = t_p(W_p^{-1}(\phi_{Y_p}(b)y_p)),$$

using the dual of item (iii) of Definition 2.3.1 in the final equality. This shows that the pair $(\pi \circ W_e^{-1}, t_p \circ W_p^{-1})$ preserves the left action of Y_p. Fixing $y_p' \in Y_p$, we also have that

$$t_p(W_p^{-1}(y_p))^* t_p(W_p^{-1}(y_p')) = \pi(\langle W_p^{-1}(y_p), W_p^{-1}(y_p') \rangle) = \pi(W_e^{-1}(\langle y_p, y_p' \rangle)),$$

using the dual of item (ii) of Definition 2.3.1 in the final equality. This shows that the pair $(\pi\circ W_e^{-1}, t_p\circ W_p^{-1})$ preserves the B-valued inner product of Y_p, and hence $(\pi\circ W_e^{-1}, t_p\circ W_p^{-1})$ is a representation of Y_p for all $p \in P$.

Finally, fix $p, q \in P, y_p \in Y_p$ and $y_q \in Y_q$. We obtain that

$$\begin{aligned} t_p(W_p^{-1}(y_p))t_q(W_q^{-1}(y_q)) &= t_{pq}((u^X_{p,q} \circ (W_p^{-1} \otimes W_q^{-1}))(y_p \otimes y_q)) \\ &= t_{pq}((W_{pq}^{-1} \circ u^Y_{p,q})(y_p \otimes y_q)) \\ &= t_{pq}(W_{pq}^{-1}(y_p y_q)), \end{aligned}$$

using the dual of item (v) of Definition 2.3.1 in the second equality. In total, we have that the family $\{(\pi \circ W_e^{-1}, t_p \circ W_p^{-1})\}_{p\in P}$ constitutes a representation of Y and thus Φ is a well-defined map.

To see that Φ is a bijection, we construct an inverse map. Specifically, we define the map

$$\Psi\colon \mathcal{S}_Y \to \mathcal{S}_X; \{(\sigma, s_p)\}_{p\in P} \mapsto \{(\sigma \circ W_e, s_p \circ W_p)\}_{p\in P} \text{ for all } \{(\sigma, s_p)\}_{p\in P} \in \mathcal{S}_Y.$$

This map is well-defined by duality. We have that Φ and Ψ are mutually inverse by definition, finishing the proof. $\square$

It follows from Proposition 2.3.4 that $\mathcal{T}_X \cong \mathcal{T}_Y$ canonically for unitarily equivalent product systems X and Y, since $\mathcal{T}_X$ and $\mathcal{T}_Y$ are universal with respect to the "same" representations.

Remark 2.3.5. Under the assumptions of Proposition 2.3.4, let (π, t) be a representation of X. Recall that $(\widetilde{\pi}, \widetilde{t}) := \{(\pi \circ W_e^{-1}, t_p \circ W_p^{-1})\}_{p\in P}$ is a representation of Y. We have that

$$\psi_p(W_p^{-1} k_p W_p) = \widetilde{\psi}_p(k_p) \text{ for all } k_p \in \mathcal{K}(Y_p), p \in P.$$

To see this, fix $p \in P$. We can take k_p to be a rank-one operator without loss of generality, so fix $y_p, y_p' \in Y_p$. We obtain that

$$\psi_p(W_p^{-1}\Theta^{Y_p}_{y_p,y_p'}W_p) = \psi_p(\Theta^{X_p}_{W_p^{-1}(y_p),W_p^{-1}(y_p')}) = t_p(W_p^{-1}(y_p))t_p(W_p^{-1}(y_p'))^* = \widetilde{\psi}_p(\Theta^{Y_p}_{y_p,y_p'}),$$

using the dual of (2.8) in the first equality. This finishes the proof of the claim.

The quintessential example of a representation that we will be using throughout is the Fock representation. Here we define this representation and collect its basic properties. Firstly, we set

$$\mathcal{F}X := \sum_{p\in P} X_p,$$

as the direct sum of right Hilbert A-modules, e.g., [40, p. 6]. For each $p \in P$, we identify X_p with the p-th direct summand of $\mathcal{F}X$. The algebraic direct sum of the fibres of X is dense in $\mathcal{F}X$; thus any bounded linear operator on $\mathcal{F}X$ is determined by its action on the fibres of X. For each $p, q \in P$, we define a map $\iota_{p,q} \colon \mathcal{L}(X_p, X_q) \to \mathcal{L}(\mathcal{F}X)$ by

$$\iota_{p,q}(S)\xi_r = \begin{cases} S\xi_p & \text{if } r = p, \\ 0 & \text{otherwise,} \end{cases}$$

for all $S \in \mathcal{L}(X_p, X_q), \xi_r \in X_r$ and $r \in P$, and note that $\iota_{p,q}(S)^* = \iota_{q,p}(S^*)$. It follows that $\iota_{p,q}$ is an isometric linear map, and hence we may identify $\mathcal{L}(X_p, X_q)$ as a subspace of $\mathcal{L}(\mathcal{F}X)$. When $p = q$, we have that $\iota_{p,p}$ is a $*$-homomorphism, and thus we may identify $\mathcal{L}(X_p)$ as a C*-subalgebra of $\mathcal{L}(\mathcal{F}X)$.

Next, recall that if Λ is an arbitrary indexing set and $(A_\lambda)_{\lambda \in \Lambda}$ is a family of C*-algebras, then the direct sum of this family is defined to be

$$\bigoplus_{\lambda \in \Lambda} A_\lambda := \{(a_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} A_\lambda \mid \sup_{\lambda \in \Lambda} \|a_\lambda\| < \infty\}.$$

It is routine to check that $\bigoplus_{\lambda \in \Lambda} A_\lambda$ is a C*-algebra under the pointwise-defined operations and supremum norm. We define an injective $*$-homomorphism ι by

$$\iota \colon \bigoplus_{p \in P} \mathcal{L}(X_p) \to \mathcal{L}(\mathcal{F}X); \iota((S_p)_{p \in P})((\xi_p)_{p \in P}) = (S_p\xi_p)_{p \in P} \text{ for all } (S_p)_{p \in P} \in \bigoplus_{p \in P} \mathcal{L}(X_p).$$

Hence we may identify $\bigoplus_{p \in P} \mathcal{L}(X_p)$ as a C*-subalgebra of $\mathcal{L}(\mathcal{F}X)$. We may also replace P by any $Q \subseteq P$ using the canonical embedding $\bigoplus_{p \in Q} \mathcal{L}(X_p) \hookrightarrow \bigoplus_{p \in P} \mathcal{L}(X_p)$ determined by $(S_p)_{p \in Q} \mapsto (T_p)_{p \in P}$, where

$$T_p = \begin{cases} S_p & \text{if } p \in Q, \\ 0 & \text{otherwise,} \end{cases}$$

for all $p \in P$ and $(S_p)_{p \in Q} \in \bigoplus_{p \in Q} \mathcal{L}(X_p)$.

The *Fock representation* $(\overline{\pi}, \overline{t})$ is the representation of X on $\mathcal{L}(\mathcal{F}X)$ determined by

$$\overline{\pi}(a)\xi_q = \phi_q(a)\xi_q \quad \text{and} \quad \overline{t}_p(\xi_p)\xi_q = \xi_p\xi_q,$$

for all $a \in A, \xi_p \in X_p, \xi_q \in X_q$ and $p, q \in P$. For $\xi_p \in X_p$ with $p \neq e$, we have that $\overline{t}_p(\xi_p)^*$ maps X_r to 0 whenever $r \notin pP$. Conversely, if $r = pq$ for some $q \in P$, then

$$\overline{t}_p(\xi_p)^*(\eta_p\eta_q) = \phi_q(\langle \xi_p, \eta_p \rangle)\eta_q \text{ for all } \eta_p \in X_p, \eta_q \in X_q.$$

The Fock representation is injective, for if $\overline{\pi}(a) = 0$ for some $a \in A$ then in particular

$$aa^* = \overline{\pi}(a)a^* = 0,$$

and thus $a = 0$ by the C*-identity.

Next we illustrate how the quotient construction for C*-correspondences extends to the setting of product systems. The following definition is motivated by the notion of positive invariance for C*-correspondences.

Definition 2.3.6. Let P be a unital subsemigroup of a discrete group G. Let X be a product system over P with coefficients in a C*-algebra A and let $I \subseteq A$ be an ideal. We say that I is *positively invariant (for X)* if it satisfies

$$X(I) := \overline{\operatorname{span}}\{\langle X_p, IX_p\rangle \mid p \in P\} \subseteq I.$$

Notice that the space $X(I)$ is an ideal of A (and in fact, this is true even when I is replaced by any subset of A), and that an ideal I is positively invariant for X if and only if it is positively invariant for every fibre of X. This observation lies at the heart of the following proposition.

Proposition 2.3.7. *Let P be a unital subsemigroup of a discrete group G. Let X be a product system over P with coefficients in a C*-algebra A and let $I \subseteq A$ be an ideal that is positively invariant for X. Set*

$$[X]_I := \{[X_p]_I\}_{p\in P}, \text{ where } [X_p]_I = X_p/X_pI \text{ for all } p \in P.$$

Then $[X]_I$ carries a canonical structure as a product system over P with coefficients in $[A]_I$, given by the multiplication maps

$$[X_p]_I \otimes_{[A]_I} [X_q]_I \to [X_{pq}]_I; [\xi_p]_I \otimes [\xi_q]_I \mapsto [\xi_p\xi_q]_I \text{ for all } \xi_p \in X_p, \xi_q \in X_q, p, q \in P.$$

Proof. We will denote the multiplication maps of X by $\{u_{p,q}\}_{p,q\in P}$. The quotient construction for C*-correspondences covered in Section 2.2 renders a canonical structure on $[X_p]_I$ as a C*-correspondence over $[A]_I$ for all $p \in P \setminus \{e\}$. Thus $[X]_I$ constitutes a family of C*-correspondences over $[A]_I$. It remains to show that $[X]_I$, together with the maps of the statement, satisfies axioms (i)-(v) of a product system.

Axiom (i) is satisfied by definition. Next we prove that the maps of the statement are well-defined. This is immediate for $p = e$ or $q = e$, from which it follows that axioms (ii) and (iii) hold. So fix $p, q \in P \setminus \{e\}$ and define the map

$$v_{p,q} \colon [X_p]_I \times [X_q]_I \to [X_{pq}]_I; ([\xi_p]_I, [\xi_q]_I) \mapsto [\xi_p\xi_q]_I \equiv [u_{p,q}(\xi_p \otimes \xi_q)]_I \text{ for all } \xi_p \in X_p, \xi_q \in X_q.$$

To see that $v_{p,q}$ is well-defined, fix $\xi_p \in X_p, \xi_q \in X_q$ and suppose that $[\xi_p]_I = [\eta_p]_I$ and $[\xi_q]_I = [\eta_q]_I$ for some $\eta_p \in X_p$ and $\eta_q \in X_q$. Then $\xi_p = \eta_p + \zeta_p a$ and $\xi_q = \eta_q + \zeta_q b$ for some $\zeta_p \in X_p$, $\zeta_q \in X_q$ and $a, b \in I$. A direct computation yields that

$$[\xi_p\xi_q]_I = [(\eta_p + \zeta_p a)(\eta_q + \zeta_q b)]_I = [\eta_p\eta_q + \eta_p(\zeta_q b) + (\zeta_p a)\eta_q + (\zeta_p a)(\zeta_q b)]_I.$$

Observe that

$$\eta_p(\zeta_q b) = (\eta_p \zeta_q) b \in X_{pq} I \quad \text{and} \quad (\zeta_p a)(\zeta_q b) = ((\zeta_p a)\zeta_q) b \in X_{pq} I.$$

Moreover, by Lemma 2.2.16 we have that

$$(\zeta_p a)\eta_q = \zeta_p(a\eta_q) \in u_{p,q}(X_p \otimes_A I X_q) \subseteq u_{p,q}(X_p \otimes_A X_q I) = X_{pq} I$$

using positive invariance of I, viewing $X_p \otimes_A I X_q, X_p \otimes_A X_q I \subseteq X_p \otimes_A X_q$. Hence

$$[\eta_p(\zeta_q b)]_I = [(\zeta_p a)\eta_q]_I = [(\zeta_p a)(\zeta_q b)]_I = 0$$

and thus $[\xi_p \xi_q]_I = [\eta_p \eta_q]_I$, as required.

It is routine to check that $v_{p,q}$ is bilinear. Additionally, fixing $\xi_p \in X_p, \xi_q \in X_q$ and $a \in A$, we have that

$$\begin{aligned} v_{p,q}([\xi_p]_I [a]_I, [\xi_q]_I) - v_{p,q}([\xi_p]_I, [a]_I [\xi_q]_I) &= v_{p,q}([\xi_p a]_I, [\xi_q]_I) - v_{p,q}([\xi_p]_I, [a\xi_q]_I) \\ &= [(\xi_p a)\xi_q]_I - [\xi_p(a\xi_q)]_I \\ &= [(\xi_p a)\xi_q - \xi_p(a\xi_q)]_I \\ &= [u_{p,q}((\xi_p a) \otimes \xi_q - \xi_p \otimes (a\xi_q))]_I = 0, \end{aligned}$$

using that $\xi_p a \otimes \xi_q = \xi_p \otimes a\xi_q$ in $X_p \otimes_A X_q$ in the last line. Thus $v_{p,q}$ is also $[A]_I$-balanced, and therefore induces a unique linear map

$$v_{p,q} \colon [X_p]_I \odot_{[A]_I} [X_q]_I \to [X_{pq}]_I ; [\xi_p]_I \otimes [\xi_q]_I \mapsto [\xi_p \xi_q]_I \text{ for all } \xi_p \in X_p, \xi_q \in X_q.$$

For $\xi_p, \eta_p \in X_p$ and $\xi_q, \eta_q \in X_q$, we have that

$$\begin{aligned} \langle v_{p,q}([\xi_p]_I \otimes [\xi_q]_I), v_{p,q}([\eta_p]_I \otimes [\eta_q]_I)\rangle &= \langle [\xi_p \xi_q]_I, [\eta_p \eta_q]_I \rangle = [\langle \xi_p \otimes \xi_q, \eta_p \otimes \eta_q \rangle]_I \\ &= [\langle \xi_q, \phi_q(\langle \xi_p, \eta_p \rangle)\eta_q \rangle]_I = \langle [\xi_p]_I \otimes [\xi_q]_I, [\eta_p]_I \otimes [\eta_q]_I \rangle , \end{aligned}$$

and thus $\langle v_{p,q}(\zeta), v_{p,q}(\zeta')\rangle = \langle \zeta, \zeta' \rangle$ for all $\zeta, \zeta' \in [X_p]_I \odot_{[A]_I} [X_q]_I$. In particular, $v_{p,q}$ is bounded with respect to the norm on $[X_p]_I \odot_{[A]_I} [X_q]_I$ induced by the $[A]_I$-valued inner product. Hence it extends to a bounded linear map

$$v_{p,q} \colon [X_p]_I \otimes_{[A]_I} [X_q]_I \to [X_{pq}]_I ; [\xi_p]_I \otimes [\xi_q]_I \mapsto [\xi_p \xi_q]_I \text{ for all } \xi_p \in X_p, \xi_q \in X_q.$$

It follows from the preceding calculations that $v_{p,q}$ is isometric. To see that $v_{p,q}$ preserves the left action, fix $\xi_p \in X_p, \xi_q \in X_q$ and $a \in A$. We have that

$$\begin{aligned} v_{p,q}([a]_I([\xi_p]_I \otimes [\xi_q]_I)) &= v_{p,q}([a\xi_p]_I \otimes [\xi_q]_I) = [(a\xi_p)\xi_q]_I = [u_{p,q}((a\xi_p) \otimes \xi_q)]_I \\ &= [u_{p,q}(a(\xi_p \otimes \xi_q))]_I = [a u_{p,q}(\xi_p \otimes \xi_q)]_I = [a]_I v_{p,q}([\xi_p]_I \otimes [\xi_q]_I), \end{aligned}$$

from which it follows that $v_{p,q}$ is a left $[A]_I$-module map by linearity and continuity of the maps involved. By similar reasoning, we have that $v_{p,q}$ is a right $[A]_I$-module map. Next, notice that

$$[X_{pq}]_I = \overline{\operatorname{span}}\{[\xi_p\xi_q]_I \mid \xi_p \in X_p, \xi_q \in X_q\},$$

using that $u_{p,q}$ is surjective. Observe that the generators $[\xi_p\xi_q]_I = v_{p,q}([\xi_p]_I \otimes [\xi_q]_I)$ are contained in the range of $v_{p,q}$ for all $\xi_p \in X_p$ and $\xi_q \in X_q$. This observation, together with the fact that $v_{p,q}$ is an isometric linear map and therefore has closed range, implies that $v_{p,q}$ is surjective. In total, we have that $v_{p,q}$ is a unitary and hence axiom (iv) holds.

Finally, we check that the multiplication maps are associative. To ease notation, we will write

$$[\xi_p]_I[\xi_q]_I \equiv v_{p,q}([\xi_p]_I \otimes [\xi_q]_I) = [\xi_p\xi_q]_I \text{ for all } \xi_p \in X_p, \xi_q \in X_q, p, q \in P.$$

Fixing $p, q, r \in P, \xi_p \in X_p, \xi_q \in X_q$ and $\xi_r \in X_r$, we have that

$$([\xi_p]_I[\xi_q]_I)[\xi_r]_I = [\xi_p\xi_q]_I[\xi_r]_I = [(\xi_p\xi_q)\xi_r]_I = [\xi_p(\xi_q\xi_r)]_I = [\xi_p]_I[\xi_q\xi_r]_I = [\xi_p]_I([\xi_q]_I[\xi_r]_I),$$

using associativity of the multiplication maps of X in the third equality. Thus axiom (v) holds and we conclude that $[X]_I$ constitutes a product system over P with coefficients in $[A]_I$, finishing the proof. □

We will use the notation $[\cdot]_I \colon X \to [X]_I$ as shorthand for the family of quotient maps $[\cdot]_I \colon X_p \to [X_p]_I$ for all $p \in P$.

2.4 Product systems over right LCM semigroups

A unital semigroup P is said to be a *right LCM semigroup* if it is left cancellative and satisfies Clifford's condition, that is:

for every $p, q \in P$ with $pP \cap qP \neq \emptyset$, there exists $w \in P$ such that $pP \cap qP = wP$.

The element w is referred to as a *right least common multiple* or *right LCM* of p and q.

Right LCM semigroups include as a special case the quasi-lattice ordered semigroups considered in [45]. Indeed, quasi-lattice ordered semigroups are right LCM semigroups with the property that the only invertible element in P is the unit. Further examples include the Artin monoids [6], the Baumslag-Solitar monoids $B(m, n)^+$ [29, 42, 57], and the semigroup $R \rtimes R^\times$ of affine transformations of an integral domain R that satisfies the GCD condition [41, 46].

Product systems over right LCM semigroups were introduced and studied by Kwaśniewski and Larsen [37, 38], extending the construction of Fowler [23]. They have been investigated further in [18, 33]. The interest lies in that they retain several of the structural properties from the single C*-correspondence case.

Let A be a C*-algebra and let P be a unital right LCM subsemigroup of a discrete group G. Let X be a product system over P with coefficients in A. We say that X is *compactly aligned* if, for all $p, q \in P \setminus \{e\}$ with the property that $pP \cap qP = wP$ for some $w \in P$, we have that

$$\iota_p^w(\mathcal{K}(X_p))\iota_q^w(\mathcal{K}(X_q)) \subseteq \mathcal{K}(X_w).$$

This condition is independent of the choice of right LCM $w \in P$ of $p, q \in P$, e.g., [18, p. 11]. Notice that we disregard the case where p or q equals e, as the compact alignment condition holds automatically in this case.

When (G, P) is totally ordered, X is automatically compactly aligned, e.g., when $P = \mathbb{Z}_+$. Moreover, it is a standard fact that if $\phi_p(A) \subseteq \mathcal{K}(X_p)$ for all $p \in P$, then X is automatically compactly aligned. We provide a short proof.

Proposition 2.4.1. *Let P be a unital right LCM subsemigroup of a discrete group G. Let X be a product system over P with coefficients in a C*-algebra A. If $\phi_p(A) \subseteq \mathcal{K}(X_p)$ for all $p \in P$, then $\iota_p^{pq}(\mathcal{K}(X_p)) \subseteq \mathcal{K}(X_{pq})$ for all $p, q \in P$, and thus X is compactly aligned.*

Proof. First note that $\iota_e^q(\phi_e(a)) = \phi_q(a) \in \mathcal{K}(X_q)$ for all $q \in P$ and $a \in A$. Now fix $p \in P \setminus \{e\}, q \in P$ and $k_p \in \mathcal{K}(X_p)$. Since $\phi_q(A) \subseteq \mathcal{K}(X_q)$ and $k_p \in \mathcal{K}(X_p)$, an application of [40, Proposition 4.7] gives that $k_p \otimes \mathrm{id}_{X_q} \in \mathcal{K}(X_p \otimes_A X_q)$. Hence an application of [40, p. 9, (1.6)] yields that $\iota_p^{pq}(k_p) = u_{p,q}(k_p \otimes \mathrm{id}_{X_q})u_{p,q}^* \in \mathcal{K}(X_{pq})$, as required. □

Compact alignment is preserved under unitary equivalence.

Proposition 2.4.2. *Let P be a unital right LCM subsemigroup of a discrete group G and let A and B be C*-algebras. Let X and Y be unitarily equivalent product systems over P with coefficients in A and B, respectively. Then X is compactly aligned if and only if Y is compactly aligned.*

Proof. We let $\{\iota_p^{pq}\}_{p,q \in P}$ (resp. $\{j_p^{pq}\}_{p,q \in P}$) denote the family of connecting $*$-homomorphisms of X (resp. Y). Let $\{W_p \colon X_p \to Y_p\}_{p \in P}$ be a family of maps that implements a unitary equivalence between X and Y. Assume that X is compactly aligned. Fix $p, q \in P \setminus \{e\}$ with the property that $pP \cap qP = wP$ for some $w \in P$. We must show that

$$j_p^w(\mathcal{K}(Y_p))j_q^w(\mathcal{K}(Y_q)) \subseteq \mathcal{K}(Y_w).$$

It suffices to prove that this holds for rank-one operators.

Fix $y_p, y_p' \in Y_p$ and $y_q, y_q' \in Y_q$. We obtain that

$$\begin{aligned} j_p^w(\Theta_{y_p,y_p'}^{Y_p})j_q^w(\Theta_{y_q,y_q'}^{Y_q}) &= W_w\iota_p^w(\Theta_{W_p^{-1}(y_p),W_p^{-1}(y_p')}^{X_p})W_w^{-1}W_w\iota_q^w(\Theta_{W_q^{-1}(y_q),W_q^{-1}(y_q')}^{X_q})W_w^{-1} \\ &= W_w\iota_p^w(\Theta_{W_p^{-1}(y_p),W_p^{-1}(y_p')}^{X_p})\iota_q^w(\Theta_{W_q^{-1}(y_q),W_q^{-1}(y_q')}^{X_q})W_w^{-1} \in \mathcal{K}(Y_w), \end{aligned}$$

using the dual of (2.11) in the first line and the compact alignment of X together with (2.7) in the last line. Hence Y is compactly aligned. The other implication is obtained via duality, completing the proof. □

We say that a representation (π, t) of a compactly aligned product system X is *Nica-covariant* if for all $p, q \in P \setminus \{e\}$, $k_p \in \mathcal{K}(X_p)$ and $k_q \in \mathcal{K}(X_q)$ we have that

$$\psi_p(k_p)\psi_q(k_q) = \begin{cases} \psi_w(\iota_p^w(k_p)\iota_q^w(k_q)) & \text{if } pP \cap qP = wP \text{ for some } w \in P, \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

This condition is independent of the choice of right LCM $w \in P$ of $p, q \in P$, e.g., [18, Proposition 2.4]. We disregard the case where p or q equals e, as (2.12) holds automatically in this case. It is straightforward to see that the Fock representation is Nica-covariant.

The Nica-covariance condition induces a Wick ordering on $\mathrm{C}^*(\pi, t)$, e.g, [18, 23, 37, 38]. Let $p, q \in P$ and suppose firstly that $pP \cap qP = wP$ for some $w \in P$. Then

$$t_p(X_p)^* t_q(X_q) \subseteq [t_{p'}(X_{p'}) t_{q'}(X_{q'})^*], \text{ where } p' = p^{-1}w, q' = q^{-1}w.$$

On the other hand, if $pP \cap qP = \emptyset$, then $t_p(X_p)^* t_q(X_q) = \{0\}$. From this it follows that

$$\mathrm{C}^*(\pi, t) = \overline{\mathrm{span}}\{t_p(X_p) t_q(X_q)^* \mid p, q \in P\}.$$

We write $\mathcal{NT}_X$ for the universal C*-algebra with respect to the Nica-covariant representations of X, and refer to it as the *Toeplitz-Nica-Pimsner algebra (of X)*. Since the Nica-covariance relations are graded, the existence of $\mathcal{NT}_X$ and its universal property follow from Proposition 2.3.3. We write $(\overline{\pi}_X, \overline{t}_X)$ for the *universal Nica-covariant representation (of X)*. If (π, t) is a Nica-covariant representation of X, we will write (in a slight abuse of notation) $\pi \times t$ for the canonical $*$-epimorphism $\mathcal{NT}_X \to \mathrm{C}^*(\pi, t)$. Injectivity of the Fock representation $(\overline{\pi}, \overline{t})$ implies that $(\overline{\pi}_X, \overline{t}_X)$ is injective. In fact, when P is contained in an amenable discrete group, the $*$-representation $\overline{\pi} \times \overline{t}$ is faithful, e.g., [33].

If X and Y are unitarily equivalent product systems, then $\mathcal{T}_X \cong \mathcal{T}_Y$ by Proposition 2.3.4. When X or Y is compactly aligned (and hence both are by Proposition 2.4.2), we moreover have that $\mathcal{NT}_X \cong \mathcal{NT}_Y$ canonically.

Proposition 2.4.3. *Let P be a unital right LCM subsemigroup of a discrete group G and let A and B be C*-algebras. Let X and Y be unitarily equivalent compactly aligned product systems over P with coefficients in A and B, respectively. Then the bijection of Proposition 2.3.4 preserves the Nica-covariant representations.*

Proof. We adopt the notation of Proposition 2.3.4 and also let $\{\iota_p^{pq}\}_{p,q \in P}$ (resp. $\{j_p^{pq}\}_{p,q \in P}$) denote the family of connecting $*$-homomorphisms of X (resp. Y). Let (π, t) be a Nica-covariant representation of X on some $\mathcal{B}(H)$. For notational convenience, we set $(\widetilde{\pi}, \widetilde{t}) := \{(\pi \circ W_e^{-1}, t_p \circ W_p^{-1})\}_{p \in P}$. By duality, it suffices to show that $(\widetilde{\pi}, \widetilde{t})$ is Nica-covariant. Fix $p, q \in P \setminus \{e\}$, $k_p \in \mathcal{K}(Y_p)$ and $k_q \in \mathcal{K}(Y_q)$. We must show that

$$\widetilde{\psi}_p(k_p)\widetilde{\psi}_q(k_q) = \begin{cases} \widetilde{\psi}_w(j_p^w(k_p) j_q^w(k_q)) & \text{if } pP \cap qP = wP \text{ for some } w \in P, \\ 0 & \text{otherwise.} \end{cases}$$

To this end, it suffices to show that the claim holds for rank-one operators.

Fix $y_p, y_p' \in Y_p$ and $y_q, y_q' \in Y_q$. We obtain that

$$\begin{aligned}\widetilde{\psi}_p(\Theta^{Y_p}_{y_p,y_p'})\widetilde{\psi}_q(\Theta^{Y_q}_{y_q,y_q'}) &= t_p(W_p^{-1}(y_p))t_p(W_p^{-1}(y_p'))^* t_q(W_q^{-1}(y_q))t_q(W_q^{-1}(y_q'))^* \\ &= \psi_p(\Theta^{X_p}_{W_p^{-1}(y_p),W_p^{-1}(y_p')})\psi_q(\Theta^{X_q}_{W_q^{-1}(y_q),W_q^{-1}(y_q')}).\end{aligned}$$

It follows that $\widetilde{\psi}_p(\Theta^{Y_p}_{y_p,y_p'})\widetilde{\psi}_q(\Theta^{Y_q}_{y_q,y_q'}) = 0$ if $pP \cap qP = \emptyset$ by Nica-covariance of (π, t). Now suppose that $pP \cap qP = wP$ for some $w \in P$. Then we have that

$$\begin{aligned}\widetilde{\psi}_p(\Theta^{Y_p}_{y_p,y_p'})\widetilde{\psi}_q(\Theta^{Y_q}_{y_q,y_q'}) &= \psi_w(\iota_p^w(\Theta^{X_p}_{W_p^{-1}(y_p),W_p^{-1}(y_p')})\iota_q^w(\Theta^{X_q}_{W_q^{-1}(y_q),W_q^{-1}(y_q')})) \\ &= \psi_w(W_w^{-1} j_p^w(\Theta^{Y_p}_{y_p,y_p'})W_w W_w^{-1} j_q^w(\Theta^{Y_q}_{y_q,y_q'})W_w) \\ &= \widetilde{\psi}_w(j_p^w(\Theta^{Y_p}_{y_p,y_p'})j_q^w(\Theta^{Y_q}_{y_q,y_q'})),\end{aligned}$$

using Nica-covariance of (π, t) in the first line, (2.11) in the second line, and Remark 2.3.5 in the last line. In total, we have that $(\widetilde{\pi}, \widetilde{t})$ is Nica-covariant, as required. □

Proposition 2.4.4. *Let P be a unital right LCM subsemigroup of a discrete group G. Let X be a compactly aligned product system over P with coefficients in a C*-algebra A and let $I \subseteq A$ be an ideal that is positively invariant for X. Let $\{\iota_p^{pq}\}_{p,q\in P}$ denote the connecting $*$-homomorphisms of X and $\{j_p^{pq}\}_{p,q\in P}$ denote the connecting $*$-homomorphisms of $[X]_I$. Then we have that*

$$j_e^q([\phi_e(a)]_I) = [\iota_e^q(\phi_e(a))]_I \text{ for all } a \in A \quad \text{and} \quad j_p^{pq}([S_p]_I) = [\iota_p^{pq}(S_p)]_I \text{ for all } S_p \in \mathcal{L}(X_p),$$

for all $p \in P \setminus \{e\}$ and $q \in P$, and thus $[X]_I$ is compactly aligned.

Proof. Fix $q \in P$ and $a \in A$, and observe that

$$j_e^q([\phi_e(a)]_I) = [\phi_q]_I([a]_I) = [\phi_q(a)]_I = [\iota_e^q(\phi_e(a))]_I.$$

Next, fix $p \in P \setminus \{e\}$ and $S_p \in \mathcal{L}(X_p)$. For $\xi_p \in X_p$ and $\xi_q \in X_q$, we have that

$$j_p^{pq}([S_p]_I)([\xi_p]_I[\xi_q]_I) = [(S_p\xi_p)\xi_q]_I = [\iota_p^{pq}(S_p)(\xi_p\xi_q)]_I = [\iota_p^{pq}(S_p)]_I([\xi_p]_I[\xi_q]_I),$$

from which it follows that $j_p^{pq}([S_p]_I) = [\iota_p^{pq}(S_p)]_I$. This proves the first claim.

Finally, take $p, q \in P \setminus \{e\}$ such that $pP \cap qP = wP$ for some $w \in P$, and fix $\dot{k}_p \in \mathcal{K}([X_p]_I)$ and $\dot{k}_q \in \mathcal{K}([X_q]_I)$. By Lemma 2.2.11, we have that

$$\dot{k}_p = [k_p]_I \text{ and } \dot{k}_q = [k_q]_I \text{ for some } k_p \in \mathcal{K}(X_p) \text{ and } k_q \in \mathcal{K}(X_q),$$

and therefore we obtain that

$$j_p^w(\dot{k}_p)j_q^w(\dot{k}_q) = j_p^w([k_p]_I)j_q^w([k_q]_I) = [\iota_p^w(k_p)]_I[\iota_q^w(k_q)]_I = [\iota_p^w(k_p)\iota_q^w(k_q)]_I.$$

Since X is compactly aligned, we have that $\iota_p^w(k_p)\iota_q^w(k_q) \in \mathcal{K}(X_w)$. Another application of Lemma 2.2.11 then gives that $j_p^w(\dot{k}_p)j_q^w(\dot{k}_q) \in \mathcal{K}([X_w]_I)$. Hence $[X]_I$ is compactly aligned and the proof is complete. □

Remark 2.4.5. Let P be a unital right LCM subsemigroup of a discrete group G. Let X be a compactly aligned product system over P with coefficients in a C*-algebra A and let $I \subseteq A$ be an ideal that is positively invariant for X. Then the maps

$$\begin{aligned} &\pi \colon A \to \mathcal{NT}_{[X]_I}; \pi(a) = \overline{\pi}_{[X]_I}([a]_I) \text{ for all } a \in A, \\ &t_p \colon X_p \to \mathcal{NT}_{[X]_I}; t_p(\xi_p) = \overline{t}_{[X]_I,p}([\xi_p]_I) \text{ for all } \xi_p \in X_p, p \in P \setminus \{e\} \end{aligned}$$

form a Nica-covariant representation of X. Indeed, it is routine to check that (π, t) is a representation. For Nica-covariance, first note that

$$\psi_p = \overline{\psi}_{[X]_I,p} \circ [\cdot]_I|_{\mathcal{K}(X_p)} \text{ for all } p \in P$$

by Lemma 2.2.11. Combining this observation with Proposition 2.4.4 and Nica-covariance of $(\overline{\pi}_{[X]_I}, \overline{t}_{[X]_I})$, we conclude that (π, t) is Nica-covariant. Additionally, note that

$$\mathrm{C}^*(\pi, t) = \mathrm{C}^*(\overline{\pi}_{[X]_I}, \overline{t}_{[X]_I}) = \mathcal{NT}_{[X]_I}$$

and therefore universality of $\mathcal{NT}_X$ yields a (unique) canonical $*$-epimorphism

$$\pi \times t \colon \mathcal{NT}_X \to \mathcal{NT}_{[X]_I}.$$

In practice we will denote this map by $[\cdot]_I$, with context distinguishing it from the other maps sharing this symbol.

We close this section by laying the groundwork for the study of $\mathrm{C}^*(\pi, t)$ for an arbitrary Nica-covariant representation (π, t). For each subset $F \subseteq P$, we define

$$B_F^{(\pi,t)} := \overline{\operatorname{span}}\{\psi_p(\mathcal{K}(X_p)) \mid p \in F\}.$$

Note that $B_\emptyset^{(\pi,t)} = \{0\}$. We refer to the spaces $B_F^{(\pi,t)}$ as the *cores* of (π, t). By imposing further structure on F, we can say more about the associated core.

A finite subset $F \subseteq P$ is said to be $\vee$-*closed* if for every $p, q \in F$ with $pP \cap qP \neq \emptyset$ there exists a unique $w \in F$ such that $pP \cap qP = wP$. When $\emptyset \neq F \subseteq P$ is finite and $\vee$-closed, we obtain a canonical partial order on F.

Lemma 2.4.6. *Let P be a unital right LCM subsemigroup of a discrete group G and let $\emptyset \neq F \subseteq P$ be finite and $\vee$-closed. Then the relation $\leq$ determined by*

$$p \leq q \iff p^{-1}q \in P,$$

where $p, q \in F$, defines a partial order on F.

Proof. Reflexivity follows from the assumption that P is unital. Now take $p, q \in F$ and suppose that $p \leq q$ and $q \leq p$. Then $p^{-1}q \in P$ and $q^{-1}p \in P$. This implies that $q = pr$ and $p = qr'$ for some $r, r' \in P$. Fixing $w \in P$, we therefore obtain that

$$pw = qr'w \in qP \quad \text{and} \quad qw = prw \in pP,$$

showing that $pP \cap qP = pP = qP$. Since F is $\vee$-closed, we deduce that $p = q$ and hence $\leq$ is antisymmetric. Finally, take $p, q, r \in F$ and assume that $p \leq q$ and $q \leq r$. Then $p^{-1}q, q^{-1}r \in P$ and we obtain that

$$p^{-1}r = p^{-1}qq^{-1}r = (p^{-1}q)(q^{-1}r) \in P,$$

showing that $p \leq r$. Hence $\leq$ is transitive and the proof is complete. □

Each finite non-empty $\vee$-closed set F contains a minimal and maximal element under the partial order of Lemma 2.4.6. We also have that $B_F^{(\pi,t)}$ is a C*-subalgebra of $\mathrm{C}^*(\pi, t)$, and we can drop the closure in the definition.

Proposition 2.4.7. *[18, Proposition 2.10] Let P be a unital right LCM subsemigroup of a discrete group G. Let X be a compactly aligned product system over P with coefficients in a C*-algebra A and let (π, t) be a Nica-covariant representation of X. Then for each finite $\vee$-closed subset $F \subseteq P$, the space $B_F^{(\pi,t)}$ is a C*-subalgebra of $\mathrm{C}^*(\pi, t)$ and*

$$B_F^{(\pi,t)} = \mathrm{span}\{\psi_p(\mathcal{K}(X_p)) \mid p \in F\}.$$

Proof. The result holds trivially when $F = \emptyset$, so fix $\emptyset \neq F \subseteq P$ finite and $\vee$-closed. We begin by showing that $B_F^{(\pi,t)}$ is a C*-subalgebra of $\mathrm{C}^*(\pi, t)$. It is clear that $B_F^{(\pi,t)}$ is a closed linear subspace of $\mathrm{C}^*(\pi, t)$ that is selfadjoint. It remains to check that $B_F^{(\pi,t)}$ is closed under multiplication. To this end, it suffices to show that the product of arbitrary generators in $B_F^{(\pi,t)}$ belongs to $B_F^{(\pi,t)}$. So fix $p, q \in F, k_p \in \mathcal{K}(X_p)$ and $k_q \in \mathcal{K}(X_q)$. If $pP \cap qP = \emptyset$, then

$$\psi_p(k_p)\psi_q(k_q) = 0 \in B_F^{(\pi,t)}$$

by Nica-covariance of (π, t). If $pP \cap qP \neq \emptyset$, then $\vee$-closure of F guarantees a unique element $w \in F$ such that $pP \cap qP = wP$. Another application of Nica-covariance gives that

$$\psi_p(k_p)\psi_q(k_q) = \psi_w(\iota_p^w(k_p)\iota_q^w(k_q)) \in B_F^{(\pi,t)}.$$

In all cases we have that $\psi_p(k_p)\psi_q(k_q) \in B_F^{(\pi,t)}$, as required. Hence $B_F^{(\pi,t)}$ is a C*-subalgebra of $\mathrm{C}^*(\pi, t)$, as claimed.

Next we show that $B_F^{(\pi,t)} = \mathrm{span}\{\psi_p(\mathcal{K}(X_p)) \mid p \in F\}$. By universality of $\mathcal{NT}_X$, there exists a (unique) canonical $*$-epimorphism $\pi \times t \colon \mathcal{NT}_X \to \mathrm{C}^*(\pi, t)$. In particular, we have that

$$(\pi \times t)(B_F^{(\overline{\pi}_X, \overline{t}_X)}) = B_F^{(\pi,t)}.$$

Therefore it suffices to show that

$$B_F^{(\overline{\pi}_X,\overline{t}_X)} = \operatorname{span}\{\overline{\psi}_{X,p}(\mathcal{K}(X_p)) \mid p \in F\}.$$

The reverse inclusion is immediate, so take $f \in B_F^{(\overline{\pi}_X,\overline{t}_X)}$. By definition there exists a net

$$(f_\lambda)_{\lambda\in\Lambda} \subseteq \operatorname{span}\{\overline{\psi}_{X,p}(\mathcal{K}(X_p)) \mid p \in F\}$$

such that $f = \|\cdot\|\text{-}\lim_\lambda f_\lambda$. For each $\lambda \in \Lambda$, we write

$$f_\lambda = \sum_{p\in F} \overline{\psi}_{X,p}(k_{p,\lambda}), \text{ where } k_{p,\lambda} \in \mathcal{K}(X_p) \text{ for all } p \in F.$$

Let $(\overline{\pi},\overline{t})$ denote the Fock representation of X. We obtain that

$$(\overline{\pi}\times\overline{t})(f) = \|\cdot\|\text{-}\lim_\lambda \sum_{p\in F} \overline{\psi}_p(k_{p,\lambda}).$$

By Lemma 2.4.6, we may choose a minimal element $p_0 \in F$. Let $Q_{p_0} \in \mathcal{L}(\mathcal{F}X)$ denote the projection onto X_{p_0} and fix $p \in F \setminus \{p_0\}$. We claim that

$$Q_{p_0}\overline{\psi}_p(k_p)Q_{p_0} = 0 \text{ for all } k_p \in \mathcal{K}(X_p).$$

To see this, we may assume that $k_p = \Theta^{X_p}_{\xi_p,\eta_p}$ for some $\xi_p, \eta_p \in X_p$ without loss of generality. Fix $(\zeta_q)_{q\in P} \in \mathcal{F}X$. We have that

$$Q_{p_0}\overline{\psi}_p(\Theta^{X_p}_{\xi_p,\eta_p})Q_{p_0}(\zeta_q)_{q\in P} = Q_{p_0}\overline{t}_p(\xi_p)\overline{t}_p(\eta_p)^*\zeta_{p_0}.$$

Note that $p \neq e$, since otherwise minimality of p_0 would force $p = p_0$, contradicting that $p \in F \setminus \{p_0\}$. Suppose that $p_0 \in pP$, so that $p \leq p_0$ by definition. By minimality of p_0, we obtain that $p = p_0$, a contradiction. Thus $p_0 \notin pP$ and hence $\overline{t}_p(\eta_p)^*\zeta_{p_0} = 0$. It follows that $Q_{p_0}\overline{\psi}_p(\Theta^{X_p}_{\xi_p,\eta_p})Q_{p_0} = 0$, as required.

Next we claim that $Q_{p_0}\overline{\psi}_{p_0}(k_{p_0})Q_{p_0} = \iota_{p_0}(k_{p_0})$ for all $k_{p_0} \in \mathcal{K}(X_{p_0})$, where ι_{p_0} denotes the canonical embedding $\mathcal{K}(X_{p_0}) \hookrightarrow \mathcal{L}(\mathcal{F}X)$. Without loss of generality, we may assume that $k_{p_0} = \Theta^{X_{p_0}}_{\xi_{p_0},\eta_{p_0}}$ for some $\xi_{p_0}, \eta_{p_0} \in X_{p_0}$. It suffices to show that the desired equality holds on X_{p_0}. Indeed, if $p \in P \setminus \{p_0\}$ and $\zeta_p \in X_p$, then we obtain that

$$Q_{p_0}\overline{\psi}_{p_0}(\Theta^{X_{p_0}}_{\xi_{p_0},\eta_{p_0}})Q_{p_0}\zeta_p = 0 = \iota_{p_0}(\Theta^{X_{p_0}}_{\xi_{p_0},\eta_{p_0}})\zeta_p,$$

as required. Fix $\zeta_{p_0} \in X_{p_0}$ and note that we may write $\zeta_{p_0} = \zeta'_{p_0}a$ for some $\zeta'_{p_0} \in X_{p_0}$ and $a \in A$ by [40, Lemma 4.4]. We have that

$$\begin{aligned} Q_{p_0}\overline{\psi}_{p_0}(\Theta^{X_{p_0}}_{\xi_{p_0},\eta_{p_0}})Q_{p_0}\zeta_{p_0} &= Q_{p_0}\overline{t}_{p_0}(\xi_{p_0})\overline{t}_{p_0}(\eta_{p_0})^*\zeta_{p_0} = Q_{p_0}\overline{t}_{p_0}(\xi_{p_0})(\langle \eta_{p_0}, \zeta'_{p_0}\rangle a) = \xi_{p_0}\langle \eta_{p_0}, \zeta'_{p_0}\rangle a \\ &= \Theta^{X_{p_0}}_{\xi_{p_0},\eta_{p_0}}(\zeta'_{p_0})a = \Theta^{X_{p_0}}_{\xi_{p_0},\eta_{p_0}}\zeta_{p_0} = \iota_{p_0}(\Theta^{X_{p_0}}_{\xi_{p_0},\eta_{p_0}})\zeta_{p_0}, \end{aligned}$$

thereby proving the claim.

Combining the preceding deductions, we obtain that

$$Q_{p_0}(\overline{\pi} \times \overline{t})(f)Q_{p_0} = \|\cdot\|\text{-}\lim_\lambda \sum_{p \in F} Q_{p_0}\overline{\psi}_p(k_{p,\lambda})Q_{p_0} = \|\cdot\|\text{-}\lim_\lambda \iota_{p_0}(k_{p_0,\lambda}).$$

Since ι_{p_0} is an isometric $*$-homomorphism, we deduce that there exists $k_{p_0} \in \mathcal{K}(X_{p_0})$ such that

$$\|\cdot\|\text{-}\lim_\lambda k_{p_0,\lambda} = k_{p_0}.$$

In turn, we have that $\|\cdot\|\text{-}\lim_\lambda \overline{\psi}_{X,p_0}(k_{p_0,\lambda}) = \overline{\psi}_{X,p_0}(k_{p_0})$.

Now we iterate the preceding argument, replacing f by $f - \overline{\psi}_{X,p_0}(k_{p_0})$, the net $(f_\lambda)_{\lambda \in \Lambda}$ by $(f_\lambda - \overline{\psi}_{X,p_0}(k_{p_0,\lambda}))_{\lambda \in \Lambda}$, and the finite $\vee$-closed set F by $F \setminus \{p_0\}$. Notice that minimality of p_0 ensures that $F \setminus \{p_0\}$ is itself $\vee$-closed. We argue in this way until all elements of the finite set F have been exhausted, and deduce that for all $p \in F$ there exists $k_p \in \mathcal{K}(X_p)$ such that $\|\cdot\|\text{-}\lim_\lambda k_{p,\lambda} = k_p$. It then follows that

$$f = \sum_{p \in F} \overline{\psi}_{X,p}(k_p) \in \operatorname{span}\{\overline{\psi}_{X,p}(\mathcal{K}(X_p)) \mid p \in F\}.$$

In total, we obtain that

$$B_F^{(\overline{\pi}_X, \overline{t}_X)} = \operatorname{span}\{\overline{\psi}_{X,p}(\mathcal{K}(X_p)) \mid p \in F\},$$

finishing the proof. □

Notice that

$$B_P^{(\pi,t)} = \overline{\bigcup\{B_F^{(\pi,t)} \mid F \subseteq P \text{ finite}\}}$$

by definition. In fact, by [18, Proposition 2.11] we can write

$$B_P^{(\pi,t)} = \overline{\bigcup\{B_F^{(\pi,t)} \mid F \subseteq P \text{ finite and } \vee\text{-closed}\}}.$$

The significance of this core will be highlighted in the next section.

2.5 Product systems over $\mathbb{Z}_+^d$

For $d \in \mathbb{N}$, we write $[d] := \{1, 2, \dots, d\}$. We denote the usual free generators of $\mathbb{Z}_+^d$ by $\underline{1}, \dots, \underline{d}$, and we set $\underline{0} = (0, \dots, 0)$. For $\underline{n} = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$, we define the *length* of $\underline{n}$ by

$$|\underline{n}| := \sum\{n_i \mid i \in [d]\}.$$

For $\emptyset \neq F \subseteq [d]$, we write

$$\underline{1}_F := \sum\{\underline{i} \mid i \in F\} \text{ and } \underline{1}_\emptyset := \underline{0}.$$

We consider the lattice structure on $\mathbb{Z}_+^d$ given by

$$\underline{n} \vee \underline{m} := (\max\{n_i, m_i\})_{i=1}^d \quad \text{and} \quad \underline{n} \wedge \underline{m} := (\min\{n_i, m_i\})_{i=1}^d.$$

The semigroup $\mathbb{Z}_+^d$ imposes a partial order on $\mathbb{Z}^d$ that is compatible with the lattice structure. Specifically, we say that $\underline{n} \leq \underline{m}$ (resp. $\underline{n} < \underline{m}$) if and only if $n_i \leq m_i$ for all $i \in [d]$ (resp. $\underline{n} \leq \underline{m}$ and $\underline{n} \neq \underline{m}$). We denote the *support* of $\underline{n}$ by

$$\operatorname{supp} \underline{n} := \{i \in [d] \mid n_i \neq 0\},$$

and we write

$$\underline{n} \perp \underline{m} \iff \operatorname{supp} \underline{n} \bigcap \operatorname{supp} \underline{m} = \emptyset.$$

We write $\underline{n} \perp F$ if $\operatorname{supp} \underline{n} \bigcap F = \emptyset$. Notice that the set $\{\underline{n} \in \mathbb{Z}_+^d \mid \underline{n} \perp F\}$ is directed; indeed, if $\underline{n}, \underline{m} \perp F$ then $\underline{n}, \underline{m} \leq \underline{n} \vee \underline{m} \perp F$. Therefore we can make sense of limits with respect to $\underline{n} \perp F$.

Many desirable properties of the fibres of a product system $X = \{X_{\underline{n}}\}_{\underline{n} \in \mathbb{Z}_+^d}$ are inherited from the corresponding properties of the fibres $X_{\underline{i}}$, where $i \in [d]$.

Proposition 2.5.1. *Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A. Fix $\emptyset \neq F \subseteq [d]$. Then the following hold:*

(i) $X_{\underline{i}}$ is injective for all $i \in F$ if and only if $X_{\underline{n}}$ is injective for all $\underline{n} \in \mathbb{Z}_+^d$ satisfying $\operatorname{supp} \underline{n} \subseteq F$.

(ii) $\phi_{\underline{i}}(A) \subseteq \mathcal{K}(X_{\underline{i}})$ for all $i \in F$ if and only if $\phi_{\underline{n}}(A) \subseteq \mathcal{K}(X_{\underline{n}})$ for all $\underline{n} \in \mathbb{Z}_+^d$ satisfying $\operatorname{supp} \underline{n} \subseteq F$.

In particular, $X_{\underline{i}}$ is regular for all $i \in [d]$ if and only if X is regular.

Proof. (i) The reverse implication is immediate, so assume that $X_{\underline{i}}$ is injective for all $i \in F$. We will prove the claim by induction on $|\underline{n}|$. First note that injectivity of $X_{\underline{0}}$ is automatic, since $\phi_{\underline{0}}$ is nothing but left multiplication in A. Next, if $\operatorname{supp} \underline{n} \subseteq F$ and $|\underline{n}| = 1$, then we must have that $\underline{n} = \underline{i}$ for some $i \in F$ and so $X_{\underline{n}}$ is injective by assumption.

Now suppose that $X_{\underline{m}}$ is injective whenever $\operatorname{supp} \underline{m} \subseteq F$ and $|\underline{m}| = N$, where $N \in \mathbb{N}$. Fix $\underline{n} \in \mathbb{Z}_+^d$ satisfying $\operatorname{supp} \underline{n} \subseteq F$ and $|\underline{n}| = N+1$. Then we can write $\underline{n} = \underline{m} + \underline{i}$ for some $\underline{m} \in \mathbb{Z}_+^d$ satisfying $\operatorname{supp} \underline{m} \subseteq F$ and $|\underline{m}| = N$, and some $i \in F$. Fix $a \in A$ and suppose that $\phi_{\underline{n}}(a) = 0$. Then we have that

$$\phi_{\underline{n}}(a) = \phi_{\underline{m}+\underline{i}}(a) = \iota_{\underline{m}}^{\underline{m}+\underline{i}}(\phi_{\underline{m}}(a)) = u_{\underline{m},\underline{i}}(\phi_{\underline{m}}(a) \otimes \mathrm{id}_{X_{\underline{i}}})u_{\underline{m},\underline{i}}^* = 0.$$

Since $u_{\underline{m},\underline{i}}$ is a unitary, it follows that $\phi_{\underline{m}}(a) \otimes \mathrm{id}_{X_{\underline{i}}} = 0$. Since $\phi_{\underline{i}}$ is injective by assumption, we have that the assignment $S \mapsto S \otimes \mathrm{id}_{X_{\underline{i}}}$ (where $S \in \mathcal{L}(X_{\underline{m}})$) is injective, e.g., [40, p. 42]. Consequently, we deduce that $\phi_{\underline{m}}(a) = 0$. By the inductive hypothesis, we have that

$\phi_{\underline{m}}$ is injective and hence $a = 0$. Thus $X_{\underline{n}}$ is injective and by induction this proves the claim.

(ii) The reverse implication is immediate, so assume that $\phi_{\underline{i}}(A) \subseteq \mathcal{K}(X_{\underline{i}})$ for all $i \in F$. We will prove the claim by induction on $|\underline{n}|$. The fact that $\phi_{\underline{0}}(A) \subseteq \mathcal{K}(A)$ is immediate. Next, if $\operatorname{supp} \underline{n} \subseteq F$ and $|\underline{n}| = 1$, then we must have that $\underline{n} = \underline{i}$ for some $i \in F$ and therefore $\phi_{\underline{n}}(A) \subseteq \mathcal{K}(X_{\underline{n}})$ by assumption.

Now suppose that $\phi_{\underline{m}}(A) \subseteq \mathcal{K}(X_{\underline{m}})$ whenever $\operatorname{supp} \underline{m} \subseteq F$ and $|\underline{m}| = N$, where $N \in \mathbb{N}$. Fix $\underline{n} \in \mathbb{Z}_+^d$ satisfying $\operatorname{supp} \underline{n} \subseteq F$ and $|\underline{n}| = N+1$. Then we can write $\underline{n} = \underline{m} + \underline{i}$ for some $\underline{m} \in \mathbb{Z}_+^d$ satisfying $\operatorname{supp} \underline{m} \subseteq F$ and $|\underline{m}| = N$, and some $i \in F$. Fixing $a \in A$, note that $\phi_{\underline{n}}(a) = u_{\underline{m},\underline{i}}(\phi_{\underline{m}}(a) \otimes \mathrm{id}_{X_{\underline{i}}}) u_{\underline{m},\underline{i}}^*$ by arguing as in (i). Since $\phi_{\underline{i}}(A) \subseteq \mathcal{K}(X_{\underline{i}})$ by assumption and $\phi_{\underline{m}}(a) \in \mathcal{K}(X_{\underline{m}})$ by the inductive hypothesis, an application of [40, Proposition 4.7] gives that $\phi_{\underline{m}}(a) \otimes \mathrm{id}_{X_{\underline{i}}} \in \mathcal{K}(X_{\underline{m}} \otimes_A X_{\underline{i}})$. Finally, we apply [40, p. 9, (1.6)] to obtain that $\phi_{\underline{n}}(a) \in \mathcal{K}(X_{\underline{n}})$. Thus $\phi_{\underline{n}}(A) \subseteq \mathcal{K}(X_{\underline{n}})$ and by induction this proves the claim.

The final claim follows by applying items (i) and (ii) in tandem in the case where $F = [d]$. This finishes the proof. ☐

The following proposition will be used frequently in the sequel.

Proposition 2.5.2. *Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A. Fix $F \subseteq [d]$ and let $I \subseteq A$ be an ideal such that*

$$I \subseteq \bigcap \{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in F^c\} \quad \text{and} \quad \langle X_{\underline{n}}, I X_{\underline{n}} \rangle \subseteq I \text{ for all } \underline{n} \perp F.$$

Then

$$\phi_{\underline{n}}(I) \subseteq \mathcal{K}(X_{\underline{n}} I) \text{ for all } \underline{n} \perp F.$$

Proof. We will proceed by induction on $|\underline{n}|$. The cases where $|\underline{n}| = 0$ and $F = [d]$ follow trivially. The statement holds when $|\underline{n}| = 1$ by (2.5). Now assume that the claim holds for all $\underline{n} \perp F$ with $|\underline{n}| = N$, where $N \geq 1$. Fix $\underline{m} \perp F$ with $|\underline{m}| = N+1$, so that $\underline{m} = \underline{n} + \underline{i}$ for some $\underline{n} \perp F$ and $i \in F^c$ such that $|\underline{n}| = N$. We must show that $\phi_{\underline{m}}(a) \in \mathcal{K}(X_{\underline{m}} I)$ for all $a \in I$. By definition we have that

$$\phi_{\underline{m}}(a) = \phi_{\underline{n}+\underline{i}}(a) = \iota_{\underline{n}}^{\underline{n}+\underline{i}}(\phi_{\underline{n}}(a)) = u_{\underline{n},\underline{i}}(\phi_{\underline{n}}(a) \otimes \mathrm{id}_{X_{\underline{i}}}) u_{\underline{n},\underline{i}}^*.$$

Consider the element $k_{\underline{n}} \otimes \mathrm{id}_{X_{\underline{i}}}$ for some $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}} I)$. From the length 1 case, we have that $\phi_{\underline{i}}(I) \subseteq \mathcal{K}(X_{\underline{i}} I)$ and an application of Corollary 2.2.14 yields that $k_{\underline{n}} \otimes \mathrm{id}_{X_{\underline{i}}} \in \mathcal{K}((X_{\underline{n}} \otimes_A X_{\underline{i}}) I)$. Fixing $\xi_{\underline{m}}, \eta_{\underline{m}} \in X_{\underline{m}}$, it follows from (2.5) that

$$\left\langle u_{\underline{n},\underline{i}}^* \xi_{\underline{m}}, (k_{\underline{n}} \otimes \mathrm{id}_{X_{\underline{i}}}) u_{\underline{n},\underline{i}}^* \eta_{\underline{m}} \right\rangle = \left\langle \xi_{\underline{m}}, u_{\underline{n},\underline{i}}(k_{\underline{n}} \otimes \mathrm{id}_{X_{\underline{i}}}) u_{\underline{n},\underline{i}}^* \eta_{\underline{m}} \right\rangle \in I.$$

Another application of (2.5) then gives that $u_{\underline{n},\underline{i}}(k_{\underline{n}} \otimes \mathrm{id}_{X_{\underline{i}}}) u_{\underline{n},\underline{i}}^* \in \mathcal{K}(X_{\underline{m}} I)$.

Returning to the proof, by the inductive hypothesis we have that $\phi_{\underline{n}}(a) \in \mathcal{K}(X_{\underline{n}} I)$ and hence, applying the preceding comment for $k_{\underline{n}} = \phi_{\underline{n}}(a)$, we deduce that $\phi_{\underline{m}}(a) \in \mathcal{K}(X_{\underline{m}} I)$.

By induction, the proof is complete. □

Let X be a compactly aligned product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A and let (π, t) be a Nica-covariant representation of X. We say that (π, t) *admits a gauge action* γ if there exists a family $\{\gamma_{\underline{z}}\}_{\underline{z}\in\mathbb{T}^d}$ of $*$-endomorphisms of $\mathrm{C}^*(\pi, t)$ satisfying

$$\gamma_{\underline{z}}(\pi(a)) = \pi(a) \text{ for all } a \in A \text{ and } \gamma_{\underline{z}}(t_{\underline{n}}(\xi_{\underline{n}})) = \underline{z}^{\underline{n}} t_{\underline{n}}(\xi_{\underline{n}}) \text{ for all } \xi_{\underline{n}} \in X_{\underline{n}}, \underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\},$$

for each $\underline{z} \in \mathbb{T}^d$. If $\underline{z} = (z_1, \dots, z_d) \in \mathbb{T}^d$ and $\underline{n} = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$, then we write $\underline{z}^{\underline{n}} := z_1^{n_1} \dots z_d^{n_d}$. We set $\overline{\underline{z}} := (\overline{z_1}, \dots, \overline{z_d}) \in \mathbb{T}^d$. When such a gauge action γ exists, it is necessarily unique. We say that an ideal $\mathfrak{J} \subseteq \mathrm{C}^*(\pi, t)$ is *gauge-invariant* or *equivariant* if $\gamma_{\underline{z}}(\mathfrak{J}) \subseteq \mathfrak{J}$ for all $\underline{z} \in \mathbb{T}^d$ (and so $\gamma_{\underline{z}}(\mathfrak{J}) = \mathfrak{J}$ for all $\underline{z} \in \mathbb{T}^d$).

Proposition 2.5.3. *Let X be a compactly aligned product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A and let (π, t) be a Nica-covariant representation of X that admits a gauge action γ. Then $\gamma_{\underline{z}}$ is a $*$-automorphism for all $\underline{z} \in \mathbb{T}^d$, the family $\{\gamma_{\underline{z}}\}_{\underline{z}\in\mathbb{T}^d}$ is point-norm continuous and we obtain a group homomorphism*

$$\gamma \colon \mathbb{T}^d \to \mathrm{Aut}(\mathrm{C}^*(\pi, t)); \underline{z} \mapsto \gamma_{\underline{z}} \textit{ for all } \underline{z} \in \mathbb{T}^d.$$

Proof. Fixing $\underline{z} \in \mathbb{T}^d$, it is routine to check that $\gamma_{\overline{\underline{z}}}$ is the inverse of $\gamma_{\underline{z}}$. This proves the first claim, and the third claim is similarly straightforward to check. Indeed, to prove that $\gamma_{\underline{z}\cdot\underline{w}} = \gamma_{\underline{z}}\gamma_{\underline{w}}$ for all $\underline{z}, \underline{w} \in \mathbb{T}^d$, it suffices to check that the equality holds on the generators of $\mathrm{C}^*(\pi, t)$. This is immediate by definition.

It remains to show that $\{\gamma_{\underline{z}}\}_{\underline{z}\in\mathbb{T}^d}$ is point-norm continuous. To this end, let $(\underline{z}_n)_{n\in\mathbb{N}} \subseteq \mathbb{T}^d$ be a sequence which converges to $\underline{z} \in \mathbb{T}^d$ in the product topology. It suffices to show that $(\gamma_{\underline{z}_n}(f))_{n\in\mathbb{N}}$ converges to $\gamma_{\underline{z}}(f)$ in the norm topology for all $f \in \mathrm{C}^*(\pi, t)$. To prove this, it will be helpful to recall that

$$\mathrm{C}^*(\pi, t) = \overline{\mathrm{span}}\{t_{\underline{n}}(X_{\underline{n}}) t_{\underline{m}}(X_{\underline{m}})^* \mid \underline{n}, \underline{m} \in \mathbb{Z}_+^d\}$$

by Nica-covariance of (π, t). We start by verifying that the convergence holds when f is a monomial in $\mathrm{C}^*(\pi, t)$, i.e., $f = t_{\underline{n}}(\xi_{\underline{n}}) t_{\underline{m}}(\xi_{\underline{m}})^*$ for some $\underline{n}, \underline{m} \in \mathbb{Z}_+^d, \xi_{\underline{n}} \in X_{\underline{n}}$ and $\xi_{\underline{m}} \in X_{\underline{m}}$. In this case, we have that

$$\|\gamma_{\underline{z}_n}(f) - \gamma_{\underline{z}}(f)\| = \|\underline{z}_n^{\underline{n}} \overline{\underline{z}_n^{\underline{m}}} f - \underline{z}^{\underline{n}} \overline{\underline{z}^{\underline{m}}} f\| = |\underline{z}_n^{\underline{n}} \overline{\underline{z}_n^{\underline{m}}} - \underline{z}^{\underline{n}} \overline{\underline{z}^{\underline{m}}}| \cdot \|f\|.$$

Note that since $\underline{z}_n \to \underline{z}$ in the product topology on $\mathbb{T}^d$, we have entrywise convergence in $\mathbb{T}$. It follows that $\underline{z}_n^{\underline{n}} \to \underline{z}^{\underline{n}}$ in $\mathbb{T}$, using that the sequences involved are bounded. In turn, continuity of the complex conjugate implies that $\overline{\underline{z}_n^{\underline{m}}} \to \overline{\underline{z}^{\underline{m}}}$ in $\mathbb{T}$, and thus $\underline{z}_n^{\underline{n}} \overline{\underline{z}_n^{\underline{m}}} \to \underline{z}^{\underline{n}} \overline{\underline{z}^{\underline{m}}}$ in $\mathbb{T}$. Hence $(\gamma_{\underline{z}_n}(f))_{n\in\mathbb{N}}$ converges to $\gamma_{\underline{z}}(f)$ in the case where f is a monomial in $\mathrm{C}^*(\pi, t)$. This extends to finite sums of monomials by continuity of addition in $\mathrm{C}^*(\pi, t)$. Finally, assume that f is the norm-limit of a sequence $(f_n)_{n\in\mathbb{N}}$ of finite sums of monomials in

$\mathrm{C}^*(\pi, t)$ and fix $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\|f - f_N\| < \varepsilon/3$. By the preceding arguments, there exists $M \in \mathbb{N}$ such that $\|\gamma_{\underline{z}_n}(f_N) - \gamma_{\underline{z}}(f_N)\| < \varepsilon/3$ for all $n \geq M$. We obtain that

$$\begin{aligned}\|\gamma_{\underline{z}_n}(f) - \gamma_{\underline{z}}(f)\| &\leq \|\gamma_{\underline{z}_n}(f) - \gamma_{\underline{z}_n}(f_N)\| + \|\gamma_{\underline{z}_n}(f_N) - \gamma_{\underline{z}}(f_N)\| + \|\gamma_{\underline{z}}(f_N) - \gamma_{\underline{z}}(f)\| \\ &\leq \|f - f_N\| + \|\gamma_{\underline{z}_n}(f_N) - \gamma_{\underline{z}}(f_N)\| + \|f_N - f\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon\end{aligned}$$

for all $n \geq M$, using that each $\gamma_{\underline{z}_n}$ and $\gamma_{\underline{z}}$ is contractive. Thus $(\gamma_{\underline{z}_n}(f))_{n\in\mathbb{N}}$ converges to $\gamma_{\underline{z}}(f)$ for all $f \in \mathrm{C}^*(\pi, t)$, completing the proof. □

We will utilise the gauge action properties of Proposition 2.5.3 frequently and without citation. The following proposition summarises the two main examples of Nica-covariant representations that admit gauge actions.

Proposition 2.5.4. *Let X be a compactly aligned product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A. Then the following hold:*

(i) the universal Nica-covariant representation $(\overline{\pi}_X, \overline{t}_X)$ of X admits a gauge action;

(ii) the Fock representation $(\overline{\pi}, \overline{t})$ of X admits a gauge action.

Proof. (i) Fix $\underline{z} \in \mathbb{T}^d$. We define maps $\pi_{\underline{z}}$ and $t_{\underline{z},\underline{n}}$ for each $\underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$ via

$$\begin{aligned}\pi_{\underline{z}} &\colon A \to \mathcal{NT}_X; a \mapsto \overline{\pi}_X(a) \text{ for all } a \in A, \\ t_{\underline{z},\underline{n}} &\colon X_{\underline{n}} \to \mathcal{NT}_X; \xi_{\underline{n}} \mapsto \underline{z}^{\underline{n}}\overline{t}_{X,\underline{n}}(\xi_{\underline{n}}) \text{ for all } \xi_{\underline{n}} \in X_{\underline{n}}, \underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}.\end{aligned}$$

It is routine to check that $(\pi_{\underline{z}}, t_{\underline{z}})$ constitutes a representation of X, using that $\underline{z}^{\underline{n}}\overline{\underline{z}^{\underline{n}}} = 1$ for all $\underline{n} \in \mathbb{Z}_+^d$ when verifying preservation of the inner product. The latter also implies that $\psi_{\underline{z},\underline{n}} = \overline{\psi}_{X,\underline{n}}$ for all $\underline{n} \in \mathbb{Z}_+^d$, from which it follows that $(\pi_{\underline{z}}, t_{\underline{z}})$ is Nica-covariant. Next we claim that $\mathrm{C}^*(\pi_{\underline{z}}, t_{\underline{z}}) = \mathcal{NT}_X$. To see this, it suffices to show that $\mathrm{C}^*(\pi_{\underline{z}}, t_{\underline{z}})$ contains the generators of $\mathcal{NT}_X = \mathrm{C}^*(\overline{\pi}_X, \overline{t}_X)$. Accordingly, for each $a \in A$ we have that $\overline{\pi}_X(a) = \pi_{\underline{z}}(a)$. Next, for each $\underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$ and $\xi_{\underline{n}} \in X_{\underline{n}}$, we have that $\overline{t}_{X,\underline{n}}(\xi_{\underline{n}}) = t_{\underline{z},\underline{n}}(\overline{\underline{z}^{\underline{n}}}\xi_{\underline{n}})$. Consequently, we have that $\mathrm{C}^*(\pi_{\underline{z}}, t_{\underline{z}}) = \mathcal{NT}_X$, as claimed.

By universality of $\mathcal{NT}_X$, we obtain a (unique) canonical $*$-epimorphism

$$\pi_{\underline{z}} \times t_{\underline{z}} \colon \mathcal{NT}_X \to \mathcal{NT}_X = \mathrm{C}^*(\pi_{\underline{z}}, t_{\underline{z}}).$$

We define $\gamma_{\underline{z}} := \pi_{\underline{z}} \times t_{\underline{z}}$. By definition we have that

$$\gamma_{\underline{z}}(\overline{\pi}_X(a)) = \overline{\pi}_X(a) \text{ and } \gamma_{\underline{z}}(\overline{t}_{X,\underline{n}}(\xi_{\underline{n}})) = \underline{z}^{\underline{n}}\overline{t}_{X,\underline{n}}(\xi_{\underline{n}}) \text{ for all } a \in A, \xi_{\underline{n}} \in X_{\underline{n}}, \underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}.$$

Hence γ constitutes a gauge action of $(\overline{\pi}_X, \overline{t}_X)$, finishing the proof of item (i).

(ii) Fix $\underline{z} \in \mathbb{T}^d$. We define a map $u_{\underline{z}}$ via

$$u_{\underline{z}} \colon \bigoplus_{\underline{n} \in \mathbb{Z}_+^d} X_{\underline{n}} \to \mathcal{F}X; (\xi_{\underline{n}})_{\underline{n} \in \mathbb{Z}_+^d} \mapsto (\underline{z}^{\underline{n}} \xi_{\underline{n}})_{\underline{n} \in \mathbb{Z}_+^d} \text{ for all } (\xi_{\underline{n}})_{\underline{n} \in \mathbb{Z}_+^d} \in \bigoplus_{\underline{n} \in \mathbb{Z}_+^d} X_{\underline{n}},$$

where $\bigoplus_{\underline{n} \in \mathbb{Z}_+^d} X_{\underline{n}}$ is the algebraic direct sum of the fibres of X. It is routine to check that $u_{\underline{z}}$ is well-defined, linear and isometric, using that $\overline{\underline{z}^{\underline{n}}} \underline{z}^{\underline{n}} = 1$ for all $\underline{n} \in \mathbb{Z}_+^d$ for the latter. Since $\bigoplus_{\underline{n} \in \mathbb{Z}_+^d} X_{\underline{n}}$ is dense in $\mathcal{F}X$, we deduce that $u_{\underline{z}}$ extends to an isometric linear map

$$u_{\underline{z}} \colon \mathcal{F}X \to \mathcal{F}X.$$

Notice also that $u_{\underline{z}}$ is an A-bimodule map, which can be seen by checking on the algebraic direct sum and then extending to $\mathcal{F}X$ using continuity of $u_{\underline{z}}$. Next, fix $(\xi_{\underline{n}})_{\underline{n} \in \mathbb{Z}_+^d} \in \bigoplus_{\underline{n} \in \mathbb{Z}_+^d} X_{\underline{n}}$. Note that

$$u_{\underline{z}}((\overline{\underline{z}^{\underline{n}}} \xi_{\underline{n}})_{\underline{n} \in \mathbb{Z}_+^d}) = (\xi_{\underline{n}})_{\underline{n} \in \mathbb{Z}_+^d}.$$

This proves that $u_{\underline{z}}$ is surjective, since it is an isometric linear map and therefore has closed range. In total, we have that $u_{\underline{z}} \in \mathcal{L}(\mathcal{F}X)$ is a unitary. A direct computation shows that $u_{\underline{z}}^* = u_{\overline{\underline{z}}}$.

Next we define a map $\gamma_{\underline{z}}$ by

$$\gamma_{\underline{z}} \colon \mathrm{C}^*(\overline{\pi}, \overline{t}) \to \mathrm{C}^*(\overline{\pi}, \overline{t}); f \mapsto u_{\underline{z}} f u_{\underline{z}}^* \text{ for all } f \in \mathrm{C}^*(\overline{\pi}, \overline{t}).$$

To see that $\gamma_{\underline{z}}$ is well-defined, it suffices to show that $u_{\underline{z}} f u_{\underline{z}}^* \in \mathrm{C}^*(\overline{\pi}, \overline{t})$ whenever f is a monomial in $\mathrm{C}^*(\overline{\pi}, \overline{t})$, recalling that

$$\mathrm{C}^*(\overline{\pi}, \overline{t}) = \overline{\mathrm{span}}\{\overline{t}_{\underline{n}}(X_{\underline{n}}) \overline{t}_{\underline{m}}(X_{\underline{m}})^* \mid \underline{n}, \underline{m} \in \mathbb{Z}_+^d\}.$$

Accordingly, fix $f = \overline{t}_{\underline{n}}(\xi_{\underline{n}}) \overline{t}_{\underline{m}}(\xi_{\underline{m}})^*$ for some $\underline{n}, \underline{m} \in \mathbb{Z}_+^d, \xi_{\underline{n}} \in X_{\underline{n}}$ and $\xi_{\underline{m}} \in X_{\underline{m}}$. A direct computation yields that

$$u_{\underline{z}} \overline{t}_{\underline{n}}(\xi_{\underline{n}}) \overline{t}_{\underline{m}}(\xi_{\underline{m}})^* u_{\underline{z}}^* = \underline{z}^{\underline{n}-\underline{m}} \overline{t}_{\underline{n}}(\xi_{\underline{n}}) \overline{t}_{\underline{m}}(\xi_{\underline{m}})^* \in \mathrm{C}^*(\overline{\pi}, \overline{t}).$$

Indeed, this follows by checking that the equality holds on the fibres of X, since all of the maps involved are adjointable and thus in particular linear and bounded. Note also that $\underline{n} - \underline{m}$ may not belong to $\mathbb{Z}_+^d$ in general, but we can nevertheless make sense of the quantity $\underline{z}^{\underline{n}-\underline{m}} \in \mathbb{T}^d$ in the natural way. Thus $\gamma_{\underline{z}}$ is well-defined and it is routine to check that it is a $*$-homomorphism. We also have that

$$\gamma_{\underline{z}}(\overline{\pi}(a)) = \overline{\pi}(a) \text{ for all } a \in A \text{ and } \gamma_{\underline{z}}(\overline{t}_{\underline{n}}(\xi_{\underline{n}})) = \underline{z}^{\underline{n}} \overline{t}_{\underline{n}}(\xi_{\underline{n}}) \text{ for all } \xi_{\underline{n}} \in X_{\underline{n}}, \underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\},$$

which can be seen by checking on fibres and then using linearity and continuity of the maps involved. In total, we have shown that γ constitutes a gauge action of $(\overline{\pi}, \overline{t})$, finishing the proof of item (ii). □

Given $\underline{m}, \underline{m}' \in \mathbb{Z}_+^d$ with $\underline{m} \leq \underline{m}'$, we write

$$[\underline{m}, \underline{m}'] := \{\underline{n} \in \mathbb{Z}_+^d \mid \underline{m} \leq \underline{n} \leq \underline{m}'\} \quad \text{and} \quad (\underline{m}, \underline{m}'] := \{\underline{n} \in \mathbb{Z}_+^d \mid \underline{m} < \underline{n} \leq \underline{m}'\}.$$

Notice that both sets are finite and $\vee$-closed, so we have that

$$B_{[\underline{m},\underline{m}']}^{(\pi,t)} = \operatorname{span}\{\psi_{\underline{n}}(\mathcal{K}(X_{\underline{n}})) \mid \underline{m} \leq \underline{n} \leq \underline{m}'\} \text{ and } B_{(\underline{m},\underline{m}']}^{(\pi,t)} = \operatorname{span}\{\psi_{\underline{n}}(\mathcal{K}(X_{\underline{n}})) \mid \underline{m} < \underline{n} \leq \underline{m}'\}$$

are C*-subalgebras of $\mathrm{C}^*(\pi, t)$ by Proposition 2.4.7. We also define

$$[\underline{m}, \infty] := \{\underline{n} \in \mathbb{Z}_+^d \mid \underline{m} \leq \underline{n}\} \quad \text{and} \quad (\underline{m}, \infty] := \{\underline{n} \in \mathbb{Z}_+^d \mid \underline{m} < \underline{n}\}.$$

Observe that the C*-algebras $B_{[\underline{m},\infty]}^{(\pi,t)}$ and $B_{(\underline{m},\infty]}^{(\pi,t)}$ can be realised as inductive limits of cores. When (π, t) admits a gauge action γ, we define the *fixed point algebra* to be

$$\mathrm{C}^*(\pi, t)^\gamma := \{f \in \mathrm{C}^*(\pi, t) \mid \gamma_{\underline{z}}(f) = f \text{ for all } \underline{z} \in \mathbb{T}^d\}.$$

Note that $\mathrm{C}^*(\pi, t)^\gamma$ is a C*-subalgebra of $\mathrm{C}^*(\pi, t)$. The following proposition clarifies the relationship between $\mathrm{C}^*(\pi, t)^\gamma$ and the cores of (π, t).

Proposition 2.5.5. *Let X be a compactly aligned product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A and let (π, t) be a Nica-covariant representation of X that admits a gauge action γ. Then we have that*

$$\mathrm{C}^*(\pi, t)^\gamma = B_{[\underline{0},\infty]}^{(\pi,t)}.$$

Proof. Fix $\underline{n} \in \mathbb{Z}_+^d$ and $\xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\underline{n}}$. Then for each $\underline{z} \in \mathbb{T}^d$, we obtain that

$$\gamma_{\underline{z}}(\psi_{\underline{n}}(\Theta_{\xi_{\underline{n}},\eta_{\underline{n}}})) = \gamma_{\underline{z}}(t_{\underline{n}}(\xi_{\underline{n}})t_{\underline{n}}(\eta_{\underline{n}})^*) = \underline{z}^{\underline{n}} t_{\underline{n}}(\xi_{\underline{n}})\overline{\underline{z}^{\underline{n}}} t_{\underline{n}}(\eta_{\underline{n}})^* = t_{\underline{n}}(\xi_{\underline{n}})t_{\underline{n}}(\eta_{\underline{n}})^* = \psi_{\underline{n}}(\Theta_{\xi_{\underline{n}},\eta_{\underline{n}}}),$$

using that $\underline{z}^{\underline{n}}\overline{\underline{z}^{\underline{n}}} = 1$ in the third equality. It follows that

$$\gamma_{\underline{z}}(\psi_{\underline{n}}(k_{\underline{n}})) = \psi_{\underline{n}}(k_{\underline{n}}) \text{ for all } k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}), \underline{n} \in \mathbb{Z}_+^d, \underline{z} \in \mathbb{T}^d.$$

We deduce that the reverse inclusion of the statement holds, since $\mathrm{C}^*(\pi, t)^\gamma$ is a C*-subalgebra of $\mathrm{C}^*(\pi, t)$.

For the forward inclusion, note that γ induces a conditional expectation

$$E_\gamma \colon \mathrm{C}^*(\pi, t) \to \mathrm{C}^*(\pi, t)^\gamma; f \mapsto \int_{\mathbb{T}^d} \gamma_{\underline{z}}(f) \, \mathrm{d}\underline{z} \text{ for all } f \in \mathrm{C}^*(\pi, t).$$

Thus we have that $\mathrm{C}^*(\pi, t)^\gamma = E_\gamma(\mathrm{C}^*(\pi, t))$, so it suffices to show that

$$E_\gamma(\mathrm{C}^*(\pi, t)) \subseteq B_{[\underline{0},\infty]}^{(\pi,t)}.$$

This in turn amounts to showing that

$$E_\gamma(t_{\underline{n}}(\xi_{\underline{n}})t_{\underline{m}}(\eta_{\underline{m}})^*) \in B_{[\underline{0},\infty]}^{(\pi,t)} \text{ for all } \xi_{\underline{n}} \in X_{\underline{n}}, \eta_{\underline{m}} \in X_{\underline{m}}, \underline{n}, \underline{m} \in \mathbb{Z}_+^d,$$

since $\mathrm{C}^*(\pi,t) = \overline{\operatorname{span}}\{t_{\underline{n}}(X_{\underline{n}})t_{\underline{m}}(X_{\underline{m}})^* \mid \underline{n},\underline{m} \in \mathbb{Z}_+^d\}$ and $B_{[\underline{0},\infty]}^{(\pi,t)}$ is a C*-subalgebra of $\mathrm{C}^*(\pi,t)$ by Nica-covariance of (π,t). Accordingly, fix $\xi_{\underline{n}} \in X_{\underline{n}}$ and $\eta_{\underline{m}} \in X_{\underline{m}}$ for some $\underline{n},\underline{m} \in \mathbb{Z}_+^d$. By definition we have that

$$E_\gamma(t_{\underline{n}}(\xi_{\underline{n}})t_{\underline{m}}(\eta_{\underline{m}})^*) = \begin{cases} t_{\underline{n}}(\xi_{\underline{n}})t_{\underline{n}}(\eta_{\underline{n}})^* & \text{if } \underline{n} = \underline{m}, \\ 0 & \text{otherwise.} \end{cases}$$

In either case we have that $E_\gamma(t_{\underline{n}}(\xi_{\underline{n}})t_{\underline{m}}(\eta_{\underline{m}})^*) \in B_{[\underline{0},\infty]}^{(\pi,t)}$, and the proof is complete. □

In the current work we will consider a subclass of compactly aligned product systems over $\mathbb{Z}_+^d$ introduced by Dor-On and Kakariadis [17]. Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A. We say that X is *strong compactly aligned* if it is compactly aligned and satisfies:

$$\iota_{\underline{n}}^{\underline{n}+\underline{i}}(\mathcal{K}(X_{\underline{n}})) \subseteq \mathcal{K}(X_{\underline{n}+\underline{i}}) \text{ whenever } \underline{n} \perp \underline{i}, \text{ where } i \in [d], \underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}. \tag{2.13}$$

We disallow $\underline{n} = \underline{0}$, as then (2.13) would imply that the left action of each fibre of X is by compact operators. Note that (2.13) does not imply compact alignment (rather, a strong compactly aligned product system is *a priori* assumed to be compactly aligned). Any C*-correspondence, when viewed as a product system over $\mathbb{Z}_+$, is vacuously strong compactly aligned. Moreover, strong compactly aligned product systems include those product systems over $\mathbb{Z}_+^d$ whose left actions are by compacts.

Corollary 2.5.6. *Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A. If $\phi_{\underline{n}}(A) \subseteq \mathcal{K}(X_{\underline{n}})$ for all $\underline{n} \in \mathbb{Z}_+^d$, then X is strong compactly aligned.*

Proof. This is immediate by Proposition 2.4.1. □

Strong compact alignment is preserved under unitary equivalence.

Proposition 2.5.7. *Let X and Y be unitarily equivalent product systems over $\mathbb{Z}_+^d$ with coefficients in C*-algebras A and B, respectively. Then X is strong compactly aligned if and only if Y is strong compactly aligned.*

Proof. Suppose that X and Y are unitarily equivalent by a collection $\{W_{\underline{n}} \colon X_{\underline{n}} \to Y_{\underline{n}}\}_{\underline{n}\in\mathbb{Z}_+^d}$. We use $\{\iota_{\underline{n}}^{\underline{n}+\underline{m}}\}_{\underline{n},\underline{m}\in\mathbb{Z}_+^d}$ to denote the connecting $*$-homomorphisms of X and $\{j_{\underline{n}}^{\underline{n}+\underline{m}}\}_{\underline{n},\underline{m}\in\mathbb{Z}_+^d}$ to denote the connecting $*$-homomorphisms of Y. Suppose that X is strong compactly aligned. Then Y is compactly aligned by Proposition 2.4.2, so it remains to check strong compact alignment. Accordingly, fix $\underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$ and $i \in [d]$ such that $\underline{n} \perp \underline{i}$. We must show that

$$j_{\underline{n}}^{\underline{n}+\underline{i}}(\mathcal{K}(Y_{\underline{n}})) \subseteq \mathcal{K}(Y_{\underline{n}+\underline{i}}).$$

It suffices to show this for rank-one operators. For all $y_{\underline{n}}, y'_{\underline{n}} \in Y_{\underline{n}}$, we have that

$$j_{\underline{n}}^{\underline{n}+\underline{i}}(\Theta^{Y_{\underline{n}}}_{y_{\underline{n}},y'_{\underline{n}}}) = W_{\underline{n}+\underline{i}}\iota_{\underline{n}}^{\underline{n}+\underline{i}}(\Theta^{X_{\underline{n}}}_{W_{\underline{n}}^{-1}(y_{\underline{n}}),W_{\underline{n}}^{-1}(y'_{\underline{n}})})W_{\underline{n}+\underline{i}}^{-1} \in W_{\underline{n}+\underline{i}}\mathcal{K}(X_{\underline{n}+\underline{i}})W_{\underline{n}+\underline{i}}^{-1} = \mathcal{K}(Y_{\underline{n}+\underline{i}}),$$

using the dual of (2.11) in the first equality, strong compact alignment of X in the membership, and (2.7) in the final equality. This proves that Y is strong compactly aligned. The converse follows by duality, completing the proof. □

Strong compact alignment is also preserved under the quotient construction of Proposition 2.3.7.

Proposition 2.5.8. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let $I \subseteq A$ be an ideal that is positively invariant for X. Then $[X]_I$ is strong compactly aligned.*

Proof. We use $\{\iota_{\underline{n}}^{\underline{n}+\underline{m}}\}_{\underline{n},\underline{m}\in\mathbb{Z}_+^d}$ (resp. $\{j_{\underline{n}}^{\underline{n}+\underline{m}}\}_{\underline{n},\underline{m}\in\mathbb{Z}_+^d}$) to denote the connecting $*$-homomorphisms of X (resp. $[X]_I$). Proposition 2.4.4 guarantees that $[X]_I$ is compactly aligned, so it remains to check strong compact alignment. To this end, fix $\underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}, i \in [d]$ and $\dot{k}_{\underline{n}} \in \mathcal{K}([X_{\underline{n}}]_I)$, and suppose that $\underline{n} \perp \underline{i}$. By Lemma 2.2.11, we have that $\dot{k}_{\underline{n}} = [k_{\underline{n}}]_I$ for some $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$. Thus, by Proposition 2.4.4, we have that

$$j_{\underline{n}}^{\underline{n}+\underline{i}}(\dot{k}_{\underline{n}}) = j_{\underline{n}}^{\underline{n}+\underline{i}}([k_{\underline{n}}]_I) = [\iota_{\underline{n}}^{\underline{n}+\underline{i}}(k_{\underline{n}})]_I \in \mathcal{K}([X_{\underline{n}+\underline{i}}]_I),$$

using the strong compact alignment of X together with Lemma 2.2.11 to establish the membership. Hence $j_{\underline{n}}^{\underline{n}+\underline{i}}(\mathcal{K}([X_{\underline{n}}]_I)) \subseteq \mathcal{K}([X_{\underline{n}+\underline{i}}]_I)$, completing the proof. □

We will require some notation and results from [17]. Henceforth we assume that X is strong compactly aligned.

Proposition 2.5.9. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Then we have that*

$$\bigcap_{i\in F} \phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) = \bigcap\{\phi_{\underline{n}}^{-1}(\mathcal{K}(X_{\underline{n}})) \mid \underline{0} \leq \underline{n} \leq \underline{1}_F\} \textit{ for all } \emptyset \neq F \subseteq [d].$$

Proof. Fix $\emptyset \neq F \subseteq [d]$. The reverse inclusion is immediate, so take $a \in \bigcap_{i\in F} \phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}}))$. It suffices to show that $\phi_{\underline{n}}(a) \in \mathcal{K}(X_{\underline{n}})$ for all $\underline{0} \leq \underline{n} \leq \underline{1}_F$. We proceed by induction on $|\underline{n}|$. When $|\underline{n}| = 0$, there is nothing to show. Likewise, when $|\underline{n}| = 1$ we must have that $\underline{n} = \underline{i}$ for some $i \in F$, and therefore $\phi_{\underline{n}}(a) \in \mathcal{K}(X_{\underline{n}})$ by assumption.

Now assume that $\phi_{\underline{m}}(a) \in \mathcal{K}(X_{\underline{m}})$ for all $\underline{0} \leq \underline{m} \leq \underline{1}_F$ satisfying $|\underline{m}| = N$ for some fixed $1 \leq N < |F|$. Take $\underline{0} \leq \underline{n} \leq \underline{1}_F$ such that $|\underline{n}| = N+1$. Then we may write $\underline{n} = \underline{m}+\underline{i}$ for some $\underline{0} \neq \underline{m} \leq \underline{1}_F$ and $i \in F$ satisfying $|\underline{m}| = N$ and $\underline{m} \perp \underline{i}$. We obtain that

$$\phi_{\underline{n}}(a) = \phi_{\underline{m}+\underline{i}}(a) = \iota_{\underline{m}}^{\underline{m}+\underline{i}}(\phi_{\underline{m}}(a)) \in \iota_{\underline{m}}^{\underline{m}+\underline{i}}(\mathcal{K}(X_{\underline{m}})) \subseteq \mathcal{K}(X_{\underline{m}+\underline{i}}) = \mathcal{K}(X_{\underline{n}}),$$

using the inductive hypothesis to establish the membership and strong compact alignment in the inclusion. By induction, the proof is complete. □

For each $\emptyset \neq F \subseteq [d]$, we define

$$\mathcal{J}_F := (\bigcap_{i\in F} \ker \phi_{\underline{i}})^\perp \cap (\bigcap_{i\in[d]} \phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}}))) \quad \text{and} \quad \mathcal{J}_\emptyset := \{0\},$$

which are ideals of A. In turn, for each $F \subseteq [d]$, we define

$$\mathcal{I}_F := \{a \in A \mid \langle X_{\underline{n}}, aX_{\underline{n}}\rangle \subseteq \mathcal{J}_F \text{ for all } \underline{n} \perp F\} = \bigcap \{X_{\underline{n}}^{-1}(\mathcal{J}_F) \mid \underline{n} \perp F\}.$$

In particular, we have that $\mathcal{I}_\emptyset = \{0\}$ and $\mathcal{I}_F \subseteq \mathcal{J}_F$ for all $F \subseteq [d]$. The ideal $\mathcal{I}_F$ is the largest ideal in $\mathcal{J}_F$ that is $F^\perp$-invariant [17, Proposition 2.7]. We include a full proof to this effect, since this fact will be used throughout.

Proposition 2.5.10. *[17, Proposition 2.7] Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Then we have that*

$$\langle X_{\underline{n}}, \mathcal{I}_F X_{\underline{n}}\rangle \subseteq \mathcal{I}_F \text{ for all } \underline{n} \perp F, F \subseteq [d].$$

Proof. The case where $F = \emptyset$ holds trivially, so fix $\emptyset \neq F \subseteq [d]$ and $\underline{n} \perp F$. Take $\xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\underline{n}}$ and $a \in \mathcal{I}_F$. It suffices to show that $b := \langle \xi_{\underline{n}}, a\eta_{\underline{n}}\rangle \in \mathcal{I}_F$. To this end, fix $\underline{m} \perp F$ and $\xi_{\underline{m}}, \eta_{\underline{m}} \in X_{\underline{m}}$. We obtain that

$$\begin{aligned}\langle \xi_{\underline{m}}, b\eta_{\underline{m}}\rangle &= \langle \xi_{\underline{m}}, \langle \xi_{\underline{n}}, a\eta_{\underline{n}}\rangle \eta_{\underline{m}}\rangle = \langle \xi_{\underline{n}} \otimes \xi_{\underline{m}}, (a\eta_{\underline{n}}) \otimes \eta_{\underline{m}}\rangle \\ &= \langle \xi_{\underline{n}}\xi_{\underline{m}}, (a\eta_{\underline{n}})\eta_{\underline{m}}\rangle = \langle \xi_{\underline{n}}\xi_{\underline{m}}, a(\eta_{\underline{n}}\eta_{\underline{m}})\rangle,\end{aligned}$$

using that the multiplication maps of X are isometric and associative. Since $\underline{n}, \underline{m} \perp F$, we have that $\underline{n} + \underline{m} \perp F$. Hence we have that

$$\langle \xi_{\underline{m}}, b\eta_{\underline{m}}\rangle \in \langle X_{\underline{n}+\underline{m}}, aX_{\underline{n}+\underline{m}}\rangle \subseteq \mathcal{J}_F,$$

since $a \in \mathcal{I}_F$. Hence $b \in \mathcal{I}_F$, finishing the proof. □

To avoid ambiguity, given two strong compactly aligned product systems X and Y, we will denote the ideals $\mathcal{J}_F$ for X and Y by $\mathcal{J}_F(X)$ and $\mathcal{J}_F(Y)$, respectively. We will use the same convention for the ideals $\mathcal{I}_F$. The ideals $\mathcal{J}_F$ and $\mathcal{I}_F$ are preserved under unitary equivalence.

Proposition 2.5.11. *Let X and Y be strong compactly aligned product systems over $\mathbb{Z}_+^d$ with coefficients in C*-algebras A and B, respectively. If X and Y are unitarily equivalent by a collection $\{W_{\underline{n}} \colon X_{\underline{n}} \to Y_{\underline{n}}\}_{\underline{n}\in\mathbb{Z}_+^d}$, then*

$$W_{\underline{0}}(\mathcal{J}_F(X)) = \mathcal{J}_F(Y) \quad \text{and} \quad W_{\underline{0}}(\mathcal{I}_F(X)) = \mathcal{I}_F(Y) \text{ for all } F \subseteq [d].$$

Proof. Both claims hold trivially when $F = \emptyset$, so fix $\emptyset \neq F \subseteq [d]$. Take $a \in \mathcal{J}_F(X)$, so that

$$a \in (\bigcap_{i \in F} \ker \phi_{X_{\underline{i}}})^\perp \cap (\bigcap_{i \in [d]} \phi_{X_{\underline{i}}}^{-1}(\mathcal{K}(X_{\underline{i}})))$$

by definition. Fixing $i \in [d]$, we have that

$$\phi_{Y_{\underline{i}}}(W_{\underline{0}}(a)) = W_{\underline{i}} \phi_{X_{\underline{i}}}(a) W_{\underline{i}}^{-1} \in W_{\underline{i}} \mathcal{K}(X_{\underline{i}}) W_{\underline{i}}^{-1} = \mathcal{K}(Y_{\underline{i}}),$$

using (2.10) in the first equality and (2.7) in the final equality. It follows that

$$W_{\underline{0}}(a) \in \bigcap_{i \in [d]} \phi_{Y_{\underline{i}}}^{-1}(\mathcal{K}(Y_{\underline{i}})).$$

Now take $b \in \bigcap_{i \in F} \ker \phi_{Y_{\underline{i}}}$. We must show that $W_{\underline{0}}(a)b = 0$. To this end, fix $a' \in A$ such that $b = W_{\underline{0}}(a')$. Fixing $i \in F$, we have that

$$\phi_{X_{\underline{i}}}(a') = W_{\underline{i}}^{-1} \phi_{Y_{\underline{i}}}(b) W_{\underline{i}} = 0,$$

using (2.10) in the first equality and the fact that $\phi_{Y_{\underline{i}}}(b) = 0$ in the second. Hence $a' \in \bigcap_{i \in F} \ker \phi_{X_{\underline{i}}}$. Consequently, we obtain that

$$W_{\underline{0}}(a)b = W_{\underline{0}}(a) W_{\underline{0}}(a') = W_{\underline{0}}(aa') = 0,$$

using that $a \in (\bigcap_{i \in F} \ker \phi_{X_{\underline{i}}})^\perp$ in the final equality. In total, we have that

$$W_{\underline{0}}(a) \in (\bigcap_{i \in F} \ker \phi_{Y_{\underline{i}}})^\perp \cap (\bigcap_{i \in [d]} \phi_{Y_{\underline{i}}}^{-1}(\mathcal{K}(Y_{\underline{i}}))) = \mathcal{J}_F(Y).$$

Hence $W_{\underline{0}}(\mathcal{J}_F(X)) \subseteq \mathcal{J}_F(Y)$. The reverse inclusion holds by duality. Hence we conclude that

$$W_{\underline{0}}(\mathcal{J}_F(X)) = \mathcal{J}_F(Y) \text{ for all } F \subseteq [d],$$

as required.

For the second claim, fix $\emptyset \neq F \subseteq [d]$. Take $a \in \mathcal{I}_F(X)$, so that

$$a \in \{a' \in A \mid \langle X_{\underline{n}}, \phi_{X_{\underline{n}}}(a') X_{\underline{n}} \rangle \subseteq \mathcal{J}_F(X) \text{ for all } \underline{n} \perp F\}$$

by definition. Fixing $\underline{n} \perp F$, we obtain that

$$\begin{aligned} \langle Y_{\underline{n}}, \phi_{Y_{\underline{n}}}(W_{\underline{0}}(a)) Y_{\underline{n}} \rangle &= \langle Y_{\underline{n}}, W_{\underline{n}} \phi_{X_{\underline{n}}}(a) W_{\underline{n}}^{-1} Y_{\underline{n}} \rangle = \langle W_{\underline{n}} W_{\underline{n}}^{-1} Y_{\underline{n}}, W_{\underline{n}} \phi_{X_{\underline{n}}}(a) W_{\underline{n}}^{-1} Y_{\underline{n}} \rangle \\ &= W_{\underline{0}}(\langle X_{\underline{n}}, \phi_{X_{\underline{n}}}(a) X_{\underline{n}} \rangle) \subseteq W_{\underline{0}}(\mathcal{J}_F(X)) = \mathcal{J}_F(Y), \end{aligned}$$

using (2.10) in the first equality, item (ii) of Definition 2.3.1 together with surjectivity of $W_{\underline{n}}^{-1}$ in the third equality, and the first assertion of the proposition in the final equality.

Hence $W_{\underline{0}}(a) \in \mathcal{I}_F(Y)$ and thus $W_{\underline{0}}(\mathcal{I}_F(X)) \subseteq \mathcal{I}_F(Y)$. The reverse inclusion is obtained by duality. Hence we conclude that

$$W_{\underline{0}}(\mathcal{I}_F(X)) = \mathcal{I}_F(Y) \text{ for all } F \subseteq [d],$$

finishing the proof. □

The ideals $\mathcal{I}_F$ emerge naturally when solving polynomial equations, originating in [15] in the case of C*-dynamical systems. In order to make this precise, we require the following notation. Following the conventions of [17, Section 3], we introduce an approximate unit $(k_{\underline{i},\lambda})_{\lambda\in\Lambda}$ of $\mathcal{K}(X_{\underline{i}})$ for each generator $\underline{i}$ of $\mathbb{Z}_+^d$. Without loss of generality, we may assume that these approximate units are indexed by the same directed set Λ, by replacing with their product.

Lemma 2.5.12. *Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A. Let $(k_{\underline{i},\lambda})_{\lambda\in\Lambda}$ be an approximate unit of $\mathcal{K}(X_{\underline{i}})$ for all $i \in [d]$. Then, fixing $\emptyset \neq F \subseteq [d]$ and $i \in F$, we have that the net $(\iota_{\underline{i}}^{\underline{1}_F}(k_{\underline{i},\lambda}))_{\lambda\in\Lambda}$ converges strictly to $\mathrm{id}_{X_{\underline{1}_F}}$.*

Proof. Since $\iota_{\underline{i}}^{\underline{1}_F}(k_{\underline{i},\lambda})$ and $\mathrm{id}_{X_{\underline{1}_F}}$ are selfadjoint for all $\lambda \in \Lambda$, it suffices to show that

$$\|\cdot\|\text{-}\lim_\lambda \iota_{\underline{i}}^{\underline{1}_F}(k_{\underline{i},\lambda})\xi_{\underline{1}_F} = \xi_{\underline{1}_F} \text{ for all } \xi_{\underline{1}_F} \in X_{\underline{1}_F}. \tag{2.14}$$

Without loss of generality, we may assume that $\xi_{\underline{1}_F} = \xi_{\underline{i}}\xi_{\underline{1}_F-\underline{i}}$ for some $\xi_{\underline{i}} \in X_{\underline{i}}$ and some $\xi_{\underline{1}_F-\underline{i}} \in X_{\underline{1}_F-\underline{i}}$, recalling that $X_{\underline{i}} \otimes_A X_{\underline{1}_F-\underline{i}} \cong X_{\underline{1}_F}$ via the multiplication map $u_{\underline{i},\underline{1}_F-\underline{i}}$. Indeed, once (2.14) has been shown for elements of this form, it is routine to check that the equality extends to finite linear combinations and their norm-limits, using an $\varepsilon/3$ argument for the latter. We have that

$$\begin{aligned}\|\iota_{\underline{i}}^{\underline{1}_F}(k_{\underline{i},\lambda})\xi_{\underline{1}_F} - \xi_{\underline{1}_F}\| &= \|\iota_{\underline{i}}^{\underline{1}_F}(k_{\underline{i},\lambda})(\xi_{\underline{i}}\xi_{\underline{1}_F-\underline{i}}) - \xi_{\underline{i}}\xi_{\underline{1}_F-\underline{i}}\| = \|(k_{\underline{i},\lambda}\xi_{\underline{i}})\xi_{\underline{1}_F-\underline{i}} - \xi_{\underline{i}}\xi_{\underline{1}_F-\underline{i}}\| \\ &= \|(k_{\underline{i},\lambda}\xi_{\underline{i}} - \xi_{\underline{i}})\xi_{\underline{1}_F-\underline{i}}\| = \|(k_{\underline{i},\lambda}\xi_{\underline{i}} - \xi_{\underline{i}}) \otimes \xi_{\underline{1}_F-\underline{i}}\| \\ &\leq \|k_{\underline{i},\lambda}\xi_{\underline{i}} - \xi_{\underline{i}}\| \cdot \|\xi_{\underline{1}_F-\underline{i}}\|.\end{aligned}$$

It follows that $\|\cdot\|\text{-}\lim_\lambda \iota_{\underline{i}}^{\underline{1}_F}(k_{\underline{i},\lambda})\xi_{\underline{1}_F} = \xi_{\underline{1}_F}$ by Lemma 2.2.1, finishing the proof. □

Proposition 2.5.13. *[17, Proposition 2.4] Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $(k_{\underline{i},\lambda})_{\lambda\in\Lambda}$ be an approximate unit of $\mathcal{K}(X_{\underline{i}})$ for all $i \in [d]$. Fix $\emptyset \neq F \subseteq [d]$ and $\underline{0} \neq \underline{n} \in \mathbb{Z}_+^d$, and set $\underline{m} = \underline{n} \vee \underline{1}_F$. Then the net $(e_{F,\lambda})_{\lambda\in\Lambda}$ defined by*

$$e_{F,\lambda} := \prod\{\iota_{\underline{i}}^{\underline{1}_F}(k_{\underline{i},\lambda}) \mid i \in F\} \text{ for all } \lambda \in \Lambda$$

is contained in $\mathcal{K}(X_{\underline{1}_F})$, and we have that

$$\|\cdot\|\text{-}\lim_\lambda \iota_{\underline{1}_F}^{\underline{m}}(e_{F,\lambda})\iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}}) = \iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}}) \text{ for all } k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}). \tag{2.15}$$

Moreover, it follows that $\iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}}) \in \mathcal{K}(X_{\underline{m}})$ *for all* $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$.

Proof. We begin with some preliminary remarks before proceeding to the proof. Firstly, we fix the usual order for the product defining each $e_{F,\lambda}$, though the ensuing arguments can be tweaked to account for any order. Fixing $i \in F$ and $\lambda \in \Lambda$, note that $\iota_{\underline{i}}^{\underline{1}_F}(k_{\underline{i},\lambda}) \in \mathcal{K}(X_{\underline{1}_F})$. This follows by an inductive argument using strong compact alignment and the fact that

$$\iota_{\underline{n}+\underline{m}}^{\underline{n}+\underline{m}+\underline{r}}\iota_{\underline{n}}^{\underline{n}+\underline{m}} = \iota_{\underline{n}}^{\underline{n}+\underline{m}+\underline{r}} \text{ for all } \underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\} \text{ and } \underline{m}, \underline{r} \in \mathbb{Z}_+^d.$$

It then follows that $(e_{F,\lambda})_{\lambda\in\Lambda} \subseteq \mathcal{K}(X_{\underline{1}_F})$. An application of Lemma 2.5.12, together with the fact that multiplication in $\mathcal{L}(X_{\underline{1}_F})$ is jointly strongly continuous on norm-bounded sets, gives that $(e_{F,\lambda})_{\lambda\in\Lambda}$ converges strictly to $\mathrm{id}_{X_{\underline{1}_F}}$.

Now fix $\underline{n} \in \mathbb{Z}_+^d$ such that $\operatorname{supp}\underline{n} \supseteq F$. We claim that

$$\|\cdot\|\text{-}\lim_{\lambda} \iota_{\underline{1}_F}^{\underline{n}}(e_{F,\lambda})k_{\underline{n}} = k_{\underline{n}} \text{ for all } k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}). \tag{2.16}$$

To see this, we may assume that $k_{\underline{n}} = \Theta^{X_{\underline{n}}}_{\xi_{\underline{n}},\eta_{\underline{n}}}$ for some $\xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\underline{n}}$ without loss of generality. Indeed, once (2.16) has been shown for elements of this form, it is routine to check that the equality extends to finite linear combinations and their norm-limits, using an $\varepsilon/3$ argument for the latter. Since $\operatorname{supp}\underline{n} \supseteq F$, we have that $\underline{n} = \underline{1}_F + \underline{m}$ for some $\underline{m} \in \mathbb{Z}_+^d$. Consequently, we may also assume that $\xi_{\underline{n}} = \xi_{\underline{1}_F}\xi_{\underline{m}}$ for some $\xi_{\underline{1}_F} \in X_{\underline{1}_F}$ and $\xi_{\underline{m}} \in X_{\underline{m}}$, recalling that $X_{\underline{1}_F} \otimes_A X_{\underline{m}} \cong X_{\underline{n}}$ via the multiplication map $u_{\underline{1}_F,\underline{m}}$. We have that

$$\begin{aligned}\|\iota_{\underline{1}_F}^{\underline{n}}(e_{F,\lambda})\Theta^{X_{\underline{n}}}_{\xi_{\underline{n}},\eta_{\underline{n}}} - \Theta^{X_{\underline{n}}}_{\xi_{\underline{n}},\eta_{\underline{n}}}\| &= \|\Theta^{X_{\underline{n}}}_{\iota_{\underline{1}_F}^{\underline{n}}(e_{F,\lambda})(\xi_{\underline{1}_F}\xi_{\underline{m}}),\eta_{\underline{n}}} - \Theta^{X_{\underline{n}}}_{\xi_{\underline{1}_F}\xi_{\underline{m}},\eta_{\underline{n}}}\| = \|\Theta^{X_{\underline{n}}}_{(e_{F,\lambda}\xi_{\underline{1}_F})\xi_{\underline{m}}-\xi_{\underline{1}_F}\xi_{\underline{m}},\eta_{\underline{n}}}\| \\ &\leq \|(e_{F,\lambda}\xi_{\underline{1}_F} - \xi_{\underline{1}_F})\xi_{\underline{m}}\| \cdot \|\eta_{\underline{n}}\| \leq \|e_{F,\lambda}\xi_{\underline{1}_F} - \xi_{\underline{1}_F}\| \cdot \|\xi_{\underline{m}}\| \cdot \|\eta_{\underline{n}}\|.\end{aligned}$$

It follows that $\|\cdot\|\text{-}\lim_\lambda \iota_{\underline{1}_F}^{\underline{n}}(e_{F,\lambda})k_{\underline{n}} = k_{\underline{n}}$ since $(e_{F,\lambda})_{\lambda\in\Lambda}$ converges strictly to $\mathrm{id}_{X_{\underline{1}_F}}$, completing the proof of the claim.

We now proceed to the proof of the proposition, and use the nomenclature of the statement. We have already argued that $(e_{F,\lambda})_{\lambda\in\Lambda} \subseteq \mathcal{K}(X_{\underline{1}_F})$, so we move on to prove (2.15). By relabeling if necessary, we may assume that $F = [q]$ for some $q \in [d]$. For each $\lambda \in \Lambda$, we have that

$$\iota_{\underline{1}_F}^{\underline{m}}(e_{F,\lambda}) = \iota_{\underline{1}_F}^{\underline{m}}(\iota_{\underline{1}}^{\underline{1}_F}(k_{\underline{1},\lambda}) \dots \iota_{\underline{q}}^{\underline{1}_F}(k_{\underline{q},\lambda})) = \iota_{\underline{1}}^{\underline{m}}(k_{\underline{1},\lambda}) \dots \iota_{\underline{q}}^{\underline{m}}(k_{\underline{q},\lambda}).$$

Fix $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ and assume that $q \in \operatorname{supp}\underline{n}$. An application of (2.16) gives that

$$\|\cdot\|\text{-}\lim_{\lambda} \iota_{\underline{q}}^{\underline{n}}(k_{\underline{q},\lambda})k_{\underline{n}} = k_{\underline{n}}.$$

Applying $\iota_{\underline{n}}^{\underline{m}}$, we obtain that

$$\|\cdot\|\text{-}\lim_{\lambda} \iota_{\underline{q}}^{\underline{m}}(k_{\underline{q},\lambda})\iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}}) = \iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}}).$$

Now assume that $q \notin \operatorname{supp} \underline{n}$. Strong compact alignment then gives that $\iota_{\underline{n}}^{\underline{n}+\underline{q}}(k_{\underline{n}}) \in \mathcal{K}(X_{\underline{n}+\underline{q}})$. Another application of (2.16) yields that

$$\|\cdot\| \text{-} \lim_{\lambda} \iota_{\underline{q}}^{\underline{n}+\underline{q}}(k_{\underline{q},\lambda}) \iota_{\underline{n}}^{\underline{n}+\underline{q}}(k_{\underline{n}}) = \iota_{\underline{n}}^{\underline{n}+\underline{q}}(k_{\underline{n}}).$$

Applying $\iota_{\underline{n}+\underline{q}}^{\underline{m}}$, we obtain that

$$\|\cdot\| \text{-} \lim_{\lambda} \iota_{\underline{q}}^{\underline{m}}(k_{\underline{q},\lambda}) \iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}}) = \iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}}).$$

Noting that we may replace q by any element of F, we deduce that

$$\|\cdot\| \text{-} \lim_{\lambda} \iota_{\underline{1}_F}^{\underline{m}}(e_{F,\lambda}) \iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}}) = \iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}})$$

via an inductive argument, as required.

For the final claim, note that if $\operatorname{supp} \underline{n} \supseteq F$ then there is nothing to show (as then $\underline{m} = \underline{n}$). Assume that $\operatorname{supp} \underline{n} \not\supseteq F$, so that there is at least one entry of $\underline{n}$ which is 0 and labelled by F. Observe that $\underline{m}$ coincides with $\underline{n}$ everywhere except at those entries of $\underline{n}$ which are 0 and indexed by F. Without loss of generality, assume that these indices are $1, \dots, r$, where $1 \leq r \leq |F|$ (by relabeling if necessary). Setting $\underline{m} = (m_1, \dots, m_d)$, we then have that $m_j = 1$ for all $j \in [r]$. By strong compact alignment, we have that $\iota_{\underline{n}}^{\underline{n}+\underline{1}}(k_{\underline{n}}) \in \mathcal{K}(X_{\underline{n}+\underline{1}})$. Another application of strong compact alignment then gives that

$$\iota_{\underline{n}+\underline{1}}^{\underline{n}+\underline{1}+\underline{2}}(\iota_{\underline{n}}^{\underline{n}+\underline{1}}(k_{\underline{n}})) = \iota_{\underline{n}}^{\underline{n}+\underline{1}+\underline{2}}(k_{\underline{n}}) \in \mathcal{K}(X_{\underline{n}+\underline{1}+\underline{2}}).$$

By iterating this argument until every element of $[r]$ has been exhausted, we obtain that

$$\iota_{\underline{n}}^{\underline{n}+\underline{1}+\dots+\underline{r}}(k_{\underline{n}}) = \iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}}) \in \mathcal{K}(X_{\underline{m}}),$$

finishing the proof. □

We emphasise that (2.15) holds independently of the order of the product defining $e_{F,\lambda}$. Let (π, t) be a Nica-covariant representation of X. Fixing $i \in [d]$ and an approximate unit $(k_{\underline{i},\lambda})_{\lambda \in \Lambda}$ of $\mathcal{K}(X_{\underline{i}})$, we define

$$p_{\underline{i},\lambda} := \psi_{\underline{i}}(k_{\underline{i},\lambda}) \text{ for all } \lambda \in \Lambda, \text{ and } p_{\underline{i}} := \text{w*-}\lim_{\lambda} p_{\underline{i},\lambda}; \tag{2.17}$$

i.e., $p_{\underline{i}}$ is the projection on the space $[\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))H]$ for the Hilbert space H on which (π, t) acts. In turn, we set

$$q_{\emptyset} := I, q_{\underline{i}} := I - p_{\underline{i}}, \text{ and } q_F := \prod_{i \in F}(I - p_{\underline{i}}) \text{ for all } \emptyset \neq F \subseteq [d]. \tag{2.18}$$

Remark 2.5.14. Note that the projections $p_{\underline{i}}$ can be defined for a (just) compactly aligned product system X. When X is in addition strong compactly aligned, the projections $p_{\underline{i}}$

commute, so there is no ambiguity regarding the order of the product defining each q_F. In particular, we claim that $p_{\underline{i}}p_{\underline{j}}$ is the projection on the space $[\psi_{\underline{i}+\underline{j}}(\mathcal{K}(X_{\underline{i}+\underline{j}}))H]$ whenever $i \neq j$; due to symmetry, we deduce that the same holds for $p_{\underline{j}}p_{\underline{i}}$, and thus $p_{\underline{i}}p_{\underline{j}} = p_{\underline{j}}p_{\underline{i}}$.

Indeed, for $i, j \in [d]$ with $i \neq j$, consider the approximate units $(k_{\underline{i},\lambda})_{\lambda\in\Lambda}$ and $(k_{\underline{j},\lambda})_{\lambda\in\Lambda}$ of $\mathcal{K}(X_{\underline{i}})$ and $\mathcal{K}(X_{\underline{j}})$, respectively. We see that the nets $(\iota_{\underline{i}}^{\underline{i}+\underline{j}}(k_{\underline{i},\lambda}))_{\lambda\in\Lambda}$ and $(\iota_{\underline{j}}^{\underline{i}+\underline{j}}(k_{\underline{j},\lambda}))_{\lambda\in\Lambda}$ are contained in $\mathcal{K}(X_{\underline{i}+\underline{j}})$ due to strong compact alignment. In particular, we claim that they provide approximate units for $\mathcal{K}(X_{\underline{i}+\underline{j}})$. Indeed, we have that $\|\iota_{\underline{i}}^{\underline{i}+\underline{j}}(k_{\underline{i},\lambda})\| \leq \|k_{\underline{i},\lambda}\| \leq 1$ for all $\lambda \in \Lambda$, and for $\xi_{\underline{i}}, \eta_{\underline{i}} \in X_{\underline{i}}$ and $\xi_{\underline{j}}, \eta_{\underline{j}} \in X_{\underline{j}}$, we have that

$$\begin{aligned}\|\iota_{\underline{i}}^{\underline{i}+\underline{j}}(k_{\underline{i},\lambda})\Theta^{X_{\underline{i}+\underline{j}}}_{\xi_{\underline{i}}\xi_{\underline{j}},\eta_{\underline{i}}\eta_{\underline{j}}} - \Theta^{X_{\underline{i}+\underline{j}}}_{\xi_{\underline{i}}\xi_{\underline{j}},\eta_{\underline{i}}\eta_{\underline{j}}}\| &= \|\Theta^{X_{\underline{i}}\otimes_A X_{\underline{j}}}_{(k_{\underline{i},\lambda}\xi_{\underline{i}}-\xi_{\underline{i}})\otimes\xi_{\underline{j}},\eta_{\underline{i}}\otimes\eta_{\underline{j}}}\| \\ &\leq \|k_{\underline{i},\lambda}\xi_{\underline{i}} - \xi_{\underline{i}}\| \cdot \|\xi_{\underline{j}}\| \cdot \|\eta_{\underline{i}}\| \cdot \|\eta_{\underline{j}}\| \xrightarrow{\lambda} 0.\end{aligned}$$

Taking finite linear combinations of rank-one operators and their norm-limits establishes the claim for $\underline{i}$, and the case of $\underline{j}$ is dealt with by symmetry. Therefore, due to Nica-covariance, we have that

$$\|\cdot\|\text{-}\lim_{\lambda} \psi_{\underline{i}}(k_{\underline{i},\lambda})\psi_{\underline{i}+\underline{j}}(k_{\underline{i}+\underline{j}}) = \|\cdot\|\text{-}\lim_{\lambda} \psi_{\underline{i}+\underline{j}}(\iota_{\underline{i}}^{\underline{i}+\underline{j}}(k_{\underline{i},\lambda})k_{\underline{i}+\underline{j}}) = \psi_{\underline{i}+\underline{j}}(k_{\underline{i}+\underline{j}})$$

for all $k_{\underline{i}+\underline{j}} \in \mathcal{K}(X_{\underline{i}+\underline{j}})$, and likewise for $\underline{j}$. Thus, for every $h \in H$ and $k_{\underline{i}+\underline{j}} \in \mathcal{K}(X_{\underline{i}+\underline{j}})$, we deduce that

$$\begin{aligned}p_{\underline{i}}\psi_{\underline{i}+\underline{j}}(k_{\underline{i}+\underline{j}})h &= (\text{w*-}\lim_{\lambda} \psi_{\underline{i}}(k_{\underline{i},\lambda}))\psi_{\underline{i}+\underline{j}}(k_{\underline{i}+\underline{j}})h \\ &= \big[\,\|\cdot\|\text{-}\lim_{\lambda} \psi_{\underline{i}}(k_{\underline{i},\lambda})\psi_{\underline{i}+\underline{j}}(k_{\underline{i}+\underline{j}})\big]h = \psi_{\underline{i}+\underline{j}}(k_{\underline{i}+\underline{j}})h.\end{aligned}$$

Likewise for $\underline{j}$, we have that $p_{\underline{j}}\psi_{\underline{i}+\underline{j}}(k_{\underline{i}+\underline{j}})h = \psi_{\underline{i}+\underline{j}}(k_{\underline{i}+\underline{j}})h$, and therefore

$$p_{\underline{i}}p_{\underline{j}}h = h \text{ for all } h \in [\psi_{\underline{i}+\underline{j}}(\mathcal{K}(X_{\underline{i}+\underline{j}}))H].$$

On the other hand, for $h \perp [\psi_{\underline{i}+\underline{j}}(\mathcal{K}(X_{\underline{i}+\underline{j}}))H]$ and $h' \in H$, we have that

$$\left\langle p_{\underline{i}}p_{\underline{j}}h, h'\right\rangle = \left\langle h, p_{\underline{j}}p_{\underline{i}}h'\right\rangle = \lim_{\lambda}\lim_{\lambda'} \left\langle h, \psi_{\underline{j}}(k_{\underline{j},\lambda})\psi_{\underline{i}}(k_{\underline{i},\lambda'})h'\right\rangle = 0,$$

where we have used that

$$\psi_{\underline{j}}(k_{\underline{j},\lambda})\psi_{\underline{i}}(k_{\underline{i},\lambda'})h' = \psi_{\underline{i}+\underline{j}}(\iota_{\underline{j}}^{\underline{i}+\underline{j}}(k_{\underline{j},\lambda})\iota_{\underline{i}}^{\underline{i}+\underline{j}}(k_{\underline{i},\lambda'}))h' \in \psi_{\underline{i}+\underline{j}}(\mathcal{K}(X_{\underline{i}+\underline{j}}))H \text{ for all } \lambda, \lambda' \in \Lambda,$$

due to Nica-covariance. Hence we have that

$$p_{\underline{i}}p_{\underline{j}}h = 0 \text{ for all } h \perp [\psi_{\underline{i}+\underline{j}}(\mathcal{K}(X_{\underline{i}+\underline{j}}))H].$$

Consequently, the operator $p_{\underline{i}}p_{\underline{j}}$ coincides with the projection on the space $[\psi_{\underline{i}+\underline{j}}(\mathcal{K}(X_{\underline{i}+\underline{j}}))H]$, as required.

We gather some useful algebraic relations proved in [17].

Proposition 2.5.15. *[17, Proposition 4.4] Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let (π, t) be a Nica-covariant representation of X and fix $F \subseteq [d]$. Then for all $\underline{m} \in \mathbb{Z}_+^d$ and $\xi_{\underline{m}} \in X_{\underline{m}}$, we have that*

$$q_F t_{\underline{m}}(\xi_{\underline{m}}) = \begin{cases} t_{\underline{m}}(\xi_{\underline{m}}) q_F & \text{if } \underline{m} \perp F, \\ 0 & \text{if } \underline{m} \not\perp F, \end{cases}$$

so that in particular $q_F \in \pi(A)'$.

Proof. There is nothing to show when $F = \emptyset$, so fix $\emptyset \neq F \subseteq [d]$. We begin by proving the result when F is a singleton, i.e., $F = \{i\}$ for some $i \in [d]$. First suppose that $\underline{m} \perp F$. By definition we have that

$$q_F t_{\underline{m}}(\xi_{\underline{m}}) = (I - p_{\underline{i}}) t_{\underline{m}}(\xi_{\underline{m}}) = t_{\underline{m}}(\xi_{\underline{m}}) - p_{\underline{i}} t_{\underline{m}}(\xi_{\underline{m}}).$$

Proving that $q_F t_{\underline{m}}(\xi_{\underline{m}}) = t_{\underline{m}}(\xi_{\underline{m}}) q_F$ therefore amounts to showing that $p_{\underline{i}}$ commutes with $t_{\underline{m}}(\xi_{\underline{m}})$. In turn, it suffices to show that $[\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))H]$ is reducing for $t_{\underline{m}}(\xi_{\underline{m}})$, where H is the Hilbert space on which (π, t) acts. Thus we must show that

$$t_{\underline{m}}(\xi_{\underline{m}})[\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))H] \subseteq [\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))H] \quad \text{and} \quad t_{\underline{m}}(\xi_{\underline{m}})^*[\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))H] \subseteq [\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))H].$$

To this end, it will be useful to collect some auxiliary facts. Firstly, it is routine to check that

$$\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}})) = [t_{\underline{i}}(X_{\underline{i}}) t_{\underline{i}}(X_{\underline{i}})^*].$$

Secondly, recall that $X_{\underline{i}}[\langle X_{\underline{i}}, X_{\underline{i}} \rangle]$ is dense in $X_{\underline{i}}$, e.g., [40, p. 5]. It follows that

$$t_{\underline{i}}(X_{\underline{i}}) \subseteq [t_{\underline{i}}(X_{\underline{i}}) t_{\underline{i}}(X_{\underline{i}})^* t_{\underline{i}}(X_{\underline{i}})].$$

Finally, we have that

$$t_{\underline{i}+\underline{m}}(X_{\underline{i}+\underline{m}}) \subseteq [t_{\underline{i}}(X_{\underline{i}}) t_{\underline{m}}(X_{\underline{m}})],$$

since $X_{\underline{i}} \otimes_A X_{\underline{m}} \cong X_{\underline{i}+\underline{m}}$ via the multiplication map $u_{\underline{i},\underline{m}}$. Combining these observations, we obtain that

$$\begin{aligned} t_{\underline{m}}(X_{\underline{m}})\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}})) &\subseteq [t_{\underline{m}}(X_{\underline{m}}) t_{\underline{i}}(X_{\underline{i}}) t_{\underline{i}}(X_{\underline{i}})^*] \subseteq [t_{\underline{i}+\underline{m}}(X_{\underline{i}+\underline{m}}) t_{\underline{i}}(X_{\underline{i}})^*] \\ &\subseteq [t_{\underline{i}}(X_{\underline{i}}) t_{\underline{m}}(X_{\underline{m}}) t_{\underline{i}}(X_{\underline{i}})^*] \subseteq [t_{\underline{i}}(X_{\underline{i}}) t_{\underline{i}}(X_{\underline{i}})^* t_{\underline{i}}(X_{\underline{i}}) t_{\underline{m}}(X_{\underline{m}}) t_{\underline{i}}(X_{\underline{i}})^*] \\ &\subseteq [\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}})) t_{\underline{i}}(X_{\underline{i}}) t_{\underline{m}}(X_{\underline{m}}) t_{\underline{i}}(X_{\underline{i}})^*]. \end{aligned}$$

It follows that

$$t_{\underline{m}}(\xi_{\underline{m}})[\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))H] \subseteq [[\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}})) t_{\underline{i}}(X_{\underline{i}}) t_{\underline{m}}(X_{\underline{m}}) t_{\underline{i}}(X_{\underline{i}})^*]H] \subseteq [\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))H].$$

Note that the corresponding statement involving $t_{\underline{m}}(\xi_{\underline{m}})^*$ holds trivially when $\underline{m} = \underline{0}$, so assume that $\underline{m} \neq \underline{0}$. Then $X_{\underline{m}} \otimes_A X_{\underline{i}} \cong X_{\underline{m}+\underline{i}}$ via the multiplication map $u_{\underline{m},\underline{i}}$, and so

$$t_{\underline{m}+\underline{i}}(X_{\underline{m}+\underline{i}}) \subseteq [t_{\underline{m}}(X_{\underline{m}})t_{\underline{i}}(X_{\underline{i}})].$$

We also have that $\underline{m} \vee \underline{i} = \underline{m} + \underline{i}$ since $\underline{m} \perp \underline{i}$. Coupling these observations with Nica-covariance gives that

$$\begin{aligned} t_{\underline{m}}(X_{\underline{m}})^*\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}})) &\subseteq [t_{\underline{m}}(X_{\underline{m}})^*t_{\underline{i}}(X_{\underline{i}})t_{\underline{i}}(X_{\underline{i}})^*] \subseteq [t_{\underline{i}}(X_{\underline{i}})t_{\underline{m}}(X_{\underline{m}})^*t_{\underline{i}}(X_{\underline{i}})^*] \\ &\subseteq [t_{\underline{i}}(X_{\underline{i}})t_{\underline{m}+\underline{i}}(X_{\underline{m}+\underline{i}})^*] \subseteq [t_{\underline{i}}(X_{\underline{i}})t_{\underline{i}}(X_{\underline{i}})^*t_{\underline{m}}(X_{\underline{m}})^*] \\ &\subseteq [\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))t_{\underline{m}}(X_{\underline{m}})^*]. \end{aligned}$$

It follows that

$$t_{\underline{m}}(\xi_{\underline{m}})^*[\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))H] \subseteq [[\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))t_{\underline{m}}(X_{\underline{m}})^*]H] \subseteq [\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))H].$$

Hence $[\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))H]$ is reducing for $t_{\underline{m}}(\xi_{\underline{m}})$, as required.

Now suppose that $\underline{m} \not\perp F$. Recalling the definition of $q_F t_{\underline{m}}(\xi_{\underline{m}})$, it suffices to show that $p_{\underline{i}} t_{\underline{m}}(\xi_{\underline{m}}) = t_{\underline{m}}(\xi_{\underline{m}})$. To this end, by assumption we have that $i \in \operatorname{supp} \underline{m}$ and so $X_{\underline{i}} \otimes_A X_{\underline{m}-\underline{i}} \cong X_{\underline{m}}$ via the multiplication map $u_{\underline{i},\underline{m}-\underline{i}}$. Without loss of generality, assume that $\xi_{\underline{m}} = \xi_{\underline{i}}\xi_{\underline{m}-\underline{i}}$ for some $\xi_{\underline{i}} \in X_{\underline{i}}$ and $\xi_{\underline{m}-\underline{i}} \in X_{\underline{m}-\underline{i}}$. We obtain that

$$p_{\underline{i}} t_{\underline{m}}(\xi_{\underline{m}}) = p_{\underline{i}} t_{\underline{m}}(\xi_{\underline{i}}\xi_{\underline{m}-\underline{i}}) = p_{\underline{i}} t_{\underline{i}}(\xi_{\underline{i}}) t_{\underline{m}-\underline{i}}(\xi_{\underline{m}-\underline{i}}).$$

Next observe that

$$t_{\underline{i}}(X_{\underline{i}})H \subseteq [t_{\underline{i}}(X_{\underline{i}})t_{\underline{i}}(X_{\underline{i}})^*t_{\underline{i}}(X_{\underline{i}})]H \subseteq [\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))t_{\underline{i}}(X_{\underline{i}})]H \subseteq [\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))H],$$

from which it follows that $p_{\underline{i}} t_{\underline{i}}(\xi_{\underline{i}}) = t_{\underline{i}}(\xi_{\underline{i}})$. Thus $p_{\underline{i}} t_{\underline{m}}(\xi_{\underline{m}}) = t_{\underline{m}}(\xi_{\underline{m}})$, as required. In total, we have proved the result in the case where F is a singleton.

Finally, fix an arbitrary $\emptyset \neq F \subseteq [d]$ and suppose that $\underline{m} \perp F$. Then $\underline{m} \perp \underline{i}$ for all $i \in F$, and iterative applications of the singleton case yield that

$$q_F t_{\underline{m}}(\xi_{\underline{m}}) = t_{\underline{m}}(\xi_{\underline{m}}) q_F.$$

Now suppose that $\underline{m} \not\perp F$. Then $\underline{m} \not\perp \underline{i}$ for some $i \in F$. Thus we have that

$$q_F t_{\underline{m}}(\xi_{\underline{m}}) = q_{F \setminus \{i\}} q_{\underline{i}} t_{\underline{m}}(\xi_{\underline{m}}) = 0$$

by the singleton case, paired with the fact that the projections $p_{\underline{j}}$ commute for all $j \in [d]$. This finishes the proof. □

Proposition 2.5.16. *[17, Section 3] Let X be a strong compactly aligned product system* *with coefficients in a C^*-algebra A and let (π, t) be a Nica-covariant representation of X. Let $p_{\underline{i},\lambda}$ and $p_{\underline{i}}$ be the associated operators of (2.17), and fix $\emptyset \neq F \subseteq [d]$. Then*

$$\|\cdot\|\text{-}\lim_{\lambda} \psi_{\underline{n}}(k_{\underline{n}}) \prod_{i \in F} p_{\underline{i},\lambda} = \psi_{\underline{n}}(k_{\underline{n}}) \prod_{i \in F} p_{\underline{i}} \textit{ for all } \underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}, k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}).$$

If $a \in \bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in F\}$, then

$$\pi(a) \prod_{i \in D} p_{\underline{i}} = \|\cdot\|\text{-}\lim_{\lambda} \pi(a) \prod_{i \in D} p_{\underline{i},\lambda} = \psi_{\underline{1}_D}(\phi_{\underline{1}_D}(a)) \textit{ for all } \emptyset \neq D \subseteq F,$$

and so

$$\pi(a) q_F = \pi(a) + \sum\{(-1)^{|\underline{n}|} \psi_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \in \mathrm{C}^*(\pi, t).$$

Proof. We fix the usual order for the product of the $p_{\underline{i},\lambda}$, though the ensuing arguments can be tweaked to account for any order. Fixing $\emptyset \neq D \subseteq F$, it will be useful to note that the w*-limit of the net $(\prod_{i \in D} p_{\underline{i},\lambda})_{\lambda \in \Lambda}$ is the projection on the space $[\psi_{\underline{1}_D}(\mathcal{K}(X_{\underline{1}_D}))H]$ and coincides with $\prod_{i \in D} p_{\underline{i}}$, where H is the Hilbert space on which (π, t) acts. This can be seen by arguing as in Remark 2.5.14.

Fix $\underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$ and $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$. Recalling the definition of the net $(e_{F,\lambda})_{\lambda \in \Lambda}$ of Proposition 2.5.13, we use Nica-covariance to obtain that

$$\psi_{\underline{n}}(k_{\underline{n}}) \prod_{i \in F} p_{\underline{i},\lambda} = \psi_{\underline{n}}(k_{\underline{n}}) \psi_{\underline{1}_F}(e_{F,\lambda}) = \psi_{\underline{m}}(\iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}}) \iota_{\underline{1}_F}^{\underline{m}}(e_{F,\lambda})) \text{ for all } \lambda \in \Lambda,$$

where $\underline{m} = \underline{n} \vee \underline{1}_F$. Applying Proposition 2.5.13 and taking adjoints in (2.15), we deduce that

$$\|\cdot\|\text{-}\lim_{\lambda} \iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}}) \iota_{\underline{1}_F}^{\underline{m}}(e_{F,\lambda}) = \iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}}).$$

In turn, we obtain that

$$\|\cdot\|\text{-}\lim_{\lambda} \psi_{\underline{m}}(\iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}}) \iota_{\underline{1}_F}^{\underline{m}}(e_{F,\lambda})) = \psi_{\underline{m}}(\iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}})).$$

Consequently, the net $(\psi_{\underline{n}}(k_{\underline{n}}) \prod_{i \in F} p_{\underline{i},\lambda})_{\lambda \in \Lambda}$ converges to $\psi_{\underline{m}}(\iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}}))$ in norm. Since the norm topology is stronger than the w*-topology, the w*-limit of $(\psi_{\underline{n}}(k_{\underline{n}}) \prod_{i \in F} p_{\underline{i},\lambda})_{\lambda \in \Lambda}$ coincides with the norm-limit. By separate continuity of multiplication in the w*-topology, we have that

$$\text{w*-}\lim_{\lambda} \psi_{\underline{n}}(k_{\underline{n}}) \prod_{i \in F} p_{\underline{i},\lambda} = \psi_{\underline{n}}(k_{\underline{n}}) \text{w*-}\lim_{\lambda} \prod_{i \in F} p_{\underline{i},\lambda} = \psi_{\underline{n}}(k_{\underline{n}}) \prod_{i \in F} p_{\underline{i}}.$$

In total, we deduce that

$$\|\cdot\|\text{-}\lim_{\lambda} \psi_{\underline{n}}(k_{\underline{n}}) \prod_{i \in F} p_{\underline{i},\lambda} = \psi_{\underline{n}}(k_{\underline{n}}) \prod_{i \in F} p_{\underline{i}},$$

as required.

For the second claim, fix $a \in \bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in F\}$ and $\emptyset \neq D \subseteq F$. Then in particular $a \in \bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in D\}$ and so we have that

$$\pi(a)\prod_{i\in D} p_{\underline{i},\lambda} = \pi(a)\psi_{\underline{1}_D}(e_{D,\lambda}) = \psi_{\underline{1}_D}(\phi_{\underline{1}_D}(a)e_{D,\lambda}) \text{ for all } \lambda \in \Lambda,$$

using Nica-covariance in the first equality. We deduce that

$$\|\cdot\|\text{-}\lim_{\lambda} \psi_{\underline{1}_D}(\phi_{\underline{1}_D}(a)e_{D,\lambda}) = \psi_{\underline{1}_D}(\|\cdot\|\text{-}\lim_{\lambda} \phi_{\underline{1}_D}(a)e_{D,\lambda}) = \psi_{\underline{1}_D}(\phi_{\underline{1}_D}(a))$$

by (2.16), using that $\phi_{\underline{1}_D}(a) \in \mathcal{K}(X_{\underline{1}_D})$ by Proposition 2.5.9. Hence we obtain that

$$\begin{aligned}\pi(a)\prod_{i\in D} p_{\underline{i}} &= \text{w*-}\lim_{\lambda} \pi(a)\prod_{i\in D} p_{\underline{i},\lambda} = \text{w*-}\lim_{\lambda} \psi_{\underline{1}_D}(\phi_{\underline{1}_D}(a)e_{D,\lambda}) \\ &= \|\cdot\|\text{-}\lim_{\lambda} \psi_{\underline{1}_D}(\phi_{\underline{1}_D}(a)e_{D,\lambda}) = \|\cdot\|\text{-}\lim_{\lambda} \pi(a)\prod_{i\in D} p_{\underline{i},\lambda} = \psi_{\underline{1}_D}(\phi_{\underline{1}_D}(a)),\end{aligned}$$

proving the second claim.

Finally, we use the second claim to deduce that

$$\begin{aligned}\pi(a)q_F = \pi(a)\prod_{i\in F}(I - p_{\underline{i}}) &= \pi(a) + \sum\{(-1)^{|D|}\pi(a)\prod_{i\in D} p_{\underline{i}} \mid \emptyset \neq D \subseteq F\} \\ &= \pi(a) + \sum\{(-1)^{|D|}\psi_{\underline{1}_D}(\phi_{\underline{1}_D}(a)) \mid \emptyset \neq D \subseteq F\} \\ &= \pi(a) + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\},\end{aligned}$$

finishing the proof. □

Let (π, t) be a Nica-covariant representation of X. In the process of studying the kernel of $\pi \times t$, one needs to solve equations of the form

$$\pi(a) \in B^{(\pi,t)}_{[\underline{0},\underline{m}]} \text{ for } a \in A, \underline{m} \in \mathbb{Z}^d_+. \tag{2.19}$$

Due to the structure of the cores, an element $\pi(a)$ satisfies (2.19) if and only if there are $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ for all $\underline{0} \neq \underline{n} \leq \underline{m}$ such that

$$\pi(a) + \sum\{\psi_{\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{m}\} = 0. \tag{2.20}$$

This observation leads to the following useful results.

Proposition 2.5.17. *[17, Proposition 3.3] Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Suppose that (π, t) is a Nica-covariant representation of X and fix $a \in A$. If $\pi(a)$ satisfies*

$$\pi(a) + \sum\{\psi_{\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{m}\} = 0$$

for some $\underline{m} \in \mathbb{Z}_+^d$*, then it satisfies*

$$\pi(a)q_F = 0 \text{ for } F := \operatorname{supp} \underline{m}.$$

Proof. There is nothing to show when $\underline{m} = \underline{0}$, so fix $\underline{m} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$ and $\underline{0} \neq \underline{n} \leq \underline{m}$. Then there exists $i \in \operatorname{supp} \underline{n} \cap F$. Using the nomenclature of Propositions 2.5.13 and 2.5.16, we have that

$$\psi_{\underline{n}}(k_{\underline{n}})p_{\underline{i}} = \|\cdot\| \text{-} \lim_\lambda \psi_{\underline{n}}(k_{\underline{n}})p_{\underline{i},\lambda} = \|\cdot\| \text{-} \lim_\lambda \psi_{\underline{n}}(k_{\underline{n}})\psi_{\underline{i}}(e_{\{i\},\lambda}) = \|\cdot\| \text{-} \lim_\lambda \psi_{\underline{n}}(k_{\underline{n}}\iota_{\underline{i}}^{\underline{n}}(e_{\{i\},\lambda})),$$

using Proposition 2.5.16 in the first equality and the fact that $\underline{n} \vee \underline{i} = \underline{n}$ since $i \in \operatorname{supp} \underline{n}$ in the final equality. Applying Proposition 2.5.13 and taking adjoints in (2.16), we deduce that

$$\|\cdot\| \text{-} \lim_\lambda k_{\underline{n}}\iota_{\underline{i}}^{\underline{n}}(e_{\{i\},\lambda}) = k_{\underline{n}},$$

from which it follows that $\psi_{\underline{n}}(k_{\underline{n}})p_{\underline{i}} = \psi_{\underline{n}}(k_{\underline{n}})$. In turn, we have that $\psi_{\underline{n}}(k_{\underline{n}})(I - p_{\underline{i}}) = 0$. Recalling that $i \in F$, we obtain that

$$\psi_{\underline{n}}(k_{\underline{n}})q_F = \psi_{\underline{n}}(k_{\underline{n}})(I - p_{\underline{i}}) \prod_{j \in F \setminus \{i\}} (I - p_{\underline{j}}) = 0,$$

using that the projections $p_{\underline{j}}$ commute for all $j \in [d]$. Hence we obtain that

$$\pi(a)q_F = \pi(a)q_F + \sum \{\psi_{\underline{n}}(k_{\underline{n}})q_F \mid \underline{0} \neq \underline{n} \leq \underline{m}\} = (\pi(a) + \sum \{\psi_{\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{m}\})q_F = 0,$$

finishing the proof. □

Proposition 2.5.18. *[17, Proposition 3.2] Let* X *be a strong compactly aligned product system with coefficients in a C*-algebra* A*. Suppose that* (π, t) *is an injective Nica-covariant representation of* X *and fix* $a \in A$*. If* $\pi(a)$ *satisfies*

$$\pi(a) + \sum \{\psi_{\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{m}\} = 0$$

for some $\underline{m} \in \mathbb{Z}_+^d$*, then it satisfies*

$$\phi_{\underline{n}}(a) \in \mathcal{K}(X_{\underline{n}}) \text{ for all } \underline{0} \leq \underline{n} \leq \underline{1}_{[d]}.$$

Proof. There is nothing to show when $\underline{m} = \underline{0}$, so fix $\underline{m} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$. Note that $\phi_{\underline{0}}(a) \in \mathcal{K}(A)$ automatically. Thus it suffices to show that

$$\phi_{\underline{i}}(a) \in \mathcal{K}(X_{\underline{i}}) \text{ for all } i \in [d],$$

as then strong compact alignment together with the fact that $\iota_{\underline{n}}^{\underline{n}+\underline{r}}(\phi_{\underline{n}}(a)) = \phi_{\underline{n}+\underline{r}}(a)$ for all $\underline{n}, \underline{r} \in \mathbb{Z}_+^d$ yields the result. Accordingly, fix $i \in [d]$. Using the nomenclature of (2.17), we have that

$$\pi(a)p_{\underline{i},\lambda} = -\sum\{\psi_{\underline{n}}(k_{\underline{n}})p_{\underline{i},\lambda} \mid \underline{0} \neq \underline{n} \leq \underline{m}\} \text{ for all } \lambda \in \Lambda.$$

Note that the right hand side converges in norm to $-\sum\{\psi_{\underline{n}}(k_{\underline{n}})p_{\underline{i}} \mid \underline{0} \neq \underline{n} \leq \underline{m}\}$ by Proposition 2.5.16. In turn, we deduce that the w*-limit $\pi(a)p_{\underline{i}}$ of the net $(\pi(a)p_{\underline{i},\lambda})_{\lambda\in\Lambda}$ coincides with the norm-limit. Observe that $(\pi(a)p_{\underline{i},\lambda})_{\lambda\in\Lambda} \subseteq \psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))$ by definition. Since $\psi_{\underline{i}}$ has closed range, we deduce that $\pi(a)p_{\underline{i}} \in \psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))$.

Fix $k_{\underline{i}} \in \mathcal{K}(X_{\underline{i}})$ such that $\pi(a)p_{\underline{i}} = \psi_{\underline{i}}(k_{\underline{i}})$ and $\xi_{\underline{i}} \in X_{\underline{i}}$. We claim that $p_{\underline{i}}t_{\underline{i}}(\xi_{\underline{i}}) = t_{\underline{i}}(\xi_{\underline{i}})$. To see this, first recall that $(p_{\underline{i},\lambda})_{\lambda\in\Lambda} \equiv (\psi_{\underline{i}}(k_{\underline{i},\lambda}))_{\lambda\in\Lambda}$ for an approximate unit $(k_{\underline{i},\lambda})_{\lambda\in\Lambda}$ of $\mathcal{K}(X_{\underline{i}})$. Observe that

$$p_{\underline{i},\lambda}t_{\underline{i}}(\xi_{\underline{i}}) = \psi_{\underline{i}}(k_{\underline{i},\lambda})t_{\underline{i}}(\xi_{\underline{i}}) = t_{\underline{i}}(k_{\underline{i},\lambda}\xi_{\underline{i}}) \text{ for all } \lambda \in \Lambda.$$

Note that the right hand side converges in norm to $t_{\underline{i}}(\xi_{\underline{i}})$ by Lemma 2.2.1 and continuity of $t_{\underline{i}}$. Thus the w*-limit $p_{\underline{i}}t_{\underline{i}}(\xi_{\underline{i}})$ of the net $(p_{\underline{i},\lambda}t_{\underline{i}}(\xi_{\underline{i}}))_{\lambda\in\Lambda}$ coincides with the norm-limit $t_{\underline{i}}(\xi_{\underline{i}})$, as claimed. In turn, we obtain that

$$t_{\underline{i}}(\phi_{\underline{i}}(a)\xi_{\underline{i}}) = \pi(a)t_{\underline{i}}(\xi_{\underline{i}}) = \pi(a)p_{\underline{i}}t_{\underline{i}}(\xi_{\underline{i}}) = \psi_{\underline{i}}(k_{\underline{i}})t_{\underline{i}}(\xi_{\underline{i}}) = t_{\underline{i}}(k_{\underline{i}}\xi_{\underline{i}}).$$

Injectivity of (π, t) implies that $t_{\underline{i}}$ is isometric by Corollary 2.2.3, and so we deduce that

$$\phi_{\underline{i}}(a)\xi_{\underline{i}} = k_{\underline{i}}\xi_{\underline{i}} \text{ for all } \xi_{\underline{i}} \in X_{\underline{i}}.$$

Thus $\phi_{\underline{i}}(a) = k_{\underline{i}} \in \mathcal{K}(X_{\underline{i}})$, as required. □

Conversely, if $\pi(a)q_F = 0$ and $\phi_{\underline{n}}(a) \in \mathcal{K}(X_{\underline{n}})$ for some $a \in A, F \subseteq [d]$ and all $\underline{0} \leq \underline{n} \leq \underline{1}_{[d]}$, then Proposition 2.5.16 gives that

$$\pi(a) + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = \pi(a)q_F = 0, \tag{2.21}$$

and so $\pi(a) \in B_{(\underline{0},\underline{m}]}^{(\pi,t)}$ for any $\underline{m} \geq \underline{1}_F$. The following proposition justifies the usage of the family $\{\mathcal{I}_F\}_{F\subseteq[d]}$.

Proposition 2.5.19. *[17, Proposition 3.4] Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Suppose that (π, t) is an injective Nica-covariant representation of X and fix $a \in A$ and $\underline{m} \in \mathbb{Z}_+^d$. If $\pi(a) \in B_{(\underline{0},\underline{m}]}^{(\pi,t)}$, then $a \in \mathcal{I}_F$ for $F := \operatorname{supp}\underline{m}$.*

Proof. There is nothing to show when $\underline{m} = \underline{0}$, so fix $\underline{m} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$. We begin by showing that $a \in \mathcal{J}_F$. By Propositions 2.5.17 and 2.5.18 respectively, we have that $\pi(a)q_F = 0$ and that $\phi_{\underline{n}}(a) \in \mathcal{K}(X_{\underline{n}})$ for all $\underline{0} \leq \underline{n} \leq \underline{1}_{[d]}$. By the latter, proving that $a \in \mathcal{J}_F$ amounts to showing that $a \in (\bigcap_{i\in F} \ker \phi_{\underline{i}})^{\perp}$. To this end, fix $i \in [d]$ and $b \in \ker \phi_{\underline{i}}$. Using the nomenclature of (2.17), observe that

$$\pi(b)p_{\underline{i}} = \text{w*-}\lim_{\lambda} \pi(b)\psi_{\underline{i}}(k_{\underline{i},\lambda}) = \text{w*-}\lim_{\lambda} \psi_{\underline{i}}(\phi_{\underline{i}}(b)k_{\underline{i},\lambda}) = 0,$$

from which it follows that $\pi(b)(I - p_{\underline{i}}) = \pi(b)$. Thus if we now take $b \in \bigcap_{i\in F} \ker \phi_{\underline{i}}$, then we obtain that

$$\pi(ba) = \pi(ba)q_F = \pi(b)\pi(a)q_F = 0,$$

using that $\bigcap_{i\in F} \ker \phi_{\underline{i}}$ is an ideal and hence $ba \in \bigcap_{i\in F} \ker \phi_{\underline{i}}$. Injectivity of π then yields that $ba = 0$ and so $a \in (\bigcap_{i\in F} \ker \phi_{\underline{i}})^{\perp}$. In total, we have that $a \in \mathcal{J}_F$, as claimed.

It remains to show that $\langle X_{\underline{r}}, aX_{\underline{r}}\rangle \subseteq \mathcal{J}_F$ for all $\underline{r} \perp F$. Accordingly, fix $\underline{r} \perp F$. Without loss of generality, we may assume that $\underline{r} \neq \underline{0}$. Using (2.21), we have that

$$\pi(a)q_F = \pi(a) + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = 0.$$

Fixing $\xi_{\underline{r}}, \eta_{\underline{r}} \in X_{\underline{r}}$, we therefore obtain that

$$\pi(\langle\xi_{\underline{r}}, a\eta_{\underline{r}}\rangle) + \sum\{(-1)^{|\underline{n}|}t_{\underline{r}}(\xi_{\underline{r}})^*\psi_{\underline{n}}(\phi_{\underline{n}}(a))t_{\underline{r}}(\eta_{\underline{r}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = t_{\underline{r}}(\xi_{\underline{r}})^*\pi(a)q_F t_{\underline{r}}(\eta_{\underline{r}}) = 0.$$

Now fix $\underline{0} \neq \underline{n} \leq \underline{1}_F$ and note that $\underline{r} \perp \underline{n}$, so that $\underline{r} \vee \underline{n} = \underline{r} + \underline{n}$. Coupling this observation with Nica-covariance gives that

$$\begin{aligned} t_{\underline{r}}(X_{\underline{r}})^*\psi_{\underline{n}}(\mathcal{K}(X_{\underline{n}}))t_{\underline{r}}(X_{\underline{r}}) &\subseteq [t_{\underline{r}}(X_{\underline{r}})^*t_{\underline{n}}(X_{\underline{n}})t_{\underline{n}}(X_{\underline{n}})^*t_{\underline{r}}(X_{\underline{r}})] \\ &\subseteq [t_{\underline{n}}(X_{\underline{n}})t_{\underline{r}}(X_{\underline{r}})^*t_{\underline{n}}(X_{\underline{n}})^*t_{\underline{r}}(X_{\underline{r}})] \\ &\subseteq [t_{\underline{n}}(X_{\underline{n}})t_{\underline{n}+\underline{r}}(X_{\underline{n}+\underline{r}})^*t_{\underline{r}}(X_{\underline{r}})] \\ &\subseteq [t_{\underline{n}}(X_{\underline{n}})\pi(A)t_{\underline{n}}(X_{\underline{n}})^*] \subseteq \psi_{\underline{n}}(\mathcal{K}(X_{\underline{n}})). \end{aligned}$$

It follows that $\pi(\langle\xi_{\underline{r}}, a\eta_{\underline{r}}\rangle) \in B^{(\pi,t)}_{(\underline{0},\underline{1}_F]} \subseteq B^{(\pi,t)}_{(\underline{0},\underline{m}]}$. Replacing a by $\langle\xi_{\underline{r}}, a\eta_{\underline{r}}\rangle$ in the first part of the proof, we obtain that $\langle\xi_{\underline{r}}, a\eta_{\underline{r}}\rangle \in \mathcal{J}_F$. Hence we have that

$$\langle X_{\underline{r}}, aX_{\underline{r}}\rangle \subseteq \mathcal{J}_F \text{ for all } \underline{r} \perp F$$

and thus $a \in \mathcal{I}_F$, finishing the proof. □

We define the *ideal of the CNP-relations with respect to* (π, t) as follows:

$$\mathfrak{J}^{(\pi,t)}_{\mathcal{I}} := \langle \pi(\mathcal{I}_F)q_F \mid F \subseteq [d]\rangle \subseteq \mathrm{C}^*(\pi, t).$$

We will say that (π, t) is a *CNP-representation (of X)* if it satisfies

$$\pi(a)q_F = \pi(a) + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = 0 \text{ for all } a \in \mathcal{I}_F, F \subseteq [d].$$

It follows that $\mathfrak{J}^{(\pi,t)}_{\mathcal{I}} = \{0\}$ if and only if (π, t) is a CNP-representation. We write $\mathcal{NO}_X$

for the universal C*-algebra with respect to the CNP-representations, and refer to it as the *Cuntz-Nica-Pimsner algebra of* X, i.e.,

$$\mathcal{NO}_X \equiv \mathcal{NT}_X / \mathfrak{J}_{\mathcal{I}}^{(\overline{\pi}_X, \overline{t}_X)}.$$

We write $(\pi_X^{\mathcal{I}}, t_X^{\mathcal{I}})$ for the *universal CNP-representation (of X)*, i.e., $(\pi_X^{\mathcal{I}}, t_X^{\mathcal{I}}) = (Q \circ \overline{\pi}_X, Q \circ \overline{t}_X)$, where $Q \colon \mathcal{NT}_X \to \mathcal{NO}_X$ is the canonical quotient map. Since $\mathcal{NO}_X$ is an equivariant quotient of $\mathcal{NT}_X$, the representation $(\pi_X^{\mathcal{I}}, t_X^{\mathcal{I}})$ admits a gauge action.

In [17] it is shown that $\mathcal{NO}_X$ coincides with the Cuntz-Nica-Pimsner algebra of Sims and Yeend [56], and thus with the strong covariance algebra of Sehnem [53]. In particular, $(\pi_X^{\mathcal{I}}, t_X^{\mathcal{I}})$ is injective by [56, Theorem 4.1], since $(\mathbb{Z}^d, \mathbb{Z}_+^d)$ satisfies [56, (3.5)]. Moreover, $\mathcal{NO}_X$ is co-universal with respect to the injective Nica-covariant representations of X that admit a gauge action [56] (see Theorem A.2.2). The co-universal property of $\mathcal{NO}_X$ has been verified in several works [11, 17, 18, 54] in more general contexts. Moreover, unitarily equivalent strong compactly aligned product systems X and Y satisfy $\mathcal{NO}_X \cong \mathcal{NO}_Y$ canonically.

Proposition 2.5.20. *Let X and Y be unitarily equivalent strong compactly aligned product systems with coefficients in C*-algebras A and B, respectively. Then the bijection of Proposition 2.3.4 preserves the CNP-representations.*

Proof. We use the same notation as in Proposition 2.3.4. Let (π, t) be a CNP-representation of X on some $\mathcal{B}(H)$. For notational convenience, we set $(\widetilde{\pi}, \widetilde{t}) := \{(\pi \circ W_{\underline{0}}^{-1}, t_{\underline{n}} \circ W_{\underline{n}}^{-1})\}_{\underline{n} \in \mathbb{Z}_+^d}$. By duality, it suffices to show that $(\widetilde{\pi}, \widetilde{t})$ is a CNP-representation of Y. By Proposition 2.4.3, we have that $(\widetilde{\pi}, \widetilde{t})$ is a Nica-covariant representation of Y. It remains to check that $(\widetilde{\pi}, \widetilde{t})$ satisfies the CNP-covariance condition. Accordingly, fix $F \subseteq [d]$ and $b \in \mathcal{I}_F(Y)$. There is nothing to show when $F = \emptyset$, so we may assume that $F \neq \emptyset$ without loss of generality. We must show that

$$\widetilde{\pi}(b) + \sum \{(-1)^{|\underline{n}|} \widetilde{\psi}_{\underline{n}}(\phi_{Y_{\underline{n}}}(b)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = 0.$$

To this end, fix $\underline{0} \neq \underline{n} \leq \underline{1}_F$. We have that

$$\widetilde{\psi}_{\underline{n}}(\phi_{Y_{\underline{n}}}(b)) = \psi_{\underline{n}}(W_{\underline{n}}^{-1} \phi_{Y_{\underline{n}}}(b) W_{\underline{n}}) = \psi_{\underline{n}}(\phi_{X_{\underline{n}}}(W_{\underline{0}}^{-1}(b))),$$

using Remark 2.3.5 in the first equality and the dual of (2.10) in the final equality. Recall that $W_{\underline{0}}^{-1}(b) \in \mathcal{I}_F(X)$ by Proposition 2.5.11. Hence we obtain that

$$\begin{aligned} &\widetilde{\pi}(b) + \sum \{(-1)^{|\underline{n}|} \widetilde{\psi}_{\underline{n}}(\phi_{Y_{\underline{n}}}(b)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = \\ &\qquad = \pi(W_{\underline{0}}^{-1}(b)) + \sum \{(-1)^{|\underline{n}|} \psi_{\underline{n}}(\phi_{X_{\underline{n}}}(W_{\underline{0}}^{-1}(b))) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = 0, \end{aligned}$$

using that (π, t) is a CNP-representation of X in the final equality. In total, we have that $(\widetilde{\pi}, \widetilde{t})$ is a CNP-representation of Y, as required. □

We finish the section with a proposition concerning canonical $*$-epimorphisms arising from injective Nica-covariant representations. This trick was used implicitly in [17, 33].

Proposition 2.5.21. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $(\widetilde{\pi}, \widetilde{t})$ and (π, t) be Nica-covariant representations of X such that (π, t) is injective, and suppose there exists a canonical $*$-epimorphism $\Phi\colon \mathrm{C}^*(\widetilde{\pi}, \widetilde{t}) \to \mathrm{C}^*(\pi, t)$. Then the following are equivalent:*

(i) $\ker \Phi \cap B^{(\widetilde{\pi},\widetilde{t})}_{[\underline{0},\infty]} = \{0\}$*;*

(ii) $\ker \Phi \cap B^{(\widetilde{\pi},\widetilde{t})}_{[\underline{0},\underline{m}]} = \{0\}$ *for all* $\underline{m} \in \mathbb{Z}_+^d$*;*

(iii) $\ker \Phi \cap B^{(\widetilde{\pi},\widetilde{t})}_{[\underline{0},\underline{1}_{[d]}]} = \{0\}$*;*

(iv) $\ker \Phi \cap B^{(\widetilde{\pi},\widetilde{t})}_{[\underline{0},\underline{1}_F]} = \{0\}$ *for all* $F \subseteq [d]$*.*

Proof. First note that $(\widetilde{\pi}, \widetilde{t})$ is injective. Indeed, if $a \in \ker \widetilde{\pi}$ then $\pi(a) = \Phi(\widetilde{\pi}(a)) = 0$ and injectivity of π gives that $a = 0$. Note that $B^{(\widetilde{\pi},\widetilde{t})}_{[\underline{0},\infty]}$ is an inductive limit of cores. More specifically, we have that

$$B^{(\widetilde{\pi},\widetilde{t})}_{[\underline{0},\infty]} = \overline{\bigcup_{m=0}^{\infty} B^{(\widetilde{\pi},\widetilde{t})}_{[\underline{0},m\cdot\underline{1}_{[d]}]}}.$$

Let K denote the kernel of the restriction of Φ to $B^{(\widetilde{\pi},\widetilde{t})}_{[\underline{0},\infty]}$. By [14, Lemma III.4.1], we have that

$$K = \overline{\bigcup_{m=0}^{\infty} (K \cap B^{(\widetilde{\pi},\widetilde{t})}_{[\underline{0},m\cdot\underline{1}_{[d]}]})}.$$

Therefore Φ is injective on $B^{(\widetilde{\pi},\widetilde{t})}_{[\underline{0},\infty]}$ if and only if $K \cap B^{(\widetilde{\pi},\widetilde{t})}_{[\underline{0},m\cdot\underline{1}_{[d]}]} = \{0\}$ for all $m \in \mathbb{Z}_+$, which is in turn equivalent to having that $\ker \Phi \cap B^{(\widetilde{\pi},\widetilde{t})}_{[\underline{0},\underline{m}]} = \{0\}$ for all $\underline{m} \in \mathbb{Z}_+^d$. This follows from the observation that for each $\underline{m} = (m_1, \dots, m_d) \in \mathbb{Z}_+^d$, we have that $\underline{m} \leq m \cdot \underline{1}_{[d]}$ for $m := \max\{m_i \mid i \in [d]\}$. This shows that item (i) is equivalent to item (ii).

If item (ii) holds then it implies item (iii) by applying for $\underline{m} = \underline{1}_{[d]}$. If item (iii) holds then it implies item (iv) since $B^{(\widetilde{\pi},\widetilde{t})}_{[\underline{0},\underline{1}_F]} \subseteq B^{(\widetilde{\pi},\widetilde{t})}_{[\underline{0},\underline{1}_{[d]}]}$ for all $F \subseteq [d]$.

It now suffices to show that item (iv) implies item (ii), so fix $\underline{m} \in \mathbb{Z}_+^d$. Without loss of generality, assume that $\underline{m}$ has at least one entry greater than 1 (the assumption deals with the case where $\underline{m}$ has no entries greater than 1). To reach contradiction, assume that $\ker \Phi \cap B^{(\widetilde{\pi},\widetilde{t})}_{[\underline{0},\underline{m}]} \neq \{0\}$. Take $0 \neq f \in \ker \Phi \cap B^{(\widetilde{\pi},\widetilde{t})}_{[\underline{0},\underline{m}]}$, so that we may write

$$f = \widetilde{\pi}(a) + \sum\{\widetilde{\psi}_{\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{m}\},$$

for some $a \in A$ and $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ for all $\underline{0} \neq \underline{n} \leq \underline{m}$. Note that we can write $\widetilde{\pi}(a) = \widetilde{\psi}_{\underline{0}}(k_{\underline{0}})$ for $k_{\underline{0}} := \phi_{\underline{0}}(a)$. Without loss of generality, we may assume that f is written irreducibly, so that we may choose a minimal $\underline{0} \leq \underline{r} \leq \underline{m}$ such that $k_{\underline{r}} \neq 0$, and $\widetilde{\psi}_{\underline{r}}(k_{\underline{r}}) \notin B^{(\widetilde{\pi},\widetilde{t})}_{(\underline{r},\underline{m}]}$. The minimality of $\underline{r}$ means that if we have $\underline{0} \leq \underline{n} \leq \underline{m}$ such that $k_{\underline{n}} \neq 0$ and $\underline{n} \leq \underline{r}$, then $\underline{n} = \underline{r}$. If $\underline{r} = \underline{m}$, then $f = \widetilde{\psi}_{\underline{m}}(k_{\underline{m}})$ and $\Phi(f) = \psi_{\underline{m}}(k_{\underline{m}}) = 0$. Injectivity of (π, t) then implies that $k_{\underline{m}} = 0$ and hence $f = 0$, a contradiction. So without loss of generality assume that $\underline{r} < \underline{m}$. Fixing $\xi_{\underline{r}}, \eta_{\underline{r}} \in X_{\underline{r}}$, we have that

$$\widetilde{t}_{\underline{r}}(\xi_{\underline{r}})^* f \widetilde{t}_{\underline{r}}(\eta_{\underline{r}}) = \widetilde{\pi}(b) + \sum\{\widetilde{\psi}_{\underline{n}}(k'_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{m} - \underline{r}\},$$

where $k'_{\underline{n}}$ is a suitably defined element of $\mathcal{K}(X_{\underline{n}})$ for all $\underline{0} \neq \underline{n} \leq \underline{m} - \underline{r}$ and $b := \langle \xi_{\underline{r}}, k_{\underline{r}} \eta_{\underline{r}} \rangle$, due to the minimality of $\underline{r}$. Applying Φ, we obtain that

$$\pi(b) + \sum\{\psi_{\underline{n}}(k'_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{m} - \underline{r}\} = 0,$$

which in turn implies that $b \in \pi^{-1}(B^{(\pi,t)}_{(\underline{0}, \underline{m}-\underline{r}]})$. Let $F := \operatorname{supp}(\underline{m} - \underline{r})$, which is non-empty since $\underline{m} \neq \underline{r}$. An application of Proposition 2.5.17 then yields that $\pi(b) q_F = 0$. Note also that $\phi_{\underline{n}}(b) \in \mathcal{K}(X_{\underline{n}})$ for all $\underline{0} \leq \underline{n} \leq \underline{1}_{[d]}$ using Proposition 2.5.18, which applies since (π, t) is injective. Therefore we obtain that

$$\widetilde{\pi}(b) \widetilde{q}_F = \widetilde{\pi}(b) + \sum\{(-1)^{|\underline{n}|} \widetilde{\psi}_{\underline{n}}(\phi_{\underline{n}}(b)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \in B^{(\widetilde{\pi}, \widetilde{t})}_{[\underline{0}, \underline{1}_F]}.$$

It then follows that

$$\begin{aligned}
\Phi(\widetilde{\pi}(b)\widetilde{q}_F) &= \Phi(\widetilde{\pi}(b) + \sum\{(-1)^{|\underline{n}|} \widetilde{\psi}_{\underline{n}}(\phi_{\underline{n}}(b)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}) \\
&= \pi(b) + \sum\{(-1)^{|\underline{n}|} \psi_{\underline{n}}(\phi_{\underline{n}}(b)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = \pi(b) q_F = 0.
\end{aligned}$$

Hence $\widetilde{\pi}(b)\widetilde{q}_F \in \ker \Phi \cap B^{(\widetilde{\pi}, \widetilde{t})}_{[\underline{0}, \underline{1}_F]}$ and so $\widetilde{\pi}(b)\widetilde{q}_F = 0$ by assumption. So we have that

$$\widetilde{t}_{\underline{r}}(\xi_{\underline{r}})^* \widetilde{\psi}_{\underline{r}}(k_{\underline{r}}) \widetilde{t}_{\underline{r}}(\eta_{\underline{r}}) = \widetilde{\pi}(b) \in B^{(\widetilde{\pi}, \widetilde{t})}_{(\underline{0}, \underline{1}_F]}$$

for all $\xi_{\underline{r}}, \eta_{\underline{r}} \in X_{\underline{r}}$. Consequently, we have that

$$\widetilde{\psi}_{\underline{r}}(\mathcal{K}(X_{\underline{r}})) \widetilde{\psi}_{\underline{r}}(k_{\underline{r}}) \widetilde{\psi}_{\underline{r}}(\mathcal{K}(X_{\underline{r}})) \subseteq [\widetilde{t}_{\underline{r}}(X_{\underline{r}}) B^{(\widetilde{\pi}, \widetilde{t})}_{(\underline{0}, \underline{1}_F]} \widetilde{t}_{\underline{r}}(X_{\underline{r}})^*] \subseteq [\widetilde{t}_{\underline{r}}(X_{\underline{r}}) B^{(\widetilde{\pi}, \widetilde{t})}_{(\underline{0}, \underline{m}-\underline{r}]} \widetilde{t}_{\underline{r}}(X_{\underline{r}})^*] \subseteq B^{(\widetilde{\pi}, \widetilde{t})}_{(\underline{r}, \underline{m}]}.$$

By using an approximate unit of $\widetilde{\psi}_{\underline{r}}(\mathcal{K}(X_{\underline{r}}))$ and the fact that $B^{(\widetilde{\pi}, \widetilde{t})}_{(\underline{r}, \underline{m}]}$ is closed in $\mathrm{C}^*(\widetilde{\pi}, \widetilde{t})$, we deduce that $\widetilde{\psi}_{\underline{r}}(k_{\underline{r}}) \in B^{(\widetilde{\pi}, \widetilde{t})}_{(\underline{r}, \underline{m}]}$, which contradicts the irreducibility assumption. Therefore we obtain that $\ker \Phi \cap B^{(\widetilde{\pi}, \widetilde{t})}_{[\underline{0}, \underline{m}]} = \{0\}$ for all $\underline{m} \in \mathbb{Z}^d_+$, as required. □

2.6 The product system IXI

In this section we present a product system construction that will be useful in Chapter 5, and which extends the results of [36, Section 9].

Let X be a C*-correspondence over a C*-algebra A and let $I \subseteq A$ be an ideal. Consider

$IXI \subseteq XI$ and recall that XI is closed. We claim that IXI is a closed linear subspace of XI, i.e., $IXI = [IXI]$. To see this, consider the map $\phi_{XI}|_I \colon I \to \mathcal{B}(XI)$. Note that $\phi_{XI}|_I$ is a bounded algebra homomorphism. The Hewitt-Cohen Factorisation Theorem then gives that $\phi_{XI}|_I(I)(XI) = [\phi_{XI}|_I(I)(XI)]$. In turn, we obtain that

$$\phi_{XI}|_I(I)(XI) = \phi_{XI}(I)(XI) = \phi_X(I)(XI).$$

It follows that $IXI = [IXI]$, as claimed. It is clear that IXI is also closed under the right action of A and hence carries the structure of a right Hilbert A-module. Next, we define a left action ϕ_{IXI} of A on IXI by

$$\phi_{IXI} \colon A \to \mathcal{L}(IXI); \phi_{IXI}(a) = \phi_X(a)|_{IXI} \text{ for all } a \in A.$$

Hence IXI inherits the structure of a C*-correspondence over A from X. By restricting ϕ_{IXI} to I, we may view IXI as a C*-correspondence over I. When I is positively invariant for X, an application of Lemma 2.2.16 gives that the IXI construction recovers the Y_I construction of [36, Section 9].

As was the case with $\mathcal{K}(XI)$, we have a natural embedding of $\mathcal{K}(IXI)$ in $\mathcal{K}(X)$.

Lemma 2.6.1. *Let X be a C*-correspondence over a C*-algebra A and let $I \subseteq A$ be an ideal. Then there exists an embedding $\iota \colon \mathcal{K}(IXI) \to \mathcal{K}(X)$ such that*

$$\iota(\Theta^{IXI}_{\xi,\eta}) = \Theta^X_{\xi,\eta} \text{ for all } \xi, \eta \in IXI.$$

Thus $\mathcal{K}(IXI)$ is $*$-isomorphic to $\overline{\operatorname{span}}\{\Theta^X_{\xi,\eta} \mid \xi, \eta \in IXI\}$ via the map ι, with

$$\iota^{-1}(k) = k|_{IXI} \text{ for all } k \in \overline{\operatorname{span}}\{\Theta^X_{\xi,\eta} \mid \xi, \eta \in IXI\}.$$

Proof. It will be helpful to recall the following elementary Hilbert C*-module results. Firstly, fixing $\xi \in X$, we have that

$$\|\xi\| = \sup\{\| \langle \xi, \xi' \rangle \| \mid \xi' \in X, \|\xi'\| \leq 1\}. \tag{2.22}$$

Secondly, fixing $T \in \mathcal{L}(X)$, we have that

$$\|T\| = \sup\{\| \langle T\xi, \eta \rangle \| \mid \xi, \eta \in X, \|\xi\|, \|\eta\| \leq 1\}. \tag{2.23}$$

Now fix $n \in \mathbb{N}$ and $\xi_j, \eta_j \in IXI$ for all $j \in [n]$. We will show that

$$\| \sum_{j=1}^n \Theta^X_{\xi_j,\eta_j} \| = \| \sum_{j=1}^n \Theta^{IXI}_{\xi_j,\eta_j} \|.$$

Recall that the A-valued inner product of IXI is nothing but the restriction of that of X. Thus we denote both by $\langle \cdot, \cdot \rangle$, and adopt an analogous convention for the norms. We obtain that

$$\|\sum_{j=1}^{n}\Theta^X_{\xi_j,\eta_j}\| = \sup\{\|\langle\sum_{j=1}^{n}\Theta^X_{\xi_j,\eta_j}(\xi),\eta\rangle\| \mid \xi,\eta\in X, \|\xi\|,\|\eta\|\leq 1\}$$

by (2.23). Notice that $\sum_{j=1}^{n}\Theta^X_{\xi_j,\eta_j}(\xi)\in IXI$ for all $\xi\in X$. From this we deduce that

$$\begin{aligned}N_1 := \sup\{\|\langle\sum_{j=1}^{n}\Theta^X_{\xi_j,\eta_j}(\xi),\eta\rangle\| \mid \xi,\eta\in X, \|\xi\|,\|\eta\|\leq 1\} = \\ = \sup\{\|\langle\sum_{j=1}^{n}\Theta^X_{\xi_j,\eta_j}(\xi),\eta\rangle\| \mid \xi\in X,\eta\in IXI, \|\xi\|,\|\eta\|\leq 1\} =: N_2.\end{aligned}$$

The fact that $N_1\geq N_2$ is clear by definition. To see that $N_1\leq N_2$, fix $\xi,\eta\in X$ such that $\|\xi\|,\|\eta\|\leq 1$. Then we have that

$$\begin{aligned}\|\langle\sum_{j=1}^{n}\Theta^X_{\xi_j,\eta_j}(\xi),\eta\rangle\| &\leq \|\sum_{j=1}^{n}\Theta^X_{\xi_j,\eta_j}(\xi)\| = \sup\{\|\langle\sum_{j=1}^{n}\Theta^X_{\xi_j,\eta_j}(\xi),\eta'\rangle\| \mid \eta'\in IXI, \|\eta'\|\leq 1\} \\ &\leq \sup\{\|\langle\sum_{j=1}^{n}\Theta^X_{\xi_j,\eta_j}(\xi'),\eta'\rangle\| \mid \xi'\in X,\eta'\in IXI, \|\xi'\|,\|\eta'\|\leq 1\} = N_2,\end{aligned}$$

using (2.22) together with the fact that $\sum_{j=1}^{n}\Theta^X_{\xi_j,\eta_j}(\xi)\in IXI$ in the first equality. It follows that $N_1\leq N_2$ and hence $N_1=N_2$, as claimed. Next, we obtain that

$$\begin{aligned}N_2 &= \sup\{\|\langle\xi,\sum_{j=1}^{n}\Theta^X_{\eta_j,\xi_j}(\eta)\rangle\| \mid \xi\in X,\eta\in IXI, \|\xi\|,\|\eta\|\leq 1\} \\ &= \sup\{\|\sum_{j=1}^{n}\Theta^X_{\eta_j,\xi_j}(\xi)\| \mid \xi\in IXI, \|\xi\|\leq 1\} =: N_3.\end{aligned}$$

The first equality follows by taking adjoints in the definition of N_2. For the second equality, first observe that the inequality $N_2\leq N_3$ is clear. Fixing $\xi\in IXI$ such that $\|\xi\|\leq 1$, we have that

$$\|\sum_{j=1}^{n}\Theta^X_{\eta_j,\xi_j}(\xi)\| = \sup\{\|\langle\eta,\sum_{j=1}^{n}\Theta^X_{\eta_j,\xi_j}(\xi)\rangle\| \mid \eta\in X, \|\eta\|\leq 1\}\leq N_2,$$

using (2.22) and taking adjoints. It follows that $N_3\leq N_2$ and hence $N_2=N_3$, as claimed. Finally, we deduce that

$$N_3 = \sup\{\|\sum_{j=1}^{n}\Theta^{IXI}_{\eta_j,\xi_j}(\xi)\| \mid \xi\in IXI, \|\xi\|\leq 1\} = \|\sum_{j=1}^{n}\Theta^{IXI}_{\eta_j,\xi_j}\| = \|\sum_{j=1}^{n}\Theta^{IXI}_{\xi_j,\eta_j}\|,$$

where the first two equalities hold by definition and the last holds by (2.23). This shows that $\|\sum_{j=1}^{n}\Theta^X_{\xi_j,\eta_j}\| = \|\sum_{j=1}^{n}\Theta^{IXI}_{\xi_j,\eta_j}\|$ for all $n\in\mathbb{N},\xi_j,\eta_j\in IXI$ and $j\in[n]$. Consequently, we may define a linear map ι via

$$\iota\colon \operatorname{span}\{\Theta^{IXI}_{\xi,\eta} \mid \xi,\eta \in IXI\} \to \mathcal{K}(X); \sum_{j=1}^{n} \Theta^{IXI}_{\xi_j,\eta_j} \mapsto \sum_{j=1}^{n} \Theta^{X}_{\xi_j,\eta_j}$$

for all $n \in \mathbb{N}, \xi_j, \eta_j \in IXI$ and $j \in [n]$. The preceding argument demonstrates that ι is a well-defined isometry. Hence ι extends to a linear isometry $\iota\colon \mathcal{K}(IXI) \to \mathcal{K}(X)$. It is routine to check that ι is a $*$-homomorphism. Thus ι implements an embedding of $\mathcal{K}(IXI)$ within $\mathcal{K}(X)$, as claimed.

The fact that $\mathcal{K}(IXI)$ and $\overline{\operatorname{span}}\{\Theta^{X}_{\xi,\eta} \mid \xi,\eta \in IXI\}$ are $*$-isomorphic via the map ι follows immediately. It is routine to check that $\iota^{-1}(k) = k|_{IXI}$ for all $k \in \overline{\operatorname{span}}\{\Theta^{X}_{\xi,\eta} \mid \xi,\eta \in IXI\}$, finishing the proof. □

Now let X be a product system over P. Recall that if I is positively invariant for X, then $IX_pI = IX_p$ for all $p \in P$ by Lemma 2.2.16. In this case we obtain a product system IXI, with a construction that is compatible with the product system structure of X.

Proposition 2.6.2. *Let P be a unital subsemigroup of a discrete group G. Let X be a product system over P with coefficients in a C*-algebra A and let $I \subseteq A$ be an ideal that is positively invariant for X. Then $IXI := \{IX_pI\}_{p\in P}$ inherits from X a canonical structure as a product system over P with coefficients in I, identifying each IX_pI as a sub-C*-correspondence of X_p.*

Proof. We will denote the multiplication maps of X by $\{u^{X}_{p,q}\}_{p,q\in P}$. The discussion preceding Lemma 2.6.1 gives that IX_pI is a C*-correspondence over I for all $p \in P$. It remains to construct multiplication maps $\{u^{IXI}_{p,q}\}_{p,q\in P}$ of IXI and check that together with these maps IXI satisfies axioms (i)-(v) of a product system.

Note that $IX_eI = IAI = I \cap A \cap I = I$, so IXI satisfies axiom (i). The maps $u^{IXI}_{e,q}$ and $u^{IXI}_{p,e}$ for all $p,q \in P$ are determined by axioms (ii) and (iii), so fix $p,q \in P \setminus \{e\}$. We must construct a unitary

$$u^{IXI}_{p,q}\colon IX_pI \otimes_I IX_qI \to IX_{pq}I.$$

To this end, first recall that IX_pI and IX_qI can be viewed as C*-correspondences over A. As such, we can take the A-balanced tensor product $IX_pI \otimes_A IX_qI$. Since IX_pI is a sub-C*-correspondence of X_p and IX_qI is a sub-C*-correspondence of X_q, it is routine to check that $IX_pI \otimes_A IX_qI$ embeds isometrically in $X_p \otimes_A X_q$ via the "inclusion" map $\iota_{p,q}$ determined by

$$\iota_{p,q}(\xi_p \otimes \xi_q) = \xi_p \otimes \xi_q \text{ for all } \xi_p \in IX_pI, \xi_q \in IX_qI.$$

Moreover, we have that

$$\operatorname{Im}(\iota_{p,q}) = \overline{\operatorname{span}}\{\xi_p \otimes \xi_q \mid \xi_p \in IX_pI, \xi_q \in IX_qI\}.$$

Next observe that $IX_pI \otimes_I IX_qI$ embeds isometrically in $\mathrm{Im}(\iota_{p,q})$ via the I-bimodule map $v_{p,q}$ determined by

$$v_{p,q}(\xi_p \otimes \xi_q) = \xi_p \otimes \xi_q \text{ for all } \xi_p \in IX_pI, \xi_q \in IX_qI.$$

We define $u_{p,q}^{IXI}$ by

$$u_{p,q}^{IXI} = u_{p,q}^X|_{\mathrm{Im}(\iota_{p,q})} \circ v_{p,q} \colon IX_pI \otimes_I IX_qI \to X_{pq}.$$

Notice that $u_{p,q}^{IXI}$ is an isometric linear I-bimodule map, which can be seen by applying the corresponding properties of $u_{p,q}^X$ and $v_{p,q}$. Thus it remains to show that $u_{p,q}^{IXI}$ is a surjection onto $IX_{pq}I$. We begin by showing that the range of $u_{p,q}^{IXI}$ is contained in $IX_{pq}I$. By linearity and continuity of $u_{p,q}^{IXI}$, it suffices to show that the simple tensors in $IX_pI \otimes_I IX_qI$ are mapped into $IX_{pq}I$. To this end, fix $a, b, c, d \in I, \xi_p \in X_p$ and $\xi_q \in X_q$. We have that

$$u_{p,q}^{IXI}((a\xi_pb) \otimes (c\xi_qd)) = u_{p,q}^X((a\xi_pb) \otimes (c\xi_qd)) = au_{p,q}^X((\xi_pb) \otimes (c\xi_q))d \in IX_{pq}I,$$

using that $u_{p,q}^X$ is an A-bimodule map in the final equality. This shows that $\mathrm{Im}(u_{p,q}^{IXI}) \subseteq IX_{pq}I$. For the reverse inclusion, note that since $u_{p,q}^{IXI}$ is in particular linear and isometric and thus has closed range, it suffices to show that $\mathrm{Im}(u_{p,q}^{IXI})$ contains all elements of the form $au_{p,q}^X(\xi_p \otimes \xi_q)b$, where $a, b \in I, \xi_p \in X_p$ and $\xi_q \in X_q$. To this end, recall that $I = II$ and thus we can write $a = cd$ for some $c, d \in I$. Consequently, we obtain that

$$au_{p,q}^X(\xi_p \otimes \xi_q)b = u_{p,q}^X((a\xi_p) \otimes (\xi_qb)) = u_{p,q}^X((cd\xi_p) \otimes (\xi_qb)),$$

using that $u_{p,q}^X$ is an A-bimodule map in the first equality. Note that $d\xi_p \in IX_p \subseteq X_pI$ by Lemma 2.2.16, which is applicable since I is positively invariant for X. Thus we can write $d\xi_p = \xi_p'f$ for some $\xi_p' \in X_p$ and $f \in I$. We may also write $f = gh$ for some $g, h \in I$. In total, we obtain that

$$u_{p,q}^X((cd\xi_p) \otimes (\xi_qb)) = u_{p,q}^X((c\xi_p'gh) \otimes (\xi_qb)) = u_{p,q}^{IXI}((c\xi_p'g) \otimes (h\xi_qb)),$$

as required. This shows that $u_{p,q}^{IXI}$ is a surjection onto $IX_{pq}I$. Combining the preceding deductions, we conclude that $u_{p,q}^{IXI} \colon IX_pI \otimes_I IX_qI \to IX_{pq}I$ is a unitary, which is in accordance with axiom (iv). Finally, associativity of the multiplication maps of IXI follows from that of the multiplication maps of X, thereby showing that IXI satisfies axiom (v). Hence IXI constitutes a product system over P with coefficients in I, as required. □

Proposition 2.6.3. *Let P be a unital subsemigroup of a discrete group G. Let X be a product system over P with coefficients in a C*-algebra A and let $I \subseteq A$ be an ideal that is positively invariant for X. Let $\{\iota_p^{pq}\}_{p,q\in P}$ denote the connecting $*$-homomorphisms of*

X and let $\{j_p^{pq}\}_{p,q\in P}$ denote the connecting $*$-homomorphisms of IXI. Then

$$j_p^{pq}(k_p|_{IX_pI}) = \iota_p^{pq}(k_p)|_{IX_{pq}I} \text{ for all } p,q\in P \text{ and } k_p \in \overline{\operatorname{span}}\{\Theta^{X_p}_{\xi_p,\eta_p} \mid \xi_p,\eta_p \in IX_pI\}.$$

Proof. Without loss of generality we may assume that $p \neq e$ and $q \neq e$, as the claim is straightforward otherwise. By Lemma 2.6.1 we have that $k_p|_{IX_pI} \in \mathcal{K}(IX_pI)$. Recalling that $IX_{pq}I$ is a closed linear subspace of X_{pq}, it suffices to show that

$$j_p^{pq}(k_p|_{IX_pI})u_{p,q}^{IXI}(\xi_p\otimes\xi_q) = \iota_p^{pq}(k_p)u_{p,q}^{IXI}(\xi_p\otimes\xi_q), \text{ for all } \xi_p\in IX_pI, \xi_q\in IX_qI.$$

Indeed, fixing $\xi_p \in IX_pI$ and $\xi_q \in IX_qI$, we have that

$$\begin{aligned} j_p^{pq}(k_p|_{IX_pI})u_{p,q}^{IXI}(\xi_p\otimes\xi_q) &= u_{p,q}^{IXI}((k_p\xi_p)\otimes\xi_q) = u_{p,q}^{X}((k_p\xi_p)\otimes\xi_q) \\ &= \iota_p^{pq}(k_p)u_{p,q}^{X}(\xi_p\otimes\xi_q) = \iota_p^{pq}(k_p)u_{p,q}^{IXI}(\xi_p\otimes\xi_q), \end{aligned}$$

as required. □

Next we show that the properties of compact alignment and strong compact alignment of IXI are inherited from X.

Proposition 2.6.4. *Let P be a unital right LCM subsemigroup of a discrete group G. Let X be a compactly aligned product system over P with coefficients in a C*-algebra A and let $I \subseteq A$ be an ideal that is positively invariant for X. Then the product system IXI is compactly aligned.*

Proof. Let $\{\iota_p^{pq}\}_{p,q\in P}$ (resp. $\{j_p^{pq}\}_{p,q\in P}$) denote the connecting $*$-homomorphisms of X (resp. IXI). It suffices to show that the compact alignment condition holds for rank-one operators. To this end, let $p,q\in P\setminus\{e\}$ be such that $pP\cap qP = wP$ for some $w\in P$, and fix $\xi_p,\eta_p\in IX_pI$ and $\xi_q,\eta_q\in IX_qI$. We must show that $j_p^w(\Theta^{IX_pI}_{\xi_p,\eta_p})j_q^w(\Theta^{IX_qI}_{\xi_q,\eta_q}) \in \mathcal{K}(IX_wI)$. Recall that

$$\Theta^{IX_pI}_{\xi_p,\eta_p} = \Theta^{X_p}_{\xi_p,\eta_p}|_{IX_pI} \quad\text{and}\quad \Theta^{IX_qI}_{\xi_q,\eta_q} = \Theta^{X_q}_{\xi_q,\eta_q}|_{IX_qI},$$

by Lemma 2.6.1. An application of Proposition 2.6.3 then gives that

$$j_p^w(\Theta^{IX_pI}_{\xi_p,\eta_p})j_q^w(\Theta^{IX_qI}_{\xi_q,\eta_q}) = [\iota_p^w(\Theta^{X_p}_{\xi_p,\eta_p})\iota_q^w(\Theta^{X_q}_{\xi_q,\eta_q})]|_{IX_wI}. \tag{2.24}$$

By Lemma 2.6.1, it suffices to show that

$$\iota_p^w(\Theta^{X_p}_{\xi_p,\eta_p})\iota_q^w(\Theta^{X_q}_{\xi_q,\eta_q}) \in \overline{\operatorname{span}}\{\Theta^{X_w}_{\xi_w,\eta_w} \mid \xi_w,\eta_w\in IX_wI\}.$$

Compact alignment of X gives that

$$\iota_p^w(\Theta^{X_p}_{\xi_p,\eta_p})\iota_q^w(\Theta^{X_q}_{\xi_q,\eta_q}) \in \mathcal{K}(X_w).$$

Next, let $(u_\lambda)_{\lambda\in\Lambda}$ denote an approximate unit of I. For each $\lambda \in \Lambda$, we have that

$$\phi_w(u_\lambda)\iota_p^w(\Theta_{\xi_p,\eta_p}^{X_p}) = \iota_p^w(\phi_p(u_\lambda)\Theta_{\xi_p,\eta_p}^{X_p}) = \iota_p^w(\Theta_{\phi_p(u_\lambda)\xi_p,\eta_p}^{X_p}).$$

Using this and the fact that $\|\cdot\|\text{-}\lim_\lambda \phi_p(u_\lambda)\xi_p = \xi_p$ (as $\xi_p \in IX_pI$), we obtain that

$$\|\cdot\|\text{-}\lim_\lambda \phi_w(u_\lambda)\iota_p^w(\Theta_{\xi_p,\eta_p}^{X_p}) = \iota_p^w(\Theta_{\xi_p,\eta_p}^{X_p}).$$

By analogous reasoning, we have that

$$\|\cdot\|\text{-}\lim_\lambda \iota_q^w(\Theta_{\xi_q,\eta_q}^{X_q})\phi_w(u_\lambda) = \iota_q^w(\Theta_{\xi_q,\eta_q}^{X_q}).$$

Therefore, we obtain that

$$\iota_p^w(\Theta_{\xi_p,\eta_p}^{X_p})\iota_q^w(\Theta_{\xi_q,\eta_q}^{X_q}) = \|\cdot\|\text{-}\lim_\lambda \phi_w(u_\lambda)\iota_p^w(\Theta_{\xi_p,\eta_p}^{X_p})\iota_q^w(\Theta_{\xi_q,\eta_q}^{X_q})\phi_w(u_\lambda).$$

Consequently, we have that $\iota_p^w(\Theta_{\xi_p,\eta_p}^{X_p})\iota_q^w(\Theta_{\xi_q,\eta_q}^{X_q})$ is expressible as the norm-limit of a net that is contained in $\phi_w(I)\mathcal{K}(X_w)\phi_w(I)$. An application of Corollary 2.2.17 then gives that

$$\iota_p^w(\Theta_{\xi_p,\eta_p}^{X_p})\iota_q^w(\Theta_{\xi_q,\eta_q}^{X_q}) \in \overline{\operatorname{span}}\{\Theta_{\xi_w,\eta_w}^{X_w} \mid \xi_w, \eta_w \in IX_wI\}, \tag{2.25}$$

as required. □

Proposition 2.6.5. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let $I \subseteq A$ be an ideal that is positively invariant for X. Then the product system IXI is strong compactly aligned.*

Proof. Let $\{\iota_{\underline{n}}^{\underline{n}+\underline{m}}\}_{\underline{n},\underline{m}\in\mathbb{Z}_+^d}$ (resp. $\{j_{\underline{n}}^{\underline{n}+\underline{m}}\}_{\underline{n},\underline{m}\in\mathbb{Z}_+^d}$) denote the connecting $*$-homomorphisms of X (resp. IXI). Proposition 2.6.4 asserts that IXI is compactly aligned, so it remains to check that IXI satisfies

$$j_{\underline{n}}^{\underline{n}+\underline{i}}(\mathcal{K}(IX_{\underline{n}}I)) \subseteq \mathcal{K}(IX_{\underline{n}+\underline{i}}I) \text{ whenever } \underline{n} \perp \underline{i}, \text{ where } i \in [d], \underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}.$$

It suffices to show that this holds for rank-one operators. To this end, fix $\xi_{\underline{n}}, \eta_{\underline{n}} \in IX_{\underline{n}}I$. An application of Proposition 2.6.3 then yields that

$$j_{\underline{n}}^{\underline{n}+\underline{i}}(\Theta_{\xi_{\underline{n}},\eta_{\underline{n}}}^{IX_{\underline{n}}I}) = j_{\underline{n}}^{\underline{n}+\underline{i}}(\Theta_{\xi_{\underline{n}},\eta_{\underline{n}}}^{X_{\underline{n}}}|_{IX_{\underline{n}}I}) = \iota_{\underline{n}}^{\underline{n}+\underline{i}}(\Theta_{\xi_{\underline{n}},\eta_{\underline{n}}}^{X_{\underline{n}}})|_{IX_{\underline{n}+\underline{i}}I}.$$

By strong compact alignment of X, we have that $\iota_{\underline{n}}^{\underline{n}+\underline{i}}(\Theta_{\xi_{\underline{n}},\eta_{\underline{n}}}^{X_{\underline{n}}}) \in \mathcal{K}(X_{\underline{n}+\underline{i}})$. By using an approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ of I and Corollary 2.2.17, we then obtain that

$$\begin{aligned}\iota_{\underline{n}}^{\underline{n}+\underline{i}}(\Theta_{\xi_{\underline{n}},\eta_{\underline{n}}}^{X_{\underline{n}}}) &= \|\cdot\|\text{-}\lim_\lambda \phi_{\underline{n}+\underline{i}}(u_\lambda)\iota_{\underline{n}}^{\underline{n}+\underline{i}}(\Theta_{\xi_{\underline{n}},\eta_{\underline{n}}}^{X_{\underline{n}}})\phi_{\underline{n}+\underline{i}}(u_\lambda) \\ &\in [\phi_{\underline{n}+\underline{i}}(I)\mathcal{K}(X_{\underline{n}+\underline{i}})\phi_{\underline{n}+\underline{i}}(I)] \\ &\subseteq \overline{\operatorname{span}}\{\Theta_{\xi_{\underline{n}+\underline{i}},\eta_{\underline{n}+\underline{i}}}^{X_{\underline{n}+\underline{i}}} \mid \xi_{\underline{n}+\underline{i}}, \eta_{\underline{n}+\underline{i}} \in IX_{\underline{n}+\underline{i}}I\}.\end{aligned}$$

An application of Lemma 2.6.1 finishes the proof. □

Next we study the representations of IXI. Given a representation (π, t) of X, we write $(\pi|_I, t|_{IXI})$ for the family $\{(\pi|_I, t_p|_{IX_pI})\}_{p\in P}$. It is routine to check that $(\pi|_I, t|_{IXI})$ is a representation of IXI. For each $p \in P$, let $\widetilde{\psi}_p \colon \mathcal{K}(IX_pI) \to \mathrm{C}^*(\pi, t)$ be the $*$-homomorphism induced by $(\pi|_I, t_p|_{IX_pI})$. Then for all $\xi_p, \eta_p \in IX_pI$, we have that

$$\widetilde{\psi}_p(\Theta^{X_p}_{\xi_p,\eta_p}|_{IX_pI}) = \widetilde{\psi}_p(\Theta^{IX_pI}_{\xi_p,\eta_p}) = t_p|_{IX_pI}(\xi_p)t_p|_{IX_pI}(\eta_p)^* = t_p(\xi_p)t_p(\eta_p)^* = \psi_p(\Theta^{X_p}_{\xi_p,\eta_p}),$$

from which it follows that

$$\widetilde{\psi}_p(k_p|_{IX_pI}) = \psi_p(k_p) \text{ for all } k_p \in \overline{\operatorname{span}}\{\Theta^{X_p}_{\xi_p,\eta_p} \mid \xi_p, \eta_p \in IX_pI\}. \tag{2.26}$$

When P is a right LCM semigroup and X is compactly aligned (and thus IXI is compactly aligned by Proposition 2.6.4), Nica-covariance of (π, t) is inherited by $(\pi|_I, t|_{IXI})$.

Proposition 2.6.6. *Let P be a unital right LCM subsemigroup of a discrete group G. Let X be a compactly aligned product system over P with coefficients in a C*-algebra A and let $I \subseteq A$ be an ideal that is positively invariant for X. Let (π, t) be a Nica-covariant representation of X. Then $(\pi|_I, t|_{IXI})$ is a Nica-covariant representation of IXI, and*

$$\mathrm{C}^*(\pi|_I, t|_{IXI}) = \overline{\operatorname{span}}\{\pi(I)t_p(X_p)\pi(I)t_q(X_q)^*\pi(I) \mid p, q \in P\}. \tag{2.27}$$

Proof. Let $\{\iota_p^{pq}\}_{p,q\in P}$ (resp. $\{j_p^{pq}\}_{p,q\in P}$) denote the connecting $*$-homomorphisms of X (resp. IXI). For each $p \in P$, let $\widetilde{\psi}_p \colon \mathcal{K}(IX_pI) \to \mathrm{C}^*(\pi, t)$ be the $*$-homomorphism induced by $(\pi|_I, t_p|_{IX_pI})$. Now fix $p, q \in P \setminus \{e\}, k_p \in \mathcal{K}(IX_pI)$ and $k_q \in \mathcal{K}(IX_qI)$. We must show that

$$\widetilde{\psi}_p(k_p)\widetilde{\psi}_q(k_q) = \begin{cases} \widetilde{\psi}_w(j_p^w(k_p)j_q^w(k_q)) & \text{if } pP \cap qP = wP \text{ for some } w \in P, \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to show this for $k_p = \Theta^{IX_pI}_{\xi_p,\eta_p}$ and $k_q = \Theta^{IX_qI}_{\xi_q,\eta_q}$ for some $\xi_p, \eta_p \in IX_pI$ and $\xi_q, \eta_q \in IX_qI$. By (2.26) and Nica-covariance of (π, t), we have that

$$\begin{aligned} \widetilde{\psi}_p(\Theta^{IX_pI}_{\xi_p,\eta_p})\widetilde{\psi}_q(\Theta^{IX_qI}_{\xi_q,\eta_q}) &= \psi_p(\Theta^{X_p}_{\xi_p,\eta_p})\psi_q(\Theta^{X_q}_{\xi_q,\eta_q}) \\ &= \begin{cases} \psi_w(\iota_p^w(\Theta^{X_p}_{\xi_p,\eta_p})\iota_q^w(\Theta^{X_q}_{\xi_q,\eta_q})) & \text{if } pP \cap qP = wP \text{ for some } w \in P, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Suppose $pP \cap qP = wP$ for some $w \in P$. Since the assumptions of Proposition 2.6.4 are satisfied, by (2.24) we have that

$$[\iota_p^w(\Theta^{X_p}_{\xi_p,\eta_p})\iota_q^w(\Theta^{X_q}_{\xi_q,\eta_q})]|_{IX_wI} = j_p^w(\Theta^{IX_pI}_{\xi_p,\eta_p})j_q^w(\Theta^{IX_qI}_{\xi_q,\eta_q}),$$

and by (2.25) we have that

$$\iota_p^w(\Theta_{\xi_p,\eta_p}^{X_p})\iota_q^w(\Theta_{\xi_q,\eta_q}^{X_q}) \in \overline{\operatorname{span}}\{\Theta_{\xi_w,\eta_w}^{X_w} \mid \xi_w,\eta_w \in IX_wI\}.$$

Another application of (2.26) then yields that

$$\widetilde{\psi}_p(\Theta_{\xi_p,\eta_p}^{IX_pI})\widetilde{\psi}_q(\Theta_{\xi_q,\eta_q}^{IX_qI}) = \begin{cases} \widetilde{\psi}_w(j_p^w(\Theta_{\xi_p,\eta_p}^{IX_pI})j_q^w(\Theta_{\xi_q,\eta_q}^{IX_qI})) & \text{if } pP\cap qP = wP \text{ for some } w\in P, \\ 0 & \text{otherwise,} \end{cases}$$

as required.

Finally, we have that

$$\begin{aligned} \mathrm{C}^*(\pi|_I, t|_{IXI}) &= \overline{\operatorname{span}}\{t_p(IX_pI)t_q(IX_qI)^* \mid p,q\in P\} \\ &= \overline{\operatorname{span}}\{\pi(I)t_p(X_p)\pi(I)t_q(X_q)^*\pi(I) \mid p,q\in P\}, \end{aligned}$$

using that $I = I^* = II$ in the second equality. This finishes the proof. □

The structure of $\mathrm{C}^*(\pi|_I, t|_{IXI})$ is linked with that of $\langle \pi(I)\rangle \subseteq \mathrm{C}^*(\pi,t)$. To clarify this link, we remind of the following notions from the theory of C*-algebras. Let A be a C*-algebra and B be a C*-subalgebra of A. We say that B is *hereditary (in A)* if whenever $a \in A_+, b \in B_+$ and $a \leq b$, we have that $a \in B$. We say that B is *full (in A)* if whenever $I \subseteq A$ is an ideal such that $I \supseteq B$, we have that $I = A$.

Proposition 2.6.7. *Let P be a unital right LCM subsemigroup of a discrete group G. Let X be a compactly aligned product system over P with coefficients in a C*-algebra A and let $I \subseteq A$ be an ideal that is positively invariant for X. Let (π,t) be a Nica-covariant representation of X. Then the following hold:*

(i) $\langle \pi(I)\rangle = \overline{\operatorname{span}}\{t_p(X_p)\pi(I)t_q(X_q)^* \mid p,q\in P\}$;

(ii) $\mathrm{C}^*(\pi|_I, t|_{IXI}) = [\pi(I)\langle\pi(I)\rangle\pi(I)]$, *and thus* $\mathrm{C}^*(\pi|_I, t|_{IXI})$ *is a hereditary, full C*-subalgebra of* $\langle\pi(I)\rangle$.

Proof. For notational convenience, we will denote the right hand side of the equality asserted in item (i) by B. The fact that $B \subseteq \langle \pi(I)\rangle$ is immediate. To see that $\langle\pi(I)\rangle \subseteq B$, it suffices to show that B is an ideal of $\mathrm{C}^*(\pi,t)$ that contains $\pi(I)$. The fact that B is a closed linear subspace is evident, and it is similarly straightforward to check that B is selfadjoint. Using the latter, showing that B is an ideal amounts to showing that $B\mathrm{C}^*(\pi,t) \subseteq B$. In turn, it suffices to show that

$$Bt_r(X_r) \subseteq B \quad \text{and} \quad Bt_r(X_r)^* \subseteq B \text{ for all } r \in P.$$

To this end, fix $p,q,r \in P$. We have that

$$t_p(X_p)\pi(I)t_q(X_q)^*t_r(X_r)^* \subseteq t_p(X_p)\pi(I)t_{rq}(X_{rq})^* \subseteq B,$$

from which it follows that $Bt_r(X_r)^* \subseteq B$. Next, suppose that $qP \cap rP = \emptyset$. Then

$$t_p(X_p)\pi(I)t_q(X_q)^*t_r(X_r) = \{0\} \subseteq B,$$

using Nica-covariance in the equality. Otherwise, assume that $qP \cap rP = wP$ for some $w \in P$. Then another application of Nica-covariance yields that

$$\begin{aligned} t_p(X_p)\pi(I)t_q(X_q)^*t_r(X_r) &\subseteq t_p(X_p)\pi(I)[t_{q'}(X_{q'})t_{r'}(X_{r'})^*] \subseteq [t_p(X_p)\pi(I)t_{q'}(X_{q'})t_{r'}(X_{r'})^*] \\ &\subseteq [t_p(X_p)t_{q'}(X_{q'})\pi(I)t_{r'}(X_{r'})^*] \subseteq [t_{pq'}(X_{pq'})\pi(I)t_{r'}(X_{r'})^*] \subseteq B, \end{aligned}$$

where $q' = q^{-1}w$ and $r' = r^{-1}w$, using Lemma 2.2.16 in the third inclusion. It follows that $Bt_r(X_r) \subseteq B$. In total, we have that B is an ideal of $\mathrm{C}^*(\pi, t)$.

It remains to check that B contains $\pi(I)$. To this end, fix $a \in I$ and an approximate unit $(u_\lambda)_{\lambda \in \Lambda}$ of A. For each $\lambda \in \Lambda$, we have that $\pi(u_\lambda)\pi(a)\pi(u_\lambda) = \pi(u_\lambda)\pi(a)\pi(u_\lambda)^* \in B$ by definition. Notice also that

$$\|\cdot\|\text{-}\lim_\lambda \pi(u_\lambda)\pi(a)\pi(u_\lambda) = \|\cdot\|\text{-}\lim_\lambda \pi(u_\lambda a u_\lambda) = \pi(a).$$

Thus $\pi(a)$ is expressible as the norm-limit of a net that is contained in B. Since B is closed in $\mathrm{C}^*(\pi, t)$, this shows that $\pi(a) \in B$. We conclude that $\pi(I) \subseteq B$ and hence $\langle \pi(I) \rangle \subseteq B$. In total we have that $\langle \pi(I) \rangle = B$, as required.

The fact that $\mathrm{C}^*(\pi|_I, t|_{IXI}) = [\pi(I) \langle \pi(I) \rangle \pi(I)]$ now follows by applying item (i) and Proposition 2.6.6 in tandem. In particular, notice that $\mathrm{C}^*(\pi|_I, t|_{IXI})$ is a C*-subalgebra of $\langle \pi(I) \rangle$. To see that $\mathrm{C}^*(\pi|_I, t|_{IXI})$ is hereditary in $\langle \pi(I) \rangle$, it suffices to show that

$$\mathrm{C}^*(\pi|_I, t|_{IXI}) \langle \pi(I) \rangle \mathrm{C}^*(\pi|_I, t|_{IXI}) \subseteq \mathrm{C}^*(\pi|_I, t|_{IXI})$$

by [44, Theorem 3.2.2]. Accordingly, we have that

$$\begin{aligned} \mathrm{C}^*(\pi|_I, t|_{IXI}) \langle \pi(I) \rangle \mathrm{C}^*(\pi|_I, t|_{IXI}) &= [\pi(I) \langle \pi(I) \rangle \pi(I)] \langle \pi(I) \rangle [\pi(I) \langle \pi(I) \rangle \pi(I)] \\ &\subseteq [\pi(I) \langle \pi(I) \rangle \pi(I) \langle \pi(I) \rangle \pi(I) \langle \pi(I) \rangle \pi(I)] \\ &\subseteq [\pi(I) \langle \pi(I) \rangle \pi(I)] = \mathrm{C}^*(\pi|_I, t|_{IXI}), \end{aligned}$$

as required.

To see that $\mathrm{C}^*(\pi|_I, t|_{IXI})$ is full, let $\mathfrak{J}$ be an ideal of $\langle \pi(I) \rangle$ such that $\mathfrak{J} \supseteq \mathrm{C}^*(\pi|_I, t|_{IXI})$. Note that since $\mathfrak{J}$ is an ideal in $\langle \pi(I) \rangle$, and $\langle \pi(I) \rangle$ is an ideal in $\mathrm{C}^*(\pi, t)$, we have that $\mathfrak{J}$ is an ideal in $\mathrm{C}^*(\pi, t)$. We also deduce that $\pi(I) \subseteq \mathrm{C}^*(\pi|_I, t|_{IXI}) \subseteq \mathfrak{J}$. In total, we have that $\mathfrak{J}$ is an ideal of $\mathrm{C}^*(\pi, t)$ that contains $\pi(I)$ and hence $\langle \pi(I) \rangle \subseteq \mathfrak{J}$. Consequently, we conclude that $\mathfrak{J} = \langle \pi(I) \rangle$ and thus $\mathrm{C}^*(\pi|_I, t|_{IXI})$ is full in $\langle \pi(I) \rangle$, finishing the proof. □

Suppose now that $P = \mathbb{Z}_+^d$ and X is strong compactly aligned. An application of the Gauge-Invariant Uniqueness Theorem permits the identification of $\mathcal{NT}_{IXI}$ and $\mathcal{NO}_{IXI}$

with C*-subalgebras of $\mathcal{NT}_X$ and $\mathcal{NO}_X$, respectively. We note that the Toeplitz-Nica-Pimsner case extends to right LCM semigroups using [33, Theorem 6.4] (but we will not require it here).

Proposition 2.6.8. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let $I \subseteq A$ be an ideal that is positively invariant for X. Then the inclusion $IXI \hookrightarrow X$ induces a canonical embedding*

$$\mathcal{NT}_{IXI} \cong \mathrm{C}^*(\overline{\pi}_X|_I, \overline{t}_X|_{IXI}) \subseteq \mathcal{NT}_X,$$

and thus $\mathcal{NT}_{IXI}$ is a hereditary, full C-subalgebra of $\langle \overline{\pi}_X(I) \rangle$.*

Proof. Note that IXI is strong compactly aligned by Proposition 2.6.5, and $(\overline{\pi}_X|_I, \overline{t}_X|_{IXI})$ is a Nica-covariant representation of IXI by Proposition 2.6.6. Injectivity of $(\overline{\pi}_X|_I, \overline{t}_X|_{IXI})$ is inherited from $(\overline{\pi}_X, \overline{t}_X)$, and $(\overline{\pi}_X|_I, \overline{t}_X|_{IXI})$ admits a gauge action by restriction. Hence it suffices to show that $\overline{\pi}_X|_I \times \overline{t}_X|_{IXI} \colon \mathcal{NT}_{IXI} \to \mathrm{C}^*(\overline{\pi}_X|_I, \overline{t}_X|_{IXI})$ is injective on the fixed point algebra; equivalently on the $[\underline{0}, \underline{1}_{[d]}]$-core by Proposition 2.5.21.

To this end, first note that we may take $(\overline{\pi}_X, \overline{t}_X)$ to be the Fock representation without loss of generality. Let $\psi_{\underline{n}} \colon \mathcal{K}(IX_{\underline{n}}I) \to \mathcal{NT}_X$ be the $*$-homomorphism induced by $(\overline{\pi}_X|_I, \overline{t}_{X,\underline{n}}|_{IX_{\underline{n}}I})$ for all $\underline{n} \in \mathbb{Z}_+^d$. Take $f \in \ker \overline{\pi}_X|_I \times \overline{t}_X|_{IXI} \cap B_{[\underline{0},\underline{1}_{[d]}]}^{(\overline{\pi}_{IXI},\overline{t}_{IXI})}$, so that we may write

$$f = \overline{\pi}_{IXI}(a) + \sum \{\overline{\psi}_{IXI,\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_{[d]}\},$$

for some $a \in I$ and $k_{\underline{n}} \in \mathcal{K}(IX_{\underline{n}}I)$ for all $\underline{0} \neq \underline{n} \leq \underline{1}_{[d]}$. Recall that we write $\overline{\pi}_{IXI}(a) = \overline{\psi}_{IXI,\underline{0}}(k_{\underline{0}})$ for $k_{\underline{0}} := \phi_{IX_{\underline{0}}I}(a)$, and in turn $\overline{\pi}_X|_I(a) = \psi_{\underline{0}}(k_{\underline{0}})$. Towards contradiction, suppose that $f \neq 0$. Then we may choose $\underline{0} \leq \underline{r} \leq \underline{1}_{[d]}$ minimal such that $k_{\underline{r}} \neq 0$. Let $P_{\underline{r}} \colon \mathcal{F}X \to X_{\underline{r}}$ be the canonical projection. Then

$$\iota(k_{\underline{r}}) = P_{\underline{r}} \left(\sum \{\psi_{\underline{n}}(k_{\underline{n}}) \mid \underline{0} \leq \underline{n} \leq \underline{1}_{[d]}\} \right) P_{\underline{r}} = 0,$$

where $\iota \colon \mathcal{K}(IX_{\underline{r}}I) \to \mathcal{K}(X_{\underline{r}})$ is the embedding guaranteed by Lemma 2.6.1. Hence $k_{\underline{r}} = 0$, which is a contradiction. We conclude that $\overline{\pi}_X|_I \times \overline{t}_X|_{IXI}$ is injective on the $[\underline{0}, \underline{1}_{[d]}]$-core, as required. Proposition 2.6.7 then completes the proof. □

To establish the corresponding result for $\mathcal{NO}_{IXI}$, we first generalise [36, Proposition 9.2].

Proposition 2.6.9. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let $I \subseteq A$ be an ideal that is positively invariant for X. Then, for all $F \subseteq [d]$, we have that:*

(i) $\mathcal{J}_F(IXI) = I \cap \mathcal{J}_F(X)$;

(ii) $\mathcal{I}_F(IXI) = I \cap \mathcal{I}_F(X)$.

Proof. (i) The claim holds trivially for $F = \emptyset$, so assume that $F \neq \emptyset$. Let $a \in \mathcal{J}_F(IXI) \subseteq I$. It suffices to show that $a \in \mathcal{J}_F(X)$. To this end, fix $i \in [d]$. By definition, we have that $a \in (\phi_{IX_{\underline{i}}I})^{-1}(\mathcal{K}(IX_{\underline{i}}I))$. Since I is positively invariant for $X_{\underline{i}}$, we may apply Lemma 2.2.16 to deduce that $IX_{\underline{i}}I = IX_{\underline{i}}$, and an application of [36, Lemma 9.1] then yields that

$$(\phi_{IX_{\underline{i}}I})^{-1}(\mathcal{K}(IX_{\underline{i}}I)) = I \cap \phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})).$$

Hence $a \in \phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}}))$ for all $i \in [d]$, as required. Moreover, since $\ker \phi_{\underline{i}} \cap I \subseteq \ker \phi_{IX_{\underline{i}}I}$ for every $i \in [d]$, we have that $(\bigcap_{i\in F} \ker \phi_{IX_{\underline{i}}I})^\perp \subseteq (\bigcap_{i\in F} \ker \phi_{\underline{i}})^\perp \cap I$. Hence $a \in (\bigcap_{i\in F} \ker \phi_{\underline{i}})^\perp$ and thus $a \in \mathcal{J}_F(X)$, as required.

For the reverse inclusion, let $a \in I \cap \mathcal{J}_F(X)$. For each $i \in [d]$, we have that

$$a \in I \cap \phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) = (\phi_{IX_{\underline{i}}I})^{-1}(\mathcal{K}(IX_{\underline{i}}I))$$

by Lemma 2.2.16 and [36, Lemma 9.1]. Thus it suffices to show that $a \in (\bigcap_{i\in F} \ker \phi_{IX_{\underline{i}}I})^\perp$. To this end, fix $b \in \bigcap_{i\in F} \ker \phi_{IX_{\underline{i}}I}$. Another application of [36, Lemma 9.1] gives that

$$b \in I \cap (\bigcap_{i\in F} \ker \phi_{\underline{i}}).$$

Since $a \in (\bigcap_{i\in F} \ker \phi_{\underline{i}})^\perp$, we therefore have that $ab = 0$, as required.

(ii) Once again, we may assume without loss of generality that $F \neq \emptyset$. Take $a \in \mathcal{I}_F(IXI) \subseteq I$ and fix $\underline{n} \perp F$. We have that

$$\begin{aligned}\langle X_{\underline{n}}, aX_{\underline{n}}\rangle &\subseteq [\langle X_{\underline{n}}, IaIX_{\underline{n}}\rangle] \subseteq [\langle IX_{\underline{n}}, aIX_{\underline{n}}\rangle] \\ &\subseteq [\big\langle IX_{\underline{n}}I, \phi_{IX_{\underline{n}}I}(a)(IX_{\underline{n}}I)\big\rangle] \subseteq \mathcal{J}_F(IXI) \subseteq \mathcal{J}_F(X),\end{aligned}$$

by using Lemma 2.2.16 and item (i). Thus $a \in I \cap \mathcal{I}_F(X)$, as required.

For the reverse inclusion, fix $a \in I \cap \mathcal{I}_F(X)$ and $\underline{n} \perp F$. We have that

$$\big\langle IX_{\underline{n}}I, \phi_{IX_{\underline{n}}I}(a)(IX_{\underline{n}}I)\big\rangle \subseteq \langle X_{\underline{n}}, aX_{\underline{n}}\rangle \subseteq I \cap \mathcal{J}_F(X) = \mathcal{J}_F(IXI),$$

using positive invariance of I, the fact that $a \in \mathcal{I}_F(X)$ and item (i). Thus $a \in \mathcal{I}_F(IXI)$, completing the proof. □

Proposition 2.6.10. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let $I \subseteq A$ be an ideal that is positively invariant for X. Then the inclusion $IXI \hookrightarrow X$ induces a canonical embedding*

$$\mathcal{NO}_{IXI} \cong \mathrm{C}^*(\pi_X^{\mathcal{I}}|_I, t_X^{\mathcal{I}}|_{IXI}) \subseteq \mathcal{NO}_X,$$

and thus $\mathcal{NO}_{IXI}$ is a hereditary, full C-subalgebra of $\big\langle \pi_X^{\mathcal{I}}(I)\big\rangle$.*

Proof. Being the restriction of an injective Nica-covariant representation of X that admits a gauge action, $(\pi_X^{\mathcal{I}}|_I, t_X^{\mathcal{I}}|_{IXI})$ is an injective Nica-covariant representation of IXI that admits a gauge action. Thus it suffices to show that $(\pi_X^{\mathcal{I}}|_I, t_X^{\mathcal{I}}|_{IXI})$ is a CNP-representation of IXI by [17, Theorem 4.2].

To this end, fix $\emptyset \neq F \subseteq [d]$ and $a \in \mathcal{I}_F(IXI)$. For each $\underline{n} \in \mathbb{Z}_+^d$, let $\widetilde{\psi}_{\underline{n}} \colon \mathcal{K}(IX_{\underline{n}}I) \to \mathcal{NO}_X$ be the $*$-homomorphism induced by $(\pi_X^{\mathcal{I}}|_I, t_{X,\underline{n}}^{\mathcal{I}}|_{IX_{\underline{n}}I})$. We must show that

$$\pi_X^{\mathcal{I}}|_I(a) + \sum\{(-1)^{|\underline{n}|}\widetilde{\psi}_{\underline{n}}(\phi_{IX_{\underline{n}}I}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = 0.$$

Fix $\underline{0} \neq \underline{n} \leq \underline{1}_F$. We claim that

$$\phi_{\underline{n}}(a) \in \overline{\operatorname{span}}\{\Theta^{X_{\underline{n}}}_{\xi_{\underline{n}},\eta_{\underline{n}}} \mid \xi_{\underline{n}}, \eta_{\underline{n}} \in IX_{\underline{n}}I\}.$$

To see this, first note that $a \in I \cap \mathcal{I}_F(X)$ by item (ii) of Proposition 2.6.9, and so $\phi_{\underline{n}}(a) \in \mathcal{K}(X_{\underline{n}})$. Let $(u_\lambda)_{\lambda \in \Lambda}$ be an approximate unit of I. Then an application of Corollary 2.2.17 yields that

$$\phi_{\underline{n}}(u_\lambda)\phi_{\underline{n}}(a)\phi_{\underline{n}}(u_\lambda) \in \overline{\operatorname{span}}\{\Theta^{X_{\underline{n}}}_{\xi_{\underline{n}},\eta_{\underline{n}}} \mid \xi_{\underline{n}}, \eta_{\underline{n}} \in IX_{\underline{n}}I\} \text{ for all } \lambda \in \Lambda.$$

Since $a \in I$, we also have that

$$\|\cdot\|\text{-}\lim_{\lambda} \phi_{\underline{n}}(u_\lambda)\phi_{\underline{n}}(a)\phi_{\underline{n}}(u_\lambda) = \phi_{\underline{n}}(a),$$

and consequently

$$\phi_{\underline{n}}(a) \in \overline{\operatorname{span}}\{\Theta^{X_{\underline{n}}}_{\xi_{\underline{n}},\eta_{\underline{n}}} \mid \xi_{\underline{n}}, \eta_{\underline{n}} \in IX_{\underline{n}}I\},$$

as claimed. Note also that $\phi_{\underline{n}}(a)|_{IX_{\underline{n}}I} = \phi_{IX_{\underline{n}}I}(a)$ by definition. An application of (2.26) then gives that

$$\widetilde{\psi}_{\underline{n}}(\phi_{IX_{\underline{n}}I}(a)) = \psi^{\mathcal{I}}_{X,\underline{n}}(\phi_{\underline{n}}(a)) \text{ for all } \underline{0} \neq \underline{n} \leq \underline{1}_F.$$

From this we obtain that

$$\begin{aligned}\pi_X^{\mathcal{I}}|_I(a) + \sum\{(-1)^{|\underline{n}|}\widetilde{\psi}_{\underline{n}}(\phi_{IX_{\underline{n}}I}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = \\ = \pi_X^{\mathcal{I}}(a) + \sum\{(-1)^{|\underline{n}|}\psi^{\mathcal{I}}_{X,\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = 0,\end{aligned}$$

where the final equality follows from the fact that $a \in \mathcal{I}_F(X)$, together with the fact that $(\pi_X^{\mathcal{I}}, t_X^{\mathcal{I}})$ is a CNP-representation of X. Thus $(\pi_X^{\mathcal{I}}|_I, t_X^{\mathcal{I}}|_{IXI})$ is a CNP-representation of IXI, as required. Proposition 2.6.7 then completes the proof. □

Chapter 3

Relative 2^d-tuples and relative Cuntz-Nica-Pimsner algebras

Parametrising the gauge-invariant ideals of the Toeplitz-Pimsner algebra of a C*-correspondence is facilitated using relative Cuntz-Pimsner algebras. Hence our first goal is to suitably adapt the relative Cuntz-Pimsner algebra construction for a strong compactly aligned product system X. We do this by mimicking the construction of $\mathcal{NO}_X$, and exploiting strong compact alignment in order to define our relative Cuntz-Nica-Pimsner algebras in terms of simple algebraic relations induced by finitely many subsets of the coefficient algebra. As an intermediate step towards the parametrisation, we study further quotients in-between $\mathcal{NT}_X$ and $\mathcal{NO}_X$.

3.1 Relative 2^d-tuples and induced ideals

We start by paring down the properties of the family $\mathcal{I} := \{\mathcal{I}_F\}_{F\subseteq[d]}$ and the CNP-relations of [17] covered in Section 2.5.

Definition 3.1.1. Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. A *2^d-tuple (of X)* is a family $\mathcal{L} := \{\mathcal{L}_F\}_{F\subseteq[d]}$ such that $\mathcal{L}_F$ is a non-empty subset of A for all $F \subseteq [d]$. A 2^d-tuple $\mathcal{L}$ of X is called *relative* if

$$\mathcal{L}_F \subseteq \bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in F\} \text{ for all } \emptyset \neq F \subseteq [d].$$

Remark 3.1.2. We stipulate that the sets $\mathcal{L}_F$ are non-empty for convenience. More specifically, the sets $\mathcal{L}_F$ are designed to generate certain "relative CNP-relations", and so $\mathcal{L}_F = \{0\}$ plays the same role as $\mathcal{L}_F = \emptyset$. In the former case the relative CNP-relations are satisfied trivially, and in the latter case they are satisfied vacuously. Functionally there is no difference, and it is more convenient to take $\mathcal{L}_F = \{0\}$ as in this case $\mathcal{L}_F$ is an ideal.

We write $\mathcal{L} \subseteq \mathcal{L}'$ for 2^d-tuples $\mathcal{L}$ and $\mathcal{L}'$ if and only if $\mathcal{L}_F \subseteq \mathcal{L}'_F$ for all $F \subseteq [d]$. This defines a partial order on the set of 2^d-tuples of X. We say that $\mathcal{L} = \mathcal{L}'$ if and only if

$\mathcal{L} \subseteq \mathcal{L}'$ and $\mathcal{L}' \subseteq \mathcal{L}$.

Let (π, t) be a Nica-covariant representation of X. The crucial property of a relative 2^d-tuple $\mathcal{L}$ is that

$$\pi(a)q_F = \pi(a) + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \in \mathrm{C}^*(\pi, t) \text{ for all } a \in \mathcal{L}_F, F \subseteq [d],$$

using Proposition 2.5.16. This allows us to extend the ideas of [17] in a natural way.

Definition 3.1.3. Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be a relative 2^d-tuple of X and let (π, t) be a Nica-covariant representation of X. We define the *ideal of the $\mathcal{L}$-relative CNP-relations with respect to* (π, t) to be

$$\mathfrak{J}_{\mathcal{L}}^{(\pi,t)} := \sum\{\mathfrak{J}_{\mathcal{L},F}^{(\pi,t)} \mid F \subseteq [d]\} \subseteq \mathrm{C}^*(\pi, t), \text{ where } \mathfrak{J}_{\mathcal{L},F}^{(\pi,t)} := \langle \pi(\mathcal{L}_F)q_F \rangle \text{ for all } F \subseteq [d].$$

We say that $\mathcal{L}$ *induces* $\mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$.

Being an algebraic sum of ideals in $\mathrm{C}^*(\pi, t)$, the space $\mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$ is itself an ideal in $\mathrm{C}^*(\pi, t)$. It follows that

$$\mathfrak{J}_{\mathcal{L}}^{(\pi,t)} = \langle \pi(\mathcal{L}_F)q_F \mid F \subseteq [d] \rangle .$$

By setting $\mathcal{L} = \mathcal{I}$, we recover $\mathfrak{J}_{\mathcal{I}}^{(\pi,t)}$ as defined in [17].

Proposition 3.1.4. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be a relative 2^d-tuple of X and let (π, t) be a Nica-covariant representation of X that admits a gauge action. Then $\mathfrak{J}_{\mathcal{L},F}^{(\pi,t)}$ is a gauge-invariant ideal of $\mathrm{C}^*(\pi, t)$ for all $F \subseteq [d]$ and hence $\mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$ is also a gauge-invariant ideal of $\mathrm{C}^*(\pi, t)$.*

Proof. Fixing $F \subseteq [d]$, it is immediate that $\mathfrak{J}_{\mathcal{L},F}^{(\pi,t)}$ is an ideal of $\mathrm{C}^*(\pi, t)$ by definition. Let γ denote the gauge action of (π, t). We must show that

$$\gamma_{\underline{z}}(\mathfrak{J}_{\mathcal{L},F}^{(\pi,t)}) \subseteq \mathfrak{J}_{\mathcal{L},F}^{(\pi,t)} \text{ for all } \underline{z} \in \mathbb{T}^d.$$

Since $\gamma_{\underline{z}} \in \mathrm{Aut}(\mathrm{C}^*(\pi, t))$ for all $\underline{z} \in \mathbb{T}^d$ and $\mathfrak{J}_{\mathcal{L},F}^{(\pi,t)}$ is an ideal, it suffices to show that

$$\gamma_{\underline{z}}(\pi(\mathcal{L}_F)q_F) \subseteq \mathfrak{J}_{\mathcal{L},F}^{(\pi,t)} \text{ for all } \underline{z} \in \mathbb{T}^d.$$

Accordingly, fix $\underline{z} \in \mathbb{T}^d$ and $a \in \mathcal{L}_F$. By the comments preceding Definition 3.1.3, we have that

$$\pi(a)q_F = \pi(a) + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}.$$

In turn, we deduce that

$$\gamma_{\underline{z}}(\pi(a)q_F) = \pi(a)q_F \in \mathfrak{J}_{\mathcal{L},F}^{(\pi,t)},$$

as required. Hence $\mathfrak{J}_{\mathcal{L},F}^{(\pi,t)}$ is gauge-invariant, proving the first claim. The second claim follows by definition of $\mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$ together with the first claim, finishing the proof. □

In many cases, we may assume that $\mathcal{L}$ is a family of ideals without loss of generality.

Proposition 3.1.5. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be a relative 2^d-tuple of X and let (π, t) be a Nica-covariant representation of X. Then $\langle \mathcal{L} \rangle := \{\langle \mathcal{L}_F \rangle\}_{F \subseteq [d]}$ is a relative 2^d-tuple of X such that $\mathcal{L} \subseteq \langle \mathcal{L} \rangle$ and $\mathfrak{J}^{(\pi,t)}_{\mathcal{L},F} = \mathfrak{J}^{(\pi,t)}_{\langle \mathcal{L} \rangle,F}$ for all $F \subseteq [d]$. In particular, we have that $\mathfrak{J}^{(\pi,t)}_{\mathcal{L}} = \mathfrak{J}^{(\pi,t)}_{\langle \mathcal{L} \rangle}$.*

Proof. It is immediate that $\langle \mathcal{L} \rangle$ is a relative 2^d-tuple of X, and that $\mathcal{L} \subseteq \langle \mathcal{L} \rangle$. It remains to see that $\mathfrak{J}^{(\pi,t)}_{\mathcal{L},F} = \mathfrak{J}^{(\pi,t)}_{\langle \mathcal{L} \rangle,F}$ for all $F \subseteq [d]$, as the final claim follows as an immediate consequence. To this end, fix $F \subseteq [d]$. Since $\mathcal{L}_F \subseteq \langle \mathcal{L}_F \rangle$, we have that $\mathfrak{J}^{(\pi,t)}_{\mathcal{L},F} \subseteq \mathfrak{J}^{(\pi,t)}_{\langle \mathcal{L} \rangle,F}$. For the reverse inclusion, it suffices to show that

$$\pi(\langle \mathcal{L}_F \rangle) q_F \subseteq \mathfrak{J}^{(\pi,t)}_{\mathcal{L},F}.$$

Recall that $\langle \mathcal{L}_F \rangle = [A \mathcal{L}_F A]$. Thus it suffices to show that

$$\pi(A \mathcal{L}_F A) q_F \subseteq \mathfrak{J}^{(\pi,t)}_{\mathcal{L},F},$$

since π is in particular linear and continuous, and $\mathfrak{J}^{(\pi,t)}_{\mathcal{L},F}$ is in particular a closed linear subspace of $\mathrm{C}^*(\pi, t)$. For $b, c \in A$ and $a \in \mathcal{L}_F$, we have that

$$\pi(bac) q_F = \pi(b)\pi(a)\pi(c) q_F = \pi(b)(\pi(a) q_F)\pi(c) \in \mathfrak{J}^{(\pi,t)}_{\mathcal{L},F},$$

using Proposition 2.5.15 in the second equality, as required. □

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be relative 2^d-tuples of X. It is straightforward to see that their sum $\mathcal{L}_1 + \mathcal{L}_2$, defined by $(\mathcal{L}_1 + \mathcal{L}_2)_F := \mathcal{L}_{1,F} + \mathcal{L}_{2,F}$ for all $F \subseteq [d]$, is also a relative 2^d-tuple. When $\mathcal{L}_1$ and $\mathcal{L}_2$ consist of ideals, summing respects the induced ideals.

Proposition 3.1.6. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be relative 2^d-tuples of X that consist of ideals and let (π, t) be a Nica-covariant representation of X. Then*

$$\mathfrak{J}^{(\pi,t)}_{\mathcal{L}_1+\mathcal{L}_2,F} = \mathfrak{J}^{(\pi,t)}_{\mathcal{L}_1,F} + \mathfrak{J}^{(\pi,t)}_{\mathcal{L}_2,F} \text{ for all } F \subseteq [d], \tag{3.1}$$

and thus $\mathfrak{J}^{(\pi,t)}_{\mathcal{L}_1+\mathcal{L}_2} = \mathfrak{J}^{(\pi,t)}_{\mathcal{L}_1} + \mathfrak{J}^{(\pi,t)}_{\mathcal{L}_2}$.

Proof. It suffices to prove the first claim, as the second then follows by definition. Accordingly, fix $F \subseteq [d]$. For the forward inclusion, it suffices to show that $\mathfrak{J}^{(\pi,t)}_{\mathcal{L}_1,F} + \mathfrak{J}^{(\pi,t)}_{\mathcal{L}_2,F}$ contains the generators of $\mathfrak{J}^{(\pi,t)}_{\mathcal{L}_1+\mathcal{L}_2,F}$, as both are ideals of $\mathrm{C}^*(\pi, t)$. To this end, fix $a \in \mathcal{L}_{1,F} + \mathcal{L}_{2,F}$. Then $a = b + c$ for some $b \in \mathcal{L}_{1,F}$ and $c \in \mathcal{L}_{2,F}$. We obtain that

$$\pi(a) q_F = \pi(b + c) q_F = \pi(b) q_F + \pi(c) q_F \in \mathfrak{J}^{(\pi,t)}_{\mathcal{L}_1,F} + \mathfrak{J}^{(\pi,t)}_{\mathcal{L}_2,F},$$

as required.

For the reverse inclusion, fix $a \in \mathcal{L}_{1,F}$. We have that

$$\pi(a)q_F = \pi(a+0)q_F \in \mathfrak{J}^{(\pi,t)}_{\mathcal{L}_1+\mathcal{L}_2,F},$$

using that $0 \in \mathcal{L}_{2,F}$ since $\mathcal{L}_2$ consists of ideals. It follows that $\mathfrak{J}^{(\pi,t)}_{\mathcal{L}_1,F} \subseteq \mathfrak{J}^{(\pi,t)}_{\mathcal{L}_1+\mathcal{L}_2,F}$. By symmetry we also have that $\mathfrak{J}^{(\pi,t)}_{\mathcal{L}_2,F} \subseteq \mathfrak{J}^{(\pi,t)}_{\mathcal{L}_1+\mathcal{L}_2,F}$. Hence we obtain the reverse inclusion of (3.1), finishing the proof. □

A first approach towards the parametrisation of the gauge-invariant ideals of $\mathcal{NT}_X$ would be to establish a correspondence between the relative 2^d-tuples of X and the gauge-invariant ideals of $\mathcal{NT}_X$ that they induce. However, this is insufficient as different relative 2^d-tuples may induce the same gauge-invariant ideal of $\mathcal{NT}_X$. We provide an example to this effect.

Remark 3.1.7. Let $\mathcal{L}$ be a relative 2^d-tuple of X that consists of ideals and (π, t) be a Nica-covariant representation. Then the relative 2^d-tuple $\mathcal{L}'$ defined by

$$\mathcal{L}'_F := \begin{cases} \mathcal{L}_\emptyset & \text{if } F = \emptyset, \\ \mathcal{L}_F + \mathcal{L}_\emptyset \cap (\bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in F\}) & \text{if } \emptyset \neq F \subseteq [d], \end{cases}$$

satisfies $\mathfrak{J}^{(\pi,t)}_{\mathcal{L}'} = \mathfrak{J}^{(\pi,t)}_{\mathcal{L}}$. Indeed, on one hand we have that $\mathfrak{J}^{(\pi,t)}_{\mathcal{L}} \subseteq \mathfrak{J}^{(\pi,t)}_{\mathcal{L}'}$ since $\mathcal{L} \subseteq \mathcal{L}'$. On the other hand, fix $\emptyset \neq F \subseteq [d]$ and $a \in \mathcal{L}_\emptyset$ satisfying $\phi_{\underline{i}}(a) \in \mathcal{K}(X_{\underline{i}})$ for all $i \in F$. We claim that

$$\psi_{\underline{n}}(\phi_{\underline{n}}(a)) \in \langle \pi(\mathcal{L}_\emptyset) \rangle = \mathfrak{J}^{(\pi,t)}_{\mathcal{L},\emptyset} \text{ for all } \underline{0} \neq \underline{n} \leq \underline{1}_F.$$

We will prove this by induction on $|\underline{n}|$. When $|\underline{n}| = 1$, we must have that $\underline{n} = \underline{i}$ for some $i \in F$. In turn, for an approximate unit $(k_{\underline{i},\lambda})_{\lambda\in\Lambda} \subseteq \mathcal{K}(X_{\underline{i}})$, we have that

$$\psi_{\underline{i}}(\phi_{\underline{i}}(a)) = \|\cdot\| \text{-} \lim_\lambda \psi_{\underline{i}}(\phi_{\underline{i}}(a)k_{\underline{i},\lambda}) = \|\cdot\| \text{-} \lim_\lambda \pi(a)\psi_{\underline{i}}(k_{\underline{i},\lambda}) \in \langle \pi(\mathcal{L}_\emptyset) \rangle,$$

as required. Now suppose that the claim holds for all $\underline{n}$ such that $\underline{0} \neq \underline{n} \leq \underline{1}_F$ and $|\underline{n}| = N$ for some $1 \leq N < |F|$. Fix $\underline{m}$ such that $\underline{0} \neq \underline{m} \leq \underline{1}_F$ and $|\underline{m}| = N + 1$. We may express $\underline{m}$ in the form $\underline{m} = \underline{n} + \underline{i}$ for some $\underline{0} \neq \underline{n} \leq \underline{1}_F$ and $i \in F$ such that $\underline{n} \perp \underline{i}$ and $|\underline{n}| = N$. We have that

$$\psi_{\underline{m}}(\phi_{\underline{m}}(a)) = \|\cdot\| \text{-} \lim_\lambda \psi_{\underline{m}}(\iota_{\underline{n}}^{\underline{m}}(\phi_{\underline{n}}(a))\iota_{\underline{i}}^{\underline{m}}(k_{\underline{i},\lambda})) = \|\cdot\| \text{-} \lim_\lambda \psi_{\underline{n}}(\phi_{\underline{n}}(a))\psi_{\underline{i}}(k_{\underline{i},\lambda}) \in \langle \pi(\mathcal{L}_\emptyset) \rangle,$$

by the inductive hypothesis, Proposition 2.5.13, and Nica-covariance (noting that $\underline{n} \vee \underline{i} = \underline{n} + \underline{i} = \underline{m}$ since $\underline{n} \perp \underline{i}$). This finishes the proof of the claim. Therefore, we have that

$$\pi(a)q_F = \pi(a) + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \in \langle \pi(\mathcal{L}_\emptyset) \rangle \subseteq \mathfrak{J}^{(\pi,t)}_{\mathcal{L}}.$$

Hence

$$\pi\left(\mathcal{L}_\emptyset \cap (\bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in F\})\right) q_F \subseteq \mathfrak{J}^{(\pi,t)}_{\mathcal{L}},$$

from which it follows that $\mathfrak{J}_{\mathcal{L}'}^{(\pi,t)} \subseteq \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$, as required.

However, notice that $\mathcal{L}$ and $\mathcal{L}'$ may differ in general. For example, assume that the left actions of the fibres of X are by compact operators. Then any 2^d-tuple of X is automatically relative. Let $\mathcal{L}_\emptyset$ be a non-zero ideal and set $\mathcal{L}_F = \{0\}$ for all $\emptyset \neq F \subseteq [d]$. Then $\mathcal{L} \neq \mathcal{L}'$, as claimed.

To remedy this issue, we will instead look for the largest relative 2^d-tuple inducing a fixed gauge-invariant ideal of $\mathcal{NT}_X$.

Definition 3.1.8. Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{M}$ be a relative 2^d-tuple of X and let (π, t) be a Nica-covariant representation of X. We say that $\mathcal{M}$ is a *maximal* 2^d*-tuple (of* X*) with respect to* (π, t) if whenever $\mathcal{L}$ is a relative 2^d-tuple of X such that $\mathfrak{J}_{\mathcal{M}}^{(\pi,t)} = \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$ and $\mathcal{M} \subseteq \mathcal{L}$, we have that $\mathcal{M} = \mathcal{L}$. When we replace (π, t) by $(\overline{\pi}_X, \overline{t}_X)$, we will refer to $\mathcal{M}$ simply as a *maximal* 2^d*-tuple (of* X*)*.

The following proposition establishes existence and uniqueness of maximal 2^d-tuples.

Proposition 3.1.9. *Let* X *be a strong compactly aligned product system with coefficients in a* C^**-algebra* A*. Let* $\mathcal{L}$ *be a relative* 2^d*-tuple of* X *and let* (π, t) *be a Nica-covariant representation of* X*. Then there exists a unique relative* 2^d*-tuple* $\mathcal{M}$ *of* X *such that:*

(i) $\mathfrak{J}_{\mathcal{L}}^{(\pi,t)} = \mathfrak{J}_{\mathcal{M}}^{(\pi,t)}$*, and*

(ii) $\mathcal{L}' \subseteq \mathcal{M}$ *for every relative* 2^d*-tuple* $\mathcal{L}'$ *of* X *satisfying* $\mathfrak{J}_{\mathcal{L}'}^{(\pi,t)} = \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$.

In particular, $\mathcal{M}$ *is a maximal* 2^d*-tuple of* X *with respect to* (π, t) *which consists of ideals.*

Proof. For each $F \subseteq [d]$, define

$$\mathcal{M}_F := \bigcup \{\mathcal{L}'_F \mid \mathcal{L}' \text{ is a relative } 2^d\text{-tuple of } X \text{ such that } \mathfrak{J}_{\mathcal{L}'}^{(\pi,t)} = \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}\}.$$

The union is well-defined, since it includes $\mathcal{L}_F$ by assumption, and the index takes values in the set $\mathcal{P}(A)^{2^d}$. Each $\mathcal{M}_F$ is a non-empty subset of A by construction, so $\mathcal{M}$ is a 2^d-tuple of X. Fix $\emptyset \neq F \subseteq [d]$ and take $a \in \mathcal{M}_F$. Then $a \in \mathcal{L}'_F$ for some relative 2^d-tuple $\mathcal{L}'$, and hence $\phi_{\underline{i}}(a) \in \mathcal{K}(X_{\underline{i}})$ for all $i \in F$. Therefore, setting $\mathcal{M} := \{\mathcal{M}_F\}_{F \subseteq [d]}$, we have that $\mathcal{M}$ is a relative 2^d-tuple of X.

Because $\mathcal{L} \subseteq \mathcal{M}$, we have that $\mathfrak{J}_{\mathcal{L}}^{(\pi,t)} \subseteq \mathfrak{J}_{\mathcal{M}}^{(\pi,t)}$ trivially. For the other inclusion, fix $F \subseteq [d]$ and $a \in \mathcal{M}_F$. It suffices to show that $\pi(a)q_F \in \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$. To this end, we have that $a \in \mathcal{L}'_F$ for some relative 2^d-tuple $\mathcal{L}'$ with the property that $\mathfrak{J}_{\mathcal{L}'}^{(\pi,t)} = \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$. By definition, we have that $\pi(a)q_F \in \mathfrak{J}_{\mathcal{L}'}^{(\pi,t)} = \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$, as required.

We have that $\mathcal{L}' \subseteq \mathcal{M}$ for every relative 2^d-tuple $\mathcal{L}'$ satisfying $\mathfrak{J}_{\mathcal{L}'}^{(\pi,t)} = \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$ by construction. It follows that $\mathcal{M}$ is maximal with respect to (π, t). For uniqueness, suppose that we have another relative 2^d-tuple $\mathcal{M}'$ satisfying (i) and (ii). Then in particular we have that $\mathcal{M} \subseteq \mathcal{M}'$ and that $\mathcal{M}' \subseteq \mathcal{M}$ using the corresponding properties of $\mathcal{M}$. Hence $\mathcal{M} = \mathcal{M}'$, showing that $\mathcal{M}$ is unique.

Finally, by Proposition 3.1.5 we have that $\langle \mathcal{M} \rangle := \{\langle \mathcal{M}_F \rangle\}_{F \subseteq [d]}$ is a relative 2^d-tuple such that $\mathcal{M} \subseteq \langle \mathcal{M} \rangle$ and $\mathfrak{J}_{\mathcal{M}}^{(\pi,t)} = \mathfrak{J}_{\langle \mathcal{M} \rangle}^{(\pi,t)}$. Applying maximality of $\mathcal{M}$, we have that $\mathcal{M} = \langle \mathcal{M} \rangle$. This shows that $\mathcal{M}$ consists of ideals, finishing the proof. □

Remark 3.1.10. The motivating example of a maximal relative 2^d-tuple is the family $\mathcal{I}$. To see this, first note that $\mathcal{I}$ is a relative 2^d-tuple by definition. Let $\mathcal{M}$ be the maximal 2^d-tuple inducing $\mathfrak{J}_{\mathcal{I}}^{(\overline{\pi}_X, \overline{t}_X)}$, so that $\mathcal{I} \subseteq \mathcal{M}$. If $a \in \mathcal{M}_F$ for $F \subseteq [d]$, then

$$\overline{\pi}_X(a)\overline{q}_{X,F} \in \mathfrak{J}_{\mathcal{M}}^{(\overline{\pi}_X, \overline{t}_X)} = \mathfrak{J}_{\mathcal{I}}^{(\overline{\pi}_X, \overline{t}_X)}.$$

Thus, letting $Q \colon \mathcal{NT}_X \to \mathcal{NO}_X$ denote the quotient map, we obtain that

$$\begin{aligned} &\pi_X^{\mathcal{I}}(a) + \sum\{(-1)^{|\underline{n}|}\psi_{X,\underline{n}}^{\mathcal{I}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = \\ &\quad = Q(\overline{\pi}_X(a) + \sum\{(-1)^{|\underline{n}|}\overline{\psi}_{X,\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}) = Q(\overline{\pi}_X(a)\overline{q}_{X,F}) = 0, \end{aligned}$$

using Proposition 2.5.16 in the second equality. Hence $\pi_X^{\mathcal{I}}(a) \in B_{(\underline{0}, \underline{1}_F]}^{(\pi_X^{\mathcal{I}}, t_X^{\mathcal{I}})}$ and an application of Proposition 2.5.19 gives that $a \in \mathcal{I}_F$. Note that the latter applies since $(\pi_X^{\mathcal{I}}, t_X^{\mathcal{I}})$ is injective. Therefore $\mathcal{M} \subseteq \mathcal{I}$, and in total $\mathcal{M} = \mathcal{I}$.

We wish to ascertain the conditions under which a relative 2^d-tuple is promoted to a maximal one. Using $\mathcal{I}$ as a prototype, we abstract two of its properties.

Definition 3.1.11. Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be a 2^d-tuple of X.

(i) We say that $\mathcal{L}$ is X-*invariant* if $[\langle X_{\underline{n}}, \mathcal{L}_F X_{\underline{n}} \rangle] \subseteq \mathcal{L}_F$, for all $\underline{n} \perp F$, $F \subseteq [d]$.

(ii) We say that $\mathcal{L}$ is *partially ordered* if $\mathcal{L}_{F_1} \subseteq \mathcal{L}_{F_2}$ whenever $F_1 \subseteq F_2 \subseteq [d]$.

When the underlying product system X is clear from the context, we will abbreviate "X-invariant" as simply "invariant". Notice that when we take $F = \emptyset$, condition (i) implies that $\mathcal{L}_\emptyset$ is positively invariant for X (provided that $\mathcal{L}_\emptyset$ is an ideal). If $\mathcal{L}_F$ is an ideal, then we may drop the closed linear span in condition (i). If $\mathcal{L}$ is a partially ordered relative 2^d-tuple, then

$$\mathcal{L}_F \subseteq \mathcal{L}_{[d]} \subseteq \bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in [d]\} \text{ for all } F \subseteq [d].$$

In particular, fixing a Nica-covariant representation (π, t), we have that $\pi(a)q_F \in \mathrm{C}^*(\pi, t)$ for all $a \in \mathcal{L}_D$ and $D, F \subseteq [d]$ by Proposition 2.5.16.

The 2^d-tuple $\mathcal{I}$ is invariant, partially ordered and consists of ideals. Since $\mathcal{I}$ is maximal by Remark 3.1.10, one may be tempted to assert that invariance and partial ordering suffice to capture maximality. However, this is not true in general, as the following counterexample shows.

Example 3.1.12. Let B be a non-zero C*-algebra and let $A = B \oplus \mathbb{C}$ be its unitisation. Consider the semigroup action $\alpha\colon \mathbb{Z}_+^2 \to \mathrm{End}(A)$ given by

$$\alpha_{(m,n)}(b,\lambda) = \begin{cases} (0,\lambda) & \text{if } n \geq 1, \\ (b,\lambda) & \text{otherwise,} \end{cases}$$

for all $(b,\lambda) \in A$. Applying the construction of Section 5.3 to the C*-dynamical system $(A, \alpha, \mathbb{Z}_+^2)$, we obtain a strong compactly aligned product system X_α over $\mathbb{Z}_+^2$ with coefficients in A. In particular, the left action of every fibre is by compacts, and so any 2^2-tuple of X_α is automatically a relative 2^2-tuple.

Next we define the relative 2^2-tuples $\mathcal{L}$ and $\mathcal{L}'$ of X_α by

$$\begin{array}{ccc} \mathcal{L}_{\{2\}} = \{0\} & \text{———} & \mathcal{L}_{\{1,2\}} = B \oplus \{0\} \\ | & & | \\ \mathcal{L}_{\emptyset} = \{0\} & \text{———} & \mathcal{L}_{\{1\}} = \{0\} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{L}'_{\{2\}} = \{0\} & \text{———} & \mathcal{L}'_{\{1,2\}} = B \oplus \{0\} \\ | & & | \\ \mathcal{L}'_{\emptyset} = \{0\} & \text{———} & \mathcal{L}'_{\{1\}} = B \oplus \{0\}. \end{array}$$

Notice that $\mathcal{L}$ and $\mathcal{L}'$ are invariant, partially ordered and consist of ideals. Let (π, t) be a Nica-covariant representation. Since $\mathcal{L} \subseteq \mathcal{L}'$, it is clear that $\mathfrak{J}_{\mathcal{L}}^{(\pi,t)} \subseteq \mathfrak{J}_{\mathcal{L}'}^{(\pi,t)}$. It is immediate that $\pi(\mathcal{L}'_F)q_F \subseteq \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$ for all $F \in \{\emptyset, \{2\}, \{1,2\}\}$. Fix $(b,0) \in \mathcal{L}'_{\{1\}}$ and observe that

$$\begin{aligned} \mathfrak{J}_{\mathcal{L}}^{(\pi,t)} \ni \pi(b,0)q_{\{1,2\}} &= \pi(b,0) - \psi_{(1,0)}(\phi_{(1,0)}(b,0)) - \psi_{(0,1)}(\phi_{(0,1)}(b,0)) + \psi_{(1,1)}(\phi_{(1,1)}(b,0)) \\ &= \pi(b,0) - \psi_{(1,0)}(\phi_{(1,0)}(b,0)) = \pi(b,0)q_{\{1\}}, \end{aligned}$$

using the fact that $\phi_{(0,1)}(b,0) = \phi_{(1,1)}(b,0) = 0$ by definition. Therefore we deduce that $\pi(\mathcal{L}'_{\{1\}})q_{\{1\}} \subseteq \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$, and thus $\mathfrak{J}_{\mathcal{L}'}^{(\pi,t)} \subseteq \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$. However, the relative 2^2-tuple $\mathcal{L}$ is not maximal with respect to (π,t) since $\mathcal{L} \subsetneq \mathcal{L}'$.

To better understand maximality, we need to explore 2^d-tuples induced by Nica-covariant representations, demonstrating the conditions of Definition 3.1.11 in action.

Definition 3.1.13. Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let (π,t) be a Nica-covariant representation of X. We define $\mathcal{L}^{(\pi,t)}$ to be the 2^d-tuple of X given by

$$\mathcal{L}_{\emptyset}^{(\pi,t)} := \ker \pi \quad \text{and} \quad \mathcal{L}_F^{(\pi,t)} := \pi^{-1}(B_{(\underline{0},\underline{1}_F]}^{(\pi,t)}) \text{ for all } \emptyset \neq F \subseteq [d].$$

Proposition 3.1.14. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let (π,t) be a Nica-covariant representation of X. Then $\mathcal{L}^{(\pi,t)}$ is invariant, partially ordered and consists of ideals.*

If (π,t) is in addition injective, then $\mathcal{L}^{(\pi,t)} \subseteq \mathcal{I}$, and thus $\mathcal{L}^{(\pi,t)}$ is relative.

Proof. It is clear that $\mathcal{L}_{\emptyset}^{(\pi,t)}$ is an ideal, so fix $\emptyset \neq F \subseteq [d]$. The fact that $\mathcal{L}_F^{(\pi,t)}$ is a C*-subalgebra of A follows from continuity of π and Proposition 2.4.7. Let $a \in \mathcal{L}_F^{(\pi,t)}$ and

$b \in A$. Then

$$\pi(a) = \sum\{\psi_{\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}$$

for some $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ for all $\underline{0} \neq \underline{n} \leq \underline{1}_F$. We have that

$$\pi(ab) = \pi(a)\pi(b) = \sum\{\psi_{\underline{n}}(k_{\underline{n}})\pi(b) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = \sum\{\psi_{\underline{n}}(k_{\underline{n}}\phi_{\underline{n}}(b)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}.$$

Hence $ab \in \mathcal{L}_F^{(\pi,t)}$, and a similar argument shows that $ba \in \mathcal{L}_F^{(\pi,t)}$. Hence $\mathcal{L}^{(\pi,t)}$ consists of ideals.

Next fix $F \subseteq [d], a \in \mathcal{L}_F^{(\pi,t)}, \underline{m} \perp F$ and $\xi_{\underline{m}}, \eta_{\underline{m}} \in X_{\underline{m}}$. By definition we may write

$$\pi(a) = \sum\{\psi_{\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}$$

for some $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ for all $\underline{0} \neq \underline{n} \leq \underline{1}_F$, with the convention that the right hand side is 0 when $F = \emptyset$. We obtain that

$$\begin{aligned}\pi(\langle \xi_{\underline{m}}, a\eta_{\underline{m}}\rangle) &= t_{\underline{m}}(\xi_{\underline{m}})^*\pi(a)t_{\underline{m}}(\eta_{\underline{m}}) \\ &= \sum\{t_{\underline{m}}(\xi_{\underline{m}})^*\psi_{\underline{n}}(k_{\underline{n}})t_{\underline{m}}(\eta_{\underline{m}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}.\end{aligned}$$

Now fix $\underline{0} \neq \underline{n} \leq \underline{1}_F$. We have that

$$\begin{aligned}t_{\underline{m}}(X_{\underline{m}})^*\psi_{\underline{n}}(\mathcal{K}(X_{\underline{n}}))t_{\underline{m}}(X_{\underline{m}}) &\subseteq [t_{\underline{m}}(X_{\underline{m}})^*t_{\underline{n}}(X_{\underline{n}})t_{\underline{n}}(X_{\underline{n}})^*t_{\underline{m}}(X_{\underline{m}})] \\ &\subseteq [t_{\underline{n}}(X_{\underline{n}})t_{\underline{n}}(X_{\underline{n}})^*] = \psi_{\underline{n}}(\mathcal{K}(X_{\underline{n}})),\end{aligned}$$

using Nica-covariance and the fact that $\underline{m} \vee \underline{n} = \underline{m} + \underline{n}$ (since $\underline{m} \perp \underline{n}$) in the second inclusion. Consequently, we deduce that

$$t_{\underline{m}}(\xi_{\underline{m}})^*\psi_{\underline{n}}(k_{\underline{n}})t_{\underline{m}}(\eta_{\underline{m}}) \in \psi_{\underline{n}}(\mathcal{K}(X_{\underline{n}})) \subseteq B_{(\underline{0},\underline{1}_F]}^{(\pi,t)} \text{ for all } \underline{0} \neq \underline{n} \leq \underline{1}_F.$$

In turn, it follows that

$$\pi(\langle \xi_{\underline{m}}, a\eta_{\underline{m}}\rangle) = \sum\{t_{\underline{m}}(\xi_{\underline{m}})^*\psi_{\underline{n}}(k_{\underline{n}})t_{\underline{m}}(\eta_{\underline{m}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \in B_{(\underline{0},\underline{1}_F]}^{(\pi,t)},$$

and hence $\langle \xi_{\underline{m}}, a\eta_{\underline{m}}\rangle \in \mathcal{L}_F^{(\pi,t)}$. This proves that $\mathcal{L}^{(\pi,t)}$ is invariant. The fact that $\mathcal{L}^{(\pi,t)}$ is partially ordered follows from the observation that

$$B_{(\underline{0},\underline{1}_F]}^{(\pi,t)} \subseteq B_{(\underline{0},\underline{1}_D]}^{(\pi,t)} \text{ for all } F \subseteq D \subseteq [d].$$

The final claim follows by Proposition 2.5.19, finishing the proof. □

In order to make further progress with maximality, we use relative 2^d-tuples to define relative Cuntz-Nica-Pimsner algebras.

Definition 3.1.15. Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be a relative 2^d-tuple of X and let (π, t) be a Nica-covariant representation of X. We say that (π, t) is an *$\mathcal{L}$-relative CNP-representation (of X)* if it satisfies

$$\pi(\mathcal{L}_F)q_F = \{0\} \text{ for all } F \subseteq [d].$$

The universal C*-algebra with respect to the $\mathcal{L}$-relative CNP-representations of X is denoted by $\mathcal{NO}(\mathcal{L}, X)$, and we refer to it as the *$\mathcal{L}$-relative Cuntz-Nica-Pimsner algebra (of X)*.

We have that (π, t) is an $\mathcal{L}$-relative CNP-representation if and only if $\mathfrak{J}_{\mathcal{L}}^{(\pi,t)} = \{0\}$; equivalently if $\pi \times t$ factors through $\mathcal{NO}(\mathcal{L}, X)$ so that the diagram

$$\begin{array}{ccc} \mathcal{NT}_X & \longrightarrow & \mathrm{C}^*(\pi, t) \\ & \searrow \quad \nearrow & \\ & \mathcal{NO}(\mathcal{L}, X) & \end{array}$$

of canonical $*$-epimorphisms is commutative. The following proposition is the analogue of Proposition 2.3.3 for relative Cuntz-Nica-Pimsner algebras, and shows that $\mathcal{NO}(\mathcal{L}, X)$ exists and admits a universal property.

Proposition 3.1.16. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let $\mathcal{L}$ be a relative 2^d-tuple of X. Then there exists a C*-algebra $\mathcal{NO}(\mathcal{L}, X)$ and an $\mathcal{L}$-relative CNP-representation $(\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}})$ of X on $\mathcal{NO}(\mathcal{L}, X)$ such that:*

(i) $\mathcal{NO}(\mathcal{L}, X) = \mathrm{C}^*(\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}})$;

(ii) if (π, t) is an $\mathcal{L}$-relative CNP-representation of X, then there exists a (unique) canonical $*$-epimorphism $\Phi \colon \mathcal{NO}(\mathcal{L}, X) \to \mathrm{C}^*(\pi, t)$.

The pair $(\mathcal{NO}(\mathcal{L}, X), (\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}}))$ is unique up to canonical $*$-isomorphism.

Proof. We define

$$\mathcal{NO}(\mathcal{L}, X) := \mathcal{NT}_X / \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)}.$$

Let $Q \colon \mathcal{NT}_X \to \mathcal{NO}(\mathcal{L}, X)$ be the quotient map and set

$$\pi_X^{\mathcal{L}} := Q \circ \overline{\pi}_X \quad \text{and} \quad t_{X,\underline{n}}^{\mathcal{L}} := Q \circ \overline{t}_{X,\underline{n}} \text{ for all } \underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}.$$

It is routine to check that $(\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}})$ is a representation. Next, we have that

$$\psi_{X,\underline{n}}^{\mathcal{L}}(\Theta_{\xi_{\underline{n}},\eta_{\underline{n}}}) = t_{X,\underline{n}}^{\mathcal{L}}(\xi_{\underline{n}}) t_{X,\underline{n}}^{\mathcal{L}}(\eta_{\underline{n}})^* = Q(\overline{t}_{X,\underline{n}}(\xi_{\underline{n}})) Q(\overline{t}_{X,\underline{n}}(\eta_{\underline{n}}))^* = Q(\overline{\psi}_{X,\underline{n}}(\Theta_{\xi_{\underline{n}},\eta_{\underline{n}}}))$$

for all $\underline{n} \in \mathbb{Z}_+^d$ and $\xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\underline{n}}$. It follows that $\psi_{X,\underline{n}}^{\mathcal{L}} = Q \circ \overline{\psi}_{X,\underline{n}}$ for all $\underline{n} \in \mathbb{Z}_+^d$. Hence, fixing $\underline{n}, \underline{m} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$, $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ and $k_{\underline{m}} \in \mathcal{K}(X_{\underline{m}})$, we obtain that

$$\begin{aligned} \psi_{X,\underline{n}}^{\mathcal{L}}(k_{\underline{n}}) \psi_{X,\underline{m}}^{\mathcal{L}}(k_{\underline{m}}) &= Q(\overline{\psi}_{X,\underline{n}}(k_{\underline{n}})) Q(\overline{\psi}_{X,\underline{m}}(k_{\underline{m}})) = Q(\overline{\psi}_{X,\underline{n}}(k_{\underline{n}}) \overline{\psi}_{X,\underline{m}}(k_{\underline{m}})) \\ &= Q(\overline{\psi}_{X,\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(k_{\underline{n}}) \iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(k_{\underline{m}}))) = \psi_{X,\underline{n}\vee\underline{m}}^{\mathcal{L}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(k_{\underline{n}}) \iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(k_{\underline{m}})), \end{aligned}$$

using Nica-covariance of $(\overline{\pi}_X, \overline{t}_X)$ in the third equality. Hence $(\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}})$ is Nica-covariant. Next, fix $F \subseteq [d]$ and $a \in \mathcal{L}_F$. We have that

$$\begin{aligned} \pi_X^{\mathcal{L}}(a) q_{X,F}^{\mathcal{L}} &= \pi_X^{\mathcal{L}}(a) + \sum \{(-1)^{|\underline{n}|} \psi_{X,\underline{n}}^{\mathcal{L}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \\ &= Q(\overline{\pi}_X(a) + \sum \{(-1)^{|\underline{n}|} \overline{\psi}_{X,\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}) \\ &= Q(\overline{\pi}_X(a) \overline{q}_{X,F}) = 0, \end{aligned}$$

using Proposition 2.5.16 in the first and third equalities and the definition of $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)}$ in the final equality. This shows that $(\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}})$ is an $\mathcal{L}$-relative CNP-representation of X.

Observe that

$$\mathcal{NO}(\mathcal{L}, X) = Q(\mathcal{NT}_X) = Q(\mathrm{C}^*(\overline{\pi}_X, \overline{t}_X)) = \mathrm{C}^*(\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}}),$$

showing that property (i) holds. Now let (π, t) be an $\mathcal{L}$-relative CNP-representation of X. Universality of $\mathcal{NT}_X$ yields a (unique) canonical $*$-epimorphism $\pi \times t \colon \mathcal{NT}_X \to \mathrm{C}^*(\pi, t)$. We have that

$$(\pi \times t)(\overline{\psi}_{X,\underline{n}}(k_{\underline{n}})) = \psi_{\underline{n}}(k_{\underline{n}}) \text{ for all } k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}), \underline{n} \in \mathbb{Z}_+^d.$$

Hence, fixing $F \subseteq [d]$ and $a \in \mathcal{L}_F$, we obtain that

$$(\pi \times t)(\overline{\pi}_X(a) \overline{q}_{X,F}) = \pi(a) q_F = 0,$$

using Proposition 2.5.16 in the first equality and the fact that (π, t) is an $\mathcal{L}$-relative CNP-representation in the second. It follows that $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)} \subseteq \ker \pi \times t$. Hence $\pi \times t$ descends to a canonical $*$-epimorphism

$$\Phi \colon \mathcal{NO}(\mathcal{L}, X) \to \mathrm{C}^*(\pi, t).$$

Canonicity ensures that Φ is unique, proving that property (ii) holds.

Finally, suppose that C is a C*-algebra and (π, t) is an $\mathcal{L}$-relative CNP-representation of X such that $(C, (\pi, t))$ satisfies properties (i) and (ii). An application of property (ii) for $(\mathcal{NO}(\mathcal{L}, X), (\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}}))$ and property (i) for $(C, (\pi, t))$ provides a canonical $*$-epimorphism

$$\Phi \colon \mathcal{NO}(\mathcal{L}, X) \to C.$$

An application of property (i) for $(\mathcal{NO}(\mathcal{L}, X), (\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}}))$ and property (ii) for $(C, (\pi, t))$ provides a canonical $*$-epimorphism

$$\Psi \colon C \to \mathcal{NO}(\mathcal{L}, X).$$

It follows that Φ and Ψ are mutually inverse $*$-homomorphisms and hence $\mathcal{NO}(\mathcal{L}, X) \cong C$ canonically, finishing the proof. □

We will refer to the representation $(\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}})$ as the *universal $\mathcal{L}$-relative CNP-representation (of X)*. Since $\mathcal{NO}(\mathcal{L}, X)$ is an equivariant quotient of $\mathcal{NT}_X$, the representation $(\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}})$ admits a gauge action. Notice that $\{\{0\}\}_{F\subseteq[d]}$ and $\mathcal{I}$ both constitute relative 2^d-tuples of X and in particular

$$\mathcal{NO}(\{\{0\}\}_{F\subseteq[d]}, X) = \mathcal{NT}_X \quad \text{and} \quad \mathcal{NO}(\mathcal{I}, X) = \mathcal{NO}_X.$$

Likewise, when X is a C*-correspondence we recover the relative Cuntz-Pimsner algebras. As per the comments of Remark 2.2.4, there is a question of whether or not we should define the relative Cuntz-Nica-Pimsner algebras with respect to ideals of A rather than just subsets. The two versions are equivalent, since $\mathcal{NO}(\mathcal{L}, X) = \mathcal{NO}(\langle\mathcal{L}\rangle, X)$ whenever $\mathcal{L}$ is a relative 2^d-tuple of X by Proposition 3.1.5.

If (π, t) is an injective Nica-covariant representation, then $\mathcal{L}^{(\pi,t)}$ is a relative 2^d-tuple by Proposition 3.1.14, and thus we can make sense of $\mathfrak{J}_{\mathcal{L}^{(\pi,t)}}^{(\overline{\pi}_X,\overline{t}_X)}$. The next proposition shows that $\mathfrak{J}_{\mathcal{L}^{(\pi,t)}}^{(\overline{\pi}_X,\overline{t}_X)}$ arises as a kernel when (π, t) admits a gauge action.

Proposition 3.1.17. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let (π, t) be an injective Nica-covariant representation of X. Then (π, t) is an $\mathcal{L}^{(\pi,t)}$-relative CNP-representation of X, and consequently there exists a (unique) canonical $*$-epimorphism*

$$\Phi\colon \mathcal{NO}(\mathcal{L}^{(\pi,t)}, X) \to \mathrm{C}^*(\pi, t).$$

Moreover, Φ is injective if and only if (π, t) admits a gauge action, in which case

$$\ker \pi \times t = \mathfrak{J}_{\mathcal{L}^{(\pi,t)}}^{(\overline{\pi}_X,\overline{t}_X)}.$$

Proof. Fix $F \subseteq [d]$ and $a \in \mathcal{L}_F^{(\pi,t)} = \pi^{-1}(B_{(\underline{0},\underline{1}_F]}^{(\pi,t)})$. Then $\pi(a)q_F = 0$ by Proposition 2.5.17. This shows that (π, t) is an $\mathcal{L}^{(\pi,t)}$-relative CNP-representation. Thus universality of $\mathcal{NO}(\mathcal{L}^{(\pi,t)}, X)$ guarantees the existence of Φ.

For the second claim, we write $(\widetilde{\pi}, \widetilde{t})$ for $(\pi_X^{\mathcal{L}^{(\pi,t)}}, t_X^{\mathcal{L}^{(\pi,t)}})$, for notational convenience. Assume that Φ is injective. Let β denote the gauge action of $(\widetilde{\pi}, \widetilde{t})$. We define

$$\gamma_{\underline{z}} := \Phi \circ \beta_{\underline{z}} \circ \Phi^{-1} \in \mathrm{Aut}(\mathrm{C}^*(\pi, t)) \text{ for all } \underline{z} \in \mathbb{T}^d.$$

It is routine to check that γ defines a gauge action of (π, t), as required.

Conversely, suppose that (π, t) admits a gauge action γ. Note that Φ intertwines the gauge actions in the sense that

$$\Phi \circ \beta_{\underline{z}} = \gamma_{\underline{z}} \circ \Phi \text{ for all } \underline{z} \in \mathbb{T}^d.$$

Thus Φ is an equivariant $*$-homomorphism and so we may apply [8, Proposition 4.5.1] to deduce that Φ is injective if and only if it is injective on $\mathcal{NO}(\mathcal{L}^{(\pi,t)}, X)^\beta = B_{[\underline{0},\infty]}^{(\widetilde{\pi},\widetilde{t})}$, using Proposition 2.5.5 in the equality. In turn, it suffices to show that Φ is injective on the $[\underline{0}, \underline{1}_{[d]}]$-core by Proposition 2.5.21. To reach contradiction, suppose that there exists $0 \neq f \in \ker \Phi \cap B^{(\widetilde{\pi},\widetilde{t})}_{[\underline{0},\underline{1}_{[d]}]}$, so that we may write

$$f = \widetilde{\pi}(a) + \sum \{\widetilde{\psi}_{\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_{[d]}\},$$

for some $a \in A$ and $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ for all $\underline{0} \neq \underline{n} \leq \underline{1}_{[d]}$. Recall that we write $\widetilde{\pi}(a) = \widetilde{\psi}_{\underline{0}}(k_{\underline{0}})$ for $k_{\underline{0}} := \phi_{\underline{0}}(a)$. Without loss of generality, we may assume that f is written irreducibly, so that we may choose $\underline{0} \leq \underline{r} \leq \underline{1}_{[d]}$ minimal such that $k_{\underline{r}} \neq 0$, and $\widetilde{\psi}_{\underline{r}}(k_{\underline{r}}) \notin B^{(\widetilde{\pi},\widetilde{t})}_{(\underline{r},\underline{1}_{[d]}]}$. If $\underline{r} = \underline{1}_{[d]}$, then $f = \widetilde{\psi}_{\underline{1}_{[d]}}(k_{\underline{1}_{[d]}})$ and $\Phi(f) = \psi_{\underline{1}_{[d]}}(k_{\underline{1}_{[d]}}) = 0$. Injectivity of (π, t) then implies that $k_{\underline{1}_{[d]}} = 0$ and hence $f = 0$, a contradiction. So without loss of generality assume that $\underline{r} < \underline{1}_{[d]}$. Then for every $\xi_{\underline{r}}, \eta_{\underline{r}} \in X_{\underline{r}}$, we have that

$$0 = t_{\underline{r}}(\xi_{\underline{r}})^* \Phi(f) t_{\underline{r}}(\eta_{\underline{r}}) = \pi(\langle \xi_{\underline{r}}, k_{\underline{r}} \eta_{\underline{r}} \rangle) + \sum \{\psi_{\underline{n}}(k'_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_{[d]} - \underline{r}\},$$

where each $k'_{\underline{n}}$ is a suitably-defined element of $\mathcal{K}(X_{\underline{n}})$ for all $\underline{0} \neq \underline{n} \leq \underline{1}_{[d]} - \underline{r}$. Hence we have that $\langle X_{\underline{r}}, k_{\underline{r}} X_{\underline{r}} \rangle \subseteq \mathcal{L}^{(\pi,t)}_F$ for $F := \mathrm{supp}(\underline{1}_{[d]} - \underline{r})$ (which is non-empty since $\underline{1}_{[d]} \neq \underline{r}$), and thus we obtain that $\widetilde{\pi}(\langle X_{\underline{r}}, k_{\underline{r}} X_{\underline{r}} \rangle) \subseteq B^{(\widetilde{\pi},\widetilde{t})}_{(\underline{0},\underline{1}_{[d]}-\underline{r}]}$ since $(\widetilde{\pi}, \widetilde{t})$ is an $\mathcal{L}^{(\pi,t)}$-relative CNP-representation. In particular, we have that

$$\begin{aligned} \widetilde{\psi}_{\underline{r}}(\mathcal{K}(X_{\underline{r}})) \widetilde{\psi}_{\underline{r}}(k_{\underline{r}}) \widetilde{\psi}_{\underline{r}}(\mathcal{K}(X_{\underline{r}})) &\subseteq [\widetilde{t}_{\underline{r}}(X_{\underline{r}}) \widetilde{\pi}(\langle X_{\underline{r}}, k_{\underline{r}} X_{\underline{r}} \rangle) \widetilde{t}_{\underline{r}}(X_{\underline{r}})^*] \\ &\subseteq [\widetilde{t}_{\underline{r}}(X_{\underline{r}}) B^{(\widetilde{\pi},\widetilde{t})}_{(\underline{0},\underline{1}_{[d]}-\underline{r}]} \widetilde{t}_{\underline{r}}(X_{\underline{r}})^*] \subseteq B^{(\widetilde{\pi},\widetilde{t})}_{(\underline{r},\underline{1}_{[d]}]}, \end{aligned}$$

and by taking an approximate unit of $\widetilde{\psi}_{\underline{r}}(\mathcal{K}(X_{\underline{r}}))$ we derive that $\widetilde{\psi}_{\underline{r}}(k_{\underline{r}}) \in B^{(\widetilde{\pi},\widetilde{t})}_{(\underline{r},\underline{1}_{[d]}]}$, which is a contradiction. Thus Φ is injective on the $[\underline{0}, \underline{1}_{[d]}]$-core, as required.

For the last assertion, we have that $\pi \times t = \Phi \circ Q$ and thus in particular

$$\ker \pi \times t = \ker \Phi \circ Q = \ker Q = \mathfrak{J}^{(\overline{\pi}_X, \overline{t}_X)}_{\mathcal{L}^{(\pi,t)}}$$

for the quotient map $Q \colon \mathcal{NT}_X \to \mathcal{NO}(\mathcal{L}^{(\pi,t)}, X)$, and the proof is complete. □

Injective Nica-covariant representations admitting a gauge action provide an ample supply of maximal relative 2^d-tuples.

Proposition 3.1.18. *Let X be a strong compactly aligned product system with coefficients in a C^*-algebra A. Let (π, t) be an injective Nica-covariant representation of X that admits a gauge action. Then $\mathcal{L}^{(\pi,t)}$ is a maximal 2^d-tuple of X that is contained in $\mathcal{I}$.*

Proof. By Proposition 3.1.14, we have that $\mathcal{L}^{(\pi,t)} \subseteq \mathcal{I}$. For maximality, let $\mathcal{L}$ be a relative 2^d-tuple of X such that $\mathfrak{J}^{(\overline{\pi}_X, \overline{t}_X)}_{\mathcal{L}} = \mathfrak{J}^{(\overline{\pi}_X, \overline{t}_X)}_{\mathcal{L}^{(\pi,t)}}$ and $\mathcal{L}^{(\pi,t)} \subseteq \mathcal{L}$. It suffices to show that $\mathcal{L} \subseteq \mathcal{L}^{(\pi,t)}$. Fix $F \subseteq [d]$ and $a \in \mathcal{L}_F$. Then

$$\overline{\pi}_X(a) \overline{q}_{X,F} \in \mathfrak{J}^{(\overline{\pi}_X, \overline{t}_X)}_{\mathcal{L}} = \mathfrak{J}^{(\overline{\pi}_X, \overline{t}_X)}_{\mathcal{L}^{(\pi,t)}}$$

by definition. Since $\mathfrak{J}_{\mathcal{L}^{(\pi,t)}}^{(\overline{\pi}_X,\overline{t}_X)} = \ker \pi \times t$ by Proposition 3.1.17, we obtain that

$$\begin{aligned}\pi(a)q_F &= \pi(a) + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \\ &= (\pi \times t)(\overline{\pi}_X(a) + \sum\{(-1)^{|\underline{n}|}\overline{\psi}_{X,\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}) \\ &= (\pi \times t)(\overline{\pi}_X(a)\overline{q}_{X,F}) = 0,\end{aligned}$$

using Proposition 2.5.16 in the first and third equalities. Thus $a \in \pi^{-1}(B_{[\underline{0},\underline{1}_F]}^{(\pi,t)}) = \mathcal{L}_F^{(\pi,t)}$ and hence $\mathcal{L} \subseteq \mathcal{L}^{(\pi,t)}$, as required. □

3.2 (E)-2^d-tuples and (M)-2^d-tuples

We are interested in a special class of 2^d-tuples that describes the relative Cuntz-Nica-Pimsner algebras $\mathcal{NO}(\mathcal{L}, X)$ in-between $\mathcal{NT}_X$ and $\mathcal{NO}_X$.

Proposition 3.2.1. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be a relative 2^d-tuple of X and let (π, t) be an $\mathcal{L}$-relative CNP-representation of X. If (π, t) is injective, then $\mathcal{L} \subseteq \mathcal{I}$. Conversely, if $\mathcal{L} \subseteq \mathcal{I}$ then the universal $\mathcal{L}$-relative CNP-representation $(\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}})$ is injective.*

Proof. First assume that (π, t) is injective. Fix $F \subseteq [d]$ and $a \in \mathcal{L}_F$. Since (π, t) is an $\mathcal{L}$-relative CNP-representation, we have that $\pi(a)q_F = 0$ and hence $\pi(a) \in B_{[\underline{0},\underline{1}_F]}^{(\pi,t)}$. Since (π, t) is assumed to be injective, an application of Proposition 2.5.19 gives that $a \in \mathcal{I}_F$ and hence $\mathcal{L} \subseteq \mathcal{I}$.

Conversely, assume that $\mathcal{L} \subseteq \mathcal{I}$. For all $F \subseteq [d]$ and $a \in \mathcal{I}_F$, we have that

$$\pi_X^{\mathcal{I}}(a)q_{X,F}^{\mathcal{I}} = 0$$

by definition. In particular, this holds for all $a \in \mathcal{L}_F$ by assumption, and therefore $(\pi_X^{\mathcal{I}}, t_X^{\mathcal{I}})$ is an $\mathcal{L}$-relative CNP-representation. Thus we obtain a (unique) canonical $*$-epimorphism

$$\Phi \colon \mathcal{NO}(\mathcal{L}, X) \to \mathcal{NO}_X$$

by universality of $\mathcal{NO}(\mathcal{L}, X)$. Fixing $a \in \ker \pi_X^{\mathcal{L}}$, we deduce that

$$\pi_X^{\mathcal{I}}(a) = \Phi(\pi_X^{\mathcal{L}}(a)) = 0.$$

Injectivity of $(\pi_X^{\mathcal{I}}, t_X^{\mathcal{I}})$ then implies that $a = 0$, finishing the proof. □

We see that the 2^d-tuples $\mathcal{L} \subseteq \mathcal{I}$ are special, and we give them their own name to reflect this.

Definition 3.2.2. Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. We say that a 2^d-tuple $\mathcal{L}$ of X is an *(E)-2^d-tuple (of X)* if $\mathcal{L} \subseteq \mathcal{I}$.

Every (E)-2^d-tuple is automatically a relative 2^d-tuple with $\mathcal{L}_\emptyset = \{0\} = \mathcal{I}_\emptyset$. The "E" of "(E)-$2^d$-tuple" stands for "Embedding", since $X \hookrightarrow \mathcal{NO}(\mathcal{L}, X)$ isometrically for a relative 2^d-tuple $\mathcal{L}$ if and only if $\mathcal{L} \subseteq \mathcal{I}$, by Proposition 3.2.1. By Proposition 3.1.14, injective Nica-covariant representations provide the quintessential supply of (E)-2^d-tuples. It is straightforward to check that the sum of two (E)-2^d-tuples is an (E)-2^d-tuple. We also make the following observation.

Proposition 3.2.3. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be a 2^d-tuple of X. If $\mathcal{L}$ is invariant and satisfies $\mathcal{L} \subseteq \mathcal{J}$, then $\mathcal{L}$ is an (E)-2^d-tuple of X.*

Proof. Fix $F \subseteq [d]$. By assumption we have that

$$\langle X_{\underline{n}}, \mathcal{L}_F X_{\underline{n}} \rangle \subseteq [\langle X_{\underline{n}}, \mathcal{L}_F X_{\underline{n}} \rangle] \subseteq \mathcal{L}_F \subseteq \mathcal{J}_F \text{ for all } \underline{n} \perp F.$$

By definition this means that $\mathcal{L}_F \subseteq \mathcal{I}_F$ and hence $\mathcal{L} \subseteq \mathcal{I}$, as required. □

Let (π, t) be a Nica-covariant representation and $\mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$ be the ideal induced by an (E)-2^d-tuple $\mathcal{L}$. Our goal in the next propositions is to show that we can choose $\mathcal{L}$ to be in addition invariant and partially ordered, and to consist of ideals.

Proposition 3.2.4. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be an (E)-2^d-tuple of X and let (π, t) be a Nica-covariant representation of X. Then the 2^d-tuple $\operatorname{Inv}(\mathcal{L})$ of X defined by*

$$\operatorname{Inv}(\mathcal{L})_F := \overline{\operatorname{span}}\{X_{\underline{n}}(\mathcal{L}_F) \mid \underline{n} \perp F\} \textit{ for all } F \subseteq [d]$$

is an invariant (E)-2^d-tuple consisting of ideals, and is such that

$$\mathcal{L} \subseteq \operatorname{Inv}(\mathcal{L}) \quad \textit{and} \quad \mathfrak{J}_{\mathcal{L},F}^{(\pi,t)} = \mathfrak{J}_{\operatorname{Inv}(\mathcal{L}),F}^{(\pi,t)} \textit{ for all } F \subseteq [d].$$

In particular, we have that $\mathfrak{J}_{\mathcal{L}}^{(\pi,t)} = \mathfrak{J}_{\operatorname{Inv}(\mathcal{L})}^{(\pi,t)}$.

Proof. Recall that $X_{\underline{n}}(\mathcal{L}_F) \equiv [\langle X_{\underline{n}}, \mathcal{L}_F X_{\underline{n}} \rangle]$ for all $\underline{n} \in \mathbb{Z}_+^d$ and $F \subseteq [d]$. In particular, we have that $X_{\underline{0}}(\mathcal{L}_F) = \langle \mathcal{L}_F \rangle$ for all $F \subseteq [d]$, from which it follows that $\mathcal{L} \subseteq \operatorname{Inv}(\mathcal{L})$.

Fix $F \subseteq [d]$. Then $X_{\underline{n}}(\mathcal{L}_F)$ is an ideal of A for all $\underline{n} \perp F$, and hence $\operatorname{Inv}(\mathcal{L})_F$ is itself an ideal of A. Since $\mathcal{L}$ is an (E)-2^d-tuple, we have that $\mathcal{L}_F \subseteq \mathcal{I}_F$ and thus $X_{\underline{n}}(\mathcal{L}_F) \subseteq \mathcal{I}_F$ for all $\underline{n} \perp F$ because $\mathcal{I}$ is invariant by Proposition 2.5.10. Thus $\operatorname{Inv}(\mathcal{L})_F \subseteq \mathcal{I}_F$, and so $\operatorname{Inv}(\mathcal{L})$ is an (E)-2^d-tuple that consists of ideals.

To see that $\operatorname{Inv}(\mathcal{L})$ is invariant, fix $F \subseteq [d], \underline{m} \perp F$ and $a \in \operatorname{Inv}(\mathcal{L})_F$. Since $\operatorname{Inv}(\mathcal{L})_F$ is an ideal, it suffices to show that $\langle X_{\underline{m}}, a X_{\underline{m}} \rangle \subseteq \operatorname{Inv}(\mathcal{L})_F$. Assume that $a = \langle \xi_{\underline{n}}, b\eta_{\underline{n}} \rangle$ for some $\underline{n} \perp F, \xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\underline{n}}$ and $b \in \mathcal{L}_F$. Then we have that

$$\langle X_{\underline{m}}, a X_{\underline{m}} \rangle = \langle X_{\underline{m}}, \langle \xi_{\underline{n}}, b\eta_{\underline{n}} \rangle X_{\underline{m}} \rangle \subseteq \langle X_{\underline{n}+\underline{m}}, b X_{\underline{n}+\underline{m}} \rangle \subseteq X_{\underline{n}+\underline{m}}(\mathcal{L}_F) \subseteq \operatorname{Inv}(\mathcal{L})_F,$$

using that $\underline{n}+\underline{m} \perp F$ in the final inclusion. It follows that $\langle X_{\underline{m}}, X_{\underline{n}}(\mathcal{L}_F)X_{\underline{m}}\rangle \subseteq \operatorname{Inv}(\mathcal{L})_F$ for all $\underline{n} \perp F$. We obtain that $\langle X_{\underline{m}}, \operatorname{Inv}(\mathcal{L})_F X_{\underline{m}}\rangle \subseteq \operatorname{Inv}(\mathcal{L})_F$ by linearity and continuity of the inner product, as required.

Since $\mathcal{L} \subseteq \operatorname{Inv}(\mathcal{L})$, we have that $\mathfrak{J}^{(\pi,t)}_{\mathcal{L},F} \subseteq \mathfrak{J}^{(\pi,t)}_{\operatorname{Inv}(\mathcal{L}),F}$ for all $F \subseteq [d]$. For the reverse inclusion, fix $F \subseteq [d]$ and $a \in \operatorname{Inv}(\mathcal{L})_F$. Assume that $a = \langle \xi_{\underline{n}}, b\eta_{\underline{n}}\rangle$ for some $\underline{n} \perp F, \xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\underline{n}}$ and $b \in \mathcal{L}_F$. We have that

$$\pi(a)q_F = t_{\underline{n}}(\xi_{\underline{n}})^*\pi(b)t_{\underline{n}}(\eta_{\underline{n}})q_F = t_{\underline{n}}(\xi_{\underline{n}})^*(\pi(b)q_F)t_{\underline{n}}(\eta_{\underline{n}}) \in \mathfrak{J}^{(\pi,t)}_{\mathcal{L},F},$$

where we have used Proposition 2.5.15 in the second equality. Therefore $\pi(X_{\underline{n}}(\mathcal{L}_F))q_F \subseteq \mathfrak{J}^{(\pi,t)}_{\mathcal{L},F}$ for all $\underline{n} \perp F$, from which it follows that $\pi(\operatorname{Inv}(\mathcal{L})_F)q_F \subseteq \mathfrak{J}^{(\pi,t)}_{\mathcal{L},F}$. In total we have that $\mathfrak{J}^{(\pi,t)}_{\mathcal{L},F} = \mathfrak{J}^{(\pi,t)}_{\operatorname{Inv}(\mathcal{L}),F}$ for all $F \subseteq [d]$, and thus in particular $\mathfrak{J}^{(\pi,t)}_{\mathcal{L}} = \mathfrak{J}^{(\pi,t)}_{\operatorname{Inv}(\mathcal{L})}$, finishing the proof. □

In order to choose $\mathcal{L}$ to be partially ordered, we need the following auxiliary proposition.

Proposition 3.2.5. *Let X be a strong compactly aligned product system with coefficients in a C^*-algebra A. Let $\mathcal{L}$ be a relative 2^d-tuple of X and let (π,t) be a Nica-covariant representation of X. Fix $D \subseteq F \subseteq [d]$. If $a \in \bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in F\}$ and $\pi(a)q_D \in \mathfrak{J}^{(\pi,t)}_{\mathcal{L}}$, then $\pi(a)q_F \in \mathfrak{J}^{(\pi,t)}_{\mathcal{L}}$.*

Proof. Without loss of generality, we may assume that $D = [k]$ (with the convention that if $k = 0$ then $D = \emptyset$) and $F = [\ell]$ for some $0 \leq k \leq \ell \leq d$. Note that we have that

$$\pi(a)q_D = \pi(a) + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_D\}$$

because $a \in \bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in F\}$, and hence in particular $a \in \bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in D\}$ since $D \subseteq F$. If $k = \ell$, then there is nothing to show. If $k < \ell$, then we have that

$$\begin{aligned}
\pi(a)q_D p_{\underline{k+1}} &= \pi(a)p_{\underline{k+1}} + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}}(\phi_{\underline{n}}(a))p_{\underline{k+1}} \mid \underline{0} \neq \underline{n} \leq \underline{1}_D\} \\
&= \|\cdot\|\text{-}\lim_\lambda \pi(a)p_{\underline{k+1},\lambda} + \sum\{(-1)^{|\underline{n}|}(\|\cdot\|\text{-}\lim_\lambda \psi_{\underline{n}}(\phi_{\underline{n}}(a))p_{\underline{k+1},\lambda}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_D\} \\
&= \|\cdot\|\text{-}\lim_\lambda(\pi(a) + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_D\})p_{\underline{k+1},\lambda} \\
&= \|\cdot\|\text{-}\lim_\lambda \pi(a)q_D p_{\underline{k+1},\lambda} \in \mathfrak{J}^{(\pi,t)}_{\mathcal{L}},
\end{aligned}$$

where in the second line we use Proposition 2.5.16 and in the last line we use that $\pi(a)q_D \in \mathfrak{J}^{(\pi,t)}_{\mathcal{L}}$ by assumption. To express $\pi(a)p_{\underline{k+1}}$ as a norm-limit, we use that $a \in \phi_{\underline{k+1}}^{-1}(\mathcal{K}(X_{\underline{k+1}}))$ and so Proposition 2.5.16 applies. Hence we have that

$$\pi(a)q_{D\cup\{k+1\}} = \pi(a)q_D - \pi(a)q_D p_{\underline{k+1}} \in \mathfrak{J}^{(\pi,t)}_{\mathcal{L}}.$$

Since $a \in \bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in D\cup\{k+1\}\}$, we can express $\pi(a)q_{D\cup\{k+1\}}$ as an alternating sum by Proposition 2.5.16. Consequently, we may apply the preceding argument for

$k+2$. The assumption that $a \in \bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in F\}$ ensures that we may argue in this way until $\{k+1, \dots, \ell\}$ has been exhausted, and we deduce that $\pi(a)q_F \in \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$, as required. □

Proposition 3.2.6. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be an (E)-2^d-tuple of X and let (π, t) be a Nica-covariant representation of X. Then the 2^d-tuple* $\mathrm{PO}(\mathcal{L})$ *of X defined by*

$$\mathrm{PO}(\mathcal{L})_F := \sum\{\langle \mathcal{L}_D \rangle \mid D \subseteq F\} \text{ for all } F \subseteq [d]$$

is a partially ordered (E)-2^d-tuple consisting of ideals, and is such that

$$\mathcal{L} \subseteq \mathrm{PO}(\mathcal{L}) \quad \text{and} \quad \mathfrak{J}_{\mathcal{L}}^{(\pi,t)} = \mathfrak{J}_{\mathrm{PO}(\mathcal{L})}^{(\pi,t)}.$$

If $\mathcal{L}$ is in addition invariant, then so is $\mathrm{PO}(\mathcal{L})$.

Proof. Every $\mathrm{PO}(\mathcal{L})_F$ is an ideal, being a finite sum of ideals in a C*-algebra. Fix $F \subseteq [d]$. For each $D \subseteq F$, we have that

$$\langle \mathcal{L}_D \rangle \subseteq \mathcal{I}_D \subseteq \mathcal{I}_F,$$

using that $\mathcal{L}$ is an (E)-2^d-tuple (and that $\mathcal{I}_D$ is an ideal) in the first inclusion, and the fact that $\mathcal{I}$ is partially ordered in the second. Hence $\mathrm{PO}(\mathcal{L})_F \subseteq \mathcal{I}_F$ for all $F \subseteq [d]$, and thus $\mathrm{PO}(\mathcal{L})$ is an (E)-2^d-tuple. It is clear that $\mathrm{PO}(\mathcal{L})$ is partially ordered, and $\mathcal{L} \subseteq \mathrm{PO}(\mathcal{L})$ by construction.

Next we check that $\mathfrak{J}_{\mathcal{L}}^{(\pi,t)} = \mathfrak{J}_{\mathrm{PO}(\mathcal{L})}^{(\pi,t)}$. Since $\mathcal{L} \subseteq \mathrm{PO}(\mathcal{L})$, we have that $\mathfrak{J}_{\mathcal{L}}^{(\pi,t)} \subseteq \mathfrak{J}_{\mathrm{PO}(\mathcal{L})}^{(\pi,t)}$. For the reverse inclusion, fix $F \subseteq [d]$ and $a \in \mathrm{PO}(\mathcal{L})_F$. It suffices to show that $\pi(a)q_F \in \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$. Let $a = \sum\{a_D \mid D \subseteq F\}$, where each $a_D \in \langle \mathcal{L}_D \rangle$, so that $\pi(a_D)q_D \in \mathfrak{J}_{\langle \mathcal{L} \rangle}^{(\pi,t)} = \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$ for all $D \subseteq F$ by Proposition 3.1.5. For each $D \subseteq F$, an application of Proposition 3.2.5 gives that $\pi(a_D)q_F \in \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$ and hence $\pi(a)q_F \in \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}$, as required.

Now suppose that $\mathcal{L}$ is in addition invariant. To establish invariance of the family $\mathrm{PO}(\mathcal{L})$, fix $F \subseteq [d], \underline{n} \perp F$ and $a \in \mathrm{PO}(\mathcal{L})_F$. By definition, we can write $a = \sum\{a_D \mid D \subseteq F\}$ for some $a_D \in \langle \mathcal{L}_D \rangle$, for each $D \subseteq F$. Note that $\underline{n} \perp D$ whenever $D \subseteq F$. Since $\mathcal{L}$ is invariant, it follows that $\langle X_{\underline{n}}, \langle \mathcal{L}_D \rangle X_{\underline{n}} \rangle \subseteq \langle \mathcal{L}_D \rangle$ and hence in particular $\langle X_{\underline{n}}, a_D X_{\underline{n}} \rangle \subseteq \langle \mathcal{L}_D \rangle$ for each $D \subseteq F$. Consequently, we have that

$$\langle X_{\underline{n}}, a X_{\underline{n}} \rangle \subseteq \sum\{\langle X_{\underline{n}}, a_D X_{\underline{n}} \rangle \mid D \subseteq F\} \subseteq \sum\{\langle \mathcal{L}_D \rangle \mid D \subseteq F\} = \mathrm{PO}(\mathcal{L})_F,$$

and therefore $\mathrm{PO}(\mathcal{L})$ is invariant, finishing the proof. □

Remark 3.2.7. Note that for an (E)-2^d-tuple $\mathcal{L}$ and a Nica-covariant representation (π, t), it is not true in general that $\mathfrak{J}_{\mathcal{L},F}^{(\pi,t)} = \mathfrak{J}_{\mathrm{PO}(\mathcal{L}),F}^{(\pi,t)}$ for all $F \subseteq [d]$. Here we provide a counterexample to this effect.

Consider the strong compactly aligned product system X_α over $\mathbb{Z}_+^2$ with coefficients in the unitisation A of a non-zero C*-algebra B considered in Example 3.1.12. Let us take

$B = \mathbb{C}$, so that $A = \mathbb{C} \oplus \mathbb{C}$. We define a 2^2-tuple $\mathcal{L}$ of X_α by

$$\begin{array}{ccc} \mathcal{L}_{\{2\}} = (\mathbb{C} \oplus \{0\})^\perp & \text{———} & \mathcal{L}_{\{1,2\}} = \{0\} \\ | & & | \\ \mathcal{L}_\emptyset = \{0\} & \text{———} & \mathcal{L}_{\{1\}} = \{0\}. \end{array}$$

Recalling that A carries the C*-structure of the unitisation, we have that

$$\begin{aligned} (\mathbb{C} \oplus \{0\})^\perp &= \{(\lambda, \mu) \in A \mid (\lambda, \mu)(\nu, 0) = (0,0) \text{ for all } \nu \in \mathbb{C}\} \\ &= \{(\lambda, \mu) \in A \mid \lambda\nu + \mu\nu = 0 \text{ for all } \nu \in \mathbb{C}\} \\ &= \{(\lambda, -\lambda) \mid \lambda \in \mathbb{C}\} \neq \{0\}. \end{aligned}$$

In turn, we have that $\mathcal{L}$ is not partially ordered since $\{2\} \subseteq \{1,2\}$ but $\mathcal{L}_{\{2\}} \not\subseteq \mathcal{L}_{\{1,2\}}$. Note also that $\mathcal{L}$ consists of ideals. By [15, p. 52] or (5.7) to come, we have that

$$\begin{array}{ccc} \mathcal{I}_{\{2\}} = (\mathbb{C} \oplus \{0\})^\perp & \text{———} & \mathcal{I}_{\{1,2\}} = A \\ | & & | \\ \mathcal{I}_\emptyset = \{0\} & \text{———} & \mathcal{I}_{\{1\}} = A. \end{array}$$

Thus $\mathcal{L} \subseteq \mathcal{I}$ and hence $\mathcal{L}$ is an (E)-2^2-tuple. By definition we have that

$$\begin{array}{ccc} \text{PO}(\mathcal{L})_{\{2\}} = (\mathbb{C} \oplus \{0\})^\perp & \text{———} & \text{PO}(\mathcal{L})_{\{1,2\}} = (\mathbb{C} \oplus \{0\})^\perp \\ | & & | \\ \text{PO}(\mathcal{L})_\emptyset = \{0\} & \text{———} & \text{PO}(\mathcal{L})_{\{1\}} = \{0\}. \end{array}$$

Let $(\overline{\pi}, \overline{t})$ denote the Fock representation. Observe that

$$\mathfrak{J}^{(\overline{\pi}, \overline{t})}_{\mathcal{L}, \{1,2\}} = \{0\} \quad \text{and} \quad \mathfrak{J}^{(\overline{\pi}, \overline{t})}_{\text{PO}(\mathcal{L}), \{1,2\}} = \left\langle \overline{\pi}((\mathbb{C} \oplus \{0\})^\perp) \overline{q}_{\{1,2\}} \right\rangle.$$

We claim that $\mathfrak{J}^{(\overline{\pi}, \overline{t})}_{\mathcal{L}, \{1,2\}} \neq \mathfrak{J}^{(\overline{\pi}, \overline{t})}_{\text{PO}(\mathcal{L}), \{1,2\}}$. Towards contradiction, suppose that $\mathfrak{J}^{(\overline{\pi}, \overline{t})}_{\mathcal{L}, \{1,2\}} = \mathfrak{J}^{(\overline{\pi}, \overline{t})}_{\text{PO}(\mathcal{L}), \{1,2\}}$ and take $a \in (\mathbb{C} \oplus \{0\})^\perp \setminus \{0\}$. Then $\overline{\pi}(a)\overline{q}_{\{1,2\}} = 0$ by assumption. Hence we obtain that

$$ab = \overline{\pi}(a)\overline{q}_{\{1,2\}} b = 0 \text{ for all } b \in A.$$

By taking $b = a^*$, we arrive at the contradiction that $a = 0$. Consequently, we have that

$$\mathfrak{J}^{(\overline{\pi}, \overline{t})}_{\mathcal{L}, \{1,2\}} \neq \mathfrak{J}^{(\overline{\pi}, \overline{t})}_{\text{PO}(\mathcal{L}), \{1,2\}},$$

as claimed.

Letting $\mathcal{L}$ be an (E)-2^d-tuple and (π, t) be a Nica-covariant representation, Propositions 3.2.4 and 3.2.6 combine to give that PO(Inv($\mathcal{L}$)) is an (E)-2^d-tuple that is invariant, partially ordered, consists of ideals and satisfies $\mathcal{L} \subseteq \text{PO}(\text{Inv}(\mathcal{L}))$ and $\mathfrak{J}^{(\pi,t)}_{\mathcal{L}} = \mathfrak{J}^{(\pi,t)}_{\text{PO}(\text{Inv}(\mathcal{L}))}$.

Next we focus on the maximal (E)-2^d-tuples.

Definition 3.2.8. Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let (π, t) be a Nica-covariant representation of X. An *(M)-2^d-tuple $\mathcal{M}$ (of X) with respect to* (π, t) is a maximal 2^d-tuple of X with respect to (π, t) that is also an (E)-2^d-tuple of X. When we replace (π, t) by $(\overline{\pi}_X, \overline{t}_X)$, we will refer to $\mathcal{M}$ simply as an *(M)-2^d-tuple (of X)*.

The "M" of "(M)-2^d-tuple" stands for "Maximal". Note that (M)-2^d-tuples (with respect to (π, t)) maximalise collections of (E)-2^d-tuples, instead of just relative 2^d-tuples. Indeed, if $\mathcal{M}$ is an (M)-2^d-tuple with respect to (π, t), then it contains every relative 2^d-tuple $\mathcal{L}$ of X satisfying $\mathfrak{J}_{\mathcal{L}}^{(\pi,t)} = \mathfrak{J}_{\mathcal{M}}^{(\pi,t)}$ by Proposition 3.1.9. Each such $\mathcal{L}$ satisfies $\mathcal{L} \subseteq \mathcal{M} \subseteq \mathcal{I}$ and is therefore an (E)-2^d-tuple, as claimed.

Remark 3.2.9. Both $\mathcal{L} := \{\{0\}\}_{F \subseteq [d]}$ and $\mathcal{I}$ are (M)-2^d-tuples. For $\mathcal{I}$ this is shown in Remark 3.1.10. On the other hand, $\mathcal{L}$ is clearly an (E)-2^d-tuple. To see that $\mathcal{L}$ is maximal, let $\mathcal{L}'$ be a relative 2^d-tuple such that $\mathfrak{J}_{\mathcal{L}'}^{(\overline{\pi}_X, \overline{t}_X)} = \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)} = \{0\}$ and $\mathcal{L} \subseteq \mathcal{L}'$. We may replace $(\overline{\pi}_X, \overline{t}_X)$ by the Fock representation $(\overline{\pi}, \overline{t})$. For the projection $P_{\underline{0}} \colon \mathcal{F}X \to X_{\underline{0}}$, we have that

$$\phi_{\underline{0}}(\mathcal{L}'_F) = P_{\underline{0}}\big(\overline{\pi}(\mathcal{L}'_F)\overline{q}_F\big)P_{\underline{0}} \subseteq P_{\underline{0}}\big(\mathfrak{J}_{\mathcal{L}'}^{(\overline{\pi},\overline{t})}\big)P_{\underline{0}} = \{0\},$$

for all $F \subseteq [d]$, and thus $\mathcal{L} = \mathcal{L}'$ as required.

Moreover, in Proposition 3.1.18 we have shown that $\mathcal{L}^{(\pi,t)}$ is an (M)-2^d-tuple whenever (π, t) is injective and admits a gauge action.

The (M)-2^d-tuples of X parametrise the gauge-invariant ideals $\mathfrak{J}$ of $\mathcal{NT}_X$ for which $\mathcal{NT}_X/\mathfrak{J}$ lies between $\mathcal{NT}_X$ and $\mathcal{NO}_X$. To prove this, we will require the following proposition.

Proposition 3.2.10. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let $\mathfrak{J} \subseteq \mathcal{NT}_X$ be a gauge-invariant ideal. Then the following are equivalent:*

(i) $\overline{\pi}_X^{-1}(\mathfrak{J}) = \{0\}$;

(ii) there exists a unique (M)-2^d-tuple of X inducing $\mathfrak{J}$;

(iii) there exists an (E)-2^d-tuple of X inducing $\mathfrak{J}$.

Proof. First assume that $\overline{\pi}_X^{-1}(\mathfrak{J}) = \{0\}$. Let $Q_{\mathfrak{J}} \colon \mathcal{NT}_X \to \mathcal{NT}_X/\mathfrak{J}$ denote the quotient map. Then the Nica-covariant representation $(Q_{\mathfrak{J}} \circ \overline{\pi}_X, Q_{\mathfrak{J}} \circ \overline{t}_X)$ is injective and admits a gauge action. By Proposition 3.1.17, we have that

$$\mathfrak{J} = \ker Q_{\mathfrak{J}} = \ker(Q_{\mathfrak{J}} \circ \overline{\pi}_X) \times (Q_{\mathfrak{J}} \circ \overline{t}_X) = \mathfrak{J}^{(\overline{\pi}_X, \overline{t}_X)}_{\mathcal{L}^{(Q_{\mathfrak{J}} \circ \overline{\pi}_X, Q_{\mathfrak{J}} \circ \overline{t}_X)}},$$

since $(Q_{\mathfrak{J}} \circ \overline{\pi}_X) \times (Q_{\mathfrak{J}} \circ \overline{t}_X) = Q_{\mathfrak{J}}$. Thus $\mathcal{L}^{(Q_{\mathfrak{J}} \circ \overline{\pi}_X, Q_{\mathfrak{J}} \circ \overline{t}_X)}$ induces $\mathfrak{J}$, and by Proposition 3.1.18 it is an (M)-2^d-tuple of X. By Proposition 3.1.9 it is also unique, proving that (i) implies (ii).

Item (ii) implies item (iii) trivially. So assume that there exists an (E)-2^d-tuple $\mathcal{L}$ such that $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)} = \mathfrak{J}$. In particular $(\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}})$ is injective by Proposition 3.2.1. Hence, letting

$$Q_{\mathfrak{J}}\colon \mathcal{NT}_X \to \mathcal{NO}(\mathcal{L}, X) = \mathcal{NT}_X/\mathfrak{J}$$

denote the quotient map, we have that

$$\{0\} = \ker \pi_X^{\mathcal{L}} = \ker Q_{\mathfrak{J}} \circ \overline{\pi}_X = \{a \in A \mid \overline{\pi}_X(a) \in \mathfrak{J}\} = \overline{\pi}_X^{-1}(\mathfrak{J}),$$

showing that item (iii) implies item (i), finishing the proof. □

Remark 3.2.11. Proposition 3.2.10 implies that the mapping

$$\{\mathcal{M} \mid \mathcal{M} \text{ is an (M)-}2^d\text{-tuple}\} \to \{\mathfrak{J} \subseteq \mathcal{NT}_X \mid \mathfrak{J} \text{ is a gauge-invariant ideal, } \overline{\pi}_X^{-1}(\mathfrak{J}) = \{0\}\}$$
$$\mathcal{M} \mapsto \mathfrak{J}_{\mathcal{M}}^{(\overline{\pi}_X,\overline{t}_X)}$$

is a bijection. More specifically, the mappings

$$\mathcal{M} \mapsto \mathfrak{J}_{\mathcal{M}}^{(\overline{\pi}_X,\overline{t}_X)} \text{ for all (M)-}2^d\text{-tuples } \mathcal{M} \text{ of } X,$$
$$\mathfrak{J} \mapsto \mathcal{L}^{(Q_{\mathfrak{J}}\circ\overline{\pi}_X, Q_{\mathfrak{J}}\circ\overline{t}_X)} \text{ for all gauge-invariant ideals } \mathfrak{J} \subseteq \mathcal{NT}_X \text{ satisfying } \overline{\pi}_X^{-1}(\mathfrak{J}) = \{0\},$$

are mutually inverse, where $Q_{\mathfrak{J}}\colon \mathcal{NT}_X \to \mathcal{NT}_X/\mathfrak{J}$ is the quotient map.

The following Gauge-Invariant Uniqueness Theorem recovers [36, Corollary 11.8] when $d = 1$, and [17, Theorem 4.2] when $\mathcal{L} = \mathcal{I}$.

Theorem 3.2.12 ($\mathbb{Z}_+^d$-GIUT for (M)-2^d-tuples)**.** *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be an (M)-2^d-tuple of X and (π, t) be a Nica-covariant representation of X. Then $\mathcal{NO}(\mathcal{L}, X) \cong \mathrm{C}^*(\pi, t)$ via a (unique) canonical $*$-isomorphism if and only if (π, t) admits a gauge action and $\mathcal{L}^{(\pi,t)} = \mathcal{L}$.*

Proof. We start with the following remark. If $\mathcal{NO}(\mathcal{L}, X) \cong \mathrm{C}^*(\pi, t)$ canonically, then (π, t) is injective as X embeds isometrically in $\mathcal{NO}(\mathcal{L}, X)$. On the other hand, if $\mathcal{L}^{(\pi,t)} = \mathcal{L}$ then $\ker \pi \equiv \mathcal{L}_\emptyset^{(\pi,t)} = \mathcal{L}_\emptyset = \{0\}$, as $\mathcal{L}$ is in particular an (E)-2^d-tuple. Hence in both implications we may use that (π, t) is injective.

Assume that $\mathcal{NO}(\mathcal{L}, X) \cong \mathrm{C}^*(\pi, t)$ via a canonical $*$-isomorphism $\Phi\colon \mathcal{NO}(\mathcal{L}, X) \to \mathrm{C}^*(\pi, t)$. For notational convenience, we will abbreviate $(\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}})$ as $(\widetilde{\pi}, \widetilde{t})$. Let β denote the gauge action of $(\widetilde{\pi}, \widetilde{t})$. By defining $\gamma_{\underline{z}} := \Phi \circ \beta_{\underline{z}} \circ \Phi^{-1}$ for each $\underline{z} \in \mathbb{T}^d$, we obtain a gauge action γ of (π, t).

Next we show that $\mathcal{L}^{(\pi,t)} = \mathcal{L}$. To this end, it suffices to show that $\mathfrak{J}_{\mathcal{L}^{(\pi,t)}}^{(\overline{\pi}_X,\overline{t}_X)} = \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)}$. Then we can apply maximality of $\mathcal{L}^{(\pi,t)}$ (which holds by Proposition 3.1.18) and of $\mathcal{L}$ (which holds by assumption), together with uniqueness from item (ii) of Proposition 3.2.10, to obtain the result. To this end, fix $F \subseteq [d]$ and $a \in \mathcal{L}_F$. Then $\widetilde{\pi}(a)\widetilde{q}_F = 0$ since $(\widetilde{\pi}, \widetilde{t})$ is an $\mathcal{L}$-relative CNP-representation. Applying Φ, we obtain that $\pi(a)q_F = 0$, and therefore $a \in \mathcal{L}_F^{(\pi,t)}$ by definition. Hence $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)} \subseteq \mathfrak{J}_{\mathcal{L}^{(\pi,t)}}^{(\overline{\pi}_X,\overline{t}_X)}$. Next, fix $F \subseteq [d]$ and $a \in \mathcal{L}_F^{(\pi,t)}$. An application of Proposition 2.5.17 yields that $\pi(a)q_F = 0$. Recall that $\widetilde{\pi} = Q \circ \overline{\pi}_X$ and $\widetilde{t}_{\underline{n}} = Q \circ \overline{t}_{X,\underline{n}}$ for all $\underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$, where $Q \colon \mathcal{NT}_X \to \mathcal{NO}(\mathcal{L}, X)$ is the quotient map. Hence we obtain that

$$\begin{aligned} Q(\overline{\pi}_X(a)\overline{q}_{X,F}) &= Q(\overline{\pi}_X(a) + \sum\{(-1)^{|\underline{n}|}\overline{\psi}_{X,\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}) \\ &= \widetilde{\pi}(a) + \sum\{(-1)^{|\underline{n}|}\widetilde{\psi}_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \\ &= \Phi^{-1}(\pi(a) + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}) \\ &= \Phi^{-1}(\pi(a)q_F) = 0, \end{aligned}$$

and thus we have that

$$\overline{\pi}_X(a)\overline{q}_{X,F} \in \ker Q = \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)}.$$

Hence we conclude that $\mathfrak{J}_{\mathcal{L}^{(\pi,t)}}^{(\overline{\pi}_X,\overline{t}_X)} \subseteq \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)}$ and so in total $\mathfrak{J}_{\mathcal{L}^{(\pi,t)}}^{(\overline{\pi}_X,\overline{t}_X)} = \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)}$, as required.

The converse is an immediate application of Proposition 3.1.17. Indeed, since (π, t) is injective and admits a gauge action, we obtain that $\mathcal{NO}(\mathcal{L}^{(\pi,t)}, X) \cong \mathrm{C}^*(\pi, t)$ canonically, and the fact that $\mathcal{L}^{(\pi,t)} = \mathcal{L}$ then gives that $\mathcal{NO}(\mathcal{L}^{(\pi,t)}, X) = \mathcal{NO}(\mathcal{L}, X)$, finishing the proof. □

3.3 Properties of ideals induced by relative 2^d-tuples

The definition of (M)-2^d-tuples itself depends extensively on the associated gauge-invariant ideals of $\mathcal{NT}_X$. Hence, to obtain a genuinely useful parametrisation, we must characterise (M)-2^d-tuples in a way that does not (explicitly) depend on any gauge-invariant ideal of $\mathcal{NT}_X$. To this end, we need to study the properties of the gauge-invariant ideals arising from relative 2^d-tuples.

When a relative 2^d-tuple $\mathcal{L}$ consists of ideals and is invariant, the ideals $\mathfrak{J}_{\mathcal{L},F}^{(\pi,t)}$ take on a particularly simple form akin to the one obtained in [17, Proposition 4.6].

Proposition 3.3.1. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be a relative 2^d-tuple of X and let (π, t) be a Nica-covariant representation of X. If $\mathcal{L}$ is invariant and consists of ideals, then*

$$\mathfrak{J}_{\mathcal{L},F}^{(\pi,t)} = \overline{\mathrm{span}}\{t_{\underline{n}}(X_{\underline{n}})\pi(\mathcal{L}_F)q_F t_{\underline{m}}(X_{\underline{m}})^* \mid \underline{n}, \underline{m} \in \mathbb{Z}_+^d\} \text{ for all } F \subseteq [d]. \tag{3.2}$$

Proof. The proof follows the arguments of [17, Proposition 4.6]. In short, fix $F \subseteq [d]$ and denote the right hand side of (3.2) by $\mathfrak{J}_F$. The fact that $\mathfrak{J}_F \subseteq \mathfrak{J}_{\mathcal{L},F}^{(\pi,t)}$ is immediate by definition. To prove that $\mathfrak{J}_{\mathcal{L},F}^{(\pi,t)} \subseteq \mathfrak{J}_F$, it suffices to show that $\mathfrak{J}_F$ is an ideal of $\mathrm{C}^*(\pi, t)$ that contains the generators of $\mathfrak{J}_{\mathcal{L},F}^{(\pi,t)}$. We begin by proving that $\mathfrak{J}_F$ is an ideal. Since $\mathfrak{J}_F$ is a selfadjoint closed linear subspace, it suffices to show that $\mathfrak{J}_F \mathrm{C}^*(\pi, t) \subseteq \mathfrak{J}_F$.

It is clear that $\mathfrak{J}_F t_{\underline{r}}(X_{\underline{r}})^* \subseteq \mathfrak{J}_F$ for all $\underline{r} \in \mathbb{Z}_+^d$. Hence it remains to show that

$\mathfrak{J}_F t_{\underline{r}}(X_{\underline{r}}) \subseteq \mathfrak{J}_F$ for all $\underline{r} \in \mathbb{Z}_+^d$. For $\underline{n}, \underline{m}, \underline{r} \in \mathbb{Z}_+^d$, we have that

$$t_{\underline{n}}(X_{\underline{n}})\pi(\mathcal{L}_F)q_F t_{\underline{m}}(X_{\underline{m}})^* t_{\underline{r}}(X_{\underline{r}}) \subseteq [t_{\underline{n}}(X_{\underline{n}})\pi(\mathcal{L}_F)q_F t_{\underline{m}'}(X_{\underline{m}'})t_{\underline{r}'}(X_{\underline{r}'})^*],$$

where $\underline{m}' = -\underline{m} + \underline{m} \vee \underline{r}$ and $\underline{r}' = -\underline{r} + \underline{m} \vee \underline{r}$, due to Nica-covariance. If $\underline{m}' \not\perp F$, then

$$t_{\underline{n}}(X_{\underline{n}})\pi(\mathcal{L}_F)q_F t_{\underline{m}'}(X_{\underline{m}'})t_{\underline{r}'}(X_{\underline{r}'})^* = \{0\} \subseteq \mathfrak{J}_F$$

by Proposition 2.5.15. If $\underline{m}' \perp F$, then positive invariance of $\mathcal{L}_F$ for $X_{\underline{m}'}$ and Lemma 2.2.16 together yield that

$$\pi(\mathcal{L}_F)t_{\underline{m}'}(X_{\underline{m}'}) = t_{\underline{m}'}(\mathcal{L}_F X_{\underline{m}'}) \subseteq t_{\underline{m}'}(X_{\underline{m}'}\mathcal{L}_F) = t_{\underline{m}'}(X_{\underline{m}'})\pi(\mathcal{L}_F).$$

Then we have that

$$\begin{aligned}[t_{\underline{n}}(X_{\underline{n}})\pi(\mathcal{L}_F)q_F t_{\underline{m}'}(X_{\underline{m}'})t_{\underline{r}'}(X_{\underline{r}'})^*] &\subseteq [t_{\underline{n}}(X_{\underline{n}})\pi(\mathcal{L}_F)t_{\underline{m}'}(X_{\underline{m}'})q_F t_{\underline{r}'}(X_{\underline{r}'})^*] \\ &\subseteq [t_{\underline{n}+\underline{m}'}(X_{\underline{n}+\underline{m}'})\pi(\mathcal{L}_F)q_F t_{\underline{r}'}(X_{\underline{r}'})^*] \subseteq \mathfrak{J}_F,\end{aligned}$$

using Proposition 2.5.15 in the first inclusion. Hence in all cases we have that

$$t_{\underline{n}}(X_{\underline{n}})\pi(\mathcal{L}_F)q_F t_{\underline{m}}(X_{\underline{m}})^* t_{\underline{r}}(X_{\underline{r}}) \subseteq \mathfrak{J}_F,$$

and thus

$$\mathfrak{J}_F t_{\underline{r}}(X_{\underline{r}}) \subseteq \mathfrak{J}_F \text{ for all } \underline{r} \in \mathbb{Z}_+^d,$$

as required.

Finally, fix $a \in \mathcal{L}_F$ and an approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ of A. Note that

$$\mathfrak{J}_F \ni \pi(u_\lambda)\pi(a)q_F\pi(u_\lambda)^* = \pi(u_\lambda a u_\lambda)q_F \text{ for all } \lambda \in \Lambda.$$

Taking the norm-limit, we deduce that $\pi(a)q_F \in \mathfrak{J}_F$. Thus $\mathfrak{J}_F$ is an ideal of $\mathrm{C}^*(\pi, t)$ that contains the generators of $\mathfrak{J}_{\mathcal{L},F}^{(\pi,t)}$, as required. □

By requiring that $\mathcal{L}$ is partially ordered, we can say more.

Proposition 3.3.2. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be a relative 2^d-tuple of X that is invariant, partially ordered, and consists of ideals. Let (π, t) be a Nica-covariant representation of X. Then*

$$q_F \mathfrak{J}_{\mathcal{L},D}^{(\pi,t)} q_F \subseteq \overline{\operatorname{span}}\{t_{\underline{n}}(X_{\underline{n}})\pi(\mathcal{L}_{F\cup D})q_{F\cup D}t_{\underline{m}}(X_{\underline{m}})^* \mid \underline{n}, \underline{m} \perp F\} \subseteq \mathfrak{J}_{\mathcal{L},F\cup D}^{(\pi,t)} \textit{ for all } F, D \subseteq [d].$$

Moreover, we have that

$$q_F \mathfrak{J}_{\mathcal{L}}^{(\pi,t)} q_F \subseteq \sum\{\mathfrak{J}_{\mathcal{L},D}^{(\pi,t)} \mid F \subseteq D \subseteq [d]\} \textit{ for all } F \subseteq [d].$$

Proof. For the first part, fix $F, D \subseteq [d]$. Then we have that

$$\begin{aligned} q_F \mathfrak{J}_{\mathcal{L},D}^{(\pi,t)} q_F &\subseteq \overline{\operatorname{span}}\{q_F t_{\underline{n}}(X_{\underline{n}})\pi(\mathcal{L}_D) q_D t_{\underline{m}}(X_{\underline{m}})^* q_F \mid \underline{n}, \underline{m} \in \mathbb{Z}_+^d\} \\ &= \overline{\operatorname{span}}\{t_{\underline{n}}(X_{\underline{n}})\pi(\mathcal{L}_D) q_{F\cup D} t_{\underline{m}}(X_{\underline{m}})^* \mid \underline{n}, \underline{m} \perp F\} \\ &\subseteq \overline{\operatorname{span}}\{t_{\underline{n}}(X_{\underline{n}})\pi(\mathcal{L}_{F\cup D}) q_{F\cup D} t_{\underline{m}}(X_{\underline{m}})^* \mid \underline{n}, \underline{m} \perp F\} \subseteq \mathfrak{J}_{\mathcal{L},F\cup D}^{(\pi,t)}, \end{aligned}$$

where we have used Propositions 2.5.15 and 3.3.1, that q_F and q_D are commuting projections, and that $\mathcal{L}$ is partially ordered (so $\mathcal{L}_D \subseteq \mathcal{L}_{F\cup D}$).

For the second part, fix $F \subseteq [d]$. Then applying the first part for all $D \subseteq [d]$ yields that

$$\begin{aligned} q_F \mathfrak{J}_{\mathcal{L}}^{(\pi,t)} q_F &= \sum\{q_F \mathfrak{J}_{\mathcal{L},D}^{(\pi,t)} q_F \mid D \subseteq [d]\} \\ &\subseteq \sum\{\mathfrak{J}_{\mathcal{L},F\cup D}^{(\pi,t)} \mid D \subseteq [d]\} = \sum\{\mathfrak{J}_{\mathcal{L},D}^{(\pi,t)} \mid F \subseteq D \subseteq [d]\}, \end{aligned}$$

and the proof is complete. □

Let I be an ideal of A and fix $F \subseteq [d]$. For $a \in A$, we write

$$\lim_{\underline{m}\perp F} \|\phi_{\underline{m}}(a) + \mathcal{K}(X_{\underline{m}} I)\| = 0$$

if for any $\varepsilon > 0$ there exists $\underline{m} \perp F$ such that

$$\|\phi_{\underline{n}}(a) + \mathcal{K}(X_{\underline{n}} I)\| < \varepsilon \text{ for all } \underline{n} \geq \underline{m} \text{ satisfying } \underline{n} \perp F.$$

This property is preserved under unitary equivalence in the following sense.

Proposition 3.3.3. *Let X and Y be strong compactly aligned product systems with coefficients in C^*-algebras A and B, respectively. Suppose that X and Y are unitarily equivalent via a family $\{W_{\underline{n}} \colon X_{\underline{n}} \to Y_{\underline{n}}\}_{\underline{n}\in\mathbb{Z}_+^d}$. Let $I \subseteq A$ be an ideal and fix $a \in A$ and $F \subseteq [d]$. Then we have that*

$$\lim_{\underline{m}\perp F} \|\phi_{X_{\underline{m}}}(a) + \mathcal{K}(X_{\underline{m}} I)\| = 0 \textit{ if and only if } \lim_{\underline{m}\perp F} \|\phi_{Y_{\underline{m}}}(W_{\underline{0}}(a)) + \mathcal{K}(Y_{\underline{m}} W_{\underline{0}}(I))\| = 0.$$

Proof. Assume that $\lim_{\underline{m}\perp F} \|\phi_{X_{\underline{m}}}(a) + \mathcal{K}(X_{\underline{m}} I)\| = 0$ and fix $\varepsilon > 0$. We must show that there exists $\underline{m} \perp F$ such that

$$\|\phi_{Y_{\underline{n}}}(W_{\underline{0}}(a)) + \mathcal{K}(Y_{\underline{n}} W_{\underline{0}}(I))\| < \varepsilon \text{ for all } \underline{n} \geq \underline{m} \text{ satisfying } \underline{n} \perp F.$$

To this end, fix $\underline{n} \in \mathbb{Z}_+^d$. We have that

$$\begin{aligned} \|\phi_{Y_{\underline{n}}}(W_{\underline{0}}(a)) + \mathcal{K}(Y_{\underline{n}} W_{\underline{0}}(I))\| &= \inf\{\|\phi_{Y_{\underline{n}}}(W_{\underline{0}}(a)) + k_{\underline{n}}\| \mid k_{\underline{n}} \in \mathcal{K}(Y_{\underline{n}} W_{\underline{0}}(I))\} \\ &= \inf\{\|W_{\underline{n}} \phi_{X_{\underline{n}}}(a) W_{\underline{n}}^{-1} + W_{\underline{n}} k_{\underline{n}} W_{\underline{n}}^{-1}\| \mid k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}} I)\} \\ &= \inf\{\|W_{\underline{n}}(\phi_{X_{\underline{n}}}(a) + k_{\underline{n}}) W_{\underline{n}}^{-1}\| \mid k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}} I)\}, \end{aligned}$$

using (2.9) and (2.10) in the second equality. Notice that

$$\|W_{\underline{n}}(\phi_{X_{\underline{n}}}(a)+k_{\underline{n}})W_{\underline{n}}^{-1}\| = \|\phi_{X_{\underline{n}}}(a)+k_{\underline{n}}\| \text{ for all } k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}I),$$

using that $W_{\underline{n}}$ and $W_{\underline{n}}^{-1}$ are isometric and surjective. Hence $\|\phi_{Y_{\underline{n}}}(W_{\underline{0}}(a))+\mathcal{K}(Y_{\underline{n}}W_{\underline{0}}(I))\|$ is a lower bound for $\{\|\phi_{X_{\underline{n}}}(a)+k_{\underline{n}}\| \mid k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}I)\}$ and thus by definition we obtain that

$$\|\phi_{Y_{\underline{n}}}(W_{\underline{0}}(a))+\mathcal{K}(Y_{\underline{n}}W_{\underline{0}}(I))\| \leq \|\phi_{X_{\underline{n}}}(a)+\mathcal{K}(X_{\underline{n}}I)\| \text{ for all } \underline{n} \in \mathbb{Z}_+^d.$$

By assumption we may choose $\underline{m} \perp F$ such that

$$\|\phi_{X_{\underline{n}}}(a)+\mathcal{K}(X_{\underline{n}}I)\| < \varepsilon \text{ for all } \underline{n} \geq \underline{m} \text{ satisfying } \underline{n} \perp F.$$

In turn, for all $\underline{n} \geq \underline{m}$ satisfying $\underline{n} \perp F$, we deduce that

$$\|\phi_{Y_{\underline{n}}}(W_{\underline{0}}(a))+\mathcal{K}(Y_{\underline{n}}W_{\underline{0}}(I))\| \leq \|\phi_{X_{\underline{n}}}(a)+\mathcal{K}(X_{\underline{n}}I)\| < \varepsilon.$$

We conclude that $\lim_{\underline{m}\perp F} \|\phi_{Y_{\underline{m}}}(W_{\underline{0}}(a))+\mathcal{K}(Y_{\underline{m}}W_{\underline{0}}(I))\| = 0$, as required. The converse is obtained by duality, finishing the proof. □

When working with a 2^d-tuple $\mathcal{L}$ satisfying certain compatibility relations, the limit condition can be recast in a simpler form.

Proposition 3.3.4. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be an invariant 2^d-tuple of X which consists of ideals and satisfies*

$$\mathcal{L}_F \subseteq \bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in [d]\} \textit{ for all } F \subseteq [d].$$

Then, for each $F \subseteq [d]$ and $a \in A$, we have that $\lim_{\underline{m}\perp F} \|\phi_{\underline{m}}(a)+\mathcal{K}(X_{\underline{m}}\mathcal{L}_F)\| = 0$ if and only if for any $\varepsilon > 0$, there exists $\underline{m} \perp F$ and $k_{\underline{m}} \in \mathcal{K}(X_{\underline{m}}\mathcal{L}_F)$ such that $\|\phi_{\underline{m}}(a)+k_{\underline{m}}\| < \varepsilon$.

Proof. The forward implication is immediate by definition. So assume that for any $\varepsilon > 0$ there exists $\underline{m} \perp F$ and $k_{\underline{m}} \in \mathcal{K}(X_{\underline{m}}\mathcal{L}_F)$ such that $\|\phi_{\underline{m}}(a)+k_{\underline{m}}\| < \varepsilon$. Fix $\varepsilon > 0$ and a corresponding $\underline{m} \perp F$ and $k_{\underline{m}} \in \mathcal{K}(X_{\underline{m}}\mathcal{L}_F)$. If $F = [d]$, then $\underline{m} \perp F$ implies that $\underline{m} = \underline{0}$ and so

$$\|\phi_{\underline{0}}(a)+\mathcal{K}(A\mathcal{L}_F)\| \leq \|\phi_{\underline{0}}(a)+k_{\underline{0}}\| < \varepsilon.$$

Note that if $\underline{n} \geq \underline{0}$ and $\underline{n} \perp [d]$, then necessarily $\underline{n} = \underline{0}$ and so $\lim_{\underline{m}\perp [d]} \|\phi_{\underline{m}}(a) + \mathcal{K}(X_{\underline{m}}\mathcal{L}_{[d]})\| = 0$ by the preceding observation, as required.

Now assume that $F \subsetneq [d]$. Without loss of generality, we may assume that $\underline{m} \neq \underline{0}$. Indeed, if $\underline{m} = \underline{0}$ then $k_{\underline{0}} \in \mathcal{K}(A\mathcal{L}_F)$ and hence $k_{\underline{0}} = \phi_{\underline{0}}(b)$ for some $b \in \mathcal{L}_F$. Note that

$$\|a+b\| = \|\phi_{\underline{0}}(a)+\phi_{\underline{0}}(b)\| = \|\phi_{\underline{0}}(a)+k_{\underline{0}}\| < \varepsilon.$$

Fix $i \in F^c$. Since $b \in \mathcal{L}_F$, we have that $\phi_{\underline{i}}(b) \in \mathcal{K}(X_{\underline{i}})$ by assumption, and $\langle X_{\underline{i}}, bX_{\underline{i}}\rangle \subseteq \mathcal{L}_F$

by invariance of $\mathcal{L}$. Thus an application of (2.5) gives that $\phi_{\underline{i}}(b) \in \mathcal{K}(X_{\underline{i}}\mathcal{L}_F)$, and

$$\|\phi_{\underline{i}}(a) + \phi_{\underline{i}}(b)\| = \|\phi_{\underline{i}}(a+b)\| \leq \|a+b\| < \varepsilon.$$

So we may assume that $\underline{m} \neq \underline{0}$.

Next, take $\underline{n} \geq \underline{m}$ with $\underline{n} \perp F$. We will show that $\|\phi_{\underline{n}}(a) + \mathcal{K}(X_{\underline{n}}\mathcal{L}_F)\| < \varepsilon$. We write $\underline{n} = \underline{m} + \underline{r}$ for some $\underline{r} \perp F$, and without loss of generality we may assume that $\underline{r} \neq \underline{0}$. By Proposition 2.5.2, we have that $\phi_{\underline{r}}(\mathcal{L}_F) \subseteq \mathcal{K}(X_{\underline{r}}\mathcal{L}_F)$. An application of Corollary 2.2.14 then yields that $k_{\underline{m}} \otimes \mathrm{id}_{X_{\underline{r}}} \in \mathcal{K}((X_{\underline{m}} \otimes_A X_{\underline{r}})\mathcal{L}_F)$, and so $\iota_{\underline{m}}^{\underline{n}}(k_{\underline{m}}) \in \mathcal{K}(X_{\underline{n}}\mathcal{L}_F)$. We then obtain that

$$\|\phi_{\underline{n}}(a) + \mathcal{K}(X_{\underline{n}}\mathcal{L}_F)\| \leq \|\phi_{\underline{n}}(a) + \iota_{\underline{m}}^{\underline{n}}(k_{\underline{m}})\| = \|\iota_{\underline{m}}^{\underline{n}}(\phi_{\underline{m}}(a) + k_{\underline{m}})\| \leq \|\phi_{\underline{m}}(a) + k_{\underline{m}}\| < \varepsilon,$$

from which it follows that $\lim_{\underline{m}\perp F} \|\phi_{\underline{m}}(a) + \mathcal{K}(X_{\underline{m}}\mathcal{L}_F)\| = 0$, as required. □

The limit condition appears naturally in the study of ideals of $\mathcal{NT}_X$ that are induced by invariant, partially ordered relative 2^d-tuples that consist of ideals.

Proposition 3.3.5. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be a relative 2^d-tuple of X that is invariant, partially ordered and consists of ideals. Fix $F \subsetneq [d]$ and let $a \in \bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in [d]\}$. Then the following are equivalent:*

(i) $\overline{\pi}_X(a)\overline{q}_{X,F} \in \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)}$;

(ii) $\overline{\pi}_X(a)\overline{q}_{X,F} \in \sum\{\mathfrak{J}_{\mathcal{L},D}^{(\overline{\pi}_X,\overline{t}_X)} \mid F \subseteq D \subseteq [d]\}$.

Furthermore, if one (and hence both) of (i) and (ii) holds, then

$$\lim_{\underline{m}\perp F} \|\phi_{\underline{m}}(a) + \mathcal{K}(X_{\underline{m}}\mathcal{L}_F)\| = 0.$$

Proof. Without loss of generality, we may replace $(\overline{\pi}_X, \overline{t}_X)$ by the Fock representation $(\overline{\pi}, \overline{t})$ and write

$$\mathfrak{J}_{\mathcal{L},F} \equiv \mathfrak{J}_{\mathcal{L},F}^{(\overline{\pi},\overline{t})} \text{ for all } F \subseteq [d] \quad \text{and} \quad \mathfrak{J}_{\mathcal{L}} \equiv \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi},\overline{t})}.$$

Indeed, this follows from the fact that

$$(\overline{\pi} \times \overline{t})(\mathfrak{J}_{\mathcal{L},F}^{(\overline{\pi}_X,\overline{t}_X)}) = \mathfrak{J}_{\mathcal{L},F} \text{ for all } F \subseteq [d] \quad \text{and} \quad (\overline{\pi} \times \overline{t})(\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)}) = \mathfrak{J}_{\mathcal{L}}.$$

The implication [(ii)⇒(i)] is immediate. Conversely, if $\overline{\pi}(a)\overline{q}_F \in \mathfrak{J}_{\mathcal{L}}$, then we have that

$$\overline{\pi}(a)\overline{q}_F = \overline{q}_F(\overline{\pi}(a)\overline{q}_F)\overline{q}_F \in \overline{q}_F\mathfrak{J}_{\mathcal{L}}\overline{q}_F \subseteq \sum\{\mathfrak{J}_{\mathcal{L},D} \mid F \subseteq D \subseteq [d]\},$$

using Proposition 2.5.15 (and the fact that $\overline{q}_F$ is a projection) in the first equality, and Proposition 3.3.2 in the final inclusion. This establishes the equivalence of (i) and (ii).

Now suppose that one (and hence both) of (i) and (ii) holds. Condition (ii) then yields that

$$\begin{aligned}\overline{\pi}(a)\overline{q}_F &\in \sum\{\overline{q}_F \mathfrak{J}_{\mathcal{L},D} \overline{q}_F \mid F \subseteq D \subseteq [d]\} \\ &\subseteq \overline{\operatorname{span}}\{\overline{t}_{\underline{n}}(X_{\underline{n}})\overline{\pi}(\mathcal{L}_D)\overline{q}_D \overline{t}_{\underline{m}}(X_{\underline{m}})^* \mid \underline{n},\underline{m} \perp F, F \subseteq D \subseteq [d]\},\end{aligned}$$

using Proposition 3.3.2 for the containment. Let γ be the gauge action of $(\overline{\pi},\overline{t})$ and let $E_\gamma \colon \mathrm{C}^*(\overline{\pi},\overline{t}) \to \mathrm{C}^*(\overline{\pi},\overline{t})^\gamma$ be the associated faithful conditional expectation. Therefore, since $\overline{\pi}(a)\overline{q}_F \in \mathrm{C}^*(\overline{\pi},\overline{t})^\gamma$, applying E_γ yields that

$$\overline{\pi}(a)\overline{q}_F \in \overline{\operatorname{span}}\{\overline{t}_{\underline{n}}(X_{\underline{n}})\overline{\pi}(\mathcal{L}_D)\overline{q}_D \overline{t}_{\underline{n}}(X_{\underline{n}})^* \mid \underline{n} \perp F, F \subseteq D \subseteq [d]\}. \tag{3.3}$$

Fix $\varepsilon > 0$. By Proposition 3.3.4, it is sufficient to find $\underline{m} \perp F$ and $k_{\underline{m}} \in \mathcal{K}(X_{\underline{m}}\mathcal{L}_F)$ such that $\|\phi_{\underline{m}}(a) + k_{\underline{m}}\| < \varepsilon$. By (3.3), for each $F \subseteq D \subseteq [d]$ there exists

$$c_D \in \operatorname{span}\{\overline{t}_{\underline{n}}(X_{\underline{n}})\overline{\pi}(\mathcal{L}_D)\overline{q}_D \overline{t}_{\underline{n}}(X_{\underline{n}})^* \mid \underline{n} \perp F\}$$

such that

$$\|\overline{\pi}(a)\overline{q}_F - \sum\{c_D \mid F \subseteq D \subseteq [d]\}\| < \varepsilon.$$

Moreover, we may assume that

$$c_D = \sum\{\sum_{j=1}^N \overline{t}_{\underline{n}}(\xi^j_{\underline{n},D})\overline{\pi}(b^j_{\underline{n},D})\overline{q}_D \overline{t}_{\underline{n}}(\eta^j_{\underline{n},D})^* \mid \underline{0} \leq \underline{n} \leq m \cdot \underline{1}_{F^c}\},$$

for some $m, N \in \mathbb{N}$, $\xi^j_{\underline{n},D}, \eta^j_{\underline{n},D} \in X_{\underline{n}}$ and $b^j_{\underline{n},D} \in \mathcal{L}_D$, for all $j \in [N]$ and $\underline{0} \leq \underline{n} \leq m \cdot \underline{1}_{F^c}$. We may use the same m and N for every c_D by padding with zeros, if necessary. Fix the projection

$$P \colon \mathcal{F}X \to X_{(m+1)\cdot \underline{1}_{F^c}}.$$

First we claim that $c_D P = 0$ whenever $F \subsetneq D \subseteq [d]$, so that

$$\begin{aligned}\|\overline{\pi}(a)\overline{q}_F P - c_F P\| &= \|(\overline{\pi}(a)\overline{q}_F - \sum\{c_D \mid F \subseteq D \subseteq [d]\})P\| \\ &\leq \|\overline{\pi}(a)\overline{q}_F - \sum\{c_D \mid F \subseteq D \subseteq [d]\}\| < \varepsilon.\end{aligned}$$

Indeed, for each $\underline{0} \leq \underline{n} \leq m \cdot \underline{1}_{F^c}$ and each $j \in [N]$, we have that $\overline{t}_{\underline{n}}(\eta^j_{\underline{n},D})^* P$ has image in $X_{\underline{r}}$ for some $\underline{r} \neq \underline{0}$ with $\operatorname{supp} \underline{r} = F^c$, since $m < m+1$. Note also that $D \cap F^c \neq \emptyset$ since $F \subsetneq D$, and thus

$$\overline{q}_D \overline{t}_{\underline{n}}(\eta^j_{\underline{n},D})^* P(\mathcal{F}X) \subseteq \overline{q}_D \sum\{X_{\underline{r}} \mid \operatorname{supp} \underline{r} = F^c\} = \{0\}.$$

Hence

$$\overline{t}_{\underline{n}}(\xi^j_{\underline{n},D})\overline{\pi}(b^j_{\underline{n},D})\overline{q}_D \overline{t}_{\underline{n}}(\eta^j_{\underline{n},D})^* P = 0, \text{ for all } \underline{0} \leq \underline{n} \leq m \cdot \underline{1}_{F^c}, j \in [N],$$

and so $c_D P = 0$.

Next we examine the terms of $c_F P$. To this end, fix $\underline{0} \neq \underline{n} \leq m \cdot \underline{1}_{F^c}$ and $j \in [N]$. Take $\zeta_{(m+1)\cdot \underline{1}_{F^c}} \in X_{(m+1)\cdot \underline{1}_{F^c}}$ such that

$$\zeta_{(m+1)\cdot \underline{1}_{F^c}} = \zeta_{\underline{n}} \zeta_{(m+1)\cdot \underline{1}_{F^c} - \underline{n}} \text{ for some } \zeta_{\underline{n}} \in X_{\underline{n}}, \zeta_{(m+1)\cdot \underline{1}_{F^c} - \underline{n}} \in X_{(m+1)\cdot \underline{1}_{F^c} - \underline{n}}.$$

Note that $\operatorname{supp}((m+1)\cdot \underline{1}_{F^c} - \underline{n}) = F^c$, and so we have that

$$\begin{aligned} \overline{t}_{\underline{n}}(\xi^j_{\underline{n},F}) \overline{\pi}(b^j_{\underline{n},F}) \overline{q}_F \overline{t}_{\underline{n}}(\eta^j_{\underline{n},F})^* P \zeta_{(m+1)\cdot \underline{1}_{F^c}} &= \overline{t}_{\underline{n}}(\xi^j_{\underline{n},F}) \overline{\pi}(b^j_{\underline{n},F}) \overline{q}_F \overline{t}_{\underline{n}}(\eta^j_{\underline{n},F})^* \zeta_{(m+1)\cdot \underline{1}_{F^c}} \\ &= \xi^j_{\underline{n},F} \big(\phi_{(m+1)\cdot \underline{1}_{F^c} - \underline{n}} (b^j_{\underline{n},F} \langle \eta^j_{\underline{n},F}, \zeta_{\underline{n}} \rangle) \zeta_{(m+1)\cdot \underline{1}_{F^c} - \underline{n}} \big) \\ &= \big(\Theta_{\xi^j_{\underline{n},F} b^j_{\underline{n},F}, \eta^j_{\underline{n},F}} (\zeta_{\underline{n}}) \big) \zeta_{(m+1)\cdot \underline{1}_{F^c} - \underline{n}} \\ &= \iota_{\underline{n}}^{(m+1)\cdot \underline{1}_{F^c}} (\Theta_{\xi^j_{\underline{n},F} b^j_{\underline{n},F}, \eta^j_{\underline{n},F}}) \zeta_{(m+1)\cdot \underline{1}_{F^c}}. \end{aligned}$$

We deduce that

$$\overline{t}_{\underline{n}}(\xi^j_{\underline{n},F}) \overline{\pi}(b^j_{\underline{n},F}) \overline{q}_F \overline{t}_{\underline{n}}(\eta^j_{\underline{n},F})^* P = \iota_{\underline{n}}^{(m+1)\cdot \underline{1}_{F^c}} (\Theta_{\xi^j_{\underline{n},F} b^j_{\underline{n},F}, \eta^j_{\underline{n},F}}),$$

where we view the right hand side as an element of $\mathcal{L}(\mathcal{F}X)$ in the way described in Section 2.3. Arguing in a similar (and simpler) way when $\underline{n} = \underline{0}$, we deduce that

$$c_F P = \phi_{(m+1)\cdot \underline{1}_{F^c}}(b) + \sum \{ \sum_{j=1}^N \iota_{\underline{n}}^{(m+1)\cdot \underline{1}_{F^c}} (\Theta_{\xi^j_{\underline{n},F} b^j_{\underline{n},F}, \eta^j_{\underline{n},F}}) \mid \underline{0} \neq \underline{n} \leq m \cdot \underline{1}_{F^c} \},$$

for some $b \in \mathcal{L}_F$. By Proposition 2.5.2, we have that

$$\phi_{(m+1)\cdot \underline{1}_{F^c}}(b) \in \mathcal{K}(X_{(m+1)\cdot \underline{1}_{F^c}} \mathcal{L}_F)$$

and

$$\phi_{(m+1)\cdot \underline{1}_{F^c} - \underline{n}}(b^j_{\underline{n},F}) \in \mathcal{K}(X_{(m+1)\cdot \underline{1}_{F^c} - \underline{n}} \mathcal{L}_F) \text{ for all } \underline{0} \neq \underline{n} \leq m \cdot \underline{1}_{F^c}, j \in [N].$$

Lemma 2.2.13 gives that all of the summands of $c_F P$ belong to $\mathcal{K}(X_{(m+1)\cdot \underline{1}_{F^c}} \mathcal{L}_F)$. Therefore we have that $c_F P \in \mathcal{K}(X_{(m+1)\cdot \underline{1}_{F^c}} \mathcal{L}_F) \hookrightarrow \mathcal{L}(\mathcal{F}X)$, which satisfies

$$\| \phi_{(m+1)\cdot \underline{1}_{F^c}}(a) - c_F P \| = \| \overline{\pi}(a) \overline{q}_F P - c_F P \| < \varepsilon,$$

where we use that

$$\phi_{(m+1)\cdot \underline{1}_{F^c}}(a) = \overline{\pi}(a) P = \overline{\pi}(a) \overline{q}_F P$$

in the identification $\mathcal{L}(X_{(m+1)\cdot \underline{1}_{F^c}}) \hookrightarrow \mathcal{L}(\mathcal{F}X)$. Hence we have found $\underline{m} := (m+1)\cdot \underline{1}_{F^c} \perp F$ and $k_{\underline{m}} := -c_F P \in \mathcal{K}(X_{\underline{m}} \mathcal{L}_F)$ such that $\| \phi_{\underline{m}}(a) + k_{\underline{m}} \| < \varepsilon$, as required. □

Remark 3.3.6. Proposition 3.3.5 also holds for $F = [d]$, though the second claim requires a different argument. Specifically, we compress $\overline{\pi}(a) \overline{q}_{[d]} \in \mathfrak{J}_{\mathcal{L},[d]}$ at the $\underline{0}$-th entry to deduce that $a \in \mathcal{L}_{[d]}$. Then $\phi_{\underline{0}}(a) \in \mathcal{K}(A \mathcal{L}_{[d]})$ and the claim immediately follows.

3.4 Constructing (M)-2^d-tuples

In this section we show how the maximal 2^d-tuple inducing $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)}$, for $\mathcal{L}$ an (E)-2^d-tuple, is constructed. The following definition is motivated in part by Proposition 3.3.5.

Definition 3.4.1. Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let $\mathcal{L}$ be a 2^d-tuple of X that consists of ideals. Fixing $\emptyset \neq F \subsetneq [d]$, we define

$$\mathcal{L}_{\text{inv},F} := \bigcap_{\underline{m}\perp F} X_{\underline{m}}^{-1}(\cap_{F\subsetneq D}\mathcal{L}_D) \quad \text{and} \quad \mathcal{L}_{\text{lim},F} := \{a \in A \mid \lim_{\underline{m}\perp F} \|\phi_{\underline{m}}(a) + \mathcal{K}(X_{\underline{m}}\mathcal{L}_F)\| = 0\}.$$

If $\mathcal{L}$ is in addition an (E)-2^d-tuple of X, then we define the 2^d-tuple $\mathcal{L}^{(1)}$ of X by

$$\mathcal{L}_F^{(1)} := \begin{cases} \{0\} & \text{if } F = \emptyset, \\ \mathcal{I}_F \cap \mathcal{L}_{\text{inv},F} \cap \mathcal{L}_{\text{lim},F} & \text{if } \emptyset \neq F \subsetneq [d], \\ \mathcal{L}_{[d]} & \text{if } F = [d]. \end{cases}$$

Observe that each $\mathcal{L}_F^{(1)}$ is non-empty, as $0 \in \mathcal{L}_F^{(1)}$. Note also that $\mathcal{L}^{(1)}$ is an (E)-2^d-tuple by construction. We will show that $\mathcal{L}^{(1)}$ consists of ideals, and so we can write

$$\mathcal{L}^{(0)} := \mathcal{L} \quad \text{and} \quad \mathcal{L}^{(k+1)} := (\mathcal{L}^{(k)})^{(1)} \text{ for all } k \in \mathbb{Z}_+.$$

To this end, first we note that $\mathcal{L}_{\text{inv},F}$ is an intersection of ideals and is thus an ideal itself. To address $\mathcal{L}_{\text{lim},F}$, we have the following proposition.

Proposition 3.4.2. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be a 2^d-tuple of X that consists of ideals and fix $\emptyset \neq F \subsetneq [d]$. Then we have that $\mathcal{L}_{\text{lim},F}$ is an ideal.*

Proof. Firstly, we define

$$B_F := \bigoplus_{\underline{m}\perp F} \mathcal{L}(X_{\underline{m}})/\mathcal{K}(X_{\underline{m}}\mathcal{L}_F) \quad \text{and} \quad c_0(B_F) := \{(S_{\underline{m}})_{\underline{m}\perp F} \in B_F \mid \lim_{\underline{m}\perp F} \|S_{\underline{m}}\| = 0\},$$

and observe that $c_0(B_F)$ is an ideal in B_F. Consider the $*$-homomorphism ψ_F defined by

$$\psi_F \colon A \to \bigoplus_{\underline{m}\perp F} \mathcal{L}(X_{\underline{m}}) \to B_F \to B_F/c_0(B_F);$$

$$\psi_F \colon a \mapsto (\phi_{\underline{m}}(a))_{\underline{m}\perp F} \mapsto (\phi_{\underline{m}}(a) + \mathcal{K}(X_{\underline{m}}\mathcal{L}_F))_{\underline{m}\perp F} \mapsto (\phi_{\underline{m}}(a) + \mathcal{K}(X_{\underline{m}}\mathcal{L}_F))_{\underline{m}\perp F} + c_0(B_F).$$

It is now a standard C*-result that

$$\ker \psi_F = \{a \in A \mid \lim_{\underline{m}\perp F} \|\phi_{\underline{m}}(a) + \mathcal{K}(X_{\underline{m}}\mathcal{L}_F)\| = 0\},$$

and thus $\mathcal{L}_{\lim,F} = \ker \psi_F$ is an ideal, as required. $\square$

Proposition 3.4.3. *Let X be a strong compactly aligned product system with coefficients in a C^*-algebra A. Let $\mathcal{L}$ be an (E)-2^d-tuple of X that consists of ideals. Then $\mathcal{L}^{(1)}$ is an (E)-2^d-tuple of X that consists of ideals. If $\mathcal{L}$ is invariant, then so is $\mathcal{L}^{(1)}$.*

Proof. We have already remarked that $\mathcal{L}^{(1)}$ is an (E)-2^d-tuple. The fact that $\mathcal{L}^{(1)}$ consists of ideals follows from Proposition 3.4.2 and the discussion preceding it.

Now assume that $\mathcal{L}$ is invariant. The invariance condition for $\mathcal{L}^{(1)}$ clearly holds when $F = \emptyset$ or $F = [d]$, so assume that $\emptyset \neq F \subsetneq [d]$. Fix $\underline{n} \perp F$, $\xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\underline{n}}$, and $a \in \mathcal{L}_F^{(1)}$. It suffices to show that

$$\langle \xi_{\underline{n}}, \phi_{\underline{n}}(a)\eta_{\underline{n}} \rangle \in \mathcal{L}_F^{(1)} = \mathcal{I}_F \cap \mathcal{L}_{\mathrm{inv},F} \cap \mathcal{L}_{\lim,F}.$$

If $\underline{n} = \underline{0}$ there is nothing to show, so assume that $\underline{n} \neq \underline{0}$. First we have that $\langle \xi_{\underline{n}}, \phi_{\underline{n}}(a)\eta_{\underline{n}} \rangle \in \mathcal{I}_F$ by Proposition 2.5.10, since $a \in \mathcal{I}_F$. Now fix $\underline{m} \perp F$ and $\xi_{\underline{m}}, \eta_{\underline{m}} \in X_{\underline{m}}$. Then

$$\langle \xi_{\underline{m}}, \phi_{\underline{m}}(\langle \xi_{\underline{n}}, \phi_{\underline{n}}(a)\eta_{\underline{n}} \rangle)\eta_{\underline{m}} \rangle = \langle \xi_{\underline{n}}\xi_{\underline{m}}, (\phi_{\underline{n}}(a)\eta_{\underline{n}})\eta_{\underline{m}} \rangle = \langle \xi_{\underline{n}}\xi_{\underline{m}}, \phi_{\underline{n}+\underline{m}}(a)(\eta_{\underline{n}}\eta_{\underline{m}}) \rangle \in \cap_{F \subsetneq D} \mathcal{L}_D,$$

since $a \in \mathcal{L}_F^{(1)}$ and thus in particular $a \in \mathcal{L}_{\mathrm{inv},F}$, noting that $\underline{n} + \underline{m} \perp F$. This proves that $\langle \xi_{\underline{n}}, \phi_{\underline{n}}(a)\eta_{\underline{n}} \rangle \in \mathcal{L}_{\mathrm{inv},F}$.

It remains to check that $\lim_{\underline{m} \perp F} \|\phi_{\underline{m}}(\langle \xi_{\underline{n}}, \phi_{\underline{n}}(a)\eta_{\underline{n}} \rangle) + \mathcal{K}(X_{\underline{m}}\mathcal{L}_F)\| = 0$. To this end, fix $\varepsilon > 0$. Since $a \in \mathcal{L}_F^{(1)} \subseteq \mathcal{L}_{\lim,F}$, there exist $\underline{m} \perp F$ and $k_{\underline{m}} \in \mathcal{K}(X_{\underline{m}}\mathcal{L}_F)$ such that

$$\|\phi_{\underline{m}}(a) + k_{\underline{m}}\| < \frac{\varepsilon}{\|\xi_{\underline{n}}\| \cdot \|\eta_{\underline{n}}\|}.$$

Without loss of generality, we may assume that $\underline{m} \neq \underline{0}$ (if $\underline{m} = \underline{0}$, then argue as in the proof of Proposition 3.3.4 to replace $\underline{m}$ by $\underline{i}$ for some $i \in F^c$). We have that

$$\iota_{\underline{m}}^{\underline{m}+\underline{n}}(k_{\underline{m}}) = u_{\underline{m},\underline{n}}(k_{\underline{m}} \otimes \mathrm{id}_{X_{\underline{n}}})u_{\underline{m},\underline{n}}^*,$$

and $k_{\underline{m}} \otimes \mathrm{id}_{X_{\underline{n}}} \in \mathcal{K}((X_{\underline{m}} \otimes_A X_{\underline{n}})\mathcal{L}_F)$ by Corollary 2.2.14, noting that $\phi_{\underline{n}}(\mathcal{L}_F) \subseteq \mathcal{K}(X_{\underline{n}}\mathcal{L}_F)$ by Proposition 2.5.2. It follows that

$$\iota_{\underline{m}}^{\underline{m}+\underline{n}}(k_{\underline{m}}) \in \mathcal{K}(X_{\underline{m}+\underline{n}}\mathcal{L}_F).$$

Next, define $\tau(\xi_{\underline{n}}) \in \mathcal{L}(X_{\underline{m}}, X_{\underline{m}+\underline{n}})$ and $\tau(\eta_{\underline{n}}) \in \mathcal{L}(X_{\underline{m}}, X_{\underline{m}+\underline{n}})$ by

$$\tau(\xi_{\underline{n}})\xi_{\underline{m}} = \xi_{\underline{n}}\xi_{\underline{m}} \quad \text{and} \quad \tau(\eta_{\underline{n}})\xi_{\underline{m}} = \eta_{\underline{n}}\xi_{\underline{m}}$$

for all $\xi_{\underline{m}} \in X_{\underline{m}}$, and observe that $\|\tau(\xi_{\underline{n}})\| \leq \|\xi_{\underline{n}}\|$ and $\|\tau(\eta_{\underline{n}})\| \leq \|\eta_{\underline{n}}\|$. We then have that

$$\tau(\xi_{\underline{n}})^* \iota_{\underline{m}}^{\underline{m}+\underline{n}}(k_{\underline{m}})\tau(\eta_{\underline{n}}) \in \mathcal{K}(X_{\underline{m}}\mathcal{L}_F). \tag{3.4}$$

Indeed, taking $\zeta_{\underline{n}}, \zeta'_{\underline{n}} \in X_{\underline{n}}, \zeta_{\underline{m}}, \zeta'_{\underline{m}} \in X_{\underline{m}}$ and $b \in \mathcal{L}_F$, we compute

$$\tau(\xi_{\underline{n}})^* \Theta^{X_{\underline{m}+\underline{n}}}_{\zeta_{\underline{n}}\zeta_{\underline{m}}b, \zeta'_{\underline{n}}\zeta'_{\underline{m}}} \tau(\eta_{\underline{n}}) = \Theta^{X_{\underline{m}}}_{\phi_{\underline{m}}(\langle \xi_{\underline{n}}, \zeta_{\underline{n}} \rangle)\zeta_{\underline{m}}b, \phi_{\underline{m}}(\langle \eta_{\underline{n}}, \zeta'_{\underline{n}} \rangle)\zeta'_{\underline{m}}} \in \mathcal{K}(X_{\underline{m}}\mathcal{L}_F).$$

By taking finite linear combinations and their norm-limits, we deduce that

$$\tau(\xi_{\underline{n}})^* \mathcal{K}(X_{\underline{m}+\underline{n}}\mathcal{L}_F)\tau(\eta_{\underline{n}}) \subseteq \mathcal{K}(X_{\underline{m}}\mathcal{L}_F),$$

which implies (3.4). Fixing $\zeta_{\underline{m}} \in X_{\underline{m}}$, we also have that

$$\begin{aligned} \tau(\xi_{\underline{n}})^* \phi_{\underline{m}+\underline{n}}(a)\tau(\eta_{\underline{n}})\zeta_{\underline{m}} &= \tau(\xi_{\underline{n}})^* \phi_{\underline{m}+\underline{n}}(a)(\eta_{\underline{n}}\zeta_{\underline{m}}) \\ &= \tau(\xi_{\underline{n}})^*((\phi_{\underline{n}}(a)\eta_{\underline{n}})\zeta_{\underline{m}}) \\ &= \phi_{\underline{m}}(\langle \xi_{\underline{n}}, \phi_{\underline{n}}(a)\eta_{\underline{n}} \rangle)\zeta_{\underline{m}}, \end{aligned}$$

from which it follows that

$$\tau(\xi_{\underline{n}})^* \phi_{\underline{m}+\underline{n}}(a)\tau(\eta_{\underline{n}}) = \phi_{\underline{m}}(\langle \xi_{\underline{n}}, \phi_{\underline{n}}(a)\eta_{\underline{n}} \rangle).$$

We then obtain that

$$\begin{aligned} \|\phi_{\underline{m}}(\langle \xi_{\underline{n}}, \phi_{\underline{n}}(a)\eta_{\underline{n}} \rangle) + \tau(\xi_{\underline{n}})^* \iota_{\underline{m}}^{\underline{m}+\underline{n}}(k_{\underline{m}})\tau(\eta_{\underline{n}})\| &= \|\tau(\xi_{\underline{n}})^*(\phi_{\underline{m}+\underline{n}}(a) + \iota_{\underline{m}}^{\underline{m}+\underline{n}}(k_{\underline{m}}))\tau(\eta_{\underline{n}})\| \\ &\leq \|\xi_{\underline{n}}\| \cdot \|\iota_{\underline{m}}^{\underline{m}+\underline{n}}(\phi_{\underline{m}}(a)) + \iota_{\underline{m}}^{\underline{m}+\underline{n}}(k_{\underline{m}})\| \cdot \|\eta_{\underline{n}}\| \\ &\leq \|\xi_{\underline{n}}\| \cdot \|\phi_{\underline{m}}(a) + k_{\underline{m}}\| \cdot \|\eta_{\underline{n}}\| \\ &< \|\xi_{\underline{n}}\| \cdot \left(\frac{\varepsilon}{\|\xi_{\underline{n}}\| \cdot \|\eta_{\underline{n}}\|} \right) \cdot \|\eta_{\underline{n}}\| < \varepsilon. \end{aligned}$$

This shows that $\langle \xi_{\underline{n}}, \phi_{\underline{n}}(a)\eta_{\underline{n}} \rangle \in \mathcal{L}_{\lim, F}$ by (3.4) and Proposition 3.3.4 (since $\mathcal{L}$ is assumed to be invariant), and the proof is complete. □

Next we explore the interaction of the $\mathcal{L}^{(1)}$ construction with the partial ordering property. To this end, we have the following auxiliary proposition.

Proposition 3.4.4. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be an (E)-2^d-tuple of X that is invariant and consists of ideals. Then for each $F \subseteq [d]$, we have that*

$$\overline{\pi}_X(\mathcal{L}_F^{(1)})\overline{q}_{X,F} \subseteq \sum\{\mathfrak{J}_{\mathcal{L},D}^{(\overline{\pi}_X, \overline{t}_X)} \mid F \subseteq D \subseteq [d]\} \subseteq \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)}.$$

Proof. It suffices to show the first inclusion of the statement, as the second holds by definition. Without loss of generality, we may replace $(\overline{\pi}_X, \overline{t}_X)$ by the Fock representation $(\overline{\pi}, \overline{t})$ and write

$$\mathfrak{J}_{\mathcal{L},F} \equiv \mathfrak{J}_{\mathcal{L},F}^{(\overline{\pi},\overline{t})} \text{ for all } F \subseteq [d] \quad \text{and} \quad \mathfrak{J}_{\mathcal{L}} \equiv \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi},\overline{t})}.$$

The claim holds trivially when $F = \emptyset$ or $F = [d]$, so fix $\emptyset \neq F \subsetneq [d]$ and $a \in \mathcal{L}_F^{(1)}$. We must show that $\overline{\pi}(a)\overline{q}_F \in \sum\{\mathfrak{J}_{\mathcal{L},D} \mid F \subseteq D \subseteq [d]\}$. Fixing $\varepsilon > 0$, it suffices to show that $\overline{\pi}(a)\overline{q}_F$ is ε-close to an element of $\sum\{\mathfrak{J}_{\mathcal{L},D} \mid F \subseteq D \subseteq [d]\}$, as the latter is closed in $\mathrm{C}^*(\overline{\pi}, \overline{t})$.

Since $a \in \mathcal{L}_F^{(1)} \subseteq \mathcal{L}_{\lim,F}$, there exists $\underline{m} \perp F$ such that

$$\|\phi_{\underline{n}}(a) + \mathcal{K}(X_{\underline{n}}\mathcal{L}_F)\| < \varepsilon \text{ for every } \underline{n} \geq \underline{m} \text{ with } \underline{n} \perp F.$$

Consider a suitable $m \in \mathbb{N}$ such that

$$\underline{n} := (m+1) \cdot \underline{1}_{F^c} \geq \underline{m},$$

and take $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}\mathcal{L}_F)$ such that $\|\phi_{\underline{n}}(a) + k_{\underline{n}}\| < \varepsilon$. Next, for each $i \in F^c$ we define the projection

$$W_i \colon \mathcal{F}X \to \sum\{X_{\underline{r}} \mid \underline{r} = (r_1, \dots, r_d) \in \mathbb{Z}_+^d, r_i \leq m\},$$

and consider the operator

$$\overline{\pi}(a)\overline{q}_F \prod_{i \in F^c}(I - W_i) = \overline{\pi}(a)\overline{q}_F + \sum_{\emptyset \neq D \subseteq F^c} (-1)^{|D|}\overline{\pi}(a)\overline{q}_F \prod_{i \in D} W_i.$$

The products can be taken in any order, since the projections W_i commute. Indeed, to see this fix distinct elements $i, j \in F^c$. Since $W_i, W_j \in \mathcal{L}(\mathcal{F}X)$, it suffices to show that W_iW_j and W_jW_i coincide on every direct summand. Accordingly, fix $\underline{r} = (r_1, \dots, r_d) \in \mathbb{Z}_+^d$ and $\zeta_{\underline{r}} \in X_{\underline{r}}$. By definition, we have that

$$W_iW_j\zeta_{\underline{r}} = \begin{cases} \zeta_{\underline{r}} & \text{if } r_i, r_j \leq m, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad W_jW_i\zeta_{\underline{r}} = \begin{cases} \zeta_{\underline{r}} & \text{if } r_j, r_i \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Due to symmetry with respect to i and j, we deduce that $W_iW_j = W_jW_i$, as claimed.

For each element $\underline{r} = (r_1, \dots, r_d) \in \mathbb{Z}_+^d$ and $\zeta_{\underline{r}} \in X_{\underline{r}}$, we have that

$$\overline{\pi}(a)\overline{q}_F \prod_{i \in F^c}(I - W_i)\zeta_{\underline{r}} = \begin{cases} \phi_{\underline{r}}(a)\zeta_{\underline{r}} & \text{if } r_i \geq m+1 \text{ for all } i \in F^c \text{ and } \underline{r} \perp F, \\ 0 & \text{if } r_i \leq m \text{ for some } i \in F^c \text{ or } \underline{r} \not\perp F. \end{cases}$$

It follows that

$$\overline{\pi}(a)\overline{q}_F \prod_{i \in F^c}(I - W_i) = \sum_{\underline{r} \geq \underline{n}, \underline{r} \perp F} \phi_{\underline{r}}(a),$$

where we view the latter as an element of $\mathcal{L}(\mathcal{F}X)$ in the usual way. Then we have that

$$\begin{aligned} \| \sum_{\underline{r} \geq \underline{n}, \underline{r} \perp F} \phi_{\underline{r}}(a) + \sum_{\underline{r} \geq \underline{n}, \underline{r} \perp F} \iota_{\underline{n}}^{\underline{r}}(k_{\underline{n}})\| &= \| \sum_{\underline{r} \geq \underline{n}, \underline{r} \perp F} \iota_{\underline{n}}^{\underline{r}}(\phi_{\underline{n}}(a) + k_{\underline{n}})\| \\ &= \sup\{\|\iota_{\underline{n}}^{\underline{r}}(\phi_{\underline{n}}(a) + k_{\underline{n}})\| \mid \underline{r} \geq \underline{n}, \underline{r} \perp F\} \\ &\leq \|\phi_{\underline{n}}(a) + k_{\underline{n}}\| < \varepsilon, \end{aligned}$$

using that each $\iota_{\underline{n}}^{\underline{r}}$ is contractive. Therefore we deduce that

$$\|\overline{\pi}(a)\overline{q}_F \prod_{i \in F^c} (I - W_i) + \sum_{\underline{r} \geq \underline{n}, \underline{r} \perp F} \iota_{\underline{n}}^{\underline{r}}(k_{\underline{n}})\| < \varepsilon. \tag{3.5}$$

We will prove two claims before proceeding to the completion of the proof.

Claim 1. The element $x_F := \sum_{\underline{r} \geq \underline{n}, \underline{r} \perp F} \iota_{\underline{n}}^{\underline{r}}(k_{\underline{n}})$ *belongs to* $\mathfrak{J}_{\mathcal{L},F}$.

Proof of Claim 1. First suppose $k_{\underline{n}} = \|\cdot\|\text{-}\lim_n k_{\underline{n},n}$, where each $k_{\underline{n},n}$ is a finite sum of rank-one operators in $\mathcal{K}(X_{\underline{n}}\mathcal{L}_F)$. Then x_F is the norm-limit of the elements $\sum_{\underline{r} \geq \underline{n}, \underline{r} \perp F} \iota_{\underline{n}}^{\underline{r}}(k_{\underline{n},n})$. Indeed, we have that

$$\begin{aligned} \|x_F - \sum_{\underline{r} \geq \underline{n}, \underline{r} \perp F} \iota_{\underline{n}}^{\underline{r}}(k_{\underline{n},n})\| &= \| \sum_{\underline{r} \geq \underline{n}, \underline{r} \perp F} \iota_{\underline{n}}^{\underline{r}}(k_{\underline{n}}) - \sum_{\underline{r} \geq \underline{n}, \underline{r} \perp F} \iota_{\underline{n}}^{\underline{r}}(k_{\underline{n},n})\| \\ &= \| \sum_{\underline{r} \geq \underline{n}, \underline{r} \perp F} \iota_{\underline{n}}^{\underline{r}}(k_{\underline{n}} - k_{\underline{n},n})\| \\ &= \sup\{\|\iota_{\underline{n}}^{\underline{r}}(k_{\underline{n}} - k_{\underline{n},n})\| \mid \underline{r} \geq \underline{n}, \underline{r} \perp F\} \\ &\leq \|k_{\underline{n}} - k_{\underline{n},n}\| \text{ for all } n \in \mathbb{N}, \end{aligned}$$

using that each $\iota_{\underline{n}}^{\underline{r}}$ is contractive in the final equality. Thus it suffices to show that each operator $\sum_{\underline{r} \geq \underline{n}, \underline{r} \perp F} \iota_{\underline{n}}^{\underline{r}}(k_{\underline{n},n})$ belongs to $\mathfrak{J}_{\mathcal{L},F}$. In turn, it suffices to show that

$$\sum_{\underline{r} \geq \underline{n}, \underline{r} \perp F} \iota_{\underline{n}}^{\underline{r}}(\Theta^{X_{\underline{n}}}_{\xi_{\underline{n}} b, \eta_{\underline{n}}}) = \overline{t}_{\underline{n}}(\xi_{\underline{n}})\overline{\pi}(b)\overline{q}_F \overline{t}_{\underline{n}}(\eta_{\underline{n}})^*, \tag{3.6}$$

where $\xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\underline{n}}$ and $b \in \mathcal{L}_F$. First note that if $\underline{s} \not\geq \underline{n}$ or $\underline{s} \not\perp F$, then

$$\overline{t}_{\underline{n}}(\xi_{\underline{n}})\overline{\pi}(b)\overline{q}_F \overline{t}_{\underline{n}}(\eta_{\underline{n}})^* \zeta_{\underline{s}} = 0 \text{ for all } \zeta_{\underline{s}} \in X_{\underline{s}}.$$

Indeed, if $\underline{s} \not\geq \underline{n}$, then $\overline{t}_{\underline{n}}(\eta_{\underline{n}})^*\zeta_{\underline{s}} = 0$ by definition. If $\underline{s} \geq \underline{n}$ but $\underline{s} \not\perp F$, then

$$\mathrm{supp}(\underline{s} - \underline{n}) \cap F \neq \emptyset,$$

and hence $\overline{q}_F \overline{t}_{\underline{n}}(\eta_{\underline{n}})^*\zeta_{\underline{s}} = 0$. Thus both sides of (3.6) map $X_{\underline{s}}$ to 0 when $\underline{s} \not\geq \underline{n}$ or $\underline{s} \not\perp F$. Next suppose that $\underline{s} \geq \underline{n}$ and $\underline{s} \perp F$, and let $\zeta_{\underline{s}} = \zeta_{\underline{n}}\zeta_{\underline{s}-\underline{n}}$ for some $\zeta_{\underline{n}} \in X_{\underline{n}}$ and $\zeta_{\underline{s}-\underline{n}} \in X_{\underline{s}-\underline{n}}$. Then we have that

$$\begin{aligned} \overline{t}_{\underline{n}}(\xi_{\underline{n}})\overline{\pi}(b)\overline{q}_F \overline{t}_{\underline{n}}(\eta_{\underline{n}})^* \zeta_{\underline{s}} &= \xi_{\underline{n}}(\phi_{\underline{s}-\underline{n}}(b \langle \eta_{\underline{n}}, \zeta_{\underline{n}} \rangle)\zeta_{\underline{s}-\underline{n}}) = \big(\Theta^{X_{\underline{n}}}_{\xi_{\underline{n}} b, \eta_{\underline{n}}}(\zeta_{\underline{n}})\big)\zeta_{\underline{s}-\underline{n}} \\ &= \iota_{\underline{n}}^{\underline{s}}(\Theta^{X_{\underline{n}}}_{\xi_{\underline{n}} b, \eta_{\underline{n}}})(\zeta_{\underline{n}}\zeta_{\underline{s}-\underline{n}}) = \sum_{\underline{r} \geq \underline{n}, \underline{r} \perp F} \iota_{\underline{n}}^{\underline{r}}(\Theta^{X_{\underline{n}}}_{\xi_{\underline{n}} b, \eta_{\underline{n}}})\zeta_{\underline{s}}, \end{aligned}$$

noting that $\underline{s} - \underline{n} \perp F$. It follows that (3.6) holds, finishing the proof of Claim 1. □

Claim 2. The element $x_{F \cup D} := \overline{\pi}(a)\overline{q}_F \prod_{i \in D} W_i$ *belongs to* $\mathfrak{J}_{\mathcal{L}, F\cup D}$ *for each* $\emptyset \neq D \subseteq F^c$.

Proof of Claim 2. Let us first consider the case where $D = F^c$. For $\underline{r} = (r_1, \dots, r_d) \in \mathbb{Z}_+^d$

and $\zeta_{\underline{r}} \in X_{\underline{r}}$, we have that

$$\overline{\pi}(a)\overline{q}_F \prod_{i \in F^c} W_i \zeta_{\underline{r}} = \begin{cases} \phi_{\underline{r}}(a)\zeta_{\underline{r}} & \text{if } \underline{0} \leq \underline{r} \leq m \cdot \underline{1}_{F^c}, \\ 0 & \text{if } r_i \geq m+1 \text{ for some } i \in F^c \text{ or } \underline{r} \not\perp F, \end{cases}$$

from which it follows that

$$\overline{\pi}(a)\overline{q}_F \prod_{i \in F^c} W_i = \sum_{\underline{0} \leq \underline{r} \leq m \cdot \underline{1}_{F^c}} \phi_{\underline{r}}(a),$$

noting that the sum is finite.

Fix $\underline{0} \leq \underline{r} \leq m \cdot \underline{1}_{F^c}$ and note that $\underline{r} \perp F$. Since $a \in \mathcal{L}_F^{(1)}$ and $\mathcal{L}^{(1)}$ is an (E)-2^d-tuple that is invariant and consists of ideals by Proposition 3.4.3, an application of Proposition 2.5.2 gives that $\phi_{\underline{r}}(a) \in \mathcal{K}(X_{\underline{r}}\mathcal{L}_F^{(1)})$. Additionally, we have that

$$\mathcal{L}_F^{(1)} \subseteq \mathcal{L}_{\text{inv},F} \subseteq A^{-1}(\cap_{F \subsetneq G} \mathcal{L}_G) = \cap_{F \subsetneq G} \mathcal{L}_G \subseteq \mathcal{L}_{[d]}$$

and so $\phi_{\underline{r}}(a) \in \mathcal{K}(X_{\underline{r}}\mathcal{L}_{[d]})$.

We claim that $\phi_{\underline{r}}(a) \in \mathfrak{J}_{\mathcal{L},[d]}$ when viewed as an operator in $\mathcal{L}(\mathcal{F}X)$. First note that

$$\phi_{\underline{0}}(a) = \overline{\pi}(a)\overline{q}_{[d]} \in \mathfrak{J}_{\mathcal{L},[d]},$$

as $a \in \mathcal{L}_F^{(1)} \subseteq \mathcal{L}_{[d]}$, so we may assume that $\underline{r} \neq \underline{0}$. We see that

$$\mathcal{K}(X_{\underline{r}}\mathcal{L}_{[d]}) \subseteq \mathfrak{J}_{\mathcal{L},[d]}, \tag{3.7}$$

when identifying $\mathcal{K}(X_{\underline{r}}\mathcal{L}_{[d]})$ within $\mathcal{L}(\mathcal{F}X)$. Indeed, it suffices to show this for rank-one operators. To this end, take $k_{\underline{r}} = \Theta^{X_{\underline{r}}}_{\xi_{\underline{r}}b,\eta_{\underline{r}}}$ for some $\xi_{\underline{r}}, \eta_{\underline{r}} \in X_{\underline{r}}$ and $b \in \mathcal{L}_{[d]}$. We claim that

$$\Theta^{X_{\underline{r}}}_{\xi_{\underline{r}}b,\eta_{\underline{r}}} = \overline{t}_{\underline{r}}(\xi_{\underline{r}})\overline{\pi}(b)\overline{q}_{[d]}\overline{t}_{\underline{r}}(\eta_{\underline{r}})^*, \tag{3.8}$$

when we view $\Theta^{X_{\underline{r}}}_{\xi_{\underline{r}}b,\eta_{\underline{r}}}$ as an operator in $\mathcal{L}(\mathcal{F}X)$. Notice that both sides of (3.8) map every summand $X_{\underline{s}}$ for which $\underline{s} \neq \underline{r}$ to 0. On the other hand, arguing as we did with x_F, we deduce that the right hand side of (3.8) coincides with $\Theta^{X_{\underline{r}}}_{\xi_{\underline{r}}b,\eta_{\underline{r}}}$ on the $X_{\underline{r}}$ summand. Hence (3.8) and in turn (3.7) hold, and applying for $\phi_{\underline{r}}(a) \in \mathcal{K}(X_{\underline{r}}\mathcal{L}_{[d]})$ gives that $\phi_{\underline{r}}(a) \in \mathfrak{J}_{\mathcal{L},[d]}$. We conclude that

$$x_{[d]} = \overline{\pi}(a)\overline{q}_F \prod_{i \in F^c} W_i \in \mathfrak{J}_{\mathcal{L},[d]},$$

being a finite sum of elements of $\mathfrak{J}_{\mathcal{L},[d]}$.

Now take $\emptyset \neq D \subsetneq F^c$. For $\underline{r} = (r_1, \dots, r_d) \in \mathbb{Z}_+^d$ and $\zeta_{\underline{r}} \in X_{\underline{r}}$, we have that

$$\overline{\pi}(a)\overline{q}_F \prod_{i \in D} W_i \zeta_{\underline{r}} = \begin{cases} \phi_{\underline{r}}(a)\zeta_{\underline{r}} & \text{if } r_i \leq m \text{ for all } i \in D \text{ and } \underline{r} \perp F, \\ 0 & \text{if } r_i \geq m+1 \text{ for some } i \in D \text{ or } \underline{r} \not\perp F. \end{cases}$$

It follows that
$$\overline{\pi}(a)\overline{q}_F \prod_{i\in D} W_i = \sum_{\underline{0}\leq\underline{\ell}\leq m\cdot\underline{1}_D} \Big(\sum_{\underline{s}\perp F\cup D} \phi_{\underline{\ell}+\underline{s}}(a) \Big),$$
noting that the sum over $\underline{\ell}$ is finite.

Fix $\underline{0} \leq \underline{\ell} \leq m \cdot \underline{1}_D$ and consider $\sum_{\underline{s}\perp F\cup D} \phi_{\underline{\ell}+\underline{s}}(a)$. Since $a \in \mathcal{L}_F^{(1)}$ and $\mathcal{L}^{(1)}$ is an (E)-2^d-tuple that is invariant and consists of ideals by Proposition 3.4.3, an application of Proposition 2.5.2 gives that $\phi_{\underline{\ell}}(a) \in \mathcal{K}(X_{\underline{\ell}}\mathcal{L}_F^{(1)})$, noting that $\underline{\ell} \perp F$. Additionally, by definition we have that
$$\mathcal{L}_F^{(1)} \subseteq \mathcal{L}_{\text{inv},F} \subseteq \cap_{F\subsetneq G}\mathcal{L}_G.$$
In particular, notice that $D \cap F^c \neq \emptyset$ and hence $F \subsetneq F \cup D$. Thus $\mathcal{L}_F^{(1)} \subseteq \mathcal{L}_{F\cup D}$, and consequently $\phi_{\underline{\ell}}(a) \in \mathcal{K}(X_{\underline{\ell}}\mathcal{L}_{F\cup D})$. Notice that
$$\sum_{\underline{s}\perp F\cup D} \phi_{\underline{\ell}+\underline{s}}(a) = \sum_{\underline{s}\perp F\cup D} \iota_{\underline{\ell}}^{\underline{\ell}+\underline{s}}(\phi_{\underline{\ell}}(a)).$$

We claim that $\sum_{\underline{s}\perp F\cup D} \iota_{\underline{\ell}}^{\underline{\ell}+\underline{s}}(\phi_{\underline{\ell}}(a)) \in \mathfrak{J}_{\mathcal{L},F\cup D}$. First note that if $\underline{\ell} = \underline{0}$, then
$$\sum_{\underline{s}\perp F\cup D} \iota_{\underline{0}}^{\underline{s}}(\phi_{\underline{0}}(a)) = \sum_{\underline{s}\perp F\cup D} \phi_{\underline{s}}(a) = \overline{\pi}(a)\overline{q}_{F\cup D} \in \mathfrak{J}_{\mathcal{L},F\cup D},$$
using that $a \in \mathcal{L}_F^{(1)} \subseteq \mathcal{L}_{F\cup D}$ in the final membership. For $\underline{\ell} \neq \underline{0}$, we see that
$$\sum_{\underline{s}\perp F\cup D} \iota_{\underline{\ell}}^{\underline{\ell}+\underline{s}}(k_{\underline{\ell}}) \in \mathfrak{J}_{\mathcal{L},F\cup D} \text{ for all } k_{\underline{\ell}} \in \mathcal{K}(X_{\underline{\ell}}\mathcal{L}_{F\cup D}). \tag{3.9}$$

It suffices to show this for $k_{\underline{\ell}} = \Theta^{X_{\underline{\ell}}}_{\xi_{\underline{\ell}}b,\eta_{\underline{\ell}}}$ for some $\xi_{\underline{\ell}}, \eta_{\underline{\ell}} \in X_{\underline{\ell}}$ and $b \in \mathcal{L}_{F\cup D}$. We claim that
$$\sum_{\underline{s}\perp F\cup D} \iota_{\underline{\ell}}^{\underline{\ell}+\underline{s}}(\Theta^{X_{\underline{\ell}}}_{\xi_{\underline{\ell}}b,\eta_{\underline{\ell}}}) = \overline{t}_{\underline{\ell}}(\xi_{\underline{\ell}})\overline{\pi}(b)\overline{q}_{F\cup D}\overline{t}_{\underline{\ell}}(\eta_{\underline{\ell}})^* \in \mathfrak{J}_{\mathcal{L},F\cup D}. \tag{3.10}$$

To see this, fix $\underline{r} \in \mathbb{Z}_+^d$ and $\zeta_{\underline{r}} \in X_{\underline{r}}$. If $\underline{r} \not\geq \underline{\ell}$ then $\underline{r} \notin \underline{\ell} + \mathbb{Z}_+^d$, and so both sides of (3.10) map $\zeta_{\underline{r}}$ to 0. If $\underline{r} \geq \underline{\ell}$ but $\underline{r} - \underline{\ell} \not\perp F \cup D$, then $\overline{q}_{F\cup D}\overline{t}_{\underline{\ell}}(\eta_{\underline{\ell}})^*\zeta_{\underline{r}} = 0$ by definition. Thus both sides of (3.10) map $\zeta_{\underline{r}}$ to 0 in this case. Finally, suppose that $\underline{r} - \underline{\ell} \perp F \cup D$ and without loss of generality write $\zeta_{\underline{r}} = \zeta_{\underline{\ell}}\zeta_{\underline{r}-\underline{\ell}}$ for some $\zeta_{\underline{\ell}} \in X_{\underline{\ell}}$ and $\zeta_{\underline{r}-\underline{\ell}} \in X_{\underline{r}-\underline{\ell}}$. Then we have that
$$\begin{aligned}\overline{t}_{\underline{\ell}}(\xi_{\underline{\ell}})\overline{\pi}(b)\overline{q}_{F\cup D}\overline{t}_{\underline{\ell}}(\eta_{\underline{\ell}})^*\zeta_{\underline{r}} &= \xi_{\underline{\ell}}(\phi_{\underline{r}-\underline{\ell}}(b\langle\eta_{\underline{\ell}},\zeta_{\underline{\ell}}\rangle)\zeta_{\underline{r}-\underline{\ell}}) = \big(\Theta^{X_{\underline{\ell}}}_{\xi_{\underline{\ell}}b,\eta_{\underline{\ell}}}(\zeta_{\underline{\ell}})\big)\zeta_{\underline{r}-\underline{\ell}} \\ &= \iota_{\underline{\ell}}^{\underline{r}}(\Theta^{X_{\underline{\ell}}}_{\xi_{\underline{\ell}}b,\eta_{\underline{\ell}}})(\zeta_{\underline{\ell}}\zeta_{\underline{r}-\underline{\ell}}) = \sum_{\underline{s}\perp F\cup D} \iota_{\underline{\ell}}^{\underline{\ell}+\underline{s}}(\Theta^{X_{\underline{\ell}}}_{\xi_{\underline{\ell}}b,\eta_{\underline{\ell}}})\zeta_{\underline{r}},\end{aligned}$$
from which it follows that (3.10) holds. In turn we deduce that (3.9) holds, and applying for $k_{\underline{\ell}} = \phi_{\underline{\ell}}(a) \in \mathcal{K}(X_{\underline{\ell}}\mathcal{L}_{F\cup D})$ yields that
$$\sum_{\underline{s}\perp F\cup D} \iota_{\underline{\ell}}^{\underline{\ell}+\underline{s}}(\phi_{\underline{\ell}}(a)) \in \mathfrak{J}_{\mathcal{L},F\cup D}.$$

Therefore, we conclude that

$$x_{F\cup D} = \overline{\pi}(a)\overline{q}_F \prod_{i\in D} W_i \in \mathfrak{J}_{\mathcal{L},F\cup D},$$

being a finite sum of elements of $\mathfrak{J}_{\mathcal{L},F\cup D}$. This finishes the proof of Claim 2. $\square$

We now conclude the proof of the proposition. By Claim 2, we have that

$$x := \sum_{\emptyset\neq D\subseteq F^c} (-1)^{|D|}\overline{\pi}(a)\overline{q}_F \prod_{i\in D} W_i = \sum_{\emptyset\neq D\subseteq F^c} (-1)^{|D|} x_{F\cup D} \in \sum_{F\subsetneq D\subseteq [d]} \mathfrak{J}_{\mathcal{L},D}.$$

In total, we have that

$$\|\overline{\pi}(a)\overline{q}_F + (x + x_F)\| = \|\overline{\pi}(a)\overline{q}_F \prod_{i\in F^c}(I - W_i) + \sum_{\underline{r}\geq \underline{n}, \underline{r}\perp F} \iota_{\underline{n}}^{\underline{r}}(k_{\underline{n}})\| < \varepsilon,$$

using (3.5) in the final inequality. So, employing Claims 1 and 2 in tandem, we have shown that for every $\varepsilon > 0$ the element $\overline{\pi}(a)\overline{q}_F$ is ε-close to an element of $\sum_{F\subseteq D\subseteq [d]} \mathfrak{J}_{\mathcal{L},D}$, as required. $\square$

We are now ready to prove that when $\mathcal{L}$ is in addition partially ordered, the (E)-2^d-tuple $\mathcal{L}^{(1)}$ is partially ordered, contains $\mathcal{L}$ and induces the same gauge-invariant ideal of $\mathcal{NT}_X$.

Proposition 3.4.5. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be an (E)-2^d-tuple of X that is invariant, partially ordered and consists of ideals. Then $\mathcal{L}^{(1)}$ is an (E)-2^d-tuple of X that is invariant, partially ordered, consists of ideals, and is such that $\mathcal{L} \subseteq \mathcal{L}^{(1)}$ and $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)} = \mathfrak{J}_{\mathcal{L}^{(1)}}^{(\overline{\pi}_X,\overline{t}_X)}$.*

Proof. By Proposition 3.4.3, the family $\mathcal{L}^{(1)}$ is an (E)-2^d-tuple that is invariant and consists of ideals. For the partial ordering property, it is immediate that $\{0\} = \mathcal{L}_{\emptyset}^{(1)} \subseteq \mathcal{L}_F^{(1)}$ for all $F \subseteq [d]$. Likewise, we have that

$$\mathcal{L}_F^{(1)} \subseteq X_{\underline{0}}^{-1}(\cap_{F\subsetneq D}\mathcal{L}_D) = \cap_{F\subsetneq D}\mathcal{L}_D \subseteq \mathcal{L}_{[d]} = \mathcal{L}_{[d]}^{(1)}$$

for all $\emptyset \neq F \subsetneq [d]$. Next fix $\emptyset \neq F \subseteq F' \subsetneq [d]$, and let $a \in \mathcal{L}_F^{(1)}$. Notice that $a \in \mathcal{L}_F^{(1)} \subseteq \mathcal{I}_F \subseteq \mathcal{I}_{F'}$ because $\mathcal{I}$ is partially ordered. Next fix $\underline{m} \perp F'$. Since $F \subseteq F'$, we have that $\underline{m} \perp F$ and thus

$$\langle X_{\underline{m}}, aX_{\underline{m}}\rangle \subseteq \cap_{F\subsetneq D}\mathcal{L}_D \subseteq \cap_{F'\subsetneq D}\mathcal{L}_D,$$

using that $a \in \mathcal{L}_{\mathrm{inv},F}$. It follows that $a \in \mathcal{L}_{\mathrm{inv},F'}$. In order to prove that $a \in \mathcal{L}_{\mathrm{lim},F'}$, we resort to Proposition 3.3.5. More specifically, it suffices to show that

$$\overline{\pi}_X(a)\overline{q}_{X,F'} \in \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)}.$$

To this end, note that $a \in \bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in F'\}$, since $\mathcal{L}^{(1)}$ is an (E)-2^d-tuple, and $\overline{\pi}_X(a)\overline{q}_{X,F} \in \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)}$ by Proposition 3.4.4. An application of Proposition 3.2.5 then gives that $\overline{\pi}_X(a)\overline{q}_{X,F'} \in \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)}$, as required. Hence $a \in \mathcal{L}_{F'}^{(1)}$ and we conclude that $\mathcal{L}^{(1)}$ is partially ordered.

Next we prove that $\mathcal{L} \subseteq \mathcal{L}^{(1)}$. Notice that $\mathcal{L}_\emptyset \subseteq \mathcal{L}_\emptyset^{(1)}$ and $\mathcal{L}_{[d]} \subseteq \mathcal{L}_{[d]}^{(1)}$ trivially, so fix $\emptyset \neq F \subsetneq [d]$ and $a \in \mathcal{L}_F$. Since $\mathcal{L}$ is an (E)-2^d-tuple, we have that $a \in \mathcal{I}_F$. Since $\mathcal{L}$ is invariant and partially ordered, we have that

$$\langle X_{\underline{m}}, aX_{\underline{m}}\rangle \subseteq \mathcal{L}_F \subseteq \cap_{F \subsetneq D}\mathcal{L}_D$$

for all $\underline{m} \perp F$, and thus $a \in \mathcal{L}_{\text{inv},F}$. Note that $\phi_{\underline{m}}(a) \in \mathcal{K}(X_{\underline{m}}\mathcal{L}_F)$ for all $\underline{m} \perp F$ by Proposition 2.5.2. Therefore, by choosing any $\underline{m} \perp F$ and $k_{\underline{m}} = -\phi_{\underline{m}}(a) \in \mathcal{K}(X_{\underline{m}}\mathcal{L}_F)$, we obtain that $\|\phi_{\underline{m}}(a) + k_{\underline{m}}\| = 0$ and thus $a \in \mathcal{L}_{\text{lim},F}$ by Proposition 3.3.4. Hence $\mathcal{L}_F \subseteq \mathcal{L}_F^{(1)}$, as required.

Finally, we show that $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)} = \mathfrak{J}_{\mathcal{L}^{(1)}}^{(\overline{\pi}_X,\overline{t}_X)}$. The forward inclusion is immediate since $\mathcal{L} \subseteq \mathcal{L}^{(1)}$. On the other hand, we have that $\overline{\pi}_X(\mathcal{L}_F^{(1)})\overline{q}_{X,F} \subseteq \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)}$ for all $F \subseteq [d]$ by Proposition 3.4.4, giving the reverse inclusion and completing the proof. □

We are now ready to provide a full characterisation of (M)-2^d-tuples.

Theorem 3.4.6. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and suppose that $\mathcal{L}$ is a 2^d-tuple of X. Then $\mathcal{L}$ is an (M)-2^d-tuple of X if and only if $\mathcal{L}$ satisfies the following four conditions:*

(i) $\mathcal{L}$ consists of ideals and $\mathcal{L} \subseteq \mathcal{J}$,

(ii) $\mathcal{L}$ is invariant,

(iii) $\mathcal{L}$ is partially ordered,

(iv) $\mathcal{L}^{(1)} \subseteq \mathcal{L}$.

Proof. Assume that $\mathcal{L}$ is an (M)-2^d-tuple. First we use Propositions 3.2.4 and 3.2.6 together with maximality of $\mathcal{L}$ to deduce that $\mathcal{L}$ consists of ideals, that $\mathcal{L} \subseteq \mathcal{I} \subseteq \mathcal{J}$, and that $\mathcal{L}$ is invariant and partially ordered. Proposition 3.4.5 together with maximality of $\mathcal{L}$ gives that $\mathcal{L}^{(1)} \subseteq \mathcal{L}$, proving the forward implication.

Now suppose that $\mathcal{L}$ is a 2^d-tuple that satisfies conditions (i)-(iv). Conditions (i) and (ii) imply that $\mathcal{L}$ is an (E)-2^d-tuple by Proposition 3.2.3, and so we can consider $\mathcal{L}^{(1)}$. It remains to check that $\mathcal{L}$ is maximal. To this end, let $\mathcal{M}$ be the (M)-2^d-tuple such that $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)} = \mathfrak{J}_{\mathcal{M}}^{(\overline{\pi}_X,\overline{t}_X)}$, as guaranteed by Proposition 3.2.10. Note also that $\mathcal{L} \subseteq \mathcal{M}$ by Proposition 3.1.9. It suffices to show that $\mathcal{M} \subseteq \mathcal{L}$.

On one hand, we have that $\mathcal{M}_\emptyset = \mathcal{L}_\emptyset = \{0\}$, since both $\mathcal{M}$ and $\mathcal{L}$ are (E)-2^d-tuples. In order to show that $\mathcal{M}_F \subseteq \mathcal{L}_F$ for all $\emptyset \neq F \subseteq [d]$, we apply strong (downward) induction on $|F|$.

For the base case, take $a \in \mathcal{M}_{[d]}$. Then $\overline{\pi}_X(a)\overline{q}_{X,[d]} \in \mathfrak{J}_{\mathcal{M}}^{(\overline{\pi}_X,\overline{t}_X)} = \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)}$. Using the fact that $\mathcal{NT}_X \cong \mathrm{C}^*(\overline{\pi},\overline{t})$ canonically for the Fock representation $(\overline{\pi},\overline{t})$, we have that $\overline{\pi}(a)\overline{q}_{[d]} \in \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi},\overline{t})}$. Conditions (i)-(iii) together with Propositions 3.3.1 and 3.3.2 give that

$$\phi_{\underline{0}}(a) = \overline{q}_{[d]} \left(\overline{\pi}(a)\overline{q}_{[d]}\right) \overline{q}_{[d]} \in \overline{q}_{[d]}^2 \left(\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi},\overline{t})}\right) \overline{q}_{[d]}^2 \subseteq \overline{q}_{[d]} \left(\mathfrak{J}_{\mathcal{L},[d]}^{(\overline{\pi},\overline{t})}\right) \overline{q}_{[d]} = \overline{q}_{[d]}\overline{\pi}(\mathcal{L}_{[d]})\overline{q}_{[d]} = \phi_{\underline{0}}(\mathcal{L}_{[d]}),$$

and thus $a \in \mathcal{L}_{[d]}$. This shows that $\mathcal{M}_{[d]} \subseteq \mathcal{L}_{[d]}$.

Next, suppose that $\mathcal{M}_F \subseteq \mathcal{L}_F$ for all $\emptyset \neq F \subseteq [d]$ with $|F| \geq n+1$, where $1 \leq n \leq d-1$. Fix $\emptyset \neq F \subsetneq [d]$ with $|F| = n$. Since $\mathcal{M}$ is an (E)-2^d-tuple, we have that $\mathcal{M}_F \subseteq \mathcal{I}_F$. Note that $\mathcal{M}$, being an (M)-2^d-tuple, satisfies conditions (i)-(iv) by the forward implication. In particular, we have that $\mathcal{M}$ is invariant, so

$$\langle X_{\underline{m}}, \mathcal{M}_F X_{\underline{m}} \rangle \subseteq \mathcal{M}_F \text{ for all } \underline{m} \perp F.$$

Likewise, we have that $\mathcal{M}$ is partially ordered, so $\langle X_{\underline{m}}, \mathcal{M}_F X_{\underline{m}} \rangle \subseteq \cap_{F \subsetneq D} \mathcal{M}_D$ for all $\underline{m} \perp F$. By the inductive hypothesis, we have that $\mathcal{M}_D = \mathcal{L}_D$ whenever $F \subsetneq D$, as $|D| \geq n+1$. Hence

$$\langle X_{\underline{m}}, \mathcal{M}_F X_{\underline{m}} \rangle \subseteq \cap_{F \subsetneq D} \mathcal{L}_D \text{ for all } \underline{m} \perp F.$$

Thus $\mathcal{M}_F \subseteq \mathcal{L}_{\mathrm{inv},F}$. Moreover, by definition we have that

$$\overline{\pi}_X(\mathcal{M}_F)\overline{q}_{X,F} \subseteq \mathfrak{J}_{\mathcal{M}}^{(\overline{\pi}_X,\overline{t}_X)} = \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)}.$$

By applying Proposition 3.3.5, we deduce that $\mathcal{M}_F \subseteq \mathcal{L}_{\mathrm{lim},F}$. In total, we have that

$$\mathcal{M}_F \subseteq \mathcal{I}_F \cap \mathcal{L}_{\mathrm{inv},F} \cap \mathcal{L}_{\mathrm{lim},F} = \mathcal{L}_F^{(1)}$$

and thus $\mathcal{M}_F \subseteq \mathcal{L}_F$ by condition (iv), as required. Induction then completes the proof. □

Iteration of the $\mathcal{L}^{(1)}$ construction constitutes the final ingredient for attaining maximality. For if we start with an (E)-2^d-tuple $\mathcal{L}$ that is invariant, partially ordered and consists of ideals, then iterative applications of Proposition 3.4.5 produce a sequence of (E)-2^d-tuples that are invariant, partially ordered, consist of ideals and satisfy

$$\mathcal{L} \subseteq \mathcal{L}^{(1)} \subseteq \cdots \subseteq \mathcal{L}^{(k)} \subseteq \dots \quad \text{and} \quad \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)} = \mathfrak{J}_{\mathcal{L}^{(k)}}^{(\overline{\pi}_X,\overline{t}_X)} \text{ for all } k \in \mathbb{N}.$$

Since each $\mathcal{L}^{(k)}$ induces the same gauge-invariant ideal of $\mathcal{NT}_X$, if the sequence eventually stabilises then it must stabilise to an (M)-2^d-tuple by Theorem 3.4.6.

Theorem 3.4.7. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be an (E)-2^d-tuple of X that is invariant, partially ordered, and consists of ideals. Fix $0 \leq k \leq d$. Then whenever $F \subseteq [d]$ satisfies $|F| = d-k$, we have that $\mathcal{L}_F^{(m)} = \mathcal{L}_F^{(k)}$ for all $m \geq k$. Consequently, $\mathcal{L}^{(d-1)}$ is the (M)-2^d-tuple that induces $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)}$.*

Proof. We proceed by applying strong induction on k. For the base case, if $F = [d]$ then $\mathcal{L}_{[d]}^{(m)} = \mathcal{L}_{[d]} = \mathcal{L}_{[d]}^{(0)}$ for all $m \geq 0$ by construction. Next, fix $0 \leq N \leq d-1$ and suppose that whenever $0 \leq n \leq N$ and $D \subseteq [d]$ satisfies $|D| = d - n$, we have that $\mathcal{L}_D^{(m)} = \mathcal{L}_D^{(n)}$ for all $m \geq n$.

Now we fix $F \subseteq [d]$ such that $|F| = d - (N+1)$. We must show that $\mathcal{L}_F^{(m)} = \mathcal{L}_F^{(N+1)}$ for all $m \geq N+1$. First note that this is clear if $F = \emptyset$ (i.e., if $N = d-1$), as each $\mathcal{L}_\emptyset^{(m)} = \{0\}$. So, without loss of generality, we may exclude the case where $N = d-1$ and consider $\emptyset \neq F \subsetneq [d]$.

Fix $m \geq N+1$. We have already argued that $\mathcal{L}_F^{(N+1)} \subseteq \mathcal{L}_F^{(m)}$, so it remains to verify the reverse inclusion. Take $a \in \mathcal{L}_F^{(m)}$. In particular, we have that $a \in \mathcal{I}_F$. Fix $\underline{m} \perp F$ and note that $\langle X_{\underline{m}}, aX_{\underline{m}} \rangle \subseteq \cap_{F \subsetneq D} \mathcal{L}_D^{(m-1)}$ by definition. Notice that whenever $F \subsetneq D$, we must have that $|D| = d - k$ for some $0 \leq k \leq N$. By the inductive hypothesis, we have that $\mathcal{L}_D^{(k)} = \mathcal{L}_D^{(N)} = \mathcal{L}_D^{(m-1)}$ and hence $\langle X_{\underline{m}}, aX_{\underline{m}} \rangle \subseteq \cap_{F \subsetneq D} \mathcal{L}_D^{(N)}$. In turn, we have that $a \in \mathcal{L}_{\mathrm{inv},F}^{(N)}$.

Notice also that $\overline{\pi}_X(a)\overline{q}_{X,F} \in \mathfrak{J}_{\mathcal{L}^{(m)}}^{(\overline{\pi}_X, \overline{t}_X)} = \mathfrak{J}_{\mathcal{L}^{(N)}}^{(\overline{\pi}_X, \overline{t}_X)}$, and hence an application of Proposition 3.3.5 yields that $a \in \mathcal{L}_{\mathrm{lim},F}^{(N)}$. In total, we have that $a \in \mathcal{I}_F \cap \mathcal{L}_{\mathrm{inv},F}^{(N)} \cap \mathcal{L}_{\mathrm{lim},F}^{(N)} = \mathcal{L}_F^{(N+1)}$. This proves that $\mathcal{L}_F^{(m)} = \mathcal{L}_F^{(N+1)}$ for all $m \geq N+1$. By induction we obtain that $\mathcal{L}_F^{(m)} = \mathcal{L}_F^{(d-|F|)}$ for all $m \geq d - |F|$, as required.

Finally, we have that $\mathcal{L}^{(d-1)} = \mathcal{L}^{(d)} = (\mathcal{L}^{(d-1)})^{(1)}$ by the first claim. Hence we conclude that $\mathcal{L}^{(d-1)}$ is the (M)-2^d-tuple inducing $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)}$ by Proposition 3.4.5 and Theorem 3.4.6, finishing the proof. □

Remark 3.4.8. Given any (E)-2^d-tuple $\mathcal{L}$, we now have an algorithm for computing the (M)-2^d-tuple that induces $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)}$. Specifically, we apply Propositions 3.2.4 and 3.2.6 to pass from $\mathcal{L}$ to the (E)-2^d-tuple $\mathrm{PO}(\mathrm{Inv}(\mathcal{L}))$, which is invariant, partially ordered, consists of ideals and satisfies

$$\mathcal{L} \subseteq \mathrm{PO}(\mathrm{Inv}(\mathcal{L})) \quad \text{and} \quad \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)} = \mathfrak{J}_{\mathrm{PO}(\mathrm{Inv}(\mathcal{L}))}^{(\overline{\pi}_X, \overline{t}_X)}.$$

We then apply the $(\mathrm{PO}(\mathrm{Inv}(\mathcal{L})))^{(1)}$ construction iteratively, and use Theorem 3.4.7 to deduce that $(\mathrm{PO}(\mathrm{Inv}(\mathcal{L})))^{(d-1)}$ is the (M)-2^d-tuple that induces $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)}$.

Theorem 3.4.9 ($\mathbb{Z}_+^d$-GIUT for (E)-2^d-tuples)**.** *Let X be a strong compactly aligned product system with coefficients in a C^*-algebra A. Let $\mathcal{L}$ be an (E)-2^d-tuple of X and (π, t) be a Nica-covariant representation of X. Then $\mathcal{NO}(\mathcal{L}, X) \cong \mathrm{C}^*(\pi, t)$ via a (unique) canonical $*$-isomorphism if and only if (π, t) admits a gauge action and*

$$\mathcal{L}^{(\pi,t)} = \Big(\mathrm{PO}\left(\mathrm{Inv}(\mathcal{L})\right) \Big)^{(d-1)}.$$

Proof. We have that $(\mathrm{PO}(\mathrm{Inv}(\mathcal{L})))^{(d-1)}$ is the (M)-2^d-tuple that induces $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)}$ by Remark 3.4.8. In particular, we have that $\mathcal{NO}((\mathrm{PO}(\mathrm{Inv}(\mathcal{L})))^{(d-1)}, X) = \mathcal{NO}(\mathcal{L}, X)$, and Theorem 3.2.12 finishes the proof. □

Chapter 4

NT-2^d-tuples and gauge-invariant ideals of $\mathcal{NT}_X$

The (M)-2^d-tuples of X parametrise the equivariant quotients that lie in-between $\mathcal{NT}_X$ and $\mathcal{NO}_X$. We now pass to the parametrisation of the quotients that may not be injective on X. We will circumvent this by "deleting the kernel", i.e., by utilising the quotient product system construction explored in Section 2.3.

4.1 NT-2^d-tuples

We begin by defining some auxiliary objects.

Definition 4.1.1. Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Fix $\emptyset \neq F \subseteq [d]$ and let $I \subseteq A$ be an ideal. We define the following subsets of A:

(i) $X_F^{-1}(I) := \bigcap\{X_{\underline{n}}^{-1}(I) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = \{a \in A \mid \langle X_{\underline{n}}, aX_{\underline{n}}\rangle \subseteq I \text{ for all } \underline{0} \neq \underline{n} \leq \underline{1}_F\}$,

(ii) $J_F(I, X) := \{a \in A \mid [\phi_{\underline{i}}(a)]_I \in \mathcal{K}([X_{\underline{i}}]_I) \text{ for all } i \in [d], aX_F^{-1}(I) \subseteq I\}$.

Notice that both $X_F^{-1}(I)$ and $J_F(I, X)$ are ideals of A, and $I \subseteq J_F(I, X)$ whenever I is positively invariant. These objects will play similar roles to the ideals $X^{-1}(I)$ and $J(I, X)$ for a C*-correspondence X over A (see the discussion succeeding Lemma 2.2.18). Let us collect some properties of $X_F^{-1}(I)$ and $J_F(I, X)$.

Proposition 4.1.2. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $I \subseteq A$ be an ideal that is positively invariant for X. Fix $\emptyset \neq F \subseteq [d]$ and $a \in A$. Then the following are equivalent:*

(i) $[a]_I \in \bigcap\{\ker[\phi_{\underline{i}}]_I \mid i \in F\}$;

(ii) $\langle X_{\underline{i}}, aX_{\underline{i}}\rangle \subseteq I$ *for all* $i \in F$;

(iii) $\langle X_{\underline{n}}, aX_{\underline{n}}\rangle \subseteq I$ *for all* $\underline{n} \neq \underline{0}$ *satisfying* $\operatorname{supp} \underline{n} \subseteq F$.

Consequently, we have that $X_F^{-1}(I) = \bigcap\{X_{\underline{i}}^{-1}(I) \mid i \in F\}$.

Proof. We will prove [(i)⇒(ii)⇒(iii)⇒(ii)⇒(i)]. First assume that $[a]_I \in \bigcap\{\ker[\phi_{\underline{i}}]_I \mid i \in F\}$. Fixing $i \in F$, we have that

$$[\phi_{\underline{i}}(a)\xi_{\underline{i}}]_I = [\phi_{\underline{i}}]_I([a]_I)[\xi_{\underline{i}}]_I = 0 \text{ for all } \xi_{\underline{i}} \in X_{\underline{i}},$$

which in turn yields that $aX_{\underline{i}} \subseteq X_{\underline{i}}I$. An application of [36, Proposition 1.3] then gives that $\langle X_{\underline{i}}, aX_{\underline{i}}\rangle \subseteq I$, as required.

Next, assume that $\langle X_{\underline{i}}, aX_{\underline{i}}\rangle \subseteq I$ for all $i \in F$. We must prove that $\langle X_{\underline{n}}, aX_{\underline{n}}\rangle \subseteq I$ for all $\underline{n} \neq \underline{0}$ satisfying $\operatorname{supp}\underline{n} \subseteq F$. We proceed by induction on $|\underline{n}|$. If $|\underline{n}| = 1$, then $\underline{n} = \underline{i}$ for some $i \in F$, in which case $\langle X_{\underline{i}}, aX_{\underline{i}}\rangle \subseteq I$ by assumption. Now suppose the claim holds for all $\underline{n} \neq \underline{0}$ satisfying $\operatorname{supp}\underline{n} \subseteq F$ and $|\underline{n}| = N$ for some $N \in \mathbb{N}$. Fix $\underline{m} \neq \underline{0}$ satisfying $\operatorname{supp}\underline{m} \subseteq F$ and $|\underline{m}| = N + 1$. We may write $\underline{m}$ in the form $\underline{m} = \underline{n} + \underline{i}$ for some $i \in F$ and some $\underline{n} \neq \underline{0}$ satisfying $\operatorname{supp}\underline{n} \subseteq F$ and $|\underline{n}| = N$. We then have that

$$\langle X_{\underline{m}}, aX_{\underline{m}}\rangle = \langle X_{\underline{n}} \otimes_A X_{\underline{i}}, aX_{\underline{n}} \otimes_A X_{\underline{i}}\rangle \subseteq [\langle X_{\underline{i}}, \phi_{\underline{i}}(\langle X_{\underline{n}}, aX_{\underline{n}}\rangle)X_{\underline{i}}\rangle] \subseteq [\langle X_{\underline{i}}, IX_{\underline{i}}\rangle] \subseteq I,$$

using the inductive hypothesis for $\underline{n}$ and positive invariance of I. Hence $\langle X_{\underline{m}}, aX_{\underline{m}}\rangle \subseteq I$ and by induction we are done.

Finally, assume that $\langle X_{\underline{n}}, aX_{\underline{n}}\rangle \subseteq I$ for all $\underline{n} \neq \underline{0}$ satisfying $\operatorname{supp}\underline{n} \subseteq F$. In particular, we have that $\langle X_{\underline{i}}, aX_{\underline{i}}\rangle \subseteq I$ for all $i \in F$. Fixing $i \in F$, we have that $aX_{\underline{i}} \subseteq X_{\underline{i}}I$ by [36, Proposition 1.3], and therefore $[\phi_{\underline{i}}]_I([a]_I)[\xi_{\underline{i}}]_I = [\phi_{\underline{i}}(a)\xi_{\underline{i}}]_I = 0$ for all $\xi_{\underline{i}} \in X_{\underline{i}}$. Hence $[a]_I \in \ker[\phi_{\underline{i}}]_I$, from which it follows that $[a]_I \in \bigcap\{\ker[\phi_{\underline{i}}]_I \mid i \in F\}$, finishing the proof of the equivalences.

The last claim follows from the equivalence of items (ii) and (iii). □

The next proposition relates the ideals $X_F^{-1}(I)$ and $J_F(I, X)$ to ideals of $[A]_I$, when I is positively invariant. This result is the higher-rank analogue of [36, Lemma 5.2].

Proposition 4.1.3. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $I \subseteq A$ be an ideal that is positively invariant for X. Then the following hold for all $\emptyset \neq F \subseteq [d]$:*

(i) $X_F^{-1}(I) = [\cdot]_I^{-1}(\bigcap\{\ker[\phi_{\underline{i}}]_I \mid i \in F\})$.

(ii) $J_F(I, X) = [\cdot]_I^{-1}(\mathcal{J}_F([X]_I))$.

(iii) $X_F^{-1}(I) \cap J_F(I, X) = I$.

Proof. Fix $\emptyset \neq F \subseteq [d]$ and $a \in A$. Note that

$$a \in [\cdot]_I^{-1}(\bigcap\{\ker[\phi_{\underline{i}}]_I \mid i \in F\}) \iff [a]_I \in \bigcap\{\ker[\phi_{\underline{i}}]_I \mid i \in F\} \iff a \in X_F^{-1}(I),$$

where the final equivalence follows by Proposition 4.1.2. This proves item (i).

Next, assume that $a \in J_F(I, X)$. We must show that

$$[a]_I \in \mathcal{J}_F([X]_I) = (\bigcap_{i \in F} \ker[\phi_{\underline{i}}]_I)^\perp \cap (\bigcap_{i \in [d]} [\phi_{\underline{i}}]_I^{-1}(\mathcal{K}([X_{\underline{i}}]_I))).$$

By definition, we have that $[\phi_{\underline{i}}]_I[a]_I = [\phi_{\underline{i}}(a)]_I \in \mathcal{K}([X_{\underline{i}}]_I)$ for all $i \in [d]$ and $aX_F^{-1}(I) \subseteq I$. In particular, we have that $[a]_I \in \bigcap_{i \in [d]} [\phi_{\underline{i}}]_I^{-1}(\mathcal{K}([X_{\underline{i}}]_I))$. Now take $[b]_I \in \bigcap_{i \in F} \ker[\phi_{\underline{i}}]_I$. By item (i), we have that $b \in [\cdot]_I^{-1}(\bigcap_{i \in F} \ker[\phi_{\underline{i}}]_I) = X_F^{-1}(I)$, and so $ab \in I$ by definition. In particular, we have that $[a]_I[b]_I = [ab]_I = 0$, which in turn implies that $[a]_I \in (\bigcap_{i \in F} \ker[\phi_{\underline{i}}]_I)^\perp$, as required.

Now assume that $a \in [\cdot]_I^{-1}(\mathcal{J}_F([X]_I))$. Then $[\phi_{\underline{i}}]_I[a]_I = [\phi_{\underline{i}}(a)]_I \in \mathcal{K}([X_{\underline{i}}]_I)$ for all $i \in [d]$ by definition. Take $b \in X_F^{-1}(I)$. By item (i), we have that $[b]_I \in \bigcap_{i \in F} \ker[\phi_{\underline{i}}]_I$. By definition, we have that $[a]_I \in (\bigcap_{i \in F} \ker[\phi_{\underline{i}}]_I)^\perp$, so $[ab]_I = [a]_I[b]_I = 0$ and hence $ab \in I$. Thus $aX_F^{-1}(I) \subseteq I$ and hence in total $a \in J_F(I, X)$, proving item (ii).

Using items (i) and (ii), and that $\mathcal{J}_F([X]_I) \subseteq (\bigcap_{i \in F} \ker[\phi_{\underline{i}}]_I)^\perp$, we obtain that

$$X_F^{-1}(I) \cap J_F(I, X) = [\cdot]_I^{-1}\bigg((\bigcap_{i \in F} \ker[\phi_{\underline{i}}]_I) \cap \mathcal{J}_F([X]_I)\bigg) = [\cdot]_I^{-1}(\{0\}) = I,$$

proving item (iii). □

We are now ready to introduce the objects that will implement the parametrisation of the gauge-invariant ideals of $\mathcal{NT}_X$.

Definition 4.1.4. Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let $\mathcal{L}$ be a 2^d-tuple of X. We say that $\mathcal{L}$ is an *NT-2^d-tuple (of X)* if the following four conditions hold:

(i) $\mathcal{L}$ consists of ideals and $\mathcal{L}_F \subseteq J_F(\mathcal{L}_\emptyset, X)$ for all $\emptyset \neq F \subseteq [d]$,

(ii) $\mathcal{L}$ is X-invariant,

(iii) $\mathcal{L}$ is partially ordered,

(iv) $[\cdot]_{\mathcal{L}_\emptyset}^{-1}\big([\mathcal{L}_F]_{\mathcal{L}_\emptyset}^{(1)}\big) \subseteq \mathcal{L}_F$ for all $F \subseteq [d]$, where $[\mathcal{L}_F]_{\mathcal{L}_\emptyset} = \mathcal{L}_F/\mathcal{L}_\emptyset \subseteq [A]_{\mathcal{L}_\emptyset}$.

To make sense of condition (iv), first note that conditions (i) and (ii) imply that $\mathcal{L}_\emptyset$ is an ideal of A that is positively invariant for X. Hence we can make sense of $[X]_{\mathcal{L}_\emptyset}$ as a strong compactly aligned product system with coefficients in $[A]_{\mathcal{L}_\emptyset}$ by Proposition 2.5.8. Condition (iii) implies in particular that $\mathcal{L}_\emptyset \subseteq \mathcal{L}_F$ for all $F \subseteq [d]$, and so by condition (i) we have that $[\mathcal{L}]_{\mathcal{L}_\emptyset} := \{[\mathcal{L}_F]_{\mathcal{L}_\emptyset}\}_{F \subseteq [d]}$ is a 2^d-tuple of $[X]_{\mathcal{L}_\emptyset}$ that consists of ideals. An application of Proposition 4.1.3 gives that $[\mathcal{L}]_{\mathcal{L}_\emptyset} \subseteq \mathcal{J}([X]_{\mathcal{L}_\emptyset})$, while item (ii) implies that $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ is $[X]_{\mathcal{L}_\emptyset}$-invariant. Hence we have that $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ is an (E)-2^d-tuple by Proposition 3.2.3, and so we can consider the family $[\mathcal{L}]_{\mathcal{L}_\emptyset}^{(1)}$. Note also that condition (iv) holds automatically for $F = \emptyset$ and $F = [d]$.

When the left action of each fibre of X is by compacts, condition (iv) simplifies as follows.

Proposition 4.1.5. *Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C^*-algebra A and suppose that $\phi_{\underline{n}}(A) \subseteq \mathcal{K}(X_{\underline{n}})$ for all $\underline{n} \in \mathbb{Z}_+^d$. Then a 2^d-tuple $\mathcal{L}$ of X is an NT-2^d-tuple of X if and only if it satisfies conditions (i)-(iii) of Definition 4.1.4 and*

$$\left(\bigcap_{\underline{n} \perp F} X_{\underline{n}}^{-1}(J_F(\mathcal{L}_\emptyset, X))\right) \cap \mathcal{L}_{\mathrm{inv},F} \cap \mathcal{L}_{\mathrm{lim},F} \subseteq \mathcal{L}_F \text{ for all } \emptyset \neq F \subsetneq [d].$$

Proof. Without loss of generality, we may assume that $\mathcal{L}$ satisfies conditions (i)-(iii) of Definition 4.1.4. Since condition (iv) of Definition 4.1.4 holds automatically for $F = \emptyset$ and $F = [d]$, and intersections are preserved under pre-images, it suffices to show that the following items hold for fixed $\emptyset \neq F \subsetneq [d]$:

(i) $[\cdot]_{\mathcal{L}_\emptyset}^{-1}(\mathcal{I}_F([X]_{\mathcal{L}_\emptyset})) = \bigcap_{\underline{n} \perp F} X_{\underline{n}}^{-1}(J_F(\mathcal{L}_\emptyset, X))$,

(ii) $[\cdot]_{\mathcal{L}_\emptyset}^{-1}([\mathcal{L}]_{\mathcal{L}_\emptyset,\mathrm{inv},F}) = \mathcal{L}_{\mathrm{inv},F}$,

(iii) $[\cdot]_{\mathcal{L}_\emptyset}^{-1}([\mathcal{L}]_{\mathcal{L}_\emptyset,\mathrm{lim},F}) = \mathcal{L}_{\mathrm{lim},F}$.

Firstly, recall that

$$\mathcal{I}_F([X]_{\mathcal{L}_\emptyset}) = \bigcap_{\underline{n} \perp F} [X_{\underline{n}}]_{\mathcal{L}_\emptyset}^{-1}(\mathcal{J}_F([X]_{\mathcal{L}_\emptyset})) \quad \text{and} \quad \mathcal{J}_F([X]_{\mathcal{L}_\emptyset}) = [J_F(\mathcal{L}_\emptyset, X)]_{\mathcal{L}_\emptyset},$$

where the latter holds by Proposition 4.1.3. Item (i) now follows as a consequence of Lemma 2.2.19, taking $I := \mathcal{L}_\emptyset$ and $J := J_F(\mathcal{L}_\emptyset, X)$.

For the second item, recall that

$$[\mathcal{L}]_{\mathcal{L}_\emptyset,\mathrm{inv},F} := \bigcap_{\underline{m} \perp F} [X_{\underline{m}}]_{\mathcal{L}_\emptyset}^{-1}(\cap_{F \subsetneq D}[\mathcal{L}_D]_{\mathcal{L}_\emptyset}).$$

It is routine to check that $\cap_{F \subsetneq D}[\mathcal{L}_D]_{\mathcal{L}_\emptyset} = [\cap_{F \subsetneq D}\mathcal{L}_D]_{\mathcal{L}_\emptyset}$. By the partial ordering property of $\mathcal{L}$, we have that $\mathcal{L}_\emptyset \subseteq \cap_{F \subsetneq D}\mathcal{L}_D$. Thus item (ii) follows as a consequence of Lemma 2.2.19, taking $I := \mathcal{L}_\emptyset$ and $J := \cap_{F \subsetneq D}\mathcal{L}_D$.

Finally, item (iii) follows by a direct application of Lemma 2.2.18, using that $\phi_{\underline{n}}(A) \subseteq \mathcal{K}(X_{\underline{n}})$ for all $\underline{n} \in \mathbb{Z}_+^d$. This completes the proof. □

Remark 4.1.6. Let X be as in Proposition 4.1.5, and let $\mathcal{L}$ be a 2^d-tuple of X that satisfies conditions (i)-(iii) of Definition 4.1.4. Note that the assumption that $\phi_{\underline{n}}(A) \subseteq \mathcal{K}(X_{\underline{n}})$ for all $\underline{n} \in \mathbb{Z}_+^d$ is only used to obtain that $[\cdot]_{\mathcal{L}_\emptyset}^{-1}([\mathcal{L}]_{\mathcal{L}_\emptyset,\mathrm{lim},F}) = \mathcal{L}_{\mathrm{lim},F}$. In other words, it is true that

$$[\cdot]_{\mathcal{L}_\emptyset}^{-1}(\mathcal{I}_F([X]_{\mathcal{L}_\emptyset})) = \bigcap_{\underline{n} \perp F} X_{\underline{n}}^{-1}(J_F(\mathcal{L}_\emptyset, X)) \quad \text{and} \quad [\cdot]_{\mathcal{L}_\emptyset}^{-1}([\mathcal{L}]_{\mathcal{L}_\emptyset,\mathrm{inv},F}) = \mathcal{L}_{\mathrm{inv},F}$$

even when X is (just) strong compactly aligned.

NT-2^d-tuples represent the higher-rank analogue of Katsura's T-pairs.

Proposition 4.1.7. *Let $X = \{X_n\}_{n \in \mathbb{Z}_+}$ be a product system with coefficients in a C*-algebra A. Then the NT-2-tuples of X are exactly the T-pairs of X_1.*

Proof. First let $\mathcal{L} = \{\mathcal{L}_\emptyset, \mathcal{L}_{\{1\}}\}$ be an NT-2-tuple of X. Then $\mathcal{L}$ consists of ideals. Since $\mathcal{L}$ is partially ordered, we have that $\mathcal{L}_\emptyset \subseteq \mathcal{L}_{\{1\}}$. Invariance of $\mathcal{L}$ for X gives in particular that $[\langle X_1, \mathcal{L}_\emptyset X_1 \rangle] \subseteq \mathcal{L}_\emptyset$ and hence $\mathcal{L}_\emptyset$ is positively invariant for X_1. Lastly, we have that

$$\mathcal{L}_{\{1\}} \subseteq J_{\{1\}}(\mathcal{L}_\emptyset, X) \equiv J(\mathcal{L}_\emptyset, X_1),$$

using that $X_{\{1\}}^{-1}(\mathcal{L}_\emptyset) \equiv X_1^{-1}(\mathcal{L}_\emptyset)$ in the equivalence. We conclude that $\mathcal{L}$ is a T-pair of X_1, as required.

Now suppose that $\mathcal{L} = \{\mathcal{L}_\emptyset, \mathcal{L}_{\{1\}}\}$ is a T-pair of X_1. By definition, this means that $\mathcal{L}$ consists of ideals, $\mathcal{L}_\emptyset$ is positively invariant for X_1 and $\mathcal{L}_\emptyset \subseteq \mathcal{L}_{\{1\}} \subseteq J(\mathcal{L}_\emptyset, X_1)$. From this it is clear that $\mathcal{L}$ is partially ordered, so condition (iii) of Definition 4.1.4 holds. Likewise, we have that

$$\mathcal{L}_{\{1\}} \subseteq J(\mathcal{L}_\emptyset, X_1) \equiv J_{\{1\}}(\mathcal{L}_\emptyset, X),$$

so condition (i) of Definition 4.1.4 holds. To see that $\mathcal{L}$ is X-invariant, it suffices to show that $\langle X_n, \mathcal{L}_\emptyset X_n \rangle \subseteq \mathcal{L}_\emptyset$ for all $n \in \mathbb{Z}_+$. We prove this by induction on n. Firstly, note that the claim holds for $n = 0$ trivially and for $n = 1$ because $\mathcal{L}_\emptyset$ is positively invariant for X_1. Now suppose that $\langle X_N, \mathcal{L}_\emptyset X_N \rangle \subseteq \mathcal{L}_\emptyset$ for some $N \in \mathbb{N}$. Recalling that $X_N \otimes_A X_1 \cong X_{N+1}$ via the multiplication map $u_{N,1}$, we obtain that

$$\begin{aligned} \langle X_{N+1}, \mathcal{L}_\emptyset X_{N+1} \rangle &= \langle X_N \otimes_A X_1, \mathcal{L}_\emptyset (X_N \otimes_A X_1) \rangle \\ &\subseteq [\langle X_1, \langle X_N, \mathcal{L}_\emptyset X_N \rangle X_1 \rangle] \subseteq [\langle X_1, \mathcal{L}_\emptyset X_1 \rangle] \subseteq \mathcal{L}_\emptyset, \end{aligned}$$

using the inductive hypothesis in the penultimate inclusion and the $n = 1$ case in the final inclusion. Thus condition (ii) of Definition 4.1.4 holds by induction. Lastly, note that condition (iv) of Definition 4.1.4 holds trivially, since there are no proper, non-empty subsets of $\{1\}$. Thus $\mathcal{L}$ is an NT-2-tuple, completing the proof. □

The following proposition links Definition 4.1.4 and Theorem 3.4.6. Moreover, it shows that (M)-2^d-tuples form a subclass of NT-2^d-tuples.

Proposition 4.1.8. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be a 2^d-tuple of X consisting of ideals satisfying $\mathcal{L}_\emptyset \subseteq \mathcal{L}_F$ for all $F \subseteq [d]$, and assume that $\mathcal{L}_\emptyset$ is positively invariant for X. Then $\mathcal{L}$ is an NT-2^d-tuple of X if and only if $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ is an (M)-2^d-tuple of $[X]_{\mathcal{L}_\emptyset}$.*

Proof. Since $\mathcal{L}_\emptyset$ is an ideal that is positively invariant for X, we can make sense of the strong compactly aligned product system $[X]_{\mathcal{L}_\emptyset}$. Moreover, the fact that $\mathcal{L}$ consists of ideals satisfying $\mathcal{L}_\emptyset \subseteq \mathcal{L}_F$ for all $F \subseteq [d]$ gives that $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ is a 2^d-tuple of $[X]_{\mathcal{L}_\emptyset}$ that consists of ideals.

Assume that $\mathcal{L}$ is an NT-2^d-tuple of X. It suffices to show that $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ satisfies the four conditions of Theorem 3.4.6. By the preceding remarks, showing that condition (i) of Theorem 3.4.6 holds amounts to proving that $[\mathcal{L}_F]_{\mathcal{L}_\emptyset} \subseteq \mathcal{J}_F([X]_{\mathcal{L}_\emptyset})$ for all $\emptyset \neq F \subseteq [d]$, noting that the inclusion holds trivially for $F = \emptyset$. To this end, fix $\emptyset \neq F \subseteq [d]$. Since $\mathcal{L}_\emptyset$ is assumed to be positively invariant for X, an application of item (ii) of Proposition 4.1.3 gives that

$$J_F(\mathcal{L}_\emptyset, X) = [\cdot]_{\mathcal{L}_\emptyset}^{-1}(\mathcal{J}_F([X]_{\mathcal{L}_\emptyset})).$$

Combining this with the assumption that $\mathcal{L}$ is an NT-2^d-tuple of X and thus in particular $\mathcal{L}_F \subseteq J_F(\mathcal{L}_\emptyset, X)$, we deduce that

$$[\mathcal{L}_F]_{\mathcal{L}_\emptyset} \subseteq \mathcal{J}_F([X]_{\mathcal{L}_\emptyset}).$$

Thus condition (i) of Theorem 3.4.6 holds. To see that $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ is $[X]_{\mathcal{L}_\emptyset}$-invariant, fix $F \subseteq [d]$ and $\underline{n} \perp F$. We have that

$$\left\langle [X_{\underline{n}}]_{\mathcal{L}_\emptyset}, [\mathcal{L}_F]_{\mathcal{L}_\emptyset}[X_{\underline{n}}]_{\mathcal{L}_\emptyset} \right\rangle = [\langle X_{\underline{n}}, \mathcal{L}_F X_{\underline{n}} \rangle]_{\mathcal{L}_\emptyset} \subseteq [\mathcal{L}_F]_{\mathcal{L}_\emptyset},$$

using that $\mathcal{L}$ is X-invariant by condition (ii) of Definition 4.1.4 in the inclusion. Hence condition (ii) of Theorem 3.4.6 holds. Conditions (iii) and (iv) of Theorem 3.4.6 follow immediately from conditions (iii) and (iv) (respectively) of Definition 4.1.4. In total, we have that $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ satisfies the four conditions of Theorem 3.4.6, as required.

Conversely, assume that $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ is an (M)-2^d-tuple of $[X]_{\mathcal{L}_\emptyset}$. It suffices to show that $\mathcal{L}$ satisfies the four conditions of Definition 4.1.4. By assumption we have that $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ satisfies the four conditions of Theorem 3.4.6. Therefore condition (i) of Definition 4.1.4 follows by employing condition (i) of Theorem 3.4.6 and item (ii) of Proposition 4.1.3 in tandem, together with the assumption that $\mathcal{L}$ consists of ideals. To see that $\mathcal{L}$ is X-invariant, fix $F \subseteq [d]$ and $\underline{n} \perp F$. We have that $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ is $[X]_{\mathcal{L}_\emptyset}$-invariant by condition (ii) of Theorem 3.4.6, and therefore

$$[\langle X_{\underline{n}}, \mathcal{L}_F X_{\underline{n}} \rangle]_{\mathcal{L}_\emptyset} = \left\langle [X_{\underline{n}}]_{\mathcal{L}_\emptyset}, [\mathcal{L}_F]_{\mathcal{L}_\emptyset}[X_{\underline{n}}]_{\mathcal{L}_\emptyset} \right\rangle \subseteq [\mathcal{L}_F]_{\mathcal{L}_\emptyset}.$$

Hence $\langle X_{\underline{n}}, \mathcal{L}_F X_{\underline{n}} \rangle \subseteq \mathcal{L}_F + \mathcal{L}_\emptyset$. Recall that $\mathcal{L}_F$ is an ideal that contains $\mathcal{L}_\emptyset$, so $\mathcal{L}_F + \mathcal{L}_\emptyset = \mathcal{L}_F$ and therefore $\langle X_{\underline{n}}, \mathcal{L}_F X_{\underline{n}} \rangle \subseteq \mathcal{L}_F$. Thus condition (ii) of Definition 4.1.4 holds. Next we check that $\mathcal{L}$ is partially ordered. To this end, fix $F \subseteq D \subseteq [d]$. By condition (iii) of Theorem 3.4.6, we have that $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ is partially ordered and therefore

$$[\mathcal{L}_F]_{\mathcal{L}_\emptyset} \subseteq [\mathcal{L}_D]_{\mathcal{L}_\emptyset}.$$

Since $\mathcal{L}_\emptyset \subseteq \mathcal{L}_D$, we obtain that $\mathcal{L}_F \subseteq \mathcal{L}_D$, showing that condition (iii) of Definition 4.1.4 holds. Finally, condition (iv) of Definition 4.1.4 follows from condition (iv) of Theorem 3.4.6 applied to $[\mathcal{L}]_{\mathcal{L}_\emptyset}$, and the proof is complete. □

It follows from Proposition 4.1.8 that the (M)-2^d-tuples of X are exactly the NT-2^d-tuples $\mathcal{L}$ of X satisfying $\mathcal{L}_\emptyset = \{0\}$. The interplay between NT-2^d-tuples and (M)-2^d-tuples allows the transferal of properties of (M)-2^d-tuples to the general setting, towards the complete parametrisation of the gauge-invariant ideals of $\mathcal{NT}_X$. To explore this further, we examine the interaction between NT-2^d-tuples and Nica-covariant representations. The following proposition extends [36, Lemma 5.10].

Proposition 4.1.9. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let (π, t) be a Nica-covariant representation of X and let $\mathcal{L}^{(\pi,t)}$ be the associated 2^d-tuple of X (see Definition 3.1.13). Then the following hold:*

(i) $\mathcal{L}_\emptyset^{(\pi,t)}$ *is positively invariant for X.*

(ii) $\ker t_{\underline{n}} = X_{\underline{n}} \mathcal{L}_\emptyset^{(\pi,t)}$ *for all $\underline{n} \in \mathbb{Z}_+^d$.*

(iii) There exists an injective Nica-covariant representation $(\dot{\pi}, \dot{t})$ of $[X]_{\mathcal{L}_\emptyset^{(\pi,t)}}$ on $\mathrm{C}^*(\pi, t)$ such that $\pi = \dot{\pi} \circ [\cdot]_{\mathcal{L}_\emptyset^{(\pi,t)}}$, $t_{\underline{n}} = \dot{t}_{\underline{n}} \circ [\cdot]_{\mathcal{L}_\emptyset^{(\pi,t)}}$ and $\psi_{\underline{n}} = \dot{\psi}_{\underline{n}} \circ [\cdot]_{\mathcal{L}_\emptyset^{(\pi,t)}}|_{\mathcal{K}(X_{\underline{n}})}$ for all $\underline{n} \in \mathbb{Z}_+^d$, and therefore $\mathrm{C}^*(\dot{\pi}, \dot{t}) = \mathrm{C}^*(\pi, t)$. If (π, t) admits a gauge action, then so does $(\dot{\pi}, \dot{t})$.

(iv) For each $\emptyset \neq F \subseteq [d]$, if $a \in \mathcal{L}_F^{(\pi,t)}$ then $[\phi_{\underline{i}}(a)]_{\mathcal{L}_\emptyset^{(\pi,t)}} \in \mathcal{K}([X_{\underline{i}}]_{\mathcal{L}_\emptyset^{(\pi,t)}})$ for all $i \in [d]$, and

$$\dot{\pi}([a]_{\mathcal{L}_\emptyset^{(\pi,t)}}) + \sum\{(-1)^{|\underline{n}|} \dot{\psi}_{\underline{n}}([\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset^{(\pi,t)}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = 0.$$

(v) For each $\emptyset \neq F \subseteq [d]$, we have that $a \in \mathcal{L}_F^{(\pi,t)}$ if and only if for every $\underline{0} \neq \underline{n} \leq \underline{1}_F$ there exists $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ such that $[\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset^{(\pi,t)}} = [k_{\underline{n}}]_{\mathcal{L}_\emptyset^{(\pi,t)}}$ satisfying

$$\pi(a) + \sum\{(-1)^{|\underline{n}|} \psi_{\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = 0.$$

(vi) For each $F \subseteq [d]$, we have that $[\mathcal{L}_F^{(\pi,t)}]_{\mathcal{L}_\emptyset^{(\pi,t)}} = \mathcal{L}_F^{(\dot{\pi},\dot{t})}$.

Proof. (i) We have that $\mathcal{L}^{(\pi,t)}$ is X-invariant by Proposition 3.1.14, and thus in particular $\mathcal{L}_\emptyset^{(\pi,t)}$ is positively invariant for X.
(ii) Fix $\underline{n} \in \mathbb{Z}_+^d$ and $\xi_{\underline{n}} \in X_{\underline{n}}$. Then $\xi_{\underline{n}} \in \ker t_{\underline{n}}$ if and only if

$$t_{\underline{n}}(\xi_{\underline{n}})^* t_{\underline{n}}(\xi_{\underline{n}}) = \pi(\langle \xi_{\underline{n}}, \xi_{\underline{n}} \rangle) = 0$$

using the C*-identity and the fact that $(\pi, t_{\underline{n}})$ is a representation of $X_{\underline{n}}$. In turn, we have that $\pi(\langle \xi_{\underline{n}}, \xi_{\underline{n}} \rangle) = 0$ if and only if $\langle \xi_{\underline{n}}, \xi_{\underline{n}} \rangle \in \mathcal{L}_\emptyset^{(\pi,t)}$ by definition. An application of [36, Proposition 1.3] then gives that

$$\langle \xi_{\underline{n}}, \xi_{\underline{n}} \rangle \in \mathcal{L}_\emptyset^{(\pi,t)} \iff \xi_{\underline{n}} \in X_{\underline{n}} \mathcal{L}_\emptyset^{(\pi,t)}.$$

Thus we conclude that $\ker t_{\underline{n}} = X_{\underline{n}} \mathcal{L}_\emptyset^{(\pi,t)}$, as required.

(iii) First note that we may make sense of the quotient product system $[X]_{\mathcal{L}_\emptyset^{(\pi,t)}}$ in the usual way by item (i). We define

$$\begin{aligned}
&\dot{\pi} : [A]_{\mathcal{L}_\emptyset^{(\pi,t)}} \to \mathrm{C}^*(\pi,t); \dot{\pi}([a]_{\mathcal{L}_\emptyset^{(\pi,t)}}) = \pi(a) \text{ for all } a \in A;\\
&\dot{t}_{\underline{n}} : [X_{\underline{n}}]_{\mathcal{L}_\emptyset^{(\pi,t)}} \to \mathrm{C}^*(\pi,t); \dot{t}_{\underline{n}}([\xi_{\underline{n}}]_{\mathcal{L}_\emptyset^{(\pi,t)}}) = t_{\underline{n}}(\xi_{\underline{n}}) \text{ for all } \xi_{\underline{n}} \in X_{\underline{n}}, \underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}.
\end{aligned}$$

The maps $\dot{\pi}$ and $\dot{t}_{\underline{n}}$ are well-defined for all $\underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$ by item (ii). It is routine to check that $(\dot{\pi}, \dot{t})$ is an injective representation of $[X]_{\mathcal{L}_\emptyset^{(\pi,t)}}$ that satisfies the stated equalities. To see that $(\dot{\pi}, \dot{t})$ is Nica-covariant, fix $\underline{n}, \underline{m} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$. By Lemma 2.2.11, it suffices to check (2.12) for $[k_{\underline{n}}]_{\mathcal{L}_\emptyset^{(\pi,t)}}$ and $[k_{\underline{m}}]_{\mathcal{L}_\emptyset^{(\pi,t)}}$, where $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ and $k_{\underline{m}} \in \mathcal{K}(X_{\underline{m}})$. Letting $\{\iota_{\underline{n}}^{\underline{n}+\underline{m}}\}_{\underline{n},\underline{m}\in\mathbb{Z}_+^d}$ denote the connecting $*$-homomorphisms of X and letting $\{j_{\underline{n}}^{\underline{n}+\underline{m}}\}_{\underline{n},\underline{m}\in\mathbb{Z}_+^d}$ denote those of $[X]_{\mathcal{L}_\emptyset^{(\pi,t)}}$, by Proposition 2.4.4 and Nica-covariance of (π,t) we have that

$$\begin{aligned}
\dot{\psi}_{\underline{n}}([k_{\underline{n}}]_{\mathcal{L}_\emptyset^{(\pi,t)}})\dot{\psi}_{\underline{m}}([k_{\underline{m}}]_{\mathcal{L}_\emptyset^{(\pi,t)}}) &= \psi_{\underline{n}}(k_{\underline{n}})\psi_{\underline{m}}(k_{\underline{m}})\\
&= \psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(k_{\underline{n}})\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(k_{\underline{m}}))\\
&= \dot{\psi}_{\underline{n}\vee\underline{m}}([\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(k_{\underline{n}})\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(k_{\underline{m}})]_{\mathcal{L}_\emptyset^{(\pi,t)}})\\
&= \dot{\psi}_{\underline{n}\vee\underline{m}}(j_{\underline{n}}^{\underline{n}\vee\underline{m}}([k_{\underline{n}}]_{\mathcal{L}_\emptyset^{(\pi,t)}})j_{\underline{m}}^{\underline{n}\vee\underline{m}}([k_{\underline{m}}]_{\mathcal{L}_\emptyset^{(\pi,t)}})),
\end{aligned}$$

as required.

If (π,t) admits a gauge action, then this is inherited by $(\dot{\pi}, \dot{t})$ since $\mathrm{C}^*(\dot{\pi}, \dot{t}) = \mathrm{C}^*(\pi,t)$, finishing the proof of item (iii).

(iv) Fix $\emptyset \neq F \subseteq [d]$ and $a \in \mathcal{L}_F^{(\pi,t)}$. Then $\pi(a) \in B_{(\underline{0},\underline{1}_F]}^{(\pi,t)}$ by definition, and thus $\dot{\pi}([a]_{\mathcal{L}_\emptyset^{(\pi,t)}}) \in B_{(\underline{0},\underline{1}_F]}^{(\dot{\pi},\dot{t})}$ by item (iii). In turn, we have that

$$\dot{\pi}([a]_{\mathcal{L}_\emptyset^{(\pi,t)}})\dot{q}_F = 0$$

by Proposition 2.5.17. An application of Proposition 2.5.18 gives that

$$[\phi_{\underline{i}}(a)]_{\mathcal{L}_\emptyset^{(\pi,t)}} = [\phi_{\underline{i}}]_{\mathcal{L}_\emptyset^{(\pi,t)}}([a]_{\mathcal{L}_\emptyset^{(\pi,t)}}) \in \mathcal{K}([X_{\underline{i}}]_{\mathcal{L}_\emptyset^{(\pi,t)}}) \text{ for all } i \in [d],$$

using the fact that $(\dot{\pi}, \dot{t})$ is injective and Nica-covariant by item (iii). Applying (2.21) for $(\dot{\pi}, \dot{t})$, we obtain that

$$\dot{\pi}([a]_{\mathcal{L}_\emptyset^{(\pi,t)}}) + \sum\{(-1)^{|\underline{n}|}\dot{\psi}_{\underline{n}}([\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset^{(\pi,t)}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = 0,$$

showing that item (iv) holds.

(v) Fix $\emptyset \neq F \subseteq [d]$. The reverse implication is immediate, so assume that $a \in \mathcal{L}_F^{(\pi,t)}$. By item (iv), we have that

$$\dot{\pi}([a]_{\mathcal{L}_\emptyset^{(\pi,t)}}) + \sum\{(-1)^{|\underline{n}|}\dot{\psi}_{\underline{n}}([\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset^{(\pi,t)}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = 0.$$

An application of Lemma 2.2.11 gives that $[\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset^{(\pi,t)}} = [k_{\underline{n}}]_{\mathcal{L}_\emptyset^{(\pi,t)}}$ for some $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$, for all $\underline{0} \neq \underline{n} \leq \underline{1}_F$. By item (iii), we have that

$$\begin{aligned} 0 &= \dot{\pi}([a]_{\mathcal{L}_\emptyset^{(\pi,t)}}) + \sum\{(-1)^{|\underline{n}|}\dot{\psi}_{\underline{n}}([\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset^{(\pi,t)}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \\ &= \dot{\pi}([a]_{\mathcal{L}_\emptyset^{(\pi,t)}}) + \sum\{(-1)^{|\underline{n}|}\dot{\psi}_{\underline{n}}([k_{\underline{n}}]_{\mathcal{L}_\emptyset^{(\pi,t)}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \\ &= \pi(a) + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}, \end{aligned}$$

showing that item (v) holds.

(vi) First recall that $\mathcal{L}_F^{(\pi,t)}$ is an ideal satisfying $\mathcal{L}_\emptyset^{(\pi,t)} \subseteq \mathcal{L}_F^{(\pi,t)}$ for all $F \subseteq [d]$ by Proposition 3.1.14. Thus we can make sense of the 2^d-tuple $[\mathcal{L}^{(\pi,t)}]_{\mathcal{L}_\emptyset^{(\pi,t)}}$ of $[X]_{\mathcal{L}_\emptyset^{(\pi,t)}}$. The claim holds trivially when $F = \emptyset$ (because $(\dot{\pi}, \dot{t})$ is injective), so fix $\emptyset \neq F \subseteq [d]$ and take $a \in \mathcal{L}_F^{(\pi,t)}$. Then item (iv) yields that $[a]_{\mathcal{L}_\emptyset^{(\pi,t)}} \in \mathcal{L}_F^{(\dot{\pi},\dot{t})}$. This shows that $[\mathcal{L}_F^{(\pi,t)}]_{\mathcal{L}_\emptyset^{(\pi,t)}} \subseteq \mathcal{L}_F^{(\dot{\pi},\dot{t})}$.

Now take $[a]_{\mathcal{L}_\emptyset^{(\pi,t)}} \in \mathcal{L}_F^{(\dot{\pi},\dot{t})}$. Then by definition and Lemma 2.2.11, for every $\underline{0} \neq \underline{n} \leq \underline{1}_F$ there exists $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ such that

$$\dot{\pi}([a]_{\mathcal{L}_\emptyset^{(\pi,t)}}) = \sum\{\dot{\psi}_{\underline{n}}([k_{\underline{n}}]_{\mathcal{L}_\emptyset^{(\pi,t)}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}.$$

Using item (iii), we obtain that

$$\pi(a) = \dot{\pi}([a]_{\mathcal{L}_\emptyset^{(\pi,t)}}) = \sum\{\dot{\psi}_{\underline{n}}([k_{\underline{n}}]_{\mathcal{L}_\emptyset^{(\pi,t)}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = \sum\{\psi_{\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}.$$

This shows that $a \in \mathcal{L}_F^{(\pi,t)}$, and hence $[a]_{\mathcal{L}_\emptyset^{(\pi,t)}} \in [\mathcal{L}_F^{(\pi,t)}]_{\mathcal{L}_\emptyset^{(\pi,t)}}$. Consequently, we have that $\mathcal{L}_F^{(\dot{\pi},\dot{t})} \subseteq [\mathcal{L}_F^{(\pi,t)}]_{\mathcal{L}_\emptyset^{(\pi,t)}}$ and hence $[\mathcal{L}_F^{(\pi,t)}]_{\mathcal{L}_\emptyset^{(\pi,t)}} = \mathcal{L}_F^{(\dot{\pi},\dot{t})}$ for all $F \subseteq [d]$, finishing the proof. □

Remark 4.1.10. Let $\mathcal{L}^{(\pi,t)}$ be the 2^d-tuple of X associated with a Nica-covariant representation (π, t) of X, and let $(\dot{\pi}, \dot{t})$ be the injective Nica-covariant representation of $[X]_{\mathcal{L}_\emptyset^{(\pi,t)}}$ defined in item (iii) of Proposition 4.1.9. By Proposition 3.1.17, we have that $(\dot{\pi}, \dot{t})$ is an $\mathcal{L}^{(\dot{\pi},\dot{t})}$-relative CNP-representation of $[X]_{\mathcal{L}_\emptyset^{(\pi,t)}}$, giving the following commutative diagram

$$\begin{array}{ccc} \mathcal{NT}_{[X]_{\mathcal{L}_\emptyset^{(\pi,t)}}} & \xrightarrow{\quad \dot{\pi}\times\dot{t} \quad} & \mathrm{C}^*(\dot{\pi}, \dot{t}) \\ & \searrow \qquad \nearrow & \\ & \mathcal{NO}(\mathcal{L}^{(\dot{\pi},\dot{t})}, [X]_{\mathcal{L}_\emptyset^{(\pi,t)}}) & \end{array}$$

of canonical $*$-epimorphisms. Using positive invariance of $\mathcal{L}_\emptyset^{(\pi,t)}$, we also obtain a canonical $*$-epimorphism

$$\mathcal{NT}_X \to \mathcal{NT}_{[X]_{\mathcal{L}_\emptyset^{(\pi,t)}}}$$

that lifts the quotient map $X \to [X]_{\mathcal{L}_\emptyset^{(\pi,t)}}$. We have that $[\mathcal{L}^{(\pi,t)}]_{\mathcal{L}_\emptyset^{(\pi,t)}} = \mathcal{L}^{(\dot{\pi},\dot{t})}$ by item (vi) of Proposition 4.1.9, and that $\mathrm{C}^*(\pi, t) = \mathrm{C}^*(\dot{\pi}, \dot{t})$ by item (iii) of Proposition 4.1.9. Hence we obtain the following commutative diagram

$$\begin{array}{ccc}
\mathcal{NT}_X & \xrightarrow{\pi\times t} & \mathrm{C}^*(\pi,t) \\
\downarrow & \nearrow_{\dot{\pi}\times\dot{t}} & \uparrow \\
\mathcal{NT}_{[X]_{\mathcal{L}_\emptyset^{(\pi,t)}}} & \longrightarrow & \mathcal{NO}([\mathcal{L}^{(\pi,t)}]_{\mathcal{L}_\emptyset^{(\pi,t)}}, [X]_{\mathcal{L}_\emptyset^{(\pi,t)}})
\end{array}$$

of canonical $*$-epimorphisms.

By Proposition 3.1.18 and Theorem 3.2.12, (M)-2^d-tuples are exactly of the form $\mathcal{L}^{(\pi,t)}$ for some injective Nica-covariant representation (π, t) that admits a gauge action. This is extended to NT-2^d-tuples by allowing (π, t) to be non-injective. To substantiate this, we introduce the following notation.

Definition 4.1.11. Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. For an NT-2^d-tuple $\mathcal{L}$ of X, we define the maps

$$\pi^{\mathcal{L}} \colon A \to \mathcal{NO}([\mathcal{L}]_{\mathcal{L}_\emptyset}, [X]_{\mathcal{L}_\emptyset}); \pi^{\mathcal{L}}(a) = \pi_{[X]_{\mathcal{L}_\emptyset}}^{[\mathcal{L}]_{\mathcal{L}_\emptyset}}([a]_{\mathcal{L}_\emptyset}) \text{ for all } a \in A,$$
$$t_{\underline{n}}^{\mathcal{L}} \colon X_{\underline{n}} \to \mathcal{NO}([\mathcal{L}]_{\mathcal{L}_\emptyset}, [X]_{\mathcal{L}_\emptyset}); t_{\underline{n}}^{\mathcal{L}}(\xi_{\underline{n}}) = t_{[X]_{\mathcal{L}_\emptyset},\underline{n}}^{[\mathcal{L}]_{\mathcal{L}_\emptyset}}([\xi_{\underline{n}}]_{\mathcal{L}_\emptyset}) \text{ for all } \xi_{\underline{n}} \in X_{\underline{n}}, \underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\},$$

where $(\pi_{[X]_{\mathcal{L}_\emptyset}}^{[\mathcal{L}]_{\mathcal{L}_\emptyset}}, t_{[X]_{\mathcal{L}_\emptyset}}^{[\mathcal{L}]_{\mathcal{L}_\emptyset}})$ denotes the universal $[\mathcal{L}]_{\mathcal{L}_\emptyset}$-relative CNP-representation of $[X]_{\mathcal{L}_\emptyset}$.

Checking that $(\pi^{\mathcal{L}}, t^{\mathcal{L}})$ is a Nica-covariant representation is routine, as it is obtained from the canonical $*$-epimorphism

$$\mathcal{NT}_X \to \mathcal{NT}_{[X]_{\mathcal{L}_\emptyset}} \to \mathcal{NO}([\mathcal{L}]_{\mathcal{L}_\emptyset}, [X]_{\mathcal{L}_\emptyset}),$$

where we use that $\mathcal{L}_\emptyset$ is positively invariant for the existence of the first map, and that $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ is an (M)-2^d-tuple (and thus in particular relative) of $[X]_{\mathcal{L}_\emptyset}$ by Proposition 4.1.8 for the existence of the second map. Additionally, notice that $(\pi^{\mathcal{L}}, t^{\mathcal{L}})$ admits a gauge action since

$$\mathrm{C}^*(\pi^{\mathcal{L}}, t^{\mathcal{L}}) = \mathrm{C}^*(\pi_{[X]_{\mathcal{L}_\emptyset}}^{[\mathcal{L}]_{\mathcal{L}_\emptyset}}, t_{[X]_{\mathcal{L}_\emptyset}}^{[\mathcal{L}]_{\mathcal{L}_\emptyset}}) = \mathcal{NO}([\mathcal{L}]_{\mathcal{L}_\emptyset}, [X]_{\mathcal{L}_\emptyset}).$$

Finally, we have that $\psi_{\underline{n}}^{\mathcal{L}} = \psi_{[X]_{\mathcal{L}_\emptyset},\underline{n}}^{[\mathcal{L}]_{\mathcal{L}_\emptyset}} \circ [\cdot]_{\mathcal{L}_\emptyset}|_{\mathcal{K}(X_{\underline{n}})}$ for all $\underline{n} \in \mathbb{Z}_+^d$.

Proposition 4.1.12. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. If $\mathcal{L}$ is an NT-2^d-tuple of X, then $\mathcal{L}^{(\pi^{\mathcal{L}}, t^{\mathcal{L}})} = \mathcal{L}$.*

Consequently, a 2^d-tuple $\mathcal{L}$ of X is an NT-2^d-tuple if and only if $\mathcal{L} = \mathcal{L}^{(\pi,t)}$ for some Nica-covariant representation (π, t) of X that admits a gauge action.

Proof. For the first claim, we denote the universal $[\mathcal{L}]_{\mathcal{L}_\emptyset}$-relative CNP-representation of $[X]_{\mathcal{L}_\emptyset}$ by $(\widetilde{\pi}, \widetilde{t})$. First we show that $\mathcal{L}_\emptyset^{(\pi^{\mathcal{L}}, t^{\mathcal{L}})} = \mathcal{L}_\emptyset$. We have that

$$a \in \mathcal{L}_\emptyset^{(\pi^{\mathcal{L}}, t^{\mathcal{L}})} \iff \pi^{\mathcal{L}}(a) = 0 \iff \widetilde{\pi}([a]_{\mathcal{L}_\emptyset}) = 0 \iff [a]_{\mathcal{L}_\emptyset} = 0 \iff a \in \mathcal{L}_\emptyset,$$

using that $(\widetilde{\pi}, \widetilde{t})$ is injective by Proposition 3.2.1, since $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ is an (M)-2^d-tuple of $[X]_{\mathcal{L}_\emptyset}$ by Proposition 4.1.8. Hence $\mathcal{L}_\emptyset^{(\pi^{\mathcal{L}}, t^{\mathcal{L}})} = \mathcal{L}_\emptyset$.

Recall that $\mathrm{C}^*(\widetilde{\pi}, \widetilde{t}) = \mathcal{NO}([\mathcal{L}]_{\mathcal{L}_\emptyset}, [X]_{\mathcal{L}_\emptyset})$ by definition, and thus by applying Theorem 3.2.12 for the (M)-2^d-tuple $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ of $[X]_{\mathcal{L}_\emptyset}$ and the Nica-covariant representation $(\widetilde{\pi}, \widetilde{t})$ of $[X]_{\mathcal{L}_\emptyset}$, we obtain that $\mathcal{L}^{(\widetilde{\pi}, \widetilde{t})} = [\mathcal{L}]_{\mathcal{L}_\emptyset}$. Hence, for $\emptyset \neq F \subseteq [d]$, we have that

$$\begin{aligned} a \in \mathcal{L}_F^{(\pi^{\mathcal{L}}, t^{\mathcal{L}})} &\iff \pi^{\mathcal{L}}(a) = \sum \{\psi_{\underline{n}}^{\mathcal{L}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \text{ for some } k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}) \\ &\iff \widetilde{\pi}([a]_{\mathcal{L}_\emptyset}) = \sum \{\widetilde{\psi}_{\underline{n}}([k_{\underline{n}}]_{\mathcal{L}_\emptyset}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \text{ for some } k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}) \\ &\iff [a]_{\mathcal{L}_\emptyset} \in \mathcal{L}_F^{(\widetilde{\pi}, \widetilde{t})} = [\mathcal{L}_F]_{\mathcal{L}_\emptyset} \\ &\iff a \in \mathcal{L}_F + \mathcal{L}_\emptyset = \mathcal{L}_F, \end{aligned}$$

and so $\mathcal{L}_F^{(\pi^{\mathcal{L}}, t^{\mathcal{L}})} = \mathcal{L}_F$, completing the proof of the first part.

For the second part, if $\mathcal{L}$ is an NT-2^d-tuple of X then $\mathcal{L} = \mathcal{L}^{(\pi,t)}$ for $(\pi, t) := (\pi^{\mathcal{L}}, t^{\mathcal{L}})$. Conversely, if $\mathcal{L} = \mathcal{L}^{(\pi,t)}$ for some Nica-covariant representation (π, t) of X that admits a gauge action, then let $(\dot{\pi}, \dot{t})$ be the injective Nica-covariant representation of $[X]_{\mathcal{L}_\emptyset}$ guaranteed by item (iii) of Proposition 4.1.9. The latter also gives that $(\dot{\pi}, \dot{t})$ admits a gauge action. Then

$$[\mathcal{L}]_{\mathcal{L}_\emptyset} = [\mathcal{L}^{(\pi,t)}]_{\mathcal{L}_\emptyset^{(\pi,t)}} = \mathcal{L}^{(\dot{\pi}, \dot{t})}$$

by item (vi) of Proposition 4.1.9. We have that $\mathcal{L}^{(\dot{\pi}, \dot{t})}$ is an (M)-2^d-tuple of $[X]_{\mathcal{L}_\emptyset}$ by Proposition 3.1.18, and thus $\mathcal{L}$ is an NT-2^d-tuple of X by Proposition 4.1.8, finishing the proof. □

Consequently, we have an extension of Proposition 3.1.17 for describing the kernels of induced $*$-representations that may not be injective on X.

Proposition 4.1.13. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let (π, t) be a Nica-covariant representation of X that admits a gauge action, and let $\mathcal{L}^{(\pi,t)}$ be the associated NT-2^d-tuple of X. Then*

$$\begin{aligned} \ker \pi \times t = \langle \overline{\pi}_X(a) + \sum_{\underline{0} \neq \underline{n} \leq \underline{1}_F} (-1)^{|\underline{n}|} \overline{\psi}_{X,\underline{n}}(k_{\underline{n}}) \mid & F \subseteq [d], a \in \mathcal{L}_F^{(\pi,t)}, k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}), \\ & [\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset^{(\pi,t)}} = [k_{\underline{n}}]_{\mathcal{L}_\emptyset^{(\pi,t)}} \text{ for all } \underline{0} \neq \underline{n} \leq \underline{1}_F \rangle. \end{aligned}$$

Proof. We denote the ideal on the right hand side by $\mathfrak{J}$. For notational convenience, we drop the superscript (π, t) and write $\mathcal{L} := \mathcal{L}^{(\pi,t)}$.

We begin by proving that $\mathfrak{J} \subseteq \ker \pi \times t$. To this end, it suffices to show that $\ker \pi \times t$ contains the generators of $\mathfrak{J}$. The generators of $\mathfrak{J}$ that are indexed by $F = \emptyset$ have the form $\overline{\pi}_X(a)$ for some $a \in \mathcal{L}_\emptyset \equiv \ker \pi$. In this case we have that $(\pi \times t)(\overline{\pi}_X(a)) = \pi(a) = 0$ trivially, so $\overline{\pi}_X(a) \in \ker \pi \times t$, as required.

Next, fix $\emptyset \neq F \subseteq [d], a \in \mathcal{L}_F$ and $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ such that $[\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset} = [k_{\underline{n}}]_{\mathcal{L}_\emptyset}$ for all $\underline{0} \neq \underline{n} \leq \underline{1}_F$. Let $(\dot{\pi}, \dot{t})$ be the injective Nica-covariant representation of $[X]_{\mathcal{L}_\emptyset}$ admitting a gauge action that is guaranteed by item (iii) of Proposition 4.1.9. We then have that

$$
\begin{aligned}
(\pi \times t)(\overline{\pi}_X(a) + \textstyle\sum\{(-1)^{|\underline{n}|}\overline{\psi}_{X,\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}) = & \\
&= \pi(a) + \textstyle\sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \\
&= \dot{\pi}([a]_{\mathcal{L}_\emptyset}) + \textstyle\sum\{(-1)^{|\underline{n}|}\dot{\psi}_{\underline{n}}([k_{\underline{n}}]_{\mathcal{L}_\emptyset}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \\
&= \dot{\pi}([a]_{\mathcal{L}_\emptyset}) + \textstyle\sum\{(-1)^{|\underline{n}|}\dot{\psi}_{\underline{n}}([\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = 0,
\end{aligned}
$$

using item (iv) of Proposition 4.1.9 to obtain the final equality. This shows that $\mathfrak{J} \subseteq \ker \pi \times t$.

We have that $[\mathcal{L}]_{\mathcal{L}_\emptyset} = \mathcal{L}^{(\dot{\pi},\dot{t})}$ by item (vi) of Proposition 4.1.9, and thus by applying Proposition 3.1.17 for $(\dot{\pi},\dot{t})$ we obtain a canonical $*$-isomorphism

$$\Phi \colon \mathcal{NO}([\mathcal{L}]_{\mathcal{L}_\emptyset}, [X]_{\mathcal{L}_\emptyset}) \to \mathrm{C}^*(\dot{\pi},\dot{t}) = \mathrm{C}^*(\pi, t).$$

By considering the representation $(\pi^{\mathcal{L}}, t^{\mathcal{L}})$ of Definition 4.1.11, and the canonical quotient map $Q_{\mathfrak{J}} \colon \mathcal{NT}_X \to \mathcal{NT}_X/\mathfrak{J}$, we obtain the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{NT}_X & \xrightarrow{\ \pi \times t\ } & \mathrm{C}^*(\pi, t) \\
{\scriptstyle Q_{\mathfrak{J}}}\big\downarrow & \searrow^{\pi^{\mathcal{L}} \times t^{\mathcal{L}}} & \big\uparrow{\scriptstyle \Phi}\ {\scriptstyle\cong} \\
\mathcal{NT}_X/\mathfrak{J} & \xrightarrow[\exists ! \Psi]{} & \mathcal{NO}([\mathcal{L}]_{\mathcal{L}_\emptyset}, [X]_{\mathcal{L}_\emptyset})
\end{array}
$$

of $*$-epimorphisms. Note that Ψ exists because

$$\mathfrak{J} \subseteq \ker \pi \times t = \ker \Phi \circ (\pi^{\mathcal{L}} \times t^{\mathcal{L}}) = \ker \pi^{\mathcal{L}} \times t^{\mathcal{L}}.$$

Suppose we have shown that Ψ is injective. Then whenever $f \in \ker \pi \times t$, we have that

$$(\Phi \circ \Psi \circ Q_{\mathfrak{J}})(f) = (\pi \times t)(f) = 0$$

and hence $Q_{\mathfrak{J}}(f) = 0$ by injectivity of Φ and Ψ. It follows that $\ker \pi \times t \subseteq \mathfrak{J}$, as required. Thus, to finish the proof, it suffices to show that Ψ is injective.

To this end, we define maps $\widetilde{\pi}$ and $\widetilde{t}_{\underline{n}}$ by

$$
\begin{aligned}
\widetilde{\pi} &\colon [A]_{\mathcal{L}_\emptyset} \to \mathcal{NT}_X/\mathfrak{J}; \widetilde{\pi}([a]_{\mathcal{L}_\emptyset}) = Q_{\mathfrak{J}}(\overline{\pi}_X(a)), \\
\widetilde{t}_{\underline{n}} &\colon [X_{\underline{n}}]_{\mathcal{L}_\emptyset} \to \mathcal{NT}_X/\mathfrak{J}; \widetilde{t}_{\underline{n}}([\xi_{\underline{n}}]_{\mathcal{L}_\emptyset}) = Q_{\mathfrak{J}}(\overline{t}_{X,\underline{n}}(\xi_{\underline{n}})),
\end{aligned}
$$

for all $a \in A, \xi_{\underline{n}} \in X_{\underline{n}}$ and $\underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$. These maps are well-defined because $\overline{\pi}_X(\mathcal{L}_\emptyset) \subseteq \mathfrak{J}$ and $\overline{t}_{X,\underline{n}}(X_{\underline{n}}\mathcal{L}_\emptyset) \subseteq \mathfrak{J}$. It is routine to check that $(\widetilde{\pi}, \widetilde{t})$ is a Nica-covariant representation of $[X]_{\mathcal{L}_\emptyset}$, since $\widetilde{\psi}_{\underline{n}} \circ [\cdot]_{\mathcal{L}_\emptyset}|_{\mathcal{K}(X_{\underline{n}})} = Q_{\mathfrak{J}} \circ \overline{\psi}_{X,\underline{n}}$ for all $\underline{n} \in \mathbb{Z}_+^d$ by definition of $(\widetilde{\pi}, \widetilde{t})$. Additionally, we have that $\mathrm{C}^*(\widetilde{\pi}, \widetilde{t}) = \mathcal{NT}_X/\mathfrak{J}$.

We claim that $(\widetilde{\pi}, \widetilde{t})$ is an $[\mathcal{L}]_{\mathcal{L}_\emptyset}$-relative CNP-representation. To see this, fix $\emptyset \neq F \subseteq [d]$ and $a \in \mathcal{L}_F$. For each $\underline{0} \neq \underline{n} \leq \underline{1}_F$, there exists $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ such that $[\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset} = [k_{\underline{n}}]_{\mathcal{L}_\emptyset}$

by item (iv) of Proposition 4.1.9 and Lemma 2.2.11. Hence we obtain that

$$\begin{aligned}\widetilde{\pi}([a]_{\mathcal{L}_\emptyset})\widetilde{q}_F &= \widetilde{\pi}([a]_{\mathcal{L}_\emptyset}) + \sum\{(-1)^{|\underline{n}|}\widetilde{\psi}_{\underline{n}}([\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \\ &= Q_{\mathfrak{J}}(\overline{\pi}_X(a)) + \sum\{(-1)^{|\underline{n}|}Q_{\mathfrak{J}}(\overline{\psi}_{X,\underline{n}}(k_{\underline{n}})) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \\ &= Q_{\mathfrak{J}}(\overline{\pi}_X(a) + \sum\{(-1)^{|\underline{n}|}\overline{\psi}_{X,\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}) = 0,\end{aligned}$$

using that $\overline{\pi}_X(a)+\sum\{(-1)^{|\underline{n}|}\overline{\psi}_{X,\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \in \mathfrak{J}$ in the final equality. This shows that $(\widetilde{\pi}, \widetilde{t})$ is an $[\mathcal{L}]_{\mathcal{L}_\emptyset}$-relative CNP-representation, and so universality of $\mathcal{NO}([\mathcal{L}]_{\mathcal{L}_\emptyset}, [X]_{\mathcal{L}_\emptyset})$ guarantees a (unique) canonical $*$-epimorphism

$$\widetilde{\Phi} \colon \mathcal{NO}([\mathcal{L}]_{\mathcal{L}_\emptyset}, [X]_{\mathcal{L}_\emptyset}) \to \mathcal{NT}_X/\mathfrak{J} = \mathrm{C}^*(\widetilde{\pi}, \widetilde{t}).$$

It is routine to check that $\widetilde{\Phi} \circ \Psi = \mathrm{id}_{\mathcal{NT}_X/\mathfrak{J}}$ and thus Ψ is injective, as required. □

4.2 Gauge-invariant ideal structure of $\mathcal{NT}_X$

We can now pass to the parametrisation of gauge-invariant ideals by NT-2^d-tuples. For an NT-2^d-tuple $\mathcal{L}$ of X, we write

$\mathfrak{J}^{\mathcal{L}} := \ker \pi^{\mathcal{L}} \times t^{\mathcal{L}}$, for the canonical $*$-epimorphism $\pi^{\mathcal{L}} \times t^{\mathcal{L}} \colon \mathcal{NT}_X \to \mathcal{NO}([\mathcal{L}]_{\mathcal{L}_\emptyset}, [X]_{\mathcal{L}_\emptyset})$.

Observe that $\mathfrak{J}^{\mathcal{L}}$ is a gauge-invariant ideal of $\mathcal{NT}_X$. Indeed, let β denote the gauge action of $(\overline{\pi}_X, \overline{t}_X)$ and let γ denote the gauge action of $(\pi^{\mathcal{L}}, t^{\mathcal{L}})$. Then we have that

$$\gamma_{\underline{z}} \circ (\pi^{\mathcal{L}} \times t^{\mathcal{L}}) = (\pi^{\mathcal{L}} \times t^{\mathcal{L}}) \circ \beta_{\underline{z}} \text{ for all } \underline{z} \in \mathbb{T}^d.$$

Fix $f \in \mathfrak{J}^{\mathcal{L}}$ and $\underline{z} \in \mathbb{T}^d$. Then we obtain that

$$(\pi^{\mathcal{L}} \times t^{\mathcal{L}})(\beta_{\underline{z}}(f)) = \gamma_{\underline{z}}((\pi^{\mathcal{L}} \times t^{\mathcal{L}})(f)) = 0$$

by definition of $\mathfrak{J}^{\mathcal{L}}$. Hence $\beta_{\underline{z}}(f) \in \mathfrak{J}^{\mathcal{L}}$, proving that $\mathfrak{J}^{\mathcal{L}}$ is gauge-invariant. On the other hand, for a gauge-invariant ideal $\mathfrak{J}$ of $\mathcal{NT}_X$, we write

$$\mathcal{L}^{\mathfrak{J}} := \mathcal{L}^{(Q_{\mathfrak{J}} \circ \overline{\pi}_X, Q_{\mathfrak{J}} \circ \overline{t}_X)}, \text{ for the canonical } *\text{-epimorphism } Q_{\mathfrak{J}} \colon \mathcal{NT}_X \to \mathcal{NT}_X/\mathfrak{J}.$$

The 2^d-tuple $\mathcal{L}^{\mathfrak{J}}$ is an NT-2^d-tuple of X by Proposition 4.1.12. The following proposition shows that these correspondences are mutually inverse.

Proposition 4.2.1. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Then the following hold:*

(i) If $\mathcal{L}$ is an NT-2^d-tuple of X, then

$$\mathcal{NT}_X/\mathfrak{J}^{\mathcal{L}} \cong \mathcal{NO}([\mathcal{L}]_{\mathcal{L}_\emptyset}, [X]_{\mathcal{L}_\emptyset}) \quad \textit{and} \quad \mathcal{L}^{\mathfrak{J}^{\mathcal{L}}} = \mathcal{L}.$$

(ii) If $\mathfrak{J} \subseteq \mathcal{NT}_X$ is a gauge-invariant ideal, then

$$\mathcal{NT}_X/\mathfrak{J} \cong \mathcal{NO}([\mathcal{L}^{\mathfrak{J}}]_{\mathcal{L}^{\mathfrak{J}}_\emptyset}, [X]_{\mathcal{L}^{\mathfrak{J}}_\emptyset}) \quad \text{and} \quad \mathfrak{J}^{\mathcal{L}^{\mathfrak{J}}} = \mathfrak{J}.$$

Proof. (i) Since $\mathfrak{J}^{\mathcal{L}} \equiv \ker \pi^{\mathcal{L}} \times t^{\mathcal{L}}$, we have that $\mathcal{NT}_X/\mathfrak{J}^{\mathcal{L}} \cong \mathcal{NO}([\mathcal{L}]_{\mathcal{L}_\emptyset}, [X]_{\mathcal{L}_\emptyset})$ by a canonical $*$-isomorphism Φ. We then have that

$$\mathcal{L}^{\mathfrak{J}^{\mathcal{L}}} \equiv \mathcal{L}^{(Q_{\mathfrak{J}^{\mathcal{L}}} \circ \overline{\pi}_X, Q_{\mathfrak{J}^{\mathcal{L}}} \circ \overline{t}_X)} = \mathcal{L}^{(\Phi^{-1} \circ \pi^{\mathcal{L}}, \Phi^{-1} \circ t^{\mathcal{L}})} = \mathcal{L}^{(\pi^{\mathcal{L}}, t^{\mathcal{L}})} = \mathcal{L}.$$

The first equality follows from the fact that $\Phi^{-1} \circ (\pi^{\mathcal{L}} \times t^{\mathcal{L}}) = Q_{\mathfrak{J}^{\mathcal{L}}}$, the second equality follows from the fact that $B_{(\underline{0},\underline{1}_F]}^{(\Phi^{-1} \circ \pi^{\mathcal{L}}, \Phi^{-1} \circ t^{\mathcal{L}})} = \Phi^{-1}(B_{(\underline{0},\underline{1}_F]}^{(\pi^{\mathcal{L}}, t^{\mathcal{L}})})$ for all $\emptyset \neq F \subseteq [d]$, and the final equality follows via an application of Proposition 4.1.12. This completes the proof of item (i).

(ii) Note that $(Q_{\mathfrak{J}} \circ \overline{\pi}_X, Q_{\mathfrak{J}} \circ \overline{t}_X)$ is a Nica-covariant representation of X that admits a gauge action and satisfies $\mathrm{C}^*(Q_{\mathfrak{J}} \circ \overline{\pi}_X, Q_{\mathfrak{J}} \circ \overline{t}_X) = \mathcal{NT}_X/\mathfrak{J}$. Hence, applying Remark 4.1.10 for the representation $(\pi, t) := (Q_{\mathfrak{J}} \circ \overline{\pi}_X, Q_{\mathfrak{J}} \circ \overline{t}_X)$, we obtain the following commutative diagram

$$\begin{array}{ccc}
\mathcal{NT}_X & \xrightarrow{\quad Q_{\mathfrak{J}} \quad} & \mathcal{NT}_X/\mathfrak{J} \\
\downarrow & \searrow^{\pi^{\mathcal{L}^{\mathfrak{J}}} \times t^{\mathcal{L}^{\mathfrak{J}}}} & \uparrow \Psi \\
\mathcal{NT}_{[X]_{\mathcal{L}^{\mathfrak{J}}_\emptyset}} & \xrightarrow[\quad Q \quad]{} & \mathcal{NO}([\mathcal{L}^{\mathfrak{J}}]_{\mathcal{L}^{\mathfrak{J}}_\emptyset}, [X]_{\mathcal{L}^{\mathfrak{J}}_\emptyset})
\end{array}$$

of canonical $*$-epimorphisms. We have that $\dot{\pi} \times \dot{t} = \Psi \circ Q$, where $(\dot{\pi}, \dot{t})$ is defined as in item (iii) of Proposition 4.1.9. Since $\mathcal{L}^{\mathfrak{J}}$ is an NT-2^d-tuple of X, we have that $[\mathcal{L}^{\mathfrak{J}}]_{\mathcal{L}^{\mathfrak{J}}_\emptyset}$ is an (M)-2^d-tuple of $[X]_{\mathcal{L}^{\mathfrak{J}}_\emptyset}$ by Proposition 4.1.8. We have that $[\mathcal{L}^{\mathfrak{J}}]_{\mathcal{L}^{\mathfrak{J}}_\emptyset} = \mathcal{L}^{(\dot{\pi}, \dot{t})}$ by item (vi) of Proposition 4.1.9, and hence an application of Theorem 3.2.12 yields that Ψ is a $*$-isomorphism.

By definition we have that $\mathfrak{J}^{\mathcal{L}^{\mathfrak{J}}} \equiv \ker \pi^{\mathcal{L}^{\mathfrak{J}}} \times t^{\mathcal{L}^{\mathfrak{J}}}$. Therefore we obtain that

$$\mathfrak{J} = \ker Q_{\mathfrak{J}} = \ker \Psi \circ (\pi^{\mathcal{L}^{\mathfrak{J}}} \times t^{\mathcal{L}^{\mathfrak{J}}}) = \ker \pi^{\mathcal{L}^{\mathfrak{J}}} \times t^{\mathcal{L}^{\mathfrak{J}}} \equiv \mathfrak{J}^{\mathcal{L}^{\mathfrak{J}}},$$

and the proof is complete. □

Using Propositions 4.1.12 and 4.1.13, we arrive at a concrete description of $\mathfrak{J}^{\mathcal{L}}$.

Proposition 4.2.2. *Let X be a strong compactly aligned product system with coefficients in a C^*-algebra A. If $\mathcal{L}$ is an NT-2^d-tuple of X, then we have that*

$$\mathfrak{J}^{\mathcal{L}} = \langle \overline{\pi}_X(a) + \sum_{\underline{0} \neq \underline{n} \leq \underline{1}_F} (-1)^{|\underline{n}|} \overline{\psi}_{X,\underline{n}}(k_{\underline{n}}) \mid F \subseteq [d], a \in \mathcal{L}_F, k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}),$$
$$[\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset} = [k_{\underline{n}}]_{\mathcal{L}_\emptyset} \text{ for all } \underline{0} \neq \underline{n} \leq \underline{1}_F \rangle.$$

If in addition $\mathcal{L}$ is a relative 2^d-tuple of X, then $\mathfrak{J}^{\mathcal{L}} = \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)}$.

Proof. For the first part we apply Proposition 4.1.13 for $(\pi, t) := (\pi^{\mathcal{L}}, t^{\mathcal{L}})$, noting that $\mathcal{L}^{(\pi^{\mathcal{L}}, t^{\mathcal{L}})} = \mathcal{L}$ by Proposition 4.1.12.

Now assume that $\mathcal{L}$ is in addition a relative 2^d-tuple of X, and recall that

$$\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)} = \langle \overline{\pi}_X(a) + \sum_{\underline{0} \neq \underline{n} \leq \underline{1}_F} (-1)^{|\underline{n}|} \overline{\psi}_{X,\underline{n}}(\phi_{\underline{n}}(a)) \mid a \in \mathcal{L}_F, F \subseteq [d] \rangle.$$

It is clear that $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)} \subseteq \mathfrak{J}^{\mathcal{L}}$ by the first part. For the reverse inclusion, first note that

$$\overline{\pi}_X(\mathcal{L}_\emptyset) \subseteq \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)}$$

by definition. Next fix $\emptyset \neq F \subseteq [d], a \in \mathcal{L}_F$ and $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ satisfying $[\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset} = [k_{\underline{n}}]_{\mathcal{L}_\emptyset}$ for all $\underline{0} \neq \underline{n} \leq \underline{1}_F$. By the first part, it suffices to show that

$$\overline{\pi}_X(a) + \sum_{\underline{0} \neq \underline{n} \leq \underline{1}_F} (-1)^{|\underline{n}|} \overline{\psi}_{X,\underline{n}}(k_{\underline{n}}) \in \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)}.$$

Fixing $\underline{0} \neq \underline{n} \leq \underline{1}_F$, we have that $[\phi_{\underline{n}}(a) - k_{\underline{n}}]_{\mathcal{L}_\emptyset} = 0$ by assumption. Since $\mathcal{L}$ is a relative 2^d-tuple of X, we also have that $\phi_{\underline{n}}(a) \in \mathcal{K}(X_{\underline{n}})$ and hence $\phi_{\underline{n}}(a) - k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$. By Lemma 2.2.11 we have that

$$\ker\{[\cdot]_{\mathcal{L}_\emptyset} : \mathcal{K}(X_{\underline{n}}) \to \mathcal{K}([X_{\underline{n}}]_{\mathcal{L}_\emptyset})\} = \mathcal{K}(X_{\underline{n}}\mathcal{L}_\emptyset),$$

and therefore $\phi_{\underline{n}}(a) - k_{\underline{n}} = k'_{\underline{n}}$ for some $k'_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}\mathcal{L}_\emptyset)$. Notice that

$$\overline{\psi}_{X,\underline{n}}(\mathcal{K}(X_{\underline{n}}\mathcal{L}_\emptyset)) \subseteq \langle \overline{\pi}_X(\mathcal{L}_\emptyset) \rangle \subseteq \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)}$$

for all $\underline{0} \neq \underline{n} \leq \underline{1}_F$. In total, we have that

$$\begin{aligned}
\overline{\pi}_X(a) + \sum_{\underline{0} \neq \underline{n} \leq \underline{1}_F} (-1)^{|\underline{n}|} \overline{\psi}_{X,\underline{n}}(k_{\underline{n}}) &= \overline{\pi}_X(a) + \sum_{\underline{0} \neq \underline{n} \leq \underline{1}_F} (-1)^{|\underline{n}|} \overline{\psi}_{X,\underline{n}}(\phi_{\underline{n}}(a) - k'_{\underline{n}}) \\
&= \Big(\overline{\pi}_X(a) + \sum_{\underline{0} \neq \underline{n} \leq \underline{1}_F} (-1)^{|\underline{n}|} \overline{\psi}_{X,\underline{n}}(\phi_{\underline{n}}(a)) \Big) - \sum_{\underline{0} \neq \underline{n} \leq \underline{1}_F} (-1)^{|\underline{n}|} \overline{\psi}_{X,\underline{n}}(k'_{\underline{n}}) \\
&\in \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)} + \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)} = \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)},
\end{aligned}$$

as required. □

We now present the main theorem.

Theorem 4.2.3. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Then the set of NT-2^d-tuples of X corresponds bijectively to the set of gauge-invariant ideals of $\mathcal{NT}_X$ by the mutually inverse maps $\mathcal{L} \mapsto \mathfrak{J}^{\mathcal{L}}$ and $\mathfrak{J} \mapsto \mathcal{L}^{\mathfrak{J}}$, for all NT-$2^d$-tuples $\mathcal{L}$ of X and all gauge-invariant ideals $\mathfrak{J}$ of $\mathcal{NT}_X$. Moreover, these maps respect inclusions.*

Proof. The fact that the maps are well-defined follows from the discussion preceding Proposition 4.2.1. The fact that the maps are mutual inverses is guaranteed by Proposition 4.2.1. It remains to see that the maps preserve inclusions.

To this end, first take NT-2^d-tuples $\mathcal{L}_1$ and $\mathcal{L}_2$ of X and suppose that $\mathcal{L}_1 \subseteq \mathcal{L}_2$. We must show that $\mathfrak{J}^{\mathcal{L}_1} \subseteq \mathfrak{J}^{\mathcal{L}_2}$. It suffices to show that $\mathfrak{J}^{\mathcal{L}_2}$ contains the generators of $\mathfrak{J}^{\mathcal{L}_1}$, recalling their form from Proposition 4.2.2. Firstly, we have that

$$\overline{\pi}_X(\mathcal{L}_{1,\emptyset}) \subseteq \overline{\pi}_X(\mathcal{L}_{2,\emptyset}) \subseteq \mathfrak{J}^{\mathcal{L}_2}$$

by definition. Next, fix $\emptyset \neq F \subseteq [d], a \in \mathcal{L}_{1,F}$ and $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ such that $[k_{\underline{n}}]_{\mathcal{L}_{1,\emptyset}} = [\phi_{\underline{n}}(a)]_{\mathcal{L}_{1,\emptyset}}$ for all $\underline{0} \neq \underline{n} \leq \underline{1}_F$. For each $\underline{0} \neq \underline{n} \leq \underline{1}_F$, we make the identification

$$\mathcal{L}([[X_{\underline{n}}]_{\mathcal{L}_{1,\emptyset}}]_{\mathcal{L}_{2,\emptyset}/\mathcal{L}_{1,\emptyset}}) \cong \mathcal{L}([X_{\underline{n}}]_{\mathcal{L}_{2,\emptyset}}),$$

so that $[\cdot]_{\mathcal{L}_{2,\emptyset}} = [\cdot]_{\mathcal{L}_{2,\emptyset}/\mathcal{L}_{1,\emptyset}} \circ [\cdot]_{\mathcal{L}_{1,\emptyset}}$, e.g., [36, p. 112]. Under this identification, we obtain that $[k_{\underline{n}}]_{\mathcal{L}_{2,\emptyset}} = [\phi_{\underline{n}}(a)]_{\mathcal{L}_{2,\emptyset}}$ for all $\underline{0} \neq \underline{n} \leq \underline{1}_F$. Since $a \in \mathcal{L}_{1,F} \subseteq \mathcal{L}_{2,F}$, it then follows that

$$\overline{\pi}_X(a) + \sum_{\underline{0} \neq \underline{n} \leq \underline{1}_F} (-1)^{|\underline{n}|} \overline{\psi}_{X,\underline{n}}(k_{\underline{n}}) \in \mathfrak{J}^{\mathcal{L}_2},$$

as required.

Finally, fix gauge-invariant ideals $\mathfrak{J}_1$ and $\mathfrak{J}_2$ of $\mathcal{NT}_X$ such that $\mathfrak{J}_1 \subseteq \mathfrak{J}_2$. Then we have that

$$\mathcal{L}_F^{\mathfrak{J}_1} \equiv \mathcal{L}_F^{(Q_{\mathfrak{J}_1} \circ \overline{\pi}_X, Q_{\mathfrak{J}_1} \circ \overline{t}_X)} \subseteq \mathcal{L}_F^{(Q_{\mathfrak{J}_2} \circ \overline{\pi}_X, Q_{\mathfrak{J}_2} \circ \overline{t}_X)} \equiv \mathcal{L}_F^{\mathfrak{J}_2} \text{ for all } F \subseteq [d].$$

This follows because

$$B_{(\underline{0},\underline{1}_F]}^{(Q_{\mathfrak{J}_1} \circ \overline{\pi}_X, Q_{\mathfrak{J}_1} \circ \overline{t}_X)} = Q_{\mathfrak{J}_1}(B_{(\underline{0},\underline{1}_F]}^{(\overline{\pi}_X, \overline{t}_X)}) \quad \text{and} \quad B_{(\underline{0},\underline{1}_F]}^{(Q_{\mathfrak{J}_2} \circ \overline{\pi}_X, Q_{\mathfrak{J}_2} \circ \overline{t}_X)} = Q_{\mathfrak{J}_2}(B_{(\underline{0},\underline{1}_F]}^{(\overline{\pi}_X, \overline{t}_X)}) \text{ for all } F \subseteq [d],$$

and $Q_{\mathfrak{J}_2}$ factors through $Q_{\mathfrak{J}_1}$. □

Remark 4.2.4. To summarise, we have that the mappings

$$\mathcal{L} \mapsto \ker \pi^{\mathcal{L}} \times t^{\mathcal{L}} \text{ for all NT-}2^d\text{-tuples } \mathcal{L} \text{ of } X,$$
$$\mathfrak{J} \mapsto \mathcal{L}^{(Q_{\mathfrak{J}} \circ \overline{\pi}_X, Q_{\mathfrak{J}} \circ \overline{t}_X)} \text{ for all gauge-invariant ideals } \mathfrak{J} \subseteq \mathcal{NT}_X,$$

are mutual inverses and respect inclusions, where $Q_{\mathfrak{J}} \colon \mathcal{NT}_X \to \mathcal{NT}_X/\mathfrak{J}$ is the quotient map. The gauge-invariant ideal $\ker \pi^{\mathcal{L}} \times t^{\mathcal{L}}$ of $\mathcal{NT}_X$ is given by

$$\ker \pi^{\mathcal{L}} \times t^{\mathcal{L}} = \langle \overline{\pi}_X(a) + \sum_{\underline{0} \neq \underline{n} \leq \underline{1}_F} (-1)^{|\underline{n}|} \overline{\psi}_{X,\underline{n}}(k_{\underline{n}}) \mid F \subseteq [d], a \in \mathcal{L}_F, k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}),$$
$$[\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset} = [k_{\underline{n}}]_{\mathcal{L}_\emptyset} \text{ for all } \underline{0} \neq \underline{n} \leq \underline{1}_F \rangle,$$

by Proposition 4.2.2, where $\mathcal{L}$ is an NT-2^d-tuple of X. The NT-2^d-tuple $\mathcal{L}^{(Q_{\mathfrak{J}} \circ \overline{\pi}_X, Q_{\mathfrak{J}} \circ \overline{t}_X)}$ of

X is given by

$$\mathcal{L}_F^{(Q_{\mathfrak{J}}\circ\overline{\pi}_X, Q_{\mathfrak{J}}\circ\overline{t}_X)} = \begin{cases} \ker Q_{\mathfrak{J}} \circ \overline{\pi}_X & \text{if } F = \emptyset, \\ (Q_{\mathfrak{J}} \circ \overline{\pi}_X)^{-1}(B_{(\underline{0},\underline{1}_F]}^{(Q_{\mathfrak{J}}\circ\overline{\pi}_X, Q_{\mathfrak{J}}\circ\overline{t}_X)}) & \text{if } \emptyset \neq F \subseteq [d], \end{cases}$$

where $\mathfrak{J}$ is a gauge-invariant ideal of $\mathcal{NT}_X$.

Note that the set of gauge-invariant ideals of $\mathcal{NT}_X$ carries a canonical lattice structure, determined by the operations

$$\mathfrak{J}_1 \vee \mathfrak{J}_2 := \mathfrak{J}_1 + \mathfrak{J}_2 \quad \text{and} \quad \mathfrak{J}_1 \wedge \mathfrak{J}_2 := \mathfrak{J}_1 \cap \mathfrak{J}_2,$$

for all gauge-invariant ideals $\mathfrak{J}_1, \mathfrak{J}_2 \subseteq \mathcal{NT}_X$. This, in tandem with Theorem 4.2.3, allows us to impose a canonical lattice structure on the set of NT-2^d-tuples of X, promoting the bijection to a lattice isomorphism.

Definition 4.2.5. Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. We equip the set of NT-2^d-tuples of X with the lattice structure determined by the operations

$$\mathcal{L}_1 \vee \mathcal{L}_2 := \mathcal{L}^{\mathfrak{J}^{\mathcal{L}_1} + \mathfrak{J}^{\mathcal{L}_2}} \quad \text{and} \quad \mathcal{L}_1 \wedge \mathcal{L}_2 := \mathcal{L}^{\mathfrak{J}^{\mathcal{L}_1} \cap \mathfrak{J}^{\mathcal{L}_2}},$$

for all NT-2^d-tuples $\mathcal{L}_1$ and $\mathcal{L}_2$ of X.

Next we describe the operations $\wedge$ and $\vee$ on the set of NT-2^d-tuples of X. The operation $\wedge$ is intersection, in accordance with [36, Proposition 5.8].

Proposition 4.2.6. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be NT-2^d-tuples of X. Then*

$$(\mathcal{L}_1 \wedge \mathcal{L}_2)_F = \mathcal{L}_{1,F} \cap \mathcal{L}_{2,F} \textit{ for all } F \subseteq [d].$$

Proof. For notational convenience we set $\mathfrak{J} := \mathfrak{J}^{\mathcal{L}_1} \cap \mathfrak{J}^{\mathcal{L}_2}$, so that $\mathcal{L}^{\mathfrak{J}} \equiv \mathcal{L}_1 \wedge \mathcal{L}_2$. For $F = \emptyset$ and $a \in A$, we have that

$$a \in \mathcal{L}_\emptyset^{\mathfrak{J}} \iff \overline{\pi}_X(a) \in \mathfrak{J} \equiv \mathfrak{J}^{\mathcal{L}_1} \cap \mathfrak{J}^{\mathcal{L}_2} \iff a \in \mathcal{L}_\emptyset^{\mathfrak{J}^{\mathcal{L}_1}} \cap \mathcal{L}_\emptyset^{\mathfrak{J}^{\mathcal{L}_2}} = \mathcal{L}_{1,\emptyset} \cap \mathcal{L}_{2,\emptyset},$$

using item (i) of Proposition 4.2.1 in the final equality. Hence $(\mathcal{L}_1 \wedge \mathcal{L}_2)_\emptyset = \mathcal{L}_{1,\emptyset} \cap \mathcal{L}_{2,\emptyset}$.

Next, fix $\emptyset \neq F \subseteq [d]$. Since the parametrisation of Theorem 4.2.3 preserves inclusions, we have that $(\mathcal{L}_1 \wedge \mathcal{L}_2)_F \subseteq \mathcal{L}_{1,F} \cap \mathcal{L}_{2,F}$. For the reverse inclusion, take $a \in \mathcal{L}_{1,F} \cap \mathcal{L}_{2,F}$. Since $(\mathcal{L}_1 \wedge \mathcal{L}_2)_F$ is an ideal, it suffices to show that $aa^* \in (\mathcal{L}_1 \wedge \mathcal{L}_2)_F$. Since $a \in \mathcal{L}_{1,F} \cap \mathcal{L}_{2,F}$, there exist $f, g \in B_{(\underline{0},\underline{1}_F]}^{(\overline{\pi}_X, \overline{t}_X)}$ such that

$$\overline{\pi}_X(a) + f \in \mathfrak{J}^{\mathcal{L}_1} \quad \text{and} \quad \overline{\pi}_X(a) + g \in \mathfrak{J}^{\mathcal{L}_2}.$$

Consider the element

$$h := \overline{\pi}_X(a)g^* + f\overline{\pi}_X(a)^* + fg^*.$$

Note that

$$\overline{\pi}_X(A)B_{(\underline{0},\underline{1}_F]}^{(\overline{\pi}_X,\overline{t}_X)} \subseteq B_{(\underline{0},\underline{1}_F]}^{(\overline{\pi}_X,\overline{t}_X)} \quad \text{and} \quad B_{(\underline{0},\underline{1}_F]}^{(\overline{\pi}_X,\overline{t}_X)}\overline{\pi}_X(A) \subseteq B_{(\underline{0},\underline{1}_F]}^{(\overline{\pi}_X,\overline{t}_X)},$$

and recall that $B_{(\underline{0},\underline{1}_F]}^{(\overline{\pi}_X,\overline{t}_X)}$ is a C*-algebra by Proposition 2.4.7. Hence $h \in B_{(\underline{0},\underline{1}_F]}^{(\overline{\pi}_X,\overline{t}_X)}$, and we obtain that

$$\overline{\pi}_X(aa^*) + h = (\overline{\pi}_X(a) + f)(\overline{\pi}_X(a) + g)^* \in \mathfrak{J}^{\mathcal{L}_1} \cap \mathfrak{J}^{\mathcal{L}_2} \equiv \mathfrak{J}.$$

By definition this means that $aa^* \in \mathcal{L}_F^{\mathfrak{J}} \equiv (\mathcal{L}_1 \wedge \mathcal{L}_2)_F$, as required. □

We have the following characterisation of the operation $\vee$.

Proposition 4.2.7. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be NT-2^d-tuples of X. Then we have that*

$$(\mathcal{L}_1 \vee \mathcal{L}_2)_\emptyset = \overline{\pi}_X^{-1}(\mathfrak{J}^{\mathcal{L}_1} + \mathfrak{J}^{\mathcal{L}_2})$$

and that

$$(\mathcal{L}_1 \vee \mathcal{L}_2)_F = [\cdot]^{-1}_{(\mathcal{L}_1 \vee \mathcal{L}_2)_\emptyset}\left[\left(\left(\mathcal{L}_{1,F} + \mathcal{L}_{2,F} + (\mathcal{L}_1 \vee \mathcal{L}_2)_\emptyset\right)/(\mathcal{L}_1 \vee \mathcal{L}_2)_\emptyset\right)^{(d-1)}\right]$$

for all $\emptyset \neq F \subseteq [d]$.

Proof. For notational convenience, we set

$$\mathcal{L} := \mathcal{L}_1 \vee \mathcal{L}_2 \quad \text{and} \quad \mathfrak{J} := \mathfrak{J}^{\mathcal{L}_1} + \mathfrak{J}^{\mathcal{L}_2},$$

so that $\mathcal{L} \equiv \mathcal{L}^{\mathfrak{J}}$. For $F = \emptyset$ and $a \in A$, we have that

$$a \in \mathcal{L}_\emptyset \iff Q_{\mathfrak{J}}(\overline{\pi}_X(a)) = 0 \iff \overline{\pi}_X(a) \in \mathfrak{J} \iff a \in \overline{\pi}_X^{-1}(\mathfrak{J}) = \overline{\pi}_X^{-1}(\mathfrak{J}^{\mathcal{L}_1} + \mathfrak{J}^{\mathcal{L}_2}).$$

Consequently, we obtain that $\mathcal{L}_\emptyset = \overline{\pi}_X^{-1}(\mathfrak{J}^{\mathcal{L}_1} + \mathfrak{J}^{\mathcal{L}_2})$, as required.

Next, in a slight abuse of notation, we denote by $\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_\emptyset$ the 2^d-tuple of X defined by

$$(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_\emptyset)_F := \mathcal{L}_{1,F} + \mathcal{L}_{2,F} + \mathcal{L}_\emptyset \text{ for all } F \subseteq [d].$$

Note that $\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_\emptyset$ consists of ideals and that

$$\mathcal{L}_\emptyset \subseteq (\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_\emptyset)_F \text{ for all } F \subseteq [d],$$

so we can make sense of the 2^d-tuple $[\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_\emptyset]_{\mathcal{L}_\emptyset}$ of $[X]_{\mathcal{L}_\emptyset}$. First we check that the family of ideals $[\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_\emptyset]_{\mathcal{L}_\emptyset}$ is an (E)-2^d-tuple of $[X]_{\mathcal{L}_\emptyset}$ that is invariant and partially ordered.

The fact that $[\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_\emptyset]_{\mathcal{L}_\emptyset}$ is invariant and partially ordered follows from the corresponding properties of the NT-2^d-tuples $\mathcal{L}_1$ and $\mathcal{L}_2$, as well as the fact that $\mathcal{L}_\emptyset$ is positively invariant for X. Next recall that $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ is an (M)-2^d-tuple, and thus it is contained in $\mathcal{I}([X]_{\mathcal{L}_\emptyset})$. Hence, in order to show that $[\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_\emptyset]_{\mathcal{L}_\emptyset}$ is an (E)-2^d-tuple, it suffices to show that

$$[\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_\emptyset]_{\mathcal{L}_\emptyset} \subseteq [\mathcal{L}]_{\mathcal{L}_\emptyset}.$$

In turn, it suffices to show that

$$\mathcal{L}_{1,F} + \mathcal{L}_{2,F} + \mathcal{L}_\emptyset \subseteq \mathcal{L}_F \text{ for all } F \subseteq [d]. \tag{4.1}$$

This is immediate, since $\mathfrak{J}^{\mathcal{L}_1}, \mathfrak{J}^{\mathcal{L}_2} \subseteq \mathfrak{J}$ and the parametrisation of Theorem 4.2.3 respects inclusions. Thus $\mathcal{L}_1, \mathcal{L}_2 \subseteq \mathcal{L}$, and by definition $\mathcal{L}_\emptyset \subseteq \mathcal{L}_F$, showing that (4.1) holds.

Hence we may consider the $(d-1)$-iteration $[\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_\emptyset]^{(d-1)}_{\mathcal{L}_\emptyset}$. For notational convenience, let $\mathcal{L}'$ be the 2^d-tuple of X defined by

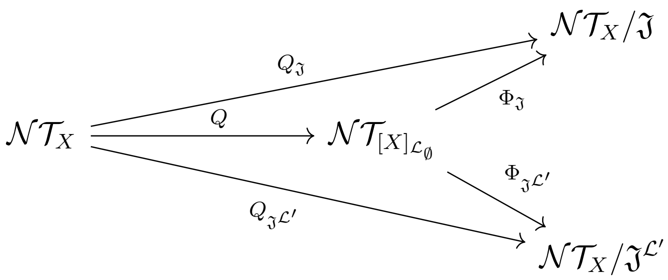
$$\mathcal{L}'_F := \begin{cases} \mathcal{L}_\emptyset & \text{if } F = \emptyset, \\ [\,\cdot\,]^{-1}_{\mathcal{L}_\emptyset}\big([\mathcal{L}_{1,F} + \mathcal{L}_{2,F} + \mathcal{L}_\emptyset]^{(d-1)}_{\mathcal{L}_\emptyset}\big) & \text{if } \emptyset \neq F \subseteq [d], \end{cases}$$

as per the statement of the proposition. Note that $\mathcal{L}'$ consists of ideals and satisfies $\mathcal{L}'_\emptyset \subseteq \mathcal{L}'_F$ for all $F \subseteq [d]$. Moreover, we have that $[\mathcal{L}']_{\mathcal{L}_\emptyset} = [\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_\emptyset]^{(d-1)}_{\mathcal{L}_\emptyset}$ is the (M)-2^d-tuple of $[X]_{\mathcal{L}_\emptyset}$ that induces $\mathfrak{J}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}, \overline{t}_{[X]_{\mathcal{L}_\emptyset}})}_{[\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_\emptyset]_{\mathcal{L}_\emptyset}}$ by Theorem 3.4.7. In particular, $\mathcal{L}'$ is an NT-2^d-tuple of X by an application of Proposition 4.1.8.

It now suffices to show that $\mathfrak{J} = \mathfrak{J}^{\mathcal{L}'}$, as the parametrisation of Theorem 4.2.3 then yields that

$$\mathcal{L} \equiv \mathcal{L}^{\mathfrak{J}} = \mathcal{L}^{\mathfrak{J}^{\mathcal{L}'}} = \mathcal{L}',$$

as required. To this end, we construct the following commutative diagram



of $*$-epimorphisms, where Q is the lift of $X \to [X]_{\mathcal{L}_\emptyset}$ guaranteed by Remark 2.4.5. The maps

$$Q_{\mathfrak{J}} \equiv (Q_{\mathfrak{J}} \circ \overline{\pi}_X) \times (Q_{\mathfrak{J}} \circ \overline{t}_X) \quad \text{and} \quad Q_{\mathfrak{J}^{\mathcal{L}'}} \equiv (Q_{\mathfrak{J}^{\mathcal{L}'}} \circ \overline{\pi}_X) \times (Q_{\mathfrak{J}^{\mathcal{L}'}} \circ \overline{t}_X)$$

are the canonical quotient maps. The map $\Phi_{\mathfrak{J}}$ is induced by the injective Nica-covariant representation of $[X]_{\mathcal{L}_\emptyset}$ obtained by applying item (iii) of Proposition 4.1.9 for $(\pi, t) := (Q_{\mathfrak{J}} \circ \overline{\pi}_X, Q_{\mathfrak{J}} \circ \overline{t}_X)$. The map $\Phi_{\mathfrak{J}^{\mathcal{L}'}}$ is obtained analogously, using $(Q_{\mathfrak{J}^{\mathcal{L}'}} \circ \overline{\pi}_X, Q_{\mathfrak{J}^{\mathcal{L}'}} \circ \overline{t}_X)$ in place of $(Q_{\mathfrak{J}} \circ \overline{\pi}_X, Q_{\mathfrak{J}} \circ \overline{t}_X)$ and the fact that $\mathcal{L}'_\emptyset = \mathcal{L}_\emptyset$. Since each map is canonical, it suffices to show that the kernel $\ker \Phi_{\mathfrak{J}} = Q(\mathfrak{J})$ coincides with the kernel $\ker \Phi_{\mathfrak{J}^{\mathcal{L}'}} = Q(\mathfrak{J}^{\mathcal{L}'})$. Then $\mathcal{NT}_X/\mathfrak{J} \cong \mathcal{NT}_X/\mathfrak{J}^{\mathcal{L}'}$ by the map $f + \mathfrak{J} \mapsto f + \mathfrak{J}^{\mathcal{L}'}$ for all $f \in \mathcal{NT}_X$ and so $\mathfrak{J} = \mathfrak{J}^{\mathcal{L}'}$. To this end, we have the following three claims.

Claim 1. With the aforementioned notation, we have that

$$Q(\mathfrak{J}^{\mathcal{L}'}) = \mathfrak{J}_{[\mathcal{L}']_{\mathcal{L}_\emptyset}}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}, \overline{t}_{[X]_{\mathcal{L}_\emptyset}})}.$$

Proof of Claim 1. By an application of item (i) of Proposition 4.2.1, we have a canonical $*$-isomorphism

$$\mathcal{NT}_X/\mathfrak{J}^{\mathcal{L}'} \cong \mathcal{NO}([\mathcal{L}']_{\mathcal{L}'_\emptyset}, [X]_{\mathcal{L}'_\emptyset}) = \mathcal{NT}_{[X]_{\mathcal{L}_\emptyset}}/\mathfrak{J}_{[\mathcal{L}']_{\mathcal{L}_\emptyset}}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}, \overline{t}_{[X]_{\mathcal{L}_\emptyset}})},$$

using that $\mathcal{L}'$ is an NT-2^d-tuple and that $\mathcal{L}'_\emptyset = \mathcal{L}_\emptyset$. This $*$-isomorphism is induced by Q in the sense that it maps $f + \mathfrak{J}^{\mathcal{L}'}$ to $Q(f) + \mathfrak{J}_{[\mathcal{L}']_{\mathcal{L}_\emptyset}}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}, \overline{t}_{[X]_{\mathcal{L}_\emptyset}})}$ for all $f \in \mathcal{NT}_X$. It follows that $Q(\mathfrak{J}^{\mathcal{L}'}) = \mathfrak{J}_{[\mathcal{L}']_{\mathcal{L}_\emptyset}}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}, \overline{t}_{[X]_{\mathcal{L}_\emptyset}})}$, as required. □

Claim 2. Let $\mathcal{L}'_1$ and $\mathcal{L}'_2$ be the 2^d-tuples of X defined by $\mathcal{L}'_{1,F} := \mathcal{L}_{1,F} + \mathcal{L}_\emptyset$ and $\mathcal{L}'_{2,F} := \mathcal{L}_{2,F} + \mathcal{L}_\emptyset$ for all $F \subseteq [d]$. Then $[\mathcal{L}'_1]_{\mathcal{L}_\emptyset}$ and $[\mathcal{L}'_2]_{\mathcal{L}_\emptyset}$ are (E)-2^d-tuples of $[X]_{\mathcal{L}_\emptyset}$ that consist of ideals, and

$$\mathfrak{J}_{[\mathcal{L}']_{\mathcal{L}_\emptyset}}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}, \overline{t}_{[X]_{\mathcal{L}_\emptyset}})} = \mathfrak{J}_{[\mathcal{L}'_1]_{\mathcal{L}_\emptyset}}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}, \overline{t}_{[X]_{\mathcal{L}_\emptyset}})} + \mathfrak{J}_{[\mathcal{L}'_2]_{\mathcal{L}_\emptyset}}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}, \overline{t}_{[X]_{\mathcal{L}_\emptyset}})}.$$

Proof of Claim 2. Both $[\mathcal{L}'_1]_{\mathcal{L}_\emptyset}$ and $[\mathcal{L}'_2]_{\mathcal{L}_\emptyset}$ consist of ideals of $[A]_{\mathcal{L}_\emptyset}$ and are (E)-2^d-tuples of $[X]_{\mathcal{L}_\emptyset}$, since both are contained in the (E)-2^d-tuple $[\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_\emptyset]_{\mathcal{L}_\emptyset}$ of $[X]_{\mathcal{L}_\emptyset}$. Moreover, it is routine to check that $[\mathcal{L}'_1]_{\mathcal{L}_\emptyset} + [\mathcal{L}'_2]_{\mathcal{L}_\emptyset} = [\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_\emptyset]_{\mathcal{L}_\emptyset}$. Hence we obtain that

$$\mathfrak{J}_{[\mathcal{L}']_{\mathcal{L}_\emptyset}}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}, \overline{t}_{[X]_{\mathcal{L}_\emptyset}})} = \mathfrak{J}_{[\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_\emptyset]_{\mathcal{L}_\emptyset}}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}, \overline{t}_{[X]_{\mathcal{L}_\emptyset}})} = \mathfrak{J}_{[\mathcal{L}'_1]_{\mathcal{L}_\emptyset}+[\mathcal{L}'_2]_{\mathcal{L}_\emptyset}}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}, \overline{t}_{[X]_{\mathcal{L}_\emptyset}})} = \mathfrak{J}_{[\mathcal{L}'_1]_{\mathcal{L}_\emptyset}}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}, \overline{t}_{[X]_{\mathcal{L}_\emptyset}})} + \mathfrak{J}_{[\mathcal{L}'_2]_{\mathcal{L}_\emptyset}}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}, \overline{t}_{[X]_{\mathcal{L}_\emptyset}})},$$

using Proposition 3.1.6 in the final equality. □

Claim 3. With the aforementioned notation, we have that

$$Q(\mathfrak{J}) = \mathfrak{J}_{[\mathcal{L}'_1]_{\mathcal{L}_\emptyset}}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}, \overline{t}_{[X]_{\mathcal{L}_\emptyset}})} + \mathfrak{J}_{[\mathcal{L}'_2]_{\mathcal{L}_\emptyset}}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}, \overline{t}_{[X]_{\mathcal{L}_\emptyset}})}.$$

Proof of Claim 3. For notational convenience, we will denote the right hand side by $\mathfrak{J}'$. For the forward inclusion, we show that $Q(f) \in \mathfrak{J}'$ for all generators f of $\mathfrak{J}^{\mathcal{L}_1}$. The same holds for $\mathfrak{J}^{\mathcal{L}_2}$ by symmetry, giving that $Q(\mathfrak{J}) \subseteq \mathfrak{J}'$. To this end, we resort to Proposition 4.2.2. Note that

$$Q(\overline{\pi}_X(\mathcal{L}_{1,\emptyset})) = \overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}(\{0\}) \subseteq \mathfrak{J}'$$

since $\mathcal{L}_{1,\emptyset} \subseteq \mathcal{L}_\emptyset$. Now fix $\emptyset \neq F \subseteq [d], a \in \mathcal{L}_{1,F}$ and $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ such that $[\phi_{\underline{n}}(a)]_{\mathcal{L}_{1,\emptyset}} =$

$[k_{\underline{n}}]_{\mathcal{L}_{1,\emptyset}}$ for all $\underline{0} \neq \underline{n} \leq \underline{1}_F$. Making the usual identification

$$\mathcal{L}([[X_{\underline{n}}]_{\mathcal{L}_{1,\emptyset}}]_{\mathcal{L}_\emptyset/\mathcal{L}_{1,\emptyset}}) \cong \mathcal{L}([X_{\underline{n}}]_{\mathcal{L}_\emptyset}) \text{ for all } \underline{0} \neq \underline{n} \leq \underline{1}_F,$$

we may write $[\cdot]_{\mathcal{L}_\emptyset} = [\cdot]_{\mathcal{L}_\emptyset/\mathcal{L}_{1,\emptyset}} \circ [\cdot]_{\mathcal{L}_{1,\emptyset}}$ and deduce that

$$[\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset} = [k_{\underline{n}}]_{\mathcal{L}_\emptyset} \text{ for all } \underline{0} \neq \underline{n} \leq \underline{1}_F.$$

Consequently, we obtain that

$$\begin{aligned} Q(\overline{\pi}_X(a) + \sum_{\underline{0}\neq\underline{n}\leq\underline{1}_F} (-1)^{|\underline{n}|}\overline{\psi}_{X,\underline{n}}(k_{\underline{n}})) &= \overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}([a]_{\mathcal{L}_\emptyset}) + \sum_{\underline{0}\neq\underline{n}\leq\underline{1}_F} (-1)^{|\underline{n}|}\overline{\psi}_{[X]_{\mathcal{L}_\emptyset},\underline{n}}([k_{\underline{n}}]_{\mathcal{L}_\emptyset}) \\ &= \overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}([a]_{\mathcal{L}_\emptyset}) + \sum_{\underline{0}\neq\underline{n}\leq\underline{1}_F} (-1)^{|\underline{n}|}\overline{\psi}_{[X]_{\mathcal{L}_\emptyset},\underline{n}}([\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset}) \\ &= \overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}([a]_{\mathcal{L}_\emptyset})\overline{q}_{[X]_{\mathcal{L}_\emptyset},F} \in \mathfrak{J}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}},\overline{t}_{[X]_{\mathcal{L}_\emptyset}})}_{[\mathcal{L}'_1]_{\mathcal{L}_\emptyset}} \subseteq \mathfrak{J}', \end{aligned}$$

using that $a \in \mathcal{L}_{1,F} \subseteq \mathcal{L}'_{1,F}$.

For the reverse inclusion, it suffices to show that $Q(\mathfrak{J})$ contains the generators of $\mathfrak{J}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}},\overline{t}_{[X]_{\mathcal{L}_\emptyset}})}_{[\mathcal{L}'_1]_{\mathcal{L}_\emptyset}}$. The same holds for $\mathfrak{J}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}},\overline{t}_{[X]_{\mathcal{L}_\emptyset}})}_{[\mathcal{L}'_2]_{\mathcal{L}_\emptyset}}$ by symmetry, concluding the proof of the claim. Fix $\emptyset \neq F \subseteq [d]$ and $a \in \mathcal{L}'_{1,F} \equiv \mathcal{L}_{1,F} + \mathcal{L}_\emptyset$. Then in particular $a \in \mathcal{L}_F$ by (4.1). By item (v) of Proposition 4.1.9, for each $\underline{0} \neq \underline{n} \leq \underline{1}_F$ there exists $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ such that $[\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset} = [k_{\underline{n}}]_{\mathcal{L}_\emptyset}$ and

$$\overline{\pi}_X(a) + \sum_{\underline{0}\neq\underline{n}\leq\underline{1}_F} (-1)^{|\underline{n}|}\overline{\psi}_{X,\underline{n}}(k_{\underline{n}}) \in \mathfrak{J} \equiv \mathfrak{J}^{\mathcal{L}}.$$

Thus we have that

$$\begin{aligned} \overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}([a]_{\mathcal{L}_\emptyset})\overline{q}_{[X]_{\mathcal{L}_\emptyset},F} &= \overline{\pi}_{[X]_{\mathcal{L}_\emptyset}}([a]_{\mathcal{L}_\emptyset}) + \sum_{\underline{0}\neq\underline{n}\leq\underline{1}_F} (-1)^{|\underline{n}|}\overline{\psi}_{[X]_{\mathcal{L}_\emptyset},\underline{n}}([\phi_{\underline{n}}(a)]_{\mathcal{L}_\emptyset}) \\ &= Q\big(\overline{\pi}_X(a) + \sum_{\underline{0}\neq\underline{n}\leq\underline{1}_F} (-1)^{|\underline{n}|}\overline{\psi}_{X,\underline{n}}(k_{\underline{n}})\big) \in Q(\mathfrak{J}), \end{aligned}$$

as required. □

Using Claims 1, 2 and 3 (and adopting the nomenclature therein), we conclude that

$$\ker \Phi_{\mathfrak{J}^{\mathcal{L}'}} = Q(\mathfrak{J}^{\mathcal{L}'}) = \mathfrak{J}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}},\overline{t}_{[X]_{\mathcal{L}_\emptyset}})}_{[\mathcal{L}']_{\mathcal{L}_\emptyset}} = \mathfrak{J}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}},\overline{t}_{[X]_{\mathcal{L}_\emptyset}})}_{[\mathcal{L}'_1]_{\mathcal{L}_\emptyset}} + \mathfrak{J}^{(\overline{\pi}_{[X]_{\mathcal{L}_\emptyset}},\overline{t}_{[X]_{\mathcal{L}_\emptyset}})}_{[\mathcal{L}'_2]_{\mathcal{L}_\emptyset}} = Q(\mathfrak{J}) = \ker \Phi_{\mathfrak{J}},$$

and the proof is complete. □

By making minor changes to Theorem 4.2.3, we can parametrise the gauge-invariant ideals of $\mathcal{NO}(\mathcal{K}, X)$ for any relative 2^d-tuple $\mathcal{K}$ of X. In particular, we can parametrise the gauge-invariant ideals of $\mathcal{NO}_X$. We begin with a definition.

Definition 4.2.8. Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{K}$ be a relative 2^d-tuple of X and let $\mathcal{L}$ be a 2^d-tuple of X. We say that $\mathcal{L}$ is a *$\mathcal{K}$-relative NO-2^d-tuple (of X)* if $\mathcal{L}$ is an NT-2^d-tuple of X and $\mathcal{K} \subseteq \mathcal{L}$. We refer to the $\mathcal{I}$-relative NO-2^d-tuples of X simply as *NO-2^d-tuples (of X)*.

It will follow from Theorem 4.2.11 that the set of $\mathcal{K}$-relative NO-2^d-tuples of X is non-empty. As we have seen in Proposition 4.1.7, NT-2^d-tuples constitute the higher-rank analogue of Katsura's T-pairs, and an analogous relationship exists between NO-2^d-tuples and Katsura's O-pairs.

Proposition 4.2.9. *Let $X = \{X_n\}_{n \in \mathbb{Z}_+}$ be a product system with coefficients in a C*-algebra A. Then the NO-2-tuples of X are exactly the O-pairs of X_1.*

Proof. This is immediate by Proposition 4.1.7, since $\mathcal{I}_{\{1\}} = \mathcal{J}_{\{1\}} = J_{X_1}$. □

The lattice operations of Definition 4.2.5 restrict to the set of $\mathcal{K}$-relative NO-2^d-tuples of X.

Proposition 4.2.10. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $\mathcal{K}$ be a relative 2^d-tuple of X and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be $\mathcal{K}$-relative NO-2^d-tuples of X. Then $\mathcal{L}_1 \vee \mathcal{L}_2$ and $\mathcal{L}_1 \wedge \mathcal{L}_2$ are $\mathcal{K}$-relative NO-2^d-tuples of X.*

Proof. For $F \subseteq [d]$, we have that $\mathcal{K}_F \subseteq \mathcal{L}_{1,F}$ and $\mathcal{K}_F \subseteq \mathcal{L}_{2,F}$ by definition. Hence

$$\mathcal{K}_F \subseteq \mathcal{L}_{1,F} \cap \mathcal{L}_{2,F} = (\mathcal{L}_1 \wedge \mathcal{L}_2)_F,$$

using Proposition 4.2.6 in the equality. Hence $\mathcal{L}_1 \wedge \mathcal{L}_2$ is a $\mathcal{K}$-relative NO-2^d-tuple of X.

Next, note that

$$\mathcal{K}_\emptyset \subseteq \mathcal{L}_{1,\emptyset} \subseteq \overline{\pi}_X^{-1}(\mathfrak{J}^{\mathcal{L}_1} + \mathfrak{J}^{\mathcal{L}_2}) = (\mathcal{L}_1 \vee \mathcal{L}_2)_\emptyset$$

by Proposition 4.2.7. Now fix $\emptyset \neq F \subseteq [d]$. Since $\mathcal{K}_F \subseteq \mathcal{L}_{1,F} + \mathcal{L}_{2,F} + (\mathcal{L}_1 \vee \mathcal{L}_2)_\emptyset$, we obtain that

$$[\mathcal{K}_F]_{(\mathcal{L}_1 \vee \mathcal{L}_2)_\emptyset} \subseteq [\mathcal{L}_{1,F} + \mathcal{L}_{2,F} + (\mathcal{L}_1 \vee \mathcal{L}_2)_\emptyset]_{(\mathcal{L}_1 \vee \mathcal{L}_2)_\emptyset} \subseteq [\mathcal{L}_{1,F} + \mathcal{L}_{2,F} + (\mathcal{L}_1 \vee \mathcal{L}_2)_\emptyset]^{(d-1)}_{(\mathcal{L}_1 \vee \mathcal{L}_2)_\emptyset},$$

and by Proposition 4.2.7 we conclude that $\mathcal{K}_F \subseteq (\mathcal{L}_1 \vee \mathcal{L}_2)_F$, completing the proof. □

For a relative 2^d-tuple $\mathcal{K}$ of X, we write $Q_\mathcal{K} \colon \mathcal{NT}_X \to \mathcal{NO}(\mathcal{K}, X)$ for the canonical quotient map. Equivariance of $Q_\mathcal{K}$ gives that $\mathfrak{J}$ is a gauge-invariant ideal of $\mathcal{NO}(\mathcal{K}, X)$ if and only if $\mathfrak{J} = Q_\mathcal{K}(\mathfrak{J}')$ for the gauge-invariant ideal $\mathfrak{J}' := Q_\mathcal{K}^{-1}(\mathfrak{J})$ of $\mathcal{NT}_X$. With this we can adapt the parametrisation of Theorem 4.2.3 to account for $\mathcal{NO}(\mathcal{K}, X)$.

Theorem 4.2.11. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let $\mathcal{K}$ be a relative 2^d-tuple of X. Equip the set of $\mathcal{K}$-relative NO-2^d-tuples of X with the lattice structure of Definition 4.2.5 (suitably restricted) and equip* *the set of gauge-invariant ideals of $\mathcal{NO}(\mathcal{K}, X)$ with the usual lattice structure. Then these sets are isomorphic as lattices via the map*

$$\mathcal{L} \mapsto Q_{\mathcal{K}}(\mathfrak{J}^{\mathcal{L}}), \text{ for the canonical quotient map } Q_{\mathcal{K}} \colon \mathcal{NT}_X \to \mathcal{NO}(\mathcal{K}, X), \tag{4.2}$$

for all $\mathcal{K}$-relative NO-2^d-tuples $\mathcal{L}$ of X. Moreover, this map preserves inclusions.

Proof. First note that the proposed lattice structure on the set of $\mathcal{K}$-relative NO-2^d-tuples of X is well-defined by Proposition 4.2.10. The comments preceding the statement of the theorem show that (4.2) constitutes a well-defined map.

Next we check that the mapping is injective and surjective. To this end, first we show that $\mathfrak{J}_{\mathcal{K}}^{(\overline{\pi}_X, \overline{t}_X)} \subseteq \mathfrak{J}^{\mathcal{L}}$ whenever $\mathcal{L}$ is a $\mathcal{K}$-relative NO-2^d-tuple of X. Fix $F \subseteq [d]$ and $a \in \mathcal{K}_F$. It suffices to show that $\overline{\pi}_X(a)\overline{q}_{X,F} \in \mathfrak{J}^{\mathcal{L}}$. Since $\mathcal{L}$ is a $\mathcal{K}$-relative NO-2^d-tuple of X, we have that $a \in \mathcal{K}_F \subseteq \mathcal{L}_F$. Likewise, because $\phi_{\underline{n}}(a) \in \mathcal{K}(X_{\underline{n}})$ for all $\underline{0} \neq \underline{n} \leq \underline{1}_F$, an application of Proposition 4.2.2 gives that

$$\overline{\pi}_X(a)\overline{q}_{X,F} = \overline{\pi}_X(a) + \sum\{(-1)^{|\underline{n}|}\overline{\psi}_{X,\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} \in \mathfrak{J}^{\mathcal{L}},$$

as required. Consequently, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{NT}_X & \xrightarrow{Q_{\mathfrak{J}^{\mathcal{L}}}} & \mathcal{NT}_X/\mathfrak{J}^{\mathcal{L}} \\ {\scriptstyle Q_{\mathcal{K}}}\downarrow & \nearrow_{\exists!\Phi} & \\ \mathcal{NO}(\mathcal{K}, X) & & \end{array}$$

of canonical $*$-epimorphisms, so that $\ker \Phi = Q_{\mathcal{K}}(\mathfrak{J}^{\mathcal{L}})$. Thus we obtain a $*$-isomorphism

$$\widetilde{\Phi} \colon \mathcal{NO}(\mathcal{K}, X)/Q_{\mathcal{K}}(\mathfrak{J}^{\mathcal{L}}) \to \mathcal{NT}_X/\mathfrak{J}^{\mathcal{L}}; \widetilde{\Phi}(f + Q_{\mathcal{K}}(\mathfrak{J}^{\mathcal{L}})) = \Phi(f) \text{ for all } f \in \mathcal{NO}(\mathcal{K}, X). \tag{4.3}$$

For injectivity of the map (4.2), suppose we have $\mathcal{K}$-relative NO-2^d-tuples $\mathcal{L}$ and $\mathcal{L}'$ of X such that $Q_{\mathcal{K}}(\mathfrak{J}^{\mathcal{L}}) = Q_{\mathcal{K}}(\mathfrak{J}^{\mathcal{L}'})$. Applying (4.3) for $\mathcal{L}$ and $\mathcal{L}'$, we obtain a $*$-isomorphism

$$\mathcal{NT}_X/\mathfrak{J}^{\mathcal{L}} \to \mathcal{NT}_X/\mathfrak{J}^{\mathcal{L}'}; Q_{\mathfrak{J}^{\mathcal{L}}}(f) \mapsto Q_{\mathfrak{J}^{\mathcal{L}'}}(f) \text{ for all } f \in \mathcal{NT}_X.$$

In turn, it follows that $\mathfrak{J}^{\mathcal{L}} = \mathfrak{J}^{\mathcal{L}'}$ and hence $\mathcal{L} = \mathcal{L}'$ by Theorem 4.2.3.

For surjectivity of the map (4.2), let $\mathfrak{J}$ be a gauge-invariant ideal of $\mathcal{NO}(\mathcal{K}, X)$. Then $Q_{\mathcal{K}}^{-1}(\mathfrak{J})$ is a gauge-invariant ideal of $\mathcal{NT}_X$ and thus $Q_{\mathcal{K}}^{-1}(\mathfrak{J}) = \mathfrak{J}^{\mathcal{L}}$ for a unique NT-2^d-tuple $\mathcal{L}$ of X by Theorem 4.2.3. It suffices to show that $\mathcal{K} \subseteq \mathcal{L}$. To this end, let $\mathcal{L}' := \mathcal{L}^{\mathfrak{J}_{\mathcal{K}}^{(\overline{\pi}_X, \overline{t}_X)}}$. Then we have that

$$\mathfrak{J}^{\mathcal{L}'} = \mathfrak{J}_{\mathcal{K}}^{(\overline{\pi}_X, \overline{t}_X)}$$

by Theorem 4.2.3. We claim that $\mathcal{K} \subseteq \mathcal{L}'$. Indeed, we have that $\overline{\pi}_X(\mathcal{K}_\emptyset) \subseteq \mathfrak{J}_{\mathcal{K}}^{(\overline{\pi}_X, \overline{t}_X)}$ by definition and therefore $\mathcal{K}_\emptyset \subseteq \mathcal{L}'_\emptyset$. Likewise, fixing $\emptyset \neq F \subseteq [d]$ and $a \in \mathcal{K}_F$, we have that $\overline{\pi}_X(a)\overline{q}_{X,F} \in \mathfrak{J}_{\mathcal{K}}^{(\overline{\pi}_X, \overline{t}_X)}$ and thus $Q_{\mathcal{K}}(\overline{\pi}_X(a)\overline{q}_{X,F}) = 0$. Using Proposition 2.5.16 to expand

$\overline{\pi}_X(a)\overline{q}_{X,F}$ as an alternating sum, we therefore obtain that $a \in \mathcal{L}'_F$ and so $\mathcal{K}_F \subseteq \mathcal{L}'_F$. In total, we have that $\mathcal{K} \subseteq \mathcal{L}'$, as claimed.

Moreover, we have that

$$\mathfrak{J}^{\mathcal{L}'} = \mathfrak{J}_{\mathcal{K}}^{(\overline{\pi}_X,\overline{t}_X)} \subseteq Q_{\mathcal{K}}^{-1}(\mathfrak{J}) = \mathfrak{J}^{\mathcal{L}},$$

and so $\mathcal{L}' \subseteq \mathcal{L}$ since the parametrisation of Theorem 4.2.3 respects inclusions. Thus $\mathcal{K} \subseteq \mathcal{L}' \subseteq \mathcal{L}$, as required.

The map (4.2) respects inclusions and the lattice structure since it is a restriction of the first parametrisation map of Theorem 4.2.3 (followed by the $*$-homomorphism $Q_{\mathcal{K}}$), which satisfies these properties. □

A direct consequence of Theorem 4.2.11 is that, if $\mathfrak{J}$ is a gauge-invariant ideal of $\mathcal{NT}_X$, then $\mathcal{L}^{\mathfrak{J}}$ is a $\mathcal{K}$-relative NO-2^d-tuple if and only if the quotient map $Q_{\mathfrak{J}} \colon \mathcal{NT}_X \to \mathcal{NT}_X/\mathfrak{J}$ factors through the quotient map $Q_{\mathcal{K}} \colon \mathcal{NT}_X \to \mathcal{NO}(\mathcal{K}, X)$.

Applying Theorem 4.2.11 for $\mathcal{K} = \mathcal{I}$ provides the parametrisation of the gauge-invariant ideals of $\mathcal{NO}_X$.

Corollary 4.2.12. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Equip the set of NO-2^d-tuples of X with the lattice structure of Definition 4.2.5 (suitably restricted) and equip the set of gauge-invariant ideals of $\mathcal{NO}_X$ with the usual lattice structure. Then these sets are isomorphic as lattices via the map*

$$\mathcal{L} \mapsto Q_{\mathcal{I}}(\mathfrak{J}^{\mathcal{L}}), \text{ for the canonical quotient map } Q_{\mathcal{I}} \colon \mathcal{NT}_X \to \mathcal{NO}_X, \tag{4.4}$$

for all NO-2^d-tuples $\mathcal{L}$ of X. Moreover, this map respects inclusions.

Theorem 4.2.3 recaptures the parametrisation of Katsura [36], as presented in Theorem 2.2.20. More generally, Theorem 4.2.11 recaptures [36, Proposition 11.9].

Chapter 5

Applications

The applications that we consider in this section pertain to product systems over $\mathbb{Z}_+^d$ that are regular, or arise from C*-dynamical systems, or from strong finitely aligned higher-rank graphs, or whose fibres (apart from the coefficient algebra) admit finite frames. We begin by exploring the situation where an ideal can be placed as the $\mathcal{L}_\emptyset$-member of an NO-2^d-tuple $\mathcal{L}$, which will be helpful for further examples.

5.1 Participation in an NO-2^d-tuple

If $I \subseteq A$ is an ideal that is positively invariant for X, then the quotient map $X \to [X]_I$ induces a canonical $*$-epimorphism

$$[\cdot]_I \colon \mathcal{NT}_X \to \mathcal{NT}_{[X]_I}$$

by Remark 2.4.5. It is well known that this map does not in general descend to a canonical $*$-epimorphism $\mathcal{NO}_X \to \mathcal{NO}_{[X]_I}$, even for $d = 1$ (see Example 5.3.13 for a counterexample). However, using the NO-2^d-tuple machinery, we can determine precisely when this occurs. To this end, we introduce the following definition, modelled after [36, Definition 4.8].

Definition 5.1.1. Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. We say that an ideal $I \subseteq A$ is *negatively invariant (for X)* if

$$\mathcal{I}_F \cap X_F^{-1}(I) \subseteq I \text{ for all } \emptyset \neq F \subseteq [d].$$

Remark 5.1.2. Note that A is negatively invariant trivially. In fact, we also have that $\{0\}$ is negatively invariant. To see this, fix $\emptyset \neq F \subseteq [d]$ and $a \in \mathcal{I}_F \cap X_F^{-1}(\{0\})$. It suffices to show that $a = 0$. Since $a \in X_F^{-1}(\{0\})$, we have that

$$\langle X_{\underline{n}}, aX_{\underline{n}} \rangle = \{0\} \text{ for all } \underline{0} \neq \underline{n} \leq \underline{1}_F$$

by definition. In turn, we deduce (in particular) that $a \in \bigcap_{i \in F} \ker \phi_{\underline{i}}$. However, we also have that $a \in \mathcal{I}_F \subseteq \mathcal{J}_F \subseteq (\bigcap_{i \in F} \ker \phi_{\underline{i}})^\perp$. Combining the preceding deductions, we conclude that $a = 0$, as required.

Definition 5.1.1 leads to the following natural extension of [36, Proposition 5.3].

Proposition 5.1.3. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let $I \subseteq A$ be an ideal. Then I is negatively invariant for X if and only if $\mathcal{I}_F \subseteq J_F(I, X)$ for all $\emptyset \neq F \subseteq [d]$.*

Proof. Assume that I is negatively invariant for X. Fix $\emptyset \neq F \subseteq [d]$ and take $a \in \mathcal{I}_F$. Then $\phi_{\underline{i}}(a) \in \mathcal{K}(X_{\underline{i}})$ for all $i \in [d]$ and so $[\phi_{\underline{i}}(a)]_I \in \mathcal{K}([X_{\underline{i}}]_I)$ for all $i \in [d]$ by Lemma 2.2.11. Moreover, we have that

$$aX_F^{-1}(I) \subseteq \mathcal{I}_F \cap X_F^{-1}(I) \subseteq I.$$

Hence $a \in J_F(I, X)$ and we conclude that $\mathcal{I}_F \subseteq J_F(I, X)$ for all $\emptyset \neq F \subseteq [d]$, as required.

Now assume that $\mathcal{I}_F \subseteq J_F(I, X)$ for all $\emptyset \neq F \subseteq [d]$. Fix $\emptyset \neq F \subseteq [d]$ and take an element $a \in \mathcal{I}_F \cap X_F^{-1}(I)$. We have that $a \in J_F(I, X)$ by assumption, so $aX_F^{-1}(I) \subseteq I$. Since $a \in X_F^{-1}(I)$, we obtain that $a \in I$ by using an approximate unit of $X_F^{-1}(I)$. It follows that I is negatively invariant, completing the proof. □

In order to motivate further the importance of the $\mathcal{L}_\emptyset$-member of an NO-2^d-tuple $\mathcal{L}$, we give the following proposition.

Proposition 5.1.4. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A and let $\mathcal{L}$ be an NO-2^d-tuple of X. If $\mathcal{L}$ satisfies*

$$\mathcal{I}_F([X]_{\mathcal{L}_\emptyset}) \subseteq (\mathcal{I}_F(X) + \mathcal{L}_\emptyset)/\mathcal{L}_\emptyset \text{ for all } F \subseteq [d],$$

then $[\mathcal{L}]_{\mathcal{L}_\emptyset} = \mathcal{I}([X]_{\mathcal{L}_\emptyset})$ and thus $\mathcal{NO}([\mathcal{L}]_{\mathcal{L}_\emptyset}, [X]_{\mathcal{L}_\emptyset}) = \mathcal{NO}_{[X]_{\mathcal{L}_\emptyset}}$.

Proof. By assumption, for all $F \subseteq [d]$ we have that

$$\mathcal{I}_F([X]_{\mathcal{L}_\emptyset}) \subseteq (\mathcal{I}_F(X) + \mathcal{L}_\emptyset)/\mathcal{L}_\emptyset \subseteq (\mathcal{L}_F + \mathcal{L}_\emptyset)/\mathcal{L}_\emptyset = [\mathcal{L}_F]_{\mathcal{L}_\emptyset} \subseteq \mathcal{I}_F([X]_{\mathcal{L}_\emptyset}),$$

using that $\mathcal{L}$ is an NO-2^d-tuple in the second inclusion and that $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ is an (M)-2^d-tuple of $[X]_{\mathcal{L}_\emptyset}$ by Proposition 4.1.8 in the final inclusion. Hence $[\mathcal{L}]_{\mathcal{L}_\emptyset} = \mathcal{I}([X]_{\mathcal{L}_\emptyset})$ and thus by definition $\mathcal{NO}([\mathcal{L}]_{\mathcal{L}_\emptyset}, [X]_{\mathcal{L}_\emptyset}) = \mathcal{NO}_{[X]_{\mathcal{L}_\emptyset}}$, as required. □

Definition 5.1.5. Let X be a strong compactly aligned product system with coefficients in a C*-algebra A, let $I \subseteq A$ be an ideal and let $\mathcal{L}$ be an NO-2^d-tuple of X. We say that *I participates in $\mathcal{L}$* if $\mathcal{L}_\emptyset = I$.

If I is an ideal that is positively invariant for X, then negative invariance of I is necessary and sufficient for the quotient map $X \to [X]_I$ to induce a canonical $*$-epimorphism between the corresponding Cuntz-Nica-Pimsner algebras.

Proposition 5.1.6. *Let X be a strong compactly aligned product system with coefficients in a C^*-algebra A and let $I \subseteq A$ be an ideal. Then the following are equivalent:*

(i) I participates in an NO-2^d-tuple of X;

(ii) I is positively and negatively invariant for X;

(iii) I is positively invariant for X and the quotient map $X \to [X]_I$ lifts to a (unique) canonical $*$-epimorphism $\mathcal{NO}_X \to \mathcal{NO}_{[X]_I}$.

Proof. [(i)⇒(ii)]: Assume that I participates in an NO-2^d-tuple $\mathcal{L}$ of X. Then in particular $I = \mathcal{L}_\emptyset$ is positively invariant for X. On the other hand, since $\mathcal{I}(X) \subseteq \mathcal{L}$ we have that

$$\mathcal{I}_F(X) \subseteq \mathcal{L}_F \subseteq J_F(\mathcal{L}_\emptyset, X) = J_F(I, X) \text{ for all } \emptyset \neq F \subseteq [d].$$

Proposition 5.1.3 then gives that I is negatively invariant for X.

[(ii)⇒(iii)]: Assume that I is positively and negatively invariant for X and let $\mathcal{L}$ be the 2^d-tuple of X (consisting of ideals) defined by

$$\mathcal{L}_F := \mathcal{I}_F(X) + I \text{ for all } F \subseteq [d].$$

Since $\mathcal{I}(X)$ is invariant and I is positively invariant for X, we have that $\mathcal{L}$ is invariant. Since $\mathcal{I}(X)$ is partially ordered, we also have that $\mathcal{L}$ is partially ordered. Moreover, by Proposition 5.1.3 we have that

$$I \subseteq \mathcal{L}_F \equiv \mathcal{I}_F(X) + I \subseteq J_F(I, X) \text{ for all } \emptyset \neq F \subseteq [d],$$

where we also use that I is positively invariant and so $I \subseteq J_F(I, X)$. Therefore, the family $[\mathcal{L}]_I$ is an invariant and partially ordered 2^d-tuple of $[X]_I$ that consists of ideals and satisfies $[\mathcal{L}]_I \subseteq \mathcal{J}([X]_I)$ by item (ii) of Proposition 4.1.3. Hence $[\mathcal{L}]_I$ is an (E)-2^d-tuple of $[X]_I$ by Proposition 3.2.3. Consequently, we have the canonical $*$-epimorphisms

$$\mathcal{NT}_X \xrightarrow{[\cdot]_I} \mathcal{NT}_{[X]_I} \xrightarrow{Q} \mathcal{NO}([\mathcal{L}]_I, [X]_I) \longrightarrow \mathcal{NO}_{[X]_I},$$

where the final $*$-epimorphism follows from the co-universal property of $\mathcal{NO}_{[X]_I}$.

In order to deduce the required $*$-epimorphism, it suffices to close the following diagram

$$\begin{array}{ccccc} \mathcal{NT}_X & \xrightarrow{[\cdot]_I} & \mathcal{NT}_{[X]_I} & & \\ \downarrow{\scriptstyle Q_{\mathcal{I}(X)}} & & \downarrow{\scriptstyle Q} & & \\ \mathcal{NO}_X & \dashrightarrow & \mathcal{NO}([\mathcal{L}]_I, [X]_I) & \longrightarrow & \mathcal{NO}_{[X]_I} \end{array}$$

by a canonical $*$-epimorphism. Hence it suffices to show that the kernel $\ker Q_{\mathcal{I}(X)} = \mathfrak{J}_{\mathcal{I}(X)}^{(\overline{\pi}_X, \overline{t}_X)}$ is contained in the kernel $\ker Q \circ [\cdot]_I$.

To this end, recall that the generators of $\ker Q_{\mathcal{I}(X)} = \mathfrak{J}_{\mathcal{I}(X)}^{(\overline{\pi}_X, \overline{t}_X)}$ are of the form $\overline{\pi}_X(a)\overline{q}_{X,F}$, where $a \in \mathcal{I}_F(X)$ and $F \subseteq [d]$. Fix $F \subseteq [d]$ and $a \in \mathcal{I}_F(X) \subseteq \mathcal{L}_F$, so that $[a]_I \in [\mathcal{L}_F]_I$. Then by definition we have that

$$[\overline{\pi}_X(a)\overline{q}_{X,F}]_I = \overline{\pi}_{[X]_I}([a]_I)\overline{q}_{[X]_I,F} \in \mathfrak{J}_{[\mathcal{L}]_I}^{(\overline{\pi}_{[X]_I}, \overline{t}_{[X]_I})} = \ker Q,$$

and thus $\overline{\pi}_X(a)\overline{q}_{X,F} \in \ker Q \circ [\cdot]_I$. It follows that $\ker Q_{\mathcal{I}(X)} \subseteq \ker Q \circ [\cdot]_I$, as required.

[(iii)⇒(i)]: Assume that I is positively invariant for X and that the quotient map $X \to [X]_I$ lifts to a (unique) canonical $*$-epimorphism $\Phi \colon \mathcal{NO}_X \to \mathcal{NO}_{[X]_I}$. Then $\ker \Phi \subseteq \mathcal{NO}_X$ is a gauge-invariant ideal and so we can consider the gauge-invariant ideal

$$\mathfrak{J} := Q_{\mathcal{I}(X)}^{-1}(\ker \Phi) \subseteq \mathcal{NT}_X.$$

In particular we have that $\mathfrak{J}^{\mathcal{I}(X)} \subseteq \mathfrak{J}$, and so $\mathcal{L}^{\mathfrak{J}}$ is an NO-2^d-tuple of X since the parametrisation of Theorem 4.2.3 respects inclusions. Using that $Q_{\mathcal{I}(X)}(\mathfrak{J}) = \ker \Phi$, we obtain a sequence of canonical $*$-isomorphisms

$$\mathcal{NO}([\mathcal{L}^{\mathfrak{J}}]_{\mathcal{L}^{\mathfrak{J}}_{\emptyset}}, [X]_{\mathcal{L}^{\mathfrak{J}}_{\emptyset}}) \cong \mathcal{NT}_X/\mathfrak{J} \cong \mathcal{NO}_X/Q_{\mathcal{I}(X)}(\mathfrak{J}) \equiv \mathcal{NO}_X/\ker \Phi \cong \mathcal{NO}_{[X]_I},$$

where the first $*$-isomorphism is given by item (ii) of Proposition 4.2.1 and the second is determined by

$$f + \mathfrak{J} \mapsto Q_{\mathcal{I}(X)}(f) + Q_{\mathcal{I}(X)}(\mathfrak{J}) \text{ for all } f \in \mathcal{NT}_X.$$

By restricting to the coefficient algebras, we have that $[A]_{\mathcal{L}^{\mathfrak{J}}_{\emptyset}} \cong [A]_I$ by the map $a + \mathcal{L}^{\mathfrak{J}}_{\emptyset} \mapsto a + I$ for all $a \in A$. Thus $\mathcal{L}^{\mathfrak{J}}_{\emptyset} = I$, as required. □

We now collect in one place the main results of this section and their consequences.

Corollary 5.1.7. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Let $I \subseteq A$ be an ideal that is positively invariant for X and satisfies*

$$\mathcal{I}_F([X]_I) = (\mathcal{I}_F(X) + I)/I \text{ for all } F \subseteq [d].$$

Let the 2^d-tuple $\mathcal{L}$ of X be defined by $\mathcal{L}_F := \mathcal{I}_F(X) + I$ for all $F \subseteq [d]$. Then the following hold:

(i) The 2^d-tuple $\mathcal{L}$ is an NO-2^d-tuple of X.

(ii) $\mathcal{NO}_X/Q_{\mathcal{I}}(\mathfrak{J}^{\mathcal{L}}) \cong \mathcal{NO}_{[X]_I}$ canonically, where $Q_{\mathcal{I}} \colon \mathcal{NT}_X \to \mathcal{NO}_X$ is the quotient map.

(iii) If $I \subseteq \bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in [d]\}$, then

$$Q_{\mathcal{I}}(\mathfrak{J}^{\mathcal{L}}) = \langle Q_{\mathcal{I}}(\overline{\pi}_X(I)) \rangle = \overline{\operatorname{span}}\{t^{\mathcal{I}}_{X,\underline{n}}(X_{\underline{n}})\pi^{\mathcal{I}}_X(I)t^{\mathcal{I}}_{X,\underline{m}}(X_{\underline{m}})^* \mid \underline{n}, \underline{m} \in \mathbb{Z}^d_+\},$$

where $(\pi^{\mathcal{I}}_X, t^{\mathcal{I}}_X)$ denotes the universal CNP-representation of X. Thus $\mathcal{NO}_{IXI}$ is a hereditary, full C-subalgebra of $Q_{\mathcal{I}}(\mathfrak{J}^{\mathcal{L}})$.*

Proof. (i) Note that $\mathcal{L}$ is a family of ideals such that $\mathcal{L}_\emptyset \subseteq \mathcal{L}_F$ for all $F \subseteq [d]$, and $\mathcal{L}_\emptyset = I$ is positively invariant for X by assumption. We also have that $[\mathcal{L}]_I = \mathcal{I}([X]_I)$ by construction. Thus $[\mathcal{L}]_I$ is an (M)-2^d-tuple of $[X]_I$ by Remark 3.2.9, and in turn $\mathcal{L}$ is an NT-2^d-tuple of X by Proposition 4.1.8. Moreover $\mathcal{I}(X) \subseteq \mathcal{L}$ by definition and thus $\mathcal{L}$ is an NO-2^d-tuple, proving item (i).

(ii) We have the canonical $*$-isomorphisms

$$\mathcal{NO}_X/Q_{\mathcal{I}}(\mathfrak{J}^{\mathcal{L}}) \cong \mathcal{NT}_X/\mathfrak{J}^{\mathcal{L}} \cong \mathcal{NO}([\mathcal{L}]_I, [X]_I) = \mathcal{NO}_{[X]_I},$$

where the first is determined by

$$Q_{\mathcal{I}}(f) + Q_{\mathcal{I}}(\mathfrak{J}^{\mathcal{L}}) \mapsto f + \mathfrak{J}^{\mathcal{L}} \text{ for all } f \in \mathcal{NT}_X$$

and the second is guaranteed by item (i) of Proposition 4.2.1. Note that the former is well-defined since $\mathcal{I}(X) \subseteq \mathcal{L}$ and therefore $\mathfrak{J}^{(\overline{\pi}_X, \overline{t}_X)}_{\mathcal{I}(X)} = \mathfrak{J}^{\mathcal{I}(X)} \subseteq \mathfrak{J}^{\mathcal{L}}$ by combining Proposition 4.2.2 with the fact that the parametrisation of Theorem 4.2.3 respects inclusions. This finishes the proof of item (ii).

(iii) Assume that $I \subseteq \bigcap\{\phi_{\underline{i}}^{-1}(\mathcal{K}(X_{\underline{i}})) \mid i \in [d]\}$. This ensures that $\mathcal{L}$ is both an NT-2^d-tuple and a relative 2^d-tuple of X. Hence $\mathfrak{J}^{\mathcal{L}} = \mathfrak{J}^{(\overline{\pi}_X, \overline{t}_X)}_{\mathcal{L}}$ by Proposition 4.2.2. Since $\pi_X^{\mathcal{I}}(\mathcal{I}_F(X)) q_{X,F}^{\mathcal{I}} = \{0\}$ for all $F \subseteq [d]$, we have that

$$\begin{aligned} Q_{\mathcal{I}}(\mathfrak{J}^{\mathcal{L}}) &= Q_{\mathcal{I}}(\left\langle \overline{\pi}_X(\mathcal{L}_F)\overline{q}_{X,F} \mid F \subseteq [d] \right\rangle) \\ &= \left\langle \pi_X^{\mathcal{I}}(\mathcal{L}_F) q_{X,F}^{\mathcal{I}} \mid F \subseteq [d] \right\rangle \\ &= \left\langle \pi_X^{\mathcal{I}}(I), \pi_X^{\mathcal{I}}(I) q_{X,F}^{\mathcal{I}} \mid \emptyset \neq F \subseteq [d] \right\rangle. \end{aligned}$$

To see that

$$\left\langle \pi_X^{\mathcal{I}}(I), \pi_X^{\mathcal{I}}(I) q_{X,F}^{\mathcal{I}} \mid \emptyset \neq F \subseteq [d] \right\rangle = \left\langle Q_{\mathcal{I}}(\overline{\pi}_X(I)) \right\rangle \equiv \left\langle \pi_X^{\mathcal{I}}(I) \right\rangle,$$

it suffices to show that the right hand side contains the generators of the left hand side. To this end, fix $a \in I$ and $\emptyset \neq F \subseteq [d]$. An application of Proposition 2.5.16 gives that

$$\pi_X^{\mathcal{I}}(a) q_{X,F}^{\mathcal{I}} = \pi_X^{\mathcal{I}}(a) + \sum\{(-1)^{|\underline{n}|} \psi_{X,\underline{n}}^{\mathcal{I}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}.$$

Fixing $\underline{0} \neq \underline{n} \leq \underline{1}_F$, we have that $\underline{n} = \underline{1}_D$ for some $\emptyset \neq D \subseteq F$. Hence we obtain that

$$\psi_{X,\underline{n}}^{\mathcal{I}}(\phi_{\underline{n}}(a)) = \|\cdot\|\text{-}\lim_\lambda \psi_{X,\underline{n}}^{\mathcal{I}}(\phi_{\underline{n}}(a) e_{D,\lambda}) = \|\cdot\|\text{-}\lim_\lambda \pi_X^{\mathcal{I}}(a) \psi_{X,\underline{n}}^{\mathcal{I}}(e_{D,\lambda}) \in \left\langle Q_{\mathcal{I}}(\overline{\pi}_X(I)) \right\rangle,$$

where we use the nomenclature of Proposition 2.5.13 in conjunction with (2.16). Hence we have that $\psi_{X,\underline{n}}^{\mathcal{I}}(\phi_{\underline{n}}(a)) \in \left\langle Q_{\mathcal{I}}(\overline{\pi}_X(I)) \right\rangle$ for all $\underline{0} \neq \underline{n} \leq \underline{1}_F$, and therefore

$$\pi_X^{\mathcal{I}}(a) q_{X,F}^{\mathcal{I}} \in \left\langle Q_{\mathcal{I}}(\overline{\pi}_X(I)) \right\rangle,$$

as required. In total, we conclude that $Q_{\mathcal{I}}(\mathfrak{J}^{\mathcal{L}}) = \langle Q_{\mathcal{I}}(\overline{\pi}_X(I)) \rangle$.

For the final equality, we have that

$$\langle Q_{\mathcal{I}}(\overline{\pi}_X(I)) \rangle \equiv \left\langle \pi_X^{\mathcal{I}}(I) \right\rangle = \overline{\operatorname{span}}\{ t_{X,\underline{n}}^{\mathcal{I}}(X_{\underline{n}}) \pi_X^{\mathcal{I}}(I) t_{X,\underline{m}}^{\mathcal{I}}(X_{\underline{m}})^* \mid \underline{n}, \underline{m} \in \mathbb{Z}_+^d \},$$

using item (i) of Proposition 2.6.7 in the second equality. The last assertion of item (iii) follows from Proposition 2.6.10, finishing the proof. □

5.2 Regular product systems

Let X be a regular product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A. Recall that X is automatically strong compactly aligned by Corollary 2.5.6. Also, an ideal $I \subseteq A$ is negatively invariant for X if and only if $X_F^{-1}(I) \subseteq I$ for all $\emptyset \neq F \subseteq [d]$, since $\mathcal{I}_F = A$ for all $\emptyset \neq F \subseteq [d]$. The gauge-invariant ideals of $\mathcal{NO}_X$ can be parametrised particularly succinctly. We start with some auxiliary results.

Proposition 5.2.1. *Let X be a regular product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A and let $I \subseteq A$ be an ideal. Then the following hold:*

(i) If I is negatively invariant for X, then $J_F(I, X) = A$ for all $\emptyset \neq F \subseteq [d]$.

(ii) If I is positively and negatively invariant for X, then $[X]_I$ is regular.

Proof. For item (i), fix $\emptyset \neq F \subseteq [d]$. We have that $\mathcal{I}_F(X) = A$ by regularity. Since I is negatively invariant, Proposition 5.1.3 yields that $A = \mathcal{I}_F(X) \subseteq J_F(I, X)$, as required.

For item (ii), Lemma 2.2.11 yields that $[\phi_{\underline{n}}]_I([A]_I) \subseteq \mathcal{K}([X_{\underline{n}}]_I)$ for all $\underline{n} \in \mathbb{Z}_+^d$, using that $\phi_{\underline{n}}(A) \subseteq \mathcal{K}(X_{\underline{n}})$ for all $\underline{n} \in \mathbb{Z}_+^d$. For injectivity, it suffices to show that $[\phi_{\underline{i}}]_I$ is injective for all $i \in [d]$ by Proposition 2.5.1. Accordingly, fix $i \in [d]$ and $[a]_I \in \ker[\phi_{\underline{i}}]_I$. Then we have that $\langle X_{\underline{i}}, aX_{\underline{i}} \rangle \subseteq I$ by Proposition 4.1.2, and so $a \in X_{\{i\}}^{-1}(I) \subseteq I$ since I is negatively invariant and X is regular. Hence $[a]_I = 0$, showing that $[\phi_{\underline{i}}]_I$ is injective. In total, we have that $[X]_I$ is regular, as required. □

Corollary 5.2.2. *Let X be a regular product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A and let $\mathcal{L}$ be a 2^d-tuple of X. Then $\mathcal{L}$ is an NO-2^d-tuple of X if and only if $\mathcal{L}_\emptyset$ is an ideal that is positively and negatively invariant for X and $\mathcal{L}_F = A$ for all $\emptyset \neq F \subseteq [d]$.*

Proof. Assume that $\mathcal{L}$ is an NO-2^d-tuple of X, so that $\mathcal{I}(X) \subseteq \mathcal{L}$. We have that $\mathcal{L}_\emptyset$ is a positively and negatively invariant ideal for X by Proposition 5.1.6. Next fix $\emptyset \neq F \subseteq [d]$. By regularity of X, we have that $A = \mathcal{I}_F(X) \subseteq \mathcal{L}_F$. Thus $\mathcal{L}_F = A$ for all $\emptyset \neq F \subseteq [d]$, proving the forward implication.

Conversely, assume that $\mathcal{L}_\emptyset$ is an ideal that is positively and negatively invariant for X and that $\mathcal{L}_F = A$ for all $\emptyset \neq F \subseteq [d]$. We start by showing that $\mathcal{L}$ is an NT-2^d-tuple of X. Note that $\mathcal{L}$ consists of ideals satisfying $\mathcal{L}_\emptyset \subseteq \mathcal{L}_F$ for all $F \subseteq [d]$, and $\mathcal{L}_\emptyset$ is in particular positively invariant for X. Hence it suffices to show that $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ is an (M)-2^d-tuple of $[X]_{\mathcal{L}_\emptyset}$

by Proposition 4.1.8. By item (ii) of Proposition 5.2.1, we have that $[X]_{\mathcal{L}_\emptyset}$ is regular and thus

$$\mathcal{I}_\emptyset([X]_{\mathcal{L}_\emptyset}) = \{0\} = [\mathcal{L}_\emptyset]_{\mathcal{L}_\emptyset} \quad \text{and} \quad \mathcal{I}_F([X]_{\mathcal{L}_\emptyset}) = [A]_{\mathcal{L}_\emptyset} = [\mathcal{L}_F]_{\mathcal{L}_\emptyset} \text{ for all } \emptyset \neq F \subseteq [d].$$

Thus $\mathcal{I}([X]_{\mathcal{L}_\emptyset}) = [\mathcal{L}]_{\mathcal{L}_\emptyset}$ and we conclude that $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ is an (M)-2^d-tuple of $[X]_{\mathcal{L}_\emptyset}$ by Remark 3.2.9. Note also that

$$\mathcal{I}_\emptyset(X) = \{0\} \subseteq \mathcal{L}_\emptyset \quad \text{and} \quad \mathcal{I}_F(X) = A \subseteq \mathcal{L}_F \text{ for all } \emptyset \neq F \subseteq [d]$$

by regularity of X. Thus $\mathcal{L}$ is an NO-2^d-tuple, finishing the proof. □

Corollary 5.2.3. *Let X be a regular product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A. Then the association*

$$I \mapsto \mathcal{L}_I, \text{ where } \mathcal{L}_{I,\emptyset} := I \text{ and } \mathcal{L}_{I,F} := A \text{ for all } \emptyset \neq F \subseteq [d], \tag{5.1}$$

defines a bijection between the set of ideals of A that are positively and negatively invariant for X and the set of NO-2^d-tuples of X. Hence the set of gauge-invariant ideals of $\mathcal{NO}_X$ corresponds bijectively to the set of ideals of A that are positively and negatively invariant for X.

Proof. By Corollary 5.2.2, the map (5.1) is well-defined and a bijection. The last assertion follows from Corollary 4.2.12. □

We obtain the following consequence of Corollary 5.2.3 when A is simple.

Corollary 5.2.4. *Let X be a regular product system over $\mathbb{Z}_+^d$ with coefficients in a non-zero simple C*-algebra A. Then $\mathcal{NO}_X$ contains no non-trivial proper gauge-invariant ideals.*

Proof. By simplicity, the only ideals of A are $\{0\}$ and A, both of which are positively and negatively invariant by Remark 5.1.2. An application of Corollary 5.2.3 finishes the proof. □

Accounting for the gauge-invariant ideals of $\mathcal{NT}_X$ when X is regular and A is simple reduces to a combinatorial argument. We begin by providing a simpler characterisation of NT-2^d-tuples in this setting.

Proposition 5.2.5. *Let X be a regular product system over $\mathbb{Z}_+^d$ with coefficients in a non-zero simple C*-algebra A. The following are equivalent for a 2^d-tuple $\mathcal{L}$ of X:*

(i) *$\mathcal{L}$ is an NT-2^d-tuple of X;*

(ii) *$\mathcal{L}$ is an (M)-2^d-tuple of X or $\mathcal{L}_F = A$ for all $F \subseteq [d]$;*

(iii) *$\mathcal{L}$ is partially ordered and consists of ideals.*

If any (and thus all) of items (i)-(iii) hold, then $\mathcal{NO}(\mathcal{L}, X)$ either factors through the quotient map $\mathcal{NT}_X \to \mathcal{NO}_X$ or $\mathcal{NO}(\mathcal{L}, X) = \{0\}$.

Proof. To see that items (i)-(iii) are equivalent, first suppose that $\mathcal{L}$ is an NT-2^d-tuple of X. If $\mathcal{L}_\emptyset = \{0\}$, then $\mathcal{L}$ is an (M)-2^d-tuple of X by Proposition 4.1.8. If $\mathcal{L}_\emptyset \neq \{0\}$, then by simplicity of A we have that $\mathcal{L}_\emptyset = A$. It follows that $\mathcal{L}_F = A$ for all $F \subseteq [d]$ by the partial ordering property of $\mathcal{L}$. This shows that item (i) implies item (ii).

Conversely, if $\mathcal{L}$ is an (M)-2^d-tuple of X then it is an NT-2^d-tuple of X by Proposition 4.1.8. On the other hand, if $\mathcal{L}_F = A$ for all $F \subseteq [d]$, then $[\mathcal{L}_F]_{\mathcal{L}_\emptyset} = \{0\}$ for all $F \subseteq [d]$. Thus $[\mathcal{L}]_{\mathcal{L}_\emptyset}$ is an (M)-2^d-tuple of $[X]_{\mathcal{L}_\emptyset}$ by Remark 3.2.9, and an application of Proposition 4.1.8 then yields that $\mathcal{L}$ is an NT-2^d-tuple of X. This shows that item (ii) implies item (i).

Clearly item (ii) implies item (iii) by Theorem 3.4.6. Finally, suppose that $\mathcal{L}$ is partially ordered and consists of ideals. By simplicity of A, we have that $\mathcal{L}_F \in \{\{0\}, A\}$ for all $F \subseteq [d]$. If $\mathcal{L}_\emptyset = A$, then $\mathcal{L}_F = A$ for all $F \subseteq [d]$ by the partial ordering property of $\mathcal{L}$. On the other hand, if $\mathcal{L}_\emptyset = \{0\}$ then $\mathcal{L} \subseteq \mathcal{J}(X)$ since $\mathcal{J}_F(X) = A$ for all $\emptyset \neq F \subseteq [d]$ by regularity of X, and thus $\mathcal{L}$ satisfies condition (i) of Theorem 3.4.6. Condition (ii) of the latter is satisfied trivially, and condition (iii) is satisfied by assumption. Moreover, by definition we have that $\mathcal{L}_F^{(1)} = \mathcal{L}_F$ when $F = \emptyset$ or $F = [d]$. For $\emptyset \neq F \subsetneq [d]$, first suppose that $\mathcal{L}_F = A$. In this case we have that $\mathcal{L}_F^{(1)} \subseteq \mathcal{L}_F$ trivially. Otherwise, if $\mathcal{L}_F = \{0\}$ then $\mathcal{K}(X_{\underline{m}}\mathcal{L}_F) = \{0\}$ for all $\underline{m} \in \mathbb{Z}_+^d$ and thus we have that

$$\mathcal{L}_{\lim, F} = \{a \in A \mid \lim_{\underline{m} \perp F} \|\phi_{\underline{m}}(a)\| = 0\} = \{a \in A \mid \lim_{\underline{m} \perp F} \|a\| = 0\} = \{0\},$$

using that X is regular and thus in particular injective in the second equality. Hence we deduce that $\mathcal{L}_F^{(1)} = \{0\} = \mathcal{L}_F$. Therefore $\mathcal{L}$ satisfies conditions (i)-(iv) of Theorem 3.4.6 and so is an (M)-2^d-tuple of X by the latter, finishing the proof of the equivalences.

For the second claim, suppose that $\mathcal{L}$ satisfies item (ii). If $\mathcal{L}$ is an (M)-2^d-tuple of X, then $\mathcal{L} \subseteq \mathcal{I}$ and thus $(\pi_X^{\mathcal{I}}, t_X^{\mathcal{I}})$ is an $\mathcal{L}$-relative CNP-representation. In turn, we have that $\mathcal{NO}(\mathcal{L}, X)$ factors through the quotient map $\mathcal{NT}_X \to \mathcal{NO}_X$ by the comments succeeding Definition 3.1.15. On the other hand, if $\mathcal{L}_F = A$ for all $F \subseteq [d]$ then in particular we have that

$$\overline{\pi}_X(A) = \overline{\pi}_X(\mathcal{L}_\emptyset) \subseteq \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)}.$$

We then deduce that

$$\overline{t}_{X,\underline{n}}(X_{\underline{n}})\overline{t}_{X,\underline{m}}(X_{\underline{m}})^* \subseteq [\overline{t}_{X,\underline{n}}(X_{\underline{n}})\overline{\pi}_X(A)\overline{t}_{X,\underline{m}}(X_{\underline{m}})^*] \subseteq \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)} \text{ for all } \underline{n}, \underline{m} \in \mathbb{Z}_+^d$$

using an approximate unit of A together with the fact that $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)}$ is an ideal. Employing this deduction in conjunction with Nica-covariance of $(\overline{\pi}_X, \overline{t}_X)$ and the fact that $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)}$ is in particular a closed linear subspace of $\mathcal{NT}_X$, we obtain that $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X, \overline{t}_X)} = \mathcal{NT}_X$. Hence we have that $\mathcal{NO}(\mathcal{L}, X) = \{0\}$ by definition, finishing the proof. ∎

Next, we consider $\mathcal{P}([d])$ with the usual partial order of inclusion. For elements $D, F \in \mathcal{P}([d])$, we say that D and F are *incomparable* if $D \not\subseteq F$ and $F \not\subseteq D$. Fixing notation, for a family $\mathcal{S} \subseteq \mathcal{P}([d])$ we write

$$\min(\mathcal{S}) := \{F \in \mathcal{S} \mid F \text{ is minimal in } \mathcal{S}\},$$

and

$$\langle \mathcal{S} \rangle := \{F \subseteq [d] \mid \text{ there exists } F' \in \mathcal{S} \text{ such that } F' \subseteq F\}.$$

It follows that $\mathcal{S} \subseteq \langle \mathcal{S} \rangle$ and that $\min(\mathcal{S})$ consists of pairwise incomparable elements of $\mathcal{S}$. In fact, a family $\mathcal{S} \subseteq \mathcal{P}([d])$ consists of pairwise incomparable elements if and only if $\mathcal{S} = \min(\mathcal{S})$. Note that if $\mathcal{S} \subseteq \mathcal{P}([d])$ is empty or a singleton, then it is a family of pairwise incomparable elements vacuously.

Lemma 5.2.6. *Let $\mathcal{S} \subseteq \mathcal{P}([d])$ be a family of subsets of $[d]$. Then we have that*

$$\langle \min(\mathcal{S}) \rangle = \langle \mathcal{S} \rangle .$$

Proof. The forward inclusion is immediate since $\min(\mathcal{S}) \subseteq \mathcal{S}$. For the reverse inclusion, fix $F \in \langle \mathcal{S} \rangle$. Then by definition there exists $F' \in \mathcal{S}$ such that $F' \subseteq F$. If F' is minimal in $\mathcal{S}$, then $F' \in \min(\mathcal{S})$ and hence $F \in \langle \min(\mathcal{S}) \rangle$, as required. If F' is not minimal in $\mathcal{S}$, then there exists $F'' \in \mathcal{S}$ such that $F'' \subsetneq F' \subseteq F$. By iterating the preceding reasoning and using that $\mathcal{S}$ is finite, we eventually obtain an element $D \in \min(\mathcal{S})$ such that $D \subseteq F$. Thus $F \in \langle \min(\mathcal{S}) \rangle$ in all cases, finishing the proof. □

Corollary 5.2.7. *Let X be a regular product system over $\mathbb{Z}_+^d$ with coefficients in a non-zero simple C*-algebra A. Then the association*

$$\mathcal{S} \mapsto \mathcal{L}_{\mathcal{S}}, \text{ where } \mathcal{L}_{\mathcal{S},F} := \begin{cases} A & \text{if } F \in \langle \mathcal{S} \rangle, \\ \{0\} & \text{otherwise,} \end{cases} \text{ for all } F \subseteq [d], \tag{5.2}$$

defines a bijection between the set of families of pairwise incomparable subsets of $[d]$ and the set of NT-2^d-tuples of X. Hence the set of gauge-invariant ideals of $\mathcal{NT}_X$ corresponds bijectively to the set of families of pairwise incomparable subsets of $[d]$.

Proof. Item (iii) of Proposition 5.2.5 implies that the map (5.2) is well-defined. Indeed, to see that $\mathcal{L}_{\mathcal{S}}$ is partially ordered, fix $F \subseteq D \subseteq [d]$. When $\mathcal{L}_{\mathcal{S},F} = \{0\}$ there is nothing to show, so assume that $\mathcal{L}_{\mathcal{S},F} = A$. Then by definition $F \in \langle \mathcal{S} \rangle$ and so there exists $F' \in \mathcal{S}$ such that

$$F' \subseteq F \subseteq D.$$

Since $F' \subseteq D$, we have that $D \in \langle \mathcal{S} \rangle$ and so $\mathcal{L}_{\mathcal{S},D} = A$. Hence in all cases we obtain that $\mathcal{L}_{\mathcal{S},F} \subseteq \mathcal{L}_{\mathcal{S},D}$, as required.

For bijectivity of the map (5.2), it suffices to construct an inverse map. To this end, fixing an NT-2^d-tuple $\mathcal{L}$ of X, we set

$$\mathcal{S}_{\mathcal{L}} := \{F \subseteq [d] \mid \mathcal{L}_F = A\}.$$

Note that $\mathcal{S}_{\mathcal{L}} = \langle \mathcal{S}_{\mathcal{L}} \rangle$ by the partial ordering property of $\mathcal{L}$. It follows that the assignment

$$\mathcal{L} \mapsto \min(\mathcal{S}_{\mathcal{L}}), \text{ where } \mathcal{L} \text{ is an NT-}2^d\text{-tuple of } X, \tag{5.3}$$

constitutes a well-defined map that takes values in the set of families of pairwise incomparable subsets of $[d]$ by the comments preceding Lemma 5.2.6. Next we check that this map is the inverse of the map (5.2).

Firstly, fix an NT-2^d-tuple $\mathcal{L}$ of X. We must show that $\mathcal{L}_{\min(\mathcal{S}_{\mathcal{L}})} = \mathcal{L}$. To this end, we have that

$$\langle \min(\mathcal{S}_{\mathcal{L}}) \rangle = \langle \mathcal{S}_{\mathcal{L}} \rangle = \mathcal{S}_{\mathcal{L}},$$

using Lemma 5.2.6 in the first equality. Hence we obtain that $\mathcal{L}_{\min(\mathcal{S}_{\mathcal{L}})} = \mathcal{L}$ as an immediate consequence of the definition of $\mathcal{S}_{\mathcal{L}}$.

Next fix a family $\mathcal{S}$ of pairwise incomparable subsets of $[d]$. We must show that $\min(\mathcal{S}_{\mathcal{L}_{\mathcal{S}}}) = \mathcal{S}$. Accordingly, note that

$$\mathcal{S}_{\mathcal{L}_{\mathcal{S}}} = \langle \mathcal{S} \rangle$$

by definition and take $F \in \min(\langle \mathcal{S} \rangle)$. Then $F \in \langle \mathcal{S} \rangle$ and F is minimal in $\langle \mathcal{S} \rangle$. Thus there exists $F' \in \mathcal{S}$ such that $F' \subseteq F$. Since $F, F' \in \langle \mathcal{S} \rangle$, minimality of F yields that $F = F'$ and so we deduce that $\min(\langle \mathcal{S} \rangle) \subseteq \mathcal{S}$. For the reverse inclusion, fix $F \in \mathcal{S} \subseteq \langle \mathcal{S} \rangle$. We must show that F is minimal in $\langle \mathcal{S} \rangle$. To this end, take $D \in \langle \mathcal{S} \rangle$ such that $D \subseteq F$. Then there exists $D' \in \mathcal{S}$ such that $D' \subseteq D \subseteq F$. Since $\mathcal{S}$ is a family of pairwise incomparable subsets of $[d]$, we must have that $D' = F$ and hence $F = D$. Consequently, we have that F is minimal in $\langle \mathcal{S} \rangle$ and thus $\mathcal{S} \subseteq \min(\langle \mathcal{S} \rangle)$. We conclude that $\min(\mathcal{S}_{\mathcal{L}_{\mathcal{S}}}) = \mathcal{S}$ and thus in total the map (5.3) is the inverse of the map (5.2), as required.

The last assertion follows from Theorem 4.2.3, and the proof is complete. □

5.3 C*-dynamical systems

In this section we interpret the parametrisation of Theorem 4.2.3 in the case of C*-dynamical systems. As a corollary we recover the well-known parametrisation for crossed products. We use this class of product systems to produce an example for which the quotient map $X \to [X]_I$ does not lift to a canonical $*$-epimorphism $\mathcal{NO}_X \to \mathcal{NO}_{[X]_I}$ for a positively invariant ideal $I \subseteq A$.

The structure of the Nica-Pimsner and Cuntz-Nica-Pimsner algebras of a dynamical system were studied in [15, 31]. In particular, the form of the CNP-relations that arose in this setting was one point of motivation for looking further into strong compactly aligned product systems in [17]. We start by establishing notation and terminology from [15].

Definition 5.3.1. A *C*-dynamical system* $(A, \alpha, \mathbb{Z}_+^d)$ consists of a C*-algebra A and a semigroup homomorphism $\alpha\colon \mathbb{Z}_+^d \to \operatorname{End}(A)$ satisfying $\alpha_{\underline{0}} = \operatorname{id}_A$. The system will be called *injective* (resp. *automorphic*) if $\alpha_{\underline{n}}$ is injective (resp. a $*$-automorphism) for all $\underline{n} \in \mathbb{Z}_+^d$. The system will be called *unital* if A is unital and $\alpha_{\underline{n}}(1_A) = 1_A$ for all $\underline{n} \in \mathbb{Z}_+^d$. The system will be called *non-degenerate* if $[\alpha_{\underline{n}}(A)A] = A$ for all $\underline{n} \in \mathbb{Z}_+^d$.

Every C*-dynamical system can be canonically associated with a product system. We present this construction in detail.

Proposition 5.3.2. *Let $(A, \alpha, \mathbb{Z}_+^d)$ be a C*-dynamical system. Fixing $\underline{n} \in \mathbb{Z}_+^d$, define the space*

$$X_{\alpha,\underline{n}} := [\alpha_{\underline{n}}(A)A].$$

Then $X_{\alpha,\underline{n}}$ carries the structure of a C-correspondence over A, where the right Hilbert A-module structure is inherited from A and the left action is given by*

$$\phi_{\underline{n}}\colon A \to \mathcal{L}(X_{\alpha,\underline{n}}); \phi_{\underline{n}}(a)\xi_{\underline{n}} = \alpha_{\underline{n}}(a)\xi_{\underline{n}} \text{ for all } a \in A, \xi_{\underline{n}} \in X_{\alpha,\underline{n}}.$$

Moreover, we have that $\alpha_{\underline{n}}(A) \subseteq X_{\alpha,\underline{n}}$ and that $\phi_{\underline{n}}(A) \subseteq \mathcal{K}(X_{\alpha,\underline{n}})$.

Proof. It is clear that $X_{\alpha,\underline{n}}$ inherits the usual right Hilbert A-module structure from A by definition. Next we check that $\phi_{\underline{n}}$ is well-defined. Accordingly, observe that

$$\alpha_{\underline{n}}(A)[\alpha_{\underline{n}}(A)A] \subseteq [\alpha_{\underline{n}}(A)\alpha_{\underline{n}}(A)A] = [\alpha_{\underline{n}}(AA)A] \subseteq [\alpha_{\underline{n}}(A)A].$$

It follows that $\phi_{\underline{n}}(a)$ is well-defined for all $a \in A$. We also have that

$$\langle \phi_{\underline{n}}(a)\xi_{\underline{n}}, \eta_{\underline{n}} \rangle = (\alpha_{\underline{n}}(a)\xi_{\underline{n}})^*\eta_{\underline{n}} = \xi_{\underline{n}}^*\alpha_{\underline{n}}(a^*)\eta_{\underline{n}} = \langle \xi_{\underline{n}}, \phi_{\underline{n}}(a^*)\eta_{\underline{n}} \rangle \text{ for all } a \in A, \xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\underline{n}},$$

and so $\phi_{\underline{n}}(a)^* = \phi_{\underline{n}}(a^*)$ for all $a \in A$. Hence $\phi_{\underline{n}}$ is a well-defined $*$-preserving map. It is routine to check that $\phi_{\underline{n}}$ is an algebra homomorphism by using that $\alpha_{\underline{n}} \in \operatorname{End}(A)$. Thus $\phi_{\underline{n}}$ is a $*$-homomorphism and so $X_{\alpha,\underline{n}}$ constitutes a C*-correspondence over A, as required.

Next fix $a \in A$ and an approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ of A. Notice that $\alpha_{\underline{n}}(a)u_\lambda \in X_{\alpha,\underline{n}}$ for all $\lambda \in \Lambda$ by definition. Thus we obtain that

$$\alpha_{\underline{n}}(a) = \|\cdot\|\text{-}\lim_{\lambda} \alpha_{\underline{n}}(a)u_\lambda \in X_{\alpha,\underline{n}},$$

using that $X_{\alpha,\underline{n}}$ is closed in A to establish the membership. This shows that $\alpha_{\underline{n}}(A) \subseteq X_{\alpha,\underline{n}}$. Fixing $\lambda \in \Lambda$ and $\xi_{\underline{n}} \in X_{\alpha,\underline{n}}$, we therefore have that

$$\|(\Theta_{\alpha_{\underline{n}}(a\sqrt{u_\lambda}),\alpha_{\underline{n}}(\sqrt{u_\lambda})} - \phi_{\underline{n}}(a))\xi_{\underline{n}}\| = \|\alpha_{\underline{n}}(au_\lambda)\xi_{\underline{n}} - \alpha_{\underline{n}}(a)\xi_{\underline{n}}\| \leq \|\alpha_{\underline{n}}(au_\lambda - a)\| \cdot \|\xi_{\underline{n}}\|.$$

It follows that

$$\|\Theta_{\alpha_{\underline{n}}(a\sqrt{u_\lambda}),\alpha_{\underline{n}}(\sqrt{u_\lambda})} - \phi_{\underline{n}}(a)\| \leq \|\alpha_{\underline{n}}(au_\lambda - a)\| \leq \|au_\lambda - a\| \text{ for all } \lambda \in \Lambda.$$

The fact that $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit of A then implies that

$$\phi_{\underline{n}}(a) = \|\cdot\|\text{-}\lim_\lambda \Theta_{\alpha_{\underline{n}}(a\sqrt{u_\lambda}),\alpha_{\underline{n}}(\sqrt{u_\lambda})} \in \mathcal{K}(X_{\alpha,\underline{n}}).$$

Thus we have that $\phi_{\underline{n}}(A) \subseteq \mathcal{K}(X_{\alpha,\underline{n}})$, finishing the proof. □

Proposition 5.3.3. *Let $(A, \alpha, \mathbb{Z}_+^d)$ be a C*-dynamical system. Set*

$$X_\alpha := \{X_{\alpha,\underline{n}}\}_{\underline{n}\in\mathbb{Z}_+^d}, \text{ where } X_{\alpha,\underline{n}} := [\alpha_{\underline{n}}(A)A] \text{ for all } \underline{n} \in \mathbb{Z}_+^d.$$

Then X_α carries a canonical structure as a product system over $\mathbb{Z}_+^d$ with coefficients in A, given by the multiplication maps

$$X_{\alpha,\underline{n}} \otimes_A X_{\alpha,\underline{m}} \to X_{\alpha,\underline{n}+\underline{m}}; \xi_{\underline{n}} \otimes \xi_{\underline{m}} \mapsto \alpha_{\underline{m}}(\xi_{\underline{n}})\xi_{\underline{m}} \text{ for all } \xi_{\underline{n}} \in X_{\alpha,\underline{n}}, \xi_{\underline{m}} \in X_{\alpha,\underline{m}}, \underline{n}, \underline{m} \in \mathbb{Z}_+^d.$$

Additionally, we have that $\phi_{\underline{n}}(A) \subseteq \mathcal{K}(X_{\alpha,\underline{n}})$ for all $\underline{n} \in \mathbb{Z}_+^d$ and hence X_α is in particular strong compactly aligned.

Proof. It suffices to show that X_α constitutes a product system over $\mathbb{Z}_+^d$ with coefficients in A, as then the final claim follows immediately from Proposition 5.3.2 and Corollary 2.5.6. To this end, first note that $X_{\alpha,\underline{n}}$ is a C*-correspondence over A for all $\underline{n} \in \mathbb{Z}_+^d$ by Proposition 5.3.2. It remains to show that X_α, together with the maps of the statement, satisfies axioms (i)-(v) of a product system.

Since $\alpha_{\underline{0}} = \mathrm{id}_A$ and $A = [AA]$, we have that $X_{\alpha,\underline{0}} = A$. Thus X_α satisfies axiom (i). Next we prove that the maps of the statement are well-defined. This is immediate for $\underline{n} = \underline{0}$ or $\underline{m} = \underline{0}$, from which it follows that axioms (ii) and (iii) hold. So fix $\underline{n}, \underline{m} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$ and define the map

$$u_{\underline{n},\underline{m}} \colon X_{\alpha,\underline{n}} \times X_{\alpha,\underline{m}} \to X_{\alpha,\underline{n}+\underline{m}}; (\xi_{\underline{n}}, \xi_{\underline{m}}) \mapsto \alpha_{\underline{m}}(\xi_{\underline{n}})\xi_{\underline{m}} \text{ for all } \xi_{\underline{n}} \in X_{\alpha,\underline{n}}, \xi_{\underline{m}} \in X_{\alpha,\underline{m}}.$$

Note that $u_{\underline{n},\underline{m}}$ is well-defined since

$$\alpha_{\underline{m}}([\alpha_{\underline{n}}(A)A])[\alpha_{\underline{m}}(A)A] \subseteq [\alpha_{\underline{m}}(\alpha_{\underline{n}}(A)A)\alpha_{\underline{m}}(A)A] \subseteq [\alpha_{\underline{n}+\underline{m}}(A)A] = X_{\alpha,\underline{n}+\underline{m}},$$

using that α is a semigroup homomorphism and that addition in $\mathbb{Z}_+^d$ is commutative in the final inclusion. It is routine to check that $u_{\underline{n},\underline{m}}$ is bilinear. Additionally, fixing $\xi_{\underline{n}} \in X_{\alpha,\underline{n}}, \xi_{\underline{m}} \in X_{\alpha,\underline{m}}$ and $a \in A$, we have that

$$\begin{aligned} u_{\underline{n},\underline{m}}(\xi_{\underline{n}}a, \xi_{\underline{m}}) - u_{\underline{n},\underline{m}}(\xi_{\underline{n}}, \phi_{\underline{m}}(a)\xi_{\underline{m}}) &= \alpha_{\underline{m}}(\xi_{\underline{n}}a)\xi_{\underline{m}} - \alpha_{\underline{m}}(\xi_{\underline{n}})(\alpha_{\underline{m}}(a)\xi_{\underline{m}}) \\ &= \alpha_{\underline{m}}(\xi_{\underline{n}})\alpha_{\underline{m}}(a)\xi_{\underline{m}} - \alpha_{\underline{m}}(\xi_{\underline{n}})\alpha_{\underline{m}}(a)\xi_{\underline{m}} = 0. \end{aligned}$$

Thus $u_{\underline{n},\underline{m}}$ is also A-balanced, and therefore induces a unique linear map

$$u_{\underline{n},\underline{m}} \colon X_{\alpha,\underline{n}} \odot_A X_{\alpha,\underline{m}} \to X_{\alpha,\underline{n}+\underline{m}}; \xi_{\underline{n}} \otimes \xi_{\underline{m}} \mapsto \alpha_{\underline{m}}(\xi_{\underline{n}})\xi_{\underline{m}} \text{ for all } \xi_{\underline{n}} \in X_{\alpha,\underline{n}}, \xi_{\underline{m}} \in X_{\alpha,\underline{m}}.$$

For $\xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\alpha,\underline{n}}$ and $\xi_{\underline{m}}, \eta_{\underline{m}} \in X_{\alpha,\underline{m}}$, we obtain that

$$\begin{aligned} \langle u_{\underline{n},\underline{m}}(\xi_{\underline{n}} \otimes \xi_{\underline{m}}), u_{\underline{n},\underline{m}}(\eta_{\underline{n}} \otimes \eta_{\underline{m}})\rangle &= \langle \alpha_{\underline{m}}(\xi_{\underline{n}})\xi_{\underline{m}}, \alpha_{\underline{m}}(\eta_{\underline{n}})\eta_{\underline{m}}\rangle = \xi_{\underline{m}}^* \alpha_{\underline{m}}(\xi_{\underline{n}}^* \eta_{\underline{n}})\eta_{\underline{m}} \\ &= \langle \xi_{\underline{m}}, \phi_{\underline{m}}(\langle \xi_{\underline{n}}, \eta_{\underline{n}}\rangle)\eta_{\underline{m}}\rangle = \langle \xi_{\underline{n}} \otimes \xi_{\underline{m}}, \eta_{\underline{n}} \otimes \eta_{\underline{m}}\rangle . \end{aligned}$$

It follows that $u_{\underline{n},\underline{m}}$ extends to an isometric linear map

$$u_{\underline{n},\underline{m}} \colon X_{\alpha,\underline{n}} \otimes_A X_{\alpha,\underline{m}} \to X_{\alpha,\underline{n}+\underline{m}}; \xi_{\underline{n}} \otimes \xi_{\underline{m}} \mapsto \alpha_{\underline{m}}(\xi_{\underline{n}})\xi_{\underline{m}} \text{ for all } \xi_{\underline{n}} \in X_{\alpha,\underline{n}}, \xi_{\underline{m}} \in X_{\alpha,\underline{m}}.$$

To see that $u_{\underline{n},\underline{m}}$ preserves the left action, fix $\xi_{\underline{n}} \in X_{\alpha,\underline{n}}, \xi_{\underline{m}} \in X_{\alpha,\underline{m}}$ and $a \in A$. Then we have that

$$\begin{aligned} u_{\underline{n},\underline{m}}(a(\xi_{\underline{n}} \otimes \xi_{\underline{m}})) &= u_{\underline{n},\underline{m}}((\phi_{\underline{n}}(a)\xi_{\underline{n}}) \otimes \xi_{\underline{m}}) = \alpha_{\underline{m}}(\alpha_{\underline{n}}(a)\xi_{\underline{n}})\xi_{\underline{m}} \\ &= \alpha_{\underline{n}+\underline{m}}(a)\alpha_{\underline{m}}(\xi_{\underline{n}})\xi_{\underline{m}} = a u_{\underline{n},\underline{m}}(\xi_{\underline{n}} \otimes \xi_{\underline{m}}), \end{aligned}$$

from which it follows that $u_{\underline{n},\underline{m}}$ is a left A-module map by linearity and continuity of the maps involved. Analogously, we obtain that

$$u_{\underline{n},\underline{m}}((\xi_{\underline{n}} \otimes \xi_{\underline{m}})a) = u_{\underline{n},\underline{m}}(\xi_{\underline{n}} \otimes (\xi_{\underline{m}}a)) = \alpha_{\underline{m}}(\xi_{\underline{n}})(\xi_{\underline{m}}a) = u_{\underline{n},\underline{m}}(\xi_{\underline{n}} \otimes \xi_{\underline{m}})a.$$

Thus $u_{\underline{n},\underline{m}}$ also preserves the right action and is thus an A-bimodule map. Since $u_{\underline{n},\underline{m}}$ is an isometric linear map and therefore has closed range, establishing surjectivity amounts to showing that $\alpha_{\underline{n}+\underline{m}}(A)A \subseteq \mathrm{Im}(u_{\underline{n},\underline{m}})$. Accordingly, fix $a, b \in A$ and an approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ of A. We deduce that

$$u_{\underline{n},\underline{m}}(\alpha_{\underline{n}}(a) \otimes (\alpha_{\underline{m}}(u_\lambda)b)) = \alpha_{\underline{m}}(\alpha_{\underline{n}}(a))\alpha_{\underline{m}}(u_\lambda)b = \alpha_{\underline{m}}(\alpha_{\underline{n}}(a)u_\lambda)b \text{ for all } \lambda \in \Lambda,$$

recalling that $\alpha_{\underline{n}}(A) \subseteq X_{\alpha,\underline{n}}$ by Proposition 5.3.2. Thus we obtain that

$$\alpha_{\underline{n}+\underline{m}}(a)b = \|\cdot\|\text{-}\lim_{\lambda} \alpha_{\underline{m}}(\alpha_{\underline{n}}(a)u_\lambda)b = \|\cdot\|\text{-}\lim_{\lambda} u_{\underline{n},\underline{m}}(\alpha_{\underline{n}}(a) \otimes (\alpha_{\underline{m}}(u_\lambda)b)) \in \mathrm{Im}(u_{\underline{n},\underline{m}}),$$

as required. In total, we have that $u_{\underline{n},\underline{m}}$ is a unitary and hence axiom (iv) holds.

It remains to check that the multiplication maps are associative. For notational clarity, we will use the symbol "$\cdot$" to denote multiplication in X_α. Fixing $\underline{n}, \underline{m}, \underline{r} \in \mathbb{Z}_+^d, \xi_{\underline{n}} \in X_{\alpha,\underline{n}}, \xi_{\underline{m}} \in X_{\alpha,\underline{m}}$ and $\xi_{\underline{r}} \in X_{\alpha,\underline{r}}$, we obtain that

$$\begin{aligned} (\xi_{\underline{n}} \cdot \xi_{\underline{m}}) \cdot \xi_{\underline{r}} &= (\alpha_{\underline{m}}(\xi_{\underline{n}})\xi_{\underline{m}}) \cdot \xi_{\underline{r}} = \alpha_{\underline{r}}(\alpha_{\underline{m}}(\xi_{\underline{n}})\xi_{\underline{m}})\xi_{\underline{r}} \\ &= \alpha_{\underline{r}+\underline{m}}(\xi_{\underline{n}})\alpha_{\underline{r}}(\xi_{\underline{m}})\xi_{\underline{r}} = \xi_{\underline{n}} \cdot (\alpha_{\underline{r}}(\xi_{\underline{m}})\xi_{\underline{r}}) = \xi_{\underline{n}} \cdot (\xi_{\underline{m}} \cdot \xi_{\underline{r}}), \end{aligned}$$

as required. Thus axiom (v) holds and we conclude that X_α constitutes a product system over $\mathbb{Z}_+^d$ with coefficients in A, finishing the proof. □

We will make liberal use of the notation in Proposition 5.3.3 throughout the section. The following two results further explore the relationship between a C*-dynamical system and its associated product system.

Proposition 5.3.4. *Let $(A, \alpha, \mathbb{Z}_+^d)$ be a C*-dynamical system. Then the following hold:*

(i) $\ker \alpha_{\underline{n}} = \ker \phi_{\underline{n}}$ *for all $\underline{n} \in \mathbb{Z}_+^d$, and thus $(A, \alpha, \mathbb{Z}_+^d)$ is injective if and only if X_α is injective.*

(ii) For an ideal $I \subseteq A$, we have that $\alpha_{\underline{n}}^{-1}(I) = X_{\alpha,\underline{n}}^{-1}(I)$ for all $\underline{n} \in \mathbb{Z}_+^d$.

Proof. (i) It suffices to prove the first claim, as the second follows as an immediate consequence. Accordingly, fix $\underline{n} \in \mathbb{Z}_+^d$ and note that the forward inclusion holds by definition of $\phi_{\underline{n}}$. Next take $a \in \ker \phi_{\underline{n}}$, so that

$$\alpha_{\underline{n}}(a)\xi_{\underline{n}} = \phi_{\underline{n}}(a)\xi_{\underline{n}} = 0 \text{ for all } \xi_{\underline{n}} \in X_{\alpha,\underline{n}}.$$

Hence, fixing an approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ of A, we obtain that

$$\alpha_{\underline{n}}(a) = \alpha_{\underline{n}}(\|\cdot\|\text{-}\lim_\lambda au_\lambda) = \|\cdot\|\text{-}\lim_\lambda \alpha_{\underline{n}}(a)\alpha_{\underline{n}}(u_\lambda) = 0,$$

using that $\alpha_{\underline{n}}(A) \subseteq X_{\alpha,\underline{n}}$ by Proposition 5.3.2 in the final equality. This shows that $\ker \phi_{\underline{n}} \subseteq \ker \alpha_{\underline{n}}$, as required.

(ii) Fix $\underline{n} \in \mathbb{Z}_+^d$ and $a \in A$. Then we have that

$$\langle X_{\alpha,\underline{n}}, aX_{\alpha,\underline{n}}\rangle = [\alpha_{\underline{n}}(A)A]^*\alpha_{\underline{n}}(a)[\alpha_{\underline{n}}(A)A]$$

by definition. Thus it is clear that $\alpha_{\underline{n}}^{-1}(I) \subseteq X_{\alpha,\underline{n}}^{-1}(I)$ since I is an ideal. Finally, suppose that $a \in X_{\alpha,\underline{n}}^{-1}(I)$. In particular, we have that

$$\alpha_{\underline{n}}(AaA) = \alpha_{\underline{n}}(A)^*\alpha_{\underline{n}}(a)\alpha_{\underline{n}}(A) \subseteq \langle X_{\alpha,\underline{n}}, aX_{\alpha,\underline{n}}\rangle \subseteq I,$$

using that $\alpha_{\underline{n}}(A) \subseteq X_{\alpha,\underline{n}}$ by Proposition 5.3.2. Since $I \subseteq A$ is closed, an application of an approximate unit of A yields that $\alpha_{\underline{n}}(a) \in I$ and hence $X_{\alpha,\underline{n}}^{-1}(I) \subseteq \alpha_{\underline{n}}^{-1}(I)$, finishing the proof. □

Proposition 5.3.5. *Let $(A, \alpha, \mathbb{Z}_+^d)$ be a C*-dynamical system and fix $\underline{n} \in \mathbb{Z}_+^d$. Then there exists a $*$-isomorphism Φ determined by*

$$\Phi \colon \mathcal{K}(X_{\alpha,\underline{n}}) \to [\alpha_{\underline{n}}(A)A\alpha_{\underline{n}}(A)]; \Theta_{\xi_{\underline{n}},\eta_{\underline{n}}} \mapsto \xi_{\underline{n}}\eta_{\underline{n}}^* \textit{ for all } \xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\alpha,\underline{n}}.$$

Additionally, if $I \subseteq A$ is an ideal then $\mathcal{K}(X_{\alpha,\underline{n}}I) \cong [\alpha_{\underline{n}}(A)I\alpha_{\underline{n}}(A)]$ via restriction of Φ.

Proof. Firstly, it is routine to check that $[\alpha_{\underline{n}}(A)I\alpha_{\underline{n}}(A)]$ is a C*-subalgebra of A whenever $I \subseteq A$ is an ideal. We begin by defining a map

$$\Phi\colon \operatorname{span}\{\Theta_{\xi_{\underline{n}},\eta_{\underline{n}}} \mid \xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\alpha,\underline{n}}\} \to [\alpha_{\underline{n}}(A)A\alpha_{\underline{n}}(A)]; \sum_{j=1}^{n} \Theta_{\xi_{\underline{n}}^{(j)},\eta_{\underline{n}}^{(j)}} \mapsto \sum_{j=1}^{n} \xi_{\underline{n}}^{(j)}\eta_{\underline{n}}^{(j)*},$$

for all $\xi_{\underline{n}}^{(j)}, \eta_{\underline{n}}^{(j)} \in X_{\alpha,\underline{n}}, j \in [n]$ and $n \in \mathbb{N}$. Note that Φ is linear by construction. To see that Φ is well-defined, first note that

$$[\alpha_{\underline{n}}(A)A][\alpha_{\underline{n}}(A)A]^* \subseteq [\alpha_{\underline{n}}(A)A][A\alpha_{\underline{n}}(A)] \subseteq [\alpha_{\underline{n}}(A)AA\alpha_{\underline{n}}(A)] \subseteq [\alpha_{\underline{n}}(A)A\alpha_{\underline{n}}(A)]$$

and so

$$\operatorname{Im}(\Phi) \subseteq [\alpha_{\underline{n}}(A)A\alpha_{\underline{n}}(A)],$$

as claimed. Next, fixing $n \in \mathbb{N}$ and $\xi_{\underline{n}}^{(j)}, \eta_{\underline{n}}^{(j)} \in X_{\alpha,\underline{n}}$ for all $j \in [n]$, we claim that

$$\|\sum_{j=1}^{n} \Theta_{\xi_{\underline{n}}^{(j)},\eta_{\underline{n}}^{(j)}}\|_{\mathcal{K}(X_{\alpha,\underline{n}})} = \|\sum_{j=1}^{n} \xi_{\underline{n}}^{(j)}\eta_{\underline{n}}^{(j)*}\|_A.$$

To see this, first note that

$$\|\sum_{j=1}^{n} \Theta_{\xi_{\underline{n}}^{(j)},\eta_{\underline{n}}^{(j)}}\|_{\mathcal{K}(X_{\alpha,\underline{n}})} = \|(\xi_{\underline{n}}^{(i)*}\xi_{\underline{n}}^{(j)})_{ij}^{\frac{1}{2}}(\eta_{\underline{n}}^{(i)*}\eta_{\underline{n}}^{(j)})_{ij}^{\frac{1}{2}}\|_{M_n(A)}$$

by [8, Lemma 4.6.1] or by arguing as in the proof of Proposition 2.2.2. Next we define matrices $M := (M_{ij})_{ij}, N := (N_{ij})_{ij} \in M_n(A)$ such that

$$M_{ij} = \begin{cases} \xi_{\underline{n}}^{(j)} & \text{if } i = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad N_{ij} = \begin{cases} \eta_{\underline{n}}^{(j)} & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $M^*M = (\xi_{\underline{n}}^{(i)*}\xi_{\underline{n}}^{(j)})_{ij}$ and $N^*N = (\eta_{\underline{n}}^{(i)*}\eta_{\underline{n}}^{(j)})_{ij}$. An application of (2.2) then gives that

$$\|(M^*M)^{\frac{1}{2}}(N^*N)^{\frac{1}{2}}\|_{M_n(A)} = \|MN^*\|_{M_n(A)}.$$

A direct computation yields that MN^* is the matrix with $(1,1)$-entry equal to $\sum_{j=1}^n \xi_{\underline{n}}^{(j)}\eta_{\underline{n}}^{(j)*}$ and all other entries equal to 0. Hence, since A embeds in $M_n(A)$ as the $(1,1)$-entry, we have that

$$\|\sum_{j=1}^{n} \Theta_{\xi_{\underline{n}}^{(j)},\eta_{\underline{n}}^{(j)}}\|_{\mathcal{K}(X_{\alpha,\underline{n}})} = \|\sum_{j=1}^{n} \xi_{\underline{n}}^{(j)}\eta_{\underline{n}}^{(j)*}\|_A,$$

as claimed. It follows that Φ is a well-defined isometric linear map. In turn, the map Φ extends to an isometric linear map $\Phi\colon \mathcal{K}(X_{\alpha,\underline{n}}) \to [\alpha_{\underline{n}}(A)A\alpha_{\underline{n}}(A)]$. It is routine to show that Φ is a $*$-homomorphism by checking on rank-one operators and then extending to the general case using linearity and continuity of Φ.

Being a $*$-homomorphism between C*-algebras, we have that $\text{Im}(\Phi)$ is a C*-subalgebra of $[\alpha_{\underline{n}}(A)A\alpha_{\underline{n}}(A)]$. Consequently, proving that Φ is surjective amounts to showing that

$$\alpha_{\underline{n}}(A)A\alpha_{\underline{n}}(A) \subseteq \text{Im}(\Phi).$$

To this end, fix $a, b, c \in A$. Recalling that $\alpha_{\underline{n}}(A) \subseteq X_{\alpha,\underline{n}}$ by Proposition 5.3.2, we obtain that

$$\alpha_{\underline{n}}(a)b\alpha_{\underline{n}}(c) = \Phi(\Theta_{\alpha_{\underline{n}}(a)b,\alpha_{\underline{n}}(c^*)}) \tag{5.4}$$

and thus Φ is surjective. Hence Φ is a $*$-isomorphism, proving the first claim.

For the final claim, fix an ideal $I \subseteq A$. By the first claim, it suffices to show that

$$\Phi(\mathcal{K}(X_{\alpha,\underline{n}}I)) = [\alpha_{\underline{n}}(A)I\alpha_{\underline{n}}(A)].$$

The forward inclusion follows by noting that

$$[\alpha_{\underline{n}}(A)A]I[\alpha_{\underline{n}}(A)A]^* \subseteq [\alpha_{\underline{n}}(A)A]I[A\alpha_{\underline{n}}(A)] \subseteq [\alpha_{\underline{n}}(A)AIA\alpha_{\underline{n}}(A)] \subseteq [\alpha_{\underline{n}}(A)I\alpha_{\underline{n}}(A)].$$

For the reverse inclusion, it suffices to show that $\alpha_{\underline{n}}(A)I\alpha_{\underline{n}}(A) \subseteq \Phi(\mathcal{K}(X_{\alpha,\underline{n}}I))$ since $\Phi(\mathcal{K}(X_{\alpha,\underline{n}}I))$ is a C*-subalgebra of $[\alpha_{\underline{n}}(A)I\alpha_{\underline{n}}(A)]$. This follows from (5.4), now taking $b \in I$. Hence $\Phi(\mathcal{K}(X_{\alpha,\underline{n}}I)) = [\alpha_{\underline{n}}(A)I\alpha_{\underline{n}}(A)]$ and the proof is complete. □

Next we consider the Nica-covariant representations of a C*-dynamical system and how they relate to the Nica-covariant representations of the associated product system.

Definition 5.3.6. Let $(A, \alpha, \mathbb{Z}_+^d)$ be a C*-dynamical system. A *covariant pair* for $(A, \alpha, \mathbb{Z}_+^d)$ is a pair (π, V) acting on a Hilbert space H such that $\pi\colon A \to \mathcal{B}(H)$ is a $*$-homomorphism, $V\colon \mathbb{Z}_+^d \to \mathcal{B}(H)$ is a semigroup homomorphism, and

$$\pi(a)V_{\underline{n}} = V_{\underline{n}}\pi(\alpha_{\underline{n}}(a)) \text{ for all } \underline{n} \in \mathbb{Z}_+^d, a \in A. \tag{5.5}$$

We say that V is *contractive/isometric* if $V_{\underline{n}}$ is contractive/isometric for all $\underline{n} \in \mathbb{Z}_+^d$. We say that V is *Nica-covariant* if it is contractive and $V_{\underline{n}}^*V_{\underline{m}} = V_{\underline{m}}V_{\underline{n}}^*$ for all $\underline{n}, \underline{m} \in \mathbb{Z}_+^d$ satisfying $\underline{n} \wedge \underline{m} = \underline{0}$.

We say that (π, V) is *contractive/isometric/Nica-covariant* if V is contractive/isometric/Nica-covariant. An isometric Nica-covariant pair (π, V) for $(A, \alpha, \mathbb{Z}_+^d)$ is called *Cuntz-Nica-Pimsner covariant* (abbrev. *CNP-covariant*) if

$$\pi(a)\prod_{i \in F}(I - V_{\underline{i}}V_{\underline{i}}^*) = 0 \text{ for all } a \in \bigcap_{\underline{n} \perp F} \alpha_{\underline{n}}^{-1}((\bigcap_{i \in F} \ker \alpha_{\underline{i}})^\perp) \text{ and } \emptyset \neq F \subseteq [d]. \tag{5.6}$$

We write $\mathrm{C}^*(\pi, V)$ for the C*-subalgebra of $\mathcal{B}(H)$ generated by the images of π and V. If V is isometric, then we have that $V_{\underline{0}} = I$ automatically. Indeed, this follows from the observation that

$$V_{\underline{0}} = V_{\underline{0}+\underline{0}} = V_{\underline{0}}^2,$$

since V is a semigroup homomorphism. Multiplying by $V_{\underline{0}}^*$ on the left then finishes the proof of the claim. Commutativity of addition in $\mathbb{Z}_+^d$ employed in tandem with the fact that V is a semigroup homomorphism ensures that the operators $V_{\underline{n}}$ commute for all $\underline{n} \in \mathbb{Z}_+^d$. We write $\mathcal{NT}(A, \alpha)$ for the C*-algebra that is universal with respect to the isometric Nica-covariant pairs for $(A, \alpha, \mathbb{Z}_+^d)$, and $\mathcal{NO}(A, \alpha)$ for the C*-algebra that is universal with respect to the CNP-covariant pairs for $(A, \alpha, \mathbb{Z}_+^d)$, both of which are generated by a copy of A and $\mathbb{Z}_+^d$.

The connection of (5.6) with the CNP-representations of X_α was established in [15], where it is shown that

$$\mathcal{J}_F = (\bigcap_{i \in F} \ker \alpha_{\underline{i}})^\perp \quad \text{and} \quad \mathcal{I}_F = \bigcap_{\underline{n} \perp F} \alpha_{\underline{n}}^{-1}((\bigcap_{i \in F} \ker \alpha_{\underline{i}})^\perp) \tag{5.7}$$

for all $\emptyset \neq F \subseteq [d]$. The isometric Nica-covariant (resp. CNP-covariant) pairs of $(A, \alpha, \mathbb{Z}_+^d)$ induce Nica-covariant representations (resp. CNP-representations) of X_α by the association $(\pi, V) \mapsto (\pi, t)$, where $t_{\underline{n}}(\alpha_{\underline{n}}(a)b) = V_{\underline{n}}\pi(\alpha_{\underline{n}}(a)b)$ for all $a, b \in A$ and $\underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$. In general, we have that

$$\mathcal{NT}_{X_\alpha} \hookrightarrow \mathcal{NT}(A, \alpha) \quad \text{and} \quad \mathcal{NO}_{X_\alpha} \hookrightarrow \mathcal{NO}(A, \alpha).$$

The embeddings are surjective when $(A, \alpha, \mathbb{Z}_+^d)$ is non-degenerate. We provide an argument to this effect in the unital case.

Proposition 5.3.7. *[15] Let $(A, \alpha, \mathbb{Z}_+^d)$ be a unital C*-dynamical system. Let $\mathcal{S}$ denote the set of isometric Nica-covariant pairs (π, V) for $(A, \alpha, \mathbb{Z}_+^d)$ with π unital and let $\mathcal{T}$ denote the set of Nica-covariant representations (π, t) of X_α with π unital. Then $\mathcal{S}$ and $\mathcal{T}$ correspond bijectively via the mutually inverse maps*

$$G\colon (\pi, V) \mapsto (\pi, t), \text{ where } t_{\underline{n}}(\xi_{\underline{n}}) = V_{\underline{n}}\pi(\xi_{\underline{n}}) \text{ for all } \xi_{\underline{n}} \in X_{\alpha,\underline{n}}, \underline{n} \in \mathbb{Z}_+^d, (\pi, V) \in \mathcal{S},$$

and

$$H\colon (\pi, t) \mapsto (\pi, V), \text{ where } V_{\underline{n}} = t_{\underline{n}}(1_{\underline{n}}) \text{ for all } \underline{n} \in \mathbb{Z}_+^d, (\pi, t) \in \mathcal{T},$$

where $1_{\underline{n}}$ denotes the unit of A for all $\underline{n} \in \mathbb{Z}_+^d$. Moreover, the maps G and H restrict to give a bijective correspondence between the set of CNP-covariant pairs (π, V) for $(A, \alpha, \mathbb{Z}_+^d)$ with π unital and the set of CNP-representations (π, t) of X_α with π unital. Consequently, we have that

$$\mathcal{NT}_{X_\alpha} \cong \mathcal{NT}(A, \alpha) \quad \text{and} \quad \mathcal{NO}_{X_\alpha} \cong \mathcal{NO}(A, \alpha)$$

by canonical $*$-isomorphisms.

Proof. For notational clarity, we will use the symbol "$\cdot$" to denote multiplication in X_α. Note that $X_{\alpha,\underline{n}} = A$ for all $\underline{n} \in \mathbb{Z}_+^d$ by unitality of $(A, \alpha, \mathbb{Z}_+^d)$. We start by showing that G and H are well-defined. Fixing $(\pi, V) \in \mathcal{S}$, we must show that $G((\pi, V)) = (\pi, t)$ is a

Nica-covariant representation of X_α, noting that π is unital by assumption. First observe that

$$t_{\underline{0}}(a) = V_{\underline{0}}\pi(a) = \pi(a) \text{ for all } a \in A,$$

using that $V_{\underline{0}} = I$ by the remarks succeeding Definition 5.3.6 in the second equality. Hence we have that $t_{\underline{0}} = \pi$. It is clear that $t_{\underline{n}}$ is a linear map for all $\underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$. Fixing $a \in A, \underline{n}, \underline{m} \in \mathbb{Z}_+^d, \xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\alpha,\underline{n}}$ and $\xi_{\underline{m}} \in X_{\alpha,\underline{m}}$, we obtain the following three items:

(i) $\pi(a)t_{\underline{n}}(\xi_{\underline{n}}) = \pi(a)V_{\underline{n}}\pi(\xi_{\underline{n}}) = V_{\underline{n}}\pi(\alpha_{\underline{n}}(a))\pi(\xi_{\underline{n}}) = V_{\underline{n}}\pi(\alpha_{\underline{n}}(a)\xi_{\underline{n}}) = t_{\underline{n}}(\phi_{\underline{n}}(a)\xi_{\underline{n}})$,

(ii) $t_{\underline{n}}(\xi_{\underline{n}})^*t_{\underline{n}}(\eta_{\underline{n}}) = \pi(\xi_{\underline{n}}^*)V_{\underline{n}}^*V_{\underline{n}}\pi(\eta_{\underline{n}}) = \pi(\xi_{\underline{n}}^*\eta_{\underline{n}}) = \pi(\langle \xi_{\underline{n}}, \eta_{\underline{n}}\rangle)$,

(iii) $t_{\underline{n}}(\xi_{\underline{n}})t_{\underline{m}}(\xi_{\underline{m}}) = V_{\underline{n}}\pi(\xi_{\underline{n}})V_{\underline{m}}\pi(\xi_{\underline{m}}) = V_{\underline{n}+\underline{m}}\pi(\alpha_{\underline{m}}(\xi_{\underline{n}})\xi_{\underline{m}}) = t_{\underline{n}+\underline{m}}(\xi_{\underline{n}} \cdot \xi_{\underline{m}})$.

For the second equality of (i) we use (5.5); for the second equality of (ii) we use that V is isometric; and for the second equality of (iii) we use (5.5) together with the fact that V is a semigroup homomorphism. Hence (π, t) constitutes a representation of X_α. To see that (π, t) is Nica-covariant, fix $\underline{n}, \underline{m} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}, k_{\underline{n}} \in \mathcal{K}(X_{\alpha,\underline{n}})$ and $k_{\underline{m}} \in \mathcal{K}(X_{\alpha,\underline{m}})$. We must show that

$$\psi_{\underline{n}}(k_{\underline{n}})\psi_{\underline{m}}(k_{\underline{m}}) = \psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(k_{\underline{n}})\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(k_{\underline{m}})) = \psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(k_{\underline{n}}))\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(k_{\underline{m}})),$$

where the second equality follows from Proposition 2.4.1. By linearity and continuity of the maps involved, we may assume that $k_{\underline{n}} = \Theta_{\xi_{\underline{n}},\eta_{\underline{n}}}^{X_{\alpha,\underline{n}}}$ and $k_{\underline{m}} = \Theta_{\xi_{\underline{m}},\eta_{\underline{m}}}^{X_{\alpha,\underline{m}}}$ for some $\xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\alpha,\underline{n}}$ and $\xi_{\underline{m}}, \eta_{\underline{m}} \in X_{\alpha,\underline{m}}$ without loss of generality. We claim that

$$\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(\Theta_{\xi_{\underline{n}},\eta_{\underline{n}}}^{X_{\alpha,\underline{n}}}) = \Theta_{\alpha_{\underline{n}\vee\underline{m}-\underline{n}}(\xi_{\underline{n}}),\alpha_{\underline{n}\vee\underline{m}-\underline{n}}(\eta_{\underline{n}})}^{X_{\alpha,\underline{n}\vee\underline{m}}},$$

noting that $\alpha_{\underline{n}\vee\underline{m}-\underline{n}}(\xi_{\underline{n}}), \alpha_{\underline{n}\vee\underline{m}-\underline{n}}(\eta_{\underline{n}}) \in A = X_{\alpha,\underline{n}\vee\underline{m}}$. To see this, it suffices to show that the equality holds on the vectors of the form $\zeta_{\underline{n}} \cdot \zeta_{\underline{n}\vee\underline{m}-\underline{n}}$ for $\zeta_{\underline{n}} \in X_{\alpha,\underline{n}}$ and $\zeta_{\underline{n}\vee\underline{m}-\underline{n}} \in X_{\alpha,\underline{n}\vee\underline{m}-\underline{n}}$, using that $X_{\alpha,\underline{n}} \otimes_A X_{\alpha,\underline{n}\vee\underline{m}-\underline{n}} \cong X_{\alpha,\underline{n}\vee\underline{m}}$ via the multiplication map $u_{\underline{n},\underline{n}\vee\underline{m}-\underline{n}}$ together with linearity and continuity of the maps involved. We obtain that

$$\begin{aligned}
\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(\Theta_{\xi_{\underline{n}},\eta_{\underline{n}}}^{X_{\alpha,\underline{n}}})(\zeta_{\underline{n}} \cdot \zeta_{\underline{n}\vee\underline{m}-\underline{n}}) &= (\xi_{\underline{n}} \langle \eta_{\underline{n}}, \zeta_{\underline{n}}\rangle) \cdot \zeta_{\underline{n}\vee\underline{m}-\underline{n}} = (\xi_{\underline{n}}\eta_{\underline{n}}^*\zeta_{\underline{n}}) \cdot \zeta_{\underline{n}\vee\underline{m}-\underline{n}} \\
&= \alpha_{\underline{n}\vee\underline{m}-\underline{n}}(\xi_{\underline{n}}\eta_{\underline{n}}^*\zeta_{\underline{n}})\zeta_{\underline{n}\vee\underline{m}-\underline{n}} = \alpha_{\underline{n}\vee\underline{m}-\underline{n}}(\xi_{\underline{n}}\eta_{\underline{n}}^*)\alpha_{\underline{n}\vee\underline{m}-\underline{n}}(\zeta_{\underline{n}})\zeta_{\underline{n}\vee\underline{m}-\underline{n}} \\
&= \alpha_{\underline{n}\vee\underline{m}-\underline{n}}(\xi_{\underline{n}})\alpha_{\underline{n}\vee\underline{m}-\underline{n}}(\eta_{\underline{n}})^*(\zeta_{\underline{n}} \cdot \zeta_{\underline{n}\vee\underline{m}-\underline{n}}) \\
&= \Theta_{\alpha_{\underline{n}\vee\underline{m}-\underline{n}}(\xi_{\underline{n}}),\alpha_{\underline{n}\vee\underline{m}-\underline{n}}(\eta_{\underline{n}})}^{X_{\alpha,\underline{n}\vee\underline{m}}}(\zeta_{\underline{n}} \cdot \zeta_{\underline{n}\vee\underline{m}-\underline{n}}),
\end{aligned}$$

as required. By analogous reasoning, we have that

$$\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(\Theta_{\xi_{\underline{m}},\eta_{\underline{m}}}^{X_{\alpha,\underline{m}}}) = \Theta_{\alpha_{\underline{n}\vee\underline{m}-\underline{m}}(\xi_{\underline{m}}),\alpha_{\underline{n}\vee\underline{m}-\underline{m}}(\eta_{\underline{m}})}^{X_{\alpha,\underline{n}\vee\underline{m}}}.$$

For notational convenience, we set $\underline{n}' := \underline{n} \vee \underline{m} - \underline{n}$ and $\underline{m}' := \underline{n} \vee \underline{m} - \underline{m}$. Thus we obtain that

$$\begin{aligned}
\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(k_{\underline{n}}))\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(k_{\underline{m}})) = \\
&= t_{\underline{n}\vee\underline{m}}(\alpha_{\underline{n}'}(\xi_{\underline{n}}))t_{\underline{n}\vee\underline{m}}(\alpha_{\underline{n}'}(\eta_{\underline{n}}))^* t_{\underline{n}\vee\underline{m}}(\alpha_{\underline{m}'}(\xi_{\underline{m}}))t_{\underline{n}\vee\underline{m}}(\alpha_{\underline{m}'}(\eta_{\underline{m}}))^* \\
&= V_{\underline{n}\vee\underline{m}}\pi(\alpha_{\underline{n}'}(\xi_{\underline{n}}))\pi(\alpha_{\underline{n}'}(\eta_{\underline{n}}^*))V_{\underline{n}\vee\underline{m}}^* V_{\underline{n}\vee\underline{m}}\pi(\alpha_{\underline{m}'}(\xi_{\underline{m}}))\pi(\alpha_{\underline{m}'}(\eta_{\underline{m}}^*))V_{\underline{n}\vee\underline{m}}^* \\
&= V_{\underline{n}\vee\underline{m}}\pi(\alpha_{\underline{n}'}(\xi_{\underline{n}}\eta_{\underline{n}}^*)\alpha_{\underline{m}'}(\xi_{\underline{m}}\eta_{\underline{m}}^*))V_{\underline{n}\vee\underline{m}}^*,
\end{aligned}$$

using that $V_{\underline{n}\vee\underline{m}}$ is an isometry in the third equality. On the other hand, we have that

$$\psi_{\underline{n}}(k_{\underline{n}})\psi_{\underline{m}}(k_{\underline{m}}) = t_{\underline{n}}(\xi_{\underline{n}})t_{\underline{n}}(\eta_{\underline{n}})^* t_{\underline{m}}(\xi_{\underline{m}})t_{\underline{m}}(\eta_{\underline{m}})^* = V_{\underline{n}}\pi(\xi_{\underline{n}})\pi(\eta_{\underline{n}}^*)V_{\underline{n}}^* V_{\underline{m}}\pi(\xi_{\underline{m}})\pi(\eta_{\underline{m}}^*)V_{\underline{m}}^*.$$

We claim that $V_{\underline{n}}^* V_{\underline{m}} = V_{\underline{n}'}V_{\underline{m}'}^*$. To see this, write $\underline{n} = (n_1, \dots, n_d)$ and $\underline{m} = (m_1, \dots, m_d)$. Then we obtain that

$$V_{\underline{n}}^* V_{\underline{m}} = (V_{\underline{d}}^*)^{n_d} \dots (V_{\underline{1}}^*)^{n_1} V_{\underline{1}}^{m_1} \dots V_{\underline{d}}^{m_d},$$

using that V is a semigroup homomorphism and the convention that raising an operator to the power of 0 yields I. Coupling the former with Nica-covariance of V, we deduce that

$$V_{\underline{n}}^* V_{\underline{m}} = (V_{\underline{1}}^*)^{n_1} V_{\underline{1}}^{m_1} \dots (V_{\underline{d}}^*)^{n_d} V_{\underline{d}}^{m_d}.$$

For each $j \in [d]$, consider the term $(V_{\underline{j}}^*)^{n_j} V_{\underline{j}}^{m_j}$. We have that

$$(V_{\underline{j}}^*)^{n_j} V_{\underline{j}}^{m_j} = \begin{cases} (V_{\underline{j}}^*)^{n_j - m_j} & \text{if } n_j \vee m_j = n_j, \\ V_{\underline{j}}^{m_j - n_j} & \text{if } n_j \vee m_j = m_j, \end{cases}$$

using that $V_{\underline{j}}$ is an isometry. In turn, we deduce that

$$(V_{\underline{j}}^*)^{n_j} V_{\underline{j}}^{m_j} = V_{\underline{j}}^{n_j \vee m_j - n_j} (V_{\underline{j}}^*)^{n_j \vee m_j - m_j}.$$

Rearranging terms as before, we obtain that

$$V_{\underline{n}}^* V_{\underline{m}} = V_{\underline{n}'} V_{\underline{m}'}^*,$$

as claimed. Hence we have that

$$\begin{aligned}
\psi_{\underline{n}}(k_{\underline{n}})\psi_{\underline{m}}(k_{\underline{m}}) &= V_{\underline{n}}\pi(\xi_{\underline{n}}\eta_{\underline{n}}^*)V_{\underline{n}'}V_{\underline{m}'}^*\pi(\xi_{\underline{m}}\eta_{\underline{m}}^*)V_{\underline{m}}^* \\
&= V_{\underline{n}}V_{\underline{n}'}\pi(\alpha_{\underline{n}'}(\xi_{\underline{n}}\eta_{\underline{n}}^*))\pi(\alpha_{\underline{m}'}(\xi_{\underline{m}}\eta_{\underline{m}}^*))V_{\underline{m}'}^* V_{\underline{m}}^* \\
&= V_{\underline{n}\vee\underline{m}}\pi(\alpha_{\underline{n}'}(\xi_{\underline{n}}\eta_{\underline{n}}^*)\alpha_{\underline{m}'}(\xi_{\underline{m}}\eta_{\underline{m}}^*))V_{\underline{n}\vee\underline{m}}^* = \psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(k_{\underline{n}}))\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(k_{\underline{m}})),
\end{aligned}$$

using (5.5) in the second equality and the fact that V is a semigroup homomorphism in the third equality. Thus (π, t) is Nica-covariant and so $G((\pi, V)) \in \mathcal{T}$. It follows that G is well-defined.

To see that H is well-defined, fix $(\pi, t) \in \mathcal{T}$. We must show that $H((\pi, t)) = (\pi, V)$ is an isometric Nica-covariant pair for $(A, \alpha, \mathbb{Z}_+^d)$, noting that π is unital by assumption. Fixing $\underline{n}, \underline{m} \in \mathbb{Z}_+^d$, we obtain that

$$V_{\underline{n}+\underline{m}} = t_{\underline{n}+\underline{m}}(1_{\underline{n}+\underline{m}}) = t_{\underline{n}+\underline{m}}(\alpha_{\underline{m}}(1_{\underline{n}})1_{\underline{m}}) = t_{\underline{n}+\underline{m}}(1_{\underline{n}} \cdot 1_{\underline{m}}) = t_{\underline{n}}(1_{\underline{n}})t_{\underline{m}}(1_{\underline{m}}) = V_{\underline{n}}V_{\underline{m}}$$

using unitality of $(A, \alpha, \mathbb{Z}_+^d)$ in the second equality. Hence V is a semigroup homomorphism. Fixing $a \in A$, we deduce that

$$\pi(a)V_{\underline{n}} = \pi(a)t_{\underline{n}}(1_{\underline{n}}) = t_{\underline{n}}(\alpha_{\underline{n}}(a)1_{\underline{n}}) = t_{\underline{n}}(1_{\underline{n}}\alpha_{\underline{n}}(a)) = t_{\underline{n}}(1_{\underline{n}})\pi(\alpha_{\underline{n}}(a)) = V_{\underline{n}}\pi(\alpha_{\underline{n}}(a)).$$

In total, we have shown that (π, V) constitutes a covariant pair of $(A, \alpha, \mathbb{Z}_+^d)$. It remains to check that (π, V) is isometric and Nica-covariant. For the former, we have that

$$V_{\underline{n}}^*V_{\underline{n}} = t_{\underline{n}}(1_{\underline{n}})^*t_{\underline{n}}(1_{\underline{n}}) = \pi(\langle 1_{\underline{n}}, 1_{\underline{n}}\rangle) = \pi(1_{\underline{n}}^*1_{\underline{n}}) = \pi(1_{\underline{n}}) = I,$$

as required. For Nica-covariance, fix $\underline{n}, \underline{m} \in \mathbb{Z}_+^d$ satisfying $\underline{n} \wedge \underline{m} = \underline{0}$. Set

$$k_{\underline{n}} := \Theta_{1_{\underline{n}}, 1_{\underline{n}}}^{X_{\alpha,\underline{n}}} = \mathrm{id}_{X_{\alpha,\underline{n}}} \quad \text{and} \quad k_{\underline{m}} := \Theta_{1_{\underline{m}}, 1_{\underline{m}}}^{X_{\alpha,\underline{m}}} = \mathrm{id}_{X_{\alpha,\underline{m}}}.$$

We obtain that

$$\begin{aligned} V_{\underline{n}}^*V_{\underline{m}} &= t_{\underline{n}}(1_{\underline{n}})^*t_{\underline{m}}(1_{\underline{m}}) = t_{\underline{n}}(k_{\underline{n}}1_{\underline{n}})^*t_{\underline{m}}(k_{\underline{m}}1_{\underline{m}}) \\ &= t_{\underline{n}}(1_{\underline{n}})^*\psi_{\underline{n}}(k_{\underline{n}})^*\psi_{\underline{m}}(k_{\underline{m}})t_{\underline{m}}(1_{\underline{m}}) = t_{\underline{n}}(1_{\underline{n}})^*\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(k_{\underline{n}})\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(k_{\underline{m}}))t_{\underline{m}}(1_{\underline{m}}), \end{aligned}$$

using Nica-covariance of (π, t) in the final equality. The fact that $\underline{n} \wedge \underline{m} = \underline{0}$ implies that $\underline{n} \perp \underline{m}$ and hence $\underline{n} \vee \underline{m} = \underline{n} + \underline{m}$. In turn, it is routine to check that

$$\iota_{\underline{n}}^{\underline{n}+\underline{m}}(k_{\underline{n}}) = \iota_{\underline{m}}^{\underline{n}+\underline{m}}(k_{\underline{m}}) = \Theta_{1_{\underline{n}+\underline{m}}, 1_{\underline{n}+\underline{m}}}^{X_{\alpha,\underline{n}+\underline{m}}} = \mathrm{id}_{X_{\alpha,\underline{n}+\underline{m}}}.$$

In total, we have that

$$\begin{aligned} V_{\underline{n}}^*V_{\underline{m}} &= t_{\underline{n}}(1_{\underline{n}})^*\psi_{\underline{n}+\underline{m}}(\Theta_{1_{\underline{n}+\underline{m}}, 1_{\underline{n}+\underline{m}}}^{X_{\alpha,\underline{n}+\underline{m}}})t_{\underline{m}}(1_{\underline{m}}) = t_{\underline{n}}(1_{\underline{n}})^*t_{\underline{n}+\underline{m}}(1_{\underline{n}+\underline{m}})t_{\underline{m}+\underline{n}}(1_{\underline{m}+\underline{n}})^*t_{\underline{m}}(1_{\underline{m}}) \\ &= t_{\underline{n}}(1_{\underline{n}})^*t_{\underline{n}}(1_{\underline{n}})t_{\underline{m}}(1_{\underline{m}})t_{\underline{n}}(1_{\underline{n}})^*t_{\underline{m}}(1_{\underline{m}})^*t_{\underline{m}}(1_{\underline{m}}) = \pi(\langle 1_{\underline{n}}, 1_{\underline{n}}\rangle)t_{\underline{m}}(1_{\underline{m}})t_{\underline{n}}(1_{\underline{n}})^*\pi(\langle 1_{\underline{m}}, 1_{\underline{m}}\rangle) \\ &= It_{\underline{m}}(1_{\underline{m}})t_{\underline{n}}(1_{\underline{n}})^*I = V_{\underline{m}}V_{\underline{n}}^*, \end{aligned}$$

using unitality of π in the fifth equality. Thus (π, V) is Nica-covariant and so $H((\pi, t)) \in \mathcal{S}$. It follows that H is well-defined.

To see that G and H are mutually inverse, fix $(\pi, V) \in \mathcal{S}$. Set $G((\pi, V)) = (\pi, t)$ and $H((\pi, t)) = (\pi, W)$. We have that

$$W_{\underline{n}} = t_{\underline{n}}(1_{\underline{n}}) = V_{\underline{n}}\pi(1_{\underline{n}}) = V_{\underline{n}} \text{ for all } \underline{n} \in \mathbb{Z}_+^d,$$

using unitality of π in the final equality. Hence $(\pi, W) = (\pi, V)$ and so $H \circ G = \mathrm{id}_{\mathcal{S}}$. Now fix $(\pi, t) \in \mathcal{T}$. Set $H((\pi, t)) = (\pi, V)$ and $G((\pi, V)) = (\pi, s)$. We obtain that

$$s_{\underline{n}}(\xi_{\underline{n}}) = V_{\underline{n}}\pi(\xi_{\underline{n}}) = t_{\underline{n}}(1_{\underline{n}})\pi(\xi_{\underline{n}}) = t_{\underline{n}}(\xi_{\underline{n}}) \text{ for all } \xi_{\underline{n}} \in X_{\alpha,\underline{n}}, \underline{n} \in \mathbb{Z}_+^d.$$

Hence $(\pi, s) = (\pi, t)$ and so $G \circ H = \mathrm{id}_{\mathcal{T}}$. Thus G and H are mutually inverse, as required.

For the second claim, fix a CNP-covariant pair (π, V) for $(A, \alpha, \mathbb{Z}_+^d)$ with π unital. We must show that $G((\pi, V)) = (\pi, t)$ is a CNP-representation of X_α, noting that π is unital by assumption. Accordingly, first observe that

$$p_{\underline{i}} = \psi_{\underline{i}}(\mathrm{id}_{X_{\alpha,\underline{i}}}) = \psi_{\underline{i}}(\Theta^{X_{\alpha,\underline{i}}}_{1_{\underline{i}},1_{\underline{i}}}) = t_{\underline{i}}(1_{\underline{i}})t_{\underline{i}}(1_{\underline{i}})^* = V_{\underline{i}}V_{\underline{i}}^* \text{ for all } i \in [d]$$

by unitality of A and π. Fixing $\emptyset \neq F \subseteq [d]$ and $a \in \mathcal{I}_F = \bigcap_{\underline{n} \perp F} \alpha_{\underline{n}}^{-1}((\bigcap_{i \in F} \ker \alpha_{\underline{i}})^\perp)$, we have that

$$\pi(a)q_F = \pi(a)\prod_{i \in F}(I - p_{\underline{i}}) = \pi(a)\prod_{i \in F}(I - V_{\underline{i}}V_{\underline{i}}^*) = 0,$$

using CNP-covariance of (π, V) in the final equality. Hence $G((\pi, V))$ is a CNP-representation of X_α, as required.

Now fix a CNP-representation (π, t) of X_α with π unital. We must show that $H((\pi, t)) = (\pi, V)$ is a CNP-covariant pair for $(A, \alpha, \mathbb{Z}_+^d)$, noting that π is unital by assumption. To this end, fix $\emptyset \neq F \subseteq [d]$ and $a \in \bigcap_{\underline{n} \perp F} \alpha_{\underline{n}}^{-1}((\bigcap_{i \in F} \ker \alpha_{\underline{i}})^\perp) = \mathcal{I}_F$. We obtain that

$$\pi(a)\prod_{i \in F}(I - V_{\underline{i}}V_{\underline{i}}^*) = \pi(a)\prod_{i \in F}(I - t_{\underline{i}}(1_{\underline{i}})t_{\underline{i}}(1_{\underline{i}})^*) = \pi(a)\prod_{i \in F}(I - p_{\underline{i}}) = \pi(a)q_F = 0,$$

using that (π, t) is a CNP-representation in the final equality. Hence $H((\pi, t))$ is a CNP-covariant pair for $(A, \alpha, \mathbb{Z}_+^d)$. Combining this with the corresponding fact for G, we deduce that G and H restrict to give the bijective correspondence asserted in the statement.

For the final claim, notice that given any isometric Nica-covariant pair (π, V) of $(A, \alpha, \mathbb{Z}_+^d)$ or any Nica-covariant representation (π, t) of X_α, we have that $\pi(1_A)$ is a unit for $\mathrm{C}^*(\pi, V)$ and $\mathrm{C}^*(\pi, t)$. Hence, by suitably restricting the codomains of π, V and each $t_{\underline{n}}$, we deduce the canonical $*$-isomorphisms. □

Using the nomenclature of Proposition 5.3.7, it is routine to check that

$$\mathrm{C}^*(\pi, V) = \mathrm{C}^*(G((\pi, V))) \quad \text{and} \quad \mathrm{C}^*(\pi, t) = \mathrm{C}^*(H((\pi, t))) \text{ for all } (\pi, V) \in \mathcal{S}, (\pi, t) \in \mathcal{T}.$$

In turn, we can view $\mathcal{NO}_{X_\alpha}$ as being generated by a CNP-covariant pair (π, V) such that π is unital and injective.

Injectivity of a unital system has an equivalent reformulation in terms of the semigroup representation in the Cuntz-Nica-Pimsner algebra.

Proposition 5.3.8. *Let $(A, \alpha, \mathbb{Z}_+^d)$ be a unital C*-dynamical system. Let $\mathcal{NO}_{X_\alpha} = \mathrm{C}^*(\pi, V)$, where (π, V) is a CNP-covariant pair for $(A, \alpha, \mathbb{Z}_+^d)$ such that π is unital and* *injective. Then $\alpha_{\underline{i}}$ is injective for all $i \in [d]$ if and only if $V_{\underline{i}}$ is a unitary for all $i \in [d]$.*

Proof. First assume that $\alpha_{\underline{i}}$ is injective for all $i \in [d]$, and so X_α is regular by Proposition 2.5.1. Hence we have that $\mathcal{I}_F = A$ for all $\emptyset \neq F \subseteq [d]$, and the CNP-covariance of (π, V) gives that

$$\pi(a)(I - V_{\underline{i}}V_{\underline{i}}^*) = 0 \text{ for all } a \in \mathcal{I}_{\{i\}} = A, i \in [d].$$

Applying for $a = 1_A$, we obtain that

$$V_{\underline{i}}V_{\underline{i}}^* = I,$$

and so $V_{\underline{i}}$ is a unitary for all $i \in [d]$.

Now assume that $V_{\underline{i}}$ is a unitary for all $i \in [d]$. Fixing $a \in A$ and $i \in [d]$, we have that

$$\pi(a) = \pi(a)V_{\underline{i}}V_{\underline{i}}^* = V_{\underline{i}}\pi(\alpha_{\underline{i}}(a))V_{\underline{i}}^*,$$

using (5.5). Thus it follows that if $a \in \ker \alpha_{\underline{i}}$, then $a \in \ker \pi = \{0\}$ and hence $\alpha_{\underline{i}}$ is injective, finishing the proof. □

Next we translate the key concepts of Chapters 3 and 4 into the language of C*-dynamical systems. Fixing a C*-dynamical system $(A, \alpha, \mathbb{Z}_+^d)$, note that all 2^d-tuples of X_α are relative since all left actions are by compacts. Let $\mathcal{L}$ be a 2^d-tuple of X_α consisting of ideals. By construction of X_α, we see that $\mathcal{L}$ is X_α-invariant if and only if

$$\mathcal{L}_F \subseteq \bigcap_{\underline{n} \perp F} \alpha_{\underline{n}}^{-1}(\mathcal{L}_F) \text{ for all } F \subseteq [d].$$

In particular, an ideal $I \subseteq A$ is positively invariant for X_α if and only if

$$I \subseteq \bigcap_{\underline{n} \in \mathbb{Z}_+^d} \alpha_{\underline{n}}^{-1}(I).$$

Also, we have that I is negatively invariant for X_α if and only if

$$\mathcal{I}_F \cap (\bigcap \{\alpha_{\underline{n}}^{-1}(I) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}) \subseteq I \text{ for all } \emptyset \neq F \subseteq [d].$$

For each $\emptyset \neq F \subseteq [d]$, we have that

$$J_F(I, X_\alpha) = \{a \in A \mid a(\bigcap \{\alpha_{\underline{n}}^{-1}(I) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}) \subseteq I\}.$$

Proposition 5.3.9. *Let $(A, \alpha, \mathbb{Z}_+^d)$ be a C*-dynamical system and let $I \subseteq A$ be an ideal. Then, fixing $\underline{m} \in \mathbb{Z}_+^d$ and $a \in A$, we have that*

$$\|\phi_{\underline{m}}(a) + \mathcal{K}(X_{\alpha,\underline{m}}I)\| = \|\alpha_{\underline{m}}(a) + [\alpha_{\underline{m}}(A)I\alpha_{\underline{m}}(A)]\|.$$

Proof. Let $\Phi \colon \mathcal{K}(X_{\alpha,\underline{m}}) \to [\alpha_{\underline{m}}(A)A\alpha_{\underline{m}}(A)]$ denote the $*$-isomorphism guaranteed by

Proposition 5.3.5. Recall that the latter also gives that $\mathcal{K}(X_{\alpha,\underline{m}}I) \cong [\alpha_{\underline{m}}(A)I\alpha_{\underline{m}}(A)]$ via restriction of Φ. Fixing an approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ of A, we obtain that

$$\phi_{\underline{m}}(a) = \|\cdot\|\text{-}\lim_\lambda \Theta_{\alpha_{\underline{m}}(a\sqrt{u_\lambda}),\alpha_{\underline{m}}(\sqrt{u_\lambda})},$$

by Proposition 5.3.2 and its proof. In turn, we have that

$$\Phi(\phi_{\underline{m}}(a)) = \|\cdot\|\text{-}\lim_\lambda \Phi(\Theta_{\alpha_{\underline{m}}(a\sqrt{u_\lambda}),\alpha_{\underline{m}}(\sqrt{u_\lambda})}) = \|\cdot\|\text{-}\lim_\lambda \alpha_{\underline{m}}(au_\lambda) = \alpha_{\underline{m}}(a).$$

Hence, fixing $k_{\underline{m}} \in \mathcal{K}(X_{\alpha,\underline{m}}I)$, we obtain that

$$\|\phi_{\underline{m}}(a) + k_{\underline{m}}\| = \|\Phi(\phi_{\underline{m}}(a) + k_{\underline{m}})\| = \|\alpha_{\underline{m}}(a) + \Phi(k_{\underline{m}})\| \geq \|\alpha_{\underline{m}}(a) + [\alpha_{\underline{m}}(A)I\alpha_{\underline{m}}(A)]\|,$$

using that $\Phi(k_{\underline{m}}) \in [\alpha_{\underline{m}}(A)I\alpha_{\underline{m}}(A)]$ in the final inequality. It follows that

$$\|\phi_{\underline{m}}(a) + \mathcal{K}(X_{\alpha,\underline{m}}I)\| \geq \|\alpha_{\underline{m}}(a) + [\alpha_{\underline{m}}(A)I\alpha_{\underline{m}}(A)]\|.$$

Analogously, fixing $b \in [\alpha_{\underline{m}}(A)I\alpha_{\underline{m}}(A)]$, note that $b = \Phi(k_{\underline{m}})$ for some $k_{\underline{m}} \in \mathcal{K}(X_{\alpha,\underline{m}}I)$ by the initial remarks. Thus we obtain that

$$\|\alpha_{\underline{m}}(a) + b\| = \|\Phi(\phi_{\underline{m}}(a) + k_{\underline{m}})\| = \|\phi_{\underline{m}}(a) + k_{\underline{m}}\| \geq \|\phi_{\underline{m}}(a) + \mathcal{K}(X_{\alpha,\underline{m}}I)\|.$$

It follows that

$$\|\alpha_{\underline{m}}(a) + [\alpha_{\underline{m}}(A)I\alpha_{\underline{m}}(A)]\| \geq \|\phi_{\underline{m}}(a) + \mathcal{K}(X_{\alpha,\underline{m}}I)\|,$$

and the proof is complete. □

Corollary 5.3.10. *Let $(A, \alpha, \mathbb{Z}_+^d)$ be a C*-dynamical system. Let $\mathcal{K}$ and $\mathcal{L}$ be 2^d-tuples of X_α. Then $\mathcal{L}$ is a $\mathcal{K}$-relative NO-2^d-tuple of X_α if and only if $\mathcal{K} \subseteq \mathcal{L}$ and the following hold:*

(i) $\mathcal{L}$ *consists of ideals and* $\mathcal{L}_F \cap (\bigcap_{i\in F} \alpha_{\underline{i}}^{-1}(\mathcal{L}_\emptyset)) \subseteq \mathcal{L}_\emptyset$ *for all* $\emptyset \neq F \subseteq [d]$,

(ii) $\mathcal{L}_F \subseteq \bigcap_{\underline{n}\perp F} \alpha_{\underline{n}}^{-1}(\mathcal{L}_F)$ *for all* $F \subseteq [d]$,

(iii) $\mathcal{L}$ *is partially ordered,*

(iv) $\mathcal{L}_{1,F} \cap \mathcal{L}_{2,F} \cap \mathcal{L}_{3,F} \subseteq \mathcal{L}_F$ *for all* $\emptyset \neq F \subsetneq [d]$, *where*

- $\mathcal{L}_{1,F} := \bigcap_{\underline{n}\perp F} \alpha_{\underline{n}}^{-1}(\{a \in A \mid a(\bigcap_{i\in F} \alpha_{\underline{i}}^{-1}(\mathcal{L}_\emptyset)) \subseteq \mathcal{L}_\emptyset\})$,
- $\mathcal{L}_{2,F} := \bigcap_{\underline{m}\perp F} \alpha_{\underline{m}}^{-1}(\cap_{F\subsetneq D}\mathcal{L}_D)$,
- $\mathcal{L}_{3,F} := \{a \in A \mid \lim_{\underline{m}\perp F} \|\alpha_{\underline{m}}(a) + [\alpha_{\underline{m}}(A)\mathcal{L}_F\alpha_{\underline{m}}(A)]\| = 0\}$.

Proof. The result follows immediately by the remarks preceding the statement and Proposition 5.3.9, applied in conjunction with Definitions 4.1.4 and 4.2.8, as well as Proposition

4.1.5. Note that the latter applies since the left action of each fibre of X_α is by compacts by Proposition 5.3.3. For item (i) and the definition of $\mathcal{L}_{1,F}$, we also use that

$$\bigcap \{\alpha_{\underline{n}}^{-1}(\mathcal{L}_\emptyset) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = \bigcap_{i \in F} \alpha_{\underline{i}}^{-1}(\mathcal{L}_\emptyset) \text{ for all } \emptyset \neq F \subseteq [d],$$

which follows from Proposition 4.1.2 since $\mathcal{L}_\emptyset$ is positively invariant. □

When $I \subseteq A$ is an ideal that is positively invariant for X_α, we define the C*-dynamical system $([A]_I, [\alpha]_I, \mathbb{Z}_+^d)$ by

$$[\alpha_{\underline{n}}]_I [a]_I = [\alpha_{\underline{n}}(a)]_I \text{ for all } a \in A, \underline{n} \in \mathbb{Z}_+^d.$$

Unitality and non-degeneracy of $(A, \alpha, \mathbb{Z}_+^d)$ are inherited by $([A]_I, [\alpha]_I, \mathbb{Z}_+^d)$. Indeed, the former is straightforward to check and the latter follows by noting that

$$[\alpha_{\underline{n}}(A)A] = \alpha_{\underline{n}}(A)A \text{ for all } \underline{n} \in \mathbb{Z}_+^d$$

by the Hewitt-Cohen Factorisation Theorem. The structure of $X_{[\alpha]_I}$ is closely related to that of $[X_\alpha]_I$, as the following two propositions show.

Proposition 5.3.11. *Let $(A, \alpha, \mathbb{Z}_+^d)$ be a C*-dynamical system and let $I \subseteq A$ be an ideal that is positively invariant for X_α. Then the following equalities hold:*

$$\ker[\phi_{\underline{n}}]_I = \ker \phi_{X_{[\alpha]_I, \underline{n}}} = \ker[\alpha_{\underline{n}}]_I = [\alpha_{\underline{n}}^{-1}(I)]_I \textit{ for all } \underline{n} \in \mathbb{Z}_+^d.$$

Proof. Fix $\underline{n} \in \mathbb{Z}_+^d$. Note that the second equality holds by item (i) of Proposition 5.3.4. For the remaining equalities, fix $a \in A$. We have that

$$\begin{aligned} [a]_I \in \ker[\phi_{\underline{n}}]_I &\iff [\phi_{\underline{n}}(a)]_I [X_{\alpha,\underline{n}}]_I = \{0\} \iff \langle X_{\alpha,\underline{n}}, \phi_{\underline{n}}(a) X_{\alpha,\underline{n}} \rangle \subseteq I \\ &\iff \xi_{\underline{n}}^* \alpha_{\underline{n}}(a) \eta_{\underline{n}} \in I \text{ for all } \xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\alpha,\underline{n}} \iff \alpha_{\underline{n}}(a) \in I \\ &\iff [a]_I \in \ker[\alpha_{\underline{n}}]_I \iff [a]_I \in [\alpha_{\underline{n}}^{-1}(I)]_I, \end{aligned}$$

where the second equivalence follows from [36, Proposition 1.3] and the fourth equivalence follows by using an approximate unit of A together with the fact that $\alpha_{\underline{n}}(A) \subseteq X_{\alpha,\underline{n}}$ by Proposition 5.3.2. The reverse implication of the final equivalence follows from the observation that

$$\alpha_{\underline{n}}(\alpha_{\underline{n}}^{-1}(I) + I) \subseteq I,$$

noting that $\alpha_{\underline{n}}(I) \subseteq I$ by positive invariance of I. This completes the proof. □

Proposition 5.3.12. *Let $(A, \alpha, \mathbb{Z}_+^d)$ be a non-degenerate C*-dynamical system and let $I \subseteq A$ be an ideal that is positively invariant for X_α. Then $X_{[\alpha]_I}$ and $[X_\alpha]_I$ are unitarily equivalent via the family of maps $\{\mathrm{id}\colon [A]_I \to [A]_I\}_{\underline{n} \in \mathbb{Z}_+^d}$.*

Proof. Recall that $([A]_I, [\alpha]_I, \mathbb{Z}_+^d)$ is non-degenerate by the comments preceding Proposition 5.3.11, and so $X_{[\alpha]_I,\underline{n}} = [A]_I$ for all $\underline{n} \in \mathbb{Z}_+^d$. We also have that $[X_{\alpha,\underline{n}}]_I = [A]_I$ for all $\underline{n} \in \mathbb{Z}_+^d$ by non-degeneracy of $(A, \alpha, \mathbb{Z}_+^d)$. It is then routine to check that $\{\mathrm{id}\colon [A]_I \to [A]_I\}_{\underline{n}\in\mathbb{Z}_+^d}$ satisfies the unitary equivalence axioms, completing the proof. □

Next we turn our attention to showing that, in general, we do not have a canonical $*$-epimorphism $\mathcal{NO}_X \to \mathcal{NO}_{[X]_I}$ for a positively invariant ideal $I \subseteq A$. In particular, injectivity of $(A, \alpha, \mathbb{Z}_+^d)$ need not imply injectivity of $([A]_I, [\alpha]_I, \mathbb{Z}_+^d)$.

Example 5.3.13. Let $(A, \alpha, \mathbb{Z}_+)$ be a unital and injective C*-dynamical system. Suppose that $I \subseteq A$ is an ideal that is positively invariant for X_α (i.e., $I \subseteq \alpha^{-1}(I)$), and so $[X_\alpha]_I \cong X_{[\alpha]_I}$ by Proposition 5.3.12.

We claim that we do not have a canonical $*$-epimorphism $\Phi\colon \mathcal{NO}_{X_\alpha} \to \mathcal{NO}_{X_{[\alpha]_I}}$, in general. To reach contradiction, assume that such a map Φ exists. Write $\mathcal{NO}_{X_\alpha} = \mathrm{C}^*(\pi, V)$ and $\mathcal{NO}_{X_{[\alpha]_I}} = \mathrm{C}^*(\sigma, W)$, where (π, V) and (σ, W) are CNP-covariant pairs for $(A, \alpha, \mathbb{Z}_+)$ and $([A]_I, [\alpha]_I, \mathbb{Z}_+)$, respectively. We may take π and σ to be unital and injective. Since α is injective, we have that V_1 is a unitary by Proposition 5.3.8, and thus $W_1 = \Phi(V_1)$ is also a unitary. Another application of Proposition 5.3.8 gives that $[\alpha]_I$ is injective. However, this leads to contradiction when we choose A, α and I so that $[\alpha]_I$ is not injective.

For such an example, let B be a unital C*-algebra and $\beta \in \mathrm{End}(B)$ be unital and non-injective. Set $A = \bigoplus_{n\in\mathbb{N}} B$ and define $\alpha \in \mathrm{End}(A)$ by $\alpha((b_n)_{n\in\mathbb{N}}) = (\beta(b_1), b_1, b_2, \dots)$ for all $(b_n)_{n\in\mathbb{N}} \in A$, noting that α is unital and injective. Set

$$I := \{(b_n)_{n\in\mathbb{N}} \in A \mid b_1 = 0\},$$

which is an ideal that is positively invariant for X_α. Define the map

$$\Psi\colon [A]_I \to B; [(b_n)_{n\in\mathbb{N}}]_I \mapsto b_1 \text{ for all } (b_n)_{n\in\mathbb{N}} \in A.$$

It is routine to check that Ψ is a $*$-isomorphism. Moreover, the diagram

$$\begin{array}{ccc} [A]_I & \xrightarrow{[\alpha]_I} & [A]_I \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\Psi} \\ B & \xrightarrow[\beta]{} & B \end{array}$$

commutes. Hence $[\alpha]_I$ is injective if and only if β is injective and thus $[\alpha]_I$ is not injective by the choice of β.

Notice that this does *not* contradict Proposition 5.1.6 because I does not participate in any NO-2-tuple of X_α. Indeed, suppose that I participates in an NO-2-tuple $\mathcal{L}$ of X_α, i.e., $\{I, \mathcal{L}_{\{1\}}\}$ is an O-pair of $X_{\alpha,1}$ by Proposition 4.2.9. Since α is injective, we have that $J_{X_{\alpha,1}} = A \subseteq \mathcal{L}_{\{1\}}$ and thus $\mathcal{L}_{\{1\}} = A$. Hence $\alpha^{-1}(I) = A \cap \alpha^{-1}(I) \subseteq I$ by item (i) of Corollary 5.3.10. However, the map β is not injective and so we can choose $0 \neq b \in \ker \beta$.

Then $\alpha(b,0,0,\dots) = (0,b,0,\dots) \in I$ but $(b,0,0,\dots) \notin I$, giving the contradiction that $\alpha^{-1}(I) \not\subseteq I$.

Next we turn to injective C*-dynamical systems $(A, \alpha, \mathbb{Z}_+^d)$, so that X_α is regular. Hence the gauge-invariant ideal structure of $\mathcal{NO}_{X_\alpha}$ falls under the purview of Corollary 5.2.3.

Corollary 5.3.14. *Let $(A, \alpha, \mathbb{Z}_+^d)$ be an injective C*-dynamical system. Then the association*

$$I \mapsto \mathcal{L}_I, \text{ where } \mathcal{L}_{I,\emptyset} := I \text{ and } \mathcal{L}_{I,F} := A \text{ for all } \emptyset \neq F \subseteq [d] \tag{5.8}$$

defines a bijection between the set of ideals $I \subseteq A$ satisfying $\alpha_{\underline{n}}(I) \subseteq I$ and $\alpha_{\underline{n}}^{-1}(I) \subseteq I$ for all $\underline{n} \in \mathbb{Z}_+^d$ and the set of NO-2^d-tuples of X_α, which in turn induces a bijection with the set of gauge-invariant ideals of $\mathcal{NO}_{X_\alpha}$.

Proof. Note that an ideal $I \subseteq A$ is positively and negatively invariant for X_α if and only if

$$I \subseteq \bigcap_{\underline{n} \in \mathbb{Z}_+^d} \alpha_{\underline{n}}^{-1}(I) \quad \text{and} \quad \bigcup_{i \in [d]} \alpha_{\underline{i}}^{-1}(I) \subseteq I. \tag{5.9}$$

Indeed, the reverse implication follows by observing that

$$\bigcup_{i \in [d]} \alpha_{\underline{i}}^{-1}(I) \subseteq I \iff \alpha_{\underline{n}}^{-1}(I) \subseteq I \text{ for all } \underline{n} \in \mathbb{Z}_+^d.$$

To see the forward implication of the latter, assume that $\bigcup_{i \in [d]} \alpha_{\underline{i}}^{-1}(I) \subseteq I$ and fix $\underline{n} = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$ and $a \in \alpha_{\underline{n}}^{-1}(I)$. Then we have that

$$\alpha_{\underline{n}}(a) = (\alpha_{\underline{1}}^{n_1} \alpha_{\underline{2}}^{n_2} \dots \alpha_{\underline{d}}^{n_d})(a) = \alpha_{\underline{1}}[(\alpha_{\underline{1}}^{n_1 - 1} \alpha_{\underline{2}}^{n_2} \dots \alpha_{\underline{d}}^{n_d})(a)] \in I,$$

using that α is a semigroup homomorphism in the first equality. Thus we have that

$$(\alpha_{\underline{1}}^{n_1 - 1} \alpha_{\underline{2}}^{n_2} \dots \alpha_{\underline{d}}^{n_d})(a) \in \alpha_{\underline{1}}^{-1}(I) \subseteq I,$$

using the assumption in the inclusion. Now we repeat the preceding argument with the element $(\alpha_{\underline{1}}^{n_1 - 1} \alpha_{\underline{2}}^{n_2} \dots \alpha_{\underline{d}}^{n_d})(a)$ in place of $\alpha_{\underline{n}}(a)$. We eliminate each occurrence of $\alpha_{\underline{1}}$ iteratively and then argue in the same way for the occurrences of $\alpha_{\underline{2}}$, now using that $\alpha_{\underline{2}}^{-1}(I) \subseteq I$. Extending this argument in the natural way, we eventually obtain that $a \in I$, as required.

Consequently, an ideal I satisfies (5.9) if and only if $\alpha_{\underline{n}}(I) \subseteq I$ and $\alpha_{\underline{n}}^{-1}(I) \subseteq I$ for all $\underline{n} \in \mathbb{Z}_+^d$. Since X_α is regular, the claim now follows from Corollary 5.2.3. □

Now let $(A, \alpha, \mathbb{Z}_+^d)$ be an automorphic C*-dynamical system. Then we can canonically extend α to a group homomorphism

$$\alpha \colon \mathbb{Z}^d \to \operatorname{Aut}(A); \underline{s} - \underline{t} \mapsto \alpha_{\underline{s} - \underline{t}} := \alpha_{\underline{s}} \circ \alpha_{\underline{t}}^{-1} \text{ for all } \underline{s}, \underline{t} \in \mathbb{Z}_+^d.$$

The fact that α is well-defined follows from the fact that $\mathbb{Z}_+^d \subseteq \mathbb{Z}^d$ is a spanning cone [15, p. 15]. In turn, we can consider the crossed product C*-algebra $A \rtimes_\alpha \mathbb{Z}^d$. We have that $\mathcal{NO}_{X_\alpha} \cong A \rtimes_\alpha \mathbb{Z}^d$ by an equivariant $*$-isomorphism, which thus preserves the lattice structure of the gauge-invariant ideals. Accordingly, Corollary 5.3.14 recovers the following well-known result from the theory of crossed products.

Corollary 5.3.15. *Let $(A, \alpha, \mathbb{Z}_+^d)$ be an automorphic C*-dynamical system. Then the bijection of Corollary 5.3.14 induces a lattice isomorphism between the set of α-invariant ideals of A and the set of gauge-invariant ideals of $A \rtimes_\alpha \mathbb{Z}^d$.*

Proof. Notice that in the automorphic case, the ideals $I \subseteq A$ satisfying $\alpha_{\underline{n}}(I) \subseteq I$ and $\alpha_{\underline{n}}^{-1}(I) \subseteq I$ for all $\underline{n} \in \mathbb{Z}_+^d$ are exactly the ideals $I \subseteq A$ satisfying $\alpha_{\underline{n}}(I) = I$ for all $\underline{n} \in \mathbb{Z}_+^d$. The set of all such ideals carries the usual lattice structure. Consider the mapping

$$I \mapsto \mathcal{L}_I, \text{ where } \mathcal{L}_{I,\emptyset} := I \text{ and } \mathcal{L}_{I,F} := A \text{ for all } \emptyset \neq F \subseteq [d],$$

from Corollary 5.3.14. In view of the latter, Corollary 4.2.12 and the comments preceding the statement, it suffices to show that if $I, J \subseteq A$ are α-invariant ideals, then

$$\mathcal{L}_{I\cap J} = \mathcal{L}_I \wedge \mathcal{L}_J \quad \text{and} \quad \mathcal{L}_{I+J} = \mathcal{L}_I \vee \mathcal{L}_J.$$

For the operation $\wedge$, we have that

$$\mathcal{L}_{I\cap J,\emptyset} = I \cap J = \mathcal{L}_{I,\emptyset} \cap \mathcal{L}_{J,\emptyset} = (\mathcal{L}_I \wedge \mathcal{L}_J)_\emptyset$$

and that

$$\mathcal{L}_{I\cap J,F} = A = A \cap A = \mathcal{L}_{I,F} \cap \mathcal{L}_{J,F} = (\mathcal{L}_I \wedge \mathcal{L}_J)_F \text{ for all } \emptyset \neq F \subseteq [d],$$

using Proposition 4.2.6 in the final equality in both cases. Hence $\mathcal{L}_{I\cap J} = \mathcal{L}_I \wedge \mathcal{L}_J$, as required.

For the operation $\vee$, first fix $\emptyset \neq F \subseteq [d]$. Since $\mathcal{L}_I \vee \mathcal{L}_J$ is an NO-2^d-tuple of the regular product system X_α, we have that

$$\mathcal{I}_F = A \subseteq (\mathcal{L}_I \vee \mathcal{L}_J)_F \subseteq A.$$

By definition of $\mathcal{L}_{I+J}$, we obtain that

$$\mathcal{L}_{I+J,F} = A = (\mathcal{L}_I \vee \mathcal{L}_J)_F,$$

as required. It remains to check that $\mathcal{L}_{I+J,\emptyset} = (\mathcal{L}_I \vee \mathcal{L}_J)_\emptyset$. By Proposition 4.2.7, this amounts to showing that

$$I + J = \pi_{X_\alpha}^{-1}(\mathfrak{J}^{\mathcal{L}_I} + \mathfrak{J}^{\mathcal{L}_J}).$$

The forward inclusion is immediate. For the reverse inclusion, we write $\mathcal{NO}_{X_\alpha} = \mathrm{C}^*(\pi, U)$

for a CNP-covariant pair (π, U) where π is injective and U is a unitary representation, and let $Q\colon \mathcal{NT}_{X_\alpha} \to \mathcal{NO}_{X_\alpha}$ denote the quotient map. Since $[X_\alpha]_I$ is regular by item (ii) of Proposition 5.2.1, an application of item (iii) of Corollary 5.1.7 gives that

$$Q(\mathfrak{J}^{\mathcal{L}_I}) = \overline{\operatorname{span}}\{U_{\underline{n}}\pi(I)U_{\underline{m}}^* \mid \underline{n}, \underline{m} \in \mathbb{Z}_+^d\},$$

and likewise for $Q(\mathfrak{J}^{\mathcal{L}_J})$. In turn, we have that

$$Q(\mathfrak{J}^{\mathcal{L}_I} + \mathfrak{J}^{\mathcal{L}_J}) = \overline{\operatorname{span}}\{U_{\underline{n}}\pi(I+J)U_{\underline{m}}^* \mid \underline{n}, \underline{m} \in \mathbb{Z}_+^d\}.$$

Let β denote the gauge action of $(\pi_{X_\alpha}^{\mathcal{I}}, t_{X_\alpha}^{\mathcal{I}})$ and let $E_\beta\colon \mathcal{NO}_{X_\alpha} \to \mathcal{NO}_{X_\alpha}^\beta$ denote the associated faithful conditional expectation. We obtain that

$$\begin{aligned} E_\beta(Q(\mathfrak{J}^{\mathcal{L}_I} + \mathfrak{J}^{\mathcal{L}_J})) &= \overline{\operatorname{span}}\{U_{\underline{n}}\pi(I+J)U_{\underline{n}}^* \mid \underline{n} \in \mathbb{Z}_+^d\} \\ &= \overline{\operatorname{span}}\{(\pi \circ \alpha_{\underline{n}}^{-1})(I+J) \mid \underline{n} \in \mathbb{Z}_+^d\} = \pi(I+J), \end{aligned}$$

using α-invariance of I and J together with (5.5). Therefore, if $a \in \overline{\pi}_{X_\alpha}^{-1}(\mathfrak{J}^{\mathcal{L}_I} + \mathfrak{J}^{\mathcal{L}_J})$ then

$$\pi(a) = Q(\overline{\pi}_{X_\alpha}(a)) \in E_\beta(Q(\mathfrak{J}^{\mathcal{L}_I} + \mathfrak{J}^{\mathcal{L}_J})) = \pi(I+J).$$

Injectivity of π then gives that $a \in I + J$, as required. Thus we have that $\mathcal{L}_{I+J} = \mathcal{L}_I \vee \mathcal{L}_J$ and the proof is complete. □

In the automorphic case, we also have that

$$(A \rtimes_\alpha \mathbb{Z}^d)/(I \rtimes_{\alpha|_I} \mathbb{Z}^d) \cong A/I \rtimes_{[\alpha]_I} \mathbb{Z}^d$$

whenever $I \subseteq A$ is an ideal satisfying $\alpha_{\underline{n}}(I) = I$ for all $\underline{n} \in \mathbb{Z}_+^d$. This is in accordance with Corollary 5.1.7, i.e.,

$$\mathcal{NO}_{X_\alpha}/\left\langle Q_{\mathcal{I}}(\overline{\pi}_{X_\alpha}(I))\right\rangle \cong \mathcal{NO}_{X_{[\alpha]_I}}.$$

5.4 Higher-rank graphs

In this section we interpret the parametrisation in the case of strong finitely aligned higher-rank graphs, which also accounts for row-finite higher-rank graphs. The parametrisation we offer is in terms of vertex sets. As a corollary, we recover the parametrisation of Raeburn, Sims and Yeend [50, Theorem 5.2] for locally convex row-finite higher-rank graphs. The concepts from the theory of higher-rank graphs that we utilise are taken from [49, 50, 51, 55]. For this section, we will reserve d for the degree map of a graph (Λ, d) of rank k.

Fix $k \in \mathbb{N}$. A *k-graph* (Λ, d) consists of a small category $\Lambda = (\mathrm{Obj}(\Lambda), \mathrm{Mor}(\Lambda), r, s)$ which is countable, together with a functor $d\colon \Lambda \to \mathbb{Z}_+^k$ (called the *degree map*) which satisfies the *factorisation property*:

> For all $\lambda \in \mathrm{Mor}(\Lambda)$ and $\underline{m}, \underline{n} \in \mathbb{Z}_+^k$ such that $d(\lambda) = \underline{m} + \underline{n}$, there exist unique $\mu, \nu \in \mathrm{Mor}(\Lambda)$ such that $d(\mu) = \underline{m}$, $d(\nu) = \underline{n}$ and $\lambda = \mu\nu$.

Here we view $\mathbb{Z}_+^k$ as a category consisting of a single object, and whose morphisms are exactly the elements of $\mathbb{Z}_+^k$ (when viewed as a set). Composition in this category is given by entrywise addition, and the identity morphism is $\underline{0}$. Therefore d being a functor means that $d(\lambda\mu) = d(\lambda) + d(\mu)$ and $d(\mathrm{id}_v) = \underline{0}$ for all $\lambda, \mu \in \mathrm{Mor}(\Lambda)$ satisfying $r(\mu) = s(\lambda)$ and $v \in \mathrm{Obj}(\Lambda)$. We view k-graphs as generalised graphs, and therefore refer to the elements of $\mathrm{Obj}(\Lambda)$ as *vertices* and the elements of $\mathrm{Mor}(\Lambda)$ as *paths*. Fixing $\lambda \in \mathrm{Mor}(\Lambda)$, the factorisation property guarantees that $d(\lambda) = \underline{0}$ if and only if $\lambda = \mathrm{id}_{s(\lambda)}$. Hence we may identify $\mathrm{Obj}(\Lambda)$ with $\{\lambda \in \mathrm{Mor}(\Lambda) \mid d(\lambda) = \underline{0}\}$, and consequently we may write $\lambda \in \Lambda$ instead of $\lambda \in \mathrm{Mor}(\Lambda)$ without any ambiguity.

Fix a k-graph (Λ, d). Given $\lambda \in \Lambda$ and $E \subseteq \Lambda$, we define

$$\lambda E := \{\lambda\mu \in \Lambda \mid \mu \in E, r(\mu) = s(\lambda)\} \quad \text{and} \quad E\lambda := \{\mu\lambda \in \Lambda \mid \mu \in E, r(\lambda) = s(\mu)\}.$$

In particular, we may replace λ by a vertex $v \in \Lambda$ and write

$$vE := \{\lambda \in E \mid r(\lambda) = v\} \quad \text{and} \quad Ev := \{\lambda \in E \mid s(\lambda) = v\}.$$

Analogously, given $E, F \subseteq \Lambda$, we define

$$EF := \{\lambda\mu \in \Lambda \mid \lambda \in E, \mu \in F, r(\mu) = s(\lambda)\}.$$

Fixing $\underline{n} \in \mathbb{Z}_+^k$, we set

$$\Lambda^{\underline{n}} := \{\lambda \in \Lambda \mid d(\lambda) = \underline{n}\} \text{ and } \Lambda^{\leq \underline{n}} := \{\lambda \in \Lambda \mid d(\lambda) \leq \underline{n} \text{ and } s(\lambda)\Lambda^{\underline{i}} = \emptyset \text{ if } d(\lambda) + \underline{i} \leq \underline{n}\}.$$

We will refer to the elements of $\Lambda^{\underline{i}}$ as *edges* for all $i \in [k]$. The following proposition collects some useful properties of the sets $v\Lambda^{\leq \underline{n}}$.

Proposition 5.4.1. *Let (Λ, d) be a k-graph. Then the following hold:*

(i) Fixing $v \in \Lambda^{\underline{0}}$ and $\underline{n} \in \mathbb{Z}_+^k$, we have that $v\Lambda^{\leq \underline{n}} \neq \emptyset$.

(ii) Fixing $v \in \Lambda^{\underline{0}}$ and $i \in [k]$, we have that

$$v\Lambda^{\leq \underline{i}} = \begin{cases} v\Lambda^{\underline{i}} & \text{if } v\Lambda^{\underline{i}} \neq \emptyset, \\ \{v\} & \text{otherwise.} \end{cases}$$

Proof. (i) We construct an element of $v\Lambda^{\leq \underline{n}}$ inductively. Start by considering $v \in \Lambda^{\underline{0}}$. If $v\Lambda^{\underline{i}} = \emptyset$ whenever $i \in [k]$ and $\underline{i} \leq \underline{n}$, then $v \in v\Lambda^{\leq \underline{n}}$ and we are done. Otherwise, suppose that there exists $i \in [k]$ such that $v\Lambda^{\underline{i}} \neq \emptyset$ and $\underline{i} \leq \underline{n}$. Take $\lambda \in v\Lambda^{\underline{i}}$ and note that $r(\lambda) = v$ and $d(\lambda) = \underline{i} \leq \underline{n}$ by construction. If $s(\lambda)\Lambda^{\underline{j}} = \emptyset$ whenever $j \in [k]$ and $d(\lambda) + \underline{j} = \underline{i} + \underline{j} \leq \underline{n}$, then $\lambda \in v\Lambda^{\leq \underline{n}}$ and we are done. Otherwise, suppose that there exists $j \in [k]$ such that $s(\lambda)\Lambda^{\underline{j}} \neq \emptyset$ and $\underline{i} + \underline{j} \leq \underline{n}$. Take $\mu \in s(\lambda)\Lambda^{\underline{j}}$ and consider the path $\lambda\mu \in \Lambda$. Note that

$$r(\lambda\mu) = r(\lambda) = v \quad \text{and} \quad d(\lambda\mu) = d(\lambda) + d(\mu) = \underline{i} + \underline{j} \leq \underline{n}.$$

Now argue by cases as before with respect to $\lambda\mu$. Proceeding inductively, at each step we append an edge to the path from the previous step. The resulting path has range v, degree at most $\underline{n}$, and the length of the degree is one greater than that of the path from the previous step. Since $|\underline{n}| < \infty$, eventually we obtain a path $\nu \in v\Lambda$ such that $d(\nu) \leq \underline{n}$ and $|d(\nu)| = |\underline{n}|$. We then necessarily have that $\nu \in v\Lambda^{\underline{n}} \subseteq v\Lambda^{\leq \underline{n}}$. By induction we conclude that $v\Lambda^{\leq \underline{n}} \neq \emptyset$, as required.

(ii) First suppose that $v\Lambda^{\underline{i}} \neq \emptyset$. It is clear that $v\Lambda^{\underline{i}} \subseteq v\Lambda^{\leq \underline{i}}$, so fix $\lambda \in v\Lambda^{\leq \underline{i}}$. Then in particular $r(\lambda) = v$ and $d(\lambda) \leq \underline{i}$. The latter ensures that either $d(\lambda) = \underline{0}$ or $d(\lambda) = \underline{i}$. Towards contradiction, suppose that $d(\lambda) = \underline{0}$. Since λ has range v, we must have that $\lambda = v$. Thus $s(\lambda)\Lambda^{\underline{i}} = v\Lambda^{\underline{i}} \neq \emptyset$ by assumption, contradicting the membership of λ to $v\Lambda^{\leq \underline{i}}$. Hence $d(\lambda) = \underline{i}$ and so $\lambda \in v\Lambda^{\underline{i}}$. This shows that $v\Lambda^{\leq \underline{i}} = v\Lambda^{\underline{i}}$, as required.

Finally, suppose that $v\Lambda^{\underline{i}} = \emptyset$. Then $v \in v\Lambda^{\leq \underline{i}}$ by definition, and so it remains to check that $v\Lambda^{\leq \underline{i}} \subseteq \{v\}$. Accordingly, take $\lambda \in v\Lambda^{\leq \underline{i}}$. Then in particular $r(\lambda) = v$ and $d(\lambda) \leq \underline{i}$. As before, we have that either $d(\lambda) = \underline{0}$ or $d(\lambda) = \underline{i}$. The latter is impossible since $v\Lambda^{\underline{i}} = \emptyset$ by assumption, so we must have that $d(\lambda) = \underline{0}$. The fact that $r(\lambda) = v$ then ensures that $\lambda = v$, as required. □

Suppose that we have $\underline{\ell}, \underline{m}, \underline{n} \in \mathbb{Z}_+^k$ satisfying $\underline{\ell} \leq \underline{m} \leq \underline{n}$, and $\lambda \in \Lambda$ satisfying $d(\lambda) = \underline{n}$. By the factorisation property, there exist unique paths $\lambda(\underline{0}, \underline{\ell}), \lambda(\underline{\ell}, \underline{m}), \lambda(\underline{m}, \underline{n}) \in \Lambda$ such that

$$d(\lambda(\underline{0}, \underline{\ell})) = \underline{\ell}, d(\lambda(\underline{\ell}, \underline{m})) = \underline{m} - \underline{\ell}, d(\lambda(\underline{m}, \underline{n})) = \underline{n} - \underline{m}, \text{ and } \lambda = \lambda(\underline{0}, \underline{\ell})\lambda(\underline{\ell}, \underline{m})\lambda(\underline{m}, \underline{n}).$$

Fixing $\mu, \nu \in \Lambda$, we define the set of *minimal common extensions* of μ and ν by

$$\mathrm{MCE}(\mu, \nu) := \{\lambda \in \Lambda \mid d(\lambda) = d(\mu) \vee d(\nu), \lambda(\underline{0}, d(\mu)) = \mu, \lambda(\underline{0}, d(\nu)) = \nu\}.$$

Notice that $\mathrm{MCE}(\mu, \nu)$ may be empty, e.g., when $r(\mu) \neq r(\nu)$. Intrinsically related to $\mathrm{MCE}(\mu, \nu)$ is the set

$$\begin{aligned}\Lambda^{\min}(\mu, \nu) &:= \{(\alpha, \beta) \in \Lambda \times \Lambda \mid \mu\alpha = \nu\beta \in \mathrm{MCE}(\mu, \nu)\} \\ &= \{(\alpha, \beta) \in \Lambda \times \Lambda \mid \mu\alpha = \nu\beta, d(\mu\alpha) = d(\mu) \vee d(\nu) = d(\nu\beta)\}.\end{aligned}$$

Note that if $\mu, \nu \in \Lambda$ satisfy $d(\mu) = d(\nu)$, then $\Lambda^{\min}(\mu, \nu)$ being non-empty implies that

$\mu = \nu$. Given a vertex $v \in \Lambda^{\underline{0}}$, a subset $E \subseteq v\Lambda$ is called *exhaustive* if for every $\lambda \in v\Lambda$ there exists $\mu \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$.

A k-graph (Λ, d) is said to be *finitely aligned* if $|\mathrm{MCE}(\mu, \nu)| < \infty$ for all $\mu, \nu \in \Lambda$; equivalently, if $|\Lambda^{\min}(\mu, \nu)| < \infty$ for all $\mu, \nu \in \Lambda$. We say that (Λ, d) is *strong finitely aligned* if (Λ, d) is finitely aligned and for every $\lambda \in \Lambda$ and $i \in [k]$ satisfying $d(\lambda) \perp \underline{i}$, there are at most finitely many edges $e \in \Lambda^{\underline{i}}$ such that $\Lambda^{\min}(\lambda, e) \neq \emptyset$, see [17, Definition 7.2]. A k-graph (Λ, d) is said to be *row-finite* if $|v\Lambda^{\underline{n}}| < \infty$ for all $v \in \Lambda^{\underline{0}}$ and $\underline{n} \in \mathbb{Z}_+^k$. Row-finite k-graphs are in particular strong finitely aligned.

We say that (Λ, d) is *locally convex* if, for all $v \in \Lambda^{\underline{0}}, i, j \in [k]$ satisfying $i \neq j, \lambda \in v\Lambda^{\underline{i}}$ and $\mu \in v\Lambda^{\underline{j}}$, we have that $s(\lambda)\Lambda^{\underline{j}}$ and $s(\mu)\Lambda^{\underline{i}}$ are non-empty. Finally, we say that (Λ, d) is *sourceless* if $v\Lambda^{\underline{n}} \neq \emptyset$ for all $v \in \Lambda^{\underline{0}}$ and $\underline{n} \in \mathbb{Z}_+^k$. Any sourceless k-graph is automatically locally convex.

Let (Λ, d) be a finitely aligned k-graph. A set of partial isometries $\{T_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{B}(H)$ is called a *Toeplitz-Cuntz-Krieger* Λ*-family* if the following hold:

(TCK1) $\{T_v\}_{v \in \Lambda^{\underline{0}}}$ is a collection of pairwise orthogonal projections;

(TCK2) $T_\lambda T_\mu = \delta_{s(\lambda), r(\mu)} T_{\lambda\mu}$ for all $\lambda, \mu \in \Lambda$;

(TCK3) $T_\lambda^* T_\mu = \sum_{(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu)} T_\alpha T_\beta^*$ for all $\lambda, \mu \in \Lambda$.

A Toeplitz-Cuntz-Krieger Λ-family $\{T_\lambda\}_{\lambda \in \Lambda}$ is called a *Cuntz-Krieger* Λ*-family* if it satisfies:

(CK) $\prod_{\lambda \in E}(T_v - T_\lambda T_\lambda^*) = 0$ for every $v \in \Lambda^{\underline{0}}$ and all non-empty finite exhaustive sets $E \subseteq v\Lambda$.

Note that $T_\lambda^* T_\mu = 0$ whenever $\lambda, \mu \in \Lambda$ satisfy $\Lambda^{\min}(\lambda, \mu) = \emptyset$ by (TCK3). Likewise, (TCK3) implies that $T_\lambda^* T_\lambda = T_{s(\lambda)}$ for all $\lambda \in \Lambda$. The C*-algebra $\mathrm{C}^*(\Lambda)$ is the universal one with respect to the Cuntz-Krieger Λ-families, and satisfies a Gauge-Invariant Uniqueness Theorem [51, Theorem 4.2].

Every k-graph (Λ, d) can be canonically associated with a product system $X(\Lambda)$ over $\mathbb{Z}_+^k$ with coefficients in $c_0(\Lambda^{\underline{0}})$, where we view $\Lambda^{\underline{0}}$ as a discrete space. We present the fundamentals of this construction.

Proposition 5.4.2. *Let (Λ, d) be a k-graph and fix $\underline{n} \in \mathbb{Z}_+^k$. Then the linear space $c_{00}(\Lambda^{\underline{n}})$ of finitely supported complex functions on $\Lambda^{\underline{n}}$ carries a canonical structure as an inner-product $c_0(\Lambda^{\underline{0}})$-module, implemented by the operations*

$$\langle \xi_{\underline{n}}, \eta_{\underline{n}} \rangle (v) = \sum_{s(\lambda)=v} \overline{\xi_{\underline{n}}(\lambda)} \eta_{\underline{n}}(\lambda) \quad \textit{and} \quad (\xi_{\underline{n}} a)(\lambda) = \xi_{\underline{n}}(\lambda) a(s(\lambda)),$$

for all $\xi_{\underline{n}}, \eta_{\underline{n}} \in c_{00}(\Lambda^{\underline{n}}), a \in c_0(\Lambda^{\underline{0}}), v \in \Lambda^{\underline{0}}$ and $\lambda \in \Lambda^{\underline{n}}$. Additionally, denoting the right Hilbert C-module completion of $c_{00}(\Lambda^{\underline{0}})$ by $X_{\underline{n}}(\Lambda)$, there exists a unique $*$-homomorphism*

$\phi_{\underline{n}} \colon c_0(\Lambda^{\underline{0}}) \to \mathcal{L}(X_{\underline{n}}(\Lambda))$ *satisfying*

$$(\phi_{\underline{n}}(a)\xi_{\underline{n}})(\lambda) = a(r(\lambda))\xi_{\underline{n}}(\lambda),$$

for all $a \in c_0(\Lambda^{\underline{0}}), \xi_{\underline{n}} \in c_{00}(\Lambda^{\underline{n}})$ *and* $\lambda \in \Lambda^{\underline{n}}$. *Hence* $X_{\underline{n}}(\Lambda)$ *carries the structure of a* C^**-correspondence over* $c_0(\Lambda^{\underline{0}})$.

Proof. The first claim is well known, e.g., [25, Example 1.2]. For the second claim, take $a \in c_0(\Lambda^{\underline{0}})$ and define the map $\phi_{\underline{n}}(a)$ by

$$\phi_{\underline{n}}(a) \colon c_{00}(\Lambda^{\underline{n}}) \to X_{\underline{n}}(\Lambda); (\phi_{\underline{n}}(a)\xi_{\underline{n}})(\lambda) = a(r(\lambda))\xi_{\underline{n}}(\lambda) \text{ for all } \xi_{\underline{n}} \in c_{00}(\Lambda^{\underline{n}}), \lambda \in \Lambda^{\underline{n}}.$$

It is routine to check that $\phi_{\underline{n}}(a)$ is well-defined and linear. To see that $\phi_{\underline{n}}(a)$ is bounded with respect to the inner product norm, fix $\xi_{\underline{n}} \in c_{00}(\Lambda^{\underline{n}})$. We have that

$$\begin{aligned}\|\phi_{\underline{n}}(a)\xi_{\underline{n}}\|^2 &= \| \langle \phi_{\underline{n}}(a)\xi_{\underline{n}}, \phi_{\underline{n}}(a)\xi_{\underline{n}} \rangle \| = \sup_{v \in \Lambda^{\underline{0}}} |\langle \phi_{\underline{n}}(a)\xi_{\underline{n}}, \phi_{\underline{n}}(a)\xi_{\underline{n}} \rangle (v)| \\ &= \sup_{v \in \Lambda^{\underline{0}}} | \sum_{s(\lambda)=v} |a(r(\lambda))\xi_{\underline{n}}(\lambda)|^2 | = \sup_{v \in \Lambda^{\underline{0}}} \sum_{s(\lambda)=v} |a(r(\lambda))\xi_{\underline{n}}(\lambda)|^2.\end{aligned}$$

Fix $v \in \Lambda^{\underline{0}}$ and observe that

$$\sum_{s(\lambda)=v} |a(r(\lambda))\xi_{\underline{n}}(\lambda)|^2 = \sum_{s(\lambda)=v} |a(r(\lambda))|^2 \cdot |\xi_{\underline{n}}(\lambda)|^2 \leq \|a\|^2 \sum_{s(\lambda)=v} |\xi_{\underline{n}}(\lambda)|^2 \leq \|a\|^2 \cdot \|\xi_{\underline{n}}\|^2.$$

It follows that

$$\|\phi_{\underline{n}}(a)\xi_{\underline{n}}\|^2 \leq \|a\|^2 \cdot \|\xi_{\underline{n}}\|^2$$

and thus $\phi_{\underline{n}}(a)$ is bounded. Hence $\phi_{\underline{n}}(a)$ extends to a unique bounded linear operator (which we also denote by $\phi_{\underline{n}}(a)$) on $X_{\underline{n}}(\Lambda)$. To see that $\phi_{\underline{n}}(a)$ is adjointable, fix $\xi_{\underline{n}}, \eta_{\underline{n}} \in c_{00}(\Lambda^{\underline{n}})$ and $v \in \Lambda^{\underline{0}}$. We obtain that

$$\begin{aligned}\langle \phi_{\underline{n}}(a)\xi_{\underline{n}}, \eta_{\underline{n}} \rangle (v) &= \sum_{s(\lambda)=v} \overline{a(r(\lambda))\xi_{\underline{n}}(\lambda)}\eta_{\underline{n}}(\lambda) \\ &= \sum_{s(\lambda)=v} \overline{\xi_{\underline{n}}(\lambda)} \cdot \overline{a(r(\lambda))}\eta_{\underline{n}}(\lambda) \\ &= \langle \xi_{\underline{n}}, \phi_{\underline{n}}(a^*)\eta_{\underline{n}} \rangle (v),\end{aligned}$$

and so $\langle \phi_{\underline{n}}(a)\xi_{\underline{n}}, \eta_{\underline{n}} \rangle = \langle \xi_{\underline{n}}, \phi_{\underline{n}}(a^*)\eta_{\underline{n}} \rangle$. Since $c_{00}(\Lambda^{\underline{n}})$ is dense in $X_{\underline{n}}(\Lambda)$, we deduce that $\phi_{\underline{n}}(a)^* = \phi_{\underline{n}}(a^*)$. Thus the map $\phi_{\underline{n}}$ of the statement is well-defined and $*$-preserving. It is routine to check that $\phi_{\underline{n}}$ is an algebra homomorphism, and uniqueness follows from the fact that $\phi_{\underline{n}}(a)$ is continuous and therefore determined by its action on $c_{00}(\Lambda^{\underline{n}})$ for all $a \in c_0(\Lambda^{\underline{0}})$. This completes the proof. □

We will make liberal use of the notation in Proposition 5.4.2 throughout the section. Notice that $X_{\underline{0}}(\Lambda) = c_0(\Lambda^{\underline{0}})$ carries the usual structure as a C*-correspondence over itself.

For each $\underline{n} \in \mathbb{Z}_+^k$ and $\lambda \in \Lambda^{\underline{n}}$, we write δ_λ for the element of $c_{00}(\Lambda^{\underline{n}})$ determined by

$$\delta_\lambda(\mu) = \begin{cases} 1 & \text{if } \mu = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\{\delta_\mu \mid \mu \in \Lambda^{\underline{n}}\}$ constitutes a basis of $c_{00}(\Lambda^{\underline{n}})$, and when $\underline{n} = \underline{0}$ the element $\delta_\lambda \in c_0(\Lambda^{\underline{0}})$ is a projection. Density of $c_{00}(\Lambda^{\underline{n}})$ in $X_{\underline{n}}(\Lambda)$ ensures that many arguments involving elements of $X_{\underline{n}}(\Lambda)$ reduce to arguments in terms of point masses. Accordingly, fixing $\lambda, \mu \in \Lambda^{\underline{n}}$ and $v \in \Lambda^{\underline{0}}$, we record three point mass properties that will be used throughout the section:

$$\delta_\lambda\delta_v = \begin{cases} \delta_\lambda & \text{if } s(\lambda) = v, \\ 0 & \text{otherwise,} \end{cases} \text{and } \delta_v\delta_\lambda = \begin{cases} \delta_\lambda & \text{if } r(\lambda) = v, \\ 0 & \text{otherwise,} \end{cases} \text{and } \langle \delta_\lambda, \delta_\mu \rangle = \begin{cases} \delta_{s(\lambda)} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

For $\underline{n} \in \mathbb{Z}_+^k$ and $\lambda, \mu, \nu \in \Lambda^{\underline{n}}$, we also have that

$$\Theta_{\delta_\lambda, \delta_\mu}^{X_{\underline{n}}(\Lambda)}(\delta_\nu) = \delta_\lambda \langle \delta_\mu, \delta_\nu \rangle = \begin{cases} \delta_\lambda\delta_{s(\mu)} & \text{if } \nu = \mu, \\ 0 & \text{otherwise.} \end{cases} \tag{5.10}$$

In turn, the map $\Theta_{\delta_\lambda, \delta_\mu}^{X_{\underline{n}}(\Lambda)}$ is non-zero if and only if $s(\lambda) = s(\mu)$, and we obtain that

$$\mathcal{K}(X_{\underline{n}}(\Lambda)) = \overline{\operatorname{span}}\{\Theta_{\delta_\lambda, \delta_\mu}^{X_{\underline{n}}(\Lambda)} \mid \lambda, \mu \in \Lambda^{\underline{n}}, s(\lambda) = s(\mu)\}. \tag{5.11}$$

We utilise this characterisation in the proof of the following proposition.

Proposition 5.4.3. *Let (Λ, d) be a k-graph. Fix $\underline{n} \in \mathbb{Z}_+^k$ and $v \in \Lambda^{\underline{0}}$. Then the following hold:*

(i) $\phi_{\underline{n}}(\delta_v) = 0$ if and only if $v\Lambda^{\underline{n}} = \emptyset$.

(ii) $\phi_{\underline{n}}(\delta_v) \in \mathcal{K}(X_{\underline{n}}(\Lambda))$ if and only if $|v\Lambda^{\underline{n}}| < \infty$.

Proof. (i) First assume that $\phi_{\underline{n}}(\delta_v) = 0$. Suppose that $v\Lambda^{\underline{n}} \neq \emptyset$ and take $\lambda \in v\Lambda^{\underline{n}}$. Then we obtain that

$$\phi_{\underline{n}}(\delta_v)\delta_\lambda = \delta_\lambda \neq 0,$$

a contradiction. Hence we must have that $v\Lambda^{\underline{n}} = \emptyset$, as required.

Now assume that $v\Lambda^{\underline{n}} = \emptyset$. To show that $\phi_{\underline{n}}(\delta_v) = 0$, it suffices to show that $\phi_{\underline{n}}(\delta_v)\delta_\lambda = 0$ for all $\lambda \in \Lambda^{\underline{n}}$, since $\phi_{\underline{n}}(\delta_v)$ is in particular linear and bounded. Accordingly, fix $\lambda \in \Lambda^{\underline{n}}$. Then we have that

$$\phi_{\underline{n}}(\delta_v)\delta_\lambda = \begin{cases} \delta_\lambda & \text{if } r(\lambda) = v, \\ 0 & \text{otherwise.} \end{cases}$$

By assumption we have that $v\Lambda^{\underline{n}} = \emptyset$ and therefore $\phi_{\underline{n}}(\delta_v)\delta_\lambda = 0$ for all $\lambda \in \Lambda^{\underline{n}}$, as required. This completes the proof of item (i).

(ii) First assume that $|v\Lambda^{\underline{n}}| < \infty$. If $v\Lambda^{\underline{n}} = \emptyset$, then $\phi_{\underline{n}}(\delta_v) = 0 \in \mathcal{K}(X_{\underline{n}}(\Lambda))$ by item (i), as required. So assume that $v\Lambda^{\underline{n}} \neq \emptyset$. Then we have that

$$\sum_{\lambda \in v\Lambda^{\underline{n}}} \Theta_{\delta_\lambda,\delta_\lambda} \in \mathcal{K}(X_{\underline{n}}(\Lambda)),$$

noting that the sum is finite by assumption. We claim that

$$\phi_{\underline{n}}(\delta_v) = \sum_{\lambda \in v\Lambda^{\underline{n}}} \Theta_{\delta_\lambda,\delta_\lambda}.$$

Since $\phi_{\underline{n}}(\delta_v)$ and $\sum_{\lambda \in v\Lambda^{\underline{n}}} \Theta_{\delta_\lambda,\delta_\lambda}$ are in particular linear and bounded, it suffices to show that the equality holds on δ_μ for arbitrary $\mu \in \Lambda^{\underline{n}}$. We obtain that

$$\sum_{\lambda \in v\Lambda^{\underline{n}}} \Theta_{\delta_\lambda,\delta_\lambda}(\delta_\mu) = \sum_{\lambda \in v\Lambda^{\underline{n}}} \delta_\lambda \langle \delta_\lambda, \delta_\mu \rangle = \begin{cases} \delta_\mu \langle \delta_\mu, \delta_\mu \rangle = \delta_\mu \delta_{s(\mu)} = \delta_\mu & \text{if } r(\mu) = v, \\ 0 & \text{otherwise,} \end{cases}$$

by (5.10). Thus we have that $\phi_{\underline{n}}(\delta_v)\delta_\mu = \sum_{\lambda \in v\Lambda^{\underline{n}}} \Theta_{\delta_\lambda,\delta_\lambda}(\delta_\mu)$ for all $\mu \in \Lambda^{\underline{n}}$, as required. In turn, we deduce that $\phi_{\underline{n}}(\delta_v) \in \mathcal{K}(X_{\underline{n}}(\Lambda))$, as required.

Finally, we prove the forward implication by contraposition. Assume that $|v\Lambda^{\underline{n}}| = \infty$, and suppose towards contradiction that $\phi_{\underline{n}}(\delta_v) \in \mathcal{K}(X_{\underline{n}}(\Lambda))$. By (5.11), we can find

$$k_{\underline{n}} \in \operatorname{span}\{\Theta_{\delta_\lambda,\delta_\mu} \mid \lambda, \mu \in \Lambda^{\underline{n}}, s(\lambda) = s(\mu)\}$$

such that $\|\phi_{\underline{n}}(\delta_v) - k_{\underline{n}}\| < 1/2$. We may write

$$k_{\underline{n}} = \sum_{j=1}^{N} c_j \Theta_{\delta_{\lambda_j},\delta_{\mu_j}}$$

for some $N \in \mathbb{N}, c_j \in \mathbb{C}$ and $\lambda_j, \mu_j \in \Lambda^{\underline{n}}$ satisfying $s(\lambda_j) = s(\mu_j)$ for all $j \in [N]$. Since $|v\Lambda^{\underline{n}}| = \infty$, we may choose a path $\nu \in v\Lambda^{\underline{n}}$ that does not coincide with λ_j or μ_j for any $j \in [N]$. Hence we have that $\phi_{\underline{n}}(\delta_v)\delta_\nu = \delta_\nu$ and that $k_{\underline{n}}\delta_\nu = 0$. Note also that

$$\|\delta_\nu\| = \| \langle \delta_\nu, \delta_\nu \rangle \|^{\frac{1}{2}} = \|\delta_{s(\nu)}\|^{\frac{1}{2}} = 1.$$

In total, we obtain that

$$\begin{aligned} 1/2 > \|\phi_{\underline{n}}(\delta_v) - k_{\underline{n}}\| &= \sup\{\|\phi_{\underline{n}}(\delta_v)\xi_{\underline{n}} - k_{\underline{n}}\xi_{\underline{n}}\| \mid \xi_{\underline{n}} \in X_{\underline{n}}(\Lambda), \|\xi_{\underline{n}}\| \leq 1\} \\ &\geq \|\phi_{\underline{n}}(\delta_v)\delta_\nu - k_{\underline{n}}\delta_\nu\| = \|\delta_\nu\| = 1, \end{aligned}$$

a contradiction. Hence $\phi_{\underline{n}}(\delta_v) \notin \mathcal{K}(X_{\underline{n}}(\Lambda))$, and by contraposition this finishes the proof. □

We impose a product system structure on $X(\Lambda) := \{X_{\underline{n}}(\Lambda)\}_{\underline{n} \in \mathbb{Z}_+^k}$ as follows.

Proposition 5.4.4. *Let (Λ, d) be a k-graph. Set $X(\Lambda) := \{X_{\underline{n}}(\Lambda)\}_{\underline{n}\in\mathbb{Z}_+^k}$, where $X_{\underline{n}}(\Lambda)$ is defined as in Proposition 5.4.2 for all $\underline{n} \in \mathbb{Z}_+^k$. Then $X(\Lambda)$ carries a canonical structure as a product system over $\mathbb{Z}_+^k$ with coefficients in $c_0(\Lambda^{\underline{0}})$, given by the multiplication maps*

$$u_{\underline{n},\underline{m}}\colon X_{\underline{n}}(\Lambda) \otimes_{c_0(\Lambda^{\underline{0}})} X_{\underline{m}}(\Lambda) \to X_{\underline{n}+\underline{m}}(\Lambda); u_{\underline{n},\underline{m}}(\delta_\lambda \otimes \delta_\mu) = \begin{cases} \delta_{\lambda\mu} & \text{if } r(\mu) = s(\lambda), \\ 0 & \text{otherwise,} \end{cases}$$

for all $\lambda \in \Lambda^{\underline{n}}, \mu \in \Lambda^{\underline{m}}$ and $\underline{n}, \underline{m} \in \mathbb{Z}_+^k$.

Proof. See [49, Proposition 3.2]. □

A k-graph (Λ, d) is finitely aligned if and only if $X(\Lambda)$ is compactly aligned [49, Theorem 5.4]. In this case, if $\lambda_1, \lambda_2 \in \Lambda^{\underline{n}}$ and $\mu_1, \mu_2 \in \Lambda^{\underline{m}}$ are non-trivial paths that satisfy $s(\lambda_1) = s(\lambda_2)$ and $s(\mu_1) = s(\mu_2)$, then we may extract the following fact from the proof of [49, Theorem 5.4]:

$$\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(\Theta^{X_{\underline{n}}(\Lambda)}_{\delta_{\lambda_1},\delta_{\lambda_2}})\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(\Theta^{X_{\underline{m}}(\Lambda)}_{\delta_{\mu_1},\delta_{\mu_2}}) = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda_2,\mu_1)} \Theta^{X_{\underline{n}\vee\underline{m}}(\Lambda)}_{\delta_{\lambda_1\alpha},\delta_{\mu_2\beta}}. \tag{5.12}$$

Similarly, (Λ, d) is strong finitely aligned if and only if $X(\Lambda)$ is strong compactly aligned [17, Proposition 7.1]. We record the following well-known result for future reference.

Proposition 5.4.5. *Let (Λ, d) be a k-graph. Then the following hold:*

(i) (Λ, d) is row-finite if and only if $\phi_{\underline{n}}(c_0(\Lambda^{\underline{0}})) \subseteq \mathcal{K}(X_{\underline{n}}(\Lambda))$ for all $\underline{n} \in \mathbb{Z}_+^k$.

(ii) (Λ, d) is sourceless if and only if $X(\Lambda)$ is injective.

(iii) (Λ, d) is row-finite and sourceless if and only if $X(\Lambda)$ is regular.

Proof. (i) Assume that (Λ, d) is row-finite. Fixing $\underline{n} \in \mathbb{Z}_+^k$, recall that $c_0(\Lambda^{\underline{0}})$ is densely spanned by point masses and that $\mathcal{K}(X_{\underline{n}}(\Lambda))$ is in particular a closed linear subspace of $\mathcal{L}(X_{\underline{n}}(\Lambda))$. Hence it suffices to show that $\phi_{\underline{n}}(\delta_v) \in \mathcal{K}(X_{\underline{n}}(\Lambda))$ for all $v \in \Lambda^{\underline{0}}$. This follows immediately from item (ii) of Proposition 5.4.3, noting that $|v\Lambda^{\underline{n}}| < \infty$ for all $v \in \Lambda^{\underline{0}}$ by row-finiteness of (Λ, d). This proves the forward implication.

Now assume that $\phi_{\underline{n}}(c_0(\Lambda^{\underline{0}})) \subseteq \mathcal{K}(X_{\underline{n}}(\Lambda))$ for all $\underline{n} \in \mathbb{Z}_+^k$. To see that (Λ, d) is row-finite, fix $v \in \Lambda^{\underline{0}}$ and $\underline{n} \in \mathbb{Z}_+^k$. By assumption we have that $\phi_{\underline{n}}(\delta_v) \in \mathcal{K}(X_{\underline{n}}(\Lambda))$. An application of item (ii) of Proposition 5.4.3 then gives that $|v\Lambda^{\underline{n}}| < \infty$. Hence (Λ, d) is row-finite, as required.

(ii) Assume that (Λ, d) is sourceless and fix $\underline{n} \in \mathbb{Z}_+^k$. Since Λ is a countable small category, we may enumerate the elements of $\Lambda^{\underline{0}}$ as $\{v_j \mid j \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, set

$$A_n := \operatorname{span}\{\delta_{v_j} \mid j \in [n]\} \subseteq c_0(\Lambda^{\underline{0}}).$$

Notice that A_n is a finite-dimensional C*-subalgebra of $c_0(\Lambda^{\underline{0}})$ satisfying $A_n \subseteq A_{n+1}$. In fact, we also have that $c_0(\Lambda^{\underline{0}}) = \overline{\bigcup_{n=1}^\infty A_n}$ and therefore $c_0(\Lambda^{\underline{0}})$ is the inductive limit of the C*-subalgebras A_n. Thus proving that $\phi_{\underline{n}}$ is injective amounts to showing that the restriction of $\phi_{\underline{n}}$ to each A_n is injective. To this end, fix $n \in \mathbb{N}$ and suppose that $\phi_{\underline{n}}(\sum_{j=1}^n c_j \delta_{v_j}) = 0$ for some $c_j \in \mathbb{C}$ for all $j \in [n]$. In particular, we have that

$$\sum_{j=1}^n c_j \phi_{\underline{n}}(\delta_{v_j}) = \phi_{\underline{n}}(\sum_{j=1}^n c_j \delta_{v_j}) = 0.$$

Fixing $i \in [n]$ and multiplying by $\phi_{\underline{n}}(\delta_{v_i})$, we obtain that

$$c_i \phi_{\underline{n}}(\delta_{v_i}) = \sum_{j=1}^n c_j \phi_{\underline{n}}(\delta_{v_j}\delta_{v_i}) = \sum_{j=1}^n c_j \phi_{\underline{n}}(\delta_{v_j})\phi_{\underline{n}}(\delta_{v_i}) = 0,$$

using that the projections $\{\delta_{v_j} \mid j \in [n]\}$ are pairwise orthogonal in the first equality. Suppose that $c_i \neq 0$ and so $\phi_{\underline{n}}(\delta_{v_i}) = 0$. By item (i) of Proposition 5.4.3, this implies that $v_i\Lambda^{\underline{n}} = \emptyset$. However, this contradicts the assumption that (Λ, d) is sourceless and so we must have that $c_i = 0$. We conclude that $c_i = 0$ for all $i \in [n]$ and hence $\sum_{j=1}^n c_j \delta_{v_j} = 0$. Thus $X_{\underline{n}}(\Lambda)$ is injective for all $\underline{n} \in \mathbb{Z}_+^k$ and so $X(\Lambda)$ is injective, as required.

Now assume that $X(\Lambda)$ is injective. Fix $v \in \Lambda^{\underline{0}}$ and $\underline{n} \in \mathbb{Z}_+^k$. Towards contradiction, suppose that $v\Lambda^{\underline{n}} = \emptyset$. By item (i) of Proposition 5.4.3, this implies that $\phi_{\underline{n}}(\delta_v) = 0$. Injectivity of $X_{\underline{n}}(\Lambda)$ then gives the contradiction that $\delta_v = 0$. Thus $v\Lambda^{\underline{n}} \neq \emptyset$ and so (Λ, d) is sourceless, as required.

Item (iii) now follows by applying items (i) and (ii) in tandem, finishing the proof. □

The following proposition implies (in particular) that $\mathrm{C}^*(\Lambda) \cong \mathcal{NO}_{X(\Lambda)}$ canonically, for finitely aligned Λ. Here $\mathcal{NO}_{X(\Lambda)}$ denotes the universal C*-algebra with respect to the Nica-covariant representations of $X(\Lambda)$ that are Cuntz-Pimsner covariant in the sense of [56, Definition 3.9]. Note that the $*$-homomorphisms $\widetilde{\phi}_q$ therein are injective by [56, Lemma 3.15], which applies since $(\mathbb{Z}^k, \mathbb{Z}_+^k)$ satisfies [56, (3.5)].

Proposition 5.4.6. *Let (λ, d) be a finitely aligned k-graph. Let $\mathcal{S}$ denote the set of Toeplitz-Cuntz-Krieger Λ-families and let $\mathcal{T}$ denote the set of Nica-covariant representations of $X(\Lambda)$. Then $\mathcal{S}$ and $\mathcal{T}$ correspond bijectively via the mutually inverse maps*

$$G\colon \{T_\lambda\}_{\lambda\in\Lambda} \mapsto (\pi, t), \text{ where } t_{\underline{n}}(\delta_\lambda) = T_\lambda \text{ for all } \lambda \in \Lambda^{\underline{n}}, \underline{n} \in \mathbb{Z}_+^k, \{T_\lambda\}_{\lambda\in\Lambda} \in \mathcal{S},$$

and

$$H\colon (\pi, t) \mapsto \{T_\lambda\}_{\lambda\in\Lambda}, \text{ where } T_\lambda = t_{d(\lambda)}(\delta_\lambda) \text{ for all } \lambda \in \Lambda, (\pi, t) \in \mathcal{T}.$$

Moreover, the maps G and H restrict to give a bijective correspondence between the set of Cuntz-Krieger Λ-families and the set of CNP-representations[1] of $X(\Lambda)$.

Proof. We start by showing that G and H are well-defined. Fixing $\{T_\lambda\}_{\lambda\in\Lambda} \in \mathcal{S}$, we must show that $G(\{T_\lambda\}_{\lambda\in\Lambda}) = (\pi, t)$ is a Nica-covariant representation of $X(\Lambda)$. It is routine

[1] In the sense of [56, Definition 3.11].

to check using the Toeplitz-Cuntz-Krieger Λ-family properties that each $t_{\underline{n}}$ (as defined in the statement) extends uniquely to a well-defined bounded linear map with domain $X_{\underline{n}}(\Lambda)$; alternatively, see [25, Example 1.2]. It is similarly straightforward to verify that $\pi := t_{\underline{0}}$ is a $*$-homomorphism. Since $t_{\underline{n}}$ is linear and continuous for all $\underline{n} \in \mathbb{Z}_+^k$, it is sufficient to check that the representation conditions hold on point masses. To this end, fix $\underline{n} \in \mathbb{Z}_+^k$, $\lambda \in \Lambda^{\underline{n}}$ and $v \in \Lambda^{\underline{0}}$. Firstly, we have that

$$\pi(\delta_v)t_{\underline{n}}(\delta_\lambda) = T_v T_\lambda = \delta_{v,r(\lambda)} T_{v\lambda} = t_{\underline{n}}(\phi_{\underline{n}}(\delta_v)\delta_\lambda),$$

using (TCK2) in the second equality. Next, fixing $\mu \in \Lambda^{\underline{n}}$ we obtain that

$$t_{\underline{n}}(\delta_\lambda)^* t_{\underline{n}}(\delta_\mu) = T_\lambda^* T_\mu = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} T_\alpha T_\beta^*,$$

using (TCK3) in the final equality. If $\Lambda^{\min}(\lambda,\mu) = \emptyset$, then in particular we have that $\lambda \neq \mu$ since

$$\Lambda^{\min}(\lambda,\lambda) = \{s(\lambda)\}.$$

Thus in this case we have that

$$t_{\underline{n}}(\delta_\lambda)^* t_{\underline{n}}(\delta_\mu) = \pi(\langle \delta_\lambda, \delta_\mu \rangle) = 0.$$

Otherwise, assume that $\Lambda^{\min}(\lambda,\mu) \neq \emptyset$. Since λ and μ have the same degree, we have that $\lambda = \mu$ by the comments succeeding Proposition 5.4.1. Consequently, we deduce that

$$t_{\underline{n}}(\delta_\lambda)^* t_{\underline{n}}(\delta_\lambda) = T_{s(\lambda)} T_{s(\lambda)}^* = T_{s(\lambda)} = \pi(\delta_{s(\lambda)}) = \pi(\langle \delta_\lambda, \delta_\lambda \rangle),$$

using (TCK1) in the second equality. Hence $(\pi, t_{\underline{n}})$ preserves the inner product of $X_{\underline{n}}(\Lambda)$ in all cases. Fixing $\underline{m} \in \mathbb{Z}_+^k$ and $\nu \in \Lambda^{\underline{m}}$, we also have that

$$t_{\underline{n}}(\delta_\lambda) t_{\underline{m}}(\delta_\nu) = T_\lambda T_\nu = \delta_{s(\lambda),r(\nu)} T_{\lambda\nu} = t_{\underline{n}+\underline{m}}(\delta_\lambda \delta_\nu),$$

using (TCK2) in the second equality and the definition of the multiplication maps of $X(\Lambda)$ in the final equality. In total, we have that (π, t) is a representation of $X(\Lambda)$. It remains to check that (π, t) is Nica-covariant. To this end, it suffices to check that (2.12) holds on rank-one operators by linearity and continuity of the maps involved. Reducing further using (5.11), we fix $\underline{n}, \underline{m} \in \mathbb{Z}_+^k \setminus \{\underline{0}\}$, $\lambda_1, \lambda_2 \in \Lambda^{\underline{n}}$ and $\mu_1, \mu_2 \in \Lambda^{\underline{m}}$ satisfying $s(\lambda_1) = s(\lambda_2)$ and $s(\mu_1) = s(\mu_2)$. We obtain that

$$\begin{aligned}
\psi_{\underline{n}}(\Theta^{X_{\underline{n}}(\Lambda)}_{\delta_{\lambda_1},\delta_{\lambda_2}})\psi_{\underline{m}}(\Theta^{X_{\underline{m}}(\Lambda)}_{\delta_{\mu_1},\delta_{\mu_2}}) &= t_{\underline{n}}(\delta_{\lambda_1}) t_{\underline{n}}(\delta_{\lambda_2})^* t_{\underline{m}}(\delta_{\mu_1}) t_{\underline{m}}(\delta_{\mu_2})^* \\
&= T_{\lambda_1} T_{\lambda_2}^* T_{\mu_1} T_{\mu_2}^* \\
&= T_{\lambda_1}\Big(\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda_2,\mu_1)} T_\alpha T_\beta^* \Big) T_{\mu_2}^*,
\end{aligned}$$

using (TCK3) in the third equality. We then deduce that

$$\begin{aligned}
\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(\Theta^{X_{\underline{n}}(\Lambda)}_{\delta_{\lambda_1},\delta_{\lambda_2}})\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(\Theta^{X_{\underline{m}}(\Lambda)}_{\delta_{\mu_1},\delta_{\mu_2}})) &= \psi_{\underline{n}\vee\underline{m}}(\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda_2,\mu_1)} \Theta^{X_{\underline{n}\vee\underline{m}}(\Lambda)}_{\delta_{\lambda_1\alpha},\delta_{\mu_2\beta}}) \\
&= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda_2,\mu_1)} t_{\underline{n}\vee\underline{m}}(\delta_{\lambda_1\alpha})t_{\underline{n}\vee\underline{m}}(\delta_{\mu_2\beta})^* \\
&= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda_2,\mu_1)} t_{\underline{n}}(\delta_{\lambda_1})t_{\underline{n}\vee\underline{m}-\underline{n}}(\delta_\alpha)t_{\underline{n}\vee\underline{m}-\underline{m}}(\delta_\beta)^*t_{\underline{m}}(\delta_{\mu_2})^* \\
&= T_{\lambda_1}(\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda_2,\mu_1)} T_\alpha T_\beta^*)T_{\mu_2}^* \\
&= \psi_{\underline{n}}(\Theta^{X_{\underline{n}}(\Lambda)}_{\delta_{\lambda_1},\delta_{\lambda_2}})\psi_{\underline{m}}(\Theta^{X_{\underline{m}}(\Lambda)}_{\delta_{\mu_1},\delta_{\mu_2}}),
\end{aligned}$$

using (5.12) in the first equality. Thus (π, t) is Nica-covariant and so $G(\{T_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{T}$. It follows that G is well-defined.

To see that H is well-defined, fix $(\pi, t) \in \mathcal{T}$. We must show that $H((\pi, t)) = \{T_\lambda\}_{\lambda\in\Lambda}$ is a Toeplitz-Cuntz-Krieger Λ-family. Firstly, fixing $\lambda \in \Lambda$ we have that

$$\begin{aligned}
T_\lambda T_\lambda^* T_\lambda &= t_{d(\lambda)}(\delta_\lambda)t_{d(\lambda)}(\delta_\lambda)^*t_{d(\lambda)}(\delta_\lambda) \\
&= t_{d(\lambda)}(\delta_\lambda)\pi(\langle\delta_\lambda, \delta_\lambda\rangle) \\
&= t_{d(\lambda)}(\delta_\lambda)\pi(\delta_{s(\lambda)}) \\
&= t_{d(\lambda)}(\delta_\lambda\delta_{s(\lambda)}) \\
&= t_{d(\lambda)}(\delta_\lambda) = T_\lambda,
\end{aligned}$$

and hence T_λ is a partial isometry. Fixing $v \in \Lambda^{\underline{0}}$, we obtain that

$$T_v^* = \pi(\delta_v)^* = \pi(\delta_v^*) = \pi(\delta_v) = T_v \quad \text{and} \quad T_v^2 = \pi(\delta_v)\pi(\delta_v) = \pi(\delta_v\delta_v) = \pi(\delta_v) = T_v,$$

and so T_v is a projection. Additionally, if $w \in \Lambda^{\underline{0}} \setminus \{v\}$ then

$$T_vT_w = \pi(\delta_v)\pi(\delta_w) = \pi(\delta_v\delta_w) = 0,$$

using that $\delta_v\delta_w = 0$ in the final equality. Hence $\{T_\lambda\}_{\lambda\in\Lambda}$ satisfies (TCK1). Next, fixing $\lambda, \mu \in \Lambda$ we have that

$$\begin{aligned}
T_\lambda T_\mu &= t_{d(\lambda)}(\delta_\lambda)t_{d(\mu)}(\delta_\mu) \\
&= t_{d(\lambda)+d(\mu)}(\delta_\lambda\delta_\mu) \\
&= \delta_{s(\lambda),r(\mu)}t_{d(\lambda\mu)}(\delta_{\lambda\mu}) \\
&= \delta_{s(\lambda),r(\mu)}T_{\lambda\mu},
\end{aligned}$$

where the penultimate equality follows by definition of the multiplication maps of $X(\Lambda)$ together with functoriality of d. Thus $\{T_\lambda\}_{\lambda\in\Lambda}$ satisfies (TCK2). Finally, notice that

(TCK3) is satisfied trivially when $d(\lambda) = \underline{0}$ or $d(\mu) = \underline{0}$, so fix $\lambda, \mu \in \Lambda \setminus \Lambda^{\underline{0}}$ and set $\underline{n} := d(\lambda)$ and $\underline{m} := d(\mu)$. We obtain that

$$\begin{aligned}
T_\lambda T_\lambda^* T_\mu T_\mu^* &= t_{\underline{n}}(\delta_\lambda) t_{\underline{n}}(\delta_\lambda)^* t_{\underline{m}}(\delta_\mu) t_{\underline{m}}(\delta_\mu)^* = \psi_{\underline{n}}(\Theta^{X_{\underline{n}}(\Lambda)}_{\delta_\lambda,\delta_\lambda}) \psi_{\underline{m}}(\Theta^{X_{\underline{m}}(\Lambda)}_{\delta_\mu,\delta_\mu}) \\
&= \psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(\Theta^{X_{\underline{n}}(\Lambda)}_{\delta_\lambda,\delta_\lambda}) \iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(\Theta^{X_{\underline{m}}(\Lambda)}_{\delta_\mu,\delta_\mu})) = \psi_{\underline{n}\vee\underline{m}}(\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} \Theta^{X_{\underline{n}\vee\underline{m}}(\Lambda)}_{\delta_{\lambda\alpha},\delta_{\mu\beta}}) \\
&= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} t_{\underline{n}\vee\underline{m}}(\delta_{\lambda\alpha}) t_{\underline{n}\vee\underline{m}}(\delta_{\mu\beta})^* \\
&= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} t_{\underline{n}}(\delta_\lambda) t_{\underline{n}\vee\underline{m}-\underline{n}}(\delta_\alpha) t_{\underline{n}\vee\underline{m}-\underline{m}}(\delta_\beta)^* t_{\underline{m}}(\delta_\mu)^* \\
&= T_\lambda(\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} T_\alpha T_\beta^*) T_\mu^*,
\end{aligned}$$

using (5.12) in the fourth equality. Notice that

$$T_\lambda^*(T_\lambda T_\lambda^* T_\mu T_\mu^*) T_\mu = T_\lambda^* T_\mu,$$

using that T_λ and T_μ are partial isometries. Hence we have that

$$T_\lambda^* T_\mu = T_\lambda^* T_\lambda(\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} T_\alpha T_\beta^*) T_\mu^* T_\mu.$$

It follows that (TCK3) holds when $\Lambda^{\min}(\lambda,\mu) = \emptyset$, so assume that $\Lambda^{\min}(\lambda,\mu) \neq \emptyset$. Fixing $(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu)$, we obtain that

$$T_\lambda^* T_\lambda T_\alpha = \pi(\delta_{s(\lambda)}) t_{\underline{n}\vee\underline{m}-\underline{n}}(\delta_\alpha) = t_{\underline{n}\vee\underline{m}-\underline{n}}(\delta_{s(\lambda)}\delta_\alpha) = t_{\underline{n}\vee\underline{m}-\underline{n}}(\delta_\alpha) = T_\alpha$$

and that

$$T_\beta^* T_\mu^* T_\mu = t_{\underline{n}\vee\underline{m}-\underline{m}}(\delta_\beta)^* \pi(\delta_{s(\mu)}) = (\pi(\delta_{s(\mu)}) t_{\underline{n}\vee\underline{m}-\underline{m}}(\delta_\beta))^* = t_{\underline{n}\vee\underline{m}-\underline{m}}(\delta_\beta)^* = T_\beta^*.$$

In total, we deduce that

$$T_\lambda^* T_\mu = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} T_\alpha T_\beta^*.$$

Thus $\{T_\lambda\}_{\lambda\in\Lambda}$ satisfies (TCK3) and so $H((\pi,t)) \in \mathcal{S}$. It follows that H is well-defined.

It is immediate that G and H are mutually inverse by definition, proving the first claim. The second claim follows from [56, Proposition 5.4], finishing the proof. □

We will use the duality between ideals of $c_0(\Lambda^{\underline{0}})$ and subsets of $\Lambda^{\underline{0}}$ given by the mutually inverse mappings

$$\begin{aligned}
I &\mapsto H_I := \{v \in \Lambda^{\underline{0}} \mid \delta_v \in I\}, \text{ for all ideals } I \subseteq c_0(\Lambda^{\underline{0}}); \\
H &\mapsto I_H := \overline{\operatorname{span}}\{\delta_v \mid v \in H\}, \text{ for all } H \subseteq \Lambda^{\underline{0}}.
\end{aligned}$$

Note that this duality implements an inclusion-preserving lattice isomorphism, and that $I_\emptyset = \{0\}$ and $I_{\Lambda^{\underline{0}}} = c_0(\Lambda^{\underline{0}})$. It follows that for any $H \subseteq \Lambda^{\underline{0}}$ and $v \in \Lambda^{\underline{0}}$, we have that $\delta_v \in I_H$ if and only if $v \in H$.

If (Λ, d) is strong finitely aligned, then we can simplify (CK) using the machinery of F-tracing vertices. Fix $\emptyset \neq F \subseteq [k]$. We say that $v \in \Lambda^{\underline{0}}$ is an *F-source* if $v\Lambda^{\underline{i}} = \emptyset$ for all $i \in F$. We say that $v \in \Lambda^{\underline{0}}$ is *F-tracing* if:

> for every $\lambda \in \Lambda$ with $d(\lambda) \perp F$ and $r(\lambda) = v$, we have that $s(\lambda)$ is not an F-source, and that $|s(\lambda)\Lambda^{\underline{i}}| < \infty$ for all $i \in [k]$.

By [17, p. 589], we have that

$$H_{\mathcal{J}_F} = \{v \in \Lambda^{\underline{0}} \mid v \text{ is not an } F\text{-source and } |v\Lambda^{\underline{i}}| < \infty \text{ for all } i \in [k]\},$$

and that

$$H_{\mathcal{I}_F} = \{v \in \Lambda^{\underline{0}} \mid v \text{ is } F\text{-tracing}\}.$$

Hence Proposition 2.5.10 implies that whenever $\lambda \in \Lambda$ satisfies $d(\lambda) \perp F$ and $r(\lambda)$ is F-tracing, we have that $s(\lambda)$ is also F-tracing.

In [17, Theorem 7.6], it is shown that a Toeplitz-Cuntz-Krieger Λ-family $\{T_\lambda\}_{\lambda\in\Lambda}$ is a Cuntz-Krieger Λ-family if and only if it satisfies

(CK′) $\prod\{T_v - T_\lambda T_\lambda^* \mid \lambda \in v\Lambda^{\underline{i}}, i \in F\} = 0$, for every F-tracing vertex v and every $\emptyset \neq F \subseteq [k]$.

Proposition 5.4.7. *Let (Λ, d) be a strong finitely aligned k-graph and let $I \subseteq c_0(\Lambda^{\underline{0}})$ be an ideal. Then we have that*

$$H_{X_{\underline{n}}(\Lambda)^{-1}(I)} = \{v \in \Lambda^{\underline{0}} \mid s(v\Lambda^{\underline{n}}) \subseteq H_I\} \textit{ for all } \underline{n} \in \mathbb{Z}_+^k.$$

Proof. Fixing $\underline{n} \in \mathbb{Z}_+^k$ and $v \in \Lambda^{\underline{0}}$, we deduce that

$$\begin{aligned} v \in H_{X_{\underline{n}}(\Lambda)^{-1}(I)} &\iff \delta_v \in X_{\underline{n}}(\Lambda)^{-1}(I) \iff \langle X_{\underline{n}}(\Lambda), \phi_{\underline{n}}(\delta_v) X_{\underline{n}}(\Lambda)\rangle \subseteq I \\ &\iff \langle \delta_\lambda, \phi_{\underline{n}}(\delta_v)\delta_\mu\rangle \in I \text{ for all } \lambda, \mu \in \Lambda^{\underline{n}} \iff \langle \delta_\lambda, \delta_\lambda\rangle \in I \text{ for all } \lambda \in v\Lambda^{\underline{n}} \\ &\iff \delta_{s(\lambda)} \in I \text{ for all } \lambda \in v\Lambda^{\underline{n}} \iff s(\lambda) \in H_I \text{ for all } \lambda \in v\Lambda^{\underline{n}} \\ &\iff s(v\Lambda^{\underline{n}}) \subseteq H_I, \end{aligned}$$

using that I is in particular a closed linear subspace of $c_0(\Lambda^{\underline{0}})$ in the third equivalence. This completes the proof. □

Proposition 5.4.7, together with the fact that the duality between ideals of $c_0(\Lambda^{\underline{0}})$ and subsets of $\Lambda^{\underline{0}}$ respects intersections, implies that

$$H_{X(\Lambda)_F^{-1}(I)} = \{v \in \Lambda^{\underline{0}} \mid s(v\Lambda^{\underline{n}}) \subseteq H_I \text{ for all } \underline{0} \neq \underline{n} \leq \underline{1}_F\} \text{ for all } \emptyset \neq F \subseteq [d]. \tag{5.13}$$

Next, let $\mathcal{L}$ be a 2^k-tuple of $X(\Lambda)$ that consists of ideals. For notational convenience, we set $H_{\mathcal{L},F} := H_{\mathcal{L}_F}$ for all $F \subseteq [k]$ and $H_{\mathcal{L}} := \{H_{\mathcal{L},F}\}_{F\subseteq[k]}$. As with (5.13), we obtain that

$$H_{\mathcal{L}_{\text{inv},F}} = \bigcap_{\underline{m}\perp F} \{v \in \Lambda^{\underline{0}} \mid s(v\Lambda^{\underline{m}}) \subseteq \cap_{F\subsetneq D} H_{\mathcal{L},D}\} \text{ for all } \emptyset \neq F \subsetneq [k]. \tag{5.14}$$

To address each $\mathcal{L}_{\text{lim},F}$, we have the following proposition.

Proposition 5.4.8. *Let (Λ, d) be a strong finitely aligned k-graph. Let $\mathcal{L}$ be a 2^k-tuple of $X(\Lambda)$ that consists of ideals and let $H_{\mathcal{L}}$ be the corresponding family of sets of vertices of Λ. Then, fixing $\emptyset \neq F \subsetneq [k]$, a vertex $v \in \Lambda^{\underline{0}}$ belongs to $H_{\mathcal{L}_{\text{lim},F}}$ if and only if there exists $\underline{m} \perp F$ such that whenever $\underline{n} \perp F$ and $\underline{n} \geq \underline{m}$, we have that $s(v\Lambda^{\underline{n}}) \subseteq H_{\mathcal{L},F}$ and $|v\Lambda^{\underline{n}}| < \infty$.*

Proof. For the forward implication, take $v \in H_{\mathcal{L}_{\text{lim},F}}$. Then we have that

$$\lim_{\underline{m}\perp F} \|\phi_{\underline{m}}(\delta_v) + \mathcal{K}(X_{\underline{m}}(\Lambda)\mathcal{L}_F)\| = 0.$$

By definition there exists $\underline{m} \perp F$ such that whenever $\underline{n} \perp F$ and $\underline{n} \geq \underline{m}$, we have that

$$\|\phi_{\underline{n}}(\delta_v) + \mathcal{K}(X_{\underline{n}}(\Lambda)\mathcal{L}_F)\| < 1/2.$$

Fixing $\underline{n} \perp F$ such that $\underline{n} \geq \underline{m}$, the fact that $\phi_{\underline{n}}(\delta_v) + \mathcal{K}(X_{\underline{n}}(\Lambda)\mathcal{L}_F)$ is a projection then implies that $\phi_{\underline{n}}(\delta_v) \in \mathcal{K}(X_{\underline{n}}(\Lambda)\mathcal{L}_F)$. By (2.5), we have that $\phi_{\underline{n}}(\delta_v) \in \mathcal{K}(X_{\underline{n}}(\Lambda))$ and that $\langle X_{\underline{n}}(\Lambda), \phi_{\underline{n}}(\delta_v)X_{\underline{n}}(\Lambda)\rangle \subseteq \mathcal{L}_F$. The former implies that $|v\Lambda^{\underline{n}}| < \infty$ by item (ii) of Proposition 5.4.3, and the latter implies that $s(v\Lambda^{\underline{n}}) \subseteq H_{\mathcal{L},F}$ by Proposition 5.4.7. Hence the forward implication holds.

Now assume that there exists $\underline{m} \perp F$ such that whenever $\underline{n} \perp F$ and $\underline{n} \geq \underline{m}$, we have that $s(v\Lambda^{\underline{n}}) \subseteq H_{\mathcal{L},F}$ and $|v\Lambda^{\underline{n}}| < \infty$. Fix $\varepsilon > 0$ and $\underline{n} \perp F$ satisfying $\underline{n} \geq \underline{m}$. By assumption we have that $|v\Lambda^{\underline{n}}| < \infty$, and so an application of item (ii) of Proposition 5.4.3 gives that $\phi_{\underline{n}}(\delta_v) \in \mathcal{K}(X_{\underline{n}}(\Lambda))$. Likewise, the assumption that $s(v\Lambda^{\underline{n}}) \subseteq H_{\mathcal{L},F}$ and Proposition 5.4.7 imply that $\langle X_{\underline{n}}(\Lambda), \phi_{\underline{n}}(\delta_v)X_{\underline{n}}(\Lambda)\rangle \subseteq \mathcal{L}_F$. An application of (2.5) then gives that $\phi_{\underline{n}}(\delta_v) \in \mathcal{K}(X_{\underline{n}}(\Lambda)\mathcal{L}_F)$, and hence

$$\|\phi_{\underline{n}}(\delta_v) + \mathcal{K}(X_{\underline{n}}(\Lambda)\mathcal{L}_F)\| = 0 < \varepsilon.$$

It follows that $\lim_{\underline{m}\perp F} \|\phi_{\underline{m}}(\delta_v) + \mathcal{K}(X_{\underline{m}}(\Lambda)\mathcal{L}_F)\| = 0$ and thus $v \in H_{\mathcal{L}_{\text{lim},F}}$, finishing the proof. □

Next we present a construction of [50] for an arbitrary k-graph (Λ, d). A subset H of $\Lambda^{\underline{0}}$ is called *hereditary (in Λ)* if whenever $v \in H$ and $v\Lambda w \neq \emptyset$ (where $w \in \Lambda^{\underline{0}}$), we have that $w \in H$. Due to duality, hereditarity is captured by positive invariance.

Proposition 5.4.9. *Let (Λ, d) be a k-graph and let H be a subset of $\Lambda^{\underline{0}}$. Then H is hereditary if and only if I_H is positively invariant for $X(\Lambda)$.*

Proof. Assume that H is hereditary. Fixing $\underline{n} \in \mathbb{Z}_+^k, \xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\underline{n}}(\Lambda)$ and $a \in I_H$, we must show that $\langle \xi_{\underline{n}}, a\eta_{\underline{n}} \rangle \in I_H$. Since I_H is in particular a closed linear subspace of $c_0(\Lambda^{\underline{0}})$, we may assume that $\xi_{\underline{n}} = \delta_\lambda, \eta_{\underline{n}} = \delta_\mu$ and $a = \delta_v$ for some $\lambda, \mu \in \Lambda^{\underline{n}}$ and $v \in H$ without loss of generality. If $r(\mu) \neq v$ or $\mu \neq \lambda$, then we have that

$$\langle \delta_\lambda, \delta_v \delta_\mu \rangle = 0 \in I_H$$

trivially. Otherwise, suppose that $r(\mu) = v$ and $\mu = \lambda$. Then we obtain that

$$\langle \delta_\lambda, \delta_v \delta_\mu \rangle = \langle \delta_\lambda, \delta_\lambda \rangle = \delta_{s(\lambda)} \in I_H.$$

where the final membership follows by noting that $\lambda \in v\Lambda s(\lambda)$ and thus $s(\lambda) \in H$ by hereditarity of H. Hence $\langle \delta_\lambda, \delta_v \delta_\mu \rangle \in I_H$ in all cases and so I_H is positively invariant.

Now assume that I_H is positively invariant. Fix $v \in H$ and $w \in \Lambda^{\underline{0}}$ such that $v\Lambda w \neq \emptyset$. Taking $\lambda \in v\Lambda w$, we have that

$$\delta_w = \langle \delta_\lambda, \delta_\lambda \rangle = \langle \delta_\lambda, \delta_v \delta_\lambda \rangle \in \left\langle X_{d(\lambda)}(\Lambda), I_H X_{d(\lambda)}(\Lambda) \right\rangle \subseteq I_H,$$

using that $r(\lambda) = v$ in the second equality and that I_H is positively invariant in the final inclusion. Thus $\delta_w \in I_H$ and so $w \in H$ by the comments succeeding Proposition 5.4.6. Hence H is hereditary, finishing the proof. □

Proposition 5.4.10. *Let (Λ, d) be a k-graph and let $H \subseteq \Lambda^{\underline{0}}$ be hereditary. Define the sets*

$$\mathrm{Obj}(\Gamma(\Lambda \setminus H)) := \Lambda^{\underline{0}} \setminus H \quad \textit{and} \quad \mathrm{Mor}(\Gamma(\Lambda \setminus H)) := \{\lambda \in \Lambda \mid s(\lambda) \notin H\}.$$

Then the quadruple

$$\Gamma(\Lambda \setminus H) := (\mathrm{Obj}(\Gamma(\Lambda \setminus H)), \mathrm{Mor}(\Gamma(\Lambda \setminus H)), r|_{\mathrm{Mor}(\Gamma(\Lambda \setminus H))}, s|_{\mathrm{Mor}(\Gamma(\Lambda \setminus H))})$$

is a sub-k-graph of (Λ, d). Additionally, if (Λ, d) is row-finite then so is $\Gamma(\Lambda \setminus H)$.

Proof. For notational convenience, we set $\Gamma := \Gamma(\Lambda \setminus H)$. First we check that Γ is a subcategory of Λ. Note that $\mathrm{Obj}(\Gamma) \subseteq \Lambda^{\underline{0}} = \mathrm{Obj}(\Lambda)$ and $\mathrm{Mor}(\Gamma) \subseteq \mathrm{Mor}(\Lambda)$ by construction. For each $v \in \mathrm{Obj}(\Gamma)$, we have that $\mathrm{id}_v \in \mathrm{Mor}(\Gamma)$ because $s(\mathrm{id}_v) = v \notin H$ by definition of $\mathrm{Obj}(\Gamma)$. For each $\lambda \in \mathrm{Mor}(\Gamma)$, we have that $s(\lambda) \in \mathrm{Obj}(\Gamma)$ by definition and $r(\lambda) \in \mathrm{Obj}(\Gamma)$ by hereditarity of H. Taking $\lambda, \mu \in \mathrm{Mor}(\Gamma)$ such that $r(\mu) = s(\lambda)$, we have that $\lambda\mu \in \mathrm{Mor}(\Gamma)$ because $s(\lambda\mu) = s(\mu) \notin H$. Hence Γ is a subcategory of Λ, and countability and smallness are inherited from the latter.

It is routine to check that d restricts to give a functor $\Gamma \to \mathbb{Z}_+^k$. It remains to verify that this functor satisfies the factorisation property. To this end, take $\lambda \in \mathrm{Mor}(\Gamma)$ and $\underline{m}, \underline{n} \in \mathbb{Z}_+^k$ such that $d(\lambda) = \underline{m} + \underline{n}$. Applying the factorisation property for d, we obtain unique paths $\mu, \nu \in \Lambda$ such that $d(\mu) = \underline{m}, d(\nu) = \underline{n}$ and $\lambda = \mu\nu$. It suffices to show that $\mu, \nu \in \mathrm{Mor}(\Gamma)$. Firstly, we have that

$$s(\lambda) = s(\mu\nu) = s(\nu) \notin H,$$

using that $\lambda \in \mathrm{Mor}(\Gamma)$ in the final assertion. Thus $\nu \in \mathrm{Mor}(\Gamma)$. Next, towards contradiction suppose that $s(\mu) \in H$. Then $s(\mu) = r(\nu) \in H$ and so $s(\nu) \in H$ by hereditarity of H, contradicting the fact that $\nu \in \mathrm{Mor}(\Gamma)$. Hence $s(\mu) \notin H$ and therefore $\mu \in \mathrm{Mor}(\Gamma)$, as required. Thus the restriction of d to Γ satisfies the factorisation property and in total we have that Γ is a sub-k-graph of (Λ, d), proving the first claim. The final claim follows from the observation that

$$v\Gamma^{\underline{n}} \subseteq v\Lambda^{\underline{n}} \text{ for all } v \in \mathrm{Obj}(\Gamma), \underline{n} \in \mathbb{Z}_+^k,$$

thereby finishing the proof. □

The nomenclature of Proposition 5.4.10 will be used throughout the remainder of the section. To avoid confusion, we will denote the corresponding generators by $\delta_{\lambda,\Lambda}$ for $\lambda \in \Lambda$, and by $\delta_{\mu,\Gamma}$ for $\mu \in \Gamma(\Lambda \setminus H)$. We can form the product systems $X(\Gamma(\Lambda \setminus H))$ and $[X(\Lambda)]_{I_H}$, using Proposition 5.4.10 for the former and Propositions 2.3.7 and 5.4.9 for the latter. We will use $\phi_{\underline{n}}^{\Gamma(\Lambda\setminus H)}$ to denote the left action of $X_{\underline{n}}(\Gamma(\Lambda \setminus H))$ for all $\underline{n} \in \mathbb{Z}_+^k$.

Proposition 5.4.11. *Let (Λ, d) be a k-graph and let $H \subseteq \Lambda^{\underline{0}}$ be hereditary. Then $X(\Gamma(\Lambda \setminus H))$ and $[X(\Lambda)]_{I_H}$ are unitarily equivalent via the family of maps $\{W_{\underline{n}}\}_{\underline{n}\in\mathbb{Z}_+^k}$ determined by*

$$W_{\underline{n}} \colon X_{\underline{n}}(\Gamma(\Lambda \setminus H)) \to [X_{\underline{n}}(\Lambda)]_{I_H}; \delta_{\lambda,\Gamma} \mapsto [\delta_{\lambda,\Lambda}]_{I_H} \text{ for all } \lambda \in \Gamma(\Lambda \setminus H)^{\underline{n}}, \underline{n} \in \mathbb{Z}_+^k.$$

Proof. For notational convenience, we set $\Gamma := \Gamma(\Lambda \setminus H)$. We must prove that the family $\{W_{\underline{n}}\}_{\underline{n}\in\mathbb{Z}_+^k}$ consists of well-defined surjective linear maps that satisfy properties (i)-(v) of Definition 2.3.1. Henceforth we will refer to these properties without explicitly mentioning the latter definition.

To this end, first we consider $\underline{n} = \underline{0}$. To see that $W_{\underline{0}}$ is well-defined, first we define the map

$$W_{\underline{0}} \colon c_{00}(\Gamma^{\underline{0}}) \to [c_0(\Lambda^{\underline{0}})]_{I_H}; \delta_{v,\Gamma} \mapsto [\delta_{v,\Lambda}]_{I_H} \text{ for all } v \in \Gamma^{\underline{0}},$$

extending linearly from the usual basis of $c_{00}(\Gamma^{\underline{0}})$. Fixing $n \in \mathbb{N}, c_j \in \mathbb{C}$ and pairwise distinct $v_j \in \Gamma^{\underline{0}}$ for all $j \in [n]$, we obtain that

$$\begin{aligned}\|W_{\underline{0}}(\sum_{j=1}^n c_j\delta_{v_j,\Gamma})\| &= \|[\sum_{j=1}^n c_j\delta_{v_j,\Lambda}]_{I_H}\| = \inf_{a\in I_H} \|\sum_{j=1}^n c_j\delta_{v_j,\Lambda} + a\| \\ &\leq \|\sum_{j=1}^n c_j\delta_{v_j,\Lambda}\| = \sup_{v\in\Lambda^{\underline{0}}} |\sum_{j=1}^n c_j\delta_{v_j,\Lambda}(v)| = \max_{j\in[n]} |c_j|.\end{aligned}$$

We also have that

$$\|\sum_{j=1}^{n} c_j\delta_{v_j,\Gamma}\| = \sup_{v\in\Gamma^{\underline{0}}} |\sum_{j=1}^{n} c_j\delta_{v_j,\Gamma}(v)| = \max_{j\in[n]} |c_j|$$

by definition. Hence we deduce that

$$\|W_{\underline{0}}(\sum_{j=1}^{n} c_j\delta_{v_j,\Gamma})\| \leq \|\sum_{j=1}^{n} c_j\delta_{v_j,\Gamma}\|.$$

Next we claim that

$$\|\sum_{j=1}^{n} c_j\delta_{v_j,\Gamma}\| \leq \|\sum_{j=1}^{n} c_j\delta_{v_j,\Lambda} + a\| \text{ for all } a \in I_H.$$

To see this, first take $a \in \operatorname{span}\{\delta_{v,\Lambda} \mid v \in H\}$. Then $a = \sum_{i=1}^{m} d_i\delta_{w_i,\Lambda}$ for some $m \in \mathbb{N}, d_i \in \mathbb{C}$ and pairwise distinct $w_i \in H$ for all $i \in [m]$. Also, we have that

$$\{v_1, \ldots, v_n\} \cap \{w_1, \ldots, w_m\} = \emptyset,$$

since $v_j \notin H$ for all $j \in [n]$ while $w_i \in H$ for all $i \in [m]$. Hence we obtain that

$$\|\sum_{j=1}^{n} c_j\delta_{v_j,\Lambda} + \sum_{i=1}^{m} d_i\delta_{w_i,\Lambda}\| = \sup_{v\in\Lambda^{\underline{0}}} |\sum_{j=1}^{n} c_j\delta_{v_j,\Lambda}(v) + \sum_{i=1}^{m} d_i\delta_{w_i,\Lambda}(v)| = \max_{\substack{j\in[n]\\ i\in[m]}}\{|c_j|, |d_i|\},$$

using that the vertices $v_1, \ldots, v_n, w_1, \ldots, w_m$ are pairwise distinct in the final equality. Observe that

$$\max_{\substack{j\in[n]\\ i\in[m]}}\{|c_j|, |d_i|\} \geq \max_{j\in[n]} |c_j| = \|\sum_{j=1}^{n} c_j\delta_{v_j,\Gamma}\|$$

and thus

$$\|\sum_{j=1}^{n} c_j\delta_{v_j,\Gamma}\| \leq \|\sum_{j=1}^{n} c_j\delta_{v_j,\Lambda} + a\| \text{ for all } a \in \operatorname{span}\{\delta_{v,\Lambda} \mid v \in H\}.$$

Now suppose that $a \in I_H$ is the norm-limit of a sequence contained in $\operatorname{span}\{\delta_{v,\Lambda} \mid v \in H\}$. Then for fixed $\varepsilon > 0$, we can find $b \in \operatorname{span}\{\delta_{v,\Lambda} \mid v \in H\}$ such that $\|a-b\| < \varepsilon$. Combining this with the preceding argument, we obtain that

$$\|\sum_{j=1}^{n} c_j\delta_{v_j,\Gamma}\| \leq \|\sum_{j=1}^{n} c_j\delta_{v_j,\Lambda} + b\| \leq \|\sum_{j=1}^{n} c_j\delta_{v_j,\Lambda} + a\| + \|b - a\| < \|\sum_{j=1}^{n} c_j\delta_{v_j,\Lambda} + a\| + \varepsilon.$$

Allowing ε to tend to 0, we deduce that

$$\|\sum_{j=1}^{n} c_j\delta_{v_j,\Gamma}\| \leq \|\sum_{j=1}^{n} c_j\delta_{v_j,\Lambda} + a\|,$$

proving the claim. By definition of the quotient norm, we have that

$$\| \sum_{j=1}^{n} c_j \delta_{v_j,\Gamma} \| \leq \| W_{\underline{0}}(\sum_{j=1}^{n} c_j \delta_{v_j,\Gamma}) \|.$$

Employing this in tandem with the preceding deductions, we ascertain that $W_{\underline{0}}$ is a linear isometry and therefore extends to a unique linear isometry

$$W_{\underline{0}} \colon c_0(\Gamma^{\underline{0}}) \to [c_0(\Lambda^{\underline{0}})]_{I_H}; \delta_{v,\Gamma} \mapsto [\delta_{v,\Lambda}]_{I_H} \text{ for all } v \in \Gamma^{\underline{0}}.$$

Thus the map $W_{\underline{0}}$ of the statement is well-defined. It is routine to check that $W_{\underline{0}}$ is a $*$-homomorphism by first checking on the $*$-subalgebra $c_{00}(\Gamma^{\underline{0}})$ and then appealing to linearity and continuity. To see that $W_{\underline{0}}$ is surjective, it suffices to show that $[\delta_{v,\Lambda}]_{I_H} \in \operatorname{Im}(W_{\underline{0}})$ for all $v \in \Lambda^{\underline{0}}$, since

$$[c_0(\Lambda^{\underline{0}})]_{I_H} = \overline{\operatorname{span}}\{[\delta_{v,\Lambda}]_{I_H} \mid v \in \Lambda^{\underline{0}}\}$$

and $\operatorname{Im}(W_{\underline{0}})$ is a C*-subalgebra of $[c_0(\Lambda^{\underline{0}})]_{I_H}$. To this end, if $v \in H$ then $W_{\underline{0}}(0) = [\delta_{v,\Lambda}]_{I_H} = 0$. If $v \notin H$, and so $v \in \Gamma^{\underline{0}}$, then $W_{\underline{0}}(\delta_{v,\Gamma}) = [\delta_{v,\Lambda}]_{I_H}$. Having accounted for all cases for $v \in \Lambda^{\underline{0}}$, we deduce that $W_{\underline{0}}$ is surjective and hence a $*$-isomorphism. In turn, property (i) is satisfied.

Next we consider $\underline{n} \neq \underline{0}$. To see that $W_{\underline{n}}$ is well-defined, first we define the map

$$W_{\underline{n}} \colon c_{00}(\Gamma^{\underline{n}}) \to [X_{\underline{n}}(\Lambda)]_{I_H}; \delta_{\lambda,\Gamma} \mapsto [\delta_{\lambda,\Lambda}]_{I_H} \text{ for all } \lambda \in \Gamma^{\underline{n}},$$

extending linearly from the usual basis of $c_{00}(\Gamma^{\underline{n}})$. Fixing $\lambda, \mu \in \Gamma^{\underline{n}}$, we have that

$$\langle W_{\underline{n}}(\delta_{\lambda,\Gamma}), W_{\underline{n}}(\delta_{\mu,\Gamma}) \rangle = \langle [\delta_{\lambda,\Lambda}]_{I_H}, [\delta_{\mu,\Lambda}]_{I_H} \rangle = [\langle \delta_{\lambda,\Lambda}, \delta_{\mu,\Lambda} \rangle]_{I_H} = \begin{cases} [\delta_{s(\lambda),\Lambda}]_{I_H} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

In either case, we deduce that

$$\langle W_{\underline{n}}(\delta_{\lambda,\Gamma}), W_{\underline{n}}(\delta_{\mu,\Gamma}) \rangle = W_{\underline{0}}(\langle \delta_{\lambda,\Gamma}, \delta_{\mu,\Gamma} \rangle).$$

It is routine to check that $\delta_{\lambda,\Gamma}$ and $\delta_{\mu,\Gamma}$ can be replaced by finite linear combinations of point masses. In turn, for each $\xi_{\underline{n}} \in c_{00}(\Gamma^{\underline{n}})$ we obtain that

$$\|W_{\underline{n}}(\xi_{\underline{n}})\|^2 = \| \langle W_{\underline{n}}(\xi_{\underline{n}}), W_{\underline{n}}(\xi_{\underline{n}}) \rangle \| = \|W_{\underline{0}}(\langle \xi_{\underline{n}}, \xi_{\underline{n}} \rangle)\| = \| \langle \xi_{\underline{n}}, \xi_{\underline{n}} \rangle \| = \|\xi_{\underline{n}}\|^2,$$

using that $W_{\underline{0}}$ is in particular an isometry in the third equality. Hence $W_{\underline{n}}$ is a linear isometry and therefore extends to a unique linear isometry

$$W_{\underline{n}} \colon X_{\underline{n}}(\Gamma) \to [X_{\underline{n}}(\Lambda)]_{I_H}; \delta_{\lambda,\Gamma} \mapsto [\delta_{\lambda,\Lambda}]_{I_H} \text{ for all } \lambda \in \Gamma^{\underline{n}}.$$

Thus the map $W_{\underline{n}}$ of the statement is well-defined. Moreover, it follows from the preceding calculations that $W_{\underline{n}}$ satisfies property (ii). To see that $W_{\underline{n}}$ is surjective, first note that $\mathrm{Im}(W_{\underline{n}})$ is a closed linear subspace of $[X_{\underline{n}}(\Lambda)]_{I_H}$ because $W_{\underline{n}}$ is a linear isometry. In turn, since

$$[X_{\underline{n}}(\Lambda)]_{I_H} = \overline{\mathrm{span}}\{[\delta_{\lambda,\Lambda}]_{I_H} \mid \lambda \in \Lambda^{\underline{n}}\},$$

it suffices to show that $[\delta_{\lambda,\Lambda}]_{I_H} \in \mathrm{Im}(W_{\underline{n}})$ for all $\lambda \in \Lambda^{\underline{n}}$. To this end, if $s(\lambda) \in H$ then $\langle \delta_{\lambda,\Lambda}, \delta_{\lambda,\Lambda}\rangle = \delta_{s(\lambda),\Lambda} \in I_H$. An application of [36, Proposition 1.3] then gives that $\delta_{\lambda,\Lambda} \in X_{\underline{n}}(\Lambda)I_H$ and hence $W_{\underline{n}}(0) = [\delta_{\lambda,\Lambda}]_{I_H} = 0$. If $s(\lambda) \notin H$, and so $\lambda \in \Gamma^{\underline{n}}$, then $W_{\underline{n}}(\delta_{\lambda,\Gamma}) = [\delta_{\lambda,\Lambda}]_{I_H}$. Having accounted for all cases for $\lambda \in \Lambda^{\underline{n}}$, we conclude that $W_{\underline{n}}$ is surjective.

To see that $W_{\underline{n}}$ satisfies property (iii), we must show that

$$[\phi_{\underline{n}}]_{I_H}(W_{\underline{0}}(a))W_{\underline{n}}(\xi_{\underline{n}}) = W_{\underline{n}}(\phi^{\Gamma}_{\underline{n}}(a)\xi_{\underline{n}}) \text{ for all } a \in c_0(\Gamma^{\underline{0}}), \xi_{\underline{n}} \in X_{\underline{n}}(\Gamma).$$

It suffices to show that the equality holds on point masses by linearity and continuity of the maps involved. To this end, fix $v \in \Gamma^{\underline{0}}$ and $\lambda \in \Gamma^{\underline{n}}$. We have that

$$\begin{aligned}
[\phi_{\underline{n}}]_{I_H}(W_{\underline{0}}(\delta_{v,\Gamma}))W_{\underline{n}}(\delta_{\lambda,\Gamma}) &= [\phi_{\underline{n}}]_{I_H}([\delta_{v,\Lambda}]_{I_H})[\delta_{\lambda,\Lambda}]_{I_H} = [\phi_{\underline{n}}(\delta_{v,\Lambda})]_{I_H}[\delta_{\lambda,\Lambda}]_{I_H} \\
&= [\phi_{\underline{n}}(\delta_{v,\Lambda})\delta_{\lambda,\Lambda}]_{I_H} = \begin{cases} [\delta_{\lambda,\Lambda}]_{I_H} & \text{if } r(\lambda) = v, \\ 0 & \text{otherwise.} \end{cases} \\
&= W_{\underline{n}}(\phi^{\Gamma}_{\underline{n}}(\delta_{v,\Gamma})\delta_{\lambda,\Gamma}),
\end{aligned}$$

as required. Hence $W_{\underline{n}}$ satisfies property (iii).

To see that $W_{\underline{n}}$ satisfies property (iv), we must show that

$$W_{\underline{n}}(\xi_{\underline{n}})W_{\underline{0}}(a) = W_{\underline{n}}(\xi_{\underline{n}}a) \text{ for all } \xi_{\underline{n}} \in X_{\underline{n}}(\Gamma), a \in c_0(\Gamma^{\underline{0}}).$$

It suffices to show that the equality holds on point masses by linearity and continuity of the maps involved. To this end, fix $\lambda \in \Gamma^{\underline{n}}$ and $v \in \Gamma^{\underline{0}}$. We have that

$$\begin{aligned}
W_{\underline{n}}(\delta_{\lambda,\Gamma})W_{\underline{0}}(\delta_{v,\Gamma}) &= [\delta_{\lambda,\Lambda}]_{I_H}[\delta_{v,\Lambda}]_{I_H} = [\delta_{\lambda,\Lambda}\delta_{v,\Lambda}]_{I_H} \\
&= \begin{cases} [\delta_{\lambda,\Lambda}]_{I_H} & \text{if } s(\lambda) = v, \\ 0 & \text{otherwise.} \end{cases} \\
&= W_{\underline{n}}(\delta_{\lambda,\Gamma}\delta_{v,\Gamma}),
\end{aligned}$$

as required. Hence $W_{\underline{n}}$ satisfies property (iv).

Finally, let $\{u_{\underline{n},\underline{m}}\}_{\underline{n},\underline{m}\in\mathbb{Z}^k_+}$ (resp. $\{v_{\underline{n},\underline{m}}\}_{\underline{n},\underline{m}\in\mathbb{Z}^k_+}$) denote the multiplication maps of $X(\Gamma)$ (resp. $[X(\Lambda)]_{I_H}$) and fix $\underline{n}, \underline{m} \in \mathbb{Z}^k_+$. We must show that

$$v_{\underline{n},\underline{m}} \circ (W_{\underline{n}} \otimes W_{\underline{m}}) = W_{\underline{n}+\underline{m}} \circ u_{\underline{n},\underline{m}}.$$

By linearity and continuity of the maps involved, it suffices to show that the equality holds on the simple tensors $\xi_{\underline{n}} \otimes \xi_{\underline{m}}$, where $\xi_{\underline{n}} \in X_{\underline{n}}(\Gamma)$ and $\xi_{\underline{m}} \in X_{\underline{m}}(\Gamma)$. Reducing further, we may assume that $\xi_{\underline{n}} = \delta_{\lambda,\Gamma}$ and $\xi_{\underline{m}} = \delta_{\mu,\Gamma}$ for some $\lambda \in \Gamma^{\underline{n}}$ and $\mu \in \Gamma^{\underline{m}}$. We obtain that

$$\begin{aligned}(v_{\underline{n},\underline{m}} \circ (W_{\underline{n}} \otimes W_{\underline{m}}))(\delta_{\lambda,\Gamma} \otimes \delta_{\mu,\Gamma}) &= v_{\underline{n},\underline{m}}(W_{\underline{n}}(\delta_{\lambda,\Gamma}) \otimes W_{\underline{m}}(\delta_{\mu,\Gamma})) = v_{\underline{n},\underline{m}}([\delta_{\lambda,\Lambda}]_{I_H} \otimes [\delta_{\mu,\Lambda}]_{I_H}) \\ &= [\delta_{\lambda,\Lambda}\delta_{\mu,\Lambda}]_{I_H} = \begin{cases} [\delta_{\lambda\mu,\Lambda}]_{I_H} & \text{if } r(\mu) = s(\lambda), \\ 0 & \text{otherwise.} \end{cases} \\ &= (W_{\underline{n}+\underline{m}} \circ u_{\underline{n},\underline{m}})(\delta_{\lambda,\Gamma} \otimes \delta_{\mu,\Gamma}),\end{aligned}$$

as required. Hence $\{W_{\underline{n}}\}_{\underline{n}\in\mathbb{Z}_+^k}$ satisfies property (v). Combining this with the preceding deductions, we conclude that $\{W_{\underline{n}}\}_{\underline{n}\in\mathbb{Z}_+^k}$ implements a unitary equivalence between $X(\Gamma)$ and $[X(\Lambda)]_{I_H}$, finishing the proof. □

Corollary 5.4.12. *Let (Λ, d) be a k-graph and let $H \subseteq \Lambda^{\underline{0}}$ be hereditary. If (Λ, d) is (strong) finitely aligned, then so is $\Gamma(\Lambda \setminus H)$.*

Proof. Since $X(\Gamma(\Lambda\setminus H)) \cong [X(\Lambda)]_{I_H}$ by Proposition 5.4.11, the result follows by combining Proposition 2.4.4 (resp. Proposition 2.5.8) with Proposition 2.4.2 (resp. Proposition 2.5.7). □

We now turn our attention to relative NO-2^k-tuples in the case of a strong finitely aligned k-graph (Λ, d). Our first aim is to describe the NT-2^k-tuples of $X(\Lambda)$ from a graph theoretic perspective, by translating items (i)-(iv) of Definition 4.1.4 into properties on vertices. We obtain the following proposition towards the characterisation of item (i) of Definition 4.1.4.

Proposition 5.4.13. *Let (Λ, d) be a strong finitely aligned k-graph and let $H \subseteq \Lambda^{\underline{0}}$ be hereditary in Λ. Then, fixing $\emptyset \neq F \subseteq [k]$, the vertex set associated with $J_F(I_H, X(\Lambda))$ is the union of H and the set*

$$\{v \in H^c \mid |v\Gamma(\Lambda \setminus H)^{\underline{i}}| < \infty \ \forall i \in [k] \text{ and } v \text{ is not an } F\text{-source in } \Gamma(\Lambda \setminus H)\}.$$

Proof. Fix $v \in \Lambda^{\underline{0}}$ and recall that $v \in H_{J_F(I_H,X(\Lambda))}$ if and only if $\delta_{v,\Lambda} \in J_F(I_H, X(\Lambda))$. Combining Propositions 4.1.3 and 5.4.9, we have that

$$J_F(I_H, X(\Lambda)) = [\cdot]_{I_H}^{-1}(\mathcal{J}_F([X(\Lambda)]_{I_H})).$$

Thus we obtain that

$$\delta_{v,\Lambda} \in J_F(I_H, X(\Lambda)) \iff [\delta_{v,\Lambda}]_{I_H} \in \mathcal{J}_F([X(\Lambda)]_{I_H}).$$

Consider the unitary equivalence $\{W_{\underline{n}} \colon X_{\underline{n}}(\Gamma(\Lambda \setminus H)) \to [X_{\underline{n}}(\Lambda)]_{I_H}\}_{\underline{n}\in\mathbb{Z}_+^k}$ of Proposition 5.4.11. Then we have that

$$[\delta_{v,\Lambda}]_{I_H} \in \mathcal{J}_F([X(\Lambda)]_{I_H})$$

if and only if

$$W_{\underline{0}}^{-1}([\delta_{v,\Lambda}]_{I_H}) \in W_{\underline{0}}^{-1}(\mathcal{J}_F([X(\Lambda)]_{I_H})) = \mathcal{J}_F(X(\Gamma(\Lambda \setminus H))),$$

using Proposition 2.5.11 in the final equality. Notice that if $v \notin H$, then $W_{\underline{0}}^{-1}([\delta_{v,\Lambda}]_{I_H}) = \delta_{v,\Gamma}$ by definition. In this case, we have that $\delta_{v,\Gamma} \in \mathcal{J}_F(X(\Gamma(\Lambda \setminus H)))$ if and only if $|v\Gamma(\Lambda \setminus H)^{\underline{i}}| < \infty$ for all $i \in [k]$ and v is not an F-source in $\Gamma(\Lambda \setminus H)$, by applying the comments succeeding Proposition 5.4.6 to $\Gamma(\Lambda \setminus H)$. Thus $v \in H_{J_F(I_H, X(\Lambda))}$ if and only if either $v \in H$ or $v \notin H$ but $|v\Gamma(\Lambda \setminus H)^{\underline{i}}| < \infty$ for all $i \in [k]$ and v is not an F-source in $\Gamma(\Lambda \setminus H)$, proving the result. □

Next we translate items (ii) and (iii) of Definition 4.1.4.

Definition 5.4.14. Let (Λ, d) be a k-graph.

(i) Given $F \subseteq [k]$, we say that $H \subseteq \Lambda^{\underline{0}}$ is *$F^\perp$-hereditary (in Λ)* if $s(H\Lambda^{\underline{n}}) \subseteq H$ for all $\underline{n} \perp F$.

(ii) We say that a family $H := \{H_F\}_{F \subseteq [k]}$ of subsets of $\Lambda^{\underline{0}}$ is *hereditary (in Λ)* if H_F is $F^\perp$-hereditary (in Λ) for all $F \subseteq [k]$.

Notice that $H \subseteq \Lambda^{\underline{0}}$ is $\emptyset^\perp$-hereditary if and only if it is hereditary in the usual sense.

Proposition 5.4.15. *Let (Λ, d) be a strong finitely aligned k-graph. Let $\mathcal{L}$ be a 2^k-tuple of $X(\Lambda)$ that consists of ideals and let $H_{\mathcal{L}}$ be the corresponding family of sets of vertices of Λ. Then $\mathcal{L}$ is $X(\Lambda)$-invariant if and only if $H_{\mathcal{L}}$ is hereditary in Λ.*

Proof. Assume that $\mathcal{L}$ is invariant. Fix $F \subseteq [k], \underline{n} \perp F$ and $\lambda \in H_{\mathcal{L},F}\Lambda^{\underline{n}}$. We have that $r(\lambda) \in H_{\mathcal{L},F}$ and hence $\delta_{r(\lambda),\Lambda} \in \mathcal{L}_F$. By assumption we have that $\langle X_{\underline{n}}(\Lambda), \mathcal{L}_F X_{\underline{n}}(\Lambda)\rangle \subseteq \mathcal{L}_F$ and thus

$$\left\langle \delta_{\lambda,\Lambda}, \delta_{r(\lambda),\Lambda}\delta_{\lambda,\Lambda}\right\rangle = \langle \delta_{\lambda,\Lambda}, \delta_{\lambda,\Lambda}\rangle = \delta_{s(\lambda),\Lambda} \in \mathcal{L}_F.$$

It follows that $s(\lambda) \in H_{\mathcal{L},F}$ and so $s(H_{\mathcal{L},F}\Lambda^{\underline{n}}) \subseteq H_{\mathcal{L},F}$ for all $\underline{n} \perp F$. Hence $H_{\mathcal{L},F}$ is $F^\perp$-hereditary for all $F \subseteq [k]$ and thus $H_{\mathcal{L}}$ is hereditary, as required.

Now assume that $H_{\mathcal{L}}$ is hereditary. Fix $F \subseteq [k]$ and $\underline{n} \perp F$. It suffices to show that the invariance condition holds on point masses, since $\mathcal{L}_F$ is in particular a closed linear subspace of $c_0(\Lambda^{\underline{0}})$. Accordingly, fix $\lambda, \mu \in \Lambda^{\underline{n}}$ and $v \in H_{\mathcal{L},F}$. Note that $\langle \delta_{\lambda,\Lambda}, \delta_{v,\Lambda}\delta_{\mu,\Lambda}\rangle \in \mathcal{L}_F$ trivially if $r(\mu) \neq v$ or $\lambda \neq \mu$, so assume that $r(\mu) = v$ and $\lambda = \mu$. In particular, notice that $\lambda \in H_{\mathcal{L},F}\Lambda^{\underline{n}}$. We obtain that

$$\langle \delta_{\lambda,\Lambda}, \delta_{v,\Lambda}\delta_{\mu,\Lambda}\rangle = \langle \delta_{\lambda,\Lambda}, \delta_{\lambda,\Lambda}\rangle = \delta_{s(\lambda),\Lambda} \in \mathcal{L}_F,$$

using $F^\perp$-hereditarity of $H_{\mathcal{L},F}$ in the final membership. Hence $\mathcal{L}$ is invariant, finishing the proof. □

Definition 5.4.16. Let (Λ, d) be a k-graph and let $H := \{H_F\}_{F \subseteq [k]}$ be a family of subsets of $\Lambda^{\underline{0}}$. We say that H is *partially ordered* if $H_{F_1} \subseteq H_{F_2}$ whenever $F_1 \subseteq F_2 \subseteq [k]$.

Proposition 5.4.17. *Let (Λ, d) be a strong finitely aligned k-graph. Let $\mathcal{L}$ be a 2^k-tuple of $X(\Lambda)$ that consists of ideals and let $H_{\mathcal{L}}$ be the corresponding family of sets of vertices of Λ. Then $\mathcal{L}$ is partially ordered if and only if $H_{\mathcal{L}}$ is partially ordered.*

Proof. The result follows immediately from the comments succeeding Proposition 5.4.6 pertaining to the duality between ideals of $c_0(\Lambda^{\underline{0}})$ and subsets of $\Lambda^{\underline{0}}$. □

Finally, to translate item (iv) of Definition 4.1.4, we will need the following definition.

Definition 5.4.18. Let (Λ, d) be a strong finitely aligned k-graph. Let $H := \{H_F\}_{F\subseteq[k]}$ be a family of subsets of $\Lambda^{\underline{0}}$. We say that H is *absorbent (in Λ)* if the following holds for every $\emptyset \neq F \subsetneq [k]$: a vertex $v \in \Lambda^{\underline{0}}$ belongs to H_F whenever it satisfies

(i) v is F-tracing,

(ii) $s(v\Lambda^{\underline{m}}) \subseteq \cap_{F\subsetneq D} H_D$ for all $\underline{m} \perp F$, and

(iii) there exists $\underline{m} \perp F$ such that whenever $\underline{n} \perp F$ and $\underline{n} \geq \underline{m}$, we have that $s(v\Lambda^{\underline{n}}) \subseteq H_F$ and $|v\Lambda^{\underline{n}}| < \infty$.

Proposition 5.4.19. *Let (Λ, d) be a strong finitely aligned k-graph. Let $\mathcal{L}$ be a 2^k-tuple of $X(\Lambda)$ that consists of ideals and let $H_{\mathcal{L}}$ be the corresponding family of sets of vertices of Λ. Then $\mathcal{L}$ is an NT-2^k-tuple of $X(\Lambda)$ if and only if the following four conditions hold:*

(i) for each $\emptyset \neq F \subseteq [k]$, the set $H_{\mathcal{L},F}$ is contained in the union of $H_{\mathcal{L},\emptyset}$ and the set

$$\{v \in H_{\mathcal{L},\emptyset}^c \mid |v\Gamma(\Lambda \setminus H_{\mathcal{L},\emptyset})^{\underline{i}}| < \infty \; \forall i \in [k] \text{ and } v \text{ is not an } F\text{-source in } \Gamma(\Lambda \setminus H_{\mathcal{L},\emptyset})\},$$

(ii) $H_{\mathcal{L}}$ is hereditary in Λ,

(iii) $H_{\mathcal{L}}$ is partially ordered,

(iv) $H_{\mathcal{L}} \setminus H_{\mathcal{L},\emptyset} := \{H_{\mathcal{L},F} \setminus H_{\mathcal{L},\emptyset}\}_{F\subseteq[k]}$ is absorbent in $\Gamma(\Lambda \setminus H_{\mathcal{L},\emptyset})$.

Proof. We have already commented on the equivalence of invariance of $\mathcal{L}$ with item (ii), and of the partial ordering of $\mathcal{L}$ with item (iii). By Proposition 5.4.9, we have that $\mathcal{L}_\emptyset$ is positively invariant for $X(\Lambda)$ if and only if $H_{\mathcal{L},\emptyset}$ is hereditary. In turn, we obtain the equivalence of item (i) with item (i) of Definition 4.1.4 by Proposition 5.4.13. To complete the proof, let $\mathcal{L}$ be a 2^k-tuple of $X(\Lambda)$ that satisfies items (i)-(iii) of Definition 4.1.4. We will show that item (iv) of Definition 4.1.4 is equivalent to item (iv) of the statement.

Since $H_{\mathcal{L},\emptyset}$ is hereditary, we can form the product system $[X(\Lambda)]_{\mathcal{L}_\emptyset}$ and the k-graph $\Gamma := \Gamma(\Lambda \setminus H_{\mathcal{L},\emptyset})$. We identify $X(\Gamma)$ with $[X(\Lambda)]_{\mathcal{L}_\emptyset}$ via the family $\{W_{\underline{n}} \colon X_{\underline{n}}(\Gamma) \to [X_{\underline{n}}(\Lambda)]_{I_H}\}_{\underline{n}\in\mathbb{Z}_+^k}$ as in Proposition 5.4.11, and for $F \subseteq [k]$ we write

$$\mathcal{L}_F^\Gamma := \overline{\operatorname{span}}\{\delta_{v,\Gamma} \mid v \in H_{\mathcal{L},F} \setminus H_{\mathcal{L},\emptyset}\} \subseteq c_0(\Gamma^{\underline{0}}).$$

Note that $\mathcal{L}_F^\Gamma$ corresponds to $[\mathcal{L}_F]_{\mathcal{L}_\emptyset}$ under the identification $X(\Gamma) \cong [X(\Lambda)]_{\mathcal{L}_\emptyset}$. In particular, we have that $\mathcal{L}^\Gamma := \{\mathcal{L}_F^\Gamma\}_{F\subseteq[k]}$ is a 2^k-tuple of $X(\Gamma)$ that consists of ideals. Now notice that

$$[\cdot]_{\mathcal{L}_\emptyset}^{-1}([\mathcal{L}_F]_{\mathcal{L}_\emptyset}^{(1)}) \subseteq \mathcal{L}_F \text{ for all } F \subseteq [k] \iff [\mathcal{L}_F]_{\mathcal{L}_\emptyset}^{(1)} \subseteq [\mathcal{L}_F]_{\mathcal{L}_\emptyset} \text{ for all } \emptyset \neq F \subsetneq [k]$$
$$\iff W_{\underline{0}}^{-1}([\mathcal{L}_F]_{\mathcal{L}_\emptyset}^{(1)}) \subseteq W_{\underline{0}}^{-1}([\mathcal{L}_F]_{\mathcal{L}_\emptyset}) \text{ for all } \emptyset \neq F \subsetneq [k].$$

Fixing $\emptyset \neq F \subsetneq [k]$, we claim that

$$W_{\underline{0}}^{-1}([\mathcal{L}_F]_{\mathcal{L}_\emptyset}^{(1)}) \equiv W_{\underline{0}}^{-1}(\mathcal{I}_F([X(\Lambda)]_{\mathcal{L}_\emptyset}) \cap [\mathcal{L}]_{\mathcal{L}_\emptyset,\mathrm{inv},F} \cap [\mathcal{L}]_{\mathcal{L}_\emptyset,\lim,F}) = \mathcal{L}_F^{\Gamma\,(1)}.$$

To see this, first note that

$$W_{\underline{0}}^{-1}([\mathcal{L}_F]_{\mathcal{L}_\emptyset}^{(1)}) = W_{\underline{0}}^{-1}(\mathcal{I}_F([X(\Lambda)]_{\mathcal{L}_\emptyset})) \cap W_{\underline{0}}^{-1}([\mathcal{L}]_{\mathcal{L}_\emptyset,\mathrm{inv},F}) \cap W_{\underline{0}}^{-1}([\mathcal{L}]_{\mathcal{L}_\emptyset,\lim,F}),$$

since $W_{\underline{0}}^{-1}$ is in particular injective. An application of Proposition 2.5.11 gives that

$$W_{\underline{0}}^{-1}(\mathcal{I}_F([X(\Lambda)]_{\mathcal{L}_\emptyset})) = \mathcal{I}_F(X(\Gamma)).$$

Additionally, we have that

$$W_{\underline{0}}^{-1}([\mathcal{L}]_{\mathcal{L}_\emptyset,\lim,F}) = \mathcal{L}_{\lim,F}^\Gamma$$

by Proposition 3.3.3. Next, fixing $\underline{m} \in \mathbb{Z}_+^k$, we claim that

$$W_{\underline{0}}^{-1}([X_{\underline{m}}(\Lambda)]_{\mathcal{L}_\emptyset}^{-1}(\cap_{F\subsetneq D}[\mathcal{L}_D]_{\mathcal{L}_\emptyset})) = X_{\underline{m}}(\Gamma)^{-1}(\cap_{F\subsetneq D}\mathcal{L}_D^\Gamma).$$

To see this, take $v \in \Gamma^{\underline{0}}$ and observe that $\delta_{v,\Gamma} \in W_{\underline{0}}^{-1}([X_{\underline{m}}(\Lambda)]_{\mathcal{L}_\emptyset}^{-1}(\cap_{F\subsetneq D}[\mathcal{L}_D]_{\mathcal{L}_\emptyset}))$ if and only if $[\delta_{v,\Lambda}]_{\mathcal{L}_\emptyset} \in [X_{\underline{m}}(\Lambda)]_{\mathcal{L}_\emptyset}^{-1}(\cap_{F\subsetneq D}[\mathcal{L}_D]_{\mathcal{L}_\emptyset})$. In turn, by arguing as in the proof of Proposition 5.4.7, we deduce that the latter holds if and only if

$$[\delta_{s(\lambda),\Lambda}]_{\mathcal{L}_\emptyset} \in \cap_{F\subsetneq D}[\mathcal{L}_D]_{\mathcal{L}_\emptyset} \text{ for all } \lambda \in v\Lambda^{\underline{m}} \iff [\delta_{s(\lambda),\Lambda}]_{\mathcal{L}_\emptyset} \in \cap_{F\subsetneq D}[\mathcal{L}_D]_{\mathcal{L}_\emptyset} \text{ for all } \lambda \in v\Gamma^{\underline{m}}$$
$$\iff W_{\underline{0}}(\delta_{s(\lambda),\Gamma}) \in \cap_{F\subsetneq D}[\mathcal{L}_D]_{\mathcal{L}_\emptyset} \text{ for all } \lambda \in v\Gamma^{\underline{m}}$$
$$\iff \delta_{s(\lambda),\Gamma} \in \cap_{F\subsetneq D}\mathcal{L}_D^\Gamma \text{ for all } \lambda \in v\Gamma^{\underline{m}}$$
$$\iff \left\langle X_{\underline{m}}(\Gamma), \phi_{\underline{m}}^\Gamma(\delta_{v,\Gamma})X_{\underline{m}}(\Gamma)\right\rangle \subseteq \cap_{F\subsetneq D}\mathcal{L}_D^\Gamma$$
$$\iff \delta_{v,\Gamma} \in X_{\underline{m}}(\Gamma)^{-1}(\cap_{F\subsetneq D}\mathcal{L}_D^\Gamma),$$

using that $[\delta_{s(\lambda),\Lambda}]_{\mathcal{L}_\emptyset} = 0$ whenever $\lambda \in v\Lambda^{\underline{m}}$ satisfies $s(\lambda) \in H_{\mathcal{L},\emptyset}$ in the second equivalence. By exploiting the duality between ideals of $c_0(\Gamma^{\underline{0}})$ and subsets of $\Gamma^{\underline{0}}$, it follows that $W_{\underline{0}}^{-1}([X_{\underline{m}}(\Lambda)]_{\mathcal{L}_\emptyset}^{-1}(\cap_{F\subsetneq D}[\mathcal{L}_D]_{\mathcal{L}_\emptyset})) = X_{\underline{m}}(\Gamma)^{-1}(\cap_{F\subsetneq D}\mathcal{L}_D^\Gamma)$, as claimed. Hence we have that

$$W_{\underline{0}}^{-1}([\mathcal{L}]_{\mathcal{L}_\emptyset,\mathrm{inv},F}) = \mathcal{L}_{\mathrm{inv},F}^\Gamma.$$

Combining the preceding deductions, we obtain that $W_{\underline{0}}^{-1}([\mathcal{L}_F]_{\mathcal{L}_\emptyset}^{(1)}) = \mathcal{L}_F^{\Gamma\,(1)}$, as claimed.

Thus we deduce that

$$[\cdot]^{-1}_{\mathcal{L}_\emptyset}([\mathcal{L}_F]^{(1)}_{\mathcal{L}_\emptyset}) \subseteq \mathcal{L}_F \text{ for all } F \subseteq [k] \iff \mathcal{L}_F^{\Gamma\,(1)} \subseteq \mathcal{L}_F^\Gamma \text{ for all } \emptyset \neq F \subsetneq [k],$$

which is in turn equivalent to having that

$$\mathcal{I}_F(X(\Gamma)) \cap \mathcal{L}^\Gamma_{\mathrm{inv},F} \cap \mathcal{L}^\Gamma_{\mathrm{lim},F} \subseteq \mathcal{L}^\Gamma_F \text{ for all } \emptyset \neq F \subsetneq [k] \tag{5.15}$$

by definition. We will show that (5.15) holds exactly when $H_\mathcal{L} \setminus H_{\mathcal{L},\emptyset}$ is absorbent in Γ, thereby completing the proof.

To this end, fix $\emptyset \neq F \subsetneq [k]$ and $v \in \Gamma^{\underline{0}}$. We have that $\delta_{v,\Gamma} \in \mathcal{I}_F(X(\Gamma))$ if and only if v is F-tracing in Γ. Likewise, by (5.14) we have that $\delta_{v,\Gamma} \in \mathcal{L}^\Gamma_{\mathrm{inv},F}$ if and only if

$$s(v\Gamma^{\underline{m}}) \subseteq \cap_{F \subsetneq D}(H_{\mathcal{L},D} \setminus H_{\mathcal{L},\emptyset}) \text{ for all } \underline{m} \perp F.$$

Finally, we have that $\delta_{v,\Gamma} \in \mathcal{L}^\Gamma_{\mathrm{lim},F}$ if and only if there exists $\underline{m} \perp F$ such that whenever $\underline{n} \perp F$ and $\underline{n} \geq \underline{m}$, we have that $s(v\Gamma^{\underline{n}}) \subseteq H_{\mathcal{L},F} \setminus H_{\mathcal{L},\emptyset}$ and $|v\Gamma^{\underline{n}}| < \infty$ by Proposition 5.4.8. It follows that (5.15) holds if and only if $H_\mathcal{L} \setminus H_{\mathcal{L},\emptyset}$ is absorbent in Γ, as required. □

In the row-finite case, the characterisation of Proposition 5.4.19 simplifies as follows.

Proposition 5.4.20. *Let (Λ, d) be a row-finite k-graph. Let $\mathcal{L}$ be a 2^k-tuple of $X(\Lambda)$ that consists of ideals and let $H_\mathcal{L}$ be the corresponding family of sets of vertices of Λ. Then $\mathcal{L}$ is an NT-2^k-tuple of $X(\Lambda)$ if and only if the following four conditions hold:*

(i) for each $\emptyset \neq F \subseteq [k]$, the set $H_{\mathcal{L},F}$ is contained in the union of $H_{\mathcal{L},\emptyset}$ and the set

$$H_F := \{v \in H^c_{\mathcal{L},\emptyset} \mid v \text{ is not an } F\text{-source in } \Gamma := \Gamma(\Lambda \setminus H_{\mathcal{L},\emptyset})\},$$

(ii) $H_\mathcal{L}$ is hereditary in Λ,

(iii) $H_\mathcal{L}$ is partially ordered,

(iv) $H_{1,F} \cap H_{2,F} \cap H_{3,F} \subseteq H_{\mathcal{L},F}$ for all $\emptyset \neq F \subsetneq [k]$, where

- $H_{1,F} := \bigcap_{\underline{n} \perp F}\{v \in \Lambda^{\underline{0}} \mid s(v\Lambda^{\underline{n}}) \subseteq H_{\mathcal{L},\emptyset} \cup H_F\}$,
- $H_{2,F} := \bigcap_{\underline{m} \perp F}\{v \in \Lambda^{\underline{0}} \mid s(v\Lambda^{\underline{m}}) \subseteq \cap_{F \subsetneq D} H_{\mathcal{L},D}\}$,
- *$H_{3,F}$ is the set of all $v \in \Lambda^{\underline{0}}$ for which there exists $\underline{m} \perp F$ such that whenever $\underline{n} \perp F$ and $\underline{n} \geq \underline{m}$, we have that $s(v\Lambda^{\underline{n}}) \subseteq H_{\mathcal{L},F}$.*

Proof. Firstly, note that Λ is in particular strong finitely aligned, so we are free to use Proposition 5.4.19. Next, assuming that $H_{\mathcal{L},\emptyset}$ is hereditary, we have that Γ inherits row-finiteness from Λ and therefore $|v\Gamma^{\underline{i}}| < \infty$ for all $v \in \Gamma^{\underline{0}}$ and $i \in [k]$ automatically. Consequently, items (i)-(iii) of the statement coincide with items (i)-(iii) of Proposition

5.4.19, which are in turn equivalent to items (i)-(iii) of Definition 4.1.4. Thus, without loss of generality, we may assume that $\mathcal{L}$ satisfies items (i)-(iii) of Definition 4.1.4. Since $\phi_{\underline{n}}(c_0(\Lambda^{\underline{0}})) \subseteq \mathcal{K}(X_{\underline{n}}(\Lambda))$ for all $\underline{n} \in \mathbb{Z}_+^k$ by item (i) of Proposition 5.4.5, by Proposition 4.1.5 it suffices to show that item (iv) of the statement is equivalent to the following condition:

$$\Big(\bigcap_{\underline{n}\perp F} X_{\underline{n}}(\Lambda)^{-1}(J_F(\mathcal{L}_\emptyset, X(\Lambda)))\Big) \cap \mathcal{L}_{\text{inv},F} \cap \mathcal{L}_{\text{lim},F} \subseteq \mathcal{L}_F \text{ for all } \emptyset \neq F \subsetneq [k].$$

To this end, fix $\emptyset \neq F \subsetneq [k]$. The vertex set associated with $\bigcap_{\underline{n}\perp F} X_{\underline{n}}(\Lambda)^{-1}(J_F(\mathcal{L}_\emptyset, X(\Lambda)))$ is nothing but $H_{1,F}$, which can be seen by combining Propositions 5.4.7 and 5.4.13. Likewise, we have that $H_{\mathcal{L}_{\text{inv},F}} = H_{2,F}$ by (5.14). Finally, we have that $H_{\mathcal{L}_{\text{lim},F}} = H_{3,F}$ by Proposition 5.4.8, noting that the stipulation that $|v\Lambda^{\underline{n}}| < \infty$ can be dropped by row-finiteness of Λ. The result now follows from the fact that the duality between ideals of $c_0(\Lambda^{\underline{0}})$ and subsets of $\Lambda^{\underline{0}}$ preserves inclusions and intersections. □

The characterisation of relative NO-2^k-tuples in the case of strong finite alignment (resp. row-finiteness) follows directly from Proposition 5.4.19 (resp. Proposition 5.4.20), as inclusion of ideals corresponds to inclusion of their associated vertex sets.

Corollary 5.4.21. *Let (Λ, d) be a strong finitely aligned (resp. row-finite) k-graph. Let $\mathcal{K}$ be a relative 2^k-tuple of $X(\Lambda)$ that consists of ideals and let $H_{\mathcal{K}}$ be the corresponding family of sets of vertices of Λ. Let $\mathcal{L}$ be a 2^k-tuple of $X(\Lambda)$ that consists of ideals and let $H_{\mathcal{L}}$ be the corresponding family of sets of vertices of Λ. Then the following are equivalent:*

- *(i) $\mathcal{L}$ is a $\mathcal{K}$-relative NO-2^k-tuple of $X(\Lambda)$;*
- *(ii) $H_{\mathcal{L}}$ satisfies* (i)-(iv) *of Proposition 5.4.19 (resp. Proposition 5.4.20) and $H_{\mathcal{K},F} \subseteq H_{\mathcal{L},F}$ for all $F \subseteq [k]$.*

In particular, the following are equivalent:

- *(i) $\mathcal{L}$ is an NO-2^k-tuple of $X(\Lambda)$;*
- *(ii) $H_{\mathcal{L}}$ satisfies* (i)-(iv) *of Proposition 5.4.19 (resp. Proposition 5.4.20) and every F-tracing vertex of Λ belongs to $H_{\mathcal{L},F}$ for all $\emptyset \neq F \subseteq [k]$.*

Notice that we restrict to the set of relative 2^k-tuples of $X(\Lambda)$ that consist of ideals in the statement of Corollary 5.4.21. This is sufficient, since for a general relative 2^k-tuple $\mathcal{K}$ of $X(\Lambda)$ we have that $\mathcal{NO}(\mathcal{K}, X(\Lambda)) = \mathcal{NO}(\langle \mathcal{K} \rangle, X(\Lambda))$ by the remarks succeeding Proposition 3.1.16.

Finally, we turn our attention to the case of a locally convex row-finite k-graph (Λ, d). In accordance with [50], a subset H of $\Lambda^{\underline{0}}$ is called *saturated* if whenever a vertex $v \in \Lambda^{\underline{0}}$ satisfies $s(v\Lambda^{\leq \underline{i}}) \subseteq H$ for some $i \in [k]$, we have that $v \in H$. Note that $\emptyset$ is vacuously saturated. When $H \subseteq \Lambda^{\underline{0}}$ is both hereditary and saturated, the row-finite k-graph $\Gamma(\Lambda \setminus H)$

is also locally convex [50, Theorem 5.2]. The *saturation* $\overline{H}^{\mathrm{s}}$ of $H \subseteq \Lambda^{\underline{0}}$ is the smallest saturated subset of $\Lambda^{\underline{0}}$ that contains H. The saturation of a hereditary set is also hereditary [50, Lemma 5.1].

Proposition 5.4.22. *[3, 50] Let (Λ, d) be a locally convex row-finite k-graph. Then the operations*

$$H_1 \vee H_2 := \overline{H_1 \cup H_2}^{s} \quad \textit{and} \quad H_1 \wedge H_2 := H_1 \cap H_2$$

for hereditary saturated subsets $H_1, H_2 \subseteq \Lambda^{\underline{0}}$ define a lattice structure on the set of hereditary saturated subsets of $\Lambda^{\underline{0}}$.

Proof. We order the set of hereditary saturated subsets of $\Lambda^{\underline{0}}$ by inclusion. Fix hereditary saturated subsets $H_1, H_2 \subseteq \Lambda^{\underline{0}}$. Note that $H_1 \vee H_2$ is saturated by definition. We also have that $H_1 \vee H_2$ is hereditary by [50, Lemma 5.1], where hereditarity of $H_1 \cup H_2$ follows from that of H_1 and H_2. It is clear that $H_1 \vee H_2$ is an upper bound for $\{H_1, H_2\}$, and in fact it is a least upper bound by the minimal property of the saturation.

The fact that $H_1 \wedge H_2$ is hereditary and saturated follows from the corresponding properties of H_1 and H_2. Additionally, it is immediate that $H_1 \wedge H_2$ is a greatest lower bound for $\{H_1, H_2\}$. In total, we conclude that the operations of the statement impose a lattice structure on the set of hereditary saturated subsets of $\Lambda^{\underline{0}}$, as required. □

Distributivity of the lattice structure of Proposition 5.4.22 will follow when we establish a lattice isomorphism between the set of hereditary saturated subsets of $\Lambda^{\underline{0}}$ and the set of gauge-invariant ideals of $\mathrm{C}^*(\Lambda)$. Next, local convexity implies that $\mathcal{J}_F(X(\Lambda))$ and $\mathcal{I}_F(X(\Lambda))$ coincide for all $F \subseteq [k]$.

Proposition 5.4.23. *Let (Λ, d) be a locally convex row-finite k-graph. Then $\mathcal{J}_F(X(\Lambda)) = \mathcal{I}_F(X(\Lambda))$ for all $F \subseteq [k]$.*

Proof. The claim holds trivially when $F = \emptyset$, so fix $\emptyset \neq F \subseteq [k]$. It suffices to show that $\mathcal{J}_F(X(\Lambda)) \subseteq \mathcal{I}_F(X(\Lambda))$. To this end, since Λ is row-finite this amounts to showing that if $v \in \Lambda^{\underline{0}}$ is not an F-source, then it is F-tracing. Recalling the definition of F-tracing vertices, we proceed by induction on the length of the degree of the paths $\lambda \in v\Lambda$ with $d(\lambda) \perp F$.

For $|d(\lambda)| = 0$, there is nothing to show (this accounts for $F = [k]$). For $|d(\lambda)| = 1$, we have that $d(\lambda) = \underline{i}$ for some $i \in F^c$. Since v is not an F-source, we can find $\mu \in v\Lambda^{\underline{j}}$ for some $j \in F$. Since $i \neq j$, $\lambda \in v\Lambda^{\underline{i}}$ and $\mu \in v\Lambda^{\underline{j}}$, local convexity of (Λ, d) gives in particular that $s(\lambda)\Lambda^{\underline{j}} \neq \emptyset$, and thus $s(\lambda)$ is not an F-source, as required.

Now assume that $s(\lambda)$ is not an F-source for all $\lambda \in v\Lambda$ satisfying $d(\lambda) \perp F$ and $|d(\lambda)| = N$ for some $N \in \mathbb{N}$. Fix $\lambda \in v\Lambda$ such that $d(\lambda) \perp F$ and $|d(\lambda)| = N + 1$. Then $d(\lambda) = \underline{n} + \underline{i}$ for some $\underline{n} \perp F$ satisfying $|\underline{n}| = N$ and some $i \in F^c$. The factorisation property gives unique paths $\mu, \nu \in \Lambda$ such that $d(\mu) = \underline{n}, d(\nu) = \underline{i}$ and $\lambda = \mu\nu$. Note that $v = r(\lambda) = r(\mu\nu) = r(\mu)$, so the inductive hypothesis implies that $s(\mu)\Lambda^{\underline{j}} \neq \emptyset$ for some

$j \in F$. We also have that $\nu \in s(\mu)\Lambda^{\underline{i}}$ and $i \neq j$, so local convexity of (Λ, d) gives that $s(\nu)\Lambda^{\underline{j}} \neq \emptyset$. In other words, $s(\nu) = s(\lambda)$ is not an F-source, as required. By induction, the proof is complete. □

Proposition 5.4.24. *Let (Λ, d) be a locally convex row-finite k-graph and let $H \subseteq \Lambda^{\underline{0}}$ be hereditary and saturated. Then*

$$\mathcal{I}_F([X(\Lambda)]_{I_H}) = (\mathcal{I}_F(X(\Lambda)) + I_H)/I_H \textit{ for all } F \subseteq [k],$$

and consequently

$$J_F(I_H, X(\Lambda)) = \mathcal{I}_F(X(\Lambda)) + I_H \textit{ for all } \emptyset \neq F \subseteq [k].$$

Proof. The first claim holds trivially when $F = \emptyset$, so fix $\emptyset \neq F \subseteq [k]$. Recall that $\Gamma := \Gamma(\Lambda \setminus H)$ is a locally convex row-finite k-graph, and that $X(\Gamma)$ and $[X(\Lambda)]_{I_H}$ are unitarily equivalent via the family $\{W_{\underline{n}} \colon X_{\underline{n}}(\Gamma(\Lambda \setminus H)) \to [X_{\underline{n}}(\Lambda)]_{I_H}\}_{\underline{n} \in \mathbb{Z}_+^k}$ of Proposition 5.4.11. We have that

$$W_{\underline{0}}(\mathcal{I}_F(X(\Gamma))) = \mathcal{I}_F([X(\Lambda)]_{I_H})$$

by Proposition 2.5.11. Moreover, since Λ and Γ are locally convex and row-finite, Proposition 5.4.23 gives that

$$\mathcal{I}_F(X(\Lambda)) = \mathcal{J}_F(X(\Lambda)) \quad \text{and} \quad \mathcal{I}_F(X(\Gamma)) = \mathcal{J}_F(X(\Gamma)).$$

Hence it suffices to show that

$$(\mathcal{J}_F(X(\Lambda)) + I_H)/I_H = W_{\underline{0}}(\mathcal{J}_F(X(\Gamma))). \tag{5.16}$$

Note that

$$(\mathcal{J}_F(X(\Lambda)) + I_H)/I_H = \overline{\operatorname{span}}\{[\delta_{v,\Lambda}]_{I_H} \mid v \notin H, v \text{ is not an } F\text{-source in } \Lambda\}. \tag{5.17}$$

Therefore, to prove the forward inclusion of (5.16), it suffices to show that $[\delta_{v,\Lambda}]_{I_H} \in W_{\underline{0}}(\mathcal{J}_F(X(\Gamma)))$ whenever $v \notin H$ and v is not an F-source in Λ.

Fix such a $v \in \Lambda^{\underline{0}}$ and note that $W_{\underline{0}}^{-1}([\delta_{v,\Lambda}]_{I_H}) = \delta_{v,\Gamma}$. We claim that v is not an F-source in Γ. Towards contradiction, suppose that $v\Gamma^{\underline{i}} = \emptyset$ for all $i \in F$. Since v is not an F-source in Λ, there exists $i \in F$ such that $v\Lambda^{\underline{i}} \neq \emptyset$. For each $\lambda \in v\Lambda^{\underline{i}}$, we must have that $s(\lambda) \in H$, as otherwise we would obtain that $v\Gamma^{\underline{i}} \neq \emptyset$. Thus $s(v\Lambda^{\leq \underline{i}}) = s(v\Lambda^{\underline{i}}) \subseteq H$. Since H is saturated, we obtain the contradiction that $v \in H$, establishing the forward inclusion of (5.16).

For the reverse inclusion of (5.16), take $v \in \Gamma^{\underline{0}}$, i.e., $v \notin H$, such that v is not an F-source in Γ. In particular, $v \in \Lambda^{\underline{0}}$ is not an F-source in Λ. Hence

$$W_{\underline{0}}(\delta_{v,\Gamma}) = [\delta_{v,\Lambda}]_{I_H} \in (\mathcal{J}_F(X(\Lambda)) + I_H)/I_H$$

by (5.17), giving (5.16).

The last claim follows by item (ii) of Proposition 4.1.3 and the fact that $I_H \subseteq J_F(I_H, X(\Lambda))$ for all $\emptyset \neq F \subseteq [k]$, as I_H is positively invariant. This finishes the proof. □

Proposition 5.4.25. *Let (Λ, d) be a locally convex row-finite k-graph and let H be a subset of $\Lambda^{\underline{0}}$. Then H is hereditary and saturated if and only if I_H is positively and negatively invariant for $X(\Lambda)$.*

Proof. Assume that H is hereditary and saturated. Then I_H is positively invariant for $X(\Lambda)$ by Proposition 5.4.9. Fix $\emptyset \neq F \subseteq [k]$. By Proposition 5.4.24, we obtain that

$$\mathcal{I}_F(X(\Lambda)) \subseteq \mathcal{I}_F(X(\Lambda)) + I_H = J_F(I_H, X(\Lambda)).$$

Hence I_H is negatively invariant for $X(\Lambda)$ by Proposition 5.1.3, as required.

Now assume that I_H is positively and negatively invariant for $X(\Lambda)$. We have that H is hereditary by Proposition 5.4.9, so it remains to check that H is saturated. Accordingly, fix $v \in \Lambda^{\underline{0}}$ and suppose that $s(v\Lambda^{\leq \underline{i}}) \subseteq H$ for some $i \in [k]$. We must show that $v \in H$. This is clear when $v\Lambda^{\underline{i}} = \emptyset$, as in this case $v\Lambda^{\leq \underline{i}} = \{v\}$ by item (ii) of Proposition 5.4.1; so assume that $v\Lambda^{\underline{i}} \neq \emptyset$. Another application of item (ii) of Proposition 5.4.1 gives that $s(v\Lambda^{\leq \underline{i}}) = s(v\Lambda^{\underline{i}}) \subseteq H$, and thus $\delta_{v,\Lambda} \in X(\Lambda)_{\{i\}}^{-1}(I_H)$ by (5.13). Since $v\Lambda^{\underline{i}} \neq \emptyset$, we have that v is not an $\{i\}$-source, and thus v is $\{i\}$-tracing by Proposition 5.4.23. Hence

$$\delta_{v,\Lambda} \in \mathcal{I}_{\{i\}}(X(\Lambda)) \cap X(\Lambda)_{\{i\}}^{-1}(I_H) \subseteq I_H,$$

using negative invariance of I_H in the final inclusion. Consequently, we obtain that $v \in H$ and hence H is saturated, finishing the proof. □

Proposition 5.4.26. *Let (Λ, d) be a locally convex row-finite k-graph. Then the association*

$$I \mapsto \mathcal{L}_I, \text{ where } \mathcal{L}_{I,F} := \mathcal{I}_F(X(\Lambda)) + I \text{ for all } F \subseteq [k],$$

defines a bijection between the set of ideals of $c_0(\Lambda^{\underline{0}})$ that are positively and negatively invariant for $X(\Lambda)$ and the set of NO-2^k-tuples of $X(\Lambda)$, which in turn induces a bijection with the set of gauge-invariant ideals of $\mathcal{NO}_{X(\Lambda)}$.

Proof. By Proposition 5.4.24 and Corollary 5.1.7, the map is well-defined and clearly injective. For surjectivity, fix an NO-2^k-tuple $\mathcal{L}$ of $X(\Lambda)$ and note that $\mathcal{L}_\emptyset$ is positively and negatively invariant for $X(\Lambda)$ by Proposition 5.1.6. It suffices to show that $\mathcal{L}_F = \mathcal{L}_{\mathcal{L}_\emptyset,F} \equiv \mathcal{I}_F(X(\Lambda)) + \mathcal{L}_\emptyset$ for every $\emptyset \neq F \subseteq [k]$ (as the equality clearly holds when $F = \emptyset$). To this end, we obtain that

$$(\mathcal{I}_F(X(\Lambda)) + \mathcal{L}_\emptyset)/\mathcal{L}_\emptyset \subseteq \mathcal{L}_F/\mathcal{L}_\emptyset \subseteq J_F(\mathcal{L}_\emptyset, X(\Lambda))/\mathcal{L}_\emptyset = (\mathcal{I}_F(X(\Lambda)) + \mathcal{L}_\emptyset)/\mathcal{L}_\emptyset$$

for all $\emptyset \neq F \subseteq [k]$, using Propositions 5.4.24 and 5.4.25 in the final equality. Thus

$\mathcal{L}_F = \mathcal{L}_{\mathcal{L}_\emptyset, F}$ for all $\emptyset \neq F \subseteq [k]$, as required. Hence the map of the statement is a bijection. The last claim follows from Corollary 4.2.12, completing the proof. □

To recover [50, Theorem 5.2], we need the following identification. Let $H \subseteq \Lambda^{\underline{0}}$ be a hereditary vertex set. Then

$$\Lambda(H) := (H, \{\lambda \in \Lambda \mid r(\lambda) \in H\}, r, s)$$

is a locally convex row-finite sub-k-graph of Λ, and $X(\Lambda(H))$ and $I_H X(\Lambda) I_H$ are unitarily equivalent via the family of maps $\{W_{\underline{n}}\}_{\underline{n} \in \mathbb{Z}_+^k}$ defined by

$$W_{\underline{n}} \colon X_{\underline{n}}(\Lambda(H)) \to I_H X_{\underline{n}}(\Lambda) I_H; \delta_{\lambda, \Lambda(H)} \mapsto \delta_{\lambda, \Lambda} \text{ for all } \lambda \in \Lambda(H)^{\underline{n}}, \underline{n} \in \mathbb{Z}_+^k.$$

The proofs of these assertions follow similar trajectories to those of Propositions 5.4.10 and 5.4.11, respectively, and so have been omitted. By Propositions 2.5.20 and 2.6.10, we obtain that

$$\mathrm{C}^*(\Lambda(H)) \cong \mathcal{NO}_{X(\Lambda(H))} \cong \mathcal{NO}_{I_H X(\Lambda) I_H} \cong I_H \mathrm{C}^*(\Lambda) I_H,$$

where we identify I_H with its faithful image inside $\mathrm{C}^*(\Lambda)$ in the final $*$-isomorphism.

Corollary 5.4.27. *[50, Theorem 5.2] Let (Λ, d) be a locally convex row-finite k-graph. Equip the set of hereditary saturated subsets of $\Lambda^{\underline{0}}$ with the lattice structure of Proposition 5.4.22, and equip the set of NO-2^k-tuples of $X(\Lambda)$ with the lattice structure of Definition 4.2.5 (suitably restricted). Let $\{T_\lambda\}_{\lambda \in \Lambda}$ be the universal Cuntz-Krieger Λ-family. Then the following hold:*

(i) The set of hereditary saturated subsets of $\Lambda^{\underline{0}}$ and the set of NO-2^k-tuples of $X(\Lambda)$ are isomorphic as lattices via the map

$$H \mapsto \mathcal{L}_{I_H}, \text{ where } \mathcal{L}_{I_H, F} := \mathcal{I}_F(X(\Lambda)) + I_H \text{ for all } F \subseteq [k]. \tag{5.18}$$

Consequently, the set of hereditary saturated subsets of $\Lambda^{\underline{0}}$ and the set of gauge-invariant ideals of $\mathrm{C}^*(\Lambda)$ are isomorphic as lattices.

(ii) Let H be a hereditary and saturated subset of $\Lambda^{\underline{0}}$ and let $Q \colon \mathcal{NT}_{X(\Lambda)} \to \mathrm{C}^*(\Lambda)$ be the canonical quotient map. Then the quotient $\mathrm{C}^*(\Lambda)/Q(\mathfrak{J}^{\mathcal{L}_{I_H}})$ is canonically $*$-isomorphic to the graph C-algebra $\mathrm{C}^*(\Gamma(\Lambda \setminus H))$.*

(iii) Let H be a hereditary subset of $\Lambda^{\underline{0}}$. Then $\mathrm{C}^*(\Lambda(H))$ is canonically $*$-isomorphic to the C-subalgebra $\mathrm{C}^*(T_\lambda \mid r(\lambda) \in H)$ of $\mathrm{C}^*(\Lambda)$, and this C*-subalgebra is a full corner of the ideal generated by $\{T_v \mid v \in H\}$ in $\mathrm{C}^*(\Lambda)$. If H is in addition saturated, then the ideal generated by $\{T_v \mid v \in H\}$ in $\mathrm{C}^*(\Lambda)$ coincides with $Q(\mathfrak{J}^{\mathcal{L}_{I_H}})$, for the canonical quotient map $Q \colon \mathcal{NT}_{X(\Lambda)} \to \mathrm{C}^*(\Lambda)$.*

Proof. (i) Firstly, note that (5.18) is the composition of the duality map $H \mapsto I_H$ and the map $I \mapsto \mathcal{L}_I$ of Proposition 5.4.26, where $H \subseteq \Lambda^{\underline{0}}$ is hereditary and saturated, and I

is an ideal of $c_0(\Lambda^{\underline{0}})$ that is positively and negatively invariant for $X(\Lambda)$. These maps are bijections by Propositions 5.4.25 and 5.4.26, respectively, and hence the map (5.18) is also a bijection. Therefore, to prove the first claim, it suffices to show that (5.18) preserves the lattice structure. To this end, fix hereditary saturated subsets H_1 and H_2 of $\Lambda^{\underline{0}}$. We must show that

$$\mathcal{L}_{I_{H_1}} \wedge \mathcal{L}_{I_{H_2}} = \mathcal{L}_{I_{H_1 \wedge H_2}} \quad \text{and} \quad \mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}} = \mathcal{L}_{I_{H_1 \vee H_2}}.$$

For the operation $\wedge$, recall that $H_1 \wedge H_2 \equiv H_1 \cap H_2$ and that $I_{H_1} I_{H_2} = I_{H_1 \cap H_2}$. Hence for each $F \subseteq [k]$ we obtain that

$$\begin{aligned}(\mathcal{L}_{I_{H_1}} \wedge \mathcal{L}_{I_{H_2}})_F &= \mathcal{L}_{I_{H_1},F} \mathcal{L}_{I_{H_2},F} = (\mathcal{I}_F(X(\Lambda)) + I_{H_1})(\mathcal{I}_F(X(\Lambda)) + I_{H_2}) \\ &= \mathcal{I}_F(X(\Lambda)) + I_{H_1} I_{H_2} = \mathcal{I}_F(X(\Lambda)) + I_{H_1 \cap H_2} = \mathcal{L}_{I_{H_1 \wedge H_2},F},\end{aligned}$$

by Proposition 4.2.6. For the operation $\vee$, we must show that

$$(\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_F = \mathcal{L}_{I_{\overline{H_1 \cup H_2}^{\mathrm{s}}},F} \text{ for all } F \subseteq [k].$$

For $F = \emptyset$, we must show that

$$(\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset \equiv \overline{\pi}_{X(\Lambda)}^{-1}(\mathfrak{J}^{\mathcal{L}_{I_{H_1}}} + \mathfrak{J}^{\mathcal{L}_{I_{H_2}}}) = I_{\overline{H_1 \cup H_2}^{\mathrm{s}}},$$

using Proposition 4.2.7 in the first identity. We have that $H_1 \subseteq \overline{H_1 \cup H_2}^{\mathrm{s}}$ and hence we obtain that $I_{H_1} \subseteq I_{\overline{H_1 \cup H_2}^{\mathrm{s}}}$. Thus $\mathcal{L}_{I_{H_1}} \subseteq \mathcal{L}_{I_{\overline{H_1 \cup H_2}^{\mathrm{s}}}}$, and so $\mathfrak{J}^{\mathcal{L}_{I_{H_1}}} \subseteq \mathfrak{J}^{\mathcal{L}_{I_{\overline{H_1 \cup H_2}^{\mathrm{s}}}}}$ using that the parametrisation of Theorem 4.2.3 respects inclusions. Likewise for H_2. In turn, we obtain that

$$\overline{\pi}_{X(\Lambda)}^{-1}(\mathfrak{J}^{\mathcal{L}_{I_{H_1}}} + \mathfrak{J}^{\mathcal{L}_{I_{H_2}}}) \subseteq \overline{\pi}_{X(\Lambda)}^{-1}(\mathfrak{J}^{\mathcal{L}_{I_{\overline{H_1 \cup H_2}^{\mathrm{s}}}}}) = \mathcal{L}_\emptyset^{\mathfrak{J}^{\mathcal{L}_{I_{\overline{H_1 \cup H_2}^{\mathrm{s}}}}}} = I_{\overline{H_1 \cup H_2}^{\mathrm{s}}}.$$

Conversely, note that $(\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset$ is positively and negatively invariant for $X(\Lambda)$ by Proposition 5.1.6, since it participates in the NO-2^k-tuple $\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}}$. An application of Proposition 5.4.25 then gives that $H_{(\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset}$ is hereditary and saturated. Additionally, we have that $I_{H_1 \cup H_2} \subseteq (\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset$ by definition. From this we deduce that $H_1 \cup H_2 \subseteq H_{(\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset}$. Minimality of the saturation then implies that

$$\overline{H_1 \cup H_2}^{\mathrm{s}} \subseteq H_{(\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset}.$$

Consequently, we have that $I_{\overline{H_1 \cup H_2}^{\mathrm{s}}} \subseteq (\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset$, as required.

Now fix $\emptyset \neq F \subseteq [k]$. Recall from the preceding argument that $\mathfrak{J}^{\mathcal{L}_{I_{H_1}}} + \mathfrak{J}^{\mathcal{L}_{I_{H_2}}} \subseteq \mathfrak{J}^{\mathcal{L}_{I_{\overline{H_1 \cup H_2}^{\mathrm{s}}}}}$ and hence

$$\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}} \equiv \mathcal{L}^{\mathfrak{J}^{\mathcal{L}_{I_{H_1}}} + \mathfrak{J}^{\mathcal{L}_{I_{H_2}}}} \subseteq \mathcal{L}^{\mathfrak{J}^{\mathcal{L}_{I_{\overline{H_1 \cup H_2}^{\mathrm{s}}}}}} = \mathcal{L}_{I_{\overline{H_1 \cup H_2}^{\mathrm{s}}}} \equiv \mathcal{L}_{I_{H_1 \vee H_2}},$$

since the parametrisation of Theorem 4.2.3 respects inclusions. In particular, we have that $(\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_F \subseteq \mathcal{L}_{I_{H_1 \vee H_2},F}$. For the reverse inclusion, first recall that

$$(\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_F = [\cdot]^{-1}_{(\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset}[((\mathcal{L}_{I_{H_1},F} + \mathcal{L}_{I_{H_2},F} + (\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset)/(\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset)^{(k-1)}],$$

by Proposition 4.2.7. Observe that

$$\begin{aligned} \mathcal{I}_F(X(\Lambda)) + (\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset &\subseteq \mathcal{L}_{I_{H_1},F} + (\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset \\ &= \mathcal{I}_F(X(\Lambda)) + I_{H_1} + (\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset \\ &\subseteq \mathcal{I}_F(X(\Lambda)) + (\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset, \end{aligned}$$

using that $I_{H_1} \subseteq (\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset$ in the final inclusion. Likewise for H_2. Therefore, we have that

$$(\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_F = [\cdot]^{-1}_{(\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset}[((\mathcal{I}_F(X(\Lambda)) + (\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset)/(\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset)^{(k-1)}].$$

We then obtain that

$$\begin{aligned} [\mathcal{L}_{I_{H_1 \vee H_2},F}]_{(\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset} &= [\mathcal{I}_F(X(\Lambda)) + I_{\overline{H_1 \cup H_2}^{\mathrm{s}}}]_{(\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset} \\ &= [\mathcal{I}_F(X(\Lambda)) + (\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset]_{(\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset} \\ &\subseteq \Big([\mathcal{I}_F(X(\Lambda)) + (\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset]_{(\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset}\Big)^{(k-1)}, \end{aligned}$$

using that $I_{\overline{H_1 \cup H_2}^{\mathrm{s}}} = (\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_\emptyset$ in the second equality. Thus $\mathcal{L}_{I_{H_1 \vee H_2},F} \subseteq (\mathcal{L}_{I_{H_1}} \vee \mathcal{L}_{I_{H_2}})_F$, as required. The second claim now follows by an application of Corollary 4.2.12.

(ii) Let $H \subseteq \Lambda^{\underline{0}}$ be hereditary and saturated. Then $\mathcal{I}_F([X(\Lambda)]_{I_H}) = (\mathcal{I}_F(X(\Lambda)) + I_H)/I_H$ for all $F \subseteq [k]$ by Proposition 5.4.24. Hence, applying items (ii) and (iii) of Corollary 5.1.7 and adopting the notation therein, we obtain that

$$\mathcal{NO}_{X(\Lambda)}/\left\langle Q_{\mathcal{I}}(\overline{\pi}_{X(\Lambda)}(I_H))\right\rangle \cong \mathcal{NO}_{[X(\Lambda)]_{I_H}} \cong \mathcal{NO}_{X(\Gamma(\Lambda \setminus H))},$$

using Propositions 2.5.20 and 5.4.11 in the final $*$-isomorphism. Note that item (iii) of Corollary 5.1.7 applies due to row-finiteness of (Λ, d). Item (ii) of the statement follows.

(iii) The first statement follows by the comments preceding the corollary, using Proposition 2.6.10 for fullness. Note also that

$$\left\langle Q_{\mathcal{I}}(\overline{\pi}_{X(\Lambda)}(I_H))\right\rangle \cong \langle T_v \mid v \in H\rangle$$

canonically. For the corner property, we use that the generators of $c_0(\Lambda^{\underline{0}})$ form a countable approximate unit of projections. The second statement is an immediate consequence of item (iii) of Corollary 5.1.7. □

5.5 Finite frames

In this section we apply the NT-2^d-tuple machinery to the case of a product system X over $\mathbb{Z}_+^d$ in which $X_{\underline{i}}$ admits a finite frame for all $i \in [d]$. Kakariadis [32] used the quotients of $\mathcal{NT}_X$ by the ideals $\left\langle \overline{\pi}_X(A)\overline{q}_{X,\underline{i}} \mid i \in F \right\rangle$ for $\emptyset \neq F \subseteq [d]$ in order to examine the structure of subinvariant KMS-states. A note was made in [32] that such quotients can be realised as the Cuntz-Nica-Pimsner algebra of an induced product system. We will see here how this is achieved from the parametrisation that we have established.

Definition 5.5.1. Let X be a right Hilbert module over a C*-algebra A. A finite non-empty subset $\{\xi^{(j)}\}_{j\in[N]}$ of X is said to be a *finite frame of* X if $\sum_{j=1}^N \Theta_{\xi^{(j)},\xi^{(j)}} = \mathrm{id}_X$. When such a subset exists, we say that X *admits a finite frame.*

We record the following well-known result for Hilbert C*-modules admitting a finite frame. For the proof, recall that a C*-algebra is said to be *σ-unital* if it admits a countable approximate unit.

Proposition 5.5.2. *Let X be a right Hilbert module over a C*-algebra A. Then X admits a finite frame if and only if* $\mathrm{id}_X \in \mathcal{K}(X)$.

Proof. The forward inclusion is immediate. For the reverse inclusion, assume that $\mathrm{id}_X \in \mathcal{K}(X)$, so that $\mathcal{K}(X)$ is a unital C*-algebra. In particular, we have that $\mathcal{K}(X)$ is σ-unital. Next, for each $\xi \in X$ we define an element $\xi^* \in \mathcal{L}(X, A)$ by

$$\xi^*(\eta) = \langle \xi, \eta \rangle \text{ for all } \eta \in X.$$

We equip the set

$$X^* := \{\xi^* \mid \xi \in X\}$$

with the usual linear space structure inherited from $\mathcal{L}(X, A)$. We also implement a right $\mathcal{K}(X)$-module structure and a $\mathcal{K}(X)$-valued inner product on X^* by

$$\xi^* \cdot k = (k^*\xi)^* \text{ for all } \xi \in X, k \in \mathcal{K}(X) \quad \text{and} \quad \langle \xi^*, \eta^* \rangle = \Theta_{\xi,\eta} \text{ for all } \xi, \eta \in X,$$

respectively. It is routine to check that X^* constitutes a right Hilbert $\mathcal{K}(X)$-module when equipped with these operations. Moreover, we have that X^* is full in the sense that

$$[\langle X^*, X^* \rangle] = \mathcal{K}(X).$$

Hence we may apply [40, Lemma 7.3] to X^* to obtain a sequence $\{\xi^{(j)}\}_{j\in\mathbb{N}} \subseteq X$ such that $\{\sum_{j=1}^n \Theta_{\xi^{(j)},\xi^{(j)}}\}_{n\in\mathbb{N}}$ is an approximate unit of $\mathcal{K}(X)$. Since $\mathrm{id}_X \in \mathcal{K}(X)$ by assumption, we have that

$$\sum_{j=1}^{\infty} \Theta_{\xi^{(j)},\xi^{(j)}} = \mathrm{id}_X,$$

where the convergence occurs in the norm topology. Thus there exists $N \in \mathbb{N}$ such that

$$\|\mathrm{id}_X - \sum_{j=1}^{N} \Theta_{\xi^{(j)},\xi^{(j)}}\| < 1.$$

Hence $k := \sum_{j=1}^{N} \Theta_{\xi^{(j)},\xi^{(j)}}$ is invertible by a standard unital Banach algebra result. In turn, we have that $k^{\frac{1}{2}}$ is invertible by using the continuous functional calculus. We then set

$$\eta^{(j)} := k^{-\frac{1}{2}}\xi^{(j)} \text{ for all } j \in [N].$$

We obtain that

$$\sum_{j=1}^{N} \Theta_{\eta^{(j)},\eta^{(j)}} = k^{-\frac{1}{2}}(\sum_{j=1}^{N} \Theta_{\xi^{(j)},\xi^{(j)}})k^{-\frac{1}{2}} = k^{-\frac{1}{2}}kk^{-\frac{1}{2}} = \mathrm{id}_X.$$

In total, we conclude that $\{\eta^{(j)}\}_{j\in[N]}$ constitutes a finite frame of X, finishing the proof. □

Proposition 5.5.2 implies that X admits a finite frame if and only if $\mathcal{K}(X) = \mathcal{L}(X)$. In total, we have that X is projective and finitely generated (see [43, Definition 1.4.4]).

Lemma 5.5.3. *Let X and Y be C*-correspondences over a C*-algebra A. Suppose that Y admits a finite frame $\{y^{(j)}\}_{j\in[N]}$. Then*

$$\Theta^X_{x_1,x_2} \otimes \mathrm{id}_Y = \sum_{j=1}^{N} \Theta^{X\otimes_A Y}_{x_1\otimes y^{(j)},x_2\otimes y^{(j)}}, \text{ for all } x_1, x_2 \in X.$$

If in addition X admits a finite frame $\{x^{(i)}\}_{i\in[M]}$, then $X \otimes_A Y$ admits the finite frame

$$\{x^{(i)} \otimes y^{(j)} \mid i \in [M], j \in [N]\}.$$

Proof. For the first claim, it suffices to show that the equality holds on simple tensors by linearity and continuity of the maps involved. Accordingly, fix $x \in X$ and $y \in Y$. We have that

$$\begin{aligned}(\Theta^X_{x_1,x_2} \otimes \mathrm{id}_Y)(x\otimes y) &= (x_1 \langle x_2, x\rangle) \otimes y = x_1 \otimes \phi_Y(\langle x_2, x\rangle)y \\ &= x_1 \otimes (\sum_{j=1}^{N} \Theta^Y_{y^{(j)},y^{(j)}}(\phi_Y(\langle x_2, x\rangle)y)) = x_1 \otimes (\sum_{j=1}^{N} y^{(j)}\langle y^{(j)}, \phi_Y(\langle x_2, x\rangle)y\rangle) \\ &= \sum_{j=1}^{N} x_1 \otimes (y^{(j)}\langle x_2 \otimes y^{(j)}, x \otimes y\rangle) = \sum_{j=1}^{N}(x_1 \otimes y^{(j)})\langle x_2 \otimes y^{(j)}, x \otimes y\rangle \\ &= (\sum_{j=1}^{N} \Theta^{X\otimes_A Y}_{x_1\otimes y^{(j)},x_2\otimes y^{(j)}})(x \otimes y),\end{aligned}$$

as required. For the second claim, we must show that

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\Theta^{X\otimes_A Y}_{x^{(i)}\otimes y^{(j)},x^{(i)}\otimes y^{(j)}} = \mathrm{id}_{X\otimes_A Y}.$$

As before, it suffices to show that the equality holds on simple tensors. Fixing $x \in X$ and $y \in Y$, we have that

$$\begin{aligned}\sum_{i=1}^{M}\sum_{j=1}^{N}\Theta^{X\otimes_A Y}_{x^{(i)}\otimes y^{(j)},x^{(i)}\otimes y^{(j)}}(x\otimes y) &= \sum_{i=1}^{M}\sum_{j=1}^{N}(x^{(i)}\otimes y^{(j)})\langle x^{(i)}\otimes y^{(j)}, x\otimes y\rangle \\ &= \sum_{i=1}^{M}\sum_{j=1}^{N} x^{(i)}\otimes (y^{(j)}\langle y^{(j)}, \phi_Y(\langle x^{(i)}, x\rangle)y\rangle) \\ &= \sum_{i=1}^{M}\sum_{j=1}^{N} x^{(i)}\otimes (\Theta^{Y}_{y^{(j)},y^{(j)}}(\phi_Y(\langle x^{(i)}, x\rangle)y)) \\ &= \sum_{i=1}^{M} x^{(i)}\otimes \phi_Y(\langle x^{(i)}, x\rangle)y \\ &= \sum_{i=1}^{M} x^{(i)}\langle x^{(i)}, x\rangle\otimes y = \sum_{i=1}^{M}\Theta^{X}_{x^{(i)},x^{(i)}}(x)\otimes y = x\otimes y,\end{aligned}$$

using that $\{y^{(j)}\}_{j\in[N]}$ is a finite frame of Y in the fourth equality and that $\{x^{(i)}\}_{i\in[M]}$ is a finite frame of X in the final equality. Thus $\{x^{(i)}\otimes y^{(j)} \mid i\in[M], j\in[N]\}$ is a finite frame of $X\otimes_A Y$, finishing the proof. □

Admission of a finite frame is preserved when we pass to quotients.

Proposition 5.5.4. *Let X be a right Hilbert module over a C*-algebra A and let $I\subseteq A$ be an ideal. If X admits a finite frame $\{\xi^{(j)}\}_{j\in[N]}$, then $[X]_I$ admits the finite frame $\{[\xi^{(j)}]_I\}_{j\in[N]}$.*

Proof. We have that

$$\sum_{j=1}^{N}\Theta^{[X]_I}_{[\xi^{(j)}]_I,[\xi^{(j)}]_I} = \sum_{j=1}^{N}[\Theta^{X}_{\xi^{(j)},\xi^{(j)}}]_I = [\sum_{j=1}^{N}\Theta^{X}_{\xi^{(j)},\xi^{(j)}}]_I = [\mathrm{id}_X]_I = \mathrm{id}_{[X]_I},$$

using Lemma 2.2.11 in the first equality. Hence the family $\{[\xi^{(j)}]_I\}_{j\in[N]}$ constitutes a finite frame of $[X]_I$, as required. □

Passing to product systems, we have the following proposition.

Proposition 5.5.5. *Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A and fix $\emptyset\neq F\subseteq[d]$. Then $X_{\underline{i}}$ admits a finite frame for all $i\in F$ if and only if $X_{\underline{n}}$ admits a finite frame for all $\underline{n}\in\mathbb{Z}_+^d\setminus\{\underline{0}\}$ satisfying $\mathrm{supp}\,\underline{n}\subseteq F$. Thus $X_{\underline{i}}$ admits a finite frame for all $i\in[d]$ if and only if $X_{\underline{n}}$ admits a finite frame for all $\underline{n}\in\mathbb{Z}_+^d\setminus\{\underline{0}\}$.*

Proof. It suffices to prove the first claim, as the second then follows by taking $F = [d]$. The reverse implication is immediate, so assume that $X_{\underline{i}}$ admits a finite frame for all $i \in F$. We will prove the claim by induction on $|\underline{n}|$. If $|\underline{n}| = 1$ and $\operatorname{supp} \underline{n} \subseteq F$, then $\underline{n} = \underline{i}$ for some $i \in F$. In this case $X_{\underline{n}}$ admits a finite frame by assumption and so the base case holds.

Now suppose that $X_{\underline{m}}$ admits a finite frame for all $\underline{m} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$ satisfying $\operatorname{supp} \underline{m} \subseteq F$ and $|\underline{m}| = N$ for fixed $N \in \mathbb{N}$. Take $\underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$ satisfying $\operatorname{supp} \underline{n} \subseteq F$ and $|\underline{n}| = N + 1$. Then we may write $\underline{n} = \underline{m} + \underline{i}$ for some $\underline{m} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$ satisfying $\operatorname{supp} \underline{m} \subseteq F$ and $|\underline{m}| = N$, and some $i \in F$. Note that $X_{\underline{m}}$ admits a finite frame by the inductive hypothesis and $X_{\underline{i}}$ admits a finite frame by assumption. Accordingly, fix finite frames $\{\xi_{\underline{m}}^{(j)}\}_{j \in [N_{\underline{m}}]}$ and $\{\xi_{\underline{i}}^{(k)}\}_{k \in [N_{\underline{i}}]}$ of $X_{\underline{m}}$ and $X_{\underline{i}}$, respectively. We claim that the family

$$\{\xi_{\underline{m}}^{(j)} \xi_{\underline{i}}^{(k)} \mid j \in [N_{\underline{m}}], k \in [N_{\underline{i}}]\} \subseteq X_{\underline{m}+\underline{i}} = X_{\underline{n}}$$

constitutes a finite frame of $X_{\underline{n}}$. To this end, we must show that

$$\sum_{j=1}^{N_{\underline{m}}} \sum_{k=1}^{N_{\underline{i}}} \Theta^{X_{\underline{n}}}_{\xi_{\underline{m}}^{(j)} \xi_{\underline{i}}^{(k)}, \xi_{\underline{m}}^{(j)} \xi_{\underline{i}}^{(k)}} = \mathrm{id}_{X_{\underline{n}}}.$$

Since $X_{\underline{m}} \otimes_A X_{\underline{i}} \cong X_{\underline{n}}$ via the multiplication map $u_{\underline{m},\underline{i}}$, it suffices to show that the equality holds on the vectors of the form $\xi_{\underline{m}} \xi_{\underline{i}}$, where $\xi_{\underline{m}} \in X_{\underline{m}}$ and $\xi_{\underline{i}} \in X_{\underline{i}}$. We obtain that

$$\begin{aligned}
\sum_{j=1}^{N_{\underline{m}}} \sum_{k=1}^{N_{\underline{i}}} \Theta^{X_{\underline{n}}}_{\xi_{\underline{m}}^{(j)} \xi_{\underline{i}}^{(k)}, \xi_{\underline{m}}^{(j)} \xi_{\underline{i}}^{(k)}} (\xi_{\underline{m}} \xi_{\underline{i}}) &= \sum_{j=1}^{N_{\underline{m}}} \sum_{k=1}^{N_{\underline{i}}} \xi_{\underline{m}}^{(j)} \xi_{\underline{i}}^{(k)} \langle \xi_{\underline{m}}^{(j)} \xi_{\underline{i}}^{(k)}, \xi_{\underline{m}} \xi_{\underline{i}} \rangle \\
&= \sum_{j=1}^{N_{\underline{m}}} \sum_{k=1}^{N_{\underline{i}}} u_{\underline{m},\underline{i}}(\xi_{\underline{m}}^{(j)} \otimes \xi_{\underline{i}}^{(k)}) \langle u_{\underline{m},\underline{i}}(\xi_{\underline{m}}^{(j)} \otimes \xi_{\underline{i}}^{(k)}), u_{\underline{m},\underline{i}}(\xi_{\underline{m}} \otimes \xi_{\underline{i}}) \rangle \\
&= u_{\underline{m},\underline{i}}\Big(\sum_{j=1}^{N_{\underline{m}}} \sum_{k=1}^{N_{\underline{i}}} (\xi_{\underline{m}}^{(j)} \otimes \xi_{\underline{i}}^{(k)}) \langle \xi_{\underline{m}}^{(j)} \otimes \xi_{\underline{i}}^{(k)}, \xi_{\underline{m}} \otimes \xi_{\underline{i}} \rangle\Big) \\
&= u_{\underline{m},\underline{i}}\Big(\sum_{j=1}^{N_{\underline{m}}} \sum_{k=1}^{N_{\underline{i}}} \Theta^{X_{\underline{m}} \otimes_A X_{\underline{i}}}_{\xi_{\underline{m}}^{(j)} \otimes \xi_{\underline{i}}^{(k)}, \xi_{\underline{m}}^{(j)} \otimes \xi_{\underline{i}}^{(k)}} (\xi_{\underline{m}} \otimes \xi_{\underline{i}})\Big) = \xi_{\underline{m}} \xi_{\underline{i}}
\end{aligned}$$

as required, using Lemma 5.5.3 in the final equality. Hence $X_{\underline{n}}$ admits a finite frame and by induction the proof is complete. □

Corollary 5.5.6. *Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C^*-algebra A, wherein $X_{\underline{i}}$ admits a finite frame for all $i \in [d]$. Then $\mathcal{K}(X_{\underline{n}}) = \mathcal{L}(X_{\underline{n}})$ for all $\underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$ and thus in particular X is strong compactly aligned.*

Proof. An application of Proposition 5.5.5 gives that $X_{\underline{n}}$ admits a finite frame for all $\underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$. It follows that $\mathcal{K}(X_{\underline{n}}) = \mathcal{L}(X_{\underline{n}})$ for all $\underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$ by Proposition 5.5.2. The final claim follows by applying Corollary 2.5.6, finishing the proof. □

We do not assume that $X_{\underline{0}} = A$ admits a finite frame, as this would force A to be unital, and the results of this section hold even when A is non-unital. Notice that X is automatically strong compactly aligned when every $X_{\underline{i}}$ admits a finite frame by Corollary 2.5.6.

Corollary 5.5.7. *Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A, wherein $X_{\underline{i}}$ admits a finite frame for all $i \in [d]$. Fix $\underline{n}, \underline{m} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$ and let $\{\xi_{\underline{m}}^{(j)}\}_{j\in[N_{\underline{m}}]}$ be a finite frame of $X_{\underline{m}}$. Then we have that*

$$\iota_{\underline{n}}^{\underline{n}+\underline{m}}(\Theta^{X_{\underline{n}}}_{\xi_{\underline{n}},\eta_{\underline{n}}}) = \sum_{j=1}^{N_{\underline{m}}} \Theta^{X_{\underline{n}+\underline{m}}}_{\xi_{\underline{n}}\xi_{\underline{m}}^{(j)},\eta_{\underline{n}}\xi_{\underline{m}}^{(j)}}, \text{ for all } \xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\underline{n}}.$$

In particular, if (π, t) is a representation of X then

$$\psi_{\underline{n}+\underline{m}}(\iota_{\underline{n}}^{\underline{n}+\underline{m}}(\Theta^{X_{\underline{n}}}_{\xi_{\underline{n}},\eta_{\underline{n}}})) = \sum_{j=1}^{N_{\underline{m}}} t_{\underline{n}}(\xi_{\underline{n}})t_{\underline{m}}(\xi_{\underline{m}}^{(j)})t_{\underline{m}}(\xi_{\underline{m}}^{(j)})^* t_{\underline{n}}(\eta_{\underline{n}})^*, \text{ for all } \xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\underline{n}}.$$

Proof. First note that the finite frame of $X_{\underline{m}}$ exists by Proposition 5.5.5. We have that

$$\iota_{\underline{n}}^{\underline{n}+\underline{m}}(\Theta^{X_{\underline{n}}}_{\xi_{\underline{n}},\eta_{\underline{n}}}) = u_{\underline{n},\underline{m}}(\Theta^{X_{\underline{n}}}_{\xi_{\underline{n}},\eta_{\underline{n}}} \otimes \mathrm{id}_{X_{\underline{m}}})u^*_{\underline{n},\underline{m}} = u_{\underline{n},\underline{m}}(\sum_{j=1}^{N_{\underline{m}}} \Theta^{X_{\underline{n}}\otimes_A X_{\underline{m}}}_{\xi_{\underline{n}}\otimes\xi_{\underline{m}}^{(j)},\eta_{\underline{n}}\otimes\xi_{\underline{m}}^{(j)}})u^*_{\underline{n},\underline{m}},$$

using Lemma 5.5.3 in the second equality. It then follows from [40, p. 9, (1.6)] that

$$\iota_{\underline{n}}^{\underline{n}+\underline{m}}(\Theta^{X_{\underline{n}}}_{\xi_{\underline{n}},\eta_{\underline{n}}}) = \sum_{j=1}^{N_{\underline{m}}} \Theta^{X_{\underline{n}+\underline{m}}}_{\xi_{\underline{n}}\xi_{\underline{m}}^{(j)},\eta_{\underline{n}}\xi_{\underline{m}}^{(j)}},$$

as required. By applying $\psi_{\underline{n}+\underline{m}}$ we obtain the final claim, finishing the proof. □

Remark 5.5.8. Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A. Suppose that every $X_{\underline{i}}$ admits a finite frame $\{\xi_{\underline{i}}^{(j)}\}_{j\in[N_{\underline{i}}]}$. If (π, t) is a Nica-covariant representation of X, then we have that

$$p_{\underline{i}} = \psi_{\underline{i}}(\mathrm{id}_{X_{\underline{i}}}) = \psi_{\underline{i}}(\sum_{j=1}^{N_{\underline{i}}} \Theta^{X_{\underline{i}}}_{\xi_{\underline{i}}^{(j)},\xi_{\underline{i}}^{(j)}}) = \sum_{j=1}^{N_{\underline{i}}} t_{\underline{i}}(\xi_{\underline{i}}^{(j)})t_{\underline{i}}(\xi_{\underline{i}}^{(j)})^* \in \mathrm{C}^*(\pi, t), \text{ for all } i \in [d].$$

Since the left action of each fibre of X is by compacts, we have that $\pi(A)q_F \subseteq \mathrm{C}^*(\pi, t)$ for all $F \subseteq [d]$ by Proposition 2.5.16. However, it may still be the case that $q_F \notin \mathrm{C}^*(\pi, t)$ (unless $\mathrm{C}^*(\pi, t)$ is unital).

Proposition 5.5.9. *Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A and suppose that $\phi_{\underline{n}}(A) \subseteq \mathcal{K}(X_{\underline{n}})$ for all $\underline{n} \in \mathbb{Z}_+^d$. Let (π, t) be a Nica-covariant representation of X and fix $\emptyset \neq F \subseteq [d]$. Then for each $\emptyset \neq D \subseteq F$, we have that*

$$\pi(A)q_D \subseteq \langle \pi(A)q_{\underline{i}} \mid i \in F\rangle \subseteq \mathrm{C}^*(\pi, t).$$

Proof. Without loss of generality, assume that $D = [m]$ and $F = [n]$ for $m \leq n$. By the Hewitt-Cohen Factorisation Theorem, for $a \in A$ there exist $b_1, \dots, b_m \in A$ such that $a = b_1 \cdots b_m$. In turn, we obtain that

$$\pi(a)q_D = (\pi(b_1)\cdots\pi(b_m))(q_{\underline{1}}\cdots q_{\underline{m}}) = (\pi(b_1)q_{\underline{1}})\cdots(\pi(b_m)q_{\underline{m}}) \in \langle \pi(A)q_{\underline{i}} \mid i \in F\rangle,$$

using that $q_{\underline{i}} \in \pi(A)'$ for all $i \in D$ by Proposition 2.5.15. This finishes the proof. □

We now pass to the study of the quotient $\mathcal{NT}_X / \left\langle \overline{\pi}_X(A)\overline{q}_{X,\underline{i}} \mid i \in F \right\rangle$ for $\emptyset \neq F \subseteq [d]$. We first turn our attention to the case of $F = [d]$, where the quotient corresponds to the finite part of the Wold decomposition of a KMS-state in the context of [32].

Proposition 5.5.10. *Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A, wherein $X_{\underline{i}}$ admits a finite frame for all $i \in [d]$. Consider the gauge-invariant ideal*

$$\mathfrak{J} := \left\langle \overline{\pi}_X(A)\overline{q}_{X,\underline{i}} \mid i \in [d] \right\rangle \subseteq \mathcal{NT}_X.$$

Then

$$\mathcal{L}_\emptyset^{\mathfrak{J}} = \{a \in A \mid \lim_{\underline{n} \in \mathbb{Z}_+^d} \|\phi_{\underline{n}}(a)\| = 0\} \quad \textit{and} \quad \mathcal{L}_F^{\mathfrak{J}} = A \textit{ for all } \emptyset \neq F \subseteq [d].$$

Moreover, the product system $[X]_{\mathcal{L}_\emptyset^{\mathfrak{J}}}$ is regular, and thus there is a canonical $*$-isomorphism

$$\mathcal{NT}_X / \mathfrak{J} \cong \mathcal{NO}_{[X]_{\mathcal{L}_\emptyset^{\mathfrak{J}}}}.$$

If X is in addition injective, then $\mathcal{L}_\emptyset^{\mathfrak{J}} = \{0\}$ and $\mathcal{NT}_X/\mathfrak{J} = \mathcal{NO}_X$.

Proof. Let $Q_{\mathfrak{J}} \colon \mathcal{NT}_X \to \mathcal{NT}_X/\mathfrak{J}$ be the canonical quotient map. The fact that

$$\mathcal{L}_\emptyset^{\mathfrak{J}} \equiv \ker Q_{\mathfrak{J}} \circ \overline{\pi}_X = \{a \in A \mid \lim_{\underline{n} \in \mathbb{Z}_+^d} \|\phi_{\underline{n}}(a)\| = 0\}$$

follows from [32, Proposition 4.3]. Note that the latter is presented in the unital setting, however this assumption can be dropped. Next fix $\emptyset \neq F \subseteq [d]$ and $a \in A$. By Proposition 5.5.9, we have that $\overline{\pi}_X(a)\overline{q}_{X,F} \in \mathfrak{J}$. Thus we obtain that

$$Q_{\mathfrak{J}}(\overline{\pi}_X(a)) + \sum\{(-1)^{|\underline{n}|}Q_{\mathfrak{J}}(\overline{\psi}_{X,\underline{n}}(\phi_{\underline{n}}(a))) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = Q_{\mathfrak{J}}(\overline{\pi}_X(a)\overline{q}_{X,F}) = 0$$

by Proposition 2.5.16. It follows that $a \in \mathcal{L}_F^{\mathfrak{J}} \equiv (Q_{\mathfrak{J}} \circ \overline{\pi}_X)^{-1}(B_{(\underline{0},\underline{1}_F]}^{(Q_{\mathfrak{J}}\circ\overline{\pi}_X, Q_{\mathfrak{J}}\circ\overline{t}_X)})$ and hence $\mathcal{L}_F^{\mathfrak{J}} = A$, as required.

Next we show that $[X]_{\mathcal{L}_\emptyset^{\mathfrak{J}}}$ is regular. Applying Lemma 2.2.11 and Corollary 5.5.6 in tandem, we deduce that the left actions of the fibres of $[X]_{\mathcal{L}_\emptyset^{\mathfrak{J}}}$ are by compacts. Thus it suffices to show that $[\phi_{\underline{i}}]_{\mathcal{L}_\emptyset^{\mathfrak{J}}}$ is injective for all $i \in [d]$ by Proposition 2.5.1. To this end, fix $i \in [d]$, a finite frame $\{\xi_{\underline{i}}^{(j)}\}_{j \in [N_{\underline{i}}]}$ of $X_{\underline{i}}$ and $[a]_{\mathcal{L}_\emptyset^{\mathfrak{J}}} \in \ker[\phi_{\underline{i}}]_{\mathcal{L}_\emptyset^{\mathfrak{J}}}$. We have that $[\phi_{\underline{i}}(a)\xi_{\underline{i}}^{(j)}]_{\mathcal{L}_\emptyset^{\mathfrak{J}}} = 0$ for every $j \in [N_{\underline{i}}]$, and hence $\phi_{\underline{i}}(a)\xi_{\underline{i}}^{(j)} \in X_{\underline{i}}\mathcal{L}_\emptyset^{\mathfrak{J}}$ for all $j \in [N_{\underline{i}}]$. For notational convenience, we write

$$\pi_{\mathfrak{J}} := Q_{\mathfrak{J}} \circ \overline{\pi}_X \quad \text{and} \quad t_{\mathfrak{J},\underline{n}} := Q_{\mathfrak{J}} \circ \overline{t}_{X,\underline{n}} \text{ for all } \underline{n} \in \mathbb{Z}_+^d \setminus \{0\}.$$

Thus we have that

$$\pi_{\mathfrak{J}}(a) t_{\mathfrak{J},\underline{i}}(\xi_{\underline{i}}^{(j)}) = t_{\mathfrak{J},\underline{i}}(\phi_{\underline{i}}(a)\xi_{\underline{i}}^{(j)}) \in t_{\mathfrak{J},\underline{i}}(X_{\underline{i}}\mathcal{L}_{\emptyset}^{\mathfrak{J}}) = t_{\mathfrak{J},\underline{i}}(X_{\underline{i}})\pi_{\mathfrak{J}}(\mathcal{L}_{\emptyset}^{\mathfrak{J}}) = \{0\}, \text{ for all } j \in [N_{\underline{i}}].$$

In turn, we obtain that

$$\pi_{\mathfrak{J}}(a) t_{\mathfrak{J},\underline{i}}(\xi_{\underline{i}}^{(j)}) t_{\mathfrak{J},\underline{i}}(\xi_{\underline{i}}^{(j)})^* = 0 \text{ for all } j \in [N_{\underline{i}}],$$

and therefore

$$\pi_{\mathfrak{J}}(a) p_{\mathfrak{J},\underline{i}} = \pi_{\mathfrak{J}}(a) \sum_{j=1}^{N_{\underline{i}}} t_{\mathfrak{J},\underline{i}}(\xi_{\underline{i}}^{(j)}) t_{\mathfrak{J},\underline{i}}(\xi_{\underline{i}}^{(j)})^* = 0$$

by Remark 5.5.8. We also have that $\overline{\pi}_X(a)\overline{q}_{X,\underline{i}} \in \mathfrak{J}$ by definition, and hence

$$\pi_{\mathfrak{J}}(a) q_{\mathfrak{J},\underline{i}} = Q_{\mathfrak{J}}(\overline{\pi}_X(a)\overline{q}_{X,\underline{i}}) = 0.$$

Consequently, we obtain that

$$\pi_{\mathfrak{J}}(a) = \pi_{\mathfrak{J}}(a) q_{\mathfrak{J},\underline{i}} + \pi_{\mathfrak{J}}(a) p_{\mathfrak{J},\underline{i}} = 0.$$

Hence $a \in \mathcal{L}_{\emptyset}^{\mathfrak{J}}$ and thus $[a]_{\mathcal{L}_{\emptyset}^{\mathfrak{J}}} = 0$, proving that $[\phi_{\underline{i}}]_{\mathcal{L}_{\emptyset}^{\mathfrak{J}}}$ is injective, as required.

Combining regularity of $[X]_{\mathcal{L}_{\emptyset}^{\mathfrak{J}}}$ with the first claim gives that $\mathcal{I}([X]_{\mathcal{L}_{\emptyset}^{\mathfrak{J}}}) = [\mathcal{L}^{\mathfrak{J}}]_{\mathcal{L}_{\emptyset}^{\mathfrak{J}}}$. Therefore we conclude that

$$\mathcal{NT}_X/\mathfrak{J} \cong \mathcal{NO}([\mathcal{L}^{\mathfrak{J}}]_{\mathcal{L}_{\emptyset}^{\mathfrak{J}}}, [X]_{\mathcal{L}_{\emptyset}^{\mathfrak{J}}}) = \mathcal{NO}_{[X]_{\mathcal{L}_{\emptyset}^{\mathfrak{J}}}},$$

using item (ii) of Proposition 4.2.1 to establish the canonical $*$-isomorphism.

If X is injective, then every $\phi_{\underline{n}}$ is isometric and hence $\mathcal{L}_{\emptyset}^{\mathfrak{J}} = \{0\}$. For the final equality, note that

$$\mathfrak{J} = \langle \overline{\pi}_X(A)\overline{q}_{X,F} \mid \emptyset \neq F \subseteq [d] \rangle = \langle \overline{\pi}_X(\mathcal{I}_F(X))\overline{q}_{X,F} \mid F \subseteq [d] \rangle \equiv \mathfrak{J}_{\mathcal{I}(X)}^{(\overline{\pi}_X, \overline{t}_X)},$$

using Proposition 5.5.9 in the first equality and regularity of X in the second. Therefore $\mathcal{NT}_X/\mathfrak{J} = \mathcal{NO}_X$, finishing the proof. □

Next we move to the characterisation of the quotient of $\mathcal{NT}_X$ by $\langle \overline{\pi}_X(A)\overline{q}_{X,\underline{i}} \mid i \in F \rangle$, for some fixed $\emptyset \neq F \subsetneq [d]$, as the Cuntz-Nica-Pimsner algebra of an appropriate quotient of a new product system Y^F over $\mathbb{Z}_+^{|F|}$. The key is that every non-trivial fibre of Y^F inherits a finite frame, and that $\mathcal{NT}_X \cong \mathcal{NT}_{Y^F}$ in a canonical way.

For ease of notation, we will assume that $F = [r]$ for some $r < d$. This is sufficient, as the ensuing arguments can be adapted to account for general $\emptyset \neq F \subsetneq [d]$ by relabelling the elements. For each $\underline{n} = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$, we define

$$\underline{n}_F := (n_1, \dots, n_r) \in \mathbb{Z}_+^r \quad \text{and} \quad \underline{n}_{F^\perp} := (n_{r+1}, \dots, n_d) \in \mathbb{Z}_+^{d-r}.$$

Therefore we can canonically express every $\underline{n} \in \mathbb{Z}_+^d$ as $\underline{n} = (\underline{n}_F, \underline{n}_{F^\perp})$. Conversely, given $\underline{k} \in \mathbb{Z}_+^r$ and $\underline{\ell} \in \mathbb{Z}_+^{d-r}$, we can form $\underline{n} := (\underline{k}, \underline{\ell}) \in \mathbb{Z}_+^d$ so that $\underline{n}_F = \underline{k}$ and $\underline{n}_{F^\perp} = \underline{\ell}$.

For the remainder of the section, we will take X to be compactly aligned and will restrict to the finite frame setting where appropriate. We identify $\mathcal{NT}_X$ with $\mathrm{C}^*(\overline{\pi}, \overline{t})$ for the Fock representation $(\overline{\pi}, \overline{t})$. We use F to split the gauge action β of $(\overline{\pi}, \overline{t})$ into two parts. More specifically, we define families β_F and $\beta_{F^\perp}$ via

$$\beta_F := \{\beta_{(\underline{x}, \underline{1}_{[d-r]})}\}_{\underline{x} \in \mathbb{T}^r} \quad \text{and} \quad \beta_{F^\perp} := \{\beta_{(\underline{1}_{[r]}, \underline{y})}\}_{\underline{y} \in \mathbb{T}^{d-r}}.$$

Note that β_F and $\beta_{F^\perp}$ are point-norm continuous families of $*$-automorphisms of $\mathcal{NT}_X$, being restrictions of β. For $\underline{n} \in \mathbb{Z}_+^d$ and $\xi_{\underline{n}} \in X_{\underline{n}}$, we have that

$$\beta_{F, \underline{x}}(\overline{t}_{\underline{n}}(\xi_{\underline{n}})) = \beta_{(\underline{x}, \underline{1}_{[d-r]})}(\overline{t}_{\underline{n}}(\xi_{\underline{n}})) = \underline{x}^{\underline{n}_F} \overline{t}_{\underline{n}}(\xi_{\underline{n}}) \text{ for all } \underline{x} \in \mathbb{T}^r,$$

and that

$$\beta_{F^\perp, \underline{y}}(\overline{t}_{\underline{n}}(\xi_{\underline{n}})) = \beta_{(\underline{1}_{[r]}, \underline{y})}(\overline{t}_{\underline{n}}(\xi_{\underline{n}})) = \underline{y}^{\underline{n}_{F^\perp}} \overline{t}_{\underline{n}}(\xi_{\underline{n}}) \text{ for all } \underline{y} \in \mathbb{T}^{d-r}.$$

In particular, we have that

$$\beta_{F, \underline{x}}(\overline{\pi}(a)) = \overline{\pi}(a) = \beta_{F^\perp, \underline{y}}(\overline{\pi}(a)) \text{ for all } \underline{x} \in \mathbb{T}^r, \underline{y} \in \mathbb{T}^{d-r}, a \in A.$$

By combining the preceding observations and using the remarks of [8, p. 133], we obtain faithful conditional expectations

$$E_{\beta_F} \colon \mathcal{NT}_X \to \mathcal{NT}_X^{\beta_F}; f \mapsto \int_{\mathbb{T}^r} \beta_{F, \underline{x}}(f) \, \mathrm{d}\underline{x} \text{ for all } f \in \mathcal{NT}_X,$$

and

$$E_{\beta_{F^\perp}} \colon \mathcal{NT}_X \to \mathcal{NT}_X^{\beta_{F^\perp}}; f \mapsto \int_{\mathbb{T}^{d-r}} \beta_{F^\perp, \underline{y}}(f) \, \mathrm{d}\underline{y} \text{ for all } f \in \mathcal{NT}_X.$$

Definition 5.5.11. Let X be a compactly aligned product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A and let $F = [r]$ for some $r < d$. We define the C*-algebra $B^{F^\perp}$ by

$$B^{F^\perp} := \mathrm{C}^*(\overline{\pi}(A), \overline{t}_{\underline{i}}(X_{\underline{i}}) \mid i \in F^c) \subseteq \mathcal{NT}_X,$$

and the collection of linear spaces $Y^F := \{Y_{\underline{n}}^F\}_{\underline{n} \in \mathbb{Z}_+^r}$ by

$$Y_{\underline{0}}^F := B^{F^\perp} \quad \text{and} \quad Y_{\underline{n}}^F := [\overline{t}_{(\underline{n}, \underline{0})}(X_{(\underline{n}, \underline{0})}) B^{F^\perp}] \subseteq \mathcal{NT}_X \text{ for all } \underline{n} \in \mathbb{Z}_+^r \setminus \{\underline{0}\}.$$

Since $\overline{\pi}(A) \subseteq B^{F^\perp}$, by using an approximate unit of A we obtain that $\overline{t}_{(\underline{n}, \underline{0})}(X_{(\underline{n}, \underline{0})}) \subseteq Y_{\underline{n}}^F$ for all $\underline{n} \in \mathbb{Z}_+^r$.

Proposition 5.5.12. *Let X be a compactly aligned product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A and let $F = [r]$ for some $r < d$. Then we have that*

$$B^{F^\perp} = \mathrm{C}^*(\overline{\pi}(A), \overline{t}_{\underline{n}}(X_{\underline{n}}) \mid \underline{n} \perp F) = \overline{\operatorname{span}}\{\overline{t}_{\underline{n}}(X_{\underline{n}})\overline{t}_{\underline{m}}(X_{\underline{m}})^* \mid \underline{n}, \underline{m} \perp F\}.$$

Proof. First we show that $B^{F^\perp} = \mathrm{C}^*(\overline{\pi}(A), \overline{t}_{\underline{n}}(X_{\underline{n}}) \mid \underline{n} \perp F)$. The forward inclusion is immediate. For the reverse inclusion, it suffices to show that

$$\overline{t}_{\underline{n}}(X_{\underline{n}}) \subseteq B^{F^\perp} \text{ for all } \underline{0} \neq \underline{n} \perp F.$$

We proceed by induction on $|\underline{n}|$. When $|\underline{n}| = 1$, we have that $\underline{n} = \underline{i}$ for some $i \in [d]$ satisfying $\underline{i} \perp F$. Thus $i \in F^c$ and so $\overline{t}_{\underline{n}}(X_{\underline{n}}) \subseteq B^{F^\perp}$ by definition.

Now suppose that $\overline{t}_{\underline{m}}(X_{\underline{m}}) \subseteq B^{F^\perp}$ whenever $\underline{m} \perp F$ and $|\underline{m}| = N$ for fixed $N \in \mathbb{N}$. Take $\underline{n} \perp F$ such that $|\underline{n}| = N + 1$, so that we may write $\underline{n} = \underline{m} + \underline{i}$ for some $\underline{m} \perp F$ satisfying $|\underline{m}| = N$ and some $i \in F^c$. We obtain that

$$\overline{t}_{\underline{n}}(X_{\underline{n}}) = \overline{t}_{\underline{m}+\underline{i}}(X_{\underline{m}+\underline{i}}) \subseteq [\overline{t}_{\underline{m}}(X_{\underline{m}})\overline{t}_{\underline{i}}(X_{\underline{i}})] \subseteq [B^{F^\perp} B^{F^\perp}] \subseteq B^{F^\perp},$$

using that $X_{\underline{m}} \otimes_A X_{\underline{i}} \cong X_{\underline{n}}$ via the multiplication map $u_{\underline{m},\underline{i}}$ in the first inclusion, and the inductive hypothesis and base case in the second inclusion. By induction we are done, and we ascertain that the first equality of the statement holds.

For notational convenience, let

$$C := \overline{\operatorname{span}}\{\overline{t}_{\underline{n}}(X_{\underline{n}})\overline{t}_{\underline{m}}(X_{\underline{m}})^* \mid \underline{n}, \underline{m} \perp F\}.$$

It remains to show that $\mathrm{C}^*(\overline{\pi}(A), \overline{t}_{\underline{n}}(X_{\underline{n}}) \mid \underline{n} \perp F) = C$. The reverse inclusion is immediate. For the forward inclusion, it suffices to show that C is a C*-subalgebra of $\mathcal{NT}_X$ that contains the generators of $\mathrm{C}^*(\overline{\pi}(A), \overline{t}_{\underline{n}}(X_{\underline{n}}) \mid \underline{n} \perp F)$. To this end, it is clear that C is a selfadjoint closed linear subspace of $\mathcal{NT}_X$. It remains to see that C is closed under multiplication. It follows from the preceding observations that it suffices to show that

$$\overline{t}_{\underline{n}}(X_{\underline{n}})\overline{t}_{\underline{m}}(X_{\underline{m}})^*\overline{t}_{\underline{k}}(X_{\underline{k}})\overline{t}_{\underline{\ell}}(X_{\underline{\ell}})^* \subseteq C \text{ for all } \underline{n}, \underline{m}, \underline{k}, \underline{\ell} \perp F.$$

We have that

$$\begin{aligned}\overline{t}_{\underline{n}}(X_{\underline{n}})\overline{t}_{\underline{m}}(X_{\underline{m}})^*\overline{t}_{\underline{k}}(X_{\underline{k}})\overline{t}_{\underline{\ell}}(X_{\underline{\ell}})^* &\subseteq [\overline{t}_{\underline{n}}(X_{\underline{n}})\overline{t}_{-\underline{m}+\underline{m}\vee\underline{k}}(X_{-\underline{m}+\underline{m}\vee\underline{k}})\overline{t}_{-\underline{k}+\underline{m}\vee\underline{k}}(X_{-\underline{k}+\underline{m}\vee\underline{k}})^*\overline{t}_{\underline{\ell}}(X_{\underline{\ell}})^*] \\ &\subseteq [\overline{t}_{\underline{n}-\underline{m}+\underline{m}\vee\underline{k}}(X_{\underline{n}-\underline{m}+\underline{m}\vee\underline{k}})\overline{t}_{\underline{\ell}-\underline{k}+\underline{m}\vee\underline{k}}(X_{\underline{\ell}-\underline{k}+\underline{m}\vee\underline{k}})^*] \subseteq C,\end{aligned}$$

using Nica-covariance in the first inclusion and that

$$\underline{n} - \underline{m} + \underline{m} \vee \underline{k} \perp F \quad \text{and} \quad \underline{\ell} - \underline{k} + \underline{m} \vee \underline{k} \perp F$$

in the final inclusion. Hence C is a C*-subalgebra of $\mathcal{NT}_X$, as claimed.

To see that C contains the generators of $\mathrm{C}^*(\overline{\pi}(A), \overline{t}_{\underline{n}}(X_{\underline{n}}) \mid \underline{n} \perp F)$, fix $\underline{n} \perp F, \xi_{\underline{n}} \in X_{\underline{n}}$ and an approximate unit $(u_\lambda)_{\lambda \in \Lambda}$ of A. We deduce that

$$\overline{t}_{\underline{n}}(\xi_{\underline{n}}) = \|\cdot\| \text{-} \lim_{\lambda} \overline{t}_{\underline{n}}(\xi_{\underline{n}} u_\lambda) = \|\cdot\| \text{-} \lim_{\lambda} \overline{t}_{\underline{n}}(\xi_{\underline{n}}) \overline{\pi}(u_\lambda)^* \in C,$$

as required. Thus $\mathrm{C}^*(\overline{\pi}(A), \overline{t}_{\underline{n}}(X_{\underline{n}}) \mid \underline{n} \perp F) = C$, finishing the proof. $\square$

Note that $\beta_F|_{B^{F^\perp}} = \mathrm{id}$. We can restrict $\beta_{F^\perp}$ to $B^{F^\perp}$ to obtain a point-norm continuous family

$$\beta_{F^\perp}|_{B^{F^\perp}} := \{\beta_{F^\perp, \underline{z}}|_{B^{F^\perp}}\}_{\underline{z} \in \mathbb{T}^{d-r}}$$

of $*$-automorphisms of $B^{F^\perp}$. In turn, the remarks of [8, p. 133] give a faithful conditional expectation

$$E_{\beta_{F^\perp}|_{B^{F^\perp}}} : B^{F^\perp} \to (B^{F^\perp})^{\beta_{F^\perp}|_{B^{F^\perp}}}; f \mapsto \int_{\mathbb{T}^{d-r}} \beta_{F^\perp, \underline{z}}(f) \, \mathrm{d}\underline{z} \text{ for all } f \in B^{F^\perp}.$$

Notice that $E_{\beta_{F^\perp}|_{B^{F^\perp}}}$ is nothing but the restriction of $E_{\beta_{F^\perp}}$ to $B^{F^\perp}$. We may characterise $B^{F^\perp}$ as the Toeplitz-Nica-Pimsner algebra of a compactly aligned product system over $\mathbb{Z}_+^{d-r}$.

Definition 5.5.13. Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A and let $F = [r]$ for some $r < d$. We define the collection of linear spaces $Z^{F^\perp} := \{Z_{\underline{n}}^{F^\perp}\}_{\underline{n} \in \mathbb{Z}_+^{d-r}}$ by

$$Z_{\underline{0}}^{F^\perp} := A \quad \text{and} \quad Z_{\underline{n}}^{F^\perp} := X_{(\underline{0}, \underline{n})} \text{ for all } \underline{n} \in \mathbb{Z}_+^{d-r} \setminus \{\underline{0}\}.$$

We can endow $Z^{F^\perp}$ with a canonical product system structure, inherited from X.

Proposition 5.5.14. *Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A and let $F = [r]$ for some $r < d$. Then $Z^{F^\perp}$ inherits a product system structure from X. Moreover, if every non-trivial fibre of X admits a finite frame (resp. X is compactly aligned, strong compactly aligned), then every non-trivial fibre of $Z^{F^\perp}$ admits a finite frame (resp. $Z^{F^\perp}$ is compactly aligned, strong compactly aligned).*

Proof. For notational convenience, we set $Z := Z^{F^\perp}$. For each $\underline{n} \in \mathbb{Z}_+^{d-r}$, we equip $Z_{\underline{n}}$ with the C*-correspondence structure of $X_{(\underline{0}, \underline{n})}$, as guaranteed by the product system structure of X. Next, let $\{u_{\underline{n}, \underline{m}}\}_{\underline{n}, \underline{m} \in \mathbb{Z}_+^d}$ denote the multiplication maps of X. Fixing $\underline{n}, \underline{m} \in \mathbb{Z}_+^{d-r}$, we set

$$v_{\underline{n}, \underline{m}} := u_{(\underline{0}, \underline{n}), (\underline{0}, \underline{m})}.$$

It follows that Z, together with the multiplication maps $\{v_{\underline{n}, \underline{m}}\}_{\underline{n}, \underline{m} \in \mathbb{Z}_+^{d-r}}$, constitutes a product system over $\mathbb{Z}_+^{d-r}$ with coefficients in A. Indeed, the product system axioms hold for Z because they hold for X by assumption. This proves the first claim.

Since Z is a subfamily of X, it is clear that every non-trivial fibre of Z admits a finite frame whenever the same is true of X. To see that (strong) compact alignment descends to Z, let $\{\iota_{\underline{n}}^{\underline{n}+\underline{m}}\}_{\underline{n},\underline{m}\in\mathbb{Z}_+^d}$ (resp. $\{j_{\underline{n}}^{\underline{n}+\underline{m}}\}_{\underline{n},\underline{m}\in\mathbb{Z}_+^{d-r}}$) denote the connecting $*$-homomorphisms of X (resp. Z). By definition we have that

$$j_{\underline{n}}^{\underline{n}+\underline{m}} \equiv \iota_{(\underline{0},\underline{n})}^{(\underline{0},\underline{n})+(\underline{0},\underline{m})} \text{ for all } \underline{n},\underline{m}\in\mathbb{Z}_+^{d-r}.$$

Assume that X is compactly aligned and fix $\underline{n},\underline{m}\in\mathbb{Z}_+^{d-r}\setminus\{\underline{0}\}$. We have that

$$\begin{aligned}
j_{\underline{n}}^{\underline{n}\vee\underline{m}}(\mathcal{K}(Z_{\underline{n}}))j_{\underline{m}}^{\underline{n}\vee\underline{m}}(\mathcal{K}(Z_{\underline{m}})) &\equiv \iota_{(\underline{0},\underline{n})}^{(\underline{0},\underline{n}\vee\underline{m})}(\mathcal{K}(X_{(\underline{0},\underline{n})}))\iota_{(\underline{0},\underline{m})}^{(\underline{0},\underline{n}\vee\underline{m})}(\mathcal{K}(X_{(\underline{0},\underline{m})}))\\
&= \iota_{(\underline{0},\underline{n})}^{(\underline{0},\underline{n})\vee(\underline{0},\underline{m})}(\mathcal{K}(X_{(\underline{0},\underline{n})}))\iota_{(\underline{0},\underline{m})}^{(\underline{0},\underline{n})\vee(\underline{0},\underline{m})}(\mathcal{K}(X_{(\underline{0},\underline{m})}))\\
&\subseteq \mathcal{K}(X_{(\underline{0},\underline{n}\vee\underline{m})}) \equiv \mathcal{K}(Z_{\underline{n}\vee\underline{m}}),
\end{aligned}$$

using compact alignment of X in the inclusion. Hence Z is compactly aligned, as required.

Finally, assume that X is strong compactly aligned. In particular X is compactly aligned and therefore Z is compactly aligned by the preceding argument. It remains to check that Z satisfies (2.13), so fix $i\in[d-r]$ and $\underline{n}\in\mathbb{Z}_+^{d-r}\setminus\{\underline{0}\}$ such that $\underline{n}\perp\underline{i}$. We have that

$$j_{\underline{n}}^{\underline{n}+\underline{i}}(\mathcal{K}(Z_{\underline{n}})) \equiv \iota_{(\underline{0},\underline{n})}^{(\underline{0},\underline{n})+(\underline{0},\underline{i})}(\mathcal{K}(X_{(\underline{0},\underline{n})})) \subseteq \mathcal{K}(X_{(\underline{0},\underline{n}+\underline{i})}) \equiv \mathcal{K}(Z_{\underline{n}+\underline{i}}),$$

using strong compact alignment of X in the inclusion, noting that $(\underline{0},\underline{n})\perp(\underline{0},\underline{i})$. Thus Z is strong compactly aligned, finishing the proof. □

Proposition 5.5.15. *Let X be a compactly aligned product system over $\mathbb{Z}_+^d$ with coefficients in a C^*-algebra A and let $F=[r]$ for some $r<d$. Then the maps*

$$\begin{aligned}
&\pi\colon A\to B^{F^\perp}; \pi(a)=\overline{\pi}(a) \text{ for all } a\in A,\\
&t_{\underline{n}}\colon Z_{\underline{n}}^{F^\perp}\to B^{F^\perp}; t_{\underline{n}}(z_{\underline{n}})=\overline{t}_{(\underline{0},\underline{n})}(z_{\underline{n}}) \text{ for all } z_{\underline{n}}\in Z_{\underline{n}}^{F^\perp}, \underline{n}\in\mathbb{Z}_+^{d-r}\setminus\{\underline{0}\}
\end{aligned}$$

form an injective Nica-covariant representation (π,t) of $Z^{F^\perp}$ which induces a canonical $*$-isomorphism $\mathcal{NT}_{Z^{F^\perp}}\cong B^{F^\perp}$.

Proof. For notational convenience, we set $B:=B^{F^\perp}$ and $Z:=Z^{F^\perp}$. Note that Z is compactly aligned by Proposition 5.5.14. The maps of the statement are well-defined by Proposition 5.5.12. It is routine to check that (π,t) constitutes an injective representation of Z. Additionally, we have that

$$\psi_{\underline{n}}=\overline{\psi}_{(\underline{0},\underline{n})} \text{ for all } \underline{n}\in\mathbb{Z}_+^{d-r}.$$

Let $\{\iota_{\underline{n}}^{\underline{n}+\underline{m}}\}_{\underline{n},\underline{m}\in\mathbb{Z}_+^d}$ (resp. $\{j_{\underline{n}}^{\underline{n}+\underline{m}}\}_{\underline{n},\underline{m}\in\mathbb{Z}_+^{d-r}}$) denote the family of connecting $*$-homomorphisms of X (resp. Z). Recall from the proof of Proposition 5.5.14 that

$$j_{\underline{n}}^{\underline{n}+\underline{m}} \equiv \iota_{(\underline{0},\underline{n})}^{(\underline{0},\underline{n})+(\underline{0},\underline{m})} \text{ for all } \underline{n},\underline{m}\in\mathbb{Z}_+^{d-r}.$$

Therefore, fixing $\underline{n}, \underline{m} \in \mathbb{Z}_+^{d-r} \setminus \{\underline{0}\}$, $k_{\underline{n}} \in \mathcal{K}(Z_{\underline{n}})$ and $k_{\underline{m}} \in \mathcal{K}(Z_{\underline{m}})$, we obtain that

$$\begin{aligned}\psi_{\underline{n}}(k_{\underline{n}})\psi_{\underline{m}}(k_{\underline{m}}) &= \overline{\psi}_{(\underline{0},\underline{n})}(k_{\underline{n}})\overline{\psi}_{(\underline{0},\underline{m})}(k_{\underline{m}}) = \overline{\psi}_{(\underline{0},\underline{n})\vee(\underline{0},\underline{m})}(\iota_{(\underline{0},\underline{n})}^{(\underline{0},\underline{n})\vee(\underline{0},\underline{m})}(k_{\underline{n}})\iota_{(\underline{0},\underline{m})}^{(\underline{0},\underline{n})\vee(\underline{0},\underline{m})}(k_{\underline{m}})) \\ &= \overline{\psi}_{(\underline{0},\underline{n}\vee\underline{m})}(\iota_{(\underline{0},\underline{n})}^{(\underline{0},\underline{n}\vee\underline{m})}(k_{\underline{n}})\iota_{(\underline{0},\underline{m})}^{(\underline{0},\underline{n}\vee\underline{m})}(k_{\underline{m}})) = \psi_{\underline{n}\vee\underline{m}}(j_{\underline{n}}^{\underline{n}\vee\underline{m}}(k_{\underline{n}})j_{\underline{m}}^{\underline{n}\vee\underline{m}}(k_{\underline{m}})),\end{aligned}$$

using Nica-covariance of $(\overline{\pi}, \overline{t})$ in the second equality. Hence (π, t) is Nica-covariant. Moreover, we have that $\mathrm{C}^*(\pi, t) = B$. Indeed, to see this it suffices to show that $\mathrm{C}^*(\pi, t)$ contains the generators of B. To this end, first note that

$$\overline{\pi}(a) = \pi(a) \in \mathrm{C}^*(\pi, t) \text{ for all } a \in A.$$

Fixing $i \in F^c$, we also have that

$$\overline{t}_{(\underline{0},\underline{i})}(\xi_{(\underline{0},\underline{i})}) = t_{\underline{i}}(\xi_{(\underline{0},\underline{i})}) \in \mathrm{C}^*(\pi, t) \text{ for all } \xi_{(\underline{0},\underline{i})} \in X_{(\underline{0},\underline{i})}.$$

Thus $\mathrm{C}^*(\pi, t)$ contains the generators of B, as required.

Applying the universal property of $\mathcal{NT}_Z$, we obtain a (unique) canonical $*$-epimorphism

$$\pi \times t \colon \mathcal{NT}_Z \to B.$$

In an abuse of notation, we will use $(\overline{\pi}_Z, \overline{t}_Z)$ to denote the Fock representation of Z, and will identify $\mathcal{NT}_Z$ with $\mathrm{C}^*(\overline{\pi}_Z, \overline{t}_Z)$. It suffices to prove that $\pi \times t$ is injective.

To this end, fix $\underline{z} \in \mathbb{T}^{d-r}$ and $a \in A$. We have that

$$\beta_{F^\perp,\underline{z}}(\pi(a)) = \beta_{F^\perp,\underline{z}}(\overline{\pi}(a)) = \overline{\pi}(a) = \pi(a).$$

Additionally, fixing $\underline{n} \in \mathbb{Z}_+^{d-r} \setminus \{\underline{0}\}$ and $\xi_{(\underline{0},\underline{n})} \in X_{(\underline{0},\underline{n})}$, we obtain that

$$\beta_{F^\perp,\underline{z}}(t_{\underline{n}}(\xi_{(\underline{0},\underline{n})})) = \beta_{F^\perp,\underline{z}}(\overline{t}_{(\underline{0},\underline{n})}(\xi_{(\underline{0},\underline{n})})) = \underline{z}^{\underline{n}}\overline{t}_{(\underline{0},\underline{n})}(\xi_{(\underline{0},\underline{n})}) = \underline{z}^{\underline{n}}t_{\underline{n}}(\xi_{(\underline{0},\underline{n})}).$$

Combining these facts with the remarks succeeding Proposition 5.5.12, we deduce that $\beta_{F^\perp}|_B$ constitutes a gauge action of (π, t). Let γ denote the gauge action of $(\overline{\pi}_Z, \overline{t}_Z)$. Observe that $\pi \times t$ intertwines the gauge actions, i.e., $\pi \times t$ is equivariant. An application of [8, Proposition 4.5.1] then gives that $\pi \times t$ is injective if and only if $(\pi \times t)|_{\mathcal{NT}_Z^\gamma}$ is injective. Thus it suffices to show that

$$(\pi \times t)|_{\mathcal{NT}_Z^\gamma} \colon \mathcal{NT}_Z^\gamma \to B^{\beta_{F^\perp}|_B}$$

is injective.

To see this, first recall that

$$\mathcal{F}Z := \sum_{\underline{n} \in \mathbb{Z}_+^{d-r}} Z_{\underline{n}} = \sum_{\underline{n} \in \mathbb{Z}_+^{d-r}} X_{(\underline{0},\underline{n})}.$$

Consequently, we have that $\mathcal{F}Z$ embeds isometrically within $\mathcal{F}X$ via the map $\iota\colon \mathcal{F}Z \to \mathcal{F}X$ defined by $\iota((z_{\underline{m}})_{\underline{m}\in\mathbb{Z}_+^{d-r}}) = (\xi_{\underline{n}})_{\underline{n}\in\mathbb{Z}_+^d}$, where

$$\xi_{\underline{n}} = \begin{cases} z_{\underline{n}_{F^\perp}} & \text{if } \underline{n}_F = \underline{0}, \\ 0 & \text{otherwise,} \end{cases}$$

for all $(z_{\underline{m}})_{\underline{m}\in\mathbb{Z}_+^{d-r}} \in \mathcal{F}Z$. Note that $\iota(\mathcal{F}Z) = \sum_{\underline{n}\perp F} X_{\underline{n}}$ and so ι implements a unitary equivalence between $\mathcal{F}Z$ and $\sum_{\underline{n}\perp F} X_{\underline{n}}$. In turn, we obtain a $*$-isomorphism Φ given by

$$\Phi\colon \mathcal{L}(\sum_{\underline{n}\perp F} X_{\underline{n}}) \to \mathcal{L}(\mathcal{F}Z); S \mapsto \iota^{-1}\circ S\circ\iota \text{ for all } S \in \mathcal{L}(\sum_{\underline{n}\perp F} X_{\underline{n}}).$$

Next, observe that

$$B^{\beta_{F^\perp}|_B} = E_{\beta_{F^\perp}}(B) = \overline{\operatorname{span}}\{\bar{t}_{(\underline{0},\underline{n})}(X_{(\underline{0},\underline{n})})\bar{t}_{(\underline{0},\underline{n})}(X_{(\underline{0},\underline{n})})^* \mid \underline{n}\in\mathbb{Z}_+^{d-r}\},$$

using Proposition 5.5.12 in the final equality. Hence $\sum_{\underline{n}\perp F} X_{\underline{n}}$ is a reducing subspace for $B^{\beta_{F^\perp}|_B}$, and in turn the map

$$\Psi\colon B^{\beta_{F^\perp}|_B} \to \mathcal{L}(\sum_{\underline{n}\perp F} X_{\underline{n}}); b\mapsto b|_{\sum_{\underline{n}\perp F} X_{\underline{n}}} \text{ for all } b\in B^{\beta_{F^\perp}|_B}$$

constitutes a well-defined $*$-homomorphism.

We now have the following sequence of $*$-homomorphisms:

$$\mathcal{NT}_Z^\gamma \xrightarrow{(\pi\times t)|_{\mathcal{NT}_Z^\gamma}} B^{\beta_{F^\perp}|_B} \xrightarrow{\Psi} \mathcal{L}(\sum_{\underline{n}\perp F} X_{\underline{n}}) \xrightarrow[\cong]{\Phi} \mathcal{L}(\mathcal{F}Z).$$

We claim that $\Phi\circ\Psi\circ(\pi\times t)|_{\mathcal{NT}_Z^\gamma} = \mathrm{id}_{\mathcal{NT}_Z^\gamma}$. To this end, recall that

$$\mathcal{NT}_Z^\gamma = \overline{\operatorname{span}}\{\bar{t}_{Z,\underline{n}}(Z_{\underline{n}})\bar{t}_{Z,\underline{n}}(Z_{\underline{n}})^* \mid \underline{n}\in\mathbb{Z}_+^{d-r}\}.$$

Since $\Phi\circ\Psi\circ(\pi\times t)|_{\mathcal{NT}_Z^\gamma}$ and $\mathrm{id}_{\mathcal{NT}_Z^\gamma}$ are in particular linear and continuous, it suffices to show that the equality holds on the generators of $\mathcal{NT}_Z^\gamma$. Accordingly, fix $\underline{n}\in\mathbb{Z}_+^{d-r}$ and $z_{\underline{n}}, z_{\underline{n}}' \in Z_{\underline{n}}$. We obtain that

$$\begin{aligned}(\Phi\circ\Psi\circ(\pi\times t)|_{\mathcal{NT}_Z^\gamma})(\bar{t}_{Z,\underline{n}}(z_{\underline{n}})\bar{t}_{Z,\underline{n}}(z_{\underline{n}}')^*) &= (\Phi\circ\Psi)(\bar{t}_{(\underline{0},\underline{n})}(z_{\underline{n}})\bar{t}_{(\underline{0},\underline{n})}(z_{\underline{n}}')^*) \\ &= \Phi((\bar{t}_{(\underline{0},\underline{n})}(z_{\underline{n}})\bar{t}_{(\underline{0},\underline{n})}(z_{\underline{n}}')^*)|_{\sum_{\underline{m}\perp F} X_{\underline{m}}}) \\ &= \iota^{-1}\circ(\bar{t}_{(\underline{0},\underline{n})}(z_{\underline{n}})\bar{t}_{(\underline{0},\underline{n})}(z_{\underline{n}}')^*)|_{\sum_{\underline{m}\perp F} X_{\underline{m}}}\circ\iota.\end{aligned}$$

It remains to show that

$$\bar{t}_{Z,\underline{n}}(z_{\underline{n}})\bar{t}_{Z,\underline{n}}(z_{\underline{n}}')^* = \iota^{-1}\circ(\bar{t}_{(\underline{0},\underline{n})}(z_{\underline{n}})\bar{t}_{(\underline{0},\underline{n})}(z_{\underline{n}}')^*)|_{\sum_{\underline{m}\perp F} X_{\underline{m}}}\circ\iota. \tag{5.19}$$

Since the maps involved are adjointable and thus in particular linear and bounded, it suffices to show that the equality holds on every direct summand of $\mathcal{F}Z$. For notational convenience, we denote the right hand side of (5.19) by S. First assume that $\underline{n} = \underline{0}$. Fixing $\underline{m} \in \mathbb{Z}_+^{d-r}$ and $z_{\underline{m}} \in Z_{\underline{m}}$, we have that

$$\bar{t}_{Z,\underline{0}}(z_{\underline{0}})\bar{t}_{Z,\underline{0}}(z'_{\underline{0}})^* z_{\underline{m}} = \phi_{(\underline{0},\underline{m})}(z_{\underline{0}} z'^*_{\underline{0}}) z_{\underline{m}} = S z_{\underline{m}}.$$

Hence (5.19) holds when $\underline{n} = \underline{0}$. Now suppose that $\underline{n} \neq \underline{0}$. If $\underline{n} \not\leq \underline{m}$, then we obtain that

$$\bar{t}_{Z,\underline{n}}(z_{\underline{n}})\bar{t}_{Z,\underline{n}}(z'_{\underline{n}})^* z_{\underline{m}} = 0 = S z_{\underline{m}},$$

using that $(\underline{0}, \underline{n}) \not\leq (\underline{0}, \underline{m})$ in the final equality. Otherwise, fixing $z''_{\underline{n}} \in Z_{\underline{n}}$ and $z_{\underline{m}-\underline{n}} \in Z_{\underline{m}-\underline{n}}$, we have that

$$\bar{t}_{Z,\underline{n}}(z_{\underline{n}})\bar{t}_{Z,\underline{n}}(z'_{\underline{n}})^*(z''_{\underline{n}} z_{\underline{m}-\underline{n}}) = z_{\underline{n}}(\phi_{(\underline{0},\underline{m}-\underline{n})}(\langle z'_{\underline{n}}, z''_{\underline{n}}\rangle) z_{\underline{m}-\underline{n}}) = S(z''_{\underline{n}} z_{\underline{m}-\underline{n}}).$$

Using that $Z_{\underline{n}} \otimes_A Z_{\underline{m}-\underline{n}} \cong Z_{\underline{m}}$ via the multiplication map $u_{(\underline{0},\underline{n}),(\underline{0},\underline{m}-\underline{n})}$, the preceding calculation implies that

$$\bar{t}_{Z,\underline{n}}(z_{\underline{n}})\bar{t}_{Z,\underline{n}}(z'_{\underline{n}})^* z_{\underline{m}} = S z_{\underline{m}}.$$

Hence (5.19) holds in all cases for $\underline{n}$, as required. In total, we have that $\Phi \circ \Psi$ is a left inverse for $(\pi \times t)|_{\mathcal{NT}_Z^\gamma}$. Hence $(\pi \times t)|_{\mathcal{NT}_Z^\gamma}$ is injective, finishing the proof. □

We isolate the following corollary from the proof of Proposition 5.5.15.

Corollary 5.5.16. *Let X be a compactly aligned product system over $\mathbb{Z}_+^d$ with coefficients in a C^*-algebra A and let $F = [r]$ for some $r < d$. Then the map*

$$\Psi \colon (B^{F^\perp})^{\beta_{F^\perp}|_{B^{F^\perp}}} \to \mathcal{L}(\sum_{\underline{n}\perp F} X_{\underline{n}}); b \mapsto b|_{\sum_{\underline{n}\perp F} X_{\underline{n}}} \text{ for all } b \in (B^{F^\perp})^{\beta_{F^\perp}|_{B^{F^\perp}}}$$

is an injective $*$-homomorphism. In particular, we have that

$$\|b|_{\sum_{\underline{n}\perp F} X_{\underline{n}}}\| = \|b\| \text{ for all } b \in (B^{F^\perp})^{\beta_{F^\perp}|_{B^{F^\perp}}}.$$

Next we focus on Y^F and explore the connection with X.

Lemma 5.5.17. *Let X be a compactly aligned product system over $\mathbb{Z}_+^d$ with coefficients in a C^*-algebra A and let $F = [r]$ for some $r < d$. Then $Y_{\underline{n}}^F$ is a sub-C^*-correspondence of $\mathcal{NT}_X$ over $B^{F^\perp}$ for all $\underline{n} \in \mathbb{Z}_+^r$.*

Proof. For notational convenience, we set $B := B^{F^\perp}$ and $Y := Y^F$. We view $Y_{\underline{0}}$ as a C*-correspondence over itself in the usual way. For $\underline{n} \in \mathbb{Z}_+^r \setminus \{\underline{0}\}$, the space $Y_{\underline{n}}$ inherits the usual inner product from $\mathcal{NT}_X$, now taking values in B. This follows by noting that

$$[\bar{t}_{(\underline{n},\underline{0})}(X_{(\underline{n},\underline{0})})B]^*[\bar{t}_{(\underline{n},\underline{0})}(X_{(\underline{n},\underline{0})})B] \subseteq [B\bar{t}_{(\underline{n},\underline{0})}(X_{(\underline{n},\underline{0})})^*][\bar{t}_{(\underline{n},\underline{0})}(X_{(\underline{n},\underline{0})})B] \subseteq [B\overline{\pi}(A)B] \subseteq B.$$

The space $Y_{\underline{n}}$ also inherits the usual right multiplication from $\mathcal{NT}_X$, now by elements of B. This follows from the observation that

$$[\overline{t}_{(\underline{n},\underline{0})}(X_{(\underline{n},\underline{0})})B]B \subseteq [\overline{t}_{(\underline{n},\underline{0})}(X_{(\underline{n},\underline{0})})BB] \subseteq [\overline{t}_{(\underline{n},\underline{0})}(X_{(\underline{n},\underline{0})})B].$$

Letting $\|\cdot\|_{Y_{\underline{n}}}$ denote the inner product norm on $Y_{\underline{n}}$ and $\|\cdot\|_{\mathcal{NT}_X}$ denote the usual C*-norm on $\mathcal{NT}_X$, we have that

$$\|y_{\underline{n}}\|_{Y_{\underline{n}}} = \| \langle y_{\underline{n}}, y_{\underline{n}} \rangle \|_{\mathcal{NT}_X}^{\frac{1}{2}} = \|y_{\underline{n}}^* y_{\underline{n}}\|_{\mathcal{NT}_X}^{\frac{1}{2}} = \|y_{\underline{n}}\|_{\mathcal{NT}_X} \text{ for all } y_{\underline{n}} \in Y_{\underline{n}}.$$

Since $Y_{\underline{n}}$ is closed in $\mathcal{NT}_X$ with respect to $\|\cdot\|_{\mathcal{NT}_X}$ by definition, the preceding computation shows that $Y_{\underline{n}}$ is complete with respect to $\|\cdot\|_{Y_{\underline{n}}}$. In total, we have that $Y_{\underline{n}}$ is a right Hilbert B-module.

It remains to equip $Y_{\underline{n}}$ with a left action $\phi_{Y_{\underline{n}}}\colon B \to \mathcal{L}(Y_{\underline{n}})$. To this end, fix $b \in B$ and define

$$\phi_{Y_{\underline{n}}}(b)\colon Y_{\underline{n}} \to Y_{\underline{n}}; \phi_{Y_{\underline{n}}}(b)y_{\underline{n}} = by_{\underline{n}} \text{ for all } y_{\underline{n}} \in Y_{\underline{n}},$$

where the multiplication is performed in $\mathcal{NT}_X$. To see that this map is well-defined, fix $\underline{k}, \underline{\ell} \in \mathbb{Z}_+^d$ satisfying $\underline{k}, \underline{\ell} \perp F$. Thus we may write $\underline{k} = (\underline{0}, \underline{k}_{F^\perp})$ and $\underline{\ell} = (\underline{0}, \underline{\ell}_{F^\perp})$. Since $(\underline{0}, \underline{\ell}_{F^\perp}) \perp (\underline{n}, \underline{0})$, Nica-covariance yields that

$$\begin{aligned}
&\overline{t}_{(\underline{0},\underline{k}_{F^\perp})}(X_{(\underline{0},\underline{k}_{F^\perp})})\overline{t}_{(\underline{0},\underline{\ell}_{F^\perp})}(X_{(\underline{0},\underline{\ell}_{F^\perp})})^*\overline{t}_{(\underline{n},\underline{0})}(X_{(\underline{n},\underline{0})})B \subseteq \\
&\qquad \subseteq [\overline{t}_{(\underline{0},\underline{k}_{F^\perp})}(X_{(\underline{0},\underline{k}_{F^\perp})})\overline{t}_{(\underline{n},\underline{0})}(X_{(\underline{n},\underline{0})})\overline{t}_{(\underline{0},\underline{\ell}_{F^\perp})}(X_{(\underline{0},\underline{\ell}_{F^\perp})})^*B] \\
&\qquad \subseteq [\overline{t}_{(\underline{n},\underline{0})}(X_{(\underline{n},\underline{0})})\overline{t}_{(\underline{0},\underline{k}_{F^\perp})}(X_{(\underline{0},\underline{k}_{F^\perp})})\overline{t}_{(\underline{0},\underline{\ell}_{F^\perp})}(X_{(\underline{0},\underline{\ell}_{F^\perp})})^*B] \\
&\qquad \subseteq [\overline{t}_{(\underline{n},\underline{0})}(X_{(\underline{n},\underline{0})})BB] \subseteq [\overline{t}_{(\underline{n},\underline{0})}(X_{(\underline{n},\underline{0})})B] \equiv Y_{\underline{n}},
\end{aligned}$$

using that $X_{(\underline{n},\underline{0})} \otimes_A X_{(\underline{0},\underline{k}_{F^\perp})} \cong X_{(\underline{n},\underline{k}_{F^\perp})}$ via the multiplication map $u_{(\underline{n},\underline{0}),(\underline{0},\underline{k}_{F^\perp})}$ in the third line and Proposition 5.5.12 in the last line. It then follows by another application of Proposition 5.5.12 that $BY_{\underline{n}} \subseteq Y_{\underline{n}}$, and hence $\phi_{Y_{\underline{n}}}(b)$ is well-defined. Fixing $y_{\underline{n}}, y'_{\underline{n}} \in Y_{\underline{n}}$, we have that

$$\left\langle \phi_{Y_{\underline{n}}}(b)y_{\underline{n}}, y'_{\underline{n}} \right\rangle = \left\langle by_{\underline{n}}, y'_{\underline{n}} \right\rangle = y_{\underline{n}}^* b^* y'_{\underline{n}} = \left\langle y_{\underline{n}}, b^* y'_{\underline{n}} \right\rangle = \left\langle y_{\underline{n}}, \phi_{Y_{\underline{n}}}(b^*)y'_{\underline{n}} \right\rangle,$$

and so $\phi_{Y_{\underline{n}}}(b)^* = \phi_{Y_{\underline{n}}}(b^*)$. Thus $\phi_{Y_{\underline{n}}}$ is a well-defined $*$-preserving map. It is routine to check that $\phi_{Y_{\underline{n}}}$ is an algebra homomorphism, and the proof is complete. □

Proposition 5.5.18. *Let X be a compactly aligned product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A and let $F = [r]$ for some $r < d$. Then Y^F carries a canonical structure as a product system over $\mathbb{Z}_+^r$ with coefficients in $B^{F^\perp}$, with the multiplication maps given by multiplication in $\mathcal{NT}_X$.*

Moreover, if $\{\xi_{(\underline{n},\underline{0})}^{(j)}\}_{j\in[N_{(\underline{n},\underline{0})}]}$ is a finite frame of $X_{(\underline{n},\underline{0})}$ for $\underline{n} \in \mathbb{Z}_+^r \setminus \{\underline{0}\}$, then we have that $\{\overline{t}_{(\underline{n},\underline{0})}(\xi_{(\underline{n},\underline{0})}^{(j)})\}_{j\in[N_{(\underline{n},\underline{0})}]}$ is a finite frame of $Y_{\underline{n}}^F$.

Proof. For notational convenience, we set $B := B^{F^\perp}$ and $Y := Y^F$. Lemma 5.5.17 gives that the fibres of Y are C*-correspondences over B, so it remains to construct multiplication maps

$$v_{\underline{n},\underline{m}} \colon Y_{\underline{n}} \otimes_B Y_{\underline{m}} \to Y_{\underline{n}+\underline{m}} \text{ for all } \underline{n}, \underline{m} \in \mathbb{Z}_+^r$$

that are compatible with axioms (i)-(v) for product systems.

Axiom (i) is satisfied by construction, and axioms (ii) and (iii) determine the maps $v_{\underline{0},\underline{n}}$ and $v_{\underline{n},\underline{0}}$, respectively, for all $\underline{n} \in \mathbb{Z}_+^r$. Note also that these maps act via multiplication in $\mathcal{NT}_X$ by definition. For $\underline{n}, \underline{m} \in \mathbb{Z}_+^r \setminus \{\underline{0}\}$, we start by defining a map

$$v_{\underline{n},\underline{m}} \colon Y_{\underline{n}} \times Y_{\underline{m}} \to Y_{\underline{n}+\underline{m}}; (y_{\underline{n}}, y_{\underline{m}}) \mapsto y_{\underline{n}} y_{\underline{m}} \text{ for all } y_{\underline{n}} \in Y_{\underline{n}}, y_{\underline{m}} \in Y_{\underline{m}},$$

where the multiplication is performed in $\mathcal{NT}_X$. To see that $v_{\underline{n},\underline{m}}$ is well-defined, observe that

$$\begin{aligned} \overline{t}_{(\underline{n},\underline{0})}(X_{(\underline{n},\underline{0})}) B \overline{t}_{(\underline{m},\underline{0})}(X_{(\underline{m},\underline{0})}) B &\subseteq [\overline{t}_{(\underline{n},\underline{0})}(X_{(\underline{n},\underline{0})}) \overline{t}_{(\underline{m},\underline{0})}(X_{(\underline{m},\underline{0})}) B] \\ &\subseteq [\overline{t}_{(\underline{n}+\underline{m},\underline{0})}(X_{(\underline{n}+\underline{m},\underline{0})}) B] \equiv Y_{\underline{n}+\underline{m}}, \end{aligned}$$

using that $BY_{\underline{m}} \subseteq Y_{\underline{m}}$ by Lemma 5.5.17. It follows that $Y_{\underline{n}} Y_{\underline{m}} \subseteq Y_{\underline{n}+\underline{m}}$, as required. It is routine to check that $v_{\underline{n},\underline{m}}$ is bilinear. Additionally, fixing $y_{\underline{n}} \in Y_{\underline{n}}, y_{\underline{m}} \in Y_{\underline{m}}$ and $b \in B$, we have that

$$v_{\underline{n},\underline{m}}(y_{\underline{n}} b, y_{\underline{m}}) - v_{\underline{n},\underline{m}}(y_{\underline{n}}, \phi_{Y_{\underline{m}}}(b) y_{\underline{m}}) = (y_{\underline{n}} b) y_{\underline{m}} - y_{\underline{n}}(b y_{\underline{m}}) = 0,$$

using associativity of multiplication in $\mathcal{NT}_X$ in the final equality. Thus $v_{\underline{n},\underline{m}}$ is B-balanced and hence linearises to a map on $Y_{\underline{n}} \odot_B Y_{\underline{m}}$, denoted by the same symbol. For all $y_{\underline{n}}, y'_{\underline{n}} \in Y_{\underline{n}}$ and $y_{\underline{m}}, y'_{\underline{m}} \in Y_{\underline{m}}$, we have that

$$\left\langle v_{\underline{n},\underline{m}}(y_{\underline{n}} \otimes y_{\underline{m}}), v_{\underline{n},\underline{m}}(y'_{\underline{n}} \otimes y'_{\underline{m}}) \right\rangle = y_{\underline{m}}^* y_{\underline{n}}^* y'_{\underline{n}} y'_{\underline{m}} = \left\langle y_{\underline{m}}, \left\langle y_{\underline{n}}, y'_{\underline{n}} \right\rangle y'_{\underline{m}} \right\rangle = \left\langle y_{\underline{n}} \otimes y_{\underline{m}}, y'_{\underline{n}} \otimes y'_{\underline{m}} \right\rangle,$$

from which it follows that

$$\langle v_{\underline{n},\underline{m}}(\zeta), v_{\underline{n},\underline{m}}(\zeta') \rangle = \langle \zeta, \zeta' \rangle \text{ for all } \zeta, \zeta' \in Y_{\underline{n}} \odot_B Y_{\underline{m}}.$$

In particular, we deduce that $v_{\underline{n},\underline{m}}$ extends to an isometric linear map

$$v_{\underline{n},\underline{m}} \colon Y_{\underline{n}} \otimes_B Y_{\underline{m}} \to Y_{\underline{n}+\underline{m}}; y_{\underline{n}} \otimes y_{\underline{m}} \mapsto y_{\underline{n}} y_{\underline{m}} \text{ for all } y_{\underline{n}} \in Y_{\underline{n}}, y_{\underline{m}} \in Y_{\underline{m}}.$$

To see that $v_{\underline{n},\underline{m}}$ preserves the left action, fix $y_{\underline{n}} \in Y_{\underline{n}}, y_{\underline{m}} \in Y_{\underline{m}}$ and $b \in B$. Then we obtain that

$$v_{\underline{n},\underline{m}}(b(y_{\underline{n}} \otimes y_{\underline{m}})) = v_{\underline{n},\underline{m}}((b y_{\underline{n}}) \otimes y_{\underline{m}}) = (b y_{\underline{n}}) y_{\underline{m}} = b(y_{\underline{n}} y_{\underline{m}}) = b v_{\underline{n},\underline{m}}(y_{\underline{n}} \otimes y_{\underline{m}}),$$

from which it follows that $v_{\underline{n},\underline{m}}$ is a left B-module map by linearity and continuity of the maps involved. Analogously, we have that

$$v_{\underline{n},\underline{m}}((y_{\underline{n}} \otimes y_{\underline{m}})b) = v_{\underline{n},\underline{m}}(y_{\underline{n}} \otimes (y_{\underline{m}}b)) = y_{\underline{n}}(y_{\underline{m}}b) = (y_{\underline{n}}y_{\underline{m}})b = v_{\underline{n},\underline{m}}(y_{\underline{n}} \otimes y_{\underline{m}})b.$$

Thus $v_{\underline{n},\underline{m}}$ also preserves the right action and is thus a B-bimodule map. Since $v_{\underline{n},\underline{m}}$ is an isometric linear map and therefore has closed range, establishing surjectivity amounts to showing that

$$\overline{t}_{(\underline{n}+\underline{m},\underline{0})}(X_{(\underline{n}+\underline{m},\underline{0})})B \subseteq \operatorname{Im}(v_{\underline{n},\underline{m}}).$$

Accordingly, fix $\xi_{(\underline{n}+\underline{m},\underline{0})} \in X_{(\underline{n}+\underline{m},\underline{0})}, b \in B$ and an approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ of A. Since $X_{(\underline{n},\underline{0})} \otimes_A X_{(\underline{m},\underline{0})} \cong X_{(\underline{n}+\underline{m},\underline{0})}$ via the multiplication map $u_{(\underline{n},\underline{0}),(\underline{m},\underline{0})}$, we may assume that

$$\xi_{(\underline{n}+\underline{m},\underline{0})} = \xi_{(\underline{n},\underline{0})}\xi_{(\underline{m},\underline{0})} \text{ for some } \xi_{(\underline{n},\underline{0})} \in X_{(\underline{n},\underline{0})}, \xi_{(\underline{m},\underline{0})} \in X_{(\underline{m},\underline{0})}$$

without loss of generality. We obtain that

$$\begin{aligned}
\overline{t}_{(\underline{n}+\underline{m},\underline{0})}(\xi_{(\underline{n}+\underline{m},\underline{0})})b &= \overline{t}_{(\underline{n}+\underline{m},\underline{0})}(\xi_{(\underline{n},\underline{0})}\xi_{(\underline{m},\underline{0})})b \\
&= \overline{t}_{(\underline{n},\underline{0})}(\xi_{(\underline{n},\underline{0})})\overline{t}_{(\underline{m},\underline{0})}(\xi_{(\underline{m},\underline{0})})b \\
&= \|\cdot\|\text{-}\lim_\lambda \overline{t}_{(\underline{n},\underline{0})}(\xi_{(\underline{n},\underline{0})})\overline{\pi}(u_\lambda)\overline{t}_{(\underline{m},\underline{0})}(\xi_{(\underline{m},\underline{0})})b \\
&= \|\cdot\|\text{-}\lim_\lambda v_{\underline{n},\underline{m}}(\overline{t}_{(\underline{n},\underline{0})}(\xi_{(\underline{n},\underline{0})})\overline{\pi}(u_\lambda) \otimes \overline{t}_{(\underline{m},\underline{0})}(\xi_{(\underline{m},\underline{0})})b) \in \operatorname{Im}(v_{\underline{n},\underline{m}}),
\end{aligned}$$

as required. In total, we have that $v_{\underline{n},\underline{m}}$ is a unitary and hence axiom (iv) holds.

To see that the multiplication maps are associative, fix $\underline{n}, \underline{m}, \underline{k} \in \mathbb{Z}_+^r, y_{\underline{n}} \in Y_{\underline{n}}, y_{\underline{m}} \in Y_{\underline{m}}$ and $y_{\underline{k}} \in Y_{\underline{k}}$. Using "$\cdot$" to denote multiplication in Y, we obtain that

$$(y_{\underline{n}} \cdot y_{\underline{m}}) \cdot y_{\underline{k}} = (y_{\underline{n}}y_{\underline{m}})y_{\underline{k}} = y_{\underline{n}}(y_{\underline{m}}y_{\underline{k}}) = y_{\underline{n}} \cdot (y_{\underline{m}} \cdot y_{\underline{k}}),$$

using associativity of multiplication in $\mathcal{NT}_X$ in the second equality. Thus axiom (v) holds and we ascertain that Y constitutes a product system over $\mathbb{Z}_+^r$ with coefficients in B whose multiplication maps are given by multiplication in $\mathcal{NT}_X$, as required.

Finally, fix $\underline{n} \in \mathbb{Z}_+^r \setminus \{\underline{0}\}$ and suppose that $\{\xi^{(j)}_{(\underline{n},\underline{0})}\}_{j\in[N_{(\underline{n},\underline{0})}]}$ is a finite frame of $X_{(\underline{n},\underline{0})}$. For $y_{\underline{n}} = \overline{t}_{(\underline{n},\underline{0})}(\xi_{(\underline{n},\underline{0})})b$, where $\xi_{(\underline{n},\underline{0})} \in X_{(\underline{n},\underline{0})}$ and $b \in B$, we have that

$$\begin{aligned}
\sum_{j=1}^{N_{(\underline{n},\underline{0})}} \Theta^{Y_{\underline{n}}}_{\overline{t}_{(\underline{n},\underline{0})}(\xi^{(j)}_{(\underline{n},\underline{0})}),\overline{t}_{(\underline{n},\underline{0})}(\xi^{(j)}_{(\underline{n},\underline{0})})}(y_{\underline{n}}) &= \sum_{j=1}^{N_{(\underline{n},\underline{0})}} \overline{t}_{(\underline{n},\underline{0})}(\xi^{(j)}_{(\underline{n},\underline{0})})\overline{t}_{(\underline{n},\underline{0})}(\xi^{(j)}_{(\underline{n},\underline{0})})^*\overline{t}_{(\underline{n},\underline{0})}(\xi_{(\underline{n},\underline{0})})b \\
&= \overline{t}_{(\underline{n},\underline{0})}(\sum_{j=1}^{N_{(\underline{n},\underline{0})}} \Theta^{X_{(\underline{n},\underline{0})}}_{\xi^{(j)}_{(\underline{n},\underline{0})},\xi^{(j)}_{(\underline{n},\underline{0})}}(\xi_{(\underline{n},\underline{0})}))b \\
&= \overline{t}_{(\underline{n},\underline{0})}(\xi_{(\underline{n},\underline{0})})b = y_{\underline{n}},
\end{aligned}$$

using that $\{\xi^{(j)}_{(\underline{n},\underline{0})}\}_{j\in[N_{(\underline{n},\underline{0})}]}$ is a finite frame of $X_{(\underline{n},\underline{0})}$ in the third equality. Since $\overline{t}_{(\underline{n},\underline{0})}(X_{(\underline{n},\underline{0})})B$

densely spans $Y_{\underline{n}}$, it follows that

$$\sum_{j=1}^{N_{(\underline{n},\underline{0})}} \Theta^{Y_{\underline{n}}}_{\overline{t}_{(\underline{n},\underline{0})}(\xi^{(j)}_{(\underline{n},\underline{0})}),\overline{t}_{(\underline{n},\underline{0})}(\xi^{(j)}_{(\underline{n},\underline{0})})} = \mathrm{id}_{Y_{\underline{n}}}$$

by linearity and continuity of the maps involved, showing that $\{\overline{t}_{(\underline{n},\underline{0})}(\xi^{(j)}_{(\underline{n},\underline{0})})\}_{j\in[N_{(\underline{n},\underline{0})}]}$ is a finite frame of $Y_{\underline{n}}$. This completes the proof. □

Next we show that $\mathcal{NT}_X \cong \mathcal{NT}_{Y^F}$ when each $X_{\underline{i}}$ admits a finite frame. Note that each $Y^F_{\underline{i}}$ admits a finite frame by Proposition 5.5.18. We start by representing Y^F in $\mathcal{NT}_X$.

Proposition 5.5.19. *Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A, wherein $X_{\underline{i}}$ admits a finite frame for all $i \in [d]$. Let $F = [r]$ for some $r < d$. Define the maps*

$$\begin{aligned} \pi\colon B^{F^\perp} &\to \mathcal{NT}_X; \pi(b) = b \text{ for all } b \in B^{F^\perp}, \\ t_{\underline{n}}\colon Y^F_{\underline{n}} &\to \mathcal{NT}_X; t_{\underline{n}}(y_{\underline{n}}) = y_{\underline{n}} \text{ for all } y_{\underline{n}} \in Y^F_{\underline{n}}, \underline{n} \in \mathbb{Z}_+^r \setminus \{\underline{0}\}. \end{aligned}$$

Then (π, t) is an injective Nica-covariant representation of Y^F satisfying $\mathrm{C}^*(\pi,t) = \mathcal{NT}_X$.

Proof. For notational convenience, we set $B := B^{F^\perp}$ and $Y := Y^F$. For $\underline{s} \in \mathbb{Z}_+^d$, let $P_{\underline{s}}$ denote the projection of $\mathcal{F}X$ onto $\sum_{\underline{k}\geq\underline{s}} X_{\underline{k}}$. Then we obtain that

$$P_{\underline{s}}\overline{t}_{\underline{s}}(\zeta_{\underline{s}}) = \overline{t}_{\underline{s}}(\zeta_{\underline{s}}) \text{ for all } \zeta_{\underline{s}} \in X_{\underline{s}}.$$

It is routine to check that (π, t) defines an injective representation of the product system Y. To see that $\mathrm{C}^*(\pi,t) = \mathcal{NT}_X$, it suffices to show that $\mathrm{C}^*(\pi,t)$ contains the generators of $\mathcal{NT}_X$. By Proposition 5.5.12 we have that

$$\overline{t}_{\underline{n}}(X_{\underline{n}}) \subseteq \mathrm{C}^*(\pi,t) \text{ for all } \underline{n} \perp F.$$

It remains to see that

$$\overline{t}_{(\underline{k},\underline{\ell})}(X_{(\underline{k},\underline{\ell})}) \subseteq \mathrm{C}^*(\pi,t) \text{ for all } \underline{k} \in \mathbb{Z}_+^r \setminus \{\underline{0}\}, \underline{\ell} \in \mathbb{Z}_+^{d-r}.$$

Fixing $\xi_{(\underline{k},\underline{0})} \in X_{(\underline{k},\underline{0})}$ and $\xi_{(\underline{0},\underline{\ell})} \in X_{(\underline{0},\underline{\ell})}$, we have that

$$\overline{t}_{(\underline{k},\underline{\ell})}(\xi_{(\underline{k},\underline{0})}\xi_{(\underline{0},\underline{\ell})}) = \overline{t}_{(\underline{k},\underline{0})}(\xi_{(\underline{k},\underline{0})})\overline{t}_{(\underline{0},\underline{\ell})}(\xi_{(\underline{0},\underline{\ell})}) = t_{\underline{k}}(\overline{t}_{(\underline{k},\underline{0})}(\xi_{(\underline{k},\underline{0})}))\pi(\overline{t}_{(\underline{0},\underline{\ell})}(\xi_{(\underline{0},\underline{\ell})})) \in \mathrm{C}^*(\pi,t).$$

Noting that $X_{(\underline{k},\underline{0})} \otimes_A X_{(\underline{0},\underline{\ell})} \cong X_{(\underline{k},\underline{\ell})}$ via the multiplication map $u_{(\underline{k},\underline{0}),(\underline{0},\underline{\ell})}$, we deduce that

$$\overline{t}_{(\underline{k},\underline{\ell})}(X_{(\underline{k},\underline{\ell})}) \subseteq [\overline{t}_{(\underline{k},\underline{0})}(X_{(\underline{k},\underline{0})})\overline{t}_{(\underline{0},\underline{\ell})}(X_{(\underline{0},\underline{\ell})})] \subseteq \mathrm{C}^*(\pi,t),$$

as required.

Next we show that (π, t) is Nica-covariant. Let $\{\iota_{\underline{n}}^{\underline{n}+\underline{m}}\}_{\underline{n},\underline{m}\in\mathbb{Z}_+^r}$ denote the connecting $*$-homomorphisms of Y. Fix $\underline{n}, \underline{m} \in \mathbb{Z}_+^r \setminus \{\underline{0}\}$, $k_{\underline{n}} \in \mathcal{K}(Y_{\underline{n}})$ and $k_{\underline{m}} \in \mathcal{K}(Y_{\underline{m}})$. We have that

$$\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(k_{\underline{n}})\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(k_{\underline{m}})) = \psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(k_{\underline{n}}))\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(k_{\underline{m}}))$$

by Proposition 2.4.1. Therefore, we must show that

$$\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(k_{\underline{n}}))\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(k_{\underline{m}})) = \psi_{\underline{n}}(k_{\underline{n}})\psi_{\underline{m}}(k_{\underline{m}}). \tag{5.20}$$

This holds trivially when $\underline{n} = \underline{n} \vee \underline{m} = \underline{m}$, so we consider the cases where $\underline{n} \neq \underline{n} \vee \underline{m}$ or $\underline{m} \neq \underline{n} \vee \underline{m}$.

Assume that $\underline{n} \neq \underline{n} \vee \underline{m}$. We proceed by showing that (5.20) holds for $k_{\underline{n}} = \Theta^{Y_{\underline{n}}}_{y_{\underline{n}},y'_{\underline{n}}}$ and $k_{\underline{m}} = \Theta^{Y_{\underline{m}}}_{y_{\underline{m}},y'_{\underline{m}}}$, where

$$y_{\underline{n}} = \overline{t}_{(\underline{n},\underline{0})}(\xi_{(\underline{n},\underline{0})})b, y'_{\underline{n}} = \overline{t}_{(\underline{n},\underline{0})}(\eta_{(\underline{n},\underline{0})}), y_{\underline{m}} = \overline{t}_{(\underline{m},\underline{0})}(\xi_{(\underline{m},\underline{0})})c, y'_{\underline{m}} = \overline{t}_{(\underline{m},\underline{0})}(\eta_{(\underline{m},\underline{0})}),$$

for some $\xi_{(\underline{n},\underline{0})}, \eta_{(\underline{n},\underline{0})} \in X_{(\underline{n},\underline{0})}, \xi_{(\underline{m},\underline{0})}, \eta_{(\underline{m},\underline{0})} \in X_{(\underline{m},\underline{0})}$ and $b, c \in B$. To this end, we compute

$$\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(\Theta^{Y_{\underline{n}}}_{y_{\underline{n}},y'_{\underline{n}}})) \quad \text{and} \quad \psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(\Theta^{Y_{\underline{m}}}_{y_{\underline{m}},y'_{\underline{m}}})).$$

For $\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(\Theta^{Y_{\underline{n}}}_{y_{\underline{n}},y'_{\underline{n}}}))$, let $\{\xi^{(j)}_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})}\}_{j\in[N]}$ be a finite frame of $X_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})}$. By Proposition 5.5.18, we have that $\{\overline{t}_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})}(\xi^{(j)}_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})})\}_{j\in[N]}$ is a finite frame of $Y_{\underline{n}\vee\underline{m}-\underline{n}}$. Note that

$$P_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})} = \sum_{j=1}^{N} \overline{t}_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})}(\xi^{(j)}_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})})\overline{t}_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})}(\xi^{(j)}_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})})^* \in \mathcal{NT}_X.$$

Observe that $P_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})}$ belongs to the commutant of $\overline{t}_{(\underline{0},\underline{\ell})}(X_{(\underline{0},\underline{\ell})})$ (and of $\overline{t}_{(\underline{0},\underline{\ell})}(X_{(\underline{0},\underline{\ell})})^*$ by taking adjoints) for all $\underline{\ell} \in \mathbb{Z}_+^{d-r}$, since for $\underline{s} \in \mathbb{Z}_+^d$ we have that

$$(\underline{n} \vee \underline{m} - \underline{n}, \underline{0}) \leq \underline{s} \iff (\underline{n} \vee \underline{m} - \underline{n}, \underline{0}) \leq \underline{s} + (\underline{0}, \underline{\ell}) \text{ for all } \underline{\ell} \in \mathbb{Z}_+^{d-r}.$$

In particular, we have that $P_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})}$ belongs to B'. Additionally, we have that

$$\overline{t}_{(\underline{n},\underline{0})}(\zeta_{(\underline{n},\underline{0})})P_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})} = P_{(\underline{n}\vee\underline{m},\underline{0})}\overline{t}_{(\underline{n},\underline{0})}(\zeta_{(\underline{n},\underline{0})}), \text{ for all } \zeta_{(\underline{n},\underline{0})} \in X_{(\underline{n},\underline{0})},$$

and by taking adjoints we obtain that

$$P_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})}\overline{t}_{(\underline{n},\underline{0})}(\zeta_{(\underline{n},\underline{0})})^* = \overline{t}_{(\underline{n},\underline{0})}(\zeta_{(\underline{n},\underline{0})})^*P_{(\underline{n}\vee\underline{m},\underline{0})}, \text{ for all } \zeta_{(\underline{n},\underline{0})} \in X_{(\underline{n},\underline{0})}.$$

This follows from the observation that for $\underline{s} \in \mathbb{Z}_+^d$ we have that

$$(\underline{n} \vee \underline{m} - \underline{n}, \underline{0}) \leq \underline{s} \iff (\underline{n} \vee \underline{m}, \underline{0}) \leq (\underline{n}, \underline{0}) + \underline{s}.$$

An application of Corollary 5.5.7 gives that

$$\begin{aligned}
\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(\Theta^{Y_{\underline{n}}}_{y_{\underline{n}},y'_{\underline{n}}})) &= \sum_{j=1}^{N} y_{\underline{n}}\overline{t}_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})}(\xi^{(j)}_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})})\overline{t}_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})}(\xi^{(j)}_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})})^*(y'_{\underline{n}})^* \\
&= \overline{t}_{(\underline{n},\underline{0})}(\xi_{(\underline{n},\underline{0})})bP_{(\underline{n}\vee\underline{m}-\underline{n},\underline{0})}\overline{t}_{(\underline{n},\underline{0})}(\eta_{(\underline{n},\underline{0})})^* \\
&= \overline{t}_{(\underline{n},\underline{0})}(\xi_{(\underline{n},\underline{0})})b\overline{t}_{(\underline{n},\underline{0})}(\eta_{(\underline{n},\underline{0})})^*P_{(\underline{n}\vee\underline{m},\underline{0})} \\
&= P_{(\underline{n}\vee\underline{m},\underline{0})}\overline{t}_{(\underline{n},\underline{0})}(\xi_{(\underline{n},\underline{0})})b\overline{t}_{(\underline{n},\underline{0})}(\eta_{(\underline{n},\underline{0})})^*.
\end{aligned}$$

For $\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(\Theta^{Y_{\underline{m}}}_{y_{\underline{m}},y'_{\underline{m}}}))$, we consider two cases. If $\underline{m}=\underline{n}\vee\underline{m}$, then we have that

$$\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(\Theta^{Y_{\underline{m}}}_{y_{\underline{m}},y'_{\underline{m}}})) = \psi_{\underline{m}}(\Theta^{Y_{\underline{m}}}_{y_{\underline{m}},y'_{\underline{m}}}) = \overline{t}_{(\underline{m},\underline{0})}(\xi_{(\underline{m},\underline{0})})c\overline{t}_{(\underline{m},\underline{0})}(\eta_{(\underline{m},\underline{0})})^*,$$

and if $\underline{m}\neq\underline{n}\vee\underline{m}$ then we have that

$$\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(\Theta^{Y_{\underline{m}}}_{y_{\underline{m}},y'_{\underline{m}}})) = P_{(\underline{n}\vee\underline{m},\underline{0})}\overline{t}_{(\underline{m},\underline{0})}(\xi_{(\underline{m},\underline{0})})c\overline{t}_{(\underline{m},\underline{0})}(\eta_{(\underline{m},\underline{0})})^*$$

by swapping $\underline{n}$ and $\underline{m}$, as well as b and c, in the preceding arguments.

Since $P_{(\underline{n}\vee\underline{m},\underline{0})}=P_{(\underline{n},\underline{0})}P_{(\underline{m},\underline{0})}$, we conclude that

$$\begin{aligned}
&\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(\Theta^{Y_{\underline{n}}}_{y_{\underline{n}},y'_{\underline{n}}}))\psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(\Theta^{Y_{\underline{m}}}_{y_{\underline{m}},y'_{\underline{m}}})) = \\
&\qquad = \overline{t}_{(\underline{n},\underline{0})}(\xi_{(\underline{n},\underline{0})})b\overline{t}_{(\underline{n},\underline{0})}(\eta_{(\underline{n},\underline{0})})^*P_{(\underline{n}\vee\underline{m},\underline{0})}\overline{t}_{(\underline{m},\underline{0})}(\xi_{(\underline{m},\underline{0})})c\overline{t}_{(\underline{m},\underline{0})}(\eta_{(\underline{m},\underline{0})})^* \\
&\qquad = \overline{t}_{(\underline{n},\underline{0})}(\xi_{(\underline{n},\underline{0})})b\left[\overline{t}_{(\underline{n},\underline{0})}(\eta_{(\underline{n},\underline{0})})^*P_{(\underline{n},\underline{0})}\right]\left[P_{(\underline{m},\underline{0})}\overline{t}_{(\underline{m},\underline{0})}(\xi_{(\underline{m},\underline{0})})\right]c\overline{t}_{(\underline{m},\underline{0})}(\eta_{(\underline{m},\underline{0})})^* \\
&\qquad = \overline{t}_{(\underline{n},\underline{0})}(\xi_{(\underline{n},\underline{0})})b\overline{t}_{(\underline{n},\underline{0})}(\eta_{(\underline{n},\underline{0})})^*\overline{t}_{(\underline{m},\underline{0})}(\xi_{(\underline{m},\underline{0})})c\overline{t}_{(\underline{m},\underline{0})}(\eta_{(\underline{m},\underline{0})})^* \\
&\qquad = \psi_{\underline{n}}(\Theta^{Y_{\underline{n}}}_{y_{\underline{n}},y'_{\underline{n}}})\psi_{\underline{m}}(\Theta^{Y_{\underline{m}}}_{y_{\underline{m}},y'_{\underline{m}}}),
\end{aligned}$$

using that $P^2_{(\underline{n}\vee\underline{m},\underline{0})}=P_{(\underline{n}\vee\underline{m},\underline{0})}$ when $\underline{m}\neq\underline{n}\vee\underline{m}$ in the first equality. By taking finite linear combinations and their norm-limits, we conclude that (5.20) holds when $\underline{n}\neq\underline{n}\vee\underline{m}$.

Since (5.20) is symmetric with respect to $\underline{n}$ and $\underline{m}$, taking adjoints (and relabelling) deals with the case where $\underline{m}\neq\underline{n}\vee\underline{m}$, showing that (π,t) is Nica-covariant. This completes the proof. □

We now arrive at the next main result of this section, namely that the decomposition of X along $\emptyset\neq F\subsetneq[d]$ induces a similar decomposition of the Toeplitz-Nica-Pimsner algebras. The following is noted in [32, Proof of Theorem 4.4 (i)].

Theorem 5.5.20. *Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A, wherein $X_{\underline{i}}$ admits a finite frame for all $i\in[d]$. Let $F=[r]$ for some $r<d$. Consider the product system Y^F over $B^{F^\perp}$ related to X and F, and define the maps*

$$\begin{aligned}
&\pi\colon B^{F^\perp}\to\mathcal{NT}_X;\pi(b)=b \text{ for all } b\in B^{F^\perp}, \\
&t_{\underline{n}}\colon Y^F_{\underline{n}}\to\mathcal{NT}_X;t_{\underline{n}}(y_{\underline{n}})=y_{\underline{n}} \text{ for all } y_{\underline{n}}\in Y^F_{\underline{n}},\underline{n}\in\mathbb{Z}_+^r\setminus\{\underline{0}\}.
\end{aligned}$$

Then the induced map $\pi \times t \colon \mathcal{NT}_{Y^F} \to \mathcal{NT}_X$ is a $*$-isomorphism.

Proof. For notational convenience, we set $B := B^{F^\perp}$ and $Y := Y^F$. By Proposition 5.5.19 we have that (π, t) is Nica-covariant and thus $\pi \times t$ is well-defined. Let γ be the gauge action of $(\overline{\pi}_Y, \overline{t}_Y)$. It is routine to check that β_F defines a gauge action of (π, t). Then $\pi \times t$ is an equivariant $*$-epimorphism, and it suffices to show that $\pi \times t$ is injective on $\mathcal{NT}_Y^\gamma$. By Proposition 2.5.21, this amounts to showing that $\pi \times t$ is injective on the $[\underline{0}, \underline{1}_{[r]}]$-core.

Towards contradiction, suppose that $\ker \pi \times t \cap B_{[\underline{0},\underline{1}_{[r]}]}^{(\overline{\pi}_Y, \overline{t}_Y)} \neq \{0\}$. We claim that we can find $0 \neq f \in \ker \pi \times t \cap B_{[\underline{0},\underline{1}_{[r]}]}^{(\overline{\pi}_Y, \overline{t}_Y)}$ of the form

$$f = \overline{\pi}_Y(b) + \sum \{ \overline{\psi}_{Y,\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_{[r]} \}, \text{ where } 0 \neq b \in B_+ \text{ and } k_{\underline{n}} \in \mathcal{K}(Y_{\underline{n}}). \tag{5.21}$$

Indeed, start by taking $0 \neq g \in \ker \pi \times t \cap B_{[\underline{0},\underline{1}_{[r]}]}^{(\overline{\pi}_Y, \overline{t}_Y)}$, so that

$$g = \overline{\pi}_Y(b') + \sum \{ \overline{\psi}_{Y,\underline{n}}(k'_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_{[r]} \}, \text{ where } b' \in B \text{ and each } k'_{\underline{n}} \in \mathcal{K}(Y_{\underline{n}}).$$

If $b' \neq 0$, then choose $f = g^* g$. If $b' = 0$, then choose $\underline{0} \neq \underline{m} \leq \underline{1}_{[r]}$ minimal such that $k'_{\underline{m}} \neq 0$. We may assume that $\underline{m} \neq \underline{1}_{[r]}$, as otherwise we would have that $g = \overline{\psi}_{Y,\underline{1}_{[r]}}(k'_{\underline{1}_{[r]}})$ and injectivity of $\psi_{\underline{1}_{[r]}}$ would give that $g = 0$, a contradiction. Since $k'_{\underline{m}} \neq 0$, we may find $0 \neq y_{\underline{m}} \in Y_{\underline{m}}$ such that $k'_{\underline{m}} y_{\underline{m}} \neq 0$. We set

$$f := \overline{t}_{Y,\underline{m}}(k'_{\underline{m}} y_{\underline{m}})^* g \overline{t}_{Y,\underline{m}}(y_{\underline{m}}) \in \ker \pi \times t \cap B_{[\underline{0},\underline{1}_{[r]}]}^{(\overline{\pi}_Y, \overline{t}_Y)},$$

and we note that

$$f = \overline{\pi}_Y(\langle k'_{\underline{m}} y_{\underline{m}}, k'_{\underline{m}} y_{\underline{m}} \rangle) + \sum \{ \overline{\psi}_{Y,\underline{n}}(k''_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_{[r]} - \underline{m} \}$$

for suitably defined $k''_{\underline{n}} \in \mathcal{K}(Y_{\underline{n}})$, for all $\underline{0} \neq \underline{n} \leq \underline{1}_{[r]} - \underline{m}$. Notice that

$$0 \neq \langle k'_{\underline{m}} y_{\underline{m}}, k'_{\underline{m}} y_{\underline{m}} \rangle \in B_+$$

by construction, and so by padding with zeroes we deduce that f has the required form.

Hence, without loss of generality, we may assume that f is of the form (5.21). We have that

$$\pi(b) + \sum \{ \psi_{\underline{n}}(k_{\underline{n}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_{[r]} \} = (\pi \times t)(f) = 0,$$

and hence $\pi(b) q_F = 0$ by Proposition 2.5.17. Fixing $i \in F$, let $\{\xi_{(\underline{i},\underline{0})}^{(j)}\}_{j \in [N]}$ be a finite frame of $X_{(\underline{i},\underline{0})}$. Then $\{\overline{t}_{(\underline{i},\underline{0})}(\xi_{(\underline{i},\underline{0})}^{(j)})\}_{j \in [N]}$ is a finite frame of $Y_{\underline{i}}$ by Proposition 5.5.18. Hence we have that

$$p_{\underline{i}} = \sum_{j=1}^{N} t_{\underline{i}}(\overline{t}_{(\underline{i},\underline{0})}(\xi_{(\underline{i},\underline{0})}^{(j)})) t_{\underline{i}}(\overline{t}_{(\underline{i},\underline{0})}(\xi_{(\underline{i},\underline{0})}^{(j)}))^* = \sum_{j=1}^{N} \overline{t}_{(\underline{i},\underline{0})}(\xi_{(\underline{i},\underline{0})}^{(j)}) \overline{t}_{(\underline{i},\underline{0})}(\xi_{(\underline{i},\underline{0})}^{(j)})^* = \overline{p}_{(\underline{i},\underline{0})}.$$

In turn, we obtain that

$$b\overline{q}_F = b\prod_{i\in F}(I-\overline{p}_{(\underline{i},\underline{0})}) = b\prod_{i\in F}(I-p_{\underline{i}}) = bq_F = \pi(b)q_F = 0.$$

Since $\overline{p}_{(\underline{i},\underline{0})} \in \mathcal{NT}_X^{\beta_{F^\perp}}$ for all $i \in F$, we have that

$$\begin{aligned} E_{\beta_{F^\perp}}(b)\overline{q}_F &= E_{\beta_{F^\perp}}(b) + \sum\{(-1)^{|D|}E_{\beta_{F^\perp}}(b)\prod_{i\in D}\overline{p}_{(\underline{i},\underline{0})} \mid \emptyset \neq D \subseteq F\} \\ &= E_{\beta_{F^\perp}}\left(b + \sum\{(-1)^{|D|}b\prod_{i\in D}\overline{p}_{(\underline{i},\underline{0})} \mid \emptyset \neq D \subseteq F\}\right) = E_{\beta_{F^\perp}}(b\overline{q}_F) = 0, \end{aligned}$$

where in the second equality we use that $E_{\beta_{F^\perp}}$ is an $\mathcal{NT}_X^{\beta_{F^\perp}}$-bimodule map. In particular, for every $\underline{n} \in \mathbb{Z}_+^d$ satisfying $\underline{n} \perp F$, we have that

$$E_{\beta_{F^\perp}}(b)\xi_{\underline{n}} = E_{\beta_{F^\perp}}(b)\overline{q}_F\xi_{\underline{n}} = 0, \text{ for all } \xi_{\underline{n}} \in X_{\underline{n}},$$

and thus $E_{\beta_{F^\perp}}(b) = 0$ by Corollary 5.5.16. Since $b \in B_+$, faithfulness of $E_{\beta_{F^\perp}}$ gives the contradiction that $b = 0$, and the proof is complete. □

We are now ready to capture the quotient of $\mathcal{NT}_X$ by the ideal $\left\langle \overline{\pi}(A)\overline{q}_{\underline{i}} \mid i \in F \right\rangle$ induced by F as a Cuntz-Nica-Pimsner algebra.

Theorem 5.5.21. *Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A, wherein $X_{\underline{i}}$ admits a finite frame for all $i \in [d]$. Let $F = [r]$ for some $r < d$. Consider the product system Y^F over $B^{F^\perp}$ related to X and F, and define the ideal*

$$I_{Y^F} := \ker\{B^{F^\perp} \to \mathcal{NT}_{Y^F}/\left\langle \overline{\pi}_{Y^F}(B^{F^\perp})\overline{q}_{Y^F,\underline{i}} \mid i \in F \right\rangle\}.$$

Then the canonical $*$-isomorphism $\mathcal{NT}_{Y^F} \cong \mathcal{NT}_X$ descends to a $*$-isomorphism

$$\mathcal{NO}_{[Y^F]_{I_{Y^F}}} \cong \mathcal{NT}_X/\left\langle \overline{\pi}(A)\overline{q}_{\underline{i}} \mid i \in F \right\rangle.$$

Proof. For notational convenience, we set $B := B^{F^\perp}$, $Y := Y^F$ and $I := I_{Y^F}$. We define

$$\mathfrak{J}_Y := \left\langle \overline{\pi}_Y(B)\overline{q}_{Y,\underline{i}} \mid i \in F \right\rangle \subseteq \mathcal{NT}_Y \quad \text{and} \quad \mathfrak{J}_X := \left\langle \overline{\pi}(A)\overline{q}_{\underline{i}} \mid i \in F \right\rangle \subseteq \mathcal{NT}_X.$$

By applying Proposition 5.5.10 to Y, we have that

$$\mathcal{NO}_{[Y]_I} \cong \mathcal{NT}_Y/\mathfrak{J}_Y.$$

Let $\pi \times t \colon \mathcal{NT}_Y \to \mathcal{NT}_X$ be the canonical $*$-isomorphism of Theorem 5.5.20. We will show that $(\pi \times t)(\mathfrak{J}_Y) = \mathfrak{J}_X$. This ensures that $\pi \times t$ descends to a $*$-isomorphism on the quotients, and thus

$$\mathcal{NO}_{[Y]_I} \cong \mathcal{NT}_Y/\mathfrak{J}_Y \cong \mathcal{NT}_X/\mathfrak{J}_X,$$

as required. To this end, fix $i \in F$. We have that $(\pi \times t)(\overline{p}_{Y,\underline{i}}) = \overline{p}_{\underline{i}}$ by use of a finite frame expansion, and thus

$$(\pi \times t)(\overline{\pi}_Y(b)\overline{q}_{Y,\underline{i}}) = b\overline{q}_{\underline{i}}, \text{ for all } b \in B.$$

Next fix $\underline{n} \in \mathbb{Z}_+^d$ such that $\underline{n} \perp F$, and observe that

$$\overline{q}_{\underline{i}}\overline{t}_{\underline{n}}(X_{\underline{n}}) = \overline{t}_{\underline{n}}(X_{\underline{n}})\overline{q}_{\underline{i}} \subseteq [\overline{t}_{\underline{n}}(X_{\underline{n}})\overline{\pi}(A)\overline{q}_{\underline{i}}] \subseteq \mathfrak{J}_X,$$

using Proposition 2.5.15 in the first equality. By taking adjoints, we also deduce that

$$\overline{t}_{\underline{n}}(X_{\underline{n}})^*\overline{q}_{\underline{i}} = \overline{q}_{\underline{i}}\overline{t}_{\underline{n}}(X_{\underline{n}})^* \subseteq \mathfrak{J}_X.$$

Hence we have that

$$\overline{t}_{\underline{n}}(X_{\underline{n}})\overline{t}_{\underline{m}}(X_{\underline{m}})^*\overline{q}_{\underline{i}} \subseteq \mathfrak{J}_X, \text{ for all } \underline{n}, \underline{m} \in \mathbb{Z}_+^d \text{ satisfying } \underline{n}, \underline{m} \perp F.$$

Thus, by taking finite linear combinations and their norm-limits and using Proposition 5.5.12, we derive that $B\overline{q}_{\underline{i}} \subseteq \mathfrak{J}_X$ and thus in particular

$$(\pi \times t)(\overline{\pi}_Y(b)\overline{q}_{Y,\underline{i}}) = b\overline{q}_{\underline{i}} \in \mathfrak{J}_X, \text{ for all } b \in B.$$

Therefore $\pi \times t$ maps the generators of $\mathfrak{J}_Y$ into $\mathfrak{J}_X$, and it follows that $(\pi \times t)(\mathfrak{J}_Y) \subseteq \mathfrak{J}_X$. For the reverse inclusion, fix $i \in F$. Then we have that

$$\overline{\pi}(a)\overline{q}_{\underline{i}} = (\pi \times t)(\overline{\pi}_Y(\overline{\pi}(a))\overline{q}_{Y,\underline{i}}), \text{ for all } a \in A.$$

Note that $\overline{\pi}_Y(\overline{\pi}(A))\overline{q}_{Y,\underline{i}} \subseteq \mathfrak{J}_Y$, and so the generators of $\mathfrak{J}_X$ are contained in $(\pi \times t)(\mathfrak{J}_Y)$. Thus $\mathfrak{J}_X \subseteq (\pi \times t)(\mathfrak{J}_Y)$, completing the proof. □

Having generalised the first part of Proposition 5.5.10, next we account for the injectivity clause. We will need the following proposition. Recall that the projections $\overline{p}_{\underline{i}}$ can be defined for a (just) compactly aligned product system by Remark 2.5.14.

Proposition 5.5.22. *Let X be a compactly aligned product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A. Let $F = [r]$ for some $r < d$ and fix $i \in F$. If $X_{\underline{i}}$ is injective, then the map*

$$\Phi \colon (B^{F^\perp})^{\beta_{F^\perp}|_{B^{F^\perp}}} \to \mathcal{L}(\mathcal{F}X); b \mapsto b\overline{p}_{\underline{i}} \textit{ for all } b \in (B^{F^\perp})^{\beta_{F^\perp}|_{B^{F^\perp}}}$$

is an injective $*$-homomorphism.

Proof. For notational convenience, we set $B := B^{F^\perp}$. Fixing $\underline{m} \in \mathbb{Z}_+^d$ and $\underline{n} \perp F$, observe that

$$i \in \operatorname{supp}(\underline{n} + \underline{m}) \iff i \in \operatorname{supp} \underline{m},$$

using that $\underline{n} \perp F$ and therefore $i \notin \operatorname{supp} \underline{n}$ in the forward implication. It follows that $\overline{p}_{\underline{i}} \in \overline{t}_{\underline{n}}(X_{\underline{n}})'$ for all $\underline{n} \perp F$. Thus $\overline{p}_{\underline{i}}$ commutes with $B^{\beta_{F^\perp}|_B}$ since

$$B^{\beta_{F^\perp}|_B} = \overline{\operatorname{span}}\{\overline{t}_{\underline{n}}(X_{\underline{n}})\overline{t}_{\underline{n}}(X_{\underline{n}})^* \mid \underline{n} \perp F\}.$$

Using this observation, it is routine to check that Φ is a well-defined $*$-homomorphism.

It remains to verify that Φ is injective. Accordingly, for each $m \in \mathbb{N}$ we define the finite $\vee$-closed[2] set $\mathcal{S}_m := \{\underline{n} \in \mathbb{Z}_+^d \mid \underline{n} \leq m \cdot \underline{1}_{F^c}\}$, noting that $B^{\beta_{F^\perp}|_B}$ is the direct limit of the C*-subalgebras

$$B_m^{\beta_{F^\perp}|_B} := \operatorname{span}\{\overline{\psi}_{\underline{n}}(\mathcal{K}(X_{\underline{n}})) \mid \underline{n} \in \mathcal{S}_m\} \text{ for all } m \in \mathbb{N}.$$

Hence, to show that Φ is injective, it suffices to show that Φ is injective on every $B_m^{\beta_{F^\perp}|_B}$.

To this end, fix $m \in \mathbb{N}$ and let

$$b := \sum_{\underline{n} \in \mathcal{S}_m} \overline{\psi}_{\underline{n}}(k_{\underline{n}}) \in \ker \Phi.$$

Then, in particular, we have that

$$b|_{\sum_{\underline{\ell} \geq \underline{i}} X_{\underline{\ell}}} = \Phi(b)|_{\sum_{\underline{\ell} \geq \underline{i}} X_{\underline{\ell}}} = 0.$$

Notice that $\underline{m} + \underline{i} \geq \underline{i}$ whenever $\underline{m} \perp F$, and therefore

$$\sup\{\|b|_{X_{\underline{m}+\underline{i}}}\| \mid \underline{m} \perp F\} \leq \|b|_{\sum_{\underline{\ell} \geq \underline{i}} X_{\underline{\ell}}}\| = 0.$$

For each $\underline{n} \in \mathcal{S}_m$, we have that $\overline{\psi}_{\underline{n}}(k_{\underline{n}}) = \sum_{\underline{\ell} \geq \underline{n}} \iota_{\underline{n}}^{\underline{\ell}}(k_{\underline{n}})$. Hence for each $\underline{m} \perp F$ we obtain that

$$\|b|_{X_{\underline{m}+\underline{i}}}\| = \|\sum_{\underline{n} \in \mathcal{S}_m} \sum_{\underline{\ell} \geq \underline{n}} \iota_{\underline{n}}^{\underline{\ell}}(k_{\underline{n}})|_{X_{\underline{m}+\underline{i}}}\| = \|\sum_{\substack{\underline{n} \in \mathcal{S}_m \\ \underline{n} \leq \underline{m}+\underline{i}}} \iota_{\underline{n}}^{\underline{m}+\underline{i}}(k_{\underline{n}})\|.$$

We then compute

$$\begin{aligned}
0 &= \sup\{\|b|_{X_{\underline{m}+\underline{i}}}\| \mid \underline{m} \perp F\} = \sup\{\|\sum_{\substack{\underline{n} \in \mathcal{S}_m \\ \underline{n} \leq \underline{m}+\underline{i}}} \iota_{\underline{n}}^{\underline{m}+\underline{i}}(k_{\underline{n}})\| \mid \underline{m} \perp F\} \\
&= \sup\{\|\sum_{\substack{\underline{n} \in \mathcal{S}_m \\ \underline{n} \leq \underline{m}}} \iota_{\underline{n}}^{\underline{m}+\underline{i}}(k_{\underline{n}})\| \mid \underline{m} \perp F\} = \sup\{\|\iota_{\underline{m}}^{\underline{m}+\underline{i}}(\sum_{\substack{\underline{n} \in \mathcal{S}_m \\ \underline{n} \leq \underline{m}}} \iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}}))\| \mid \underline{m} \perp F\} \\
&= \sup\{\|\sum_{\substack{\underline{n} \in \mathcal{S}_m \\ \underline{n} \leq \underline{m}}} \iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}})\| \mid \underline{m} \perp F\} = \|\sum_{\underline{m} \perp F} \sum_{\substack{\underline{n} \in \mathcal{S}_m \\ \underline{n} \leq \underline{m}}} \iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}})\| \\
&= \|\sum_{\underline{n} \in \mathcal{S}_m} \sum_{\substack{\underline{m} \perp F \\ \underline{m} \geq \underline{n}}} \iota_{\underline{n}}^{\underline{m}}(k_{\underline{n}})\| = \|b|_{\sum_{\underline{m} \perp F} X_{\underline{m}}}\| = \|b\|.
\end{aligned}$$

[2] See the comments preceding Lemma 2.4.6.

In the second line we use that $\underline{n} \leq \underline{m} + \underline{i}$ if and only if $\underline{n} \leq \underline{m}$, whenever $\underline{n} \in \mathcal{S}_m$ and $\underline{m} \perp F$. In the third line we use injectivity of $X_{\underline{i}}$ to deduce that $\iota_{\underline{m}}^{\underline{m}+\underline{i}}$ is isometric for all $\underline{m} \perp F$. In the final line we apply Corollary 5.5.16. In total, we have that $b = 0$ and the proof is complete. □

Proposition 5.5.23. *Let X be a compactly aligned product system over $\mathbb{Z}_+^d$ with coefficients in a C^*-algebra A. Let $F = [r]$ for some $r < d$ and let $i \in F$. If $X_{\underline{i}}$ admits a finite frame and is injective, then $Y_{\underline{i}}^F$ admits a finite frame and is injective.*

Proof. For notational convenience, we set $B := B^{F^\perp}$ and $Y := Y^F$. Let $\{\xi_{\underline{i}}^{(j)}\}_{j \in [N_{\underline{i}}]}$ be a finite frame of $X_{\underline{i}}$. Then $\{\overline{t}_{\underline{i}}(\xi_{\underline{i}}^{(j)})\}_{j \in [N_{\underline{i}}]}$ defines a finite frame of $Y_{\underline{i}}$ by Proposition 5.5.18. Next let $b \in \ker \phi_{Y_{\underline{i}}}$, so that

$$b^* b \overline{p}_{\underline{i}} = \sum_{j=1}^{N_{\underline{i}}} b^* b \overline{t}_{\underline{i}}(\xi_{\underline{i}}^{(j)}) \overline{t}_{\underline{i}}(\xi_{\underline{i}}^{(j)})^* = \sum_{j=1}^{N_{\underline{i}}} b^* \left(\phi_{Y_{\underline{i}}}(b) \overline{t}_{\underline{i}}(\xi_{\underline{i}}^{(j)}) \right) \overline{t}_{\underline{i}}(\xi_{\underline{i}}^{(j)})^* = 0.$$

Noting that $\overline{p}_{\underline{i}} \in \mathcal{NT}_X^{\beta_{F^\perp}}$, we obtain that $E_{\beta_{F^\perp}}(b^* b) \overline{p}_{\underline{i}} = 0$ by using that $E_{\beta_{F^\perp}}$ is a bimodule map over $\mathcal{NT}_X^{\beta_{F^\perp}}$. An application of Proposition 5.5.22 gives that $E_{\beta_{F^\perp}}(b^* b) = 0$, and since $E_{\beta_{F^\perp}}$ is faithful we obtain that $b^* b = 0$. We conclude that $b = 0$, and the proof is complete. □

Corollary 5.5.24. *Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C^*-algebra A, wherein $X_{\underline{i}}$ admits a finite frame for all $i \in [d]$. Let $F = [r]$ for some $r < d$. If $X_{\underline{i}}$ is injective for all $i \in F$, then Y^F is regular, and the canonical $*$-isomorphism $\mathcal{NT}_{Y^F} \cong \mathcal{NT}_X$ descends to a $*$-isomorphism*

$$\mathcal{NO}_{Y^F} \cong \mathcal{NT}_X / \left\langle \overline{\pi}(A) \overline{q}_{\underline{i}} \mid i \in F \right\rangle.$$

Proof. By Propositions 2.5.1 and 5.5.23, we have that Y^F is regular. Next, recall the definition of the ideal I_{Y^F} of Theorem 5.5.21. An application of the last part of Proposition 5.5.10 to Y^F gives that $I_{Y^F} = \{0\}$. Thus the final claim follows by a direct application of Theorem 5.5.21. □

The description in Theorem 5.5.21 first applies the Y^F construction to X and then passes to a quotient. We can have an alternative route by first considering a quotient of X and then applying the Y^F construction. To avoid confusion, we will denote by Y_X^F the product system induced by $F = [r]$ and X, as in Definition 5.5.11.

Theorem 5.5.25. *Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C^*-algebra A, wherein $X_{\underline{i}}$ admits a finite frame for all $i \in [d]$. Let $F = [r]$ for some $r < d$, and let*

$$\mathfrak{J} := \left\langle \overline{\pi}_X(A) \overline{q}_{X,\underline{i}} \mid i \in F \right\rangle \subseteq \mathcal{NT}_X \quad \textit{and} \quad I_X^F := \mathcal{L}_\emptyset^{\mathfrak{J}} \subseteq A.$$

Then the following hold:

(i) $I_X^F = \{a \in A \mid \lim_{\underline{n} \in \mathbb{Z}_+^r} \|\phi_{(\underline{n},\underline{0})}(a)\| = 0\}$.

(ii) $[X_{\underline{i}}]_{I_X^F}$ *is injective for all* $i \in F$.

(iii) *The product system* $Y^F_{[X]_{I_X^F}}$ *is regular.*

(iv) *The association* $X \to [X]_{I_X^F} \to Y^F_{[X]_{I_X^F}}$ *induces a* $*$*-isomorphism* $\mathcal{NT}_X/\mathfrak{J} \cong \mathcal{NO}_{Y^F_{[X]_{I_X^F}}}$.

Proof. To ease notation, we write $I := I_X^F$. Item (i) follows by [32, Proposition 4.3]. By following the same arguments as in the proof of Proposition 5.5.10, but for $i \in F$ instead of $i \in [d]$, we obtain that $[X_{\underline{i}}]_I$ is injective for all $i \in F$. An application of Corollary 5.5.24 yields that $Y^F_{[X]_I}$ is regular, and that there is a canonical $*$-isomorphism

$$\mathcal{NT}_{[X]_I} / \left\langle \overline{\pi}_{[X]_I}([A]_I)\overline{q}_{[X]_I,\underline{i}} \mid i \in F \right\rangle \cong \mathcal{NO}_{Y^F_{[X]_I}}.$$

It suffices to show that the canonical map $[\,\cdot\,]_I \colon \mathcal{NT}_X \to \mathcal{NT}_{[X]_I}$ descends to a $*$-isomorphism

$$\Phi \colon \mathcal{NT}_X/\mathfrak{J} \to \mathcal{NT}_{[X]_I} / \left\langle \overline{\pi}_{[X]_I}([A]_I)\overline{q}_{[X]_I,\underline{i}} \mid i \in F \right\rangle.$$

To this end, we have that $[\overline{\pi}_X(a)\overline{q}_{X,\underline{i}}]_I = \overline{\pi}_{[X]_I}([a]_I)\overline{q}_{[X]_I,\underline{i}}$ for all $a \in A$ and $i \in F$. Therefore

$$[\mathfrak{J}]_I = \left\langle \overline{\pi}_{[X]_I}([A]_I)\overline{q}_{[X]_I,\underline{i}} \mid i \in F \right\rangle,$$

and thus Φ is a well-defined $*$-epimorphism. On the other hand, by applying item (ii) of Proposition 4.2.1 to $\mathfrak{J}$ and noting that $I \equiv \mathcal{L}_\emptyset^{\mathfrak{J}}$, we obtain a canonical $*$-epimorphism

$$\Psi \colon \mathcal{NT}_{[X]_I} \to \mathcal{NO}([\mathcal{L}^{\mathfrak{J}}]_I, [X]_I) \cong \mathcal{NT}_X/\mathfrak{J},$$

such that

$$\Psi(\overline{\pi}_{[X]_I}([a]_I)) = \overline{\pi}_X(a) + \mathfrak{J} \quad \text{and} \quad \Psi(\overline{t}_{[X]_I,\underline{n}}([\xi_{\underline{n}}]_I) = \overline{t}_{X,\underline{n}}(\xi_{\underline{n}}) + \mathfrak{J}$$

for all $a \in A, \xi_{\underline{n}} \in X_{\underline{n}}$ and $\underline{n} \in \mathbb{Z}_+^d$. In particular, we have that $\Psi(\overline{p}_{[X]_I,\underline{i}}) = \overline{p}_{X,\underline{i}} + \mathfrak{J}$ for all $i \in F$ using a finite frame expansion, and thus

$$\Psi(\overline{\pi}_{[X]_I}([a]_I)\overline{q}_{[X]_I,\underline{i}}) = \overline{\pi}_X(a)\overline{q}_{X,\underline{i}} + \mathfrak{J} = 0, \text{ for all } a \in A, i \in F,$$

by definition of $\mathfrak{J}$. Hence Ψ descends to a canonical $*$-epimorphism

$$\widetilde{\Psi} \colon \mathcal{NT}_{[X]_I} / \left\langle \overline{\pi}_{[X]_I}([A]_I)\overline{q}_{[X]_I,\underline{i}} \mid i \in F \right\rangle \to \mathcal{NT}_X/\mathfrak{J}.$$

By definition $\widetilde{\Psi}$ is a left inverse of Φ and thus Φ is a $*$-isomorphism, as required. □

Theorems 5.5.21 and 5.5.25 show that there is no difference as when to consider the quotient product system.

Corollary 5.5.26. *Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C^*-algebra A, wherein $X_{\underline{i}}$ admits a finite frame for all $i \in [d]$. Let $F = [r]$ for some $r < d$. On the one hand, define the positively invariant ideal*

$$I_{Y_X^F} := \ker\{Y_{X,\underline{0}}^F \to \mathcal{NT}_{Y_X^F} / \left\langle \overline{\pi}_{Y_X^F}(Y_{X,\underline{0}}^F)\overline{q}_{Y_X^F,\underline{i}} \mid i \in F \right\rangle\}$$

for the product system Y_X^F related to X and F. On the other hand, define the positively invariant ideal

$$I_X^F := \ker\{A \to \mathcal{NT}_X / \left\langle \overline{\pi}_X(A)\overline{q}_{X,\underline{i}} \mid i \in F \right\rangle\}$$

for X, and consider the product system $Y_{[X]_{I_X^F}}^F$ related to $[X]_{I_X^F}$ and F. Then there are canonical $*$-isomorphisms

$$\mathcal{NO}_{[Y_X^F]_{I_{Y_X^F}}} \cong \mathcal{NT}_X / \left\langle \overline{\pi}_X(A)\overline{q}_{X,\underline{i}} \mid i \in F \right\rangle \cong \mathcal{NO}_{Y_{[X]_{I_X^F}}^F}.$$

If, in addition, $X_{\underline{i}}$ is injective for all $i \in F$, then Y_X^F is regular, $I_{Y_X^F} = \{0\}$ and $I_X^F = \{0\}$.

Proof. It suffices to comment on the last part. If $X_{\underline{i}}$ is injective for all $i \in F$, then Y_X^F is regular by Corollary 5.5.24 and

$$\mathcal{NT}_{Y_X^F} / \left\langle \overline{\pi}_{Y_X^F}(Y_{X,\underline{0}}^F)\overline{q}_{Y_X^F,\underline{i}} \mid i \in F \right\rangle = \mathcal{NO}_{Y_X^F}$$

by Proposition 5.5.10. Thus $I_{Y_X^F} = \{0\}$ by injectivity of the universal CNP-representation of Y_X^F. Finally, every $\phi_{(\underline{n},\underline{0})}$ with $\underline{n} \in \mathbb{Z}_+^r$ is isometric, and thus $I_X^F = \{0\}$ by item (i) of Theorem 5.5.25. □

Chapter 6

Proper product systems

We close by studying further the product systems X over $\mathbb{Z}_+^d$ satisfying $\phi_{\underline{n}}(A) \subseteq \mathcal{K}(X_{\underline{n}})$ for all $\underline{n} \in \mathbb{Z}_+^d$. More specifically, we show that the parametrisation result of Bilich [4, Theorem 4.15] aligns with Theorem 4.2.3 and Corollary 4.2.12.

6.1 T-families and O-families

We begin by presenting the definitions/results of [4] that we will need going forward.

Definition 6.1.1. Let X be a product system over $\mathbb{Z}_+^d$ with coefficients in a C*-algebra A. We say that X is *proper* if $\phi_{\underline{n}}(A) \subseteq \mathcal{K}(X_{\underline{n}})$ for all $\underline{n} \in \mathbb{Z}_+^d$.

For the remainder of the section, we will take X to be a proper product system with coefficients in a C*-algebra A. Note that X is strong compactly aligned by Corollary 2.5.6 and that all 2^d-tuples of X are automatically relative. Given a Nica-covariant representation (π, t) of X, we have that $\pi(a)q_F$ can be written as an alternating sum for all $a \in A$ and $F \subseteq [d]$ by Proposition 2.5.16. In turn, we have that

$$\pi(a)q_F = 0 \iff a \in \mathcal{L}_F^{(\pi,t)}, \tag{6.1}$$

where the reverse implication follows by Proposition 2.5.17. Lastly, given an ideal $I \subseteq A$, we have that

$$\mathcal{J}_F = (\bigcap_{i \in F} \ker \phi_{\underline{i}})^\perp \quad \text{and} \quad J_F(I, X) = \{a \in A \mid aX_F^{-1}(I) \subseteq I\} \text{ for all } \emptyset \neq F \subseteq [d].$$

Definition 6.1.2. [4, Definition 4.2] Let X be a proper product system with coefficients in a C*-algebra A. A 2^d-tuple $\mathcal{L}$ of X is said to be a *T-family (of X)* if it consists of ideals and satisfies

$$\mathcal{L}_F = X_{\underline{i}}^{-1}(\mathcal{L}_F) \cap \mathcal{L}_{F \cup \{i\}} \text{ for all } F \subsetneq [d], i \in [d] \setminus F. \tag{6.2}$$

A T-family $\mathcal{L}$ of X is said to be an *O-family (of X)* if $\mathcal{I} \subseteq \mathcal{L}$.

According to [4], T-families admit the following Gauge-Invariant Uniqueness Theorem.

Theorem 6.1.3 ($\mathbb{Z}_+^d$-GIUT for T-families [4, Corollary 4.14])**.** *Let X be a proper product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be a T-family of X and (π, t) be an $\mathcal{L}$-relative CNP-representation of X. Then $\mathcal{NO}(\mathcal{L}, X) \cong \mathrm{C}^*(\pi, t)$ via a (unique) canonical $*$-isomorphism if and only if $\pi(a)q_F = 0$ implies that $a \in \mathcal{L}_F$ (for any $a \in A$ and $F \subseteq [d]$) and (π, t) admits a gauge action.*

It will be useful to rephrase Theorem 6.1.3 via the following lemma.

Lemma 6.1.4. *Let X be a proper product system with coefficients in a C*-algebra A. Let $\mathcal{L}$ be a T-family of X and (π, t) be an $\mathcal{L}$-relative CNP-representation of X. Then $\pi(a)q_F = 0$ implies that $a \in \mathcal{L}_F$ (for any $a \in A$ and $F \subseteq [d]$) if and only if $\mathcal{L} = \mathcal{L}^{(\pi,t)}$.*

Proof. Immediate by (6.1) and the fact that (π, t) is an $\mathcal{L}$-relative CNP-representation. □

We present the main result of [4] in the following slightly modified form.

Theorem 6.1.5. *[4, Theorem 4.15] Let X be a proper product system with coefficients in a C*-algebra A. Then the following hold:*

(i) There exists an order-preserving bijection between the set of T-families of X and the set of gauge-invariant ideals of $\mathcal{NT}_X$.

(ii) There exists an order-preserving bijection between the set of O-families of X and the set of gauge-invariant ideals of $\mathcal{NO}_X$.

We will show that Theorem 4.2.3 (resp. Corollary 4.2.12) aligns with item (i) (resp. item (ii)) of Theorem 6.1.5 by proving that the NT-2^d-tuples (resp. NO-2^d-tuples) of X are exactly the T-families (resp. O-families) of X.

6.2 Connection with NT-2^d-tuples and NO-2^d-tuples

To pass from NT-2^d-tuples to T-families, we will require the following proposition for general strong compactly aligned product systems.

Proposition 6.2.1. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Then we have that*

$$X_{\underline{i}}^{-1}(\mathcal{J}_F) \cap \mathcal{J}_{F \cup \{i\}} \subseteq \mathcal{J}_F \text{ for all } F \subsetneq [d], i \in [d] \setminus F.$$

Proof. First we prove the claim for $F = \emptyset$. To this end, take $i \in [d]$ and $a \in X_{\underline{i}}^{-1}(\mathcal{J}_\emptyset) \cap \mathcal{J}_{\{i\}}$. In particular, we have that

$$\langle X_{\underline{i}}, aX_{\underline{i}} \rangle \subseteq \mathcal{J}_\emptyset = \{0\}$$

and thus $a \in \ker \phi_{\underline{i}}$. Since $a \in \mathcal{J}_{\{i\}} \subseteq (\ker \phi_{\underline{i}})^\perp$, it follows that $a = 0$, as required.

Now fix $\emptyset \neq F \subsetneq [d], i \in [d] \setminus F$ and $a \in X_{\underline{i}}^{-1}(\mathcal{J}_F) \cap \mathcal{J}_{F \cup \{i\}}$. Then by definition we have that

$$\langle X_{\underline{i}}, aX_{\underline{i}} \rangle \subseteq (\bigcap_{j \in F} \ker \phi_{\underline{j}})^{\perp} \cap (\bigcap_{j \in [d]} \phi_{\underline{j}}^{-1}(\mathcal{K}(X_{\underline{j}}))) \text{ and } a \in (\bigcap_{j \in F \cup \{i\}} \ker \phi_{\underline{j}})^{\perp} \cap (\bigcap_{j \in [d]} \phi_{\underline{j}}^{-1}(\mathcal{K}(X_{\underline{j}}))).$$

Showing that $a \in \mathcal{J}_F$ amounts to proving that $a \in (\bigcap_{j \in F} \ker \phi_{\underline{j}})^{\perp}$. To this end, fix $b \in \bigcap_{j \in F} \ker \phi_{\underline{j}}$. We claim that

$$\langle X_{\underline{i}}, bX_{\underline{i}} \rangle \subseteq \bigcap_{j \in F} \ker \phi_{\underline{j}}.$$

To see this, fix $\xi_{\underline{i}}, \eta_{\underline{i}} \in X_{\underline{i}}$ and $j \in F$. Recall from the proof of Proposition 3.4.3 that

$$\phi_{\underline{j}}(\langle \xi_{\underline{i}}, \phi_{\underline{i}}(b)\eta_{\underline{i}} \rangle) = \tau(\xi_{\underline{i}})^* \phi_{\underline{j}+\underline{i}}(b) \tau(\eta_{\underline{i}}),$$

where now $\tau(\xi_{\underline{i}}), \tau(\eta_{\underline{i}}) \in \mathcal{L}(X_{\underline{j}}, X_{\underline{j}+\underline{i}})$. In turn, we obtain that

$$\tau(\xi_{\underline{i}})^* \phi_{\underline{j}+\underline{i}}(b) \tau(\eta_{\underline{i}}) = \tau(\xi_{\underline{i}})^* \iota_{\underline{j}}^{\underline{j}+\underline{i}}(\phi_{\underline{j}}(b)) \tau(\eta_{\underline{i}}) = 0,$$

using that $b \in \bigcap_{j \in F} \ker \phi_{\underline{j}}$ in the final equality. Thus $\langle X_{\underline{i}}, bX_{\underline{i}} \rangle \subseteq \bigcap_{j \in F} \ker \phi_{\underline{j}}$, as claimed. Hence we have that

$$\langle X_{\underline{i}}, aX_{\underline{i}} \rangle \langle X_{\underline{i}}, bX_{\underline{i}} \rangle = \{0\}.$$

Fixing $\xi_{\underline{i}}, \eta_{\underline{i}}, \zeta_{\underline{i}}, \nu_{\underline{i}} \in X_{\underline{i}}$, we compute that

$$\left\langle \xi_{\underline{i}}, (\phi_{\underline{i}}(a) \Theta_{\eta_{\underline{i}}, \zeta_{\underline{i}}} \phi_{\underline{i}}(b)) \nu_{\underline{i}} \right\rangle = \left\langle \xi_{\underline{i}}, a(\Theta_{\eta_{\underline{i}}, \zeta_{\underline{i}}}(b\nu_{\underline{i}})) \right\rangle = \langle \xi_{\underline{i}}, a(\eta_{\underline{i}} \langle \zeta_{\underline{i}}, b\nu_{\underline{i}} \rangle) \rangle = \langle \xi_{\underline{i}}, a\eta_{\underline{i}} \rangle \langle \zeta_{\underline{i}}, b\nu_{\underline{i}} \rangle = 0.$$

Since $\xi_{\underline{i}}, \nu_{\underline{i}} \in X_{\underline{i}}$ are arbitrary, we deduce that

$$\phi_{\underline{i}}(a) \Theta_{\eta_{\underline{i}}, \zeta_{\underline{i}}} \phi_{\underline{i}}(b) = 0 \text{ for all } \eta_{\underline{i}}, \zeta_{\underline{i}} \in X_{\underline{i}}.$$

In turn, because $\eta_{\underline{i}}, \zeta_{\underline{i}} \in X_{\underline{i}}$ are arbitrary, it follows that

$$\phi_{\underline{i}}(a) \mathcal{K}(X_{\underline{i}}) \phi_{\underline{i}}(b) = \{0\}.$$

Since $\phi_{\underline{i}}(a) \in \mathcal{K}(X_{\underline{i}})$, an application of an approximate unit of $\mathcal{K}(X_{\underline{i}})$ gives that $\phi_{\underline{i}}(ab) = 0$ and hence $ab \in \bigcap_{j \in F \cup \{i\}} \ker \phi_{\underline{j}}$. However, we also have that $ab \in (\bigcap_{j \in F \cup \{i\}} \ker \phi_{\underline{j}})^{\perp}$ since $a \in \mathcal{J}_{F \cup \{i\}}$. Thus $ab = 0$ and so $a \in (\bigcap_{j \in F} \ker \phi_{\underline{j}})^{\perp}$, completing the proof. □

Observe that Proposition 6.2.1 generalises [15, Lemma 4.3.4]. Using the nomenclature therein, the latter is recovered by using Proposition 5.3.4 and taking $F = [d] \setminus \{1\}$ and $i = 1$.

Remark 6.2.2. Note that $\mathcal{J}$ is not a T-family in general. This is because the reverse inclusion in the statement of Proposition 6.2.1 may not hold, since $\mathcal{J}$ may not be invariant and so $\mathcal{J}_F \not\subseteq X_{\underline{i}}^{-1}(\mathcal{J}_F)$. To see this, let B be a non-zero C*-algebra and set $A = B \oplus B$. We define a $*$-endomorphism

$$\alpha\colon A \to A; (b, b') \mapsto (0, b) \text{ for all } (b, b') \in A.$$

Note that $\ker \alpha = \{0\} \oplus B$. By setting $\alpha_{(1,0)} := \alpha$ and $\alpha_{(0,1)} := \alpha$, we obtain a canonical unital semigroup homomorphism $\mathbb{Z}_+^2 \to \mathrm{End}(A)$ which we also denote by α. Applying Proposition 5.3.3, we obtain a proper product system X_α. We have that

$$\mathcal{J}_{\{1\}} = (\ker \alpha_{(1,0)})^\perp = B \oplus \{0\},$$

using item (i) of Proposition 5.3.4 in the first equality. Fixing $b \in B \setminus \{0\}$, we obtain that

$$\alpha_{(0,1)}(b, 0) = (0, b) \notin \mathcal{J}_{\{1\}}.$$

It then follows that $\mathcal{J}_{\{1\}} \not\subseteq X_{\alpha,(0,1)}^{-1}(\mathcal{J}_{\{1\}})$ by item (ii) of Proposition 5.3.4.

We are now ready to prove that all NT-2^d-tuples are T-families. We proceed directly, using the definitions alone.

Proposition 6.2.3. *Let X be a proper product system with coefficients in a C*-algebra A. Then every NT-2^d-tuple of X is a T-family of X.*

Proof. Let $\mathcal{L}$ be an NT-2^d-tuple of X. Then $\mathcal{L}$ consists of ideals by item (i) of Definition 4.1.4. It remains to check that $\mathcal{L}$ satisfies (6.2).

We begin by addressing the case where $F = \emptyset$. Fixing $i \in [d]$, note that $\mathcal{L}_\emptyset \subseteq X_{\underline{i}}^{-1}(\mathcal{L}_\emptyset)$ since $\mathcal{L}$ is invariant by item (ii) of Definition 4.1.4. We also have that $\mathcal{L}_\emptyset \subseteq \mathcal{L}_{\{i\}}$ because $\mathcal{L}$ is partially ordered by item (iii) of Definition 4.1.4. This shows that $\mathcal{L}_\emptyset \subseteq X_{\underline{i}}^{-1}(\mathcal{L}_\emptyset) \cap \mathcal{L}_{\{i\}}$. For the reverse inclusion, take $a \in X_{\underline{i}}^{-1}(\mathcal{L}_\emptyset) \cap \mathcal{L}_{\{i\}}$. An application of item (i) of Definition 4.1.4 gives that $a \in J_{\{i\}}(\mathcal{L}_\emptyset, X)$ and hence $aX_{\underline{i}}^{-1}(\mathcal{L}_\emptyset) \subseteq \mathcal{L}_\emptyset$. Since $a \in X_{\underline{i}}^{-1}(\mathcal{L}_\emptyset)$ by assumption, using an approximate unit of $X_{\underline{i}}^{-1}(\mathcal{L}_\emptyset)$ yields that $a \in \mathcal{L}_\emptyset$. Hence we have that

$$\mathcal{L}_\emptyset = X_{\underline{i}}^{-1}(\mathcal{L}_\emptyset) \cap \mathcal{L}_{\{i\}} \text{ for all } i \in [d].$$

To account for $F \neq \emptyset$, we proceed by strong (downward) induction on $|F|$. For the base case, fix $\emptyset \neq F \subsetneq [d]$ such that $|F| = d - 1$ and $i \in [d] \setminus F$. Note that $\mathcal{L}_F \subseteq X_{\underline{i}}^{-1}(\mathcal{L}_F) \cap \mathcal{L}_{[d]}$ since $\mathcal{L}$ is invariant and partially ordered. For the reverse inclusion, take $a \in X_{\underline{i}}^{-1}(\mathcal{L}_F) \cap \mathcal{L}_{[d]}$. By Proposition 4.1.5, it suffices to show that

$$a \in \left(\bigcap_{\underline{n} \perp F} X_{\underline{n}}^{-1}(J_F(\mathcal{L}_\emptyset, X)) \right) \cap \mathcal{L}_{\mathrm{inv},F} \cap \mathcal{L}_{\mathrm{lim},F}.$$

To this end, fix $\underline{n} = (n_1, \dots, n_d) \perp F$. First suppose that $n_i > 0$. Then we may write $\underline{n} = \underline{i} + \underline{m}$ for some $\underline{m} \perp F$. Since $X_{\underline{i}} \otimes_A X_{\underline{m}} \cong X_{\underline{n}}$ via the multiplication map $u_{\underline{i},\underline{m}}$, we obtain that

$$\begin{aligned}\langle X_{\underline{n}}, aX_{\underline{n}}\rangle &= \langle X_{\underline{i}} \otimes_A X_{\underline{m}}, a(X_{\underline{i}} \otimes_A X_{\underline{m}})\rangle \\ &\subseteq [\langle X_{\underline{m}}, \langle X_{\underline{i}}, aX_{\underline{i}}\rangle X_{\underline{m}}\rangle] \\ &\subseteq \mathcal{L}_F \subseteq J_F(\mathcal{L}_\emptyset, X),\end{aligned} \tag{6.3}$$

using that $a \in X_{\underline{i}}^{-1}(\mathcal{L}_F)$ and $\mathcal{L}$ is invariant in the second inclusion and item (i) of Definition 4.1.4 in the final inclusion. Thus $a \in X_{\underline{n}}^{-1}(J_F(\mathcal{L}_\emptyset, X))$. Now suppose that $n_i = 0$, so that $\underline{n} = \underline{0}$ because $|F| = d-1$. We must show that $a \in J_F(\mathcal{L}_\emptyset, X)$. This is equivalent to showing that $[a]_{\mathcal{L}_\emptyset} \in \mathcal{J}_F([X]_{\mathcal{L}_\emptyset})$ by item (ii) of Proposition 4.1.3, which applies since $\mathcal{L}$ is invariant. To this end, note that

$$\langle X_{\underline{i}}, aX_{\underline{i}}\rangle \subseteq \mathcal{L}_F \subseteq J_F(\mathcal{L}_\emptyset, X) = [\cdot]_{\mathcal{L}_\emptyset}^{-1}(\mathcal{J}_F([X]_{\mathcal{L}_\emptyset}))$$

and that

$$a \in \mathcal{L}_{[d]} \subseteq J_{[d]}(\mathcal{L}_\emptyset, X) = [\cdot]_{\mathcal{L}_\emptyset}^{-1}(\mathcal{J}_{[d]}([X]_{\mathcal{L}_\emptyset}))$$

by assumption. In other words, we have that

$$[a]_{\mathcal{L}_\emptyset} \in [X_{\underline{i}}]_{\mathcal{L}_\emptyset}^{-1}(\mathcal{J}_F([X]_{\mathcal{L}_\emptyset})) \cap \mathcal{J}_{[d]}([X]_{\mathcal{L}_\emptyset})$$

and so $[a]_{\mathcal{L}_\emptyset} \in \mathcal{J}_F([X]_{\mathcal{L}_\emptyset})$ by Proposition 6.2.1, which applies since $[X]_{\mathcal{L}_\emptyset}$ is proper by Lemma 2.2.11 (and so $[X]_{\mathcal{L}_\emptyset}$ is strong compactly aligned by Corollary 2.5.6). In total, we deduce that

$$a \in \bigcap_{\underline{n}\perp F} X_{\underline{n}}^{-1}(J_F(\mathcal{L}_\emptyset, X)).$$

To see that $a \in \mathcal{L}_{\text{inv},F} \equiv \bigcap_{\underline{n}\perp F} X_{\underline{n}}^{-1}(\mathcal{L}_{[d]})$, fix $\underline{n} = (n_1, \dots, n_d) \perp F$. If $n_i > 0$ then we may argue as in (6.3), replacing $J_F(\mathcal{L}_\emptyset, X)$ by $\mathcal{L}_{[d]}$ and invoking the partial ordering of $\mathcal{L}$, to obtain that $a \in X_{\underline{n}}^{-1}(\mathcal{L}_{[d]})$. If $n_i = 0$ and so $\underline{n} = \underline{0}$, then there is nothing to show since $a \in \mathcal{L}_{[d]}$ by assumption. Hence $a \in \mathcal{L}_{\text{inv},F}$.

Next, since $a \in X_{\underline{i}}^{-1}(\mathcal{L}_F)$ and X is proper, we may apply (2.5) to obtain that $\phi_{\underline{i}}(a) \in \mathcal{K}(X_{\underline{i}}\mathcal{L}_F)$. By Proposition 3.3.4, it follows that $a \in \mathcal{L}_{\text{lim},F}$. Combining the preceding deductions, we ascertain that $a \in \mathcal{L}_F$, establishing the base case.

Now fix $1 \leq N \leq d-2$ and suppose we have proved that $\mathcal{L}$ satisfies (6.2) for all $\emptyset \neq F \subsetneq [d]$ satisfying $|F| = d-n$, for all $1 \leq n \leq N$. Fix $\emptyset \neq F \subsetneq [d]$ such that $|F| = d-(N+1)$ and $i \in [d] \setminus F$. We must show that

$$\mathcal{L}_F = X_{\underline{i}}^{-1}(\mathcal{L}_F) \cap \mathcal{L}_{F\cup\{i\}}.$$

The forward inclusion is immediate since $\mathcal{L}$ is invariant and partially ordered. For the reverse inclusion, take $a \in X_{\underline{i}}^{-1}(\mathcal{L}_F) \cap \mathcal{L}_{F\cup\{i\}}$. As in the base case, an application of

Proposition 4.1.5 ensures that it suffices to show that

$$a \in \left(\bigcap_{\underline{n} \perp F} X_{\underline{n}}^{-1}(J_F(\mathcal{L}_\emptyset, X)) \right) \cap \mathcal{L}_{\text{inv},F} \cap \mathcal{L}_{\text{lim},F}.$$

Accordingly, fix $\underline{n} = (n_1, \dots, n_d) \perp F$. If $n_i > 0$, then we argue as in (6.3) to obtain that $a \in X_{\underline{n}}^{-1}(J_F(\mathcal{L}_\emptyset, X))$. Now suppose that $n_i = 0$, so that $\underline{n} \perp F \cup \{i\}$. Applying invariance of $\mathcal{L}$ in tandem with the fact that $a \in \mathcal{L}_{F \cup \{i\}}$, we obtain that

$$\langle X_{\underline{n}}, aX_{\underline{n}} \rangle \subseteq \mathcal{L}_{F \cup \{i\}} \subseteq J_{F \cup \{i\}}(\mathcal{L}_\emptyset, X). \tag{6.4}$$

Note also that

$$\begin{aligned} \langle X_{\underline{i}}, \langle X_{\underline{n}}, aX_{\underline{n}} \rangle X_{\underline{i}} \rangle &\subseteq \langle X_{\underline{n}} \otimes_A X_{\underline{i}}, a(X_{\underline{n}} \otimes_A X_{\underline{i}}) \rangle \subseteq \langle X_{\underline{n}+\underline{i}}, aX_{\underline{n}+\underline{i}} \rangle \\ &= \langle X_{\underline{i}+\underline{n}}, aX_{\underline{i}+\underline{n}} \rangle = \langle X_{\underline{i}} \otimes_A X_{\underline{n}}, a(X_{\underline{i}} \otimes_A X_{\underline{n}}) \rangle \\ &\subseteq [\langle X_{\underline{n}}, \langle X_{\underline{i}}, aX_{\underline{i}} \rangle X_{\underline{n}} \rangle] \subseteq \mathcal{L}_F \subseteq J_F(\mathcal{L}_\emptyset, X), \end{aligned} \tag{6.5}$$

arguing as in (6.3). In other words, we have that

$$\langle X_{\underline{n}}, aX_{\underline{n}} \rangle \subseteq X_{\underline{i}}^{-1}(J_F(\mathcal{L}_\emptyset, X)).$$

At this point we argue as in the base case, working over $[X]_{\mathcal{L}_\emptyset}$ and applying Proposition 6.2.1 to obtain that $a \in X_{\underline{n}}^{-1}(J_F(\mathcal{L}_\emptyset, X))$. In total, we deduce that

$$a \in \bigcap_{\underline{n} \perp F} X_{\underline{n}}^{-1}(J_F(\mathcal{L}_\emptyset, X)).$$

To see that $a \in \mathcal{L}_{\text{inv},F} \equiv \bigcap_{\underline{n} \perp F} X_{\underline{n}}^{-1}(\cap_{F \subsetneq D} \mathcal{L}_D)$, fix $\underline{n} = (n_1, \dots, n_d) \perp F$. If $n_i > 0$, then an application of (6.3) gives that

$$\langle X_{\underline{n}}, aX_{\underline{n}} \rangle \subseteq \mathcal{L}_F \subseteq \cap_{F \subsetneq D} \mathcal{L}_D,$$

where the final inclusion follows from the partial ordering of $\mathcal{L}$. If $n_i = 0$, then $\underline{n} \perp F \cup \{i\}$ and so $\langle X_{\underline{n}}, aX_{\underline{n}} \rangle \subseteq \mathcal{L}_{F \cup \{i\}}$ by (6.4). Fix $D \supsetneq F$ and suppose that $i \in D$. Then $F \cup \{i\} \subseteq D$ and so $\langle X_{\underline{n}}, aX_{\underline{n}} \rangle \subseteq \mathcal{L}_D$ by the partial ordering of $\mathcal{L}$. Now suppose that $i \notin D$. Observe that $|F| < |D|$ and so $|D| = d - n$ for some $1 \leq n \leq N$. By the inductive hypothesis, we have that

$$\mathcal{L}_D = X_{\underline{i}}^{-1}(\mathcal{L}_D) \cap \mathcal{L}_{D \cup \{i\}}.$$

Note that $\langle X_{\underline{n}}, aX_{\underline{n}} \rangle \subseteq \mathcal{L}_{D \cup \{i\}}$ by the partial ordering of $\mathcal{L}$. We also obtain that $\langle X_{\underline{n}}, aX_{\underline{n}} \rangle \subseteq X_{\underline{i}}^{-1}(\mathcal{L}_D)$ by arguing as in (6.5) until the final inclusion, at which point we use that $\mathcal{L}_F \subseteq \mathcal{L}_D$ by the partial ordering of $\mathcal{L}$. Hence $\langle X_{\underline{n}}, aX_{\underline{n}} \rangle \subseteq \mathcal{L}_D$. Since our choice of $D \supsetneq F$ was arbitrary, we ascertain that $\langle X_{\underline{n}}, aX_{\underline{n}} \rangle \subseteq \cap_{F \subsetneq D} \mathcal{L}_D$ in all cases. Thus $a \in \mathcal{L}_{\text{inv},F}$.

Finally, since $a \in X_{\underline{i}}^{-1}(\mathcal{L}_F)$ and X is proper, we may apply (2.5) to obtain that

$\phi_{\underline{i}}(a) \in \mathcal{K}(X_{\underline{i}}\mathcal{L}_F)$. By Proposition 3.3.4, it follows that $a \in \mathcal{L}_{\lim,F}$. Combining the preceding deductions, we ascertain that $a \in \mathcal{L}_F$ and in total we obtain that

$$\mathcal{L}_F = X_{\underline{i}}^{-1}(\mathcal{L}_F) \cap \mathcal{L}_{F\cup\{i\}}.$$

By strong induction, the proof is complete. □

Next we turn to the passage from T-families to NT-2^d-tuples, which is encompassed by the following proposition.

Proposition 6.2.4. *Let X be a proper product system with coefficients in a C^*-algebra A and suppose that $\mathcal{L}$ is a 2^d-tuple of X. Then the following are equivalent:*

(i) $\mathcal{L}$ is a maximal 2^d-tuple of X;

(ii) $\mathcal{L} = \mathcal{L}^{(\pi,t)}$ for some Nica-covariant representation (π, t) of X that admits a gauge action;

(iii) $\mathcal{L}$ is an NT-2^d-tuple of X;

(iv) $\mathcal{L}$ is a T-family of X.

Proof. First we comment on our strategy for the proof. We will begin by showing [(i) $\iff$ (ii) $\iff$ (iii)]. To account for (iv), we will then show [(iii) $\implies$ (iv) $\implies$ (ii)].

[(i) $\iff$ (ii)]: Assume that $\mathcal{L}$ is a maximal 2^d-tuple of X and set $\mathfrak{J} := \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)}$. Let $Q_{\mathfrak{J}} \colon \mathcal{NT}_X \to \mathcal{NT}_X/\mathfrak{J}$ denote the quotient map. Recall that $(Q_{\mathfrak{J}} \circ \overline{\pi}_X, Q_{\mathfrak{J}} \circ \overline{t}_X)$ is a Nica-covariant representation of X that admits a gauge action. We claim that $\mathcal{L} = \mathcal{L}^{(Q_{\mathfrak{J}}\circ\overline{\pi}_X,Q_{\mathfrak{J}}\circ\overline{t}_X)}$. First note that $\mathcal{L} \subseteq \mathcal{L}^{(Q_{\mathfrak{J}}\circ\overline{\pi}_X,Q_{\mathfrak{J}}\circ\overline{t}_X)}$ by definition of $\mathfrak{J}$. Maximality of $\mathcal{L}$ then implies that it suffices to show that

$$\mathfrak{J} = \mathfrak{J}^{(\overline{\pi}_X,\overline{t}_X)}_{\mathcal{L}^{(Q_{\mathfrak{J}}\circ\overline{\pi}_X,Q_{\mathfrak{J}}\circ\overline{t}_X)}}.$$

To this end, observe that the forward inclusion is immediate by the preceding reasoning. For the reverse inclusion, fix $F \subseteq [d]$ and $a \in \mathcal{L}_F^{(Q_{\mathfrak{J}}\circ\overline{\pi}_X,Q_{\mathfrak{J}}\circ\overline{t}_X)}$. It suffices to show that $\overline{\pi}_X(a)\overline{q}_{X,F} \in \mathfrak{J}$. Accordingly, an application of Proposition 2.5.17 (and Proposition 2.5.16) yields that

$$Q_{\mathfrak{J}}(\overline{\pi}_X(a)\overline{q}_{X,F}) = Q_{\mathfrak{J}}(\overline{\pi}_X(a)) + \sum\{(-1)^{|\underline{n}|}Q_{\mathfrak{J}}(\overline{\psi}_{X,\underline{n}}(\phi_{\underline{n}}(a))) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = 0.$$

Thus $\overline{\pi}_X(a)\overline{q}_{X,F} \in \mathfrak{J}$, as required.

For the converse, assume that $\mathcal{L} = \mathcal{L}^{(\pi,t)}$ for some Nica-covariant representation (π, t) of X that admits a gauge action. Let $\mathcal{L}'$ be a 2^d-tuple of X such that $\mathcal{L} \subseteq \mathcal{L}'$ and $\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)} = \mathfrak{J}_{\mathcal{L}'}^{(\overline{\pi}_X,\overline{t}_X)}$. We must show that $\mathcal{L}' \subseteq \mathcal{L}$. To this end, fix $F \subseteq [d]$ and $a \in \mathcal{L}'_F$. By definition we have that $\overline{\pi}_X(a)\overline{q}_{X,F} \in \mathfrak{J}_{\mathcal{L}'}^{(\overline{\pi}_X,\overline{t}_X)} = \mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)}$. Observe that

$$(\pi \times t)(\mathfrak{J}_{\mathcal{L}}^{(\overline{\pi}_X,\overline{t}_X)}) = \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}.$$

In turn, we have that

$$\pi(a)q_F = (\pi \times t)(\overline{\pi}_X(a)\overline{q}_{X,F}) \in \mathfrak{J}_{\mathcal{L}}^{(\pi,t)}.$$

However, since (π, t) is an $\mathcal{L}^{(\pi,t)}$-relative CNP-representation (this can be seen by arguing as in the proof of Proposition 3.1.17) and $\mathcal{L} = \mathcal{L}^{(\pi,t)}$ by assumption, we obtain that

$$\mathfrak{J}_{\mathcal{L}}^{(\pi,t)} = \mathfrak{J}_{\mathcal{L}^{(\pi,t)}}^{(\pi,t)} = \{0\}.$$

Thus $\pi(a)q_F = 0$ and so $a \in \mathcal{L}_F^{(\pi,t)} = \mathcal{L}_F$ by Proposition 2.5.16. Hence $\mathcal{L}' \subseteq \mathcal{L}$ and we conclude that $\mathcal{L}$ is maximal, as required.

[(ii) $\iff$ (iii)]: This is provided by Proposition 4.1.12.

[(iii) $\implies$ (iv)]: This is provided by Proposition 6.2.3.

[(iv) $\implies$ (ii)]: Assume that $\mathcal{L}$ is a T-family of X. Consider the universal $\mathcal{L}$-relative CNP-representation $(\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}})$. Since $\mathcal{NO}(\mathcal{L}, X)$ is canonically $*$-isomorphic to itself via the identity map, combining Theorem 6.1.3 and Lemma 6.1.4 yields that $\mathcal{L} = \mathcal{L}^{(\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}})}$ and $(\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}})$ admits a gauge action. Thus, taking $(\pi, t) := (\pi_X^{\mathcal{L}}, t_X^{\mathcal{L}})$, the proof is complete. □

Notice that the (ii) $\implies$ (i) implication of Proposition 6.2.4 did not use that (π, t) admits a gauge action. This is also the case for the (ii) $\implies$ (iv) implication. We provide an independent proof to this effect.

Proposition 6.2.5. *Let X be a proper product system with coefficients in a C^*-algebra A and let (π, t) be a Nica-covariant representation of X. Then $\mathcal{L}^{(\pi,t)}$ is a T-family of X.*

Proof. Recall that $\mathcal{L}^{(\pi,t)}$ consists of ideals by Proposition 3.1.14. Thus, fixing $F \subsetneq [d]$ and $i \in [d] \setminus F$, it suffices to show that

$$\mathcal{L}_F^{(\pi,t)} = X_{\underline{i}}^{-1}(\mathcal{L}_F^{(\pi,t)}) \cap \mathcal{L}_{F\cup\{i\}}^{(\pi,t)}.$$

The forward inclusion is immediate since $\mathcal{L}^{(\pi,t)}$ is invariant and partially ordered by Proposition 3.1.14. For the reverse inclusion, fix $a \in X_{\underline{i}}^{-1}(\mathcal{L}_F^{(\pi,t)}) \cap \mathcal{L}_{F\cup\{i\}}^{(\pi,t)}$. Applying (6.1), we obtain that

$$t_{\underline{i}}(X_{\underline{i}})^*\pi(a)q_F t_{\underline{i}}(X_{\underline{i}}) = t_{\underline{i}}(X_{\underline{i}})^*\pi(a)t_{\underline{i}}(X_{\underline{i}})q_F = \pi(\langle X_{\underline{i}}, aX_{\underline{i}}\rangle)q_F = \{0\} \text{ and } \pi(a)q_{F\cup\{i\}} = 0,$$

where we also use Proposition 2.5.15 in the first equality. In particular, it follows that

$$\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}}))\pi(a)q_F\psi_{\underline{i}}(\mathcal{K}(X_{\underline{i}})) = \{0\}.$$

Fixing $k_{\underline{i}}, k'_{\underline{i}} \in \mathcal{K}(X_{\underline{i}})$ and writing $\pi(a)q_F$ as an alternating sum using Proposition 2.5.16, we obtain that

$$\psi_{\underline{i}}(k_{\underline{i}})\pi(a)\psi_{\underline{i}}(k'_{\underline{i}}) + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{i}}(k_{\underline{i}})\psi_{\underline{n}}(\phi_{\underline{n}}(a))\psi_{\underline{i}}(k'_{\underline{i}}) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = 0.$$

For $\underline{0} \neq \underline{n} \leq \underline{1}_F$, we have that

$$\psi_{\underline{i}}(k_{\underline{i}})\psi_{\underline{n}}(\phi_{\underline{n}}(a))\psi_{\underline{i}}(k'_{\underline{i}}) = \psi_{\underline{n}+\underline{i}}(\iota_{\underline{i}}^{\underline{n}+\underline{i}}(k_{\underline{i}})\iota_{\underline{n}}^{\underline{n}+\underline{i}}(\phi_{\underline{n}}(a))\iota_{\underline{i}}^{\underline{n}+\underline{i}}(k'_{\underline{i}}))$$

by Nica-covariance of (π, t), noting that $\underline{n} \vee \underline{i} = \underline{n} + \underline{i}$ since $\underline{n} \perp \underline{i}$. Since $k_{\underline{i}}, k'_{\underline{i}} \in \mathcal{K}(X_{\underline{i}})$ are arbitrary, we may replace them by an approximate unit of $\mathcal{K}(X_{\underline{i}})$ and use Proposition 2.5.13 to obtain that

$$\psi_{\underline{i}}(\phi_{\underline{i}}(a)) + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}+\underline{i}}(\phi_{\underline{n}+\underline{i}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = 0. \tag{6.6}$$

Recalling that $\pi(a)q_{F\cup\{i\}} = 0$, we also have that

$$\pi(a) + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_{F\cup\{i\}}\} = 0. \tag{6.7}$$

By summing (6.6) and (6.7), we deduce that

$$\pi(a) + \sum\{(-1)^{|\underline{n}|}\psi_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = 0.$$

In other words, we have that $a \in \mathcal{L}_F^{(\pi,t)}$, completing the proof. □

In total, we obtain the desired result.

Corollary 6.2.6. *Let X be a proper product system with coefficients in a C*-algebra A. Then the NT-2^d-tuples (resp. NO-2^d-tuples) of X are exactly the T-families (resp. O-families) of X.*

Proof. Immediate by Proposition 6.2.4 and Definition 4.2.8. □

Remark 6.2.7. Let $\mathcal{L}$ be a T-family of X. We have not yet been able to prove that $\mathcal{L}$ is an NT-2^d-tuple directly (i.e., using the definition alone). However, it is straightforward to check that $\mathcal{L}$ satisfies conditions (i)-(iii) of Definition 4.1.4.

We start by showing that $\mathcal{L}$ is invariant. To this end, fix $F \subseteq [d]$ and $\underline{n} \perp F$. Since $\mathcal{L}$ consists of ideals, it suffices to show that

$$\langle X_{\underline{n}}, \mathcal{L}_F X_{\underline{n}}\rangle \subseteq \mathcal{L}_F.$$

Without loss of generality, we may assume that $F \neq [d]$ and $\underline{n} \neq \underline{0}$. We proceed by induction on $|\underline{n}|$. First suppose that $|\underline{n}| = 1$, so that $\underline{n} = \underline{i}$ for some $i \in [d] \setminus F$. Since $\mathcal{L}$ is a T-family, we have that $\mathcal{L}_F \subseteq X_{\underline{i}}^{-1}(\mathcal{L}_F)$ and hence $\langle X_{\underline{i}}, \mathcal{L}_F X_{\underline{i}}\rangle \subseteq \mathcal{L}_F$, as required. Now suppose that we have proved the claim for all $\underline{n} \perp F$ satisfying $|\underline{n}| = N$ for some $N \in \mathbb{N}$. Fix $\underline{n} \perp F$ such that $|\underline{n}| = N + 1$. Then we may write $\underline{n} = \underline{m} + \underline{i}$ for some $\underline{m} \perp F$ satisfying $|\underline{m}| = N$ and some $i \in [d] \setminus F$. We obtain that

$$\langle X_{\underline{n}}, \mathcal{L}_F X_{\underline{n}}\rangle = \langle X_{\underline{m}} \otimes_A X_{\underline{i}}, \mathcal{L}_F(X_{\underline{m}} \otimes_A X_{\underline{i}})\rangle \subseteq [\langle X_{\underline{i}}, \langle X_{\underline{m}}, \mathcal{L}_F X_{\underline{m}}\rangle X_{\underline{i}}\rangle] \subseteq \mathcal{L}_F,$$

where we appeal to the inductive hypothesis in tandem with the base case in the final inclusion. Thus, by induction, we have that $\mathcal{L}$ is invariant.

Next we show that $\mathcal{L}$ is partially ordered. Accordingly, fix $F \subseteq D \subseteq [d]$. Without loss of generality, we may assume that $F = [k]$ (with the convention that if $k = 0$ then $F = \emptyset$) and $D = [\ell]$ for some $0 \leq k \leq \ell \leq d$. Since $k+1 \notin F$ and $\mathcal{L}$ is a T-family, we have that $\mathcal{L}_F \subseteq \mathcal{L}_{F\cup\{k+1\}}$. Analogously, since $k+2 \notin F \cup \{k+1\}$ we have that $\mathcal{L}_{F\cup\{k+1\}} \subseteq \mathcal{L}_{F\cup\{k+1,k+2\}}$. Arguing iteratively until $D \setminus F$ has been exhausted, we obtain a sequence of inclusions

$$\mathcal{L}_F \subseteq \mathcal{L}_{F\cup\{k+1\}} \subseteq \mathcal{L}_{F\cup\{k+1,k+2\}} \subseteq \cdots \subseteq \mathcal{L}_{F\cup(D\setminus F)} \equiv \mathcal{L}_D.$$

Thus $\mathcal{L}_F \subseteq \mathcal{L}_D$ and hence $\mathcal{L}$ is partially ordered.

To see that $\mathcal{L}$ satisfies condition (i) of Definition 4.1.4, fix $\emptyset \neq F \subseteq [d]$ and $a \in \mathcal{L}_F$. We must show that $a \in J_F(\mathcal{L}_\emptyset, X)$, which amounts to showing that $aX_F^{-1}(\mathcal{L}_\emptyset) \subseteq \mathcal{L}_\emptyset$. Note that invariance of $\mathcal{L}$ implies that $X_F^{-1}(\mathcal{L}_\emptyset) = \bigcap_{i\in F} X_{\underline{i}}^{-1}(\mathcal{L}_\emptyset)$ by Proposition 4.1.2. Fix $b \in X_F^{-1}(\mathcal{L}_\emptyset)$. Without loss of generality, we may assume that $F = [k]$ for some $0 \leq k \leq d$. We start by setting

$$F_1 := F \setminus \{k\}.$$

Since $a \in \mathcal{L}_F$, we have that $ab \in \mathcal{L}_F \equiv \mathcal{L}_{F_1\cup\{k\}}$. Additionally, we have that $b \in X_F^{-1}(\mathcal{L}_\emptyset) \subseteq X_{\underline{k}}^{-1}(\mathcal{L}_\emptyset)$ and hence

$$ab \in X_{\underline{k}}^{-1}(\mathcal{L}_\emptyset) \subseteq X_{\underline{k}}^{-1}(\mathcal{L}_{F_1}),$$

where the inclusion follows by [36, p. 117] together with the fact that $\mathcal{L}$ is partially ordered. Hence $ab \in X_{\underline{k}}^{-1}(\mathcal{L}_{F_1}) \cap \mathcal{L}_{F_1\cup\{k\}}$ and so $ab \in \mathcal{L}_{F_1}$ since $\mathcal{L}$ is a T-family. Next we set

$$F_2 := F_1 \setminus \{k-1\} = F \setminus \{k-1, k\}.$$

We have that $ab \in \mathcal{L}_{F_1} \equiv \mathcal{L}_{F_2\cup\{k-1\}}$ and $b \in X_F^{-1}(\mathcal{L}_\emptyset) \subseteq X_{\underline{k-1}}^{-1}(\mathcal{L}_\emptyset)$, so that

$$ab \in X_{\underline{k-1}}^{-1}(\mathcal{L}_\emptyset) \subseteq X_{\underline{k-1}}^{-1}(\mathcal{L}_{F_2}),$$

arguing as with $\mathcal{L}_{F_1}$. Hence $ab \in X_{\underline{k-1}}^{-1}(\mathcal{L}_{F_2}) \cap \mathcal{L}_{F_2\cup\{k-1\}}$ and so $ab \in \mathcal{L}_{F_2}$ since $\mathcal{L}$ is a T-family. We iterate the preceding argument until all elements of F have been exhausted, eventually yielding that $ab \in \mathcal{L}_\emptyset$. This shows that $a \in J_F(\mathcal{L}_\emptyset, X)$ and so $\mathcal{L}_F \subseteq J_F(\mathcal{L}_\emptyset, X)$, as required.

Remark 6.2.8. Let $\mathcal{L}$ be a T-family of X. To complete the proof that $\mathcal{L}$ is an NT-2^d-tuple, it suffices to show that

$$\mathcal{L}'_F := \Big(\bigcap_{\underline{n}\perp F} X_{\underline{n}}^{-1}(J_F(\mathcal{L}_\emptyset, X))\Big) \cap \mathcal{L}_{\text{inv},F} \cap \mathcal{L}_{\text{lim},F} \subseteq \mathcal{L}_F \text{ for all } \emptyset \neq F \subsetneq [d]$$

by Proposition 4.1.5. Accordingly, take $a \in \mathcal{L}'_F$. We claim that it is sufficient to show that $\phi_{\underline{n}}(a) \in \mathcal{K}(X_{\underline{n}}\mathcal{L}_F)$ for some $\underline{n} \perp F$. Indeed, suppose we have shown this. If $\underline{n} = \underline{0}$, then we have that $AaA \subseteq \mathcal{L}_F$ by (2.5) and hence $a \in \mathcal{L}_F$ by using an approximate unit of A. Now suppose that $\underline{n} = (n_1, \dots, n_d) \neq \underline{0}$. Then we may write

$$\underline{n} = \sum\{n_i\underline{i} \mid i \in [d] \setminus F\},$$

since $\underline{n} \perp F$. As $\underline{n} \neq \underline{0}$, we have that $n_i \in \mathbb{N}$ for some $i \in [d] \setminus F$. It follows that $\underline{n} = \underline{i} + \underline{m}$, where

$$\underline{m} = \sum\{n_j\underline{j} \mid j \in [d] \setminus (F \cup \{i\})\} + (n_i - 1)\underline{i}.$$

We claim that $\langle X_{\underline{m}}, aX_{\underline{m}}\rangle \subseteq \mathcal{L}_F$. To see this, first observe that

$$\langle X_{\underline{i}}, \langle X_{\underline{m}}, aX_{\underline{m}}\rangle X_{\underline{i}}\rangle \subseteq \langle X_{\underline{m}} \otimes_A X_{\underline{i}}, a(X_{\underline{m}} \otimes_A X_{\underline{i}})\rangle \subseteq \langle X_{\underline{n}}, aX_{\underline{n}}\rangle \subseteq \mathcal{L}_F,$$

where the final inclusion follows by (2.5). This shows that $\langle X_{\underline{m}}, aX_{\underline{m}}\rangle \subseteq X_{\underline{i}}^{-1}(\mathcal{L}_F)$. Since $\underline{m} \perp F$ and $a \in \mathcal{L}_{\mathrm{inv},F}$, we have that $a \in X_{\underline{m}}^{-1}(\cap_{F \subsetneq D}\mathcal{L}_D)$ by definition. In particular, we obtain that $\langle X_{\underline{m}}, aX_{\underline{m}}\rangle \subseteq \mathcal{L}_{F\cup\{i\}}$ since $F \subsetneq F \cup \{i\}$. In total, we deduce that

$$\langle X_{\underline{m}}, aX_{\underline{m}}\rangle \subseteq X_{\underline{i}}^{-1}(\mathcal{L}_F) \cap \mathcal{L}_{F\cup\{i\}} = \mathcal{L}_F,$$

using that $\mathcal{L}$ is a T-family in the final equality. An application of (2.5) then yields that $\phi_{\underline{m}}(a) \in \mathcal{K}(X_{\underline{m}}\mathcal{L}_F)$. Now we iterate the preceding argument, replacing $\underline{n}$ by $\underline{m}$ in the next iteration. In this way we progressively decrease each non-zero entry of $\underline{n}$ by 1 until we eventually obtain that $\phi_{\underline{0}}(a) \in \mathcal{K}(A\mathcal{L}_F)$. It then follows that $a \in \mathcal{L}_F$ by the $\underline{n} = \underline{0}$ case, as required.

The veracity of the claim that $\phi_{\underline{n}}(a) \in \mathcal{K}(X_{\underline{n}}\mathcal{L}_F)$ for some $\underline{n} \perp F$ is unclear in general. Nevertheless, we can make progress when restricting to product systems induced by row-finite k-graphs. Accordingly, henceforth we assume that $X \equiv X(\Lambda)$ for some row-finite k-graph (Λ, d) (see Section 5.4). Note that $X(\Lambda)$ is proper by item (i) of Proposition 5.4.5. Let $H_{\mathcal{L}}$ be the family of sets of vertices of Λ corresponding to $\mathcal{L}$ and let $H_{\mathcal{L}',F}$ be the vertex set associated with $\mathcal{L}'_F$. It is enough to show that $H_{\mathcal{L}',F} \subseteq H_{\mathcal{L},F}$, since the duality between ideals of $c_0(\Lambda^{\underline{0}})$ and subsets of $\Lambda^{\underline{0}}$ respects inclusions.

Accordingly, take $v \in H_{\mathcal{L}',F}$. We must show that $\delta_v \in \mathcal{L}_F$. By the preceding reasoning, it is sufficient to show that $\phi_{\underline{n}}(\delta_v) \in \mathcal{K}(X_{\underline{n}}(\Lambda)\mathcal{L}_F)$ for some $\underline{n} \perp F$. To this end, note that $\delta_v \in \mathcal{L}_{\mathrm{lim},F}$ by construction and so $\lim_{\underline{n}\perp F} \|\phi_{\underline{n}}(\delta_v) + \mathcal{K}(X_{\underline{n}}(\Lambda)\mathcal{L}_F)\| = 0$. In particular, there exists $\underline{n} \perp F$ such that

$$\|\phi_{\underline{n}}(\delta_v) + \mathcal{K}(X_{\underline{n}}(\Lambda)\mathcal{L}_F)\| < 1/2.$$

Since $\phi_{\underline{n}}(\delta_v) + \mathcal{K}(X_{\underline{n}}(\Lambda)\mathcal{L}_F)$ is a projection, it follows that $\phi_{\underline{n}}(\delta_v) \in \mathcal{K}(X_{\underline{n}}(\Lambda)\mathcal{L}_F)$, as required. We conclude that $\mathcal{L}$ is an NT-2^k-tuple.

In total, in the setting of row-finite k-graphs we can show that every T-family is an

NT-2^k-tuple (and vice versa) directly. The general case is less clear since $\phi_{\underline{n}}(a)+\mathcal{K}(X_{\underline{n}}\mathcal{L}_F)$ may not be a projection (for $a \in \mathcal{L}'_F$ and $\underline{n} \perp F$).

Appendix A

Appendix

A.1 The Hewitt-Cohen Factorisation Theorem

In this section we provide a proof of Theorem 2.1.1. We begin with a definition. Let X be a linear space and A be a C*-algebra. Suppose also that X is a left A-module with compatible scalar multiplication, i.e., we have that

$$\lambda(a\xi) = (\lambda a)\xi = a(\lambda\xi) \text{ for all } \lambda \in \mathbb{C}, a \in A, \xi \in X.$$

We say that X is a *Banach A-module* if X is a Banach space and there exists $M \geq 0$ such that

$$\|a\xi\| \leq M \cdot \|a\| \cdot \|\xi\| \text{ for all } a \in A, \xi \in X. \tag{A.1}$$

Henceforth assume that X is a Banach A-module. For simplicity we assume that $M = 1$, with the understanding that the ensuing arguments require only minor tweaks to hold in the general case. We say that X is *non-degenerate* if $[AX] = X$.

Lemma A.1.1. *Let X be a non-degenerate Banach A-module, where A is a C*-algebra. Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit of A. Then we have that*

$$\|\cdot\| \text{-}\lim_\lambda u_\lambda \xi = \xi \text{ for all } \xi \in X.$$

Proof. First suppose that $\xi = a\eta$ for some $a \in A$ and $\eta \in X$. Then we have that

$$\|u_\lambda\xi - \xi\| = \|u_\lambda a\eta - a\eta\| = \|(u_\lambda a - a)\eta\| \leq \|u_\lambda a - a\| \cdot \|\eta\| \text{ for all } \lambda \in \Lambda,$$

using (A.1) in the inequality. It follows that $\|\cdot\|\text{-}\lim_\lambda u_\lambda\xi = \xi$ since $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit of A. In turn, an application of the triangle inequality yields that the claim holds for all $\xi \in \operatorname{span} AX$. Finally, suppose that $\xi \in [AX]$, so that ξ is the norm-limit of a sequence $(\xi_n)_{n\in\mathbb{N}} \subseteq \operatorname{span} AX$. Hence, fixing $\varepsilon > 0$, there exists $N \in \mathbb{N}$ so that

$$\|\xi - \xi_N\| < \varepsilon/3.$$

By the preceding reasoning, there exists $\lambda_0 \in \Lambda$ such that for all $\lambda \in \Lambda$ satisfying $\lambda \geq \lambda_0$, we have that

$$\|u_\lambda \xi_N - \xi_N\| < \varepsilon/3.$$

Thus, for all $\lambda \in \Lambda$ satisfying $\lambda \geq \lambda_0$, we obtain that

$$\begin{aligned}\|u_\lambda \xi - \xi\| &\leq \|u_\lambda \xi - u_\lambda \xi_N\| + \|u_\lambda \xi_N - \xi_N\| + \|\xi_N - \xi\| \\ &= \|u_\lambda(\xi - \xi_N)\| + \|u_\lambda \xi_N - \xi_N\| + \|\xi_N - \xi\| \\ &\leq \|\xi - \xi_N\| + \|u_\lambda \xi_N - \xi_N\| + \|\xi_N - \xi\| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,\end{aligned}$$

using (A.1) in the second inequality. Hence the claim holds for all $\xi \in [AX]$ and non-degeneracy of X completes the proof. □

Lemma A.1.2. *Let X be a right Hilbert module over a C*-algebra A. Then for each $\xi \in X$ there exists a unique $\eta \in X$ such that $\xi = \eta \langle \eta, \eta \rangle$.*

Proof. See [52, Proposition 2.31]. □

For a C*-algebra A, viewed as a right Hilbert module over itself, we set $H_A := \sum_{n \in \mathbb{Z}_+} A$ as the direct sum of right Hilbert A-modules, e.g., [40, p. 6].

Proposition A.1.3. *[52, Proposition 2.33] Let X be a non-degenerate Banach A-module, where A is a C*-algebra. Then we have that $X = AX$.*

Proof. The reverse inclusion is immediate, so fix $\xi \in X$. Setting $\xi_0 := \xi$, iterative applications of Lemma A.1.1 yield sequences $(\xi_n)_{n \in \mathbb{Z}_+} \subseteq X$ and $(u_n)_{n \in \mathbb{Z}_+} \subseteq A_{sa}$ such that $\|u_n\| \leq 1$ and $\xi_{n+1} := \xi_n - u_n \xi_n$ has norm at most 2^{-2n-2} for all $n \in \mathbb{Z}_+$. Next we set $v_n := 2^{-n} u_n$ for all $n \in \mathbb{Z}_+$. We claim that $v := (v_n)_{n \in \mathbb{Z}_+} \in H_A$. This amounts to showing that $\sum_{n \in \mathbb{Z}_+} v_n^* v_n \in A$. Since A is in particular a Banach space, it suffices to show that $\sum_{n \in \mathbb{Z}_+} \|v_n^* v_n\| < \infty$. To this end, fix $n \in \mathbb{Z}_+$. We have that

$$\|v_n^* v_n\| = \|(2^{-n} u_n^*)(2^{-n} u_n)\| = 2^{-2n} \|u_n^* u_n\| = 2^{-2n} \|u_n\|^2 \leq 2^{-2n} = (1/4)^n.$$

Since $\sum_{n \in \mathbb{Z}_+} (1/4)^n < \infty$, an application of the comparison test for series convergence gives that $\sum_{n \in \mathbb{Z}_+} \|v_n^* v_n\| < \infty$, as required.

Next set $\eta_n := 2^n \xi_n$ for all $n \in \mathbb{Z}_+$. We claim that $\sum_{n \in \mathbb{Z}_+} (\xi_n - \xi_{n+1}) = \xi_0$. To see this, fix $\ell \in \mathbb{Z}_+$. We obtain that

$$\|\sum_{n=0}^{\ell} (\xi_n - \xi_{n+1}) - \xi_0\| = \| - \xi_{\ell+1}\| = \|\xi_{\ell+1}\| \leq 2^{-2\ell-2}.$$

Since the right hand side converges to 0, we have that

$$\sum_{n \in \mathbb{Z}_+} v_n \eta_n = \sum_{n \in \mathbb{Z}_+} u_n \xi_n = \sum_{n \in \mathbb{Z}_+} (\xi_n - \xi_{n+1}) = \xi_0,$$

as claimed. Applying Lemma A.1.2 to the Hilbert A-module H_A, we obtain a unique $w = (w_n)_{n\in\mathbb{Z}_+} \in H_A$ such that $v = w \langle w, w\rangle$. For each $n \in \mathbb{Z}_+$, observe that

$$0 \leq w_n^* w_n \leq \sum_{m\in\mathbb{Z}_+} w_m^* w_m.$$

Taking norms, we deduce that $\|w_n\| \leq \|w\|$. Next we claim that $\eta := \sum_{n\in\mathbb{Z}_+} w_n^*\eta_n \in X$. Since X is in particular a Banach space, it suffices to show that $\sum_{n\in\mathbb{Z}_+} \|w_n^*\eta_n\| < \infty$. To this end, fix $n \in \mathbb{N}$. We deduce that

$$\|w_n^*\eta_n\| \leq \|w_n\| \cdot \|2^n\xi_n\| \leq 2^n\|w\| \cdot 2^{-2n} = \|w\| \cdot 2^{-n} = \|w\|(1/2)^n,$$

using (A.1) in the first inequality. Since $\sum_{n\in\mathbb{Z}_+} \|w\|(1/2)^n < \infty$, an application of the comparison test for series convergence gives that $\sum_{n\in\mathbb{Z}_+} \|w_n^*\eta_n\| < \infty$, as required. Finally, setting $a := \langle w, w\rangle$, we obtain that

$$a\eta = \langle w, w\rangle \sum_{n\in\mathbb{Z}_+} w_n^*\eta_n = \sum_{n\in\mathbb{Z}_+} (\langle w, w\rangle\, w_n^*)\eta_n = \sum_{n\in\mathbb{Z}_+} (w_n \langle w, w\rangle)^*\eta_n = \sum_{n\in\mathbb{Z}_+} v_n\eta_n = \xi.$$

This shows that $X \subseteq AX$, finishing the proof. □

Proof of Theorem 2.1.1: Observe that X is a Banach A-module under the left A-module multiplication determined by

$$(a, \xi) \mapsto \pi(a)\xi \text{ for all } a \in A, \xi \in X.$$

In turn, we have that $[\pi(A)X]$ is a closed left A-submodule of X and is therefore a Banach A-module in its own right. Since $[\pi(A)X]$ is non-degenerate, the result follows immediately by Proposition A.1.3. □

A.2 Co-universality of $\mathcal{NO}_X$

In this section we provide a proof of the well-known co-universality result for $\mathcal{NO}_X$ [11, 17, 18, 54], in the particular case where X is a strong compactly aligned product system. Definitions, nomenclature and results from the preceding sections will be used liberally and oftentimes without citation. We proceed via the C*-envelope, which will require a sojourn into the theory of nonselfadjoint operator algebras. We present only the aspects of the theory that we will need, and the reader is addressed to [5, 47] for the full details.

We refer to a norm-closed subalgebra $\mathfrak{A} \subseteq \mathcal{B}(H)$ as an *operator algebra*. Notice that $M_n(\mathfrak{A}) \subseteq M_n(\mathcal{B}(H)) \cong \mathcal{B}(H^n)$ is also an operator algebra for each $n \in \mathbb{N}$, where H^n is the usual Hilbert space direct sum. Given an operator algebra $\mathfrak{B}$, let $\phi\colon \mathfrak{A} \to \mathfrak{B}$ be a linear map. We say that ϕ is *completely contractive* (resp. *completely isometric*) if the ampliation map

$$\phi^{(n)}\colon M_n(\mathfrak{A}) \to M_n(\mathfrak{B}); (a_{ij})_{ij} \mapsto (\phi(a_{ij}))_{ij} \text{ for all } (a_{ij})_{ij} \in M_n(\mathfrak{A})$$

is contractive (resp. isometric) for all $n \in \mathbb{N}$. We use the abbreviation "c.c." for "completely contractive" and "c.is." for "completely isometric". When $\mathfrak{A}$ and $\mathfrak{B}$ are C*-algebras, we say that ϕ is *completely positive*[1] (abbrev. c.p.) if $\phi^{(n)}$ is positive for all $n \in \mathbb{N}$. If ϕ is a $*$-homomorphism then it is in particular c.p., as each $\phi^{(n)}$ is also a $*$-homomorphism and therefore preserves positivity. Analogous reasoning gives that all $*$-homomorphisms are c.c. maps and all injective $*$-homomorphisms are c.is. maps.

A pair (ι, C) where C is a C*-algebra and $\iota\colon \mathfrak{A} \to C$ is a c.is. algebra homomorphism satisfying $C = \mathrm{C}^*(\iota(\mathfrak{A}))$ is called a *C*-cover (of $\mathfrak{A}$)*. There exists a co-universal C*-cover $(\iota, \mathrm{C}^*_{\mathrm{env}}(\mathfrak{A}))$ of $\mathfrak{A}$ such that for any C*-cover (j, C) of $\mathfrak{A}$, there exists a (unique) $*$-epimorphism $\Phi\colon C \to \mathrm{C}^*_{\mathrm{env}}(\mathfrak{A})$ satisfying $\Phi \circ j = \iota$. We refer to the C*-algebra $\mathrm{C}^*_{\mathrm{env}}(\mathfrak{A})$ as the *C*-envelope (of $\mathfrak{A}$)*. The existence of the C*-envelope was established by Hamana [27] through the existence of the injective envelope. An independent proof was established by Dritschel and McCullough [19] through the existence of maximal dilations.

Let P be a unital subsemigroup of a discrete group G and let X be a product system over P with coefficients in a C*-algebra A. Given a representation (π, t) of X acting on a Hilbert space H, we write $\overline{\mathrm{alg}}(\pi, t)$ for the norm-closed subalgebra of $\mathcal{B}(H)$ generated by $\pi(A)$ and every $t_p(X_p)$. When $P = \mathbb{Z}_+^d$ and X is compactly aligned, we define the *tensor algebra (of X)* to be

$$\mathcal{NT}_X^+ := \overline{\mathrm{alg}}(\overline{\pi}_X, \overline{t}_X).$$

Reverting to the case of general P and X, consider the family $t(X) := \{t_p(X_p)\}_{p\in P}$. If (π, t) is injective, then $t(X)$ carries a canonical structure as a product system over P with coefficients in $\pi(A)$, where the C*-correspondence operations are inherited from $\mathcal{B}(H)$ and the multiplication maps are induced by the usual multiplication in $\mathcal{B}(H)$. Moreover, we have that $X \cong t(X)$. Both claims follow quickly from the representation properties of (π, t), and injectivity is required in order for each $t_p(X_p)$ to be complete with respect to the corresponding inner product norm.

Finally, given C*-algebras A and B, we write $A \otimes B$ to denote the spatial tensor product. We will require some basic properties of the spatial tensor product going forward, all of which can be found in [8, Chapter 3].

Proposition A.2.1. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A, and let (π, t) be an injective Nica-covariant representation of X that admits a gauge action. Then $(\pi \times t)|_{\mathcal{NT}_X^+}$ is completely isometric.*

Proof. Let H denote the Hilbert space on which (π, t) acts and let $(e_{\underline{n}})_{\underline{n}\in\mathbb{Z}_+^d}$ denote the usual orthonormal basis of $\ell^2(\mathbb{Z}_+^d)$. For each $\underline{n} \in \mathbb{Z}_+^d$, we write $V_{\underline{n}}$ for the shift operator in

[1] More generally, we can make sense of complete positivity for linear maps between operator systems [47, p. 9].

$\mathcal{B}(\ell^2(\mathbb{Z}_+^d))$ determined by

$$V_{\underline{n}} e_{\underline{m}} = e_{\underline{n}+\underline{m}} \text{ for all } \underline{m} \in \mathbb{Z}_+^d.$$

Observe that

$$V_{\underline{n}}^* e_{\underline{m}} = \begin{cases} e_{\underline{m}-\underline{n}} & \text{if } \underline{n} \leq \underline{m}, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for all } \underline{m} \in \mathbb{Z}_+^d,$$

from which it follows that $V_{\underline{n}}^* V_{\underline{n}} = I$ (but $V_{\underline{n}} V_{\underline{n}}^* \neq I$). Hence we obtain a unital isometric semigroup homomorphism V as follows:

$$V \colon \mathbb{Z}_+^d \to \mathcal{B}(\ell^2(\mathbb{Z}_+^d)); \underline{n} \mapsto V_{\underline{n}} \text{ for all } \underline{n} \in \mathbb{Z}_+^d.$$

We write $\mathrm{C}_\lambda^*(\mathbb{Z}_+^d)$ for the C*-subalgebra of $\mathcal{B}(\ell^2(\mathbb{Z}_+^d))$ generated by each $V_{\underline{n}}$. We define the maps

$$\begin{aligned} &\widehat{\pi} \colon A \to \mathrm{C}^*(\pi, t) \otimes \mathrm{C}_\lambda^*(\mathbb{Z}_+^d); a \mapsto \pi(a) \otimes I \text{ for all } a \in A, \\ &\widehat{t}_{\underline{n}} \colon X_{\underline{n}} \to \mathrm{C}^*(\pi, t) \otimes \mathrm{C}_\lambda^*(\mathbb{Z}_+^d); \xi_{\underline{n}} \mapsto t_{\underline{n}}(\xi_{\underline{n}}) \otimes V_{\underline{n}} \text{ for all } \xi_{\underline{n}} \in X_{\underline{n}}, \underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}. \end{aligned}$$

It is routine to check that $(\widehat{\pi}, \widehat{t})$ is a representation of X. Note that preservation of inner products follows because V is isometric and preservation of the multiplicative structure follows because V is a semigroup homomorphism. Next, take $a \in A$ and suppose that $\widehat{\pi}(a) = 0$. Then we have that $\pi(a) = 0$ and hence $a = 0$ by injectivity of π. Thus $(\widehat{\pi}, \widehat{t})$ is injective.

For each $\underline{n} \in \mathbb{Z}_+^d$ and $\xi_{\underline{n}}, \eta_{\underline{n}} \in \mathbb{Z}_+^d$, we obtain that

$$\widehat{\psi}_{\underline{n}}(\Theta_{\xi_{\underline{n}}, \eta_{\underline{n}}}) = \widehat{t}_{\underline{n}}(\xi_{\underline{n}}) \widehat{t}_{\underline{n}}(\eta_{\underline{n}})^* = (t_{\underline{n}}(\xi_{\underline{n}}) \otimes V_{\underline{n}})(t_{\underline{n}}(\eta_{\underline{n}})^* \otimes V_{\underline{n}}^*) = \psi_{\underline{n}}(\Theta_{\xi_{\underline{n}}, \eta_{\underline{n}}}) \otimes V_{\underline{n}} V_{\underline{n}}^*.$$

It follows that

$$\widehat{\psi}_{\underline{n}}(k_{\underline{n}}) = \psi_{\underline{n}}(k_{\underline{n}}) \otimes V_{\underline{n}} V_{\underline{n}}^* \text{ for all } k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}), \underline{n} \in \mathbb{Z}_+^d.$$

Next, fixing $\underline{n}, \underline{m}, \underline{r} \in \mathbb{Z}_+^d$, we have that

$$V_{\underline{n}}^* V_{\underline{m}} e_{\underline{r}} = V_{\underline{n}}^* e_{\underline{m}+\underline{r}} = \begin{cases} e_{(\underline{m}+\underline{r})-\underline{n}} & \text{if } \underline{n} \leq \underline{m} + \underline{r}, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, we obtain that

$$\begin{aligned} V_{\underline{n} \vee \underline{m} - \underline{n}} V_{\underline{n} \vee \underline{m} - \underline{m}}^* e_{\underline{r}} &= \begin{cases} V_{\underline{n} \vee \underline{m} - \underline{n}} e_{\underline{r} - (\underline{n} \vee \underline{m} - \underline{m})} & \text{if } \underline{n} \vee \underline{m} - \underline{m} \leq \underline{r}, \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} e_{(\underline{m}+\underline{r})-\underline{n}} & \text{if } \underline{n} \vee \underline{m} \leq \underline{m} + \underline{r}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since $\underline{n} \leq \underline{m} + \underline{r}$ if and only if $\underline{n} \vee \underline{m} \leq \underline{m} + \underline{r}$, we deduce that $V_{\underline{n}}^* V_{\underline{m}} = V_{\underline{n}\vee\underline{m}-\underline{n}} V_{\underline{n}\vee\underline{m}-\underline{m}}^*$. Now take $\underline{n}, \underline{m} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}, k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ and $k_{\underline{m}} \in \mathcal{K}(X_{\underline{m}})$. Combining the preceding observations, we have that

$$\begin{aligned}\widehat{\psi}_{\underline{n}}(k_{\underline{n}})\widehat{\psi}_{\underline{m}}(k_{\underline{m}}) &= \psi_{\underline{n}}(k_{\underline{n}})\psi_{\underline{m}}(k_{\underline{m}}) \otimes V_{\underline{n}} V_{\underline{n}}^* V_{\underline{m}} V_{\underline{m}}^* \\ &= \psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(k_{\underline{n}})\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(k_{\underline{m}})) \otimes V_{\underline{n}} V_{\underline{n}\vee\underline{m}-\underline{n}} V_{\underline{n}\vee\underline{m}-\underline{m}}^* V_{\underline{m}}^* \\ &= \psi_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(k_{\underline{n}})\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(k_{\underline{m}})) \otimes V_{\underline{n}\vee\underline{m}} V_{\underline{n}\vee\underline{m}}^* = \widehat{\psi}_{\underline{n}\vee\underline{m}}(\iota_{\underline{n}}^{\underline{n}\vee\underline{m}}(k_{\underline{n}})\iota_{\underline{m}}^{\underline{n}\vee\underline{m}}(k_{\underline{m}})),\end{aligned}$$

using Nica-covariance of (π, t) in the second equality and that V is a semigroup homomorphism in the third equality. Hence $(\widehat{\pi}, \widehat{t})$ is Nica-covariant.

Let γ denote the gauge action of (π, t) and fix $\underline{z} \in \mathbb{T}^d$. Since $\gamma_{\underline{z}}$ and $\mathrm{id}_{\mathrm{C}_\lambda^*(\mathbb{Z}_+^d)}$ are $*$-homomorphisms and thus in particular c.p., an application of [8, Theorem 3.5.3] gives a $*$-homomorphism

$$\gamma_{\underline{z}} \otimes \mathrm{id}_{\mathrm{C}_\lambda^*(\mathbb{Z}_+^d)} \colon \mathrm{C}^*(\pi, t) \otimes \mathrm{C}_\lambda^*(\mathbb{Z}_+^d) \to \mathrm{C}^*(\pi, t) \otimes \mathrm{C}_\lambda^*(\mathbb{Z}_+^d); f \otimes g \mapsto \gamma_{\underline{z}}(f) \otimes g,$$

for all $f \in \mathrm{C}^*(\pi, t)$ and $g \in \mathrm{C}_\lambda^*(\mathbb{Z}_+^d)$. Setting

$$\beta_{\underline{z}} := (\gamma_{\underline{z}} \otimes \mathrm{id}_{\mathrm{C}_\lambda^*(\mathbb{Z}_+^d)})|_{\mathrm{C}^*(\widehat{\pi},\widehat{t})} \text{ for all } \underline{z} \in \mathbb{T}^d,$$

it is routine to check that we obtain a gauge action β of $(\widehat{\pi}, \widehat{t})$.

We claim that $\mathcal{L}_F^{(\widehat{\pi},\widehat{t})} = \{0\}$ for all $F \subseteq [d]$. Injectivity of $(\widehat{\pi}, \widehat{t})$ accounts for $F = \emptyset$, so fix $\emptyset \neq F \subseteq [d]$ and $a \in \mathcal{L}_F^{(\widehat{\pi},\widehat{t})}$. Then we have that

$$\widehat{\pi}(a) - \sum_{\underline{0}\neq\underline{n}\leq\underline{1}_F} \widehat{\psi}_{\underline{n}}(k_{\underline{n}}) = (\pi(a) \otimes I) - \sum_{\underline{0}\neq\underline{n}\leq\underline{1}_F} (\psi_{\underline{n}}(k_{\underline{n}}) \otimes V_{\underline{n}} V_{\underline{n}}^*) = 0 \tag{A.2}$$

for some $k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}})$ for all $\underline{0} \neq \underline{n} \leq \underline{1}_F$. Note that we may view

$$\mathrm{C}^*(\pi, t) \otimes \mathrm{C}_\lambda^*(\mathbb{Z}_+^d) \subseteq \mathcal{B}(H) \otimes \mathcal{B}(\ell^2(\mathbb{Z}_+^d)) \subseteq \mathcal{B}(H \otimes \ell^2(\mathbb{Z}_+^d)),$$

using [8, Proposition 3.6.1] in the first inclusion and [8, Lemma 3.3.9, Proposition 3.3.11] in the second inclusion. Let $P_{\underline{0}} \in \mathcal{B}(\ell^2(\mathbb{Z}_+^d))$ be the projection onto the $\underline{0}$-th entry, i.e., $P_{\underline{0}}((\lambda_{\underline{n}})_{\underline{n}\in\mathbb{Z}_+^d}) = \lambda_{\underline{0}} e_{\underline{0}}$ for all $(\lambda_{\underline{n}})_{\underline{n}\in\mathbb{Z}_+^d} \in \ell^2(\mathbb{Z}_+^d)$. Consider the operator $I_H \otimes P_{\underline{0}} \in \mathcal{B}(H \otimes \ell^2(\mathbb{Z}_+^d))$, e.g., [8, Proposition 3.2.3]. Taking (A.2) and composing on the right by $I_H \otimes P_{\underline{0}}$, we obtain that

$$(\pi(a) \otimes P_{\underline{0}}) - \sum_{\underline{0}\neq\underline{n}\leq\underline{1}_F} (\psi_{\underline{n}}(k_{\underline{n}}) \otimes V_{\underline{n}} V_{\underline{n}}^* P_{\underline{0}}) = 0.$$

Fixing $\underline{0} \neq \underline{n} \leq \underline{1}_F$, notice that $V_{\underline{n}} V_{\underline{n}}^* P_{\underline{0}} = 0$ since $\underline{n} \not\leq \underline{0}$. It follows that $\pi(a) \otimes P_{\underline{0}}$=0 and thus $\pi(a) = 0$. Injectivity of (π, t) then yields that $a = 0$, as required.

In total, we have shown that $(\widehat{\pi}, \widehat{t})$ is an injective Nica-covariant representation of X

that admits a gauge action and satisfies $\mathcal{L}^{(\widehat{\pi},\widehat{t})} = \{\{0\}\}_{F\subseteq[d]}$. An application of Theorem 3.2.12 (with $\{\{0\}\}_{F\subseteq[d]}$ in place of $\mathcal{L}$ and $(\widehat{\pi},\widehat{t})$ in place of (π,t)) gives that

$$\widehat{\pi}\times\widehat{t}\colon \mathcal{NT}_X \to \mathrm{C}^*(\widehat{\pi},\widehat{t})$$

is a $*$-isomorphism and hence in particular a c.is. map by the comments preceding the statement. By restricting, we obtain that

$$\phi_1 := (\widehat{\pi}\times\widehat{t})|_{\mathcal{NT}_X^+}\colon \mathcal{NT}_X^+ \to \overline{\mathrm{alg}}(\widehat{\pi},\widehat{t})$$

is a c.is. algebra homomorphism. We also define

$$\phi_2 := (\pi\times t)|_{\mathcal{NT}_X^+}\colon \mathcal{NT}_X^+ \to \overline{\mathrm{alg}}(\pi,t),$$

which is a c.c. algebra homomorphism.

Next we repeat the preceding argument, this time working over $\mathbb{Z}^d$. Let $(e'_{\underline{n}})_{\underline{n}\in\mathbb{Z}^d}$ denote the usual orthonormal basis of $\ell^2(\mathbb{Z}^d)$. For each $\underline{n}\in\mathbb{Z}^d$, we write $U_{\underline{n}}$ for the shift operator in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$ determined by

$$U_{\underline{n}}e'_{\underline{m}} = e'_{\underline{n}+\underline{m}} \text{ for all } \underline{m}\in\mathbb{Z}^d.$$

Observe that

$$U_{\underline{n}}^*e'_{\underline{m}} = e'_{\underline{m}-\underline{n}} \text{ for all } \underline{m}\in\mathbb{Z}^d,$$

from which it follows that $U_{\underline{n}}$ is a unitary. Letting $\mathcal{U}(\ell^2(\mathbb{Z}^d))$ denote the group of unitary operators in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$, we obtain a group homomorphism U as follows:

$$U\colon \mathbb{Z}^d \to \mathcal{U}(\ell^2(\mathbb{Z}^d)); \underline{n}\mapsto U_{\underline{n}} \text{ for all } \underline{n}\in\mathbb{Z}^d.$$

We write $\mathrm{C}^*_\lambda(\mathbb{Z}^d)$ for the C*-subalgebra of $\mathcal{B}(\ell^2(\mathbb{Z}^d))$ generated by each $U_{\underline{n}}$. We define the maps

$$\widetilde{\pi}\colon A\to \mathrm{C}^*(\pi,t)\otimes \mathrm{C}^*_\lambda(\mathbb{Z}^d); a\mapsto \pi(a)\otimes I \text{ for all } a\in A,$$
$$\widetilde{t}_{\underline{n}}\colon X_{\underline{n}}\to \mathrm{C}^*(\pi,t)\otimes \mathrm{C}^*_\lambda(\mathbb{Z}^d); \xi_{\underline{n}}\mapsto t_{\underline{n}}(\xi_{\underline{n}})\otimes U_{\underline{n}} \text{ for all } \xi_{\underline{n}}\in X_{\underline{n}}, \underline{n}\in\mathbb{Z}_+^d\setminus\{\underline{0}\}.$$

Arguing as before, we deduce that $(\widetilde{\pi},\widetilde{t})$ is an injective Nica-covariant representation of X that admits a gauge action. Checking Nica-covariance is made simpler via the observation that

$$\widetilde{\psi}_{\underline{n}}(k_{\underline{n}}) = \psi_{\underline{n}}(k_{\underline{n}})\otimes I \text{ for all } k_{\underline{n}}\in\mathcal{K}(X_{\underline{n}}), \underline{n}\in\mathbb{Z}_+^d.$$

We claim that

$$\mathcal{L}_F^{(\widetilde{\pi},\widetilde{t})} = \mathcal{L}_F^{(\pi,t)} \text{ for all } F\subseteq[d].$$

Injectivity of $(\widetilde{\pi},\widetilde{t})$ and (π,t) accounts for $F=\emptyset$, so fix $\emptyset\neq F\subseteq[d]$ and $a\in A$. We have that

$$\begin{aligned} a \in \mathcal{L}_F^{(\widetilde{\pi},\widetilde{t})} &\iff \widetilde{\pi}(a) - \sum_{\underline{0}\neq\underline{n}\leq\underline{1}_F} \widetilde{\psi}_{\underline{n}}(k_{\underline{n}}) = 0 \text{ for some } k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}) \\ &\iff (\pi(a)\otimes I) - \sum_{\underline{0}\neq\underline{n}\leq\underline{1}_F} (\psi_{\underline{n}}(k_{\underline{n}})\otimes I) = 0 \text{ for some } k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}) \\ &\iff (\pi(a) - \sum_{\underline{0}\neq\underline{n}\leq\underline{1}_F} \psi_{\underline{n}}(k_{\underline{n}}))\otimes I = 0 \text{ for some } k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}) \\ &\iff \pi(a) - \sum_{\underline{0}\neq\underline{n}\leq\underline{1}_F} \psi_{\underline{n}}(k_{\underline{n}}) = 0 \text{ for some } k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}) \iff a \in \mathcal{L}_F^{(\pi,t)}, \end{aligned}$$

using that the spatial C*-norm is a cross norm in the fourth equivalence.

In total, we have shown that $(\widetilde{\pi},\widetilde{t})$ is an injective Nica-covariant representation of X that admits a gauge action and satisfies $\mathcal{L}^{(\widetilde{\pi},\widetilde{t})} = \mathcal{L}^{(\pi,t)}$. Recalling that $\mathcal{L}^{(\pi,t)}$ is an (M)-2^d-tuple by Proposition 3.1.18, we apply Theorem 3.2.12 (with $\mathcal{L}^{(\pi,t)}$ in place of $\mathcal{L}$ and $(\widetilde{\pi},\widetilde{t})$ in place of (π,t)) to deduce that $\mathcal{NO}(\mathcal{L}^{(\pi,t)},X) \cong \mathrm{C}^*(\widetilde{\pi},\widetilde{t})$ canonically. Another application of Theorem 3.2.12 (this time only replacing $\mathcal{L}$ by $\mathcal{L}^{(\pi,t)}$) gives that $\mathcal{NO}(\mathcal{L}^{(\pi,t)},X) \cong \mathrm{C}^*(\pi,t)$ canonically. Hence we have a canonical $*$-isomorphism $\mathrm{C}^*(\pi,t) \to \mathrm{C}^*(\widetilde{\pi},\widetilde{t})$ and by restricting we obtain a c.is. algebra homomorphism

$$\phi_3 \colon \overline{\mathrm{alg}}(\pi,t) \to \overline{\mathrm{alg}}(\widetilde{\pi},\widetilde{t}).$$

Next we illustrate the connection between $(\widehat{\pi},\widehat{t})$ and $(\widetilde{\pi},\widetilde{t})$. We identify $\ell^2(\mathbb{Z}_+^d)$ inside $\ell^2(\mathbb{Z}^d)$ in the obvious way. Let $P_{\ell^2(\mathbb{Z}_+^d)} \in \mathcal{B}(\ell^2(\mathbb{Z}^d))$ denote the projection onto $\ell^2(\mathbb{Z}_+^d)$. We define the compression map

$$\Phi \colon \mathcal{B}(\ell^2(\mathbb{Z}^d)) \to \mathcal{B}(\ell^2(\mathbb{Z}_+^d)); S \mapsto P_{\ell^2(\mathbb{Z}_+^d)} S|_{\ell^2(\mathbb{Z}_+^d)} \text{ for all } S \in \mathcal{B}(\ell^2(\mathbb{Z}^d)).$$

Notice that

$$\Phi(U_{\underline{n}}) = V_{\underline{n}} \text{ for all } \underline{n} \in \mathbb{Z}_+^d.$$

It is routine to check that Φ is a c.p. linear map satisfying $\|\Phi\| \leq 1$. In turn, we may apply [8, Theorem 3.5.3] to obtain a c.p. linear map

$$\mathrm{id}_{\mathcal{B}(H)} \otimes \Phi \colon \mathcal{B}(H)\otimes\mathcal{B}(\ell^2(\mathbb{Z}^d)) \to \mathcal{B}(H)\otimes\mathcal{B}(\ell^2(\mathbb{Z}_+^d)); S\otimes T \mapsto S\otimes\Phi(T),$$

for all $S \in \mathcal{B}(H)$ and $T \in \mathcal{B}(\ell^2(\mathbb{Z}^d))$. We also have that

$$\|(\mathrm{id}_{\mathcal{B}(H)}\otimes\Phi)^{(n)}\| \leq \|\mathrm{id}_{\mathcal{B}(H)}\|\cdot\|\Phi\| \leq 1 \text{ for all } n \in \mathbb{N},$$

where the first inequality is justified by [8, Theorem 3.5.3] and its proof. Hence $\mathrm{id}_{\mathcal{B}(H)}\otimes\Phi$ is a c.c. map. Recall that we may view

$$\mathrm{C}^*(\pi,t)\otimes\mathrm{C}^*_\lambda(\mathbb{Z}_+^d) \subseteq \mathcal{B}(H)\otimes\mathcal{B}(\ell^2(\mathbb{Z}_+^d)) \quad \text{and} \quad \mathrm{C}^*(\pi,t)\otimes\mathrm{C}^*_\lambda(\mathbb{Z}^d) \subseteq \mathcal{B}(H)\otimes\mathcal{B}(\ell^2(\mathbb{Z}^d)).$$

In turn, we can consider the restriction

$$\phi_4 := (\mathrm{id}_{\mathcal{B}(H)} \otimes \Phi)|_{\overline{\mathrm{alg}}(\widetilde{\pi},\widetilde{t})} \colon \overline{\mathrm{alg}}(\widetilde{\pi},\widetilde{t}) \to \mathcal{B}(H) \otimes \mathcal{B}(\ell^2(\mathbb{Z}_+^d)),$$

which is also a c.c. linear map. We claim that ϕ_4 is in fact an algebra homomorphism. Since ϕ_4 is in particular linear and continuous, it suffices to show that the homomorphism condition holds on the generators of $\overline{\mathrm{alg}}(\widetilde{\pi},\widetilde{t})$. Accordingly, fix $\underline{n}, \underline{m} \in \mathbb{Z}_+^d, \xi_{\underline{n}} \in X_{\underline{n}}$ and $\xi_{\underline{m}} \in X_{\underline{m}}$. First note that

$$\phi_4(\widetilde{t}_{\underline{n}}(\xi_{\underline{n}})) = \phi_4(t_{\underline{n}}(\xi_{\underline{n}}) \otimes U_{\underline{n}}) = t_{\underline{n}}(\xi_{\underline{n}}) \otimes \Phi(U_{\underline{n}}) = t_{\underline{n}}(\xi_{\underline{n}}) \otimes V_{\underline{n}} = \widehat{t}_{\underline{n}}(\xi_{\underline{n}}).$$

We then obtain that

$$\phi_4(\widetilde{t}_{\underline{n}}(\xi_{\underline{n}})\widetilde{t}_{\underline{m}}(\xi_{\underline{m}})) = \phi_4(\widetilde{t}_{\underline{n}+\underline{m}}(\xi_{\underline{n}}\xi_{\underline{m}})) = \widehat{t}_{\underline{n}+\underline{m}}(\xi_{\underline{n}}\xi_{\underline{m}}) = \phi_4(\widetilde{t}_{\underline{n}}(\xi_{\underline{n}}))\phi_4(\widetilde{t}_{\underline{m}}(\xi_{\underline{m}})),$$

as required. It follows that $\phi_4(\overline{\mathrm{alg}}(\widetilde{\pi},\widetilde{t})) \subseteq \overline{\mathrm{alg}}(\widehat{\pi},\widehat{t})$.

In total, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{NT}_X^+ & \xrightarrow{\ \phi_1\ } & \overline{\mathrm{alg}}(\widehat{\pi},\widehat{t}) \\ {\scriptstyle \phi_2}\downarrow & & \uparrow{\scriptstyle \phi_4} \\ \overline{\mathrm{alg}}(\pi,t) & \xrightarrow[\ \phi_3\ ]{} & \overline{\mathrm{alg}}(\widetilde{\pi},\widetilde{t}) \end{array}$$

of algebra homomorphisms. The horizontal arrows are c.is., the vertical arrows are c.c., and commutativity follows from canonicity of the maps involved. It follows that

$$\phi_1^{(n)} = \phi_4^{(n)} \circ \phi_3^{(n)} \circ \phi_2^{(n)} \text{ for all } n \in \mathbb{N}.$$

Fixing $n \in \mathbb{N}$ and $x \in M_n(\mathcal{NT}_X^+)$, we deduce that

$$\|x\| \geq \|\phi_2^{(n)}(x)\| \geq \|(\phi_4^{(n)} \circ \phi_3^{(n)} \circ \phi_2^{(n)})(x)\| = \|\phi_1^{(n)}(x)\| = \|x\|,$$

using that ϕ_2 is c.c. in the first inequality, that ϕ_4 and ϕ_3 are c.c. in the second inequality and that ϕ_1 is c.is. in the final equality. Thus $\phi_2 \equiv (\pi \times t)|_{\mathcal{NT}_X^+}$ is c.is., completing the proof. □

Theorem A.2.2. *Let X be a strong compactly aligned product system with coefficients in a C*-algebra A. Then the following hold:*

(i) $\mathcal{NO}_X \cong \mathrm{C}^*_{\mathrm{env}}(\mathcal{NT}_X^+)$ canonically.

(ii) For each injective Nica-covariant representation (π, t) of X that admits a gauge action, there exists a (unique) canonical $*$-epimorphism $\mathrm{C}^*(\pi,t) \to \mathcal{NO}_X$.

Proof. Let (π, t) be an injective Nica-covariant representation of X that admits a gauge action. An application of Proposition A.2.1 gives that

$$(\pi \times t)|_{\mathcal{NT}_X^+} : \mathcal{NT}_X^+ \to \mathrm{C}^*(\pi, t)$$

is a c.is. algebra homomorphism. Additionally, we have that

$$\mathrm{C}^*((\pi \times t)(\mathcal{NT}_X^+)) = \mathrm{C}^*(\overline{\mathrm{alg}}(\pi, t)) = \mathrm{C}^*(\pi, t).$$

In other words, the pair $((\pi \times t)|_{\mathcal{NT}_X^+}, \mathrm{C}^*(\pi, t))$ is a C*-cover of $\mathcal{NT}_X^+$. Let $(\iota, \mathrm{C}^*_{\mathrm{env}}(\mathcal{NT}_X^+))$ denote the co-universal C*-cover of $\mathcal{NT}_X^+$. An application of the co-universal property of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{NT}_X^+)$ yields a (unique) $*$-epimorphism

$$\Phi^{(\pi,t)} : \mathrm{C}^*(\pi, t) \to \mathrm{C}^*_{\mathrm{env}}(\mathcal{NT}_X^+)$$

such that $\Phi^{(\pi,t)} \circ (\pi \times t)|_{\mathcal{NT}_X^+} = \iota$. Hence item (ii) follows as a consequence of item (i).

To see that item (i) holds, we set

$$\Phi := \Phi^{(\pi_X^{\mathcal{I}}, t_X^{\mathcal{I}})} : \mathcal{NO}_X \to \mathrm{C}^*_{\mathrm{env}}(\mathcal{NT}_X^+).$$

It suffices to show that Φ is injective. To this end, we set

$$\widetilde{\pi} := \Phi|_{\pi_X^{\mathcal{I}}(A)} \quad \text{and} \quad \widetilde{t}_{\underline{n}} := \Phi|_{t^{\mathcal{I}}_{X,\underline{n}}(X_{\underline{n}})} \text{ for all } \underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}.$$

Recall that $X \cong t^{\mathcal{I}}(X)$ by the comments preceding Proposition A.2.1, and so $t^{\mathcal{I}}(X)$ is strong compactly aligned by Proposition 2.5.7. It is routine to check that $(\widetilde{\pi}, \widetilde{t})$ constitutes an injective representation of $t^{\mathcal{I}}(X)$. We also obtain that $(\widetilde{\pi}, \widetilde{t})$ is Nica-covariant by combining Remarks 2.3.2 and 2.3.5. Notice that

$$\begin{aligned}\mathrm{C}^*(\widetilde{\pi}, \widetilde{t}) &= \mathrm{C}^*(\Phi(\pi_X^{\mathcal{I}}(A)), \Phi(t^{\mathcal{I}}_{X,\underline{n}}(X_{\underline{n}})) \mid \underline{n} \in \mathbb{Z}_+^d) = \mathrm{C}^*(\iota(\overline{\pi}_X(A)), \iota(\overline{t}_{X,\underline{n}}(X_{\underline{n}})) \mid \underline{n} \in \mathbb{Z}_+^d) \\ &= \mathrm{C}^*(\iota(\mathcal{NT}_X^+)) = \mathrm{C}^*_{\mathrm{env}}(\mathcal{NT}_X^+).\end{aligned}$$

We claim that $(\widetilde{\pi}, \widetilde{t})$ admits a gauge action. To see this, let γ denote the gauge action of $(\overline{\pi}_X, \overline{t}_X)$. Since each $\gamma_{\underline{z}}$ is a $*$-automorphism, the restriction $\gamma_{\underline{z}}|_{\mathcal{NT}_X^+} : \mathcal{NT}_X^+ \to \mathcal{NT}_X^+$ is a c.is. algebra epimorphism. We set $\widetilde{\beta}_{\underline{z}} := \iota \circ \gamma_{\underline{z}}|_{\mathcal{NT}_X^+}$ for each $\underline{z} \in \mathbb{T}^d$. Observe that each $\widetilde{\beta}_{\underline{z}}$ is a c.is. algebra homomorphism and

$$\mathrm{C}^*(\widetilde{\beta}_{\underline{z}}(\mathcal{NT}_X^+)) = \mathrm{C}^*(\iota(\gamma_{\underline{z}}(\mathcal{NT}_X^+))) = \mathrm{C}^*(\iota(\mathcal{NT}_X^+)) = \mathrm{C}^*_{\mathrm{env}}(\mathcal{NT}_X^+).$$

In other words, the pair $(\widetilde{\beta}_{\underline{z}}, \mathrm{C}^*_{\mathrm{env}}(\mathcal{NT}_X^+))$ is a C*-cover of $\mathcal{NT}_X^+$ for all $\underline{z} \in \mathbb{T}^d$. Applying the co-universal property of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{NT}_X^+)$, for each $\underline{z} \in \mathbb{T}^d$ we obtain a (unique) $*$-epimorphism

$$\beta_{\underline{z}} : \mathrm{C}^*_{\mathrm{env}}(\mathcal{NT}_X^+) \to \mathrm{C}^*_{\mathrm{env}}(\mathcal{NT}_X^+)$$

such that $\beta_{\underline{z}} \circ \widetilde{\beta}_{\underline{z}} = \iota$. Observe that

$$\begin{aligned}\beta_{\underline{z}}(\widetilde{\pi}(\pi_X^{\mathcal{I}}(a))) &= \beta_{\underline{z}}(\Phi(\pi_X^{\mathcal{I}}(a))) = \beta_{\underline{z}}(\iota(\overline{\pi}_X(a))) = \beta_{\underline{z}}(\iota(\gamma_{\overline{\underline{z}}}(\overline{\pi}_X(a))))\\ &= \beta_{\underline{z}}(\widetilde{\beta}_{\underline{z}}(\overline{\pi}_X(a))) = \iota(\overline{\pi}_X(a)) = \Phi(\pi_X^{\mathcal{I}}(a)) = \widetilde{\pi}(\pi_X^{\mathcal{I}}(a))\end{aligned}$$

for all $a \in A$. We also have that

$$\begin{aligned}\beta_{\underline{z}}(\widetilde{t}_{\underline{n}}(t_{X,\underline{n}}^{\mathcal{I}}(\xi_{\underline{n}}))) &= \beta_{\underline{z}}(\Phi(t_{X,\underline{n}}^{\mathcal{I}}(\xi_{\underline{n}}))) = \beta_{\underline{z}}(\iota(\overline{t}_{X,\underline{n}}(\xi_{\underline{n}}))) = \beta_{\underline{z}}(\iota(\gamma_{\overline{\underline{z}}}(\underline{z}^{\underline{n}}\overline{t}_{X,\underline{n}}(\xi_{\underline{n}}))))\\ &= \beta_{\underline{z}}(\widetilde{\beta}_{\underline{z}}(\underline{z}^{\underline{n}}\overline{t}_{X,\underline{n}}(\xi_{\underline{n}}))) = \iota(\underline{z}^{\underline{n}}\overline{t}_{X,\underline{n}}(\xi_{\underline{n}})) = \Phi(\underline{z}^{\underline{n}}t_{X,\underline{n}}^{\mathcal{I}}(\xi_{\underline{n}})) = \underline{z}^{\underline{n}}\widetilde{t}_{\underline{n}}(t_{X,\underline{n}}^{\mathcal{I}}(\xi_{\underline{n}}))\end{aligned}$$

for all $\xi_{\underline{n}} \in X_{\underline{n}}$ and $\underline{n} \in \mathbb{Z}_+^d \setminus \{\underline{0}\}$. It follows that we obtain a gauge action β of $(\widetilde{\pi}, \widetilde{t})$.

Next we claim that $(\widetilde{\pi}, \widetilde{t})$ is a CNP-representation of $t^{\mathcal{I}}(X)$. To see this, for each $\underline{n} \in \mathbb{Z}_+^d$ let $W_{\underline{n}} \colon X_{\underline{n}} \to t_{X,\underline{n}}^{\mathcal{I}}(X_{\underline{n}})$ be the map determined by $\xi_{\underline{n}} \mapsto t_{X,\underline{n}}^{\mathcal{I}}(\xi_{\underline{n}})$ for all $\xi_{\underline{n}} \in X_{\underline{n}}$. It is straightforward to check that $\{W_{\underline{n}}\}_{\underline{n}\in\mathbb{Z}_+^d}$ implements a unitary equivalence between X and $t^{\mathcal{I}}(X)$. Recall that $(\widehat{\pi}, \widehat{t}) := \{(\widetilde{\pi} \circ W_{\underline{0}}, \widetilde{t}_{\underline{n}} \circ W_{\underline{n}})\}_{\underline{n}\in\mathbb{Z}_+^d}$ is a Nica-covariant representation of X by Proposition 2.4.3. By Proposition 2.5.20, it suffices to show that $(\widehat{\pi}, \widehat{t})$ is a CNP-representation. To this end, fix $\emptyset \neq F \subseteq [d]$ and $a \in \mathcal{I}_F(X)$. We must show that

$$\widehat{\pi}(a) + \sum\{(-1)^{|\underline{n}|}\widehat{\psi}_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} = 0.$$

To this end, first fix $\underline{n} \in \mathbb{Z}_+^d$ and $\xi_{\underline{n}}, \eta_{\underline{n}} \in X_{\underline{n}}$. On the one hand, we have that

$$\widetilde{\psi}_{\underline{n}}(\Theta^{t_{X,\underline{n}}^{\mathcal{I}}(X_{\underline{n}})}_{t_{X,\underline{n}}^{\mathcal{I}}(\xi_{\underline{n}}), t_{X,\underline{n}}^{\mathcal{I}}(\eta_{\underline{n}})}) = \widetilde{t}_{\underline{n}}(t_{X,\underline{n}}^{\mathcal{I}}(\xi_{\underline{n}}))\widetilde{t}_{\underline{n}}(t_{X,\underline{n}}^{\mathcal{I}}(\eta_{\underline{n}}))^* = \Phi(\psi_{X,\underline{n}}^{\mathcal{I}}(\Theta^{X_{\underline{n}}}_{\xi_{\underline{n}},\eta_{\underline{n}}})).$$

On the other hand, we have that

$$\widetilde{\psi}_{\underline{n}}(\Theta^{t_{X,\underline{n}}^{\mathcal{I}}(X_{\underline{n}})}_{t_{X,\underline{n}}^{\mathcal{I}}(\xi_{\underline{n}}), t_{X,\underline{n}}^{\mathcal{I}}(\eta_{\underline{n}})}) = \widetilde{\psi}_{\underline{n}}(\Theta^{t_{X,\underline{n}}^{\mathcal{I}}(X_{\underline{n}})}_{W_{\underline{n}}(\xi_{\underline{n}}), W_{\underline{n}}(\eta_{\underline{n}})}) = \widetilde{\psi}_{\underline{n}}(W_{\underline{n}}\Theta^{X_{\underline{n}}}_{\xi_{\underline{n}},\eta_{\underline{n}}}W_{\underline{n}}^{-1}).$$

It follows that

$$\Phi(\psi_{X,\underline{n}}^{\mathcal{I}}(k_{\underline{n}})) = \widetilde{\psi}_{\underline{n}}(W_{\underline{n}}k_{\underline{n}}W_{\underline{n}}^{-1}) \text{ for all } k_{\underline{n}} \in \mathcal{K}(X_{\underline{n}}).$$

Applying for $\underline{0} \neq \underline{n} \leq \underline{1}_F$ and $k_{\underline{n}} = \phi_{\underline{n}}(a)$, we obtain that

$$\widehat{\psi}_{\underline{n}}(\phi_{\underline{n}}(a)) = \widetilde{\psi}_{\underline{n}}(W_{\underline{n}}\phi_{\underline{n}}(a)W_{\underline{n}}^{-1}) = \Phi(\psi_{X,\underline{n}}^{\mathcal{I}}(\phi_{\underline{n}}(a))),$$

using Remark 2.3.5 in the first equality. Thus we have that

$$\begin{aligned}\widehat{\pi}(a) + \sum\{(-1)^{|\underline{n}|}\widehat{\psi}_{\underline{n}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\} &=\\ &= \Phi(\pi_X^{\mathcal{I}}(a)) + \sum\{(-1)^{|\underline{n}|}\Phi(\psi_{X,\underline{n}}^{\mathcal{I}}(\phi_{\underline{n}}(a))) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}\\ &= \Phi(\pi_X^{\mathcal{I}}(a) + \sum\{(-1)^{|\underline{n}|}\psi_{X,\underline{n}}^{\mathcal{I}}(\phi_{\underline{n}}(a)) \mid \underline{0} \neq \underline{n} \leq \underline{1}_F\}) = 0,\end{aligned}$$

as required, using that $(\pi_X^{\mathcal{I}}, t_X^{\mathcal{I}})$ is a CNP-representation in the final equality.

In total, we have shown that $(\widetilde{\pi}, \widetilde{t})$ is an injective CNP-representation of $t^{\mathcal{I}}(X)$ that admits a gauge action. An application of [17, Theorem 4.2] gives that $\mathcal{NO}_{t^{\mathcal{I}}(X)} \cong \mathrm{C}^*(\widetilde{\pi}, \widetilde{t}) = \mathrm{C}^*_{\mathrm{env}}(\mathcal{NT}_X^+)$ canonically. Since $\mathcal{NO}_X \cong \mathcal{NO}_{t^{\mathcal{I}}(X)}$ canonically via Proposition 2.5.20, we deduce that $\mathcal{NO}_X \cong \mathrm{C}^*_{\mathrm{env}}(\mathcal{NT}_X^+)$ canonically. Canonicity ensures that this $*$-isomorphism is nothing but Φ, finishing the proof. $\square$

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