Function theory of the pentablock

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Abstract

The *pentablock* is the set in $\mathbb{C}^3$

$$\mathcal{P} = \{(a_{21}, \text{tr } A, \det A) : A = [a_{ij}]_{i,j=1}^2 \in \mathbb{B}^{2\times 2}\},$$

where $\mathbb{B}^{2\times 2}$ denotes the open unit ball in the space of 2×2 complex matrices. The closure of $\mathcal{P}$ is denoted by $\overline{\mathcal{P}}$. The sets $\mathcal{P}$ and $\overline{\mathcal{P}}$ are polynomially convex and starlike about (0,0,0), but not convex. In this thesis we identify the singular set of $\mathcal{P}$ which is $\mathcal{S}_{\mathcal{P}} = \{(0, s, p) \in \mathcal{P} : s^2 = 4p\}$ and show that $\mathcal{S}_{\mathcal{P}}$ is invariant under the automorphism group Aut $\mathcal{P}$ of $\mathcal{P}$ and is a complex geodesic in $\mathcal{P}$. We provide a description of rational maps from the unit disc $\mathbb{D}$ to $\overline{\mathcal{P}}$ that map the unit circle $\mathbb{T}$ to the distinguished boundary $b\overline{\mathcal{P}}$ of $\overline{\mathcal{P}}$, where $b\overline{\mathcal{P}} = \left\{(a, s, p) \in \mathbb{C}^3 : |s| \leq 2,\ |p| = 1,\ s = \bar{s}p \textit{ and } |a| = \sqrt{1 - \frac{1}{4}|s|^2}\right\}$. These functions are called *rational* $\overline{\mathcal{P}}$*-inner functions*. We establish relations between $\overline{\mathcal{P}}$-inner functions and Γ-inner functions from $\mathbb{D}$ to the symmetrized bidisc Γ. We give a method of constructing rational $\overline{\mathcal{P}}$-inner functions starting from a rational Γ-inner function. We describe an algorithm to construct rational $\overline{\mathcal{P}}$-inner functions $x = (a, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ of prescribed degree from the zeros of a, s and $s^2 - 4p$. We use a result of Agler and Young to construct an interpolating rational $\overline{\mathcal{P}}$-inner function $x : \mathbb{D} \to \overline{\mathcal{P}}$ such that $x(0) = (0, 0, 0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$ for suitable points $\lambda_0 \in \mathbb{D}$ and $(a_0, s_0, p_0) \in \overline{\mathcal{P}}$. We prove a Schwarz lemma for the pentablock.

Declaration on collaborative work

My dissertation is the result of a collaboration with my supervisors Dr Z. A. Lykova and Prof. N. J. Young. Lykova and I have published a joint paper [18]. Lykova provided me with the primary issues and solutions for these issues. Lykova and I have met weekly to discuss my thesis-related mathematics, methods, new ideas, and research papers. We have conducted research jointly.

The remainder of each week was spent independently working on my thesis. I performed the calculations necessary for each step of the proofs, searched for research materials pertinent to our research project, and organised all of the research materials in the thesis. I have presented multiple talks on the subject matter of my dissertation at workshops in Newcastle, Bristol, Birmingham and to the 34^{th} International Workshop on Operator Theory and its Applications which was held at the University of Helsinki, Finland (IWOTA 2023).

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Contents

Chapter 1. Introduction and historical remarks

1.1 Introduction

The *pentablock* $\mathcal{P}$ is the domain defined by

$$\mathcal{P} = \{(a_{21}, \text{tr } A, \det A) : A = [a_{ij}]_{i,j=1}^{2} \in \mathbb{B}^{2\times 2}\} \tag{1.1}$$

where $\mathbb{B}^{2\times 2}$ denotes the open unit ball $\{A \in \mathbb{C}^{2\times 2} : \|A\| < 1\}$ in the space of 2×2 complex matrices. The polynomial map implicit in the above definition can be written as

$$\pi(A) = (a_{21}, \text{tr } A, \det A), \text{ where } A = [a_{ij}]_{i,j=1}^{2} \in \mathbb{C}^{2\times 2}. \tag{1.2}$$

Thus $\mathcal{P} = \pi(\mathbb{B}^{2\times 2})$. In this thesis we consider the normed space of complex matrices with the operator norm arising from the standard inner product on $\mathbb{C}^2$. The pentablock was introduced in [4] in 2015. It is a bounded nonconvex domain in $\mathbb{C}^3$ which arises naturally in connection with a certain problem of μ-synthesis. The group of automorphisms of the pentablock was studied in [4] and [38]. The pentablock $\mathcal{P}$ is a region in 3-dimensional complex space which intersects $\mathbb{R}^3$ in a convex body bounded by five faces, comprising two triangles, an elliptic region and two curved surfaces [4]. $\mathcal{P}$ is a $\mathbb{C}$-convex domain [45].

1.2 Main results

The closure of $\mathcal{P}$ is denoted by $\overline{\mathcal{P}}$. We analyse $\overline{\mathcal{P}}$-*inner functions* which are analytic maps from the unit disc $\mathbb{D}$ to $\overline{\mathcal{P}}$ whose radial limits almost everywhere on the unit circle $\mathbb{T}$ lie in the distinguished boundary $b\overline{\mathcal{P}}$ of $\mathcal{P}$. The distinguished boundary $b\overline{\mathcal{P}}$ of $\mathcal{P}$ is

$$b\overline{\mathcal{P}} = \left\{(a, s, p) \in \mathbb{C}^3 : |s| \le 2,\ |p| = 1,\ s = \bar{s}p \textit{ and } |a| = \sqrt{1 - \frac{1}{4}|s|^2}\right\},$$

see [4]. The degree of a rational $\overline{\mathcal{P}}$-inner function $x = (a, s, p)$ is defined to be the pair of numbers $(\deg a, \deg p)$, where $\deg a$ and $\deg p$ are degrees of rational functions a and p correspondently. We say that $\deg x \le (m, n)$ if $\deg a \le m$ and $\deg p \le n$.

1.2.1 On rational $\mathcal{P}$-inner functions

We develop a concrete structure theory for the rational $\overline{\mathcal{P}}$-inner functions. We give relations between $\overline{\mathcal{P}}$-inner functions and inner functions from $\mathbb{D}$ to the symmetrized bidisc Γ. We should mention papers on the construction of rational inner functions from $\mathbb{D}$ to Γ [6, 5]. One of main results on the description of $\overline{\mathcal{P}}$-inner functions is the following theorem.

Theorem 4.5.2. Let $x = (a, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ be a rational $\overline{\mathcal{P}}$-inner function of degree $(m + n, n)$. Let $a \neq 0$ and let an inner-outer factorization of a be given by $a = a_{in}a_{out}$, where a_{in} is an inner function of degree m and a_{out} is an outer function. Then there exist polynomials A, E, D such that

(1) $\deg(A), \deg(E), \deg(D) \leq n$,

(2) $E^{\sim n} = E, \quad \left(E^{\sim n}(\lambda) \stackrel{\text{def}}{=} \lambda^n \overline{E(1/\overline{\lambda})}\right)$

(3) $D(\lambda) \neq 0$ for all $\lambda \in \overline{\mathbb{D}}$,

(4) $|E(\lambda)| \leq 2|D(\lambda)|$ for all $\lambda \in \overline{\mathbb{D}}$,

(5) A is an outer polynomial such that $|A(\lambda)|^2 = |D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2$ for $\lambda \in \mathbb{T}$,

(6) $a = a_{in}\dfrac{A}{D}$ on $\overline{\mathbb{D}}$,

(7) $s = \dfrac{E}{D}$ on $\overline{\mathbb{D}}$,

(8) $p = \dfrac{D^{\sim n}}{D}$ on $\overline{\mathbb{D}}$. $\quad \left(D^{\sim n}(\lambda) \stackrel{\text{def}}{=} \lambda^n \overline{D(1/\overline{\lambda})}\right)$

Theorem 4.5.6. (Converse to Theorem 4.5.2) Suppose polynomials A, E, D satisfy

(1) $\deg(A), \deg(E), \deg(D) \leq n$,

(2) $E^{\sim n} = E$,

(3) $D(\lambda) \neq 0$ for all $\lambda \in \overline{\mathbb{D}}$,

(4) $|E(\lambda)| \leq 2|D(\lambda)|$ for all $\lambda \in \overline{\mathbb{D}}$,

(5) A is an outer polynomial such that $|A(\lambda)|^2 = |D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2$ for $\lambda \in \mathbb{T}$,

(6) a_{in} is a rational inner function on $\mathbb{D}$ of degree $\leq m$.

Let a, s, p be defined by

$$a = a_{in}\frac{A}{D}, \quad s = \frac{E}{D} \quad \text{and} \quad p = \frac{D^{\sim n}}{D} \quad \text{on } \overline{\mathbb{D}}.$$

Then $x = (a, s, p)$ is a rational $\overline{\mathcal{P}}$-inner function of degree less than or equal $(m + n, n)$.

The pentablock $\mathcal{P}$ is closely related to the symmetrized bidisc

$$\mathbb{G} \stackrel{\text{def}}{=} \{(z+w, zw) : |z| < 1,\ |w| < 1\}.$$

Note that $\mathcal{P}$ is fibred over $\mathbb{G}$ by the map

$$(a, s, p) \mapsto (s, p).$$

We denote by Γ the closure of $\mathbb{G}$. Recall that a Γ-inner function h is an analytic map from $\mathbb{D}$ to Γ whose radial limit

$$\lim_{r \to 1^-} h(r\lambda)$$

almost everywhere on $\mathbb{T}$ lies in the distinguished boundary $b\Gamma$ of Γ [6]. It was shown in [6] that any rational Γ-inner function h of degree n can be presented as $h = \left(\dfrac{E}{D}, \dfrac{D^{\sim n}}{D}\right)$, where E, D are polynomials of degree $\leq n$ and the following conditions are satisfied: $E^{\sim n} = E$, $|E(\lambda)| \leq 2|D(\lambda)|$ on $\overline{\mathbb{D}}$ and $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$. In [6] the *royal polynomial* R_h of h was introduced. It is defined by

$$R_h(\lambda) = 4D(\lambda)D^{\sim n}(\lambda) - E(\lambda)^2.$$

We call the points $\sigma \in \overline{\mathbb{D}}$ such that $R_h(\sigma) = 0$ the *royal nodes* of h.

Proposition 4.5.7. Let $x = (0, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ be a rational $\overline{\mathcal{P}}$-inner function. Then $x = (0, 2\varphi, \varphi^2)$, for some rational inner function $\varphi : \mathbb{D} \to \overline{\mathbb{D}}$. Moreover, $x(\lambda) \in \mathcal{R}_{\overline{\mathcal{P}}} \cap \overline{\mathcal{P}}$ for all $\lambda \in \mathbb{D}$ and $x(\lambda) \in b\overline{\mathcal{P}} \cap \mathcal{R}_{\overline{\mathcal{P}}}$ for almost all $\lambda \in \mathbb{T}$.

Definition 1.2.1. *The royal variety $\mathcal{R}_\Gamma$ of the symmetrized bidisc is*

$$\mathcal{R}_\Gamma = \{(s, p) \in \mathbb{C}^2 : s^2 = 4p\}.$$

Every rational Γ-inner function $h = \left(\dfrac{E}{D}, \dfrac{D^{\sim n}}{D}\right)$ such that $h(\overline{\mathbb{D}}) \not\subseteq \mathcal{R}_\Gamma \cap \Gamma$ allows us to construct a family of rational $\overline{\mathcal{P}}$-inner functions.

Theorem 4.5.5. Let $h = \left(\dfrac{E}{D}, \dfrac{D^{\sim n}}{D}\right)$ be a rational Γ-inner function, where E, D are polynomials such that $\deg(E)$, $\deg(D) \leq n$, $E^{\sim n} = E$, $|E(\lambda)| \leq 2|D(\lambda)|$ on $\overline{\mathbb{D}}$ and $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$. Let A be an outer polynomial such that

$$|A(\lambda)|^2 = |D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2.$$

Then, for every finite Blaschke product B and $|c| = 1$, $x = \left(cB\dfrac{A}{D}, \dfrac{E}{D}, \dfrac{D^{\sim n}}{D}\right)$ is a rational $\overline{\mathcal{P}}$-inner function.

The next theorem provides an algorithm for the construction of rational $\overline{\mathcal{P}}$-inner functions from the zeros of a, s and the royal nodes of (s, p).

Theorem 4.6.5. Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_{k_0} \in \mathbb{D}$ and $\eta_1, \eta_2, \ldots, \eta_{k_1} \in \mathbb{T}$, where $2k_0 + k_1 = n$ and suppose that $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{D}$. Suppose that $\sigma_1, \ldots, \sigma_n$ in $\overline{\mathbb{D}}$ are distinct from $\eta_1, \ldots, \eta_{k_1}$. Then there exists a rational $\overline{\mathcal{P}}$-inner function $x = (a, s, p)$ of degree $\leq (m+n, n)$ such that

(1) the zeros of a in $\mathbb{D}$, repeated according to multiplicity, are $\beta_1, \beta_2, \ldots, \beta_m$,

(2) the zeros of s in $\overline{\mathbb{D}}$, repeated according to multiplicity, are $\alpha_1, \alpha_2, \ldots, \alpha_{k_0}$ and $\eta_1, \eta_2, \ldots, \eta_{k_1}$,

(3) the royal nodes of (s, p) are $\sigma_1, \ldots, \sigma_n$.

Such a function x can be constructed as follows. Let $t_+ > 0$ and let $t \in \mathbb{R} \setminus \{0\}$. Let R and E be defined by

$$R(\lambda) = t_+ \prod_{j=1}^{n} (\lambda - \sigma_j)(1 - \overline{\sigma_j}\lambda),$$

$$E(\lambda) = t \prod_{j=1}^{k_0} (\lambda - \alpha_j)(1 - \overline{\alpha_j}\lambda) \prod_{j=1}^{k_1} i\mathrm{e}^{-i\theta_j/2}(\lambda - \eta_j)$$

where $\eta_j = \mathrm{e}^{i\theta_j}$, $0 \leq \theta_j < 2\pi$. Let $a_{in} : \mathbb{D} \to \overline{\mathbb{D}}$ be defined by

$$a_{in}(\lambda) = c \prod_{i=1}^{m} B_{\beta_i}(\lambda),$$

where $|c| = 1$, $\beta_i \in \mathbb{D}$ and $B_{\beta_i}(z) = \dfrac{z - \beta_i}{1 - \overline{\beta}_i z}$ for $z \in \mathbb{D}$.

(i) There exist outer polynomials D and A of degree at most n such that

$$\lambda^{-n} R(\lambda) + |E(\lambda)|^2 = 4|D(\lambda)|^2$$

and

$$\lambda^{-n} R(\lambda) = 4|A(\lambda)|^2$$

for all $\lambda \in \mathbb{T}$.

(ii) The function x defined by

$$x = (a, s, p) = \left(a_{in} \frac{A}{D}, \frac{E}{D}, \frac{D^{\sim n}}{D} \right)$$

is a rational $\overline{\mathcal{P}}$-inner function such that $\deg(x) \leq (m+n, n)$ and conditions (1), (2) and (3) hold. The royal polynomial of (s, p) is R.

1.2.2 On a Schwarz lemma for the pentablock $\mathcal{P}$

In Chapter 5 we prove a Schwarz lemma for $\mathcal{P}$. There is a well developed theory of Schwarz lemmas for various domains by many authors. Agler, Lykova and Young proved the following result for the pentablock $\mathcal{P}$.

Proposition 5.3.1. [4, Proposition 11.1] Let $\lambda_0 \in \mathbb{D} \setminus \{0\}, (a_0, s_0, p_0) \in \overline{\mathcal{P}}$. If $x \in \mathrm{Hol}(\mathbb{D}, \mathcal{P})$ satisfies $x(0) = (0, 0, 0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$ then

$$|s_0| < 2, \qquad \frac{2|s_0 - \overline{s}_0 p_0| + |s_0^2 - 4p_0|}{4 - |s_0|^2} \leq |\lambda_0|$$

and

$$|a_0| \Big/ \left| 1 - \frac{\frac{1}{2} s_0 \overline{\beta}}{1 + \sqrt{1 - |\beta|^2}} \right| \leq |\lambda_0|$$

where $\beta = (s_0 - \overline{s}_0 p_0)/(1 - |p_0|^2)$ when $|p_0| < 1$ and $\beta = \frac{1}{2} s_0$ when $|p_0| = 1$.

We have proved Theorem 5.1.6 and Theorem 5.3.4.

Theorem 5.1.6. Let $\lambda_0 \in \mathbb{D} \setminus \{0\}$, and $(a_0, s_0, p_0) \in \overline{\mathcal{P}}$, where $s_0 = \lambda_1 + \lambda_2,\ p_0 = \lambda_1 \lambda_2$, for some $\lambda_1, \lambda_2 \in \mathbb{D}$. Then the following are equivalent:

(i) $|\lambda_1| \leq |\lambda_0|,\ |\lambda_2| \leq |\lambda_0|$, and

$$|a_0| \leq |\lambda_0| \left(1 - \left|\frac{\lambda_1}{\lambda_0}\right|^2\right)^{\frac{1}{2}} \left(1 - \left|\frac{\lambda_2}{\lambda_0}\right|^2\right)^{\frac{1}{2}};$$

(ii) there exists an analytic map $F : \mathbb{D} \to \overline{\mathbb{B}^{2\times 2}}$ such that

$$F(0) = 0 \text{ and } F(\lambda_0) = \begin{bmatrix} \lambda_1 & 0 \\ a_0 & \lambda_2 \end{bmatrix}.$$

Furthermore, $x(\lambda) = \pi \circ F(\lambda)$, for $\lambda \in \mathbb{D}$, is an analytic map from $\mathbb{D}$ to $\overline{\mathcal{P}}$ such that $x(0) = (0, 0, 0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$.

Recall that the map π is defined by equation (1.2).

To prove a Schwarz lemma for the pentablock we need Theorem 1.1 and Theorem 1.4 from [11] on a Schwarz lemma for the symmetrized bidisc. Recall [11, Theorem 1.1].

Theorem 1.2.2. [11, Theorem 1.1] *Let $\lambda_0 \in \mathbb{D}$ and $(s_0, p_0) \in \Gamma$. The following conditions are equivalent:*

(1) *there exists an analytic function $\varphi : \mathbb{D} \to \Gamma$ such that $\varphi(0) = (0, 0)$ and $\varphi(\lambda_0) = (s_0, p_0)$;*

(2) $|s_0| < 2$ *and*
$$\frac{2|s_0 - p_0\overline{s_0}| + |s_0^2 - 4p_0|}{4 - |s_0|^2} \leq |\lambda_0|;$$

(3) $\big||\lambda_0|^2 s_0 - p_0 \overline{s_0}\big| + |p_0|^2 + (1 - |\lambda_0|^2)\dfrac{|s_0|^2}{4} - |\lambda_0|^2 \leq 0;$

(4)
$$|s_0| \leq \frac{2}{1-|\lambda_0|^2}(|\lambda_0||1-p_0\overline{\omega}^2| - \big||\lambda_0|^2 - p_0\overline{\omega}^2\big|)$$
where ω is a complex number of unit modulus such that $s_0 = |s_0|\omega$.

Moreover, for any analytic function $\varphi = (\varphi_1, \varphi_2) : \mathbb{D} \to \Gamma$ such that $\varphi(0) = (0,0)$,
$$\frac{1}{2}|\varphi_1'(0)| + |\varphi_2'(0)| \leq 1.$$

Theorem 5.3.4. Let $\lambda_0 \in \mathbb{D} \setminus \{0\}$, and $(a_0, s_0, p_0) \in \overline{\mathcal{P}}$. Then the following conditions are equivalent:

(1) there exists a rational $\overline{\mathcal{P}}$-inner function $x = (a, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ such that $x(0) = (0,0,0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$;

(2) there exists an analytic function $x = (a, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ such that $x(0) = (0,0,0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$, and $|a_0| \leq |\lambda_0|\sqrt{1 - \frac{1}{4}|s_0|^2}$;

(3)
$$|s_0| < 2, \qquad \frac{2|s_0 - p_0\overline{s}_0| + |s_0^2 - 4p_0|}{4 - |s_0|^2} \leq |\lambda_0|$$
and
$$|a_0| \leq |\lambda_0|\sqrt{1 - \frac{1}{4}|s_0|^2}.$$

The construction of an interpolating function $x = (a, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ such that $x(0) = (0,0,0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$ is given in Theorem 5.3.2.

1.3 Description of results by section

In Chapter 2 we recall definitions of the symmetrized bidisc $\mathbb{G}$ and its closure Γ. In Proposition 2.1.3 from [3], we recollect the characteristics of the distinguished boundary of $\mathbb{G}$. The definition of a Γ-inner function and the definition of the degree of such function are also included in this chapter. Finally, Proposition 2.1.12 from [6] provides a description of rational Γ-inner functions $h = (s, p)$ of prescribed degree n, and Theorem 2.1.18 from [6] describes how to construct all such functions from the zeros of s and the royal nodes of h.

In Chapter 3, we recall the main properties of the pentablock. The majority of the definitions and outcomes in this chapter are from [4]. Theorems 3.2.3 and 3.2.4 provide the characterisations of points in $\mathcal{P}$ and $\overline{\mathcal{P}}$ respectively. Theorem 3.4.2 gives a description of the distinguished boundary of $\mathcal{P}$. In Section 3.5, we identify the singular set of $\mathcal{P}$ which is $\mathcal{R}_{\overline{\mathcal{P}}} \cap \overline{\mathcal{P}} = \{(0, s, p) \in \overline{\mathcal{P}} : s^2 = 4p\}$, see Proposition 3.5.1. We recall the automorphism group of $\mathcal{P}$ and show that $\mathcal{R}_{\overline{\mathcal{P}}} \cap \mathcal{P}$ is invariant under Aut $\mathcal{P}$ and is a complex geodesic in $\mathcal{P}$.

Chapter 4 begins with definitions of inner and outer functions in $H^p(\mathbb{D})$, where $0 < p \leq \infty$, from [44]. We define the $\overline{\mathcal{P}}$-inner functions. Then, in Section 4.2, we construct numerous examples of $\overline{\mathcal{P}}$-inner functions. In Section 4.3, we establish relations between Γ-inner functions and $\overline{\mathcal{P}}$-inner functions, see Lemma 4.3.1 and Proposition 4.3.2. In Theorem 4.5.2, we provide a description of rational $\overline{\mathcal{P}}$-inner functions. In Theorem 4.5.5, we show the construction of rational $\overline{\mathcal{P}}$-inner functions from a rational Γ-inner function. In Theorem 4.6.5 we describe the construction of rational $\overline{\mathcal{P}}$-inner functions from zeroes of a, s and $s^2 - 4p$.

Chapter 5 begins with the statement of the classical Schwarz lemma. In Theorem 5.1.6 we prove a special case of a Schwarz lemma for $\mathcal{P}$. In particular, we consider the case when $x = \pi \circ F$ is an analytic map from $\mathbb{D}$ to $\overline{\mathcal{P}}$ such that $x(0) = (0, 0, 0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$, where $F : \mathbb{D} \to \mathbb{B}^{2\times 2}$ is an analytic map such that

$$F(0) = 0 \text{ and } F(\lambda_0) = \begin{bmatrix} \lambda_1 & 0 \\ a_0 & \lambda_2 \end{bmatrix}.$$

In Theorem 5.3.2 we give sufficient conditions for a Schwarz lemma for $\mathcal{P}$. Namely, we provide sufficent conditions on the pairs $\lambda_0 \in \mathbb{D}$ and $(a_0, s_0, p_0) \in \overline{\mathcal{P}}$ that ensure the existence of a function $x \in \mathrm{Hol}(\mathbb{D}, \overline{\mathcal{P}})$ such that $x(0) = (0, 0, 0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$. In Theorem 5.3.4 we prove a Schwarz lemma for $\mathcal{P}$.

1.4 Historical remarks

For an $m \times n$-matrix A and a linear subspace E of $\mathbb{C}^{m\times n}$, the *structured singular value* of A relative to E is

$$\mu_E(A) = \left(\inf\{\|X\| : X \in E, 1 - AX \text{ is singular}\}\right)^{-1}. \tag{1.3}$$

The μ_E-synthesis problem can be stated as follows: for given distinct points $\lambda_1, \dots, \lambda_\ell \in \mathbb{D}$ and target matrices $W_1, \dots, W_\ell \in \mathbb{C}^{m\times n}$, does there exists an analytic $m \times n$ matrix-valued function F on $\mathbb{D}$ such that

$$F(\lambda_j) = W_j \ \text{ for } j = 1, \dots, \ell, \text{ and}$$

$$\mu_E(F(\lambda)) < 1, \text{ for all } \lambda \in \mathbb{D}?$$

There are several papers on inner functions from $\mathbb{D}$ to various domains associated with μ-synthesis problem. The symmetrized bidisc $\mathbb{G}$ (1.4), the pentablock $\mathcal{P}$ and the tetrablock $\mathbb{E}$ (1.5) are examples of domains in $\mathbb{C}^d$ which arise in connection with μ-synthesis problems.

In [9] Agler and Young introduced the *symmetrized bidisc* which is defined to be the set

$$\mathbb{G} \stackrel{\text{def}}{=} \{(z+w, zw) : |z| < 1, |w| < 1\}. \tag{1.4}$$

We denote its closure by Γ. Let r be the *spectral radius* defined, for $A \in \mathbb{C}^{2\times 2}$, by

$$r(A) = \max\big\{|\lambda| : \lambda \text{ is an eigenvalue of } A\big\}.$$

Note that $r(\cdot)$ is the special case of $\mu_E(\cdot)$ when $E = \mathbb{C}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \subset \mathbb{C}^{2\times 2}$.
The μ_E-synthesis problem in the case of 2×2 matrices with $\mu_E(A) = r(A)$ becomes the 2×2 spectral Nevanlinna-Pick problem that can be stated as follows.

Question 1.4.1. *Let $\lambda_1, \ldots, \lambda_k$ be distinct points in $\mathbb{D}$. Let $W_1, \ldots, W_k \in \mathbb{C}^{2\times 2}$ be such that $r(W_j) \leq 1$ for $j = 1, \ldots, n$. Does there exist a holomorphic 2×2 matrix function F on $\mathbb{D}$ such that $F(\lambda_j) = W_j$ for all $j = 1 \ldots, k$, and $r(F(\lambda)) \leq 1$ for all $\lambda \in \mathbb{D}$?*

Agler and Young showed in [13] that this question is equivalent to an interpolation problem in the set of holomorphic functions from the disc to the symmetrised bidisc. Costara showed in [24] that the symmetrised bidisc is not biholomorphic to a convex set. Agler and Young in [11] determined that the Carathéodory distance $C_{\mathbb{G}}$ and the Kobayashi distance $K_{\mathbb{G}}$ in $\mathbb{G}$ are equal. Lempert's Theorem asserts that for any bounded convex domain $\Omega \subset \mathbb{C}^n$, $C_\Omega = K_\Omega$, see [39] and Chapter C. The symmetrized bidisc $\mathbb{G}$ was the first example of a domain of holomorphy which is not biholomorphic to a convex domain and for which Carathéodory and Kobayashi distances coincide (see Chapter C). Edigarian improved on this result in [30], showing that the symmetrised bidisc cannot be exhausted by domains biholomoprhic to convex ones. A rational Γ-inner function is an analytic function $h : \mathbb{D} \to \Gamma$ with the property that h maps the unit circle $\mathbb{T}$ to the distinguished boundary $b\Gamma$ of Γ. In [6] the authors developed an explicit and detailed structure theory for the rational Γ-inner functions. In [11] Agler and Young proved an explicit, sharp Schwarz lemma for the symmetrized bidisc.

The pentablock $\mathcal{P}$, defined in Difinition 1.1, arose in connection with the μ_E-synthesis problem in the case of 2×2 matrices A with $\mu_E(A)$, where

$$E \stackrel{\text{def}}{=} \text{span}\left\{ \text{id}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

is the space of 2×2 matrices spanned by the identity matrix *id* and a Jordan cell. $\mathcal{P}$ was first introduced in [4] by Agler, Lykova and Young. The authors establish the basic complex geometry and function theory of $\mathcal{P}$. They showed the close relation between the symmetrized bidisc Γ and the pentablock $\mathcal{P}$. The distinguished boundary of $\mathcal{P}$ and a group of automorphisms of $\mathcal{P}$ were described in [4]. In [38] Kosiński proved that this group is the full group of automorphisms of $\mathcal{P}$. The fact that $\mathcal{P}$ is a $\mathbb{C}$-convex domain was proved by Su in [45].

The *tetrablock* is the domain defined as

$$\mathbb{E} = \{x \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 zw \neq 0 \text{ for } |z| \leq 1, |w| \leq 1\} \tag{1.5}$$

and its closure is denoted by $\overline{\mathbb{E}}$. This domain was introduced by Abouhajar, White and Young in [1]. The μ-synthesis problem connected to the tetrablock is the μ_{Diag}-synthesis problem from $\mathbb{D}$ to $\mathbb{C}^{2\times 2}$, where

$$Diag := \left\{ \begin{bmatrix} z & 0 \\ 0 & w \end{bmatrix} : z, w \in \mathbb{C} \right\}.$$

In [27] Edigarian, Kosiński and Zwonek showed that the equality between the Lempert function and the Carathéodory distance stays true in the tetrablock. An explicit and detailed structure theory for rational tetra-inner functions was developed by Alsalhi and Lykova in [16]. In [17] Alshammari and Lykova gave a prescription for the construction of a general rational tetra-inner function of degree n. A Schwarz lemma for $\mathbb{E}$ was proved in [1]. In [29] Edigarian and Zwonek gave the form of all extremals in the Schwarz Lemma for the tetrablock.

Since Agler and Young's first paper on the subject, the study has led to other domains related to cases of μ-synthesis. Under the supervision of Young, D. J. Ogle studied the symmetrised n-disc in his thesis, see [41]. In his thesis he provided criteria for the existence of a solution to the $n \times n$ spectral Nevanlinna-Pick problem.

In [19], Bharali presented a large family of domains related to the μ-synthesis problem, called $\mu_{1,n}$*-quotients*. This family contains some known domains, such as the symmetrized polydisc and the tetrablock. The author studied analytic interpolation from $\mathbb{D}$ into the space of $n \times n$ matrices A with structured singular value $\mu_{1,n}(A)$ less than 1. In [47], Zapałowski introduced the generalized tetrablock. In his paper he investigated the geometric properties of this domain containing the family of the $\mu_{1,n}$-quotients $\mathbb{E}_n, n \geq 2$. Zapałowski proved that the generalized tetrablock cannot be exhausted by domains biholomorphic to convex ones. Additionally, the author showed that the Carathéodory distance and the Lempert function are not equal on a large subfamily of the generalized tetrablocks for $\mathbb{E}_n, n \geq 4$.

Aside from their use in the study of μ-synthesis, these domains have turned out to have many properties of interest to specialists in several complex variables and operator theory.

A set V in a domain U in $\mathbb{C}^n$ has the *norm-preserving extension property* if every bounded analytic function on V has an analytic extension to U with the same supremum norm. In [7], Agler, Lykova and Young proved that an algebraic subset V of the symmetrized bidisc $\mathbb{G}$ has the norm-preserving extension property if and only if V is either a singleton, $\mathbb{G}$ itself, a complex geodesic of $\mathbb{G}$, or the union of the set $\{(2z, z^2) : |z| < 1\}$ and a complex geodesic of degree 1 in $\mathbb{G}$. They showed in [7, Theorem 15.3] that the tetrablock and the pentablock contain sets having the norm-preserving extension property which are not retracts in the respective domains.

In [21], Bhattacharyya, Pal and Shyam Roy show the existence and uniqueness of solution to the operator equation for a Γ-contraction (S, P) and construct an explicit Γ-isometric dilation of a Γ-contraction (S, P). Here, for a contraction P and a bounded commutant S of P, a solution X of the operator equation

$$S - S^*P = (I - P^*P)^{\frac{1}{2}} X (I - P^*P)^{\frac{1}{2}},$$

where X is a bounded operator on $\overline{\mathrm{Ran}}(I - P^*P)^{\frac{1}{2}}$ with numerical radius of X being not greater than 1. A pair of bounded operators (S, P) which has Γ as a spectral set, is called a Γ-contraction.

Chapter 2. The symmetrized bidisc

2.1 Γ-inner functions

Definition 2.1.1. *The symmetrized bidisc is the set*

$$\mathbb{G} \stackrel{\text{def}}{=} \{(z+w, zw) : |z| < 1, \ |w| < 1\}, \tag{2.1}$$

and its closure is

$$\Gamma = \{(z+w, zw) : |z| \le 1, \ |w| \le 1\}.$$

Remark 2.1.2. *The pentablock is closely related to the symmetrized bidisc. Indeed, from the definition* (1.1)*, $\mathcal{P}$ is fibred over $\mathbb{G}$ by the map $(a, s, p) \mapsto (s, p)$, since if $A \in \mathbb{B}^{2\times 2}$ then the eigenvalues of A lie in $\mathbb{D}$ and so* $(\mathrm{tr}\ A, \det A) \in \mathbb{G}$. See [4, Section 2, Page 510].

In 1999 Agler and Young introduced the symmetrized bidisc in [9]. There is a strong connection between the symmetrized bidisc and the pentablock. Following [9] we shall often use the co-ordinates (s, p) for points in the symmetrized bidisc $\mathbb{G}$, chosen to suggest 'sum' and 'product'. The following results afford useful criteria for membership of $\mathbb{G}$, of the distinguished boundary $b\Gamma$ of Γ (see Section 3.4), and of the topological boundary $\partial\Gamma$ of Γ, see [3].

Proposition 2.1.3. [3, Proposition 3.2] *Let (s, p) belong to $\mathbb{C}^2$. Then*

(1) (s, p) *belongs to $\mathbb{G}$ if and only if*

$$|s - \bar{s}p| < 1 - |p|^2;$$

(2) (s, p) *belongs to Γ if and only if*

$$|s| \le 2 \text{ and } |s - \bar{s}p| \le 1 - |p|^2;$$

(3) (s, p) *lies in $b\Gamma$ if and only if*

$$|p| = 1, |s| \le 2 \text{ and } s - \bar{s}p = 0;$$

(4) $(s,p) \in \partial\Gamma$ *if and only if*

$$|s| \le 2 \text{ and } |s - \bar{s}p| = 1 - |p|^2.$$

For $w \in \mathbb{T}$, define the function $\Phi_w : \Gamma \to \mathbb{C}$ by

$$\Phi_w(s,p) = \Phi(w,s,p) = \frac{2wp - s}{2 - ws} \quad \text{for } (s,p) \in \Gamma \text{ such that } ws \neq 2. \tag{2.2}$$

Theorem 2.1.4. [12, Theorem 2.1] *Let $s, p \in \mathbb{C}$. The following statements are equivalent:*

(1) $(s,p) \in \mathbb{G}$;

(2) *the roots of the equation $z^2 - sz + p = 0$ lie in $\mathbb{D}$;*

(3) $|s - \bar{s}p| < 1 - |p|^2$;

(4) $|s| < 2$ *and, for all $z \in \overline{\mathbb{D}}$,*

$$\left| \frac{2zp - s}{2 - zs} \right| < 1;$$

(5) $|p| < 1$ *and there exists $\beta \in \mathbb{D}$ such that $s = \beta p + \bar{\beta}$;*

(6) $2|s - \bar{s}p| + |s^2 - 4p| + |s|^2 < 4.$

In this thesis an *automorphism* of a domain Ω is an analytic bijective self-map of Ω having an analytic inverse. Note that if $f : \Omega \to \Omega$ is analytic and bijective, then f^{-1} is automatically analytic. The following is well known.
For $\alpha \in \mathbb{D}$ define

$$B_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha}z}. \tag{2.3}$$

The rational function B_α is called a *Blaschke factor*. A Möbius function is a function of the form cB_α for some $\alpha \in \mathbb{D}$ and $c \in \mathbb{T}$. The set of all Möbius functions is *the automorphism group* Aut $\mathbb{D}$ *of* $\mathbb{D}$.

The group of automorphisms of $\mathbb{G}$ was announced by Agler and Young in [12, Section 6]. A shorter proof was found by M. Jarnicki and P. Pflug [35] and the result has been extended to the symmetrized polydisc by A. Edigarian and W. Zwonek [28].

Theorem 2.1.5. [14, Theorem 4.1] *The automorphisms of the symmetrized bidisc $\mathbb{G}$ are of the form*

$$\tau_v(z_1 + z_2, z_1 z_2) = (v(z_1) + v(z_2), v(z_1)v(z_2)), \ \ z_1, z_2 \in \mathbb{D},$$

for some automorphism v of the unit disc $\mathbb{D}$.

Definition 2.1.6. *The royal variety $\mathcal{R}_\Gamma$ of the symmetrized bidisc is*

$$\mathcal{R}_\Gamma = \{(s,p) \in \mathbb{C}^2 : s^2 = 4p\}.$$

Lemma 2.1.7. [14, Lemma 4.3] *Every automorphism of $\mathbb{G}$ maps the royal variety to itself.*

See Chapter D for detailed proofs.

Γ-inner functions were defined and studied in [3].

Definition 2.1.8. *A Γ-inner function is an analytic function $h : \mathbb{D} \to \Gamma$ such that the radial limit*

$$\lim_{r \to 1^-} h(r\lambda) \tag{2.4}$$

exists and belongs to $b\Gamma$ for almost all $\lambda \in \mathbb{T}$ with respect to Lebesgue measure.

By Fatou's Theorem, the limit (2.4) exists for almost all $\lambda \in \mathbb{T}$.

Definition 2.1.9. *Let f be a polynomial of degree less than or equal to n, where $n \geq 0$. Then we define the polynomial $f^{\sim n}$ by*

$$f^{\sim n}(\lambda) = \lambda^n \overline{f(1/\overline{\lambda})}.$$

In the next definition we use the fundamental group $\pi_1(X)$ of a topological space X, see Chapter B.

Definition 2.1.10. [6, Definition 3.1] *The degree $\deg(h)$ of a rational Γ-inner function h is defined to be $h_*(1)$, where $h_* : \mathbb{Z} = \pi_1(\mathbb{T}) \to \pi_1(b\Gamma)$ is the homomorphism of fundamental groups induced by h when it is regarded as a continuous map from $\mathbb{T}$ to $b\Gamma$.*

Proposition 2.1.11. [6, Proposition 3.3] *For any rational Γ-inner function $h = (s,p)$, $\deg(h)$ is the degree $\deg(p)$ (in the usual sense) of the finite Blaschke product p.*

Note that p is a rational inner function on $\mathbb{D}$ of degree n (that is, a Blaschke product with n factors) if and only if there exists a polynomial D of degree less than or equal to n such that $D(\lambda) \neq 0$ for all $\lambda \in \overline{\mathbb{D}}$ and $p(\lambda) = \dfrac{D^{\sim n}(\lambda)}{D(\lambda)}$, see [6].

Proposition 2.1.12. [6, Proposition 2.2] *If $h = (s,p)$ is a rational Γ-inner function of degree n then there exist polynomials E and D such that*

(1) $\deg(E), \deg(D) \leq n$,

(2) $E^{\sim n} = E$,

(3) $D(\lambda) \neq 0$ *for all* $\lambda \in \overline{\mathbb{D}}$,

(4) $|E(\lambda)| \leq 2|D(\lambda)|$ *for all* $\lambda \in \overline{\mathbb{D}}$,

(5) $s = \dfrac{E}{D}$ *on* $\overline{\mathbb{D}}$,

(6) $p = \dfrac{D^{\sim n}}{D}$ *on* $\overline{\mathbb{D}}$.

Furthermore, E_1 *and* D_1 *is a second pair of polynomials that satisfy* (1)-(6) *if and only if there exists a nonzero* $t \in \mathbb{R}$ *such that*

$$E_1 = tE \text{ and } D_1 = tD.$$

Conversely, if E *and* D *are polynomials that satisfy* (1), (2), (4), $D(\lambda) \neq 0$ *for all* $\lambda \in \mathbb{D}$, *and* s *and* p *are defined by* (5) *and* (6), *then* $h = (s, p)$ *is a rational* Γ*-inner function of degree less than or equal to* n.

Definition 2.1.13. [6, Page 140] *Let* $h = (s, p)$ *be a rational* Γ*-inner function of degree* n. *Let* E *and* D *be as in Proposition 2.1.12. The royal polynomial* R_h *of* h *is defined by*

$$R_h(\lambda) = 4D(\lambda)D^{\sim n}(\lambda) - E(\lambda)^2. \tag{2.5}$$

We call the points $\lambda \in \overline{\mathbb{D}}$ such that $h(\lambda) \in \mathcal{R}_\Gamma$ the *royal nodes* of h and, for such λ, we call $h(\lambda)$ a *royal point* of h, that is, $4p(\lambda) - s(\lambda)^2 = 0$. There exists a special class of rational Γ-inner functions h such that $h(\mathbb{D}) \subset \mathcal{R}_\Gamma$. These are precisely the rational Γ-inner functions of the form $h = (2f, f^2)$ for some finite Blaschke product f. The royal polynomials of $h = (2f, f^2)$ are identically zero. For completeness, we shall define the degree of the zero polynomial to be $-\infty$.

Remark 2.1.14. *Since* $D(\lambda) \neq 0$ *for all* $\lambda \in \overline{\mathbb{D}}$ (see Proposition 2.1.12 (3)), *the royal nodes of* h *exactly correspond to the zeros of the royal polynomial* R_h. *Hence,* $\lambda \in \overline{\mathbb{D}}$ *is a royal node of* h *if and only if* $R_h(\lambda) = 0$.

Proposition 2.1.15. [6, Proposition 3.5] *Let* h *be a rational* Γ*-inner function of degree* n *and let* R_h *be the royal polynomial of* h *as defined by equation* (2.5). *Then* R_h *is* $2n$*-symmetric and the zeros of* R_h *that lie on* $\mathbb{T}$ *have either even or infinite order.*

Definition 2.1.16. [6, Definition 3.6] *Let* h *be a rational* Γ*-inner function such that* $h(\overline{\mathbb{D}}) \nsubseteq \mathcal{R}_\Gamma \cap \Gamma$ *and let* R_h *be the royal polynomial of* h. *If* σ *is a zero of* R_h *of order* ℓ, *we define the multiplicity* $\#\sigma$ *of* σ *(as a royal node of* h*) by*

$$\#\sigma = \begin{cases} \ell & \text{if } \sigma \in \mathbb{D} \\ \frac{1}{2}\ell & \text{if } \sigma \in \mathbb{T}. \end{cases}$$

We define the type of h *to be the ordered pair* (n, k) *where* n *is the sum of the multiplicities of the royal nodes of* h *that lie in* $\overline{\mathbb{D}}$ *and* k *is the sum of the multiplicities of the royal* *nodes of* h *that lie in* $\mathbb{T}$. *We define* $\mathcal{R}_\Gamma^{n,k}$ *to be the collection of rational* Γ*-inner functions* h *of type* (n,k).

Proposition 2.1.17. [6, Proposition 4.5] *Let the royal nodes of a rational* Γ*-inner function* h *be* $\sigma_1,\dots,\sigma_n$, *with repetitions according to multiplicity of the nodes as described in Definition 2.1.16. The royal polynomial* R_h *of* h, *up to a positive multiple, is*

$$R_h(\lambda)=\prod_{j=1}^{n} Q_{\sigma_j}(\lambda), \quad \text{for } \lambda\in\mathbb{C},$$

where the polynomial Q_σ *is defined by* $Q_\sigma(\lambda)=(\lambda-\sigma)(1-\overline{\sigma}\lambda)$, *for* $\sigma\in\overline{\mathbb{D}}$.

Theorem 2.1.18. [6, Theorem 4.8] *Let* n *be a positive integer and suppose points* $\alpha_1,\alpha_2,\dots,\alpha_{k_0}\in\mathbb{D}$ *and* $\tau_1,\tau_2,\dots,\tau_{k_1}\in\mathbb{T}$ *are given, where* $2k_0+k_1=n$, *and points* $\sigma_1,\dots,\sigma_n$ *in* $\overline{\mathbb{D}}$ *are distinct from* $\tau_1,\dots,\tau_{k_1}$.

There exists a rational Γ*-inner function* $h=(s,p)$ *of degree* n *such that*

(1) *the zeros of* s *in* $\overline{\mathbb{D}}$, *repeated according to multiplicity, are* $\alpha_1,\alpha_2,\dots,\alpha_{k_0}$ *and* $\tau_1,\tau_2,\dots,\tau_{k_1}$,

(2) *the royal nodes of* h *are* $\sigma_1,\dots,\sigma_n$.

Such a function h *can be constructed as follows. Let* $t_+>0$ *and let* $t\in\mathbb{R}\setminus\{0\}$. *Let* R *and* E *be defined by*

$$R(\lambda)=t_+\prod_{j=1}^{n}(\lambda-\sigma_j)(1-\overline{\sigma_j}\lambda)$$

and

$$E(\lambda)=t\prod_{j=1}^{k_0}(\lambda-\alpha_j)(1-\overline{\alpha_j}\lambda)\prod_{j=1}^{k_1}\mathrm{i}\mathrm{e}^{-i\theta_j/2}(\lambda-\tau_j)$$

where $\tau_j=\mathrm{e}^{i\theta_j}$, $0\le\theta_j<2\pi$.

(i) *There exists an outer polynomial* D *of degree at most* n *such that*

$$\lambda^{-n}R(\lambda)+|E(\lambda)|^2=4|D(\lambda)|^2$$

for all $\lambda\in\mathbb{T}$.

(ii) *The function* h *defined by*

$$h=(s,p)=\left(\frac{E}{D},\frac{D^{\sim n}}{D}\right)$$

is a rational Γ*-inner function such that* $\deg(h)=n$ *and conditions* (1) *and* (2) *hold. The royal polynomial of* h *is* R.

Proposition 2.1.19. [6, Proposition 4.9] *Let* $h=(s,p)$ *be a rational* Γ*-inner function of degree* n *such that*

(1) *the zeros of* s *in* $\overline{\mathbb{D}}$, *repeated according to multiplicity, are* $\alpha_1, \alpha_2, \ldots, \alpha_{k_0} \in \mathbb{D}$, $\tau_1, \tau_2, \ldots, \tau_{k_1} \in \mathbb{T}$, *where* $2k_0 + k_1 = n$, *and*

(2) *the royal nodes of* h *are* $\sigma_1, \ldots, \sigma_n$.

There exists some choice of $t_+ > 0$, $t \in \mathbb{R} \setminus \{0\}$ *and* $\omega \in \mathbb{T}$ *such that the recipe in Theorem 2.1.18 with these choices produces the function* h.

2.2 The two-by-two spectral Nevanlinna-Pick problem and the Γ-interpolation problem

Agler and Young provided proofs of the connection between the two-by-two spectral Nevanlinna-Pick problem and the Γ-interpolation problem in [10]. Instead of interpolating from $\mathbb{D}$ into the 4-dimensional space of 2×2 complex matrices, they studied the interpolation problem from $\mathbb{D}$ into the symmetrized bidisc Γ, which is a compact subset of $\mathbb{C}^2$.

Theorem 2.2.1. [10, Theorem 2.1] *Let* $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$ *be distinct and* $A_1, \ldots, A_n \in \mathbb{C}^{2\times 2}$. *Suppose that either all or none of* $A_1, \ldots, A_n$ *are scalar matrices. The following are equivalent:*

(1) there exists an analytic 2×2 *matrix function* $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$ *such that*

$$r\big(F(\lambda)\big) \leq 1 \quad \text{for all } \lambda \in \mathbb{D} \quad \text{and} \quad F(\lambda_k) = A_k,\ k = 1, \ldots, n;$$

(2) there exists an analytic function $h : \mathbb{D} \to \Gamma$ *such that*

$$h(\lambda_k) = (\operatorname{tr} A_k, \det A_k), \quad k = 1, \ldots, n.$$

Theorem 2.2.2. [2, Theorem 8.1] *Let* $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$ *be distinct and let* $(s_k, p_k) \in \Gamma$ *for* $k = 1, \ldots, n$. *The following are equivalent:*

(1) there exists an analytic function $h : \mathbb{D} \to \Gamma$ *such that*

$$h(\lambda_k) = (s_k, p_k), \quad k = 1, \ldots, n.$$

(2) there exists a rational Γ*-inner function* $h : \mathbb{D} \to \Gamma$ *satisfying*

$$h(\lambda_k) = (s_k, p_k), \quad k = 1, \ldots, n.$$

Chapter 3. The pentablock $\mathcal{P}$

3.1 The pentablock $\mathcal{P}$

Definition 3.1.1. [4] *The pentablock is the domain defined by*

$$\mathcal{P} = \{(a_{21}, \text{tr } A, \det A) : A = [a_{ij}]_{i,j=1}^{2} \in \mathbb{B}^{2\times 2}\} \tag{3.1}$$

where $\mathbb{B}^{2\times 2}$ denotes the open unit ball in the space of 2×2 complex matrices.

Recall that $\mathcal{P} = \pi\left(\mathbb{B}^{2\times 2}\right)$, where the polynomial map implicit in the definition (3.1) can be written as

$$\pi(A) = (a_{21}, \text{tr } A, \det A), \quad \text{where } A = [a_{ij}]_{i,j=1}^{2} \in \mathbb{C}^{2\times 2}.$$

The pentablock, which is a bounded nonconvex domain in $\mathbb{C}^3$, was introduced in [4] in 2015 by Agler, Lykova and Young.

Definition 3.1.2. [4, Definition 4.1] *For $z \in \mathbb{D}$ and $(a, s, p) \in \mathbb{C}^3$ such that $1 - sz + pz^2 \neq 0$ define $\Psi_z(a, s, p)$ by*

$$\Psi_z(a, s, p) = \frac{a(1 - |z|^2)}{1 - sz + pz^2} \tag{3.2}$$

and define $\kappa(s, p)$ by

$$\kappa(s, p) = \sup_{z\in\mathbb{D}} \frac{1 - |z|^2}{|1 - sz + pz^2|}.$$

Theorem 3.1.3. [4, Theorem 1.1] *Let*

$$(s, p) = (\lambda_1 + \lambda_2, \lambda_1\lambda_2)$$

where λ_1, $\lambda_2 \in \mathbb{D}$. Let $a \in \mathbb{C}$ and let

$$\beta = \frac{s - \bar{s}p}{1 - |p|^2}.$$

Then $|\beta| < 1$ and the following statements are equivalent:

(1) $(a, s, p) \in \mathcal{P}$*;*

(2) *there exists $A \in \mathbb{C}^{2\times 2}$ such that $\mu_E(A) < 1$ and $\pi(A) = (a, s, p)$;* ($\mu_E(A)$ is defined in equation (1.3))

(3) $|a| < |1 - \frac{\frac{1}{2}s\overline{\beta}}{1+\sqrt{1-|\beta|^2}}|$;

(4) $|a| < \frac{1}{2}|1 - \overline{\lambda_2}\lambda_1| + \frac{1}{2}(1 - |\lambda_1|^2)^{\frac{1}{2}}(1 - |\lambda_2|^2)^{\frac{1}{2}}$;

(5) $\sup_{z\in\mathbb{D}} |\Psi_z(a, s, p)| < 1$.

Proposition 3.1.4. [4, Proposition 4.2] *For $\beta \in \mathbb{D}$ and $(s, p) = (\beta + \overline{\beta}p, p) \in \mathbb{G}$,*

$$\kappa(s,p) = \left|1 - \frac{\frac{1}{2}s\overline{\beta}}{1+\sqrt{1-|\beta|^2}}\right|^{-1}.$$

Moreover the supremum of $\frac{1-|z|^2}{|1-sz+pz^2|}$ over $z \in \mathbb{D}$ is attained uniquely at the point

$$z = \frac{\overline{\beta}}{1+\sqrt{1-|\beta|^2}}.$$

A domain Ω is said to be polynomially convex if for each compact subset $\mathbf{K}$ of Ω, the polynomial hull $\widehat{\mathbf{K}}$ of $\mathbf{K}$ is contained in Ω, where $\widehat{\mathbf{K}}$ is defined as

$$\widehat{\mathbf{K}} = \{z \in \Omega : |p(z)| \le \max_{k\in\mathbf{K}} |p(k)| \text{ for all polynomials}\}.$$

Theorem 3.1.5. [4, Theorem 6.3] *$\mathcal{P}$ and $\overline{\mathcal{P}}$ are polynomially convex.*

The pentablock is a Hartogs Domain [38]. Following the description of the pentablock $\mathcal{P}$, we can learn that the pentablock $\mathcal{P}$ is closely related to the symmetrized bidisc $\mathbb{G}$. In fact, the pentablock $\mathcal{P}$ can be seen as a Hartogs domain in $\mathbb{C}^3$ over the symmetrized bidisc $\mathbb{G}$, that is,

$$\mathcal{P} = \left\{(a, s, p) \in \mathbb{D}\times\mathbb{G} : |a|^2 < e^{-\varphi(s,p)}\right\},$$

where

$$\varphi(s,p) = -2\log\left|1 - \frac{\frac{1}{2}s\overline{\beta}}{1+\sqrt{1-|\beta|^2}}\right|,$$

$(s, p) \in \mathbb{G}$ and $\beta = \frac{s-\overline{s}p}{1-|p|^2}$.

Hartogs domains are important objects in several complex variables.

Definition 3.1.6. [36, page 259] *A domain $D \subset \mathbb{C}^n$ is called $\mathbb{C}$-convex if for any complex line $\ell = a + b\mathbb{C}$, $0 \ne a, b \in \mathbb{C}^n$ such that $\ell \cap D \ne \emptyset$, this intersection $\ell \cap D$ is connected and simply connected.*

Theorem 3.1.7. [45, Theorem 1.1] *The pentablock $\mathcal{P}$ is a $\mathbb{C}$-convex domain.*

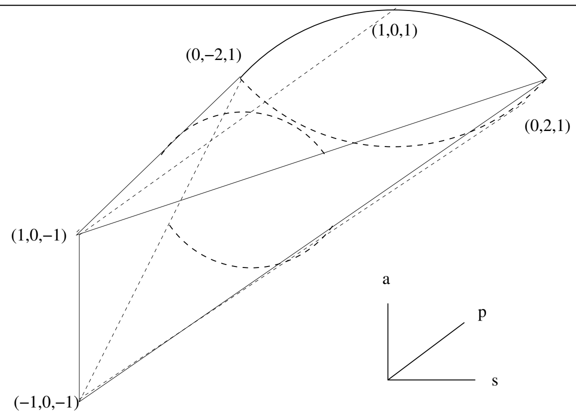

Figure 3.1: The real pentablock

Theorem 3.1.8. [4, Theorem 9.2] *The real pentablock $\mathcal{P} \cap \mathbb{R}^3$ is convex.*

Theorem 3.1.9. [4, Theorem 9.3] *$\mathcal{P} \cap \mathbb{R}^3$ is a convex open domain with five faces and with the four vertices $(0, -2, 1)$, $(0, 2, 1)$, $(1, 0, -1)$ and $(-1, 0, -1)$. The faces are the following sets:*

(1) *the triangle with vertices $(0, 2, 1)$, $(1, 0, -1)$ and $(-1, 0, -1)$ together with its interior;*

(2) *the triangle with vertices $(0, -2, 1)$, $(1, 0, -1)$ and $(-1, 0, -1)$ together with its interior;*

(3) *the ellipse*
$$\{(a, s, 1) : a^2 + s^2/4 = 1, -2 \leq s \leq 2\}$$
with centre at $(0, 0, 1)$, with major axis joining the points $(0, 2, 1)$ and $(0, -2, 1)$ and with minor axis joining the points $(1, 0, 1)$ and $(-1, 0, 1)$, together with its interior;

(4) *a surface with vertices $(1, 0, -1)$ and $(0, -2, 1)$, $(0, 2, 1)$ and boundaries:*

- $\{(a, s, 1) : a = \sqrt{1 - s^2/4}, -2 \leq s \leq 2\}$;
- *the straight line segment joining $(0, -2, 1)$ and $(1, 0, -1)$;*
- *the straight line segment joining $(0, 2, 1)$ and $(1, 0, -1)$;*

(5) *a surface with vertices $(-1, 0, -1)$ and $(0, -2, 1)$, $(0, 2, 1)$ and boundaries:*

- $\{(a, s, 1) : a = -\sqrt{1 - s^2/4}, -2 \leq s \leq 2\}$;
- *the straight line segment joining $(0, -2, 1)$ and $(-1, 0, -1)$;*
- *the straight line segment joining $(0, 2, 1)$ and $(-1, 0, -1)$.*

3.2 The pentablock and the μ_E-synthesis problem

The pentablock is connected to the following μ-synthesis notion:
for a 2×2-matrix A,

$$\mu_E(A) = \left(\inf\{\|X\| : X \in E, 1 - AX \text{ is singular}\}\right)^{-1},$$

where

$$E \stackrel{\text{def}}{=} \text{span}\left\{\text{id}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$$

is the space of 2×2 matrices spanned by the identity matrix id and a Jordan cell.

Proposition 3.2.1. [4, Proposition 4.3] *For any matrix $A = [a_{ij}] \in \mathbb{C}^{2\times 2}$,*

$$\mu_E(A) < 1 \ \text{ if and only if } \ (s,p) \in \mathbb{G} \ \text{ and } \ |a_{21}| < \left|1 - \frac{\frac{1}{2}s\overline{\beta}}{1 + \sqrt{1 - |\beta|^2}}\right|$$

and

$$\mu_E(A) \ \leq \ 1 \ \text{ if and only if } \ (s,p) \in \Gamma \ \text{ and } \ |a_{21}| \ \leq \ \left|1 - \frac{\frac{1}{2}s\overline{\beta}}{1 + \sqrt{1 - |\beta|^2}}\right|$$

where $s = \text{tr } A$, $p = \det A$ and $\beta = (s - \overline{s}p)/(1 - |p|^2)$.

Definition 3.2.2. [4, Definition 3.3] *$\mathbb{B}_\mu$ is the domain in $\mathbb{C}^{2\times 2}$ given by*

$$\mathbb{B}_\mu = \{A \in \mathbb{C}^{2\times 2} : \mu_E(A) < 1\}.$$

$\mathcal{P}_\mu$ is the domain in $\mathbb{C}^3$ given by

$$\mathcal{P}_\mu = \left\{(a_{21}, \text{tr}A, \det A) : A \in \mathbb{C}^{2\times 2}, \mu_E(A) < 1\right\} \subset \mathbb{C}^3.$$

Theorem 3.2.3. [4, Theorem 5.2] *Let*

$$(s,p) = (\beta + \overline{\beta}p, p) = (\lambda_1 + \lambda_2, \lambda_1\lambda_2) \in \mathbb{G}$$

and let $a \in \mathbb{C}$. The following statements are equivalent:

(1) $(a, s, p) \in \mathcal{P}$;

(2) $(a, s, p) \in \mathcal{P}_\mu$;

(3) $|a| < |1 - \frac{\frac{1}{2}s\overline{\beta}}{1+\sqrt{1-|\beta|^2}}|$;

(4) $|a| < \frac{1}{2}|1 - \overline{\lambda}_2\lambda_1| + \frac{1}{2}(1 - |\lambda_1|^2)^{\frac{1}{2}}(1 - |\lambda_2|^2)^{\frac{1}{2}}$;

(5) $\sup_{z\in\mathbb{D}}|\Psi_z(a, s, p)| < 1$, *where Ψ_z is defined by equation* (3.2).

Theorem 3.2.4. [4, Theorem 5.3] *Let*

$$(s,p) = (\beta + \overline{\beta}p, p) = (\lambda_1 + \lambda_2, \lambda_1\lambda_2) \in \Gamma$$

where $|\beta| \leq 1$ *and if* $|p| = 1$ *then* $\beta = \frac{1}{2}s$. *Let* $a \in \mathbb{C}$. *The following statements are equivalent:*

(1) $(a, s, p) \in \overline{\mathcal{P}}$;

(2) $(a, s, p) \in \overline{\mathcal{P}}_\mu$;

(3) $|a| \leq |1 - \frac{\frac{1}{2}s\overline{\beta}}{1+\sqrt{1-|\beta|^2}}|$;

(4) $|a| \leq \frac{1}{2}|1 - \overline{\lambda}_2\lambda_1| + \frac{1}{2}(1 - |\lambda_1|^2)^{\frac{1}{2}}(1 - |\lambda_2|^2)^{\frac{1}{2}}$;

(5) $|\Psi_z(a, s, p)| \leq 1$ *for all* $z \in \mathbb{D}$, *where* Ψ_z *is defined by equation* (3.2);

(6) *there exists* $A \in \mathbb{C}^{2\times 2}$ *such that* $\|A\| \leq 1$ *and* $\pi(A) = (a, s, p)$;

(7) *there exists* $A \in \mathbb{C}^{2\times 2}$ *such that* $\mu_E(A) \leq 1$ *and* $\pi(A) = (a, s, p)$.

3.3 Analytic lifting

In the present context the μ-synthesis problem is an interpolation problem for analytic functions from $\mathbb{D}$ to $\mathbb{B}_\mu$. If $H : \mathbb{D} \to \mathbb{B}_\mu$ is an analytic function satisfying interpolation conditions $H(\lambda_j) = W_j$ for given points $\lambda_1, \dots, \lambda_n \in \mathbb{D}$ and target points $W_1, \dots, W_n \in \mathbb{B}_\mu$, then $h \stackrel{\text{def}}{=} \pi \circ H : \mathbb{D} \to \mathcal{P}$ is an analytic function that satisfies

$$h(\lambda_j) = \pi(W_j) \text{ for } j = 1, \dots, n.$$

Let U be a domain in $\mathbb{C}^n$. $\text{Hol}(\mathbb{D}, U)$ denotes the space of analytic functions from $\mathbb{D}$ to U.

We say that $H \in \text{Hol}(\mathbb{D}, \mathbb{C}^{2\times 2})$ is an *analytic lifting* of $h \in \text{Hol}(\mathbb{D}, \overline{\mathcal{P}})$ if $\pi \circ H = h$. We say that H is a *Schur lifting* of h if $\pi \circ H = h$ and H belongs to the matricial Schur class

$$\mathcal{S}_{2\times 2} \stackrel{\text{def}}{=} \{F \in \text{Hol}(\mathbb{D}, \mathbb{C}^{2\times 2}) : \|F(\lambda)\| \leq 1 \text{ for all } \lambda \in \mathbb{D}\}.$$

If H is an analytic lifting of h then $H \in \text{Hol}(\mathbb{D}, \overline{\mathbb{B}}_\mu)$.

Example 3.3.1. [4, Example 12.1] Let $h(\lambda) = (\lambda, 0, \lambda)$. This $h \in \text{Hol}(\mathbb{D}, \mathcal{P})$ lifts to $H \in \mathcal{S}_{2\times 2}$ given by

$$H(\lambda) = \begin{bmatrix} 0 & -1 \\ \lambda & 0 \end{bmatrix}.$$

Here $H(\lambda)$ does not belong to the open matrix ball $\mathbb{B}$ for any $\lambda \in \mathbb{D}$. The construction in Theorem 3.2.3 (4) gives the following non-analytic lifting of $(\lambda, 0, \lambda) \in \mathcal{P}$ to $\mathbb{B}$:

$$H(\lambda) = \begin{bmatrix} i(1-|\lambda|)^{\frac{1}{2}}\zeta & -|\lambda| \\ \lambda & -i(1-|\lambda|)^{\frac{1}{2}}\zeta \end{bmatrix}$$

where ζ is a square root of λ.

Example 3.3.2. [4, Example 12.2] Let $h(\lambda) = (\lambda^2, 0, \lambda)$. Then $h \in \mathrm{Hol}(\mathbb{D}, \mathcal{P})$, but there is no $H \in \mathrm{Hol}(\mathbb{D}, \mathbb{C}^{2\times 2})$ such that $h = \pi \circ H$.

Proof. Suppose that H is such that $h = \pi \circ H$. We can write

$$H = \begin{bmatrix} -\eta & g \\ \lambda^2 & \eta \end{bmatrix}$$

for some g, η in $\mathrm{Hol}(\mathbb{D}, \mathbb{C})$. Since $\det H = \lambda$ we must have

$$\eta(\lambda)^2 = -\lambda - \lambda^2 g(\lambda)$$

for $\lambda \in \mathbb{D}$. This is a contradiction, since the right hand side has a simple zero at 0, while the left hand side has a zero of multiplicity at least 2. $\square$

Proposition 3.3.3. [4, Proposition 12.4] *A function $h = (a, s, p)$ lifts to* $\mathrm{Hol}(\mathbb{D}, \mathbb{C}^{2\times 2})$ *if and only if there is no point $\alpha \in \mathbb{D}$ such that, for some odd positive integer n,*

(1) *α is a zero of $\frac{1}{4}s^2 - p$ of multiplicity n and*

(2) *α is a zero of a of multiplicity greater than n.*

3.4 The distinguished boundary of $\mathcal{P}$

Let Ω be a domain in $\mathbb{C}^n$ with closure $\overline{\Omega}$ and let $A(\Omega)$ be the algebra of continuous scalar functions on $\overline{\Omega}$ that are holomorphic on Ω. A *boundary* for Ω is a subset C of $\overline{\Omega}$ such that every function in $A(\Omega)$ attains its maximum modulus on C.
Since $\overline{\mathcal{P}}$ is polynomially convex, there is a smallest closed boundary of $\mathcal{P}$, contained in all the closed boundaries of $\mathcal{P}$, called the *distinguished boundary of* $\mathcal{P}$ and denoted by $b\mathcal{P}$. The distinguished boundary of $\mathcal{P}$ is the same thing as the Shilov boundary of the function algebra $A(\mathcal{P})$. If there is a function $g \in A(\mathcal{P})$ and a point $u \in \overline{\mathcal{P}}$ such that $g(u) = 1$ and $|g(x)| < 1$ for all $x \in \overline{\mathcal{P}} \backslash \{u\}$, then u must belong to $b\mathcal{P}$. Such a point u is called a *peak point* of $\overline{\mathcal{P}}$ and the function g a *peaking function* for u.

Define

$$K_0 \stackrel{\text{def}}{=} \left\{ (a, s, p) \in \mathbb{C}^3 : (s, p) \in b\Gamma,\ |a| = \sqrt{1 - \frac{1}{4}|s|^2} \right\}.$$

and

$$K_1 \stackrel{\text{def}}{=} \left\{(a,s,p) \in \mathbb{C}^3 : (s,p) \in b\Gamma,\ |a| \leq \sqrt{1 - \frac{1}{4}|s|^2}\right\}. \tag{3.3}$$

Proposition 3.4.1. [4, Proposition 8.3] *The subsets K_0 and K_1 of $\overline{\mathcal{P}}$ are closed boundaries for $A(\mathcal{P})$.*

Theorem 3.4.2. [4, Theorem 8.4] *For $x \in \mathbb{C}^3$, the following are equivalent:*
(1) *$x \in K_0$;*
(2) *x is a peak point of $\overline{\mathcal{P}}$;*
(3) *$x \in b\mathcal{P}$, the distinguished boundary of $\mathcal{P}$.*

Therefore

$$b\mathcal{P} = \left\{(a,s,p) \in \mathbb{C}^3 : (s,p) \in b\Gamma,\ |a| = \sqrt{1 - \frac{1}{4}|s|^2}\right\}$$

and so

$$b\mathcal{P} = \left\{(a,s,p) \in \mathbb{C}^3 : |s| \leq 2,\ |p| = 1,\ s = \bar{s}p \text{ and } |a| = \sqrt{1 - \frac{1}{4}|s|^2}\right\}.$$

Theorem 3.4.3. [4, Theorem 8.5] *The distinguished boundary $b\mathcal{P}$ is homeomorphic to*

$$\{(\sqrt{1-x^2}w, x, \theta) : -1 \leq x \leq 1,\ 0 \leq \theta \leq 2\pi,\ w \in \mathbb{T}\}$$

with the two points $(\sqrt{1-x^2}w, x, 0)$ and $(\sqrt{1-x^2}w, -x, 2\pi)$ identified for every $w \in \mathbb{T}$ and $x \in [-1,1]$.

3.5 The singular set of $\mathcal{P}$ and Aut $\mathcal{P}$

Recall that $\mathcal{P} = \pi(\mathbb{B}^{2\times 2})$, where $\pi : \mathbb{C}^{2\times 2} \to \mathbb{C}^3$ is defined as $\pi : A \mapsto (a_{21}, \operatorname{tr} A, \det A)$. We define the *singular set* of $\mathcal{P} = \pi(\mathbb{B}^{2\times 2})$ to be the image under π of the set of critical points of π. Recall that the set of critical points of the map π is the set $\pi(\{A \in \mathbb{B}^{2\times 2} : \mathbf{J}_\pi(A)$ is not of full rank$\})$, where $\mathbf{J}_\pi(A)$ is the Jacobian matrix of π.

Proposition 3.5.1. *The singular set of the pentablock is $\mathcal{S}_{\mathcal{P}} = \{(0,s,p) \in \mathcal{P} : s^2 = 4p\}$.*

Proof. The Jacobian matrix of π, $\mathbf{J}_\pi(A)$, is defined by

$$\mathbf{J}_\pi(A) = \begin{bmatrix} \frac{\partial \pi_1}{\partial a_{11}} & \frac{\partial \pi_1}{\partial a_{12}} & \frac{\partial \pi_1}{\partial a_{21}} & \frac{\partial \pi_1}{\partial a_{22}} \\ \\ \frac{\partial \pi_2}{\partial a_{11}} & \frac{\partial \pi_2}{\partial a_{12}} & \frac{\partial \pi_2}{\partial a_{21}} & \frac{\partial \pi_2}{\partial a_{22}} \\ \\ \frac{\partial \pi_3}{\partial a_{11}} & \frac{\partial \pi_3}{\partial a_{12}} & \frac{\partial \pi_3}{\partial a_{21}} & \frac{\partial \pi_3}{\partial a_{22}} \end{bmatrix}$$

for

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{C}^{2\times 2}.$$

Thus,

$$\mathbf{J}_\pi(A) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ a_{22} & -a_{21} & -a_{12} & a_{11} \end{bmatrix}.$$

Note that $\mathbf{J}_\pi(A)$ is not of full rank if and only if rank $\mathbf{J}_\pi(A) \leq 2$. That means all 3×3 minors of $\mathbf{J}_\pi(A)$ are zero. Let us find all 3×3 minors of $\mathbf{J}_\pi(A)$.

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ a_{22} & -a_{21} & -a_{12} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ a_{22} & -a_{21} \end{vmatrix} = -a_{21},$$

$$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_{21} & -a_{12} & a_{11} \end{vmatrix} = -\begin{vmatrix} 0 & 1 \\ -a_{21} & a_{11} \end{vmatrix} = -a_{21},$$

$$\begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ a_{22} & -a_{21} & a_{11} \end{vmatrix} = 0,$$

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ a_{22} & -a_{12} & a_{11} \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ a_{22} & a_{11} \end{vmatrix} = -a_{11} + a_{22}.$$

Thus $\mathbf{J}_\pi(A)$ is not of full rank if and only if $a_{21} = 0$ and $a_{11} = a_{22}$.
Therefore,

$$\begin{aligned} \mathcal{S}_{\mathcal{P}} = \pi(\{A \in \mathbb{B}^{2\times 2} : \mathbf{J}_\pi(A) \text{ is not of full rank}\}) &= \pi\left(A = \begin{bmatrix} a & * \\ 0 & a \end{bmatrix} \in \mathbb{B}^{2\times 2}\right) \\ &= \left\{\left(0, 2a, a^2\right) : a \in \mathbb{D}\right\} \\ &= \left\{\left(0, s, p\right) : (s,p) \in \mathbb{G}, s^2 = 4p\right\}. \end{aligned}$$

Here $s = \text{tr } A$ and $p = \det A$. □

By analogy with the established terminology for the symmetrized bidisc, we define the *royal variety* of the pentablock as

$$\mathcal{R}_{\overline{\mathcal{P}}} = \{(0, s, p) \in \mathbb{C}^3 : s^2 = 4p\}.$$

Remark 3.5.2. *The singular set of the pentablock can be presented as* $\mathcal{S}_{\mathcal{P}} = \{(0, s, p) \in$

$\mathcal{P} : (s,p) \in \mathcal{R}_\Gamma \cap \mathbb{G}\}$.

Lemma 3.5.3. *Let $(a,s,p) \in b\overline{\mathcal{P}}$. Then the following conditions are equivalent:*

(i) $a = 0$;

(ii) $(a,s,p) \in b\overline{\mathcal{P}} \cap \mathcal{R}_{\overline{\mathcal{P}}}$;

(iii) $|s| = 2$.

Proof. By assumption, $(a,s,p) \in b\overline{\mathcal{P}}$, that is, $|s| \leq 2$, $|p| = 1$, $s = \overline{s}p$ and $|a| = \sqrt{1 - \frac{1}{4}|s|^2}$.
(i) $\Leftrightarrow$ (iii) Suppose $a = 0$, then $\sqrt{1 - \frac{1}{4}|s|^2} = 0$. Hence $|s| = 2$. The converse is obvious.
(i) $\Rightarrow$ (ii) Suppose $a = 0$, it implies that $|s|^2 = 4$, and so $s\overline{s} = 4$. Thus $s\overline{s}p = 4p$. Since $s = \overline{s}p$, we have $s^2 = 4p$. Recall that $\mathcal{R}_{\overline{\mathcal{P}}} = \{(0,s,p) : (s,p) \in \mathbb{G},\ s^2 = 4p\}$. Hence $(0,s,p) \in b\overline{\mathcal{P}} \cap \mathcal{R}_{\overline{\mathcal{P}}}$.
(ii) $\Rightarrow$ (iii) Suppose $(a,s,p) \in b\overline{\mathcal{P}} \cap \mathcal{R}_{\overline{\mathcal{P}}}$. Thus $a = 0$. Since $(a,s,p) \in b\overline{\mathcal{P}}$ and $a = 0$, we get $0 = |a| = \sqrt{1 - \frac{1}{4}|s|^2}$. Hence $|s| = 2$. □

The automorphism group of $\mathcal{P}$. Recall the known information on the automorphism group Aut $\mathcal{P}$ of $\mathcal{P}$. For $w \in \mathbb{T}$ and $v \in$ Aut $\mathbb{D}$, let

$$f_{wv}(a,s,p) = \left(\frac{w\eta(1-|\alpha|^2)a}{1-\overline{\alpha}s+\overline{\alpha}^2p}, \tau_v(s,p)\right) \tag{3.4}$$

where $v = \eta B_\alpha$ for $\alpha \in \mathbb{D}$, $\eta \in \mathbb{T}$, $B_\alpha(z) = \dfrac{z-\alpha}{1-\overline{\alpha}z}$ is a Blaschke factor and $\tau_v \in$ Aut $\mathbb{G}$ is defined by

$$\tau_v(z+w, zw) = \big(v(z)+v(w), v(z)v(w)\big).$$

Theorem 3.5.4. [4, Theorem 7.1] *The maps f_{wv}, for $w \in \mathbb{T}$ and $v \in$ Aut $\mathbb{D}$, constitute a group of automorphisms of $\mathcal{P}$ under composition. Each automorphism f_{wv} extends analytically to a neighbourhood of $\overline{\mathcal{P}}$.*
Moreover, for all $w_1, w_2 \in \mathbb{T}$, $v_1, v_2 \in$ Aut $\mathbb{D}$,

$$f_{w_1v_1} \circ f_{w_2v_2} = f_{(w_1w_2)(v_1\circ v_2)},$$

and, for all $w \in \mathbb{T}$, $v \in$ Aut $\mathbb{D}$,

$$(f_{wv})^{-1} = f_{\overline{w}v^{-1}}.$$

L. Kosiński [38] has proven that the set $\{f_{wv} : w \in \mathbb{T}, v \in \text{Aut } \mathbb{D}\}$ is the full group of automorphisms of $\mathcal{P}$.

Lemma 3.5.5. *$\mathcal{R}_{\overline{\mathcal{P}}} \cap \mathcal{P}$ is invariant under* Aut $\mathcal{P}$.

Proof. Every element of $\mathcal{R}_{\overline{\mathcal{P}}} \cap \mathcal{P}$ is of the form $(0, s, p) \in \mathcal{P}$ where $s^2 = 4p$. By Theorem 3.5.4, any element of Aut $\mathcal{P}$ has the form: for $w \in \mathbb{T}$ and $v \in$ Aut $\mathbb{D}$,

$$f_{wv}(a, s, p) = \Big(\frac{w\eta(1 - |\alpha|^2)a}{1 - \overline{\alpha}s + \overline{\alpha}^2 p}, \tau_v(s, p)\Big),$$

where $v = \eta B_\alpha$ for $\alpha \in \mathbb{D}$, $\eta \in \mathbb{T}$, $B_\alpha(z) = \dfrac{z - \alpha}{1 - \overline{\alpha}z}$ is a Blaschke factor and $\tau_v \in$ Aut $\mathbb{G}$. It is easy to see that when $a = 0$, the first component of $f_{wv}(a, s, p)$, namely, $\dfrac{w\eta(1 - |\alpha|^2)a}{1 - \overline{\alpha}s + \overline{\alpha}^2 p} = 0.$
Thus

$$f_{wv}(0, s, p) = \Big(0, \tau_v(s, p)\Big).$$

Since $\tau_v \in$ Aut $\mathbb{G}$ and $(s, p) \in \mathcal{R}_\Gamma \cap \mathbb{G}$, by Lemma 2.1.7, $\tau_v(s, p) \in \mathcal{R}_\Gamma \cap \mathbb{G}$. By definition, $\mathcal{R}_{\overline{\mathcal{P}}} \cap \overline{\mathcal{P}} = \big\{(0, s, p) \in \overline{\mathcal{P}} : s^2 = 4p\big\}$. Therefore, $f_{wv}(0, s, p) \in \mathcal{R}_{\overline{\mathcal{P}}} \cap \mathcal{P}$. □

Definition 3.5.6. *Let U be a domain in $\mathbb{C}^n$ and let $\mathcal{D} \subset U$. We say $\mathcal{D}$ is a complex geodesic in U if there exists a function $k \in \mathrm{Hol}(\mathbb{D}, U)$ and a function $C \in \mathrm{Hol}(U, \mathbb{D})$ such that $C \circ k = \mathrm{id}_{\mathbb{D}}$ and $\mathcal{D} = k(\mathbb{D})$.*

For a geometric classification of complex geodesics in the symmetrized bidisc $\mathbb{G}$, see [7].

Lemma 3.5.7. *$\mathcal{R}_{\overline{\mathcal{P}}} \cap \mathcal{P}$ is a complex geodesic in $\mathcal{P}$.*

Proof. Define the analytic functions k and c by

$$k : \mathbb{D} \to \mathcal{P},\ k(\lambda) = (0, -2\lambda, \lambda^2)$$

and

$$c : \mathcal{P} \to \mathbb{D},\ c(a, s, p) = \Phi_w(s, p) = \frac{2wp - s}{2 - ws},\quad \omega \in \mathbb{T}.$$

For $\lambda \in \mathbb{D}$,

$$(c \circ k)(\lambda) = c(k(\lambda)) = c(0, -2\lambda, \lambda^2) = \frac{2w\lambda^2 + 2\lambda}{2 + 2w\lambda} = \frac{2\lambda(w\lambda + 1)}{2(w\lambda + 1)} = \lambda, \text{ which means}$$

$$c \circ k = \mathrm{id}_{\mathbb{D}}\,.$$

By the definition of $\mathcal{R}_{\overline{\mathcal{P}}} \cap \mathcal{P}$, it is easy to see that, $\mathcal{R}_{\overline{\mathcal{P}}} \cap \mathcal{P} = k(\mathbb{D})$. Therefore $\mathcal{R}_{\overline{\mathcal{P}}} \cap \mathcal{P}$ is a complex geodesic in $\mathcal{P}$. □

Chapter 4. $\overline{\mathcal{P}}$-inner functions

4.1 Inner and outer functions in $H^\infty(\mathbb{D})$

In this section we recall definitions of inner and outer functions in $H^p(\mathbb{D}),\ 0 < p \le \infty$ from [44, Chapter III].

Definition 4.1.1. *$H^p(\mathbb{D}),\ 0 < p \le \infty$ is the Hardy class of analytic functions u on $\mathbb{D}$ such that the corresponding norm*

$$\|u\|_p = \begin{cases} \sup\limits_{0<r<1} \left[\frac{1}{2\pi}\int_0^{2\pi} |u(re^{it})|^p \, dt\right]^{\frac{1}{p}} & \text{if } 0 < p < \infty, \\ \sup\limits_{\lambda\in\mathbb{D}} |u(\lambda)| & \text{if } p = \infty \end{cases}$$

is finite.

Definition 4.1.2. *We call* outer function *every function on $\mathbb{D}$ which admits a representation of the form*

$$u(\lambda) = c\exp\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+\lambda}{e^{it}-\lambda}\log k(t)\, dt\right], \quad \lambda \in \mathbb{D},$$

where

$$k(t) \ge 0, \quad \log k(t) \in L^1, \ \text{(integrable function)}$$

and c is a complex number of modulus 1.

The outer function u in Definition 4.1.2 belongs to $H^p(\mathbb{D}),\ 0 < p \le \infty$ if and only if k belongs to the Lebesgue space L^p; in this case

$$|u(e^{it})| = k(t) \ \text{ a.e.}$$

Recall that a *rational inner function* is a rational map f from the open unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$ to its closure $\overline{\mathbb{D}}$ with the property that f maps the unit circle $\mathbb{T}$ into itself. See [23] for a survey of results, linking inner functions and operator theory.

Definition 4.1.3. *An* inner function *is an analytic map $f : \mathbb{D} \to \overline{\mathbb{D}}$ such that the radial limit*

$$\lim_{r\to 1^-} f(r\lambda)$$

exists and belongs to $\mathbb{T}$ *for almost all* $\lambda \in \mathbb{T}$ *with respect to Lebesgue measure.*

Definition 4.1.4. [8, page 2] *A* finite Blaschke product *is a function of the form*

$$B(z) = c \prod_{i=1}^{n} \frac{z - \alpha_i}{1 - \overline{\alpha_i} z} \qquad for\ z \in \mathbb{C} \setminus \{1/\overline{\alpha_1}, \dots, 1/\overline{\alpha_n}\},$$

where $|c| = 1$ *and* $\alpha_1, \dots, \alpha_n \in \mathbb{D}$.

Theorem 4.1.5. [32, Theorem 3] *The rational inner functions on* $\mathbb{D}$ *are precisely the finite Blaschke products.*

One can see that the only functions which are at the same time inner and outer are the constant functions of modulus 1.

The class of the outer functions belonging to $H^p(\mathbb{D})$ will be denoted by E^p.

Theorem 4.1.6. [44] *Every function* $u \in H^p(\mathbb{D})$, $0 < p \leq \infty$ *such that* $u \neq 0$, *has a "canonical" factorization*

$$u = u_{in} u_{out}$$

into the product of an inner function u_{in} *and an outer function* u_{out}, *which are uniquely determined up to constant factors of modulus 1. The function* u_{out} *belongs to the class* E^p *and is given by the formula*

$$u_{out}(\lambda) = c \exp\left[\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + \lambda}{e^{it} - \lambda} \log |u(e^{it})|\, dt\right], \quad \lambda \in \mathbb{D}, \tag{4.1}$$

where $|c| = 1$; u_{in} *and* u_{out} *are called the* inner factor *and the* outer factor *of* u, *respectively.*

Remark 4.1.7. *From equation* (4.1) *it follows easily that if* u, v *and* uv *belong to Hardy classes and do not vanish identically, then*

$$(uv)_{out} = u_{out} v_{out} \quad and \quad (uv)_{in} = u_{in} v_{in};$$

this holds in particular if $u \in H^\infty(\mathbb{D})$ *and* $v \in H^p(\mathbb{D})$, *because then* $uv \in H^p(\mathbb{D})$.

In this thesis we will study properties of inner functions from $\mathbb{D}$ to $\overline{\mathcal{P}}$.

Definition 4.1.8. *A* $\overline{\mathcal{P}}$*-inner or penta-inner function is an analytic map* $f : \mathbb{D} \to \overline{\mathcal{P}}$ *such that the radial limit*

$$\lim_{r \to 1^-} f(r\lambda)$$

exists and belongs to $b\overline{\mathcal{P}}$ *for almost all* $\lambda \in \mathbb{T}$ *with respect to Lebesgue measure.*

Remark 4.1.9. *Let* $f : \mathbb{D} \to \overline{\mathcal{P}}$ *be a rational* $\overline{\mathcal{P}}$*-inner function. Since* f *is rational and bounded on* $\mathbb{D}$ *it has no poles in* $\overline{\mathbb{D}}$ *and hence* f *extends to a continuous function on* $\overline{\mathbb{D}}$. *Thus one can consider the continuous function*

$$\tilde{f} : \mathbb{T} \to b\overline{\mathcal{P}}, \ \ where\ \tilde{f}(\lambda) = \lim_{r \to 1^-} f(r\lambda)\ for\ all\ \lambda \in \mathbb{T}.$$

4.2 Examples of $\overline{\mathcal{P}}$-inner functions

Example 4.2.1. Let us consider an example of an analytic function $f : \mathbb{D} \to \overline{\mathcal{P}}$. Consider the analytic map $h : \mathbb{D} \to \mathbb{B}^{2\times 2}$ defined by

$$h(\lambda) = \begin{bmatrix} \varphi(\lambda) & 0 \\ 0 & \psi(\lambda) \end{bmatrix} \quad \text{for } \lambda \in \mathbb{D}, \tag{4.2}$$

where $\varphi, \psi \in H^\infty(\mathbb{D})$ are nonconstant inner functions.
Note that $\|h(\lambda)\| = \max\{|\varphi(\lambda)|, |\psi(\lambda)|\} < 1$ for $\lambda \in \mathbb{D}$. Recall that $\pi : \mathbb{C}^{2\times 2} \to \mathbb{C}^3$ is defined by $\pi(A) = (a_{21}, \text{tr } A, \det A)$. By Definition 3.1.1, for all $\lambda \in \mathbb{D}$,

$$f(\lambda) = \pi(h(\lambda)) = (0, \text{tr } h(\lambda), \det h(\lambda)) \in \mathcal{P}.$$

Let, for $\lambda \in \mathbb{D}$, $a(\lambda) = 0$, $s(\lambda) = \text{tr } h(\lambda) = \varphi(\lambda) + \psi(\lambda)$ and $p(\lambda) = \det h(\lambda) = \varphi(\lambda)\psi(\lambda)$. Clearly, $f : \mathbb{D} \to \overline{\mathcal{P}}$ is an analytic function.

Let us check when f is $\overline{\mathcal{P}}$-inner. We need to check that $f(\lambda) \in b\overline{\mathcal{P}}$ for almost every $\lambda \in \mathbb{T}$, that is, $\big(s(\lambda), p(\lambda)\big) \in b\Gamma$ and $\sqrt{1 - \frac{1}{4}|s(\lambda)|^2} = 0$ for almost every $\lambda \in \mathbb{T}$. Note that, for almost every $\lambda \in \mathbb{T}$,

$$|p(\lambda)| = |\varphi(\lambda)\psi(\lambda)| = |\varphi(\lambda)||\psi(\lambda)| = 1, \quad \text{since } \ |\varphi(\lambda)| = 1 \text{ and } \ |\psi(\lambda)| = 1,$$

$$|s(\lambda)| = |\varphi(\lambda) + \psi(\lambda)| \leq |\varphi(\lambda)| + |\psi(\lambda)| = 2,$$

and

$$\begin{aligned} (\bar{s}p)(\lambda) &= \big(\overline{\varphi(\lambda) + \psi(\lambda)}\big)\big(\varphi(\lambda)\psi(\lambda)\big) = \varphi(\lambda)\overline{\varphi(\lambda)}\psi(\lambda) + \varphi(\lambda)\psi(\lambda)\overline{\psi(\lambda)} \\ &= |\varphi(\lambda)|^2\psi(\lambda) + \varphi(\lambda)|\psi(\lambda)|^2 = \varphi(\lambda) + \psi(\lambda) = s(\lambda). \end{aligned}$$

Hence for almost every $\lambda \in \mathbb{T}$, $|p(\lambda)| = 1$, $|s(\lambda)| \leq 2$ and $(\bar{s}p)(\lambda) = s(\lambda)$, and so $\big(\text{tr } h(\lambda), \det h(\lambda)\big) \in b\Gamma$.
Now, for almost every $\lambda \in \mathbb{T}$,

$$\begin{aligned} 1 - \frac{1}{4}|s(\lambda)|^2 &= 1 - \frac{1}{4}|\varphi(\lambda) + \psi(\lambda)|^2 = 1 - \frac{1}{4}\Big(\big(\varphi(\lambda) + \psi(\lambda)\big)\big(\overline{\varphi(\lambda) + \psi(\lambda)}\big)\Big) \\ &= 1 - \frac{1}{4}\Big(1 + 1 + 2\text{Re}\big(\varphi(\lambda)\overline{\psi(\lambda)}\big)\Big) = 1 - \frac{1}{2} - \frac{1}{2}\text{Re}\big(\varphi(\lambda)\overline{\psi(\lambda)}\big) \\ &= \frac{1}{2} - \frac{1}{2}\text{Im}\big(i\varphi(\lambda)\overline{\psi(\lambda)}\big) \end{aligned}$$

Hence $|a| = \sqrt{1 - \frac{1}{4}|s|^2}$ almost everywhere on $\mathbb{T}$ if and only if

$$\frac{1}{2} - \frac{1}{2}\text{Im}\big(i\varphi(\lambda)\overline{\psi(\lambda)}\big) = 0, \text{ for almost every } \lambda \in \mathbb{T},$$

if and only if $\operatorname{Im}(i\varphi(\lambda)\overline{\psi(\lambda)}) = 1$ for almost every $\lambda \in \mathbb{T}$. Therefore, $|a| = \sqrt{1 - \frac{1}{4}|s|^2}$ almost everywhere on $\mathbb{T}$ if and only if $\varphi(\lambda)\overline{\psi(\lambda)} = 1$ almost everywhere on $\mathbb{T}$, and so, $\varphi(\lambda) = \psi(\lambda)$ almost everywhere on $\mathbb{T}$. Thus the function f is a $\overline{\mathcal{P}}$-inner function only when $\varphi = \psi$.

Example 4.2.2. Let $h_1 : \mathbb{D} \to \mathbb{C}^{2\times 2}$ be defined by $h_1 = Uh$, where h is defined by equation (4.2) and

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i \end{bmatrix} \text{ is a unitary matrix.}$$

Then, for $\lambda \in \mathbb{D}$,

$$\begin{aligned} h_1(\lambda) &= Uh(\lambda) \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} \varphi(\lambda) & 0 \\ 0 & \psi(\lambda) \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} \varphi(\lambda) & \psi(\lambda) \\ i\varphi(\lambda) & -i\psi(\lambda) \end{bmatrix}. \end{aligned}$$

Note that, for all $\lambda \in \mathbb{D}$,

$$\|h_1(\lambda)\| \le \|h(\lambda)\| = \max\{|\varphi(\lambda)|, |\psi(\lambda)|\} < 1,$$

since U is unitary.
Hence $h_1(\lambda) \in \mathbb{B}^{2\times 2}$ for all $\lambda \in \mathbb{D}$. Define $f_1 = \pi \circ h_1$ on $\mathbb{D}$. Then, for $\lambda \in \mathbb{D}$,

$$\begin{aligned} f_1(\lambda) &= \pi(h_1(\lambda)) \\ &= \pi\left(\frac{1}{\sqrt{2}} \begin{bmatrix} \varphi(\lambda) & \psi(\lambda) \\ i\varphi(\lambda) & -i\psi(\lambda) \end{bmatrix} \right) \\ &= \left(\frac{i\varphi(\lambda)}{\sqrt{2}}, \frac{\varphi(\lambda) - i\psi(\lambda)}{\sqrt{2}}, -i\varphi(\lambda)\psi(\lambda) \right). \end{aligned} \tag{4.3}$$

Clearly, $f_1 : \mathbb{D} \to \overline{\mathcal{P}}$ is an analytic function since φ, ψ are analytic on $\mathbb{D}$.

Let us check when the function f_1 is $\overline{\mathcal{P}}$-inner. We need to find conditions when f_1 maps $\mathbb{T}$ into the distinguished boundary $b\overline{\mathcal{P}}$ of $\mathcal{P}$. Since φ, ψ are inner functions, they have unit modulus almost everywhere on $\mathbb{T}$. Thus one can see that, for $s = \dfrac{\varphi - i\psi}{\sqrt{2}}$ and $p = -i\varphi\psi$, for almost every $\lambda \in \mathbb{T}$,

$$|p(\lambda)| = |-i\varphi(\lambda)\psi(\lambda)| = |\varphi(\lambda)||\psi(\lambda)| = 1.$$

$$|s(\lambda)| = \left| \frac{\varphi(\lambda) - i\psi(\lambda)}{\sqrt{2}} \right| = \frac{|\varphi(\lambda) + (-i\psi(\lambda))|}{\sqrt{2}} \le \frac{|\varphi(\lambda)| + |-i\psi(\lambda)|}{\sqrt{2}} = \frac{2}{\sqrt{2}}.$$

$$\begin{aligned}(\bar{s}p)(\lambda) &= \frac{\overline{\varphi(\lambda) - i\psi(\lambda)}}{\sqrt{2}}(-i\varphi(\lambda)\psi(\lambda)) = \frac{\overline{\varphi(\lambda)} - i\overline{\psi(\lambda)}}{\sqrt{2}}(-i\varphi(\lambda)\psi(\lambda)) \\ &= \frac{-i\varphi(\lambda)\overline{\varphi(\lambda)}\psi(\lambda) + \overline{\psi(\lambda)}\psi(\lambda)\varphi(\lambda)}{\sqrt{2}} = \frac{-i|\varphi(\lambda)|^2\psi(\lambda) + \varphi(\lambda)|\psi(\lambda)|^2}{\sqrt{2}} \\ &= \frac{\varphi(\lambda) - i\psi(\lambda)}{\sqrt{2}} = s(\lambda).\end{aligned}$$

Therefore, for almost all $\lambda \in \mathbb{T}, |p(\lambda)| = 1$, $|s(\lambda)| \le 2$ and $(\bar{s}p)(\lambda) = s(\lambda)$ and so $(s(\lambda), p(\lambda)) \in b\Gamma$. Finally,

$$\begin{aligned}\sqrt{1 - \frac{1}{4}|s(\lambda)|^2} &= \sqrt{1 - \frac{1}{4}\left(\frac{1}{2}|\varphi(\lambda) - i\psi(\lambda)|^2\right)} = \sqrt{1 - \frac{1}{8}\left(1 + 1 + 2\mathrm{Re}(\varphi(\lambda)i\overline{\psi(\lambda)})\right)} \\ &= \frac{1}{2}\sqrt{4 - 1 - \mathrm{Re}(i\varphi(\lambda)\overline{\psi(\lambda)})} = \frac{1}{2}\sqrt{3 + \mathrm{Im}(\varphi(\lambda)\overline{\psi(\lambda)})}.\end{aligned}$$

We want $|a| = \sqrt{1 - \frac{1}{4}|s|^2}$ almost everywhere on $\mathbb{T}$, that is, for $\lambda \in \mathbb{T}$,

$$\frac{1}{\sqrt{2}} = \frac{|\varphi(\lambda)|}{\sqrt{2}} = \frac{1}{2}\sqrt{3 + \mathrm{Im}(\varphi(\lambda)\overline{\psi(\lambda)})}.$$

Hence $|a| = \sqrt{1 - \frac{1}{4}|s|^2}$ almost everywhere on $\mathbb{T}$ if and only if $\sqrt{3 + \mathrm{Im}(\varphi(\lambda)\overline{\psi(\lambda)})} = \sqrt{2}$, or equivalently $\mathrm{Im}(\varphi(\lambda)\overline{\psi(\lambda)}) = -1$. Thus $|a| = \sqrt{1 - \frac{1}{4}|s|^2}$ almost everywhere on $\mathbb{T}$ if and only if $\varphi\overline{\psi} = -i$ almost everywhere on $\mathbb{T}$. Therefore f_1 given by equation (4.3) is a $\overline{\mathcal{P}}$-inner function if and only if $\varphi = -i\psi$ almost everywhere on $\mathbb{T}$.

Example 4.2.3. Let v, φ and ψ be inner functions on $\mathbb{D}$. Consider

$$V(\lambda) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & v(\lambda) \\ -1 & v(\lambda)\end{bmatrix} \quad \text{and} \quad h(\lambda) = \begin{bmatrix}\varphi(\lambda) & 0 \\ 0 & \psi(\lambda)\end{bmatrix}, \text{ for } \lambda \in \mathbb{D}.$$

Define

$$\begin{aligned}U(\lambda) &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ \overline{v(\lambda)} & \overline{v(\lambda)}\end{bmatrix}\begin{bmatrix}\varphi(\lambda) & 0 \\ 0 & \psi(\lambda)\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & v(\lambda) \\ -1 & v(\lambda)\end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ \overline{v(\lambda)} & \overline{v(\lambda)}\end{bmatrix}\begin{bmatrix}\varphi(\lambda) & \varphi(\lambda)v(\lambda) \\ -\psi(\lambda) & \psi(\lambda)v(\lambda)\end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix}\varphi(\lambda) + \psi(\lambda) & (\varphi(\lambda) - \psi(\lambda))v(\lambda) \\ (\varphi(\lambda) - \psi(\lambda))\overline{v(\lambda)} & \varphi(\lambda) + \psi(\lambda)\end{bmatrix}, \quad \text{for } \lambda \in \mathbb{D}.\end{aligned}$$

Note that

$$\|V^*(\lambda)\| = \left\|\begin{bmatrix} 1 & 0 \\ 0 & \overline{v(\lambda)}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1\end{bmatrix}\right\| \le 1$$

since $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is unitary and $|v(\lambda)| \leq 1$ for $\lambda \in \mathbb{D}$. Hence

$$\|U(\lambda)\| \leq \|V(\lambda)\|^2 \|h(\lambda)\| < 1 \text{ for } \lambda \in \mathbb{D}.$$

Define $f : \mathbb{D} \to \overline{\mathcal{P}}$ by $f = \pi \circ U$. Then, for $\lambda \in \mathbb{D}$,

$$f(\lambda) = (a(\lambda), s(\lambda), p(\lambda)) = \Big(\frac{1}{2}(\varphi - \psi)\overline{v}, \varphi + \psi, \frac{1}{4}\big((\varphi + \psi)^2 - (\varphi - \psi)^2 |v|^2\big)\Big)(\lambda). \quad (4.4)$$

Note that f is analytic on $\mathbb{D}$ if and only if v is constant or $\varphi = \psi$.
(i) Consider the case when v is constant. As v is inner, $|v| = 1$. Let us check that $f : \mathbb{D} \to \overline{\mathcal{P}}$ is $\overline{\mathcal{P}}$-inner, that is, $f(\mathbb{T}) \subset b\overline{\mathcal{P}}$. By definition

$$s(\lambda) = (\varphi + \psi)(\lambda)$$

and

$$p(\lambda) = \frac{1}{4}\big((\varphi + \psi)^2 - (\varphi - \psi)^2 |v|^2\big)(\lambda) = (\varphi\psi)(\lambda).$$

Thus, for almost all $\lambda \in \mathbb{T}$, $(\varphi + \psi, \varphi\psi)(\lambda) \in b\Gamma$ as in Example 4.2.1. For $\lambda \in \mathbb{T}$,

$$|a|^2(\lambda) = \frac{1}{4}|\varphi - \psi|^2(\lambda) = \frac{1}{4}\Big(1 + 1 - 2\mathrm{Re}(\overline{\varphi}\psi)\Big)(\lambda) = \frac{1}{2} - \frac{1}{2}\mathrm{Re}(\overline{\varphi}\psi)(\lambda).$$

$$\begin{aligned} \Big(1 - \frac{1}{4}|s|^2\Big)(\lambda) &= \Big(1 - \frac{1}{4}|\varphi + \psi|^2\Big)(\lambda) = 1 - \frac{1}{4}\Big(1 + 1 + 2\mathrm{Re}(\overline{\varphi}\psi)\Big)(\lambda) \\ &= \frac{1}{2} - \frac{1}{2}\mathrm{Re}(\overline{\varphi}\psi)(\lambda) = |a|^2(\lambda). \end{aligned}$$

Thus $|a|^2 = 1 - \frac{1}{4}|s|^2$ almost everywhere on $\mathbb{T}$, and so, f given by equation (4.4) is a $\overline{\mathcal{P}}$-inner function in the case that v is constant.
(ii) Consider the case when $\varphi = \psi$. Then

$$f(\lambda) = \Big(0, 2\varphi, \frac{1}{4}(2\varphi)^2\Big)(\lambda) = (0, 2\varphi, \varphi^2)(\lambda).$$

Thus, for almost every $\lambda \in \mathbb{T}$,

$$|p(\lambda)| = |\varphi(\lambda)^2| = 1,$$

$$|s(\lambda)| = 2|\varphi(\lambda)| = 2,$$

$$(\overline{s}p)(\lambda) = 2\overline{\varphi}(\lambda)\varphi(\lambda)^2 = 2|\varphi(\lambda)|^2\varphi(\lambda) = 2\varphi(\lambda) = s(\lambda), \text{ and}$$

$$1 - \frac{1}{4}|s(\lambda)|^2 = 1 - \frac{1}{4}(4) = 0 = |a(\lambda)|^2.$$

Thus for almost all $\lambda \in \mathbb{T}$, $f(\lambda) \in b\overline{\mathcal{P}}$, and so, f is a $\overline{\mathcal{P}}$-inner function in the case that $\varphi = \psi$.

Example 4.2.4. Define the function $x(\lambda) = (\lambda^m, 0, \lambda) : \mathbb{D} \to \overline{\mathcal{P}}$. First we need to show that for all $\lambda \in \mathbb{D}$, $x(\lambda) \in \overline{\mathcal{P}}$.
By Proposition 2.1.3, $(s,p) \in \Gamma$ if and only if

$$|s| \leq 2 \ \text{ and } |s - \bar{s}p| \leq 1 - |p|^2.$$

It is easy to see that $(0, \lambda) \in \Gamma$. By Theorem 3.2.4, if $(s,p) \in \Gamma$, then for $a \in \mathbb{C}$, $(a, s, p) \in \overline{\mathcal{P}}$ if and only if

$$|a| \leq |1 - \frac{\frac{1}{2}s\overline{\beta}}{1 + \sqrt{1 - |\beta|^2}}|, \quad \text{where} \quad \beta = \frac{s - \bar{s}p}{1 - |p|^2}. \tag{4.5}$$

In the case $s = 0$, equation (4.5) is equivalent to $|a| \leq 1$.
Note that $x(\lambda) = \big(a(\lambda), s(\lambda), p(\lambda)\big) = (\lambda^m, 0, \lambda)$, $\lambda \in \mathbb{D}$, is analytic in $\mathbb{D}$, and

$$|a(\lambda)| = |\lambda^m| \leq 1, \quad \text{for all} \quad \lambda \in \mathbb{D}.$$

Thus for $\lambda \in \mathbb{D}$, $x(\lambda) \in \overline{\mathcal{P}}$.

Now, let us check if x is a $\overline{\mathcal{P}}$-inner function, that is, x maps $\mathbb{T}$ into the distinguished boundary $b\overline{\mathcal{P}}$ of $\mathcal{P}$. For all $\lambda \in \mathbb{T}$,

$$|p(\lambda)| = |\lambda| = 1, \ |s(\lambda)| = |0| \leq 2,$$

$$(\bar{s}p)(\lambda) = 0 = s(\lambda) \ \text{ and}$$

$$|a(\lambda)| = |\lambda^m| = \sqrt{1 - \frac{1}{4}|s(\lambda)|^2} = 1.$$

Therefore for every $\lambda \in \mathbb{T}$, $x(\lambda) \in b\overline{\mathcal{P}}$ and hence $x : \mathbb{D} \to \overline{\mathcal{P}} : \lambda \longmapsto (\lambda^m, 0, \lambda)$ is a rational $\overline{\mathcal{P}}$-inner function.

Example 4.2.5. For $\lambda \in \mathbb{D}$, define the function $x(\lambda) = (\lambda, 0, \lambda^n)$, where $n = 1, 2, \ldots$. As in the previous example, for all $\lambda \in \mathbb{D}$, by Proposition 2.1.3, $(0, \lambda^n) \in \Gamma$. For $a \in \mathbb{C}$, we want

$$|a| \leq |1 - \frac{\frac{1}{2}s\overline{\beta}}{1 + \sqrt{1 - |\beta|^2}}| \tag{4.6}$$

Since $s = 0$, the condition (4.6) is equivalent to $|a| \leq 1$. Note that

$$|a(\lambda)| = |\lambda| \leq 1, \ \text{for all} \ \ \lambda \in \mathbb{D}.$$

Thus, by Theorem 3.2.4, for $\lambda \in \mathbb{D}$, $x(\lambda) \in \overline{\mathcal{P}}$.

Now, let us check if x is a $\overline{\mathcal{P}}$-inner function, that is, x maps $\mathbb{T}$ into the distinguished boundary $b\overline{\mathcal{P}}$ of $\mathcal{P}$. For all $\lambda \in \mathbb{T}$,

$$|p(\lambda)| = |\lambda^n| = 1, \; |s(\lambda)| = |0| \leq 2,$$

$$(\bar{s}p)(\lambda) = 0 = s(\lambda) \;\; \text{and}$$

$$|a(\lambda)| = |\lambda| = \sqrt{1 - \frac{1}{4}|s(\lambda)|^2} = 1.$$

Therefore for every $\lambda \in \mathbb{T}$, $x(\lambda) \in b\overline{\mathcal{P}}$ and hence $x : \mathbb{D} \to \overline{\mathcal{P}} : \lambda \longmapsto (\lambda, 0, \lambda^n)$ is a rational $\overline{\mathcal{P}}$-inner function.

4.3 Some properties of analytic functions $x : \mathbb{D} \to \overline{\mathcal{P}}$

Lemma 4.3.1. (i) *Let $x = (a, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ be an analytic function. Then $h = (s, p) : \mathbb{D} \to \Gamma$ is an analytic function.*
(ii) *Let $x = (a, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ be a $\overline{\mathcal{P}}$-inner function. Then $h = (s, p) : \mathbb{D} \to \Gamma$ is a Γ-inner function.*

Proof. (i) By assumption, $x = (a, s, p)$ is analytic on $\mathbb{D}$ and for all $\lambda \in \mathbb{D}$, $x(\lambda) = (a(\lambda), s(\lambda), p(\lambda)) \in \overline{\mathcal{P}}$. By Remark 2.1.2, for all $\lambda \in \mathbb{D}$, $(s(\lambda), p(\lambda)) \in \Gamma$. Thus $h = (s, p) : \mathbb{D} \to \Gamma$, where $h(\lambda) = (s(\lambda), p(\lambda))$, for $\lambda \in \mathbb{D}$, is well-defined and analytic from $\mathbb{D}$ to Γ.
(ii) By assumption $x = (a, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ is a penta-inner function, and so, for almost all $\lambda \in \mathbb{T}$, $x(\lambda) \in b\overline{\mathcal{P}}$. Recall $b\overline{\mathcal{P}} = \Big\{(a, s, p) \in \mathbb{C}^3 : (s, p) \in b\Gamma, \; |a| = \sqrt{1 - \frac{1}{4}|s|^2}\Big\}$. By Theorem 3.4.2, for almost all $\lambda \in \mathbb{T}$, $h(\lambda) = \big(s(\lambda), p(\lambda)\big) \in b\Gamma$. Hence h is a Γ-inner function. □

Recall that, by Proposition 3.4.1, $K_1 = \left\{(a, s, p) \in \overline{\mathcal{P}} : (s, p) \in b\Gamma, \; |a| \leq \sqrt{1 - \frac{1}{4}|s|^2}\right\}$ is a closed boundary of $A(\mathcal{P})$.

Proposition 4.3.2. (i) *Let $h = (s, p) : \mathbb{D} \to \Gamma$ be an analytic function. Then $x = (0, s, p)$ is an analytic function from $\mathbb{D}$ to $\overline{\mathcal{P}}$.*
(ii) *Let $h = (s, p) : \mathbb{D} \to \Gamma$ be a Γ-inner function. Then $x = (0, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ is an analytic function such that, for almost all $\lambda \in \mathbb{T}$, $x(\lambda) \in K_1$.*

Proof. (i) It follows from Theorem 3.2.4 that, for all $\lambda \in \mathbb{D}$, $\big(0, s(\lambda), p(\lambda)\big) \in \overline{\mathcal{P}}$.
(ii) Suppose that h is a Γ-inner function. By Proposition 2.1.3, $|p(\lambda)| = 1, \; |s(\lambda)| \leq 2$ and $(\bar{s}p)(\lambda) = s(\lambda)$, for almost all $\lambda \in \mathbb{T}$. Since $a = 0$ and $\sqrt{1 - \frac{1}{4}|s(\lambda)|^2} \geq 0$ for almost all $\lambda \in \mathbb{T}$, $x(\mathbb{T}) \subset K_1$. □

Proposition 4.3.3. *Let $x = (a, s, p) \in \mathrm{Hol}(\mathbb{D}, \overline{\mathcal{P}})$. Let $x_1 = (a_{out}, s, p)$. Then $x_1 \in \mathrm{Hol}(\mathbb{D}, \overline{\mathcal{P}})$, where $a_{in}\, a_{out}$ is the inner-outer factorization of a.*

Proof. By assumption, for each $\lambda \in \mathbb{D}$, $(a(\lambda), s(\lambda), p(\lambda)) \in \overline{\mathcal{P}}$. By Theorem 3.2.4, for all $\lambda \in \mathbb{D}$, $|\Psi_z(a(\lambda), s(\lambda), p(\lambda))| \leq 1$ for all $z \in \mathbb{D}$. Thus

$$\left| \frac{a(\lambda)(1-|z|^2)}{1 - s(\lambda)z + p(\lambda)z^2} \right| \leq 1, \quad \text{for all } \lambda, z \in \mathbb{D}.$$

Recall that $a = a_{in}a_{out}$, and so

$$\left| a_{in}(\lambda) \frac{a_{out}(\lambda)(1-|z|^2)}{1 - s(\lambda)z + p(\lambda)z^2} \right| \leq 1, \quad \text{for all } \lambda, z \in \mathbb{D}.$$

The function a_{in} is inner, and so $|a_{in}(\lambda)| = 1$ for almost all $\lambda \in \mathbb{T}$. Therefore, for every $z \in \mathbb{D}$, and for almost all $\lambda \in \mathbb{T}$,

$$\left| \frac{a_{out}(\lambda)(1-|z|^2)}{1 - s(\lambda)z + p(\lambda)z^2} \right| \leq 1.$$

Note that, for every $z \in \mathbb{D}$, the function

$$\lambda \mapsto \frac{a_{out}(\lambda)(1-|z|^2)}{1 - s(\lambda)z + p(\lambda)z^2}$$

is analytic on $\mathbb{D}$. By the maximum principle, for every $z \in \mathbb{D}$, $\left| \frac{a_{out}(\lambda)(1-|z|^2)}{1 - s(\lambda)z + p(\lambda)z^2} \right| \leq 1$, for all $\lambda \in \mathbb{D}$. Hence, by Theorem 3.2.4, for each $\lambda \in \mathbb{D}$, $(a_{out}(\lambda), s(\lambda), p(\lambda)) \in \overline{\mathcal{P}}$. Therefore, $x_1 = (a_{out}, s, p) \in \mathrm{Hol}(\mathbb{D}, \overline{\mathcal{P}})$. $\square$

Example 4.3.4. Let $x(\lambda) = (\lambda, 0, \lambda)$. Then, by Example 4.2.5, $x \in \mathrm{Hol}(\mathbb{D}, \overline{\mathcal{P}})$. By Proposition 4.3.3, $(1, 0, \lambda) \in \mathrm{Hol}(\mathbb{D}, \overline{\mathcal{P}})$.

Proposition 4.3.5. *Let $x = (a, s, p)$ be a $\overline{\mathcal{P}}$-inner function. Let $a_{in}\ a_{out}$ be the inner-outer factorization of a. Then $\widetilde{x} = (a_{out}, s, p)$ is a $\overline{\mathcal{P}}$-inner function.*

Proof. By assumption $x = (a, s, p)$ is a $\overline{\mathcal{P}}$-inner function. Then, by Proposition 4.3.3, $\widetilde{x} = (a_{out}, s, p) \in \mathrm{Hol}(\mathbb{D}, \overline{\mathcal{P}})$. To prove the statement, we must show that, for almost all $\lambda \in \mathbb{T}$, $\widetilde{x}(\lambda) = (a_{out}(\lambda), s(\lambda), p(\lambda)) \in b\overline{\mathcal{P}}$. Recall that

$$b\overline{\mathcal{P}} = \left\{ (a, s, p) \in \mathbb{C}^3 : (s, p) \in b\Gamma, \ |a| = \sqrt{1 - \frac{1}{4}|s|^2} \right\}.$$

By Lemma 4.3.1, for almost all $\lambda \in \mathbb{T}$, $(s(\lambda), p(\lambda)) \in b\Gamma$. Since $x = (a, s, p)$ is a $\overline{\mathcal{P}}$-inner function, we have, for almost all $\lambda \in \mathbb{T}$,

$$|a(\lambda)| = \sqrt{1 - \frac{1}{4}|s(\lambda)|^2}.$$

Since $a_{in}a_{out} = a$ is the inner-outer factorization of a and $|a_{in}(\lambda)| = 1$ for almost all $\lambda \in \mathbb{T}$,

$$|a_{out}(\lambda)| = \sqrt{1 - \frac{1}{4}|s(\lambda)|^2} \text{ for almost all } \lambda \in \mathbb{T}.$$

Therefore $\widetilde{x} = (a_{out}, s, p)$ is a $\overline{\mathcal{P}}$-inner function. $\square$

Example 4.3.6. We have shown in Example 4.2.5 that $x : \mathbb{D} \to \overline{\mathcal{P}} : \lambda \longmapsto (\lambda, 0, \lambda^n)$ is a $\overline{\mathcal{P}}$-inner function. By Proposition 4.3.5, $x_1 = (1, 0, \lambda^n)$ is a $\overline{\mathcal{P}}$-inner function.

4.4 The degree of a rational $\overline{\mathcal{P}}$-inner function

Lemma 4.4.1. *$b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}}$ is homotopic to $\mathbb{T} \times \mathbb{T}$ and $\pi_1(b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}}) = \mathbb{Z} \times \mathbb{Z}$.*

Proof. Recall that if $(a, s, p) \in b\overline{\mathcal{P}}$, then $|s| \leq 2$, $|p| = 1$, $s = \bar{s}p$ and $|a| = \sqrt{1 - \frac{1}{4}|s|^2}$. By the definition, $\mathcal{R}_{\overline{\mathcal{P}}} = \{(0, s, p) \in \overline{\mathcal{P}} : s^2 = 4p\}$, hence $b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}} = \{(a, s, p) \in b\overline{\mathcal{P}} : a \neq 0, (s, p) \in b\Gamma\}$. Define the maps f and g by

$$f : b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}} \to \mathbb{T} \times \mathbb{T},\ f(a, s, p) = (\frac{a}{|a|}, p)$$

and

$$g : \mathbb{T} \times \mathbb{T} \to b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}},\ g(z, w) = (z, 0, w).$$

We need to show that

$$f \circ g : \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}$$

is homotopic to the identity map

$$id : \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T},$$

and

$$g \circ f : b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}} \to b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}}$$

is homotopic to

$$id : b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}} \to b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}}.$$

For $(z, w) \in \mathbb{T} \times \mathbb{T}$,

$$(f \circ g)(z, w) = f(g(z, w)) = f(z, 0, w) = (z, w), \text{ which means } f \circ g = id_{\mathbb{T}\times\mathbb{T}}.$$

For $(a, s, p) \in b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}}$,

$$(g \circ f)(a, s, p) = g(f(a, s, p)) = g(\frac{a}{|a|}, p) = (\frac{a}{|a|}, 0, p).$$

Now, if $(a,s,p) \in b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}}$ and $t \in [0,1]$ we have

$$\left| a \frac{\sqrt{1-\frac{1}{4}\left|ts\right|^2}}{|a|} \right| = \frac{|a| \left| \sqrt{1-\frac{1}{4}\left|ts\right|^2} \right|}{|a|} = \sqrt{1-\frac{1}{4}|ts|^2}. \tag{4.7}$$

Since $(s,p) \in b\Gamma$ and $t \in [0,1]$, we have

$$|p| = 1, \ |s| \le 2, \ s = \bar{s}p, \text{ and so, } |p| = 1, \ ts = t\bar{s}p, \ |ts| \le 2.$$

Therefore

$$\left(a\frac{\sqrt{1-\frac{1}{4}|ts|^2}}{|a|}, ts, p \right) \in b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}}.$$

Let $I = [0,1]$, consider the continuous map

$$h : b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}} \times I \to b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}} \text{ which is defined by}$$

$$h(a,s,p,t) = \left(a\frac{\sqrt{1-\frac{1}{4}|ts|^2}}{|a|}, ts, p \right).$$

For $(a,s,p) \in b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}}$,

$$h(a,s,p,0) = (\frac{a}{|a|}, 0, p) = (g \circ f)(a,s,p) \text{ and}$$

$$h(a,s,p,1) = \left(a\frac{\sqrt{1-\frac{1}{4}|s|^2}}{|a|}, s, p \right) = (a,s,p) = id_{b\overline{\mathcal{P}}\setminus\mathcal{R}_{\overline{\mathcal{P}}}}(a,s,p).$$

Therefore, h defines a homotopy between $g \circ f$ and $id_{b\overline{\mathcal{P}}\setminus\mathcal{R}_{\overline{\mathcal{P}}}}$, that is, $g \circ f \simeq id_{b\overline{\mathcal{P}}\setminus\mathcal{R}_{\overline{\mathcal{P}}}}$. Hence $b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}}$ is homotopically equivalent to $\mathbb{T} \times \mathbb{T}$ and it follows that $\pi_1(b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}}) = \pi_1(\mathbb{T} \times \mathbb{T}) = \mathbb{Z} \times \mathbb{Z}$. □

Define $\widetilde{b\mathcal{P}} = b\mathcal{P} \setminus \{(0, 2w, w^2) : w \in \mathbb{T}\}$. By Lemma 4.4.1, $\widetilde{b\mathcal{P}}$ is homotopic to $\mathbb{T} \times \mathbb{T}$.

Definition 4.4.2. *The degree of a rational $\overline{\mathcal{P}}$-inner function $x = (a,s,p)$ is defined to be the pair of numbers $(\deg a, \deg p)$. We say that $\deg x \le (m,n)$ if $\deg a \le m$ and $\deg p \le n$.*

Proposition 4.4.3. *Let $x = (a,s,p)$ be a rational $\overline{\mathcal{P}}$-inner function such that $x(\mathbb{T})$ does not meet $\mathcal{R}_{\overline{\mathcal{P}}}$. Then $\deg(x)$ is equal to $x_*(1)$, where $x_* : \mathbb{Z} = \pi_1(\mathbb{T}) \to \pi_1(b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}})$ is the homomorphism of fundamental groups induced by x when x is regarded as a continuous map from $\mathbb{T}$ to $b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}}$.*

Proof. By assumption, $x(\mathbb{T}) \cap \mathcal{R}_{\overline{\mathcal{P}}} = \emptyset$ and x is a rational $\overline{\mathcal{P}}$-inner function, and so $x(\mathbb{T}) \subset b\overline{\mathcal{P}}$. By Lemma 3.5.3, $a(\lambda) \neq 0$ on $\mathbb{T}$. Consider two functions $x = (a, s, p)$ and $y = \left(\frac{a}{|a|}, 0, p\right)$ as continuous maps from $\mathbb{T}$ to $b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}}$. x and y are said to be homotopic if there exists a continuous mapping

$$f : \mathbb{T} \times I \to b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}}$$

such that

$$f(\lambda, 0) = y(\lambda) \;\; \textit{and} \;\; f(\lambda, 1) = x(\lambda), \quad \text{for all } \lambda \in \mathbb{T}.$$

Let

$$x^t(\lambda) = \left(\left(\frac{a}{|a|}\sqrt{1 - \frac{1}{4}|ts|^2}\right)(\lambda), ts(\lambda), p(\lambda)\right)$$

for $\lambda \in \mathbb{T}$ and $t \in [0, 1]$. Note that $x^t(\lambda)$ is a continuous function of $(t, \lambda) \in I \times \overline{\mathbb{D}}$.
Since $x(\lambda) \in b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}}$ for all $\lambda \in \mathbb{T}$, by Theorem 3.4.2, Proposition 3.5.1, and by equation (4.7),

$$x^t(\lambda) \in b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}} \text{ for almost all } \lambda \in \mathbb{T}.$$

Hence x^t is a homotopy between $x = (a, s, p) = x^1$ and $y = (a, 0, p) = x^0$.
It follows that the homomorphism

$$x_* : \pi_1(\mathbb{T}) = \mathbb{Z} \to \pi_1(b\overline{\mathcal{P}} \setminus \mathcal{R}_{\overline{\mathcal{P}}}) = \mathbb{Z} \times \mathbb{Z}$$

coincides with

$$(x^0)_* = \left(\frac{a}{|a|}, 0, p\right)_*.$$

Therefore the degree $\deg x = (\deg a, \deg p) = \left(\left(\frac{a}{|a|}\right)_*(1), p_*(1)\right)$ is equal to $x_*(1)$. $\square$

4.5 Connections between rational Γ-inner and rational $\overline{\mathcal{P}}$-inner functions

Theorem 4.5.1. (Fejér-Riesz theorem) [42, Section 53] *If $f(\lambda) = \sum_{i=-n}^{n} a_i \lambda^i$ is a trigonometric polynomial of degree n such that $f(\lambda) \geq 0$ for all $\lambda \in \mathbb{T}$, then there exists an analytic polynomial $D(\lambda) = \sum_{i=0}^{n} b_i \lambda^i$ of degree n such that D is outer (that is, $D(\lambda) \neq 0$ for all $\lambda \in \mathbb{D}$) and*

$$f(\lambda) = |D(\lambda)|^2$$

for all $\lambda \in \mathbb{T}$.

Recall that for every $a \neq 0$ in $H^\infty(\mathbb{D})$ there is an outer-inner factorization. Rational inner functions can be written in the form $c \prod_{i=1}^{n} B_{\alpha_i}$ for some $n \geq 1$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{D}$ and $c \in \mathbb{T}$, see equation (2.3) for the definition of B_α.

The next theorem provides a description of the structure of rational penta-inner functions of prescribed degree.

Theorem 4.5.2. *Let $x = (a, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ be a rational penta-inner function of degree (m, n). Let $a \neq 0$ and let an inner-outer factorization of a be given by $a = a_{in}a_{out}$, where a_{in} is an inner function and a_{out} is an outer function. Then there exist polynomials A, E, D such that*

(1) $\deg(A), \deg(E), \deg(D) \leq n$,

(2) $E^{\sim n} = E$,

(3) $D(\lambda) \neq 0$ *for all* $\lambda \in \overline{\mathbb{D}}$,

(4) $|E(\lambda)| \leq 2|D(\lambda)|$ *for all* $\lambda \in \overline{\mathbb{D}}$,

(5) A *is an outer polynomial such that* $|A(\lambda)|^2 = |D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2$ *for* $\lambda \in \mathbb{T}$,

(6) $a = a_{in}\dfrac{A}{D}$ *on* $\overline{\mathbb{D}}$,

(7) $s = \dfrac{E}{D}$ *on* $\overline{\mathbb{D}}$,

(8) $p = \dfrac{D^{\sim n}}{D}$ *on* $\overline{\mathbb{D}}$.

Proof. Suppose that $x = (a, s, p)$ is a rational penta-inner function. By Lemma 4.3.1, $h = (s, p)$ is a rational Γ-inner function. By [3, Corollary 6.10], p can be written in the form

$$p(\lambda) = c\frac{\lambda^k D^{\sim(n-k)}(\lambda)}{D(\lambda)}$$

where $|c| = 1$, $0 \leq k \leq n$ and D is a polynomial of degree $n - k$ such that $D(0) = 1$. Therefore, by Proposition 2.1.12, there exist polynomials E and D such that

(1) deg(E), deg(D) $\leq n$,

(2) $E^{\sim n} = E$,

(3) $D(\lambda) \neq 0$ for all $\lambda \in \overline{\mathbb{D}}$,

(4) $|E(\lambda)| \leq 2|D(\lambda)|$ for all $\lambda \in \overline{\mathbb{D}}$,

(5) $s = \dfrac{E}{D}$ on $\overline{\mathbb{D}}$,

(6) $p = \dfrac{D^{\sim n}}{D}$ on $\overline{\mathbb{D}}$.

By assumption $x = (a, s, p)$ is a $\overline{\mathcal{P}}$-inner function, and so, for almost all $\lambda \in \mathbb{T}$, $(a(\lambda), s(\lambda), p(\lambda)) \in b\overline{\mathcal{P}}$, which implies

$$|a_{out}(\lambda)|^2 = 1 - \frac{1}{4}|s(\lambda)|^2, \text{ since } |a_{in}| = 1 \text{ almost everywhere on } \mathbb{T}.$$

Thus

$$|a_{out}(\lambda)|^2 = 1 - \frac{1}{4}\frac{|E(\lambda)|^2}{|D(\lambda)|^2} \quad \text{since } s(\lambda) = \frac{E(\lambda)}{D(\lambda)},$$

and so,

$$|a_{out}(\lambda)|^2 |D(\lambda)|^2 = |D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2. \tag{4.8}$$

By Proposition 2.1.12, $|E(\lambda)| \leq 2|D(\lambda)|$. By the Fejér-Riesz Theorem, since $|D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2 \geq 0$, there exists an analytic polynomial A of degree $\leq n$ such that A is outer and

$$|A(\lambda)|^2 = |D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2 \tag{4.9}$$

for all $\lambda \in \mathbb{T}$.

From equations (4.8) and (4.9) we have, $|A(\lambda)|^2 = |a_{out}(\lambda)|^2 |D(\lambda)|^2$. Note that $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$. Thus $|a_{out}(\lambda)| = \left|\frac{A}{D}(\lambda)\right|$ for $\lambda \in \mathbb{T}$, and so $\frac{A}{D}$ is an outer function such that $|a(\lambda)| = \left|\frac{A}{D}(\lambda)\right|$ for almost all $\lambda \in \mathbb{T}$. Since outer factors are unique up to unimodular constant multiples, there exists $\omega \in \mathbb{T}$ such that

$$a_{out}(\lambda) = \omega \frac{A(\lambda)}{D(\lambda)}.$$

Therefore $a = a_{in}\dfrac{A}{D}$ on $\overline{\mathbb{D}}$. $\square$

Remark 4.5.3. *Results similar to Theorem 4.5.2 were proved in* [37] *using different methods.*

Example 4.5.4. Let $x = (a, s, p)$ be a rational $\overline{\mathcal{P}}$-inner function such that $x(\lambda) = (\lambda^m, 0, \lambda)$ for $\lambda \in \mathbb{D}$. It is easy to see that $E(\lambda) = 0$, since $s(\lambda) = \dfrac{E(\lambda)}{D(\lambda)} = 0$.

By Theorem 4.5.2,

$$p(\lambda) = \frac{D^{\sim 1}(\lambda)}{D(\lambda)}, \text{ for } \lambda \in \overline{\mathbb{D}}.$$

We are given, $p(\lambda) = \lambda$, and so, deg $p = 1$. Thus $p(\lambda) = \dfrac{D^{\sim 1}(\lambda)}{D(\lambda)}$ implies $D(\lambda) = 1$.

Note, for $D(\lambda) = 1$, $D^{\sim 1}(\lambda) = \lambda^1 \cdot 1 = \lambda$.

By assumption $x = (a, s, p)$ is a $\overline{\mathcal{P}}$-inner function, and so, for all $\lambda \in \mathbb{T}$,

$$|a(\lambda)|^2 = 1 - \frac{1}{4}|s(\lambda)|^2 = 1.$$

Since $a(\lambda) = \lambda^m$ is inner, $a_{in} = a$ and so $a_{in}(\lambda) = \lambda^m$ and $A(\lambda) = 1$, for $\lambda \in \overline{\mathbb{D}}$.

Every rational Γ-inner function $h = \left(\frac{E}{D}, \frac{D^{\sim n}}{D}\right)$ such that $h(\overline{\mathbb{D}}) \nsubseteq \mathcal{R}_\Gamma \cap \Gamma$ allows us to construct a family of rational $\overline{\mathcal{P}}$-inner functions.

Theorem 4.5.5. *Let $h = \left(\frac{E}{D}, \frac{D^{\sim n}}{D}\right)$ be a rational Γ-inner function, where E, D are polynomials such that $\deg(E)$, $\deg(D) \le n$, $E^{\sim n} = E, |E(\lambda)| \le 2|D(\lambda)|$ on $\overline{\mathbb{D}}$ and $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$. Let A be an outer polynomial such that*

$$|A(\lambda)|^2 = |D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2. \tag{4.10}$$

Then, for every finite Blaschke product B and $|c| = 1$, $x = \left(cB\frac{A}{D}, \frac{E}{D}, \frac{D^{\sim n}}{D}\right)$ is a rational $\overline{\mathcal{P}}$-inner function.

Proof. Let a, s, p be defined by

$$a = cB\frac{A}{D}, \quad s = \frac{E}{D} \quad \text{and } p = \frac{D^{\sim n}}{D}.$$

Let us show that $x = (a, s, p)$ is a rational $\overline{\mathcal{P}}$-inner function. We have to prove that $x : \mathbb{D} \to \overline{\mathcal{P}}$ and, for almost all $\lambda \in \mathbb{T}$, $x(\lambda) \in b\overline{\mathcal{P}}$.
By assumption $h = (s, p) : \mathbb{D} \to \Gamma$ is a rational Γ-inner function, which means $|p(\lambda)| = 1$, $|s(\lambda)| \le 2$ and $(\overline{s}p)(\lambda) = s(\lambda)$, for almost all $\lambda \in \mathbb{T}$. Now we need to show that for almost all $\lambda \in \mathbb{T}$, $|a(\lambda)| = \sqrt{1 - \frac{1}{4}|s(\lambda)|^2}$. For almost all $\lambda \in \mathbb{T}$,

$$\begin{aligned}
|a(\lambda)|^2 &= \left|cB(\lambda)\frac{A(\lambda)}{D(\lambda)}\right|^2 = \frac{|A(\lambda)|^2}{|D(\lambda)|^2} \quad (\text{since } |c| = 1 \text{ and } |B(\lambda)| = 1 \text{ on } \mathbb{T}) \\
&= \frac{|D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2}{|D(\lambda)|^2} = 1 - \frac{1}{4}\left|\frac{E(\lambda)}{D(\lambda)}\right|^2 \\
&= 1 - \frac{1}{4}|s(\lambda)|^2.
\end{aligned}$$

Let us show that $x = (a, s, p) = \left(cB\frac{A}{D}, \frac{E}{D}, \frac{D^{\sim n}}{D}\right)$ maps $\mathbb{D}$ to $\overline{\mathcal{P}}$, that is, $x(\lambda) = (a(\lambda), s(\lambda), p(\lambda)) \in \overline{\mathcal{P}}$ for all $\lambda \in \mathbb{D}$. By Theorem 3.2.4, for each $\lambda \in \mathbb{D}$, $x(\lambda) \in \overline{\mathcal{P}}$ if and only if $|\Psi_z(x(\lambda))| \le 1$ for all $z \in \mathbb{D}$, where

$$\begin{aligned}
&\Psi_z(x(.)) : \mathbb{D} \to \mathbb{C} \\
&\qquad \lambda \mapsto (1 - |z|^2)\, \frac{a(\lambda)}{1 - s(\lambda)z + p(\lambda)z^2}.
\end{aligned}$$

Note that $1 - s(\lambda)z + p(\lambda)z^2 \neq 0$ for all $z \in \mathbb{D}$ since $(s(\lambda), p(\lambda)) \in \Gamma$. By the construction, $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$, and so $(a(\lambda), s(\lambda), p(\lambda))$ is analytic on $\mathbb{D}$. Hence, for every $z \in \mathbb{D}$, $\Psi_z(x(.))$ is analytic on $\mathbb{D}$. By the maximum principle, to prove that $|\Psi_z(x(\lambda))| \le 1$ for all $\lambda \in \mathbb{D}$, it suffices to show that $|\Psi_z(x(\lambda))| \leq 1$ for all $\lambda \in \mathbb{T}$. We have shown above that, for almost all $\lambda \in \mathbb{T}$, $(a(\lambda), s(\lambda), p(\lambda)) \in b\overline{\mathcal{P}}$. Thus, for all $\lambda \in \mathbb{T}$, $|a(\lambda)| = \sqrt{1 - \frac{1}{4}|s(\lambda)|^2}$, $|p(\lambda)| = 1$, $|s(\lambda)| \leq 2$ and $s(\lambda) = \overline{s(\lambda)}p(\lambda)$, and so $(s(\lambda), p(\lambda)) = (\beta + \overline{\beta}p, p)(\lambda) \in b\Gamma$, where $\beta(\lambda) = \frac{1}{2}s(\lambda)$. One can see that, for all $\lambda \in \mathbb{T}$,

$$\begin{aligned}
\left|1 - \frac{\frac{1}{2}s(\lambda)\overline{\beta}(\lambda)}{1 + \sqrt{1 - |\beta(\lambda)|^2}}\right| &= \left|1 - \frac{\frac{1}{4}|s(\lambda)|^2}{1 + \sqrt{1 - \frac{1}{4}|s(\lambda)|^2}}\right| \\
&= \left|\frac{1 + \sqrt{1 - \frac{1}{4}|s(\lambda)|^2} - \frac{1}{4}|s(\lambda)|^2}{1 + \sqrt{1 - \frac{1}{4}|s(\lambda)|^2}}\right| \\
&= \left|\frac{\sqrt{1 - \frac{1}{4}|s(\lambda)|^2}\left(1 + \sqrt{1 - \frac{1}{4}|s(\lambda)|^2}\right)}{1 + \sqrt{1 - \frac{1}{4}|s(\lambda)|^2}}\right| \\
&= \sqrt{1 - \frac{1}{4}|s(\lambda)|^2} = |a(\lambda)|.
\end{aligned}$$

By Theorem 3.2.4 (3) $\Leftrightarrow$ (5), for each $\lambda \in \mathbb{T}$,

$$|a(\lambda)| \leq \left|1 - \frac{\frac{1}{2}s(\lambda)\overline{\beta}(\lambda)}{1 + \sqrt{1 - |\beta(\lambda)|^2}}\right| \quad \text{if and only if } |\Psi_z(a(\lambda), s(\lambda), p(\lambda))| \leq 1.$$

Hence, by the maximum principle, for all $z, \lambda \in \mathbb{D}$, $|\Psi_z(a(\lambda), s(\lambda), p(\lambda))| \leq 1$. Thus, by Theorem 3.2.4, $x(\lambda) = (a(\lambda), s(\lambda), p(\lambda)) \in \overline{\mathcal{P}}$ for all $\lambda \in \mathbb{D}$. $\square$

Theorem 4.5.6. (Converse to Theorem 4.5.2) *Suppose polynomials A, E, D satisfy*

(1) $\deg(A), \deg(E), \deg(D) \leq n$,

(2) $E^{\sim n} = E$,

(3) $D(\lambda) \neq 0$ *for all* $\lambda \in \overline{\mathbb{D}}$,

(4) $|E(\lambda)| \leq 2|D(\lambda)|$ *for all* $\lambda \in \overline{\mathbb{D}}$,

(5) A *is an outer polynomial such that* $|A(\lambda)|^2 = |D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2$ *for* $\lambda \in \mathbb{T}$,

(6) a_{in} *is a rational inner function on* $\mathbb{D}$ *of degree* $\leq m$.

Let a, s, p be defined by

$$a = a_{in}\frac{A}{D}, \quad s = \frac{E}{D} \quad \textit{and} \quad p = \frac{D^{\sim n}}{D} \quad \textit{on } \overline{\mathbb{D}}.$$

Then $x = (a, s, p)$ is a rational $\overline{\mathcal{P}}$-inner function of degree less than or equal $(m + n, n)$.

Proof. By the converse of Proposition 2.1.12, $h = (s, p)$, where

$$s = \frac{E}{D} \quad \text{and} \quad p = \frac{D^{\sim n}}{D},$$

is a rational Γ-inner function of degree at most n. Since the rational inner functions on $\mathbb{D}$ are precisely the finite Blaschke products, the statement of the theorem follows from Theorem 4.5.5. $\square$

Proposition 4.5.7. *Let $x = (0, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ be a rational $\overline{\mathcal{P}}$-inner function. Then $x = (0, 2\varphi, \varphi^2)$, for some rational inner function $\varphi : \mathbb{D} \to \overline{\mathbb{D}}$. Moreover, $x(\lambda) \in \mathcal{R}_{\overline{\mathcal{P}}} \cap \overline{\mathcal{P}}$ for all $\lambda \in \mathbb{D}$ and $x(\lambda) \in b\overline{\mathcal{P}} \cap \mathcal{R}_{\overline{\mathcal{P}}}$ for almost all $\lambda \in \mathbb{T}$.*

Proof. By assumption $x = (0, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ is a rational $\overline{\mathcal{P}}$-inner function. Hence for almost all $\lambda \in \mathbb{T}$, $x(\lambda) \in b\overline{\mathcal{P}}$. By Lemma 3.5.3, since $a = 0$, for almost all $\lambda \in \mathbb{T}$, $x(\lambda) = (a, s, p)(\lambda) \in b\overline{\mathcal{P}} \cap \mathcal{R}_{\overline{\mathcal{P}}}$ and $|s(\lambda)| = 2$.
Thus, for almost all $\lambda \in \mathbb{T}$, the following conditions hold: $s(\lambda)\overline{s(\lambda)} = 4$, $|p(\lambda)| = 1$ and $s(\lambda) = \overline{s(\lambda)}p(\lambda)$. These imply that $s(\lambda)\overline{s(\lambda)}p(\lambda) = 4p(\lambda)$ and $s(\lambda)^2 = 4p(\lambda)$ for almost all $\lambda \in \mathbb{T}$. Hence $p(\lambda) = \frac{1}{4}s(\lambda)^2$ for all $\lambda \in \mathbb{D}$. Since p is a rational inner function from $\mathbb{D} \to \overline{\mathbb{D}}$, $\frac{1}{2}s$ is also a rational inner function from $\mathbb{D} \to \overline{\mathbb{D}}$. Thus there exists a rational inner function

$$\varphi : \mathbb{D} \to \overline{\mathbb{D}} \ \text{ such that } \ \frac{1}{2}s = \varphi \ \text{ and } \ p = \varphi^2.$$

Thus, $x(\lambda) = (0, 2\varphi(\lambda), \varphi(\lambda)^2)$ for all $\lambda \in \mathbb{D}$ and $x(\lambda) \in \mathcal{R}_{\overline{\mathcal{P}}} \cap \overline{\mathcal{P}}$ for all $\lambda \in \mathbb{D}$. $\square$

Proposition 4.5.8. *Let $x = (0, s, p)$ be a $\overline{\mathcal{P}}$-inner function. Then $x(\lambda) \in \mathcal{R}_{\overline{\mathcal{P}}}$ for all $\lambda \in \overline{\mathbb{D}}$ and $x(\lambda) \in b\overline{\mathcal{P}} \cap \mathcal{R}_{\overline{\mathcal{P}}}$ for all $\lambda \in \mathbb{T}$.*

Proof. By assumption $x = (0, s, p)$ is a $\overline{\mathcal{P}}$-inner function. Hence, for almost all $\lambda \in \mathbb{T}$, $x(\lambda) \in b\overline{\mathcal{P}}$. By Lemma 3.5.3, since $a = 0$, for almost all $\lambda \in \mathbb{T}$, $x(\lambda) \in b\overline{\mathcal{P}} \cap \mathcal{R}_{\overline{\mathcal{P}}}$. Let $f = s^2 - 4p$. Then f is analytic on $\mathbb{D}$. Since, for almost all $\lambda \in \mathbb{T}$, $x(\lambda) = (0, s(\lambda), p(\lambda)) \in \mathcal{R}_{\overline{\mathcal{P}}}$, we have $f(\lambda) = (s^2 - 4p)(\lambda) = 0$ almost everywhere on $\mathbb{T}$. By Cauchy's integral formula, for any point $z_0 \in \mathbb{D}$,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(z)}{z - z_0} dz \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{s^2(z) - 4p(z)}{z - z_0} dz \\ &= 0. \end{aligned}$$

Therefore, for all $\lambda \in \overline{\mathbb{D}}$, $(s^2 - 4p)(\lambda) = 0$ and $x(\lambda) \in \mathcal{R}_{\overline{\mathcal{P}}} \cap \overline{\mathcal{P}}$. $\square$

4.6 Construction of rational $\overline{\mathcal{P}}$-inner functions

In this section we describe an algorithm for the construction of rational $\overline{\mathcal{P}}$-inner function from certain interpolation data.

Definition 4.6.1. [6, Definition 3.4] *We say that a polynomial f is n-symmetric if* $\deg(f) \leq n$ *and $f^{\sim n} = f$. For any set $E \subset \mathbb{C}$,* $\operatorname{ord}_E(f)$ *will denote the number of zeros of f in E, counted with multiplicity, and* $\operatorname{ord}_0(f)$ *will mean the same as* $\operatorname{ord}_{\{0\}}(f)$.

Definition 4.6.2. [6, Definition 4.1] *A nonzero polynomial R is n-balanced if* $\deg(R) \leq 2n$, *R is $2n$-symmetric and $\lambda^{-n}R(\lambda) \geq 0$ for all $\lambda \in \mathbb{T}$.*

Lemma 4.6.3. [6, Lemma 4.4] *For $\sigma \in \overline{\mathbb{D}}$, let the polynomial Q_σ be defined by the formula*

$$Q_\sigma(\lambda) = (\lambda - \sigma)(1 - \overline{\sigma}\lambda).$$

Let n be a positive integer and let R be a nonzero polynomial. R is n-balanced if and only if there exist points $\sigma_1, \sigma_2, \ldots, \sigma_n \in \overline{\mathbb{D}}$ and $t_+ > 0$ such that

$$R(\lambda) = t_+ \prod_{j=1}^{n} Q_{\sigma_j}(\lambda), \quad \lambda \in \mathbb{C}.$$

Lemma 4.6.4. [6, Lemma 4.6] *For $\tau = \mathrm{e}^{i\theta}$, $0 \leq \theta < 2\pi$, let L_τ be defined by*

$$L_\tau(\lambda) = i\mathrm{e}^{-i\frac{\theta}{2}}(\lambda - \tau).$$

Let n be a positive integer. A polynomial E is n-symmetric if and only if there exist points $\alpha_1, \alpha_2, \ldots, \alpha_{k_0} \in \mathbb{D}$, points $\tau_1, \tau_2, \ldots, \tau_{k_1} \in \mathbb{T}$ and $t \in \mathbb{R}$ such that

$$k_0 = \mathrm{ord}_0(E) + \mathrm{ord}_{\mathbb{D}\setminus\{0\}}(E),$$

$$k_1 = \mathrm{ord}_{\mathbb{T}}(E),$$

$$2k_0 + k_1 = n \quad \textit{and}$$

$$E(\lambda) = t \prod_{j=1}^{k_0} Q_{\alpha_j}(\lambda) \prod_{j=1}^{k_1} L_{\tau_j}(\lambda).$$

We next present a description of rational penta-inner functions (a, s, p) in terms of the zeros of a, s and $s^2 - 4p$.

Theorem 4.6.5. *Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_{k_0} \in \mathbb{D}$ and $\eta_1, \eta_2, \ldots, \eta_{k_1} \in \mathbb{T}$, where $2k_0 + k_1 = n$ and suppose that $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{D}$. Suppose that $\sigma_1, \ldots, \sigma_n$ in $\overline{\mathbb{D}}$ are distinct from $\eta_1, \ldots, \eta_{k_1}$. Then there exists a rational $\overline{\mathcal{P}}$-inner function $x = (a, s, p)$ of degree less than or equal $(m + n, n)$ such that*

(1) *the zeros of a in $\mathbb{D}$, repeated according to multiplicity, are $\beta_1, \beta_2, \ldots, \beta_m$,*

(2) *the zeros of s in $\overline{\mathbb{D}}$, repeated according to multiplicity, are $\alpha_1, \alpha_2, \ldots, \alpha_{k_0}$ and $\eta_1, \eta_2, \ldots, \eta_{k_1}$,*

(3) *the royal nodes of (s, p) are $\sigma_1, \ldots, \sigma_n$.*

Such a function x can be constructed as follows. Let $t_+ > 0$ and let $t \in \mathbb{R} \setminus \{0\}$. Let R and E be defined by

$$R(\lambda) = t_+ \prod_{j=1}^{n} (\lambda - \sigma_j)(1 - \overline{\sigma_j}\lambda),$$

$$E(\lambda) = t \prod_{j=1}^{k_0} (\lambda - \alpha_j)(1 - \overline{\alpha_j}\lambda) \prod_{j=1}^{k_1} i e^{-i\theta_j/2}(\lambda - \eta_j)$$

where $\eta_j = e^{i\theta_j}$, $0 \le \theta_j < 2\pi$ *for* $j = 1, \ldots, k_1$. *Let* $a_{in} : \mathbb{D} \to \overline{\mathbb{D}}$ *be defined by*

$$a_{in}(\lambda) = c \prod_{i=1}^{m} B_{\beta_i}(\lambda), \tag{4.11}$$

where $|c| = 1$, $\beta_i \in \mathbb{D}$, $i = 1, \ldots, m$ *and* $B_{\beta_i}(z) = \dfrac{z - \beta_i}{1 - \overline{\beta}_i z}$.

(i) *There exist outer polynomials* D *and* A *of degree at most* n *such that*

$$\lambda^{-n} R(\lambda) + |E(\lambda)|^2 = 4|D(\lambda)|^2 \tag{4.12}$$

and

$$\lambda^{-n} R(\lambda) = 4|A(\lambda)|^2 \tag{4.13}$$

for all $\lambda \in \mathbb{T}$.

(ii) *The function* x *defined by*

$$x = (a, s, p) = \left(a_{in} \frac{A}{D}, \frac{E}{D}, \frac{D^{\sim n}}{D} \right) \tag{4.14}$$

is a rational $\overline{\mathcal{P}}$*-inner function such that* $\deg(x) \le (m+n, n)$ *and conditions* (1), (2) *and* (3) *hold. The royal polynomial of* (s,p) *is* R.

Proof. (i) By Lemma 4.6.3, R is n-balanced, and so $\lambda^{-n} R(\lambda) \ge 0$ for all $\lambda \in \mathbb{T}$. Therefore

$$\lambda^{-n} R(\lambda) + |E(\lambda)|^2 \ge 0 \text{ for all } \lambda \in \mathbb{T}.$$

By the Fejér-Riesz theorem, there exist outer polynomials A and D of degree at most n such that

$$\lambda^{-n} R(\lambda) = 4|A(\lambda)|^2 \ \text{ for all } \lambda \in \mathbb{T}$$

and

$$\lambda^{-n} R(\lambda) + |E(\lambda)|^2 = 4|D(\lambda)|^2 \ \text{ for all } \lambda \in \mathbb{T}.$$

(ii) By Theorem 2.1.18, the function h defined by

$$h = (s, p) = \left(\frac{E}{D}, \frac{D^{\sim n}}{D} \right)$$

is a rational Γ-inner function such that $\deg(h) = n$ and conditions (2) and (3) hold. The royal polynomial of h is R.

By equations (4.12) and (4.13),

$$\begin{aligned} |A(\lambda)|^2 &= |D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2 \\ &= \frac{1}{4}\lambda^{-n}R(\lambda). \end{aligned}$$

Therefore, by Theorem 4.5.5,

$$x = \left(a_{in}\frac{A}{D}, \frac{E}{D}, \frac{D^{\sim n}}{D}\right)$$

is a rational $\overline{\mathcal{P}}$- inner function. We need to show that the inner and outer parts, a_{in} and $a_{out} = \omega\dfrac{A(\lambda)}{D(\lambda)}$, for some $\omega \in \mathbb{T}$, respectively, of the function a defined in equation (4.14), satisfy condition (1) of Theorem 4.6.5. By the definition (4.11), the zeros of a_{in} are $\beta_1, \ldots, \beta_m$, while, since A is an outer polynomial, A has no zeros in $\mathbb{D}$. Hence the zeros of $a = a_{in}\dfrac{A}{D}$ in $\mathbb{D}$ are $\beta_1, \ldots, \beta_m$, as required for (1). □

Example 4.6.6. Let $n = 1$, $\beta_1 = 0$, $\eta_1 = 1$ and $\sigma_1 = 0$. Let us construct a rational $\overline{\mathcal{P}}$-inner function $x = (a, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ such that β_1 is a zero of a in $\mathbb{D}$, η_1 is a zero of s and σ_1 is a royal node of (s, p).
As in Theorem 4.6.5, for $\lambda \in \mathbb{T}$, let

$$R(\lambda) = t_+\,\lambda, \qquad t_+ \text{ is a positive real number, and}$$

$$E(\lambda) = ti(\lambda - 1), \qquad t \in \mathbb{R}\setminus\{0\}.$$

Let

$$a_{in}(\lambda) = c\lambda, \; |c| = 1.$$

The equation (4.12) for the polynomial D is the following, for all $\lambda \in \mathbb{T}$,

$$\begin{aligned} |D(\lambda)|^2 &= \frac{1}{4}\{\lambda^{-1}R(\lambda) + |E(\lambda)|^2\} \\ &= \frac{1}{4}\{\overline{\lambda}t_+\lambda + |ti(\lambda-1)|^2\} \\ &= \frac{1}{4}\{t_+ + |t|^2(\lambda-1)(\overline{\lambda}-1)\} \\ &= \frac{1}{4}\{t_+ + |t|^2(2-\lambda-\overline{\lambda})\} \\ &= \frac{1}{4}t_+ + \frac{1}{2}|t|^2 - \frac{1}{4}|t|^2\lambda - \frac{1}{4}|t|^2\overline{\lambda}. \end{aligned} \tag{4.15}$$

Since the degree of D is at most 1, $D(\lambda) = a_1 + a_2\lambda$, where $a_1, a_2 \in \mathbb{C}$ and $\lambda \in \mathbb{T}$,

$$\begin{aligned} D(\lambda)\overline{D(\lambda)} &= |a_1 + a_2\lambda|^2 = (a_1 + a_2\lambda)(\overline{a_1} + \overline{a_2}\overline{\lambda}) \\ &= |a_1|^2 + |a_2|^2 + a_1\overline{a_2}\overline{\lambda} + \overline{a_1}a_2\lambda. \end{aligned} \tag{4.16}$$

Compare equations (4.15) and (4.16). We have

$$\begin{cases} \overline{a_1}a_2 = -\frac{1}{4}|t|^2, \\ a_1\overline{a_2} = -\frac{1}{4}|t|^2, \\ |a_1|^2 + |a_2|^2 = \frac{1}{4}t_+ + \frac{1}{2}|t|^2. \end{cases} \tag{4.17}$$

The equation (4.13) for the polynomial A is, for all $\lambda \in \mathbb{T}$,

$$\begin{aligned} |A(\lambda)|^2 &= \frac{1}{4}\lambda^{-1}R(\lambda) \\ &= \frac{1}{4}\overline{\lambda}\; t_+\; \lambda = \frac{1}{4}t_+. \end{aligned} \tag{4.18}$$

Since the degree of A is at most 1, $A(\lambda) = b_1 + b_2\lambda$, where $b_1, b_2 \in \mathbb{C}$ and $\lambda \in \mathbb{T}$,

$$A(\lambda)\overline{A(\lambda)} = |b_1|^2 + |b_2|^2 + b_1\overline{b_2}\;\overline{\lambda} + \overline{b_1}b_2\lambda. \tag{4.19}$$

Compare equations (4.18) and (4.19). We have

$$\begin{cases} \overline{b_1}b_2 = 0, \\ b_1\overline{b_2} = 0, \\ |b_1|^2 + |b_2|^2 = \frac{1}{4}t_+. \end{cases} \tag{4.20}$$

Finally the function x can be written in the form

$$x = \Big(a_{in}\frac{A}{D}, \frac{E}{D}, \frac{D^{\sim 1}}{D}\Big),$$

with

$$\begin{aligned} a(\lambda) &= a_{in}\frac{A}{D}(\lambda) = c\lambda\frac{b_1 + b_2\lambda}{a_1 + a_2\lambda}, \\ s(\lambda) &= \frac{E}{D}(\lambda) = \frac{ti(\lambda - 1)}{a_1 + a_2\lambda}, \\ p(\lambda) &= \frac{D^{\sim 1}}{D}(\lambda) = \frac{\overline{a_1}\lambda + \overline{a_2}}{a_1 + a_2\lambda}, \end{aligned}$$

where $|c| = 1$ and $a_i, b_i,\ i = 1, 2$ satisfy equations (4.17) and (4.20).

Theorem 4.6.7. *Let $x = (a, s, p)$ be a rational $\overline{\mathcal{P}}$-inner function of degree $(m+n, n)$ such that*

(1) *the zeros of a, repeated according to multiplicity, are $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{D}$,*

(2) *the zeros of s, repeated according to multiplicity, are $\alpha_1, \alpha_2, \ldots, \alpha_{k_0} \in \mathbb{D}$ and $\eta_1, \eta_2, \ldots, \eta_{k_1} \in \mathbb{T}$, where $2k_0 + k_1 = n$,*

(3) *the royal nodes of (s, p) are $\sigma_1, \ldots, \sigma_n$.*

There exists some choice of $c \in \mathbb{T}$, $t_+ > 0$, $t \in \mathbb{R} \setminus \{0\}$ *and* $\omega \in \mathbb{T}$ *such that the recipe in Theorem 4.6.5 with these choices produces the function* x.

Proof. By Lemma 4.3.1, $h = (s, p)$ is a rational Γ-inner function of degree n. As in Proposition 2.1.19, there exists some choice of $t_+ > 0$, $t \in \mathbb{R} \setminus \{0\}$ and $\omega \in \mathbb{T}$ such that the recipe of Theorem 2.1.18 produces the function h. Let us give those steps.
By Proposition 2.1.12, there exist polynomials E_1 and D_1 such that $\deg(E_1), \deg(D_1) \leq n$, E_1 is n-symmetric, $D_1(\lambda) \neq 0$ on $\overline{\mathbb{D}}$, and

$$s = \frac{E_1}{D_1} \quad \text{and} \quad p = \frac{D_1^{\sim n}}{D_1} \text{ on } \overline{\mathbb{D}}.$$

By hypothesis, the zeros of s, repeated according to multiplicity, are $\alpha_1, \alpha_2, \dots, \alpha_{k_0}$ and $\eta_1, \eta_2, \dots, \eta_{k_1}$, where $2k_0 + k_1 = n$. Since E_1 is n-symmetric, by Lemma 4.6.4, there exists $t \in \mathbb{R} \setminus \{0\}$ such that

$$E_1(\lambda) = t \prod_{j=1}^{k_0} (\lambda - \alpha_j)(1 - \overline{\alpha_j}\lambda) \prod_{j=1}^{k_1} i\mathrm{e}^{-i\theta_j/2}(\lambda - \eta_j),$$

where $\eta_j = \mathrm{e}^{i\theta_j}$ for $j = 1, \dots, k_1$. The royal nodes of h are assumed to be $\sigma_1, \dots, \sigma_n$. By Proposition 2.1.17, for the royal polynomial R_1 of h, there exists $t_+ > 0$ such that

$$R_1(\lambda) = t_+ \prod_{j=1}^{n} Q_{\sigma_j}(\lambda).$$

Since E_1 and R_1 coincide with E and R in the construction of Theorem 4.6.5, for a suitable choice of $t_+ > 0$ and $t \in \mathbb{R} \setminus \{0\}$, D_1 is a permissible choice for ωD for some $\omega \in \mathbb{T}$, as a solution of the equation (4.12).
By assumption the zeros of a, repeated according to multiplicity, are $\beta_1, \beta_2, \dots, \beta_m \in \mathbb{D}$. Then the inner part of a will be equal to $a_{in}^1 = c_1 \prod_{i=1}^{m} B_{\beta_i}$ where $|c_1| = 1$. For the outer part of a there is an outer polynomial A_1 such that

$$\begin{aligned} |A_1(\lambda)|^2 &= |D_1(\lambda)|^2 - \frac{1}{4}|E_1(\lambda)|^2 \\ &= |D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2 \\ &= \lambda^{-n} R(\lambda), \end{aligned}$$

for $\lambda \in \mathbb{T}$. By equation (4.13), $A_1 = c_2 A$ up to a constant c_2 such that $|c_2| = 1$. Also, a_{in}^1 coincides with a_{in} for a suitable choice of $c \in \mathbb{T}$. Hence the construction of Theorem 4.6.5 yields $x = (a, s, p)$ for the appropriate choices of $t_+ > 0$, $t \in \mathbb{R} \setminus \{0\}$, ω and $c \in \mathbb{T}$. $\square$

Chapter 5. A Schwarz lemma for the pentablock $\mathcal{P}$

There is a well developed theory of Schwarz lemmas for various domains by many authors, including Dineen and Harris [26, 33]. In particular, for the symmetrized bidisc and the tetrablock, see [11, 1].

Lemma 5.0.1. (Classical Schwarz lemma) *Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic map and $f(0) = 0$ then*

(1) $|f(z)| \leq |z|$ *for all* $z \in \mathbb{D}$,

(2) $|f'(0)| \leq 1$.

Moreover, if $|f(z)| = |z|$ for some non-zero $z \in \mathbb{D}$ or $|f'(0)| = 1$, then $f(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$ for all z in $\mathbb{D}$.

The classical Schwarz lemma gives a solvability criterion for a two-point interpolation problem in $\mathbb{D}$. In [4] a simple analogue of the Schwarz lemma for two-point μ-synthesis was given. We consider a general linear subspace E of $\mathbb{C}^{n\times m}$ and the corresponding μ_E on $\mathbb{C}^{m\times n}$,

$$\mu_E(A) = \left(\inf\{\|X\| : X \in E \text{ and } \det(1 - AX) = 0\}\right)^{-1}.$$

We shall denote by N the Nevanlinna class of functions on the disc [43] and if F is a matricial function on $\mathbb{D}$ then we write $F \in N$ to mean that each entry of F belongs to N, see Subsection A.0.1. It then follows from Fatou's Theorem that if $F \in N$ is an $m \times n$-matrix-valued function then

$$\lim_{r\to 1^-} F(r\lambda) \text{ exists for almost all } \lambda \in \mathbb{T}.$$

The following Schwarz lemma was proved in [4, Proposition 10.3].

Proposition 5.0.2. [4, Proposition 10.3] *Let $\lambda_0 \in \mathbb{D} \setminus \{0\}$, let $W \in \mathbb{C}^{m\times n}$ and let E be a subset of $\mathbb{C}^{n\times m}$. There exists $F \in N \cap \mathrm{Hol}(\mathbb{D}, \mathbb{C}^{m\times n})$ such that*

(1) $F(0) = 0$ *and* $F(\lambda_0) = W$,

(2) $\mu_E(F(\lambda)) < 1$ *for all* $\lambda \in \mathbb{D}$

if and only if $\mu_E(W) \leq |\lambda_0|$.

5.1 A special case of a Schwarz lemma for $\mathcal{P}$

In this section in Theorem 5.1.6 we consider a simple case of a Schwarz lemma for the pentablock. We will need the following elementary technical lemma.

Lemma 5.1.1. *Let* $A = \begin{bmatrix} \lambda_1 & 0 \\ a & \lambda_2 \end{bmatrix}$, *where* $\lambda_1,\ \lambda_2,\ a \in \mathbb{C}$. *Then the following are equivalent:*

(i) $\lambda_1, \lambda_2 \in \overline{\mathbb{D}},\ |a| \le (1-|\lambda_1|^2)^{\frac{1}{2}}(1-|\lambda_2|^2)^{\frac{1}{2}}$;

(ii) $\|A\| \le 1$;

(iii) $1 - A^*A \ge 0$.

Proof. (ii) $\Leftrightarrow$ (iii) By Proposition A.0.6, $\|A\| \le 1$ if and only if $1 - A^*A \ge 0$.
(i) $\Leftrightarrow$ (iii) Note that

$$A^*A = \begin{bmatrix} \overline{\lambda_1} & \overline{a} \\ 0 & \overline{\lambda_2} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ a & \lambda_2 \end{bmatrix} = \begin{bmatrix} |\lambda_1|^2 + |a|^2 & \overline{a}\lambda_2 \\ \overline{\lambda_2}a & |\lambda_2|^2 \end{bmatrix}$$

$$1 - A^*A = 1 - \begin{bmatrix} \overline{\lambda_1} & \overline{a} \\ 0 & \overline{\lambda_2} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ a & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 - |\lambda_1|^2 - |a|^2 & -\overline{a}\lambda_2 \\ -\overline{\lambda_2}a & 1 - |\lambda_2|^2 \end{bmatrix}.$$

Let μ_1, μ_2 be the eigenvalues of A^*A, then the characteristic polynomial of A^*A is

$$P_{A^*A}(t) = \det(tI - A^*A) = (t-\mu_1)(t-\mu_2).$$

Therefore

$$\begin{aligned} \det(I - A^*A) &= (1-\mu_1)(1-\mu_2) \\ &= 1 - (\mu_1 + \mu_2) + \mu_1\mu_2 \\ &= 1 - \text{tr } A^*A + \det A^*A \qquad (\text{tr } A^*A = \sum_{i=1}^{2} \mu_i,\ \det A^*A = \prod_{i=1}^{2} \mu_i) \\ &= 1 - \text{tr } A^*A + |\det A|^2. \end{aligned}$$

Here $\text{tr } A^*A = |\lambda_1|^2 + |\lambda_2|^2 + |a|^2,\ \det A = \lambda_1\lambda_2$. Then,

$$\begin{aligned} \det(1 - A^*A) &= 1 - (|\lambda_1|^2 + |\lambda_2|^2 + |a|^2) + |\lambda_1\lambda_2|^2 \\ &= (1-|\lambda_1|^2)(1-|\lambda_2|^2) - |a|^2. \end{aligned}$$

Note that

$$(1 - A^*A)^* = \begin{bmatrix} 1 - |\lambda_1|^2 - |a|^2 & -\overline{a}\lambda_2 \\ -\overline{\lambda_2}a & 1 - |\lambda_2|^2 \end{bmatrix} = 1 - A^*A.$$

Therefore $1 - A^*A$ is Hermitian.

Let us write down the principal minors of $1 - A^*A$. They are:

$$D_1 = 1 - |\lambda_1|^2 - |a|^2, \;\; D_2 = 1 - |\lambda_2|^2,$$

$$D_3 = \begin{vmatrix} 1 - |\lambda_1|^2 - |a|^2 & -\bar{a}\lambda_2 \\ -\overline{\lambda_2}a & 1 - |\lambda_2|^2 \end{vmatrix} = (1 - |\lambda_1|^2)(1 - |\lambda_2|^2) - |a|^2.$$

By Theorem A.0.5 (ii), a matrix $1 - A^*A$ is positive if and only if it is Hermitian and all its principal minors are nonnegative.

(i) $\Rightarrow$ (iii) Suppose $\lambda_1, \lambda_2 \in \overline{\mathbb{D}}$ and $|a|^2 \le (1 - |\lambda_1|^2)(1 - |\lambda_2|^2)$. Then $D_2 = 1 - |\lambda_2|^2 \ge 0$ and $D_3 \ge 0$. Note that $0 \le (1 - |\lambda_2|^2) \le 1$, hence $|a|^2 \le (1 - |\lambda_1|^2)(1 - |\lambda_2|^2) \le (1 - |\lambda_1|^2)$. Therefore $D_1 = 1 - |\lambda_1|^2 - |a|^2 \ge 0$. By Theorem A.0.5 (ii), $1 - A^*A \ge 0$.

(iii) $\Rightarrow$ (i) Note that $D_3 \ge 0$ if and only if $|a|^2 \le (1 - |\lambda_1|^2)(1 - |\lambda_2|^2)$. Since $D_2 = 1 - |\lambda_2|^2 \ge 0$, we have $\lambda_2 \in \overline{\mathbb{D}}$. $D_1 = 1 - |\lambda_1|^2 - |a|^2 \ge 0$ implies that $1 - |\lambda_1|^2 \ge |a|^2 \ge 0$. Hence $|\lambda_1|^2 \le 1$. Thus conditions (i) are satisfied.

Therefore, $1 - A^*A \ge 0$ if and only if the following holds: $|a| \le (1 - |\lambda_1|^2)^{\frac{1}{2}}(1 - |\lambda_2|^2)^{\frac{1}{2}}$, $|\lambda_1| \le 1$, $|\lambda_2| \le 1$. □

Definition 5.1.2. $H^\infty(\mathbb{D}, \mathbb{C}^{2\times 2})$ *denotes the space of bounded analytic* 2×2 *matrix-valued functions on* $\mathbb{D}$ *with the supremum norm:*

$$\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} \|f(z)\|_{\mathbb{C}^{2\times 2}}.$$

Definition 5.1.3. $L^\infty(\mathbb{T}, \mathbb{C}^{2\times 2})$ *denotes the space of essentially bounded Lebesgue-measurable* 2×2 *matrix-valued functions on* $\mathbb{T}$ *with the essential supremum norm:*

$$\|f\|_{L^\infty} = \operatorname*{ess\,sup}_{|z|=1} \|f(z)\|_{\mathbb{C}^{2\times 2}}.$$

Lemma 5.1.4. *If* $g \in H^\infty(\mathbb{D}, \mathbb{C}^{2\times 2})$ *and* $\lambda_0 \in \mathbb{D}$ *then* $\|g(\lambda_0)\|_{\mathbb{C}^{2\times 2}} \le \|g\|_{L^\infty}$.

Proof. Consider any unit vectors $x, y \in \mathbb{C}^2$ and the scalar function

$$\begin{aligned} f \;&:\; \mathbb{D} \to \mathbb{C} \\ &:\; \lambda \longmapsto \langle g(\lambda)x, y\rangle_{\mathbb{C}^2}. \end{aligned}$$

Note that, for every $\lambda \in \mathbb{D}$, since $\|x\|_{\mathbb{C}^2} = \|y\|_{\mathbb{C}^2} = 1$

$$\begin{aligned} |f(\lambda)| = |\langle g(\lambda)x, y\rangle_{\mathbb{C}^2}| \;&\le\; \|g(\lambda)x\|_{\mathbb{C}^2}\,\|y\|_{\mathbb{C}^2} \qquad \text{(Cauchy-Schwarz inequality)} \\ &\le\; \|g(\lambda)\|_{\mathbb{C}^{2\times 2}}\,\|x\|_{\mathbb{C}^2}\,\|y\|_{\mathbb{C}^2} \\ &\le\; \|g\|_{H^\infty}. \end{aligned}$$

Thus f is bounded on $\mathbb{D}$. Since g is analytic on $\mathbb{D}$, let us show that f is analytic on $\mathbb{D}$ too. We claim that, for every $z_0 \in \mathbb{D}$, $f'(z_0) = \langle g'(z_0)x, y\rangle_{\mathbb{C}^2}$, that is,

$$\lim_{z\to z_0}\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| = 0.$$

Note that

$$\begin{aligned} f(z)-f(z_0) &= \langle g(z)x,y\rangle_{\mathbb{C}^2} - \langle g(z_0)x,y\rangle_{\mathbb{C}^2} \\ &= \langle (g(z)-g(z_0))x,y\rangle_{\mathbb{C}^2}, \end{aligned}$$

and so

$$\begin{aligned} \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) &= \left\langle \frac{1}{z-z_0}(g(z)-g(z_0))x, y\right\rangle_{\mathbb{C}^2} - \langle g'(z_0)x,y\rangle_{\mathbb{C}^2} \\ &= \left\langle \left(\frac{g(z)-g(z_0)}{z-z_0} - g'(z_0)\right)x, y\right\rangle_{\mathbb{C}^2}. \end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} \left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| &= \left|\left\langle \left(\frac{g(z)-g(z_0)}{z-z_0} - g'(z_0)\right)x, y\right\rangle_{\mathbb{C}^2}\right| \\ &\le \left\|\left(\frac{g(z)-g(z_0)}{z-z_0} - g'(z_0)\right)x\right\|_{\mathbb{C}^2}\|y\|_{\mathbb{C}^2} \\ &\le \left\|\frac{g(z)-g(z_0)}{z-z_0} - g'(z_0)\right\|_{\mathbb{C}^{2\times2}}\|x\|_{\mathbb{C}^2}\|y\|_{\mathbb{C}^2} \\ &= \left\|\frac{g(z)-g(z_0)}{z-z_0} - g'(z_0)\right\|_{\mathbb{C}^{2\times2}} \to 0 \quad \text{as } z\to z_0 \end{aligned}$$

since g is analytic on $\mathbb{D}$. Therefore, $f'(z_0) = \langle g'(z_0)x, y\rangle$ for every $z_0 \in \mathbb{D}$.

By the maximum principle for scalar analytic functions, for every $\lambda_0 \in \mathbb{D}$, $|f(\lambda_0)| \le \operatorname*{ess\,sup}_{z\in\mathbb{T}} |f(z)|$, and so

$$\begin{aligned} |\langle g(\lambda_0)x,y\rangle_{\mathbb{C}^2}| &\le \operatorname*{ess\,sup}_{z\in\mathbb{T}} |\langle g(z)x,y\rangle| \\ &\le \operatorname*{ess\,sup}_{z\in\mathbb{T}} \|g(z)\|_{\mathbb{C}^{2\times2}} = \|g\|_{L^\infty}. \end{aligned}$$

Take the supremum of both sides in this inequality over unit vectors x, y to get

$$\|g(\lambda_0)\|_{\mathbb{C}^{2\times2}} \le \|g\|_{L^\infty}.$$

□

Corollary 5.1.5. *If $F \in H^\infty(\mathbb{D}, \mathbb{C}^{2\times2})$ and $F(0) = 0$ then, for any $\lambda_0 \in \mathbb{D}$,*

$$\|F(\lambda_0)\|_{\mathbb{C}^{2\times2}} \le |\lambda_0|\ \|F\|_{H^\infty}.$$

Proof. Let $g(\lambda) = F(\lambda)/\lambda$ for $\lambda \in \overline{\mathbb{D}} \setminus \{0\}$. Then g is analytic on $\mathbb{D} \setminus \{0\}$. By assumption, F is analytic and $F(0) = 0$. Thus the Taylor expansion of F on an open disc $D(0, r)$ for some r, is

$$\begin{aligned}
F(\lambda) &= \sum_{n=0}^{\infty} \lambda^n \frac{F^{(n)}(0)}{n!} = F(0) + \lambda F'(0) + \frac{\lambda^2}{2!} F''(0) + \dots \\
&= \lambda F'(0) + \frac{\lambda^2}{2!} F''(0) + \dots \\
&= \lambda \left(F'(0) + \frac{\lambda}{2!} F''(0) + \dots \right) \\
&= \lambda\, h(\lambda).
\end{aligned}$$

Here the above series converges with respect to $\|.\|_{\mathbb{C}^{2\times2}}$. Therefore h is analytic on $\mathbb{D}$. Since $h(\lambda) = g(\lambda)$ for $\lambda \neq 0$, then $g(\lambda)$ extends to be analytic on $\mathbb{D}$, and, for $\lambda \in \mathbb{T}$,

$$\|g(\lambda)\|_{\mathbb{C}^{2\times2}} = \|F(\lambda)\|_{\mathbb{C}^{2\times2}} \leq \|F\|_{H^\infty}.$$

Hence, by Lemma 5.1.4,

$$\|g(\lambda_0)\|_{\mathbb{C}^{2\times2}} \leq \|g\|_{H^\infty_{2\times2}} \leq \|F\|_{H^\infty}.$$

Then

$$\frac{\|F(\lambda_0)\|_{\mathbb{C}^{2\times2}}}{|\lambda_0|} \leq \|F\|_{H^\infty}, \text{ and so, } \|F(\lambda_0)\|_{\mathbb{C}^{2\times2}} \leq |\lambda_0|\ \|F\|_{H^\infty}.$$

□

Theorem 5.1.6. *Let $\lambda_0 \in \mathbb{D} \setminus \{0\}$, and $(a_0, s_0, p_0) \in \overline{\mathcal{P}}$, where $s_0 = \lambda_1 + \lambda_2$, $p_0 = \lambda_1\lambda_2$, for some $\lambda_1, \lambda_2 \in \mathbb{D}$. Then the following are equivalent:*

(i) $|\lambda_1| \leq |\lambda_0|$, $|\lambda_2| \leq |\lambda_0|$, *and*

$$|a_0| \leq |\lambda_0| \left(1 - \left|\frac{\lambda_1}{\lambda_0}\right|^2\right)^{\frac{1}{2}} \left(1 - \left|\frac{\lambda_2}{\lambda_0}\right|^2\right)^{\frac{1}{2}}; \tag{5.1}$$

(ii) *there exists an analytic map $F : \mathbb{D} \to \overline{\mathbb{B}^{2\times2}}$ such that*

$$F(0) = 0 \text{ and } F(\lambda_0) = \begin{bmatrix} \lambda_1 & 0 \\ a_0 & \lambda_2 \end{bmatrix}.$$

Furthermore, if conditions (i) *and* (ii) *are satisfied and $x(\lambda) = \pi \circ F(\lambda)$, for $\lambda \in \mathbb{D}$, then $x : \mathbb{D} \to \overline{\mathcal{P}}$ is an analytic map on $\mathbb{D}$ such that $x(0) = (0, 0, 0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$.*

Proof. (i) $\Rightarrow$ (ii) By assumption, $|\lambda_1| \leq |\lambda_0|$, $|\lambda_2| \leq |\lambda_0|$ and

$$|a_0| \leq |\lambda_0| \left(1 - \left|\frac{\lambda_1}{\lambda_0}\right|^2\right)^{\frac{1}{2}} \left(1 - \left|\frac{\lambda_2}{\lambda_0}\right|^2\right)^{\frac{1}{2}}.$$

Define

$$F(\lambda) = \frac{\lambda}{\lambda_0} \begin{bmatrix} \lambda_1 & 0 \\ a_0 & \lambda_2 \end{bmatrix} = \lambda \begin{bmatrix} \lambda_1/\lambda_0 & 0 \\ a_0/\lambda_0 & \lambda_2/\lambda_0 \end{bmatrix}. \tag{5.2}$$

By Lemma 5.1.1, $\|F(\lambda)\| = |\lambda| \left\| \begin{bmatrix} \lambda_1/\lambda_0 & 0 \\ a_0/\lambda_0 & \lambda_2/\lambda_0 \end{bmatrix} \right\| \leq |\lambda|$, for all $\lambda \in \mathbb{D}$, and so

$$\|F\|_\infty = \sup_{\lambda \in \mathbb{D}} \|F(\lambda)\|_{\mathbb{C}^{2\times 2}} \leq \sup_{\lambda \in \mathbb{D}} |\lambda| = 1.$$

From the definition (5.2) of F, we have

$$F(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } F(\lambda_0) = \begin{bmatrix} \lambda_1 & 0 \\ a_0 & \lambda_2 \end{bmatrix}.$$

(ii) $\Rightarrow$ (i) Suppose (ii) is satisfied. By Corollary 5.1.5,

$$\left\| \begin{bmatrix} \lambda_1 & 0 \\ a_0 & \lambda_2 \end{bmatrix} \right\|_{\mathbb{C}^{2\times 2}} = \|F(\lambda_0)\|_{\mathbb{C}^{2\times 2}} \leq |\lambda_0| \ \|F\|_{H^\infty}.$$

By assumption $\|F\|_{H^\infty} \leq 1$, and so $\left\| \begin{bmatrix} \lambda_1 & 0 \\ a_0 & \lambda_2 \end{bmatrix} \right\|_{\mathbb{C}^{2\times 2}} \leq |\lambda_0|$, hence

$$\left\| \begin{bmatrix} \frac{\lambda_1}{\lambda_0} & 0 \\ \frac{a_0}{\lambda_0} & \frac{\lambda_2}{\lambda_0} \end{bmatrix} \right\|_{\mathbb{C}^{2\times 2}} \leq 1. \quad (\|cA\| = |c| \ \|A\|, \text{ for all } c \in \mathbb{C}, A \in \mathbb{C}^{n\times n})$$

By Lemma 5.1.1,

$$\left\| \begin{bmatrix} \frac{\lambda_1}{\lambda_0} & 0 \\ \frac{a_0}{\lambda_0} & \frac{\lambda_2}{\lambda_0} \end{bmatrix} \right\|_{\mathbb{C}^{2\times 2}} \leq 1 \text{ if and only if}$$

$$|a_0| \leq |\lambda_0| \left(1 - \left|\frac{\lambda_1}{\lambda_0}\right|^2\right)^{\frac{1}{2}} \left(1 - \left|\frac{\lambda_2}{\lambda_0}\right|^2\right)^{\frac{1}{2}}, \quad |\lambda_1| \leq |\lambda_0| \text{ and } |\lambda_2| \leq |\lambda_0|.$$

Let us consider $x = \pi \circ F$ on $\mathbb{D}$. By assumption $(a_0, s_0, p_0) \in \overline{\mathcal{P}}$. By Theorem 3.2.4 (6), since $\|F(\lambda)\| \leq 1$ for each $\lambda \in \mathbb{D}$, $x(\lambda) = \pi(F(\lambda)) = \dfrac{\lambda}{\lambda_0}(a_0, s_0, p_0)$ maps $\mathbb{D}$ to $\overline{\mathcal{P}}$. Therefore $x : \mathbb{D} \to \overline{\mathcal{P}}$ is analytic on $\mathbb{D}$ and maps 0 to $(0,0,0)$ and λ_0 to (a_0, s_0, p_0). $\square$

5.2 A Schwarz lemma for the symmetrized bidisc Γ

In [11] Agler and Young proved the following theorems.

Theorem 5.2.1. [11, Theorem 1.1] *Let $\lambda_0 \in \mathbb{D} \setminus \{0\}$ and $(s_0, p_0) \in \Gamma$. The following conditions are equivalent:*

(1) *there exists an analytic function $\varphi : \mathbb{D} \to \Gamma$ such that $\varphi(0) = (0,0)$ and $\varphi(\lambda_0) = (s_0, p_0)$;*

(2) $|s_0| < 2$ *and*
$$\frac{2|s_0 - p_0\overline{s_0}| + |s_0^2 - 4p_0|}{4 - |s_0|^2} \leq |\lambda_0|;$$

(3) $\big||\lambda_0|^2 s_0 - p_0\overline{s_0}\big| + |p_0|^2 + (1 - |\lambda_0|^2)\dfrac{|s_0|^2}{4} - |\lambda_0|^2 \leq 0;$

(4)
$$|s_0| \leq \frac{2}{1 - |\lambda_0|^2}(|\lambda_0||1 - p_0\overline{\omega}^2| - \big||\lambda_0|^2 - p_0\overline{\omega}^2\big|)$$
where ω is a complex number of unit modulus such that $s_0 = |s_0|\omega$.

Moreover, for any analytic function $\varphi = (\varphi_1, \varphi_2) : \mathbb{D} \to \Gamma$ such that $\varphi(0) = (0,0)$,
$$\frac{1}{2}|\varphi_1'(0)| + |\varphi_2'(0)| \leq 1.$$

The following Theorem shows the construction of an interpolating function φ satisfying the inequalities of Theorem 5.2.1 with equality.

Theorem 5.2.2. [11, Theorem 1.4] *Let $\lambda_0 \in \mathbb{D}$, and $(s_0, p_0) \in \Gamma$ be such that $\lambda_0 \neq 0$, $|s_0| < 2$ and*
$$\frac{2|s_0 - p_0\bar{s}_0| + |s_0^2 - 4p_0|}{4 - |s_0|^2} = |\lambda_0|.$$
Then there exists an analytic function $\varphi : \mathbb{D} \to \Gamma$ such that $\varphi(0) = (0,0)$ and $\varphi(\lambda_0) = (s_0, p_0)$, given explicitly as follows.
If $|p_0| = |\lambda_0|$, then $\varphi(\lambda) = (0, \omega\lambda)$ where ω is a complex number of unit modulus such that $\omega\lambda_0 = p_0$.
If $|p_0| < |\lambda_0|$, then $\varphi = (\varphi_1, \varphi_2)$ where
$$\varphi_1(\lambda) = \frac{c\zeta\lambda}{(1 - \bar{\lambda}_0\lambda)(1 + \bar{p}_1\zeta^2 v(\lambda))}, \tag{5.3}$$
$$v(\lambda) = \frac{\lambda - \lambda_0}{1 - \bar{\lambda}_0\lambda}, \qquad \zeta\lambda_0|s_0| = |\lambda_0|s_0, \qquad |\zeta| = 1,$$
$$p_1 = \frac{p_0}{\lambda_0}, \qquad c = \frac{2}{|\lambda_0|}\{|\bar{\lambda}_0 - \bar{p}_0\lambda_0\zeta^2| - |\lambda_0^2\zeta^2 - p_0|\},$$
$$\varphi_2(\lambda) = \frac{\lambda(\zeta^2 v(\lambda) + p_1)}{1 + \bar{p}_1\zeta^2 v(\lambda)}. \tag{5.4}$$

Lemma 5.2.3. *Consider the rational function $\varphi = (\varphi_1, \varphi_2)$, where φ_1, φ_2 are defined as in equations (5.3) and (5.4) above. Define the polynomials E and D by the equations:*

$$E(\lambda) = c\lambda,$$

$$D(\lambda) = \overline{\zeta}\{(1 - \overline{\lambda_0}\lambda) + \overline{p_1}\zeta^2(\lambda - \lambda_0)\},$$

where

$$|\zeta| = 1, \quad p_1 = \frac{p_0}{\lambda_0}, \qquad c = \frac{2}{|\lambda_0|}\{|\overline{\lambda}_0 - \overline{p}_0\lambda_0\zeta^2| - |\lambda_0^2\zeta^2 - p_0|\}.$$

Then $\varphi_1 = \dfrac{E}{D}$ and $\varphi_2 = \dfrac{D^{\sim 2}}{D}$. Moreover, $E^{\sim 2} = E$ and $|E(\lambda)| \leq 2|D(\lambda)|$ on $\overline{\mathbb{D}}$.

Proof. Let us check that $\varphi_1(\lambda) = \dfrac{E(\lambda)}{D(\lambda)}$.

$$\begin{aligned}
\frac{E(\lambda)}{D(\lambda)} &= \frac{c\lambda}{\overline{\zeta}\{(1 - \overline{\lambda_0}\lambda) + \overline{p_1}\zeta^2(\lambda - \lambda_0)\}} \times \frac{\zeta}{\zeta} \\
&= \frac{c\zeta\lambda}{(1 - \overline{\lambda_0}\lambda) + \overline{p_1}\zeta^2(\lambda - \lambda_0)} \\
&= \frac{c\zeta\lambda}{(1 - \overline{\lambda_0}\lambda)(1 + \overline{p_1}\zeta^2 v(\lambda))} = \varphi_1(\lambda).
\end{aligned}$$

To check that $\varphi_2(\lambda) = \dfrac{D^{\sim 2}(\lambda)}{D(\lambda)}$, we need to find $D^{\sim 2}(\lambda)$.

$$\begin{aligned}
D^{\sim 2}(\lambda) = \lambda^2\overline{D(1/\overline{\lambda})} &= \lambda^2\overline{\overline{\zeta}\left\{\left(1 - \frac{\overline{\lambda_0}}{\overline{\lambda}}\right) + \overline{p_1}\zeta^2\left(\frac{1}{\overline{\lambda}} - \lambda_0\right)\right\}} \\
&= \lambda^2\zeta\left\{\left(1 - \frac{\lambda_0}{\lambda}\right) + p_1\overline{\zeta}^2\left(\frac{1}{\lambda} - \overline{\lambda_0}\right)\right\} \\
&= \lambda\zeta(\lambda - \lambda_0) + \lambda p_1\overline{\zeta}(1 - \overline{\lambda_0}\lambda).
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{D^{\sim 2}(\lambda)}{D(\lambda)} &= \frac{\lambda\zeta(\lambda - \lambda_0) + \lambda p_1\overline{\zeta}(1 - \overline{\lambda_0}\lambda)}{\overline{\zeta}\{(1 - \overline{\lambda_0}\lambda) + \overline{p_1}\zeta^2(\lambda - \lambda_0)\}} \times \frac{\left(\dfrac{\zeta}{1 - \overline{\lambda_0}\lambda}\right)}{\left(\dfrac{\zeta}{1 - \overline{\lambda_0}\lambda}\right)} \\
&= \frac{\lambda\zeta^2\left(\dfrac{\lambda - \lambda_0}{1 - \overline{\lambda_0}\lambda}\right) + \lambda p_1}{1 + \overline{p_1}\zeta^2\left(\dfrac{\lambda - \lambda_0}{1 - \overline{\lambda_0}\lambda}\right)} \\
&= \frac{\lambda(\zeta^2 v(\lambda) + p_1)}{1 + \overline{p_1}\zeta^2 v(\lambda)} = \varphi_2(\lambda),
\end{aligned}$$

where $v(\lambda) = \dfrac{\lambda - \lambda_0}{1 - \overline{\lambda_0}\lambda}$.

We would like to show that $E^{\sim 2} = E$. For $\lambda \in \mathbb{D}$,

$$\begin{aligned} E^{\sim 2}(\lambda) = \lambda^2 \overline{E(1/\overline{\lambda})} &= \lambda^2 \overline{\left(c\,\frac{1}{\overline{\lambda}}\right)} \\ &= c\,\lambda = E(\lambda), \quad \text{since } c \in \mathbb{R}. \end{aligned}$$

Since $(\varphi_1, \varphi_2) : \mathbb{D} \to \Gamma$, we have $|\varphi_1(\lambda)| \le 2$. Thus $|\varphi_1(\lambda)| = \left|\frac{E(\lambda)}{D(\lambda)}\right| \le 2$, for $\lambda \in \overline{\mathbb{D}}$. □

Proposition 5.2.4. *Let $h = (s, p)$ be the function from $\mathbb{D}$ to Γ defined by*

$$s(\lambda) = \varphi_1(\lambda),\ p(\lambda) = \varphi_2(\lambda),\ \lambda \in \mathbb{D},$$

as in equations (5.3) *and* (5.4). *Then h is a rational Γ-inner function of degree 2.*

Proof. One can easily see that $h = (s, p)$ is a rational function, and so there are only finitely many singularities of this function. Hence we can extend h continuously to almost all points in $\mathbb{T}$.

Let us show that for almost all $\lambda \in \mathbb{T}$, $h(\lambda) \in b\Gamma$. We need to show that, for almost all $\lambda \in \mathbb{T}$, $|p(\lambda)| = 1$, $|s(\lambda)| \le 2$ and $s(\lambda) = \overline{s(\lambda)}p(\lambda)$. Since $v(\lambda) = \dfrac{\lambda - \lambda_0}{1 - \overline{\lambda_0}\lambda}$ is an inner function from $\mathbb{D}$ to $\overline{\mathbb{D}}$, for almost all $\lambda \in \mathbb{T}$, $|v(\lambda)| = 1$.

$$\begin{aligned} |p(\lambda)| &= \frac{|\lambda||\zeta^2 v(\lambda) + p_1|}{|1 + \overline{p_1}\zeta^2 v(\lambda)|} = \frac{|\zeta^2 v(\lambda) + p_1|}{|\zeta^2\overline{\zeta^2}(v\overline{v})(\lambda) + \overline{p_1}\zeta^2 v(\lambda)|} \\ &= \frac{|\zeta^2 v(\lambda) + p_1|}{|\zeta^2 v(\lambda)\overline{(\zeta^2 v(\lambda) + p_1)}|} = \frac{|\zeta^2 v(\lambda) + p_1|}{|\zeta^2 v(\lambda) + p_1|} = 1, \end{aligned}$$

for almost all $\lambda \in \mathbb{T}$.

In [11, Theorem 1.5] it was proved that, for all $\lambda \in \mathbb{D}$,

$$\left||\lambda|^2 s(\lambda) - \overline{s(\lambda)}p(\lambda)\right| + |p(\lambda)|^2 + (1 - |\lambda|^2)\frac{|s(\lambda)|^2}{4} - |\lambda|^2 = 0. \tag{5.5}$$

Choose a sequence $(r_n)_{n\ge1}$ of numbers such that $0 < r_n < 1$ for each n and $\lim_{n\to\infty} r_n = 1$. Consider $\mu \in \mathbb{T}$ and let $\lambda = r_n\mu$ in equation (5.5). On letting $n \to \infty$ we find that, for almost all $\mu \in \mathbb{T}$,

$$\left||\mu|^2 s(\mu) - \overline{s(\mu)}p(\mu)\right| + |p(\mu)|^2 + (1 - |\mu|^2)\frac{|s(\mu)|^2}{4} - |\mu|^2 = 0. \tag{5.6}$$

Note that $|\mu| = 1$ and $|p(\mu)|^2 = 1$, and so equation (5.6) is equivalent to

$$|s(\mu) - \overline{s(\mu)}p(\mu)| = 0.$$

Hence $s(\mu) = \overline{s(\mu)}p(\mu)$ for almost all $\mu \in \mathbb{T}$.

Note that for all $\lambda \in \mathbb{D}$, $h(\lambda) = (s(\lambda), p(\lambda)) \in \Gamma$, which means $|s(\lambda)| \le 2$ for all $\lambda \in \mathbb{D}$. Since for every $\mu \in \mathbb{T}$, $\lim_{n\to\infty} \lambda_n = \mu$, then for almost all $\mu \in \mathbb{T}$, $|s(\mu)| \le 2$.

By Proposition 2.1.11, $\deg(h) = \deg(p)$.

$$p(\lambda) = \varphi_2(\lambda) = \frac{D^{\sim 2}(\lambda)}{D(\lambda)} = \frac{\lambda\zeta(\lambda - \lambda_0) + \lambda p_1 \overline{\zeta}(1 - \overline{\lambda_0}\lambda)}{\overline{\zeta}\{(1 - \overline{\lambda_0}\lambda) + \overline{p_1}\zeta^2(\lambda - \lambda_0)\}},$$

where $D(\lambda) = \overline{\zeta}\{(1 - \overline{\lambda_0}\lambda) + \overline{p_1}\zeta^2(\lambda - \lambda_0)\}$. Since $\deg(D^{\sim 2}) = 2$ and $\deg(D) = 1$, then $\deg(p) = 2$. Therefore $\deg(h) = 2$. □

5.3 Some necessary and sufficient conditions for a Schwarz lemma for $\mathcal{P}$

Some necessary conditions for a Schwarz lemma for $\mathcal{P}$ were given in the following Proposition. That is, for which pairs $\lambda_0 \in \mathbb{D}$ and $(a, s, p) \in \overline{\mathcal{P}}$ does there exist $x \in \mathrm{Hol}(\mathbb{D}, \mathcal{P})$ such that $x(0) = (0, 0, 0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$?

Proposition 5.3.1. [4, Proposition 11.1] *Let $\lambda_0 \in \mathbb{D} \setminus \{0\}$ and let $(a_0, s_0, p_0) \in \overline{\mathcal{P}}$. If $x \in \mathrm{Hol}(\mathbb{D}, \overline{\mathcal{P}})$ satisfies $x(0) = (0, 0, 0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$ then*

$$|s_0| < 2, \qquad \frac{2|s_0 - \overline{s}_0 p_0| + |s_0^2 - 4p_0|}{4 - |s_0|^2} \leq |\lambda_0| \tag{5.7}$$

and

$$|a_0| \Big/ \left| 1 - \frac{\frac{1}{2}s_0\overline{\beta}}{1 + \sqrt{1 - |\beta|^2}} \right| \leq |\lambda_0| \tag{5.8}$$

where $\beta = (s_0 - \overline{s}_0 p_0)/(1 - |p_0|^2)$ when $|p_0| < 1$ and $\beta = \frac{1}{2}s_0$ when $|p_0| = 1$.

Proof. If $x = (x_1, x_2, x_3)$ then $(x_2, x_3) \in \mathrm{Hol}(\mathbb{D}, \mathbb{G})$ maps 0 to $(0, 0)$ and λ_0 to (s_0, p_0). By the Schwarz lemma for $\mathbb{G}$ (Theorem 5.2.1), the inequality (5.7) holds.

By Theorem 3.2.3, for every $z \in \mathbb{D}$, the function

$$\Psi_z(a, s, p) = \frac{a(1 - |z|^2)}{1 - sz + pz^2}$$

maps $\mathcal{P}$ analytically to $\mathbb{D}$. It also maps $(0, 0, 0)$ to 0. Hence $\Psi_z \circ x$ is an analytic self-map of $\mathbb{D}$ that maps 0 to 0 and λ_0 to $\Psi_z(a_0, s_0, p_0)$. By the classical Schwarz lemma 5.0.1 we have

$$|\Psi_z(a_0, s_0, p_0)| \leq |\lambda_0| \text{ for all } z \in \mathbb{D}.$$

On taking the supremum of the left hand side over $z \in \mathbb{D}$ and invoking Proposition 3.1.4 we obtain the inequality (5.8). □

Theorem 5.3.2. *Let $\lambda_0 \in \mathbb{D} \setminus \{0\}$, and $(a_0, s_0, p_0) \in \overline{\mathcal{P}}$ be such that $|s_0| < 2$,*

$$\frac{2|s_0 - p_0\overline{s}_0| + |s_0^2 - 4p_0|}{4 - |s_0|^2} = |\lambda_0| \tag{5.9}$$

and

$$|a_0| \le |\lambda_0|\sqrt{1-\frac{1}{4}|s_0|^2}. \tag{5.10}$$

Then there exists a rational $\overline{\mathcal{P}}$-inner function $x : \mathbb{D} \to \overline{\mathcal{P}}$ such that $x(0) = (0,0,0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$ given explicitly as follows.

(i) *If $|p_0| = |\lambda_0|$, then $s_0 = 0$ and $x(\lambda) = (\lambda\,\varphi(\lambda), 0, \omega\lambda)$, where $\omega\lambda_0 = p_0$, $\omega \in \mathbb{T}$ and*

  (a) *if $|a_0| = |\lambda_0|$, then, for $\lambda \in \mathbb{D}$, $\varphi(\lambda) = \kappa$, where $|\kappa| = 1$ and $\kappa\lambda_0 = a_0$;*

  (b) *if $|a_0| < |\lambda_0|$, then*

$$\varphi(\lambda) = \frac{\lambda - \lambda_0 + \eta_0(1-\overline{\lambda_0}\lambda)}{1-\overline{\lambda_0}\lambda + \overline{\eta_0}(\lambda-\lambda_0)}, \ \lambda \in \mathbb{D},$$

  and $\eta_0 = \dfrac{a_0}{\lambda_0}$.

(ii) *If $|p_0| < |\lambda_0|$, then $x(\lambda) = (a(\lambda), \varphi_1(\lambda), \varphi_2(\lambda)$, where φ_1 and φ_2 are defined by the equations*

$$\varphi_1(\lambda) = \frac{c\zeta\lambda}{(1-\overline{\lambda}_0\lambda)(1+\overline{p}_1\zeta^2 v(\lambda))},$$

$$v(\lambda) = \frac{\lambda-\lambda_0}{1-\overline{\lambda}_0\lambda}, \qquad \zeta\lambda_0|s_0| = |\lambda_0|s_0, \qquad |\zeta| = 1,$$

$$p_1 = \frac{p_0}{\lambda_0}, \qquad c = \frac{2}{|\lambda_0|}\{|\overline{\lambda}_0 - \overline{p}_0\lambda_0\zeta^2| - |\lambda_0^2\zeta^2 - p_0|\},$$

$$\varphi_2(\lambda) = \frac{\lambda(\zeta^2 v(\lambda) + p_1)}{1+\overline{p}_1\zeta^2 v(\lambda)},$$

and

  (a) *if $|a_0| = |\lambda_0|\sqrt{1-\frac{1}{4}|s_0|^2}$, then, for $\lambda \in \mathbb{D}$,*

$$a(\lambda) = \gamma\lambda\frac{A(\lambda)}{D(\lambda)},$$

  where $|\gamma| = 1$ such that $a_0 = \gamma\lambda_0\sqrt{1-\frac{1}{4}|s_0|^2}$, $A(\lambda) = b_0(1+b_1\lambda)$ is an outer polynomial of degree 1 such that

$$|A(\lambda)|^2 = |D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2, \tag{5.11}$$

  and

$$E(\lambda) = c\lambda, \ \ D(\lambda) = \overline{\zeta}\{(1-\overline{\lambda_0}\lambda) + \overline{p_1}\zeta^2(\lambda-\lambda_0)\}. \tag{5.12}$$

  (b) *if $|a_0| < |\lambda_0|\sqrt{1-\frac{1}{4}|s_0|^2}$, then, for $\lambda \in \mathbb{D}$,*

$$a(\lambda) = \lambda\frac{\lambda-\lambda_0+\mu_0(1-\overline{\lambda_0}\lambda)}{1-\overline{\lambda_0}\lambda+\overline{\mu_0}(\lambda-\lambda_0)}\frac{A(\lambda)}{D(\lambda)},$$

where $\mu_0 = \dfrac{a_0}{\lambda_0\sqrt{1-\frac{1}{4}|s_0|^2}}$ *and polynomials* A, E, D *defined by equations* (5.11) *and* (5.12).

Proof. Since $(a_0, s_0, p_0) \in \overline{\mathcal{P}}$, we have $(s_0, p_0) \in \Gamma$. By assumption the equality (5.9) holds. Hence, by Theorem 5.2.2, there exists a rational analytic function

$$\varphi : \mathbb{D} \to \Gamma : \lambda \longmapsto (s(\lambda), p(\lambda))$$

such that $\varphi(0) = (0,0)$ and $\varphi(\lambda_0) = (s_0, p_0)$. By Proposition 5.2.4, the function $\varphi = (s,p)$ is a rational Γ-inner function of degree 1 or 2.
It is known from [6, Proposition 2.2], see also Proposition 2.1.12, that, for the rational Γ-inner function $\varphi = (s,p)$ there exist polynomials E and D such that

(1) $\deg(E), \deg(D) \leq \deg(\varphi)$,

(2) $E^{\sim 2} = E$,

(3) $D(\lambda) \neq 0$ *for all* $\lambda \in \overline{\mathbb{D}}$,

(4) $|E(\lambda)| \leq 2|D(\lambda)|$ *for all* $\lambda \in \overline{\mathbb{D}}$,

(5) $s = \dfrac{E}{D}$ *on* $\overline{\mathbb{D}}$,

(6) $p = \dfrac{D^{\sim 2}}{D}$ *on* $\overline{\mathbb{D}}$.

Then, by Theorem 4.5.5, we can construct a rational $\overline{\mathcal{P}}$-inner function

$x = \left(cB\dfrac{A}{D}, \dfrac{E}{D}, \dfrac{D^{\sim n}}{D}\right)$, for an arbitrary Blaschke product B and any $c \in \mathbb{C}$ such that $|c| = 1$,

where A is an outer polynomial such that

$$|A(\lambda)|^2 = |D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2.$$

We would like to construct $a : \mathbb{D} \to \mathbb{C}$ of the form $a = cB\dfrac{A}{D}$ such that $a(0) = 0$ and $a(\lambda_0) = a_0$.
As in Theorem 5.2.2, it follows from equation (5.9) that $|p_0| \leq |\lambda_0| < 1$. Then we consider two cases when $|p_0| = |\lambda_0|$ and $|p_0| < |\lambda_0|$.
Case (i). If $|p_0| = |\lambda_0|$, then $s(\lambda) = 0$ and $p(\lambda) = \omega\lambda$, for some ω such that $|\omega| = 1$. Since $s(\lambda) = \dfrac{E(\lambda)}{D(\lambda)} = 0$, then, for $\lambda \in \mathbb{D}$,

$$\frac{|A(\lambda)|^2}{|D(\lambda)|^2} = 1.$$

It is easy to see that $p(\lambda) = \frac{D^{\sim 1}}{D}(\lambda)$ where $D(\lambda) = \overline{\omega}_1$ such that $\omega_1^2 = \omega$. Therefore $|A(\lambda)|^2 = |\overline{\omega}_1|^2 = 1$ and A is a non-zero constant. Let $B(\lambda) = \lambda\tilde{B}(\lambda)$ with some finite Blaschke product $\tilde{B}$. Then $B(0) = 0$, and so $a(0) = 0$.
Recall we require $a(\lambda_0) = a_0$, and so

$$a_0 = a(\lambda_0) = cB(\lambda_0) = c\lambda_0\tilde{B}(\lambda_0).$$

By assumption (5.10), $|a_0| \le |\lambda_0|$, and so

(a) if $|a_0| = |\lambda_0|$, we define $a(\lambda) = \kappa\lambda$, where $\kappa \in \mathbb{T}$ is such that $a_0 = \kappa\lambda_0$. For $\lambda \in \mathbb{D}$, $a(\lambda) = \kappa\lambda$ satisfies the conditions $a(0) = 0$ and $a(\lambda_0) = a_0$.

(b) if $|a_0| < |\lambda_0|$, then $\frac{|a_0|}{|\lambda_0|} < 1$. Let
$$\eta_0 \overset{\text{def}}{=} \frac{a_0}{\lambda_0},$$
it is clear that $\eta_0 \in \mathbb{D}$. Consider the following Blaschke factors
$$B_{\eta_0}^{-1}(z) = \frac{z+\eta_0}{1+\overline{\eta_0}z} \text{ and } B_{\lambda_0}(z) = \frac{z-\lambda_0}{1-\overline{\lambda_0}z}.$$
Then define
$$\tilde{B}(z) = B_{\eta_0}^{-1} \circ B_{\lambda_0}(z) = \frac{z-\lambda_0+\eta_0(1-\overline{\lambda_0}z)}{1-\overline{\lambda_0}z+\overline{\eta_0}(z-\lambda_0)}.$$
Note that $\tilde{B}(\lambda_0) = \frac{\eta_0(1-\overline{\lambda_0}\lambda_0)}{1-\overline{\lambda_0}\lambda_0} = \eta_0$. Let us define $a : \mathbb{D} \to \mathbb{C}$, for $\lambda \in \mathbb{D}$, by
$$a(\lambda) = \lambda\tilde{B}(\lambda) = \lambda\frac{\lambda-\lambda_0+\eta_0(1-\overline{\lambda_0}\lambda)}{1-\overline{\lambda_0}\lambda+\overline{\eta_0}(\lambda-\lambda_0)},$$
where $\eta_0 = \frac{a_0}{\lambda_0}$. Note that $a(0) = 0$.
$$a(\lambda_0) = \lambda_0\,\eta_0 = \lambda_0\,\frac{a_0}{\lambda_0} = a_0.$$
Define a rational $\overline{\mathcal{P}}$-inner function $x : \mathbb{D} \to \overline{\mathcal{P}}$ by $x(\lambda) = (\lambda\varphi(\lambda), 0, \omega\lambda)$, where
$$\varphi(\lambda) = \frac{\lambda-\lambda_0+\eta_0(1-\overline{\lambda_0}\lambda)}{1-\overline{\lambda_0}\lambda+\overline{\eta_0}(\lambda-\lambda_0)},\ \lambda \in \mathbb{D},$$
and $\eta_0 = \frac{a_0}{\lambda_0}$, $\omega\lambda_0 = p_0$, $\omega \in \mathbb{T}$. This function x satisfies the conditions $x(0) = (0,0,0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$.

Case (ii). If $|p_0| < |\lambda_0|$. Thus, for $\lambda \in \mathbb{D}$,
$$\frac{|A(\lambda)|^2}{|D(\lambda)|^2} = 1 - \frac{1}{4}|s(\lambda)|^2.$$

Let $B(\lambda) = \lambda\tilde{B}(\lambda)$ with some finite Blaschke product $\tilde{B}$. Then $B(0) = 0$, and so $a(0) = 0$. Note that

$$a(\lambda_0) = cB(\lambda_0)\frac{A}{D}(\lambda_0), \qquad \text{where } \left|\frac{A(\lambda)}{D(\lambda)}\right|^2 = 1 - \frac{1}{4}|s(\lambda)|^2.$$

Hence

$$\begin{aligned} a_0 &= cB(\lambda_0)\sqrt{1 - \frac{1}{4}|s(\lambda_0)|^2} \\ &= c\lambda_0\tilde{B}(\lambda_0)\sqrt{1 - \frac{1}{4}|s_0|^2}. \end{aligned} \tag{5.13}$$

By assumption (5.10), $|a_0| \leq |\lambda_0|\sqrt{1 - \frac{1}{4}|s_0|^2}$.

(a) if $|a_0| = |\lambda_0|\sqrt{1 - \frac{1}{4}|s_0|^2}$, then by (5.13), it is easy to see that, for $\lambda \in \mathbb{D}$,

$$a(\lambda) = \gamma\lambda\frac{A}{D}(\lambda),$$

where $\gamma \in \mathbb{T}$ is such that $a_0 = \gamma\lambda_0\sqrt{1 - \frac{1}{4}|s_0|^2}$.

(b) if

$$|a_0| < |\lambda_0|\sqrt{1 - \frac{1}{4}|s_0|^2}, \qquad \text{then } \frac{|a_0|}{|\lambda_0|\sqrt{1 - \frac{1}{4}|s_0|^2}} < 1.$$

Let

$$\mu_0 \stackrel{\text{def}}{=} \frac{a_0}{\lambda_0\sqrt{1 - \frac{1}{4}|s_0|^2}},$$

it is clear that $\mu_0 \in \mathbb{D}$. We need to find $\tilde{B} : \mathbb{D} \to \mathbb{D}$ such that $\tilde{B}(\lambda_0) = \mu_0$. Consider the following Blaschke factors

$$B_{\mu_0}^{-1}(z) = \frac{z + \mu_0}{1 + \overline{\mu_0}z} \text{ and } B_{\lambda_0}(z) = \frac{z - \lambda_0}{1 - \overline{\lambda_0}z}.$$

Let

$$\begin{aligned} \tilde{B}(z) &= B_{\mu_0}^{-1} \circ B_{\lambda_0}(z) = \frac{\dfrac{z - \lambda_0}{1 - \overline{\lambda_0}z} + \mu_0}{1 + \overline{\mu_0}\left(\dfrac{z - \lambda_0}{1 - \overline{\lambda_0}z}\right)} \\ &= \frac{z - \lambda_0 + \mu_0(1 - \overline{\lambda_0}z)}{1 - \overline{\lambda_0}z} \Big/ \frac{1 - \overline{\lambda_0}z + \overline{\mu_0}(z - \lambda_0)}{1 - \overline{\lambda_0}z} \\ &= \frac{z - \lambda_0 + \mu_0(1 - \overline{\lambda_0}z)}{1 - \overline{\lambda_0}z + \overline{\mu_0}(z - \lambda_0)}. \end{aligned}$$

Note that $\tilde{B}(\lambda_0) = \dfrac{\mu_0(1-\overline{\lambda_0}\lambda_0)}{1-\overline{\lambda_0}\lambda_0} = \mu_0$. Let us define $a : \mathbb{D} \to \mathbb{C}$, for $\lambda \in \mathbb{D}$, by

$$a(\lambda) = \lambda\tilde{B}(\lambda)\frac{A(\lambda)}{D(\lambda)} = \lambda\frac{\lambda-\lambda_0+\mu_0(1-\overline{\lambda_0}\lambda)}{1-\overline{\lambda_0}\lambda+\overline{\mu_0}(\lambda-\lambda_0)}\frac{A(\lambda)}{D(\lambda)},$$

where $\mu_0 = \dfrac{a_0}{\lambda_0\sqrt{1-\frac{1}{4}|s_0|^2}}$. Note that $a(0)=0$.

$$\begin{aligned} a(\lambda_0) &= \lambda_0\mu_0\frac{A(\lambda_0)}{D(\lambda_0)} \\ &= \lambda_0\frac{a_0}{\lambda_0\sqrt{1-\frac{1}{4}|s_0|^2}}\sqrt{1-\frac{1}{4}|s_0|^2} \\ &= a_0. \end{aligned}$$

□

Remark 5.3.3. *Let A, E, D be the polynomials defined in the previous theorem such that*

$$|A(\lambda)|^2 = |D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2.$$

Then a polynomial A has a form $A(\lambda) = b_0(1+b_1\lambda)$, where

$$|b_0|^2 = |1-\overline{p_1}\zeta^2\lambda_0|^2 \quad \textit{and} \quad |b_1|^2 = 2\,\frac{|\lambda_0\zeta^2-p_1|}{|1-\overline{p_1}\zeta^2\lambda_0|} - 1.$$

Proof. We would like to find $A(\lambda)$. Since E and D are polynomials of degree 1, then A is also a degree 1 polynomial such that $A(\lambda) = b_0(1+b_1\lambda)$. Note that A is outer, that is, $A(\lambda) \neq 0$ for all $\lambda \in \mathbb{D}$. Note that $A(\lambda) = 0$ if $(1+b_1\lambda) = 0$, that is, $\lambda = -1/b_1$.
$-1/b_1 \in \mathbb{D}$ if $|b_1| > 1$, and so, we need $|b_1| \leq 1$ in order for $A(\lambda)$ to be outer. Note that

$$|A(\lambda)|^2 = |D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2. \tag{5.14}$$

By Lemma 5.2.3

$$E(\lambda) = c\lambda,$$

$$D(\lambda) = \overline{\zeta}\{(1-\overline{\lambda_0}\lambda) + \overline{p_1}\zeta^2(\lambda-\lambda_0)\},$$

where

$$|\zeta| = 1, \quad p_1 = \frac{p_0}{\lambda_0}, \qquad c = \frac{2}{|\lambda_0|}\{|\overline{\lambda}_0 - \overline{p}_0\lambda_0\zeta^2| - |\lambda_0^2\zeta^2 - p_0|\}.$$

$$|E(\lambda)|^2 = |c\,\lambda|^2 = |c|^2|\lambda|^2.$$

$$\begin{aligned} |D(\lambda)|^2 &= \left|\bar{\zeta}\left\{\left(1-\overline{\lambda_0}\lambda\right)+\overline{p_1}\zeta^2(\lambda-\lambda_0)\right\}\right|^2 \\ &= |(1-\overline{p_1}\zeta^2\lambda_0)+\lambda(-\overline{\lambda_0}+\overline{p_1}\zeta^2)|^2 \\ &= \left|1-\overline{p_1}\zeta^2\lambda_0\right|^2+2\mathrm{Re}\left\{\overline{(1-\overline{p_1}\zeta^2\lambda_0)}(-\overline{\lambda_0}+\overline{p_1}\zeta^2)\lambda\right\}+|\lambda|^2\left|-\overline{\lambda_0}+\overline{p_1}\zeta^2\right|^2. \end{aligned}$$

Thus

$$\begin{aligned} |D(\lambda)|^2-\frac{1}{4}|E(\lambda)|^2 = \left|1-\overline{p_1}\zeta^2\lambda_0\right|^2+2\mathrm{Re}\left\{\overline{(1-\overline{p_1}\zeta^2\lambda_0)}(-\overline{\lambda_0}+\overline{p_1}\zeta^2)\lambda\right\} \\ +|\lambda|^2\left|-\overline{\lambda_0}+\overline{p_1}\zeta^2\right|^2-\frac{1}{4}|c|^2|\lambda|^2. \end{aligned} \tag{5.15}$$

By taking $\lambda=0$, we have on the left side of equation (5.14)

$$|A(0)|^2=|b_0(1+b_1 0)|^2=|b_0|^2,$$

and so, by equation (5.15),

$$|b_0|^2=|1-\overline{p_1}\zeta^2\lambda_0|^2.$$

Thus $b_0=\alpha(1-\overline{p_1}\zeta^2\lambda_0)$, for some $\alpha\in\mathbb{T}$.
By replacing λ with $-\lambda$ in (5.15), we obtain

$$\begin{aligned} |D(-\lambda)|^2-\frac{1}{4}|E(-\lambda)|^2 = \left|1-\overline{p_1}\zeta^2\lambda_0\right|^2-2\mathrm{Re}\left\{\overline{(1-\overline{p_1}\zeta^2\lambda_0)}(-\overline{\lambda_0}+\overline{p_1}\zeta^2)\lambda\right\} \\ +|\lambda|^2\left|-\overline{\lambda_0}+\overline{p_1}\zeta^2\right|^2-\frac{1}{4}|c|^2|\lambda|^2, \end{aligned} \tag{5.16}$$

$$|A(\lambda)|^2=|b_0(1+b_1\lambda)|^2=|b_0|^2|1+b_1\lambda|^2=|b_0|^2(1+2\mathrm{Re}(b_1\lambda)+|b_1|^2|\lambda|^2). \tag{5.17}$$

Note that

$$|A(-\lambda)|^2=|b_0|^2(1-2\mathrm{Re}(b_1\lambda)+|b_1|^2|\lambda|^2), \tag{5.18}$$

and so

$$|A(\lambda)|^2+|A(-\lambda)|^2=|b_0|^2(2+2|b_1|^2|\lambda|^2). \tag{5.19}$$

Note that

$$|A(\lambda)|^2+|A(-\lambda)|^2=|D(\lambda)|^2-\frac{1}{4}|E(\lambda)|^2+|D(-\lambda)|^2-\frac{1}{4}|E(-\lambda)|^2.$$

Adding (5.15) and (5.16) gives

$$|A(\lambda)|^2+|A(-\lambda)|^2=2\left|1-\overline{p_1}\zeta^2\lambda_0\right|^2+2\left|\lambda\right|^2\left|-\overline{\lambda_0}+\overline{p_1}\zeta^2\right|^2-\frac{1}{2}|c|^2|\lambda|^2. \tag{5.20}$$

By equations (5.19) and (5.20),

$$|b_0|^2\left(2+2|b_1|^2|\lambda|^2\right)=2\left|1-\overline{p_1}\zeta^2\lambda_0\right|^2+2|\lambda|^2\left|-\overline{\lambda_0}+\overline{p_1}\zeta^2\right|^2+2\left(-\frac{1}{4}|c|^2|\lambda|^2\right). \tag{5.21}$$

Recall that $|b_0|^2 = |1 - \overline{p_1}\zeta^2\lambda_0|^2$, and so, for all λ,

$$|b_0|^2\,|b_1|^2\,|\lambda|^2 = \left(\left|-\overline{\lambda_0} + \overline{p_1}\zeta^2\right|^2 - \frac{1}{4}|c|^2\right)|\lambda|^2.$$

Therefore,

$$|b_1|^2 = \frac{\left|-\overline{\lambda_0} + \overline{p_1}\zeta^2\right|^2 - \frac{1}{4}|c|^2}{|b_0|^2}.$$

Now, we would like to show that $|b_1|^2 \le 1$. First we need to find $\frac{1}{4}|c|^2$.

$$\begin{aligned}
\frac{1}{4}|c|^2 &= \frac{1}{|\lambda_0|^2}\left(\left|\overline{\lambda_0} - \overline{p_1}|\lambda_0|^2\zeta^2\right|^2 - 2|\overline{\lambda_0} - \overline{p_1}|\lambda_0|^2\zeta^2|\,|\lambda_0^2\zeta^2 - p_1\lambda_0| + |\lambda_0^2\zeta^2 - p_1\lambda_0|^2\right)\\
&= \frac{1}{|\lambda_0|^2}\,|\overline{\lambda_0}|^2\left|1 - \overline{p_1}\lambda_0\zeta^2\right|^2 - \frac{2}{|\lambda_0|^2}\,|\overline{\lambda_0}|\,|1 - \overline{p_1}\lambda_0\zeta^2|\,|\lambda_0|\,|\lambda_0\zeta^2 - p_1|\\
&\quad + \frac{1}{|\lambda_0|^2}\,|\lambda_0|^2|\lambda_0\zeta^2 - p_1|^2\\
&= \left|1 - \overline{p_1}\lambda_0\zeta^2\right|^2 - 2|1 - \overline{p_1}\lambda_0\zeta^2|\,|\lambda_0\zeta^2 - p_1| + |\lambda_0\zeta^2 - p_1|^2.
\end{aligned}$$

Then

$$|b_1|^2 = \frac{1}{|1 - \overline{p_1}\zeta^2\lambda_0|^2}\left(|\overline{p_1}\zeta^2 - \overline{\lambda_0}|^2 - \left|1 - \overline{p_1}\lambda_0\zeta^2\right|^2 + 2|1 - \overline{p_1}\lambda_0\zeta^2|\,|\lambda_0\zeta^2 - p_1| - |\lambda_0\zeta^2 - p_1|^2\right).$$

Note that

$$|\lambda_0\zeta^2 - p_1| = |\overline{\lambda_0}\ \overline{\zeta^2} - \overline{p_1}| = |\overline{\lambda_0} - \overline{p_1}\zeta^2| = |\overline{p_1}\zeta^2 - \overline{\lambda_0}|,$$

and so,

$$\begin{aligned}
|b_1|^2 &= \frac{1}{|1 - \overline{p_1}\zeta^2\lambda_0|^2}\left(2|1 - \overline{p_1}\lambda_0\zeta^2|\,|\lambda_0\zeta^2 - p_1| - \left|1 - \overline{p_1}\lambda_0\zeta^2\right|^2\right)\\
&= \frac{1}{|1 - \overline{p_1}\zeta^2\lambda_0|^2}\left\{|1 - \overline{p_1}\lambda_0\zeta^2|\left(2|\lambda_0\zeta^2 - p_1| - |1 - \overline{p_1}\lambda_0\zeta^2|\right)\right\}\\
&= \frac{1}{|1 - \overline{p_1}\zeta^2\lambda_0|}\left(2|\lambda_0\zeta^2 - p_1| - |1 - \overline{p_1}\lambda_0\zeta^2|\right)\\
&= 2\,\frac{|\lambda_0\zeta^2 - p_1|}{|1 - \overline{p_1}\zeta^2\lambda_0|} - 1.
\end{aligned}$$

We claim that

$$0 \le 2\,\frac{|\lambda_0\zeta^2 - p_1|}{|1 - \overline{p_1}\zeta^2\lambda_0|} - 1 \le 1,$$

so that

$$\frac{|\lambda_0\zeta^2 - p_1|}{|1 - \overline{p_1}\zeta^2\lambda_0|} \le 1.$$

Note that

$$\left|\frac{\lambda_0\zeta^2 - p_1}{1 - \overline{p_1}\zeta^2\lambda_0}\right| = |B_{p_1}(\lambda_0\zeta^2)| < 1.$$

Recall that by Theorem 5.2.2, $p_1 = \dfrac{p_0}{\lambda_0}$ and $\zeta\lambda_0|s_0| = |\lambda_0|s_0$. In this case we consider $|p_0| < |\lambda_0|$, and so, $|p_1| = \left|\dfrac{p_0}{\lambda_0}\right| < 1$.
Also,

$$\lambda_0\zeta^2 = \lambda_0\frac{|\lambda_0|^2 s_0^2}{\lambda_0^2|s_0|^2} = \frac{|\lambda_0|^2 s_0^2}{\lambda_0|s_0|^2}.$$

Thus $|\lambda_0\zeta^2| = |\lambda_0| < 1$. □

Theorem 5.3.4. *Let $\lambda_0 \in \mathbb{D} \setminus \{0\}$, and $(a_0, s_0, p_0) \in \overline{\mathcal{P}}$. Then the following conditions are equivalent:*

(1) *there exists a rational $\overline{\mathcal{P}}$-inner function $x = (a, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ such that $x(0) = (0, 0, 0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$;*

(2) *there exists an analytic function $x = (a, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ such that $x(0) = (0, 0, 0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$, and $|a_0| \leq |\lambda_0|\sqrt{1 - \frac{1}{4}|s_0|^2}$;*

(3)
$$|s_0| < 2, \qquad \frac{2|s_0 - p_0\bar{s}_0| + |s_0^2 - 4p_0|}{4 - |s_0|^2} \leq |\lambda_0| \tag{5.22}$$

and

$$|a_0| \leq |\lambda_0|\sqrt{1 - \frac{1}{4}|s_0|^2}. \tag{5.23}$$

Proof. We shall prove that (1) ⇔ (3) and (1) ⇔ (2).

(1) ⇒ (3) Suppose (1) holds, that is, there exists a rational $\overline{\mathcal{P}}$-inner function $x = (a, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ such that $x(0) = (0, 0, 0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$. Let $\deg(x) = (m, n)$, for some positive integers m and n. By Lemma 4.3.1(ii), $h = (s, p) : \mathbb{D} \to \Gamma$ is a rational Γ-inner function. Note that $h(0) = (0, 0)$ and $h(\lambda_0) = (s_0, p_0)$. By Theorem 5.2.1,

$$|s_0| < 2 \quad \text{and} \quad \frac{2|s_0 - p_0\overline{s_0}| + |s_0^2 - 4p_0|}{4 - |s_0|^2} \leq |\lambda_0|.$$

Hence condition (5.22) holds.
By Proposition 2.1.12, there are polynomials E, D such that $\deg(E), \deg(D) \leq n$, $E^{\sim n} = E$, $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$, $|E(\lambda)| \leq 2|D(\lambda)|$ on $\overline{\mathbb{D}}$, such that $s = \frac{E}{D}$ on $\overline{\mathbb{D}}$ and $p = \frac{D^{\sim n}}{D}$ on $\overline{\mathbb{D}}$. By Theorem 4.5.2, there exists an outer polynomial A such that

$$|A(\lambda)|^2 = |D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2 \text{ for all } \lambda \in \mathbb{T}$$

and the given

$$x = (a, s, p) = \left(cB\frac{A}{D}, \frac{E}{D}, \frac{D^{\sim n}}{D}\right) \text{ on } \overline{\mathbb{D}},$$

for some finite Blaschke product B and some c such that $|c| = 1$.
The function $\lambda \mapsto a(\lambda) = cB(\lambda)\dfrac{A}{D}(\lambda)$ is an analytic map from $\mathbb{D} \to \overline{\mathbb{D}}$ and $a(0) = 0$ and

$a(\lambda_0) = a_0$. Note, for all $\lambda \in \overline{\mathbb{D}}$

$$\left|\frac{A}{D}(\lambda)\right|^2 = 1 - \frac{1}{4}\left|\frac{E}{D}(\lambda)\right|^2, \quad \text{and so}$$

$$\left|\frac{A}{D}(\lambda)\right|^2 = 1 - \frac{1}{4}\left|s(\lambda)\right|^2.$$

By construction, A and D are outer polynomials on $\overline{\mathbb{D}}$. Let us consider the function

$$f(\lambda) = \frac{a(\lambda)}{\left(\frac{A}{D}(\lambda)\right)} = cB(\lambda).$$

Thus f is an analytic map from $\mathbb{D} \to \overline{\mathbb{D}}$. By assumption, $a(0) = 0$, and so $f(0) = 0$. By classical Schwarz lemma 5.0.1,

$$|f(\lambda)| \leq |\lambda|, \text{ for } \lambda \in \mathbb{D}.$$

Thus

$$|f(\lambda)| = \left|\frac{a(\lambda)}{\sqrt{1 - \frac{1}{4}|s(\lambda)|^2}}\right| \leq |\lambda|, \text{ for } \lambda \in \mathbb{D}.$$

Therefore,

$$|a(\lambda)| \leq |\lambda|\sqrt{1 - \frac{1}{4}|s(\lambda)|^2}, \quad \text{for } \lambda \in \mathbb{D}.$$

Since $a(\lambda_0) = a_0$, we have

$$|a_0| = |a(\lambda_0)| \leq |\lambda_0|\sqrt{1 - \frac{1}{4}|s_0|^2}.$$

Hence condition (5.23) is satisfied.

(1) $\Rightarrow$ (2) Suppose (1) holds. A rational $\overline{\mathcal{P}}$-inner function $x = (a, s, p) : \mathbb{D} \to \overline{\mathcal{P}}$ is analytic and conditions $x(0) = (0, 0, 0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$ are satisfied. Since (1) $\Rightarrow$ (3), we have $|a_0| \leq |\lambda_0|\sqrt{1 - \frac{1}{4}|s_0|^2}$. Hence (2) holds.

(2) $\Rightarrow$ (1) Suppose (2) holds, namely, there exists an analytic function $x_1 = (a_1, s_1, p_1) : \mathbb{D} \to \overline{\mathcal{P}}$ such that $x_1(0) = (0, 0, 0)$ and $x_1(\lambda_0) = (a_0, s_0, p_0)$. By Lemma 4.3.1(i), $h_1 = (s_1, p_1) : \mathbb{D} \to \Gamma$ is an analytic function with the conditions $h_1(0) = (0, 0)$ and $h_1(\lambda_0) = (s_0, p_0)$. By [25, Theorem 4], there is a rational Γ-inner function $h : \mathbb{D} \to \Gamma$ such that $h(0) = (0, 0)$ and $h(\lambda_0) = (s_0, p_0)$. Let $n = \deg(h)$. By Proposition 2.1.12, there are polynomials E, D satisfying $\deg(E), \deg(D) \leq n$, $E^{\sim n} = E$, $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$ and $|E(\lambda)| \leq 2|D(\lambda)|$ on $\overline{\mathbb{D}}$ such that $h = \left(\frac{E}{D}, \frac{D^{\sim n}}{D}\right)$. By Theorem 4.5.5, we can construct a rational

$\overline{\mathcal{P}}$-inner function

$x = \left(cB\frac{A}{D}, \frac{E}{D}, \frac{D^{\sim n}}{D}\right)$, for some finite Blaschke product B and some $c \in \mathbb{C}$ such that $|c| = 1$,

where A is an outer polynomial such that

$$|A(\lambda)|^2 = |D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2, \text{ for all } \lambda \in \mathbb{T}.$$

Let

$$a(\lambda) = cB(\lambda)\frac{A}{D}(\lambda).$$

We need to find a finite Blaschke product B such that $a(0) = 0$ and $a(\lambda_0) = a_0$. Note that since

$$\frac{|A(\lambda)|^2}{|D(\lambda)|^2} = 1 - \frac{1}{4}|s(\lambda)|^2,$$

we have

$$\left|\frac{A}{D}(\lambda_0)\right|^2 = 1 - \frac{1}{4}|s(\lambda_0)|^2 = 1 - \frac{1}{4}|s_0|^2.$$

Note that since $B(0) = 0$, there is a finite Blaschke product $\tilde{B}$ such that $B(\lambda) = \lambda\tilde{B}(\lambda)$ for all $\lambda \in \mathbb{D}$. Hence

$$a(\lambda) = c\lambda\tilde{B}(\lambda)\frac{A}{D}(\lambda).$$

By assumption (2), $|a_0| \le |\lambda_0|\sqrt{1 - \frac{1}{4}|s_0|^2}$. As in Theorem 5.3.2, to meet the conditions $a(0) = 0$ and $a(\lambda_0) = a_0$, we have two cases:

(i) if $|a_0| = |\lambda_0|\sqrt{1 - \frac{1}{4}|s_0|^2}$, then, for $\lambda \in \mathbb{D}$,

$$a(\lambda) = \gamma\lambda\frac{A}{D}(\lambda),$$

where $|\gamma| = 1$ such that $a_0 = \gamma\lambda_0\sqrt{1 - \frac{1}{4}|s_0|^2}$.

(ii) if $|a_0| < |\lambda_0|\sqrt{1 - \frac{1}{4}|s_0|^2}$, then

$$\frac{|a_0|}{|\lambda_0|\sqrt{1 - \frac{1}{4}|s_0|^2}} < 1.$$

Let

$$\mu_0 \stackrel{\text{def}}{=} \frac{a_0}{\lambda_0\sqrt{1 - \frac{1}{4}|s_0|^2}},$$

it is clear that $\mu_0 \in \mathbb{D}$. We need to find $\tilde{B} : \mathbb{D} \to \mathbb{D}$ such that $\tilde{B}(\lambda_0) = \mu_0$. Consider the following Blaschke factors

$$B_{\mu_0}^{-1}(z) = \frac{z + \mu_0}{1 + \overline{\mu_0}z} \text{ and } B_{\lambda_0}(z) = \frac{z - \lambda_0}{1 - \overline{\lambda_0}z}.$$

Then

$$\tilde{B}(z) = B_{\mu_0}^{-1} \circ B_{\lambda_0}(z) = \frac{z - \lambda_0 + \mu_0(1 - \overline{\lambda_0} z)}{1 - \overline{\lambda_0} z + \overline{\mu_0}(z - \lambda_0)}.$$

Note that $\tilde{B}(\lambda_0) = \dfrac{\mu_0(1 - \overline{\lambda_0}\lambda_0)}{1 - \overline{\lambda_0}\lambda_0} = \mu_0$. Let us define $a : \mathbb{D} \to \mathbb{C}$, for $\lambda \in \mathbb{D}$, by the formula

$$a(\lambda) = \lambda \tilde{B}(\lambda) \frac{A(\lambda)}{D(\lambda)} = \lambda \frac{\lambda - \lambda_0 + \mu_0(1 - \overline{\lambda_0}\lambda)}{1 - \overline{\lambda_0}\lambda + \overline{\mu_0}(\lambda - \lambda_0)} \frac{A(\lambda)}{D(\lambda)},$$

where $\mu_0 = \dfrac{a_0}{\lambda_0 \sqrt{1 - \frac{1}{4}|s_0|^2}}$. Note that $a(0) = 0$.

$$\begin{aligned} a(\lambda_0) &= \lambda_0 \mu_0 \frac{A(\lambda_0)}{D(\lambda_0)} \\ &= \lambda_0 \frac{a_0}{\lambda_0 \sqrt{1 - \frac{1}{4}|s_0|^2}} \sqrt{1 - \frac{1}{4}|s_0|^2} \\ &= a_0. \end{aligned}$$

Therefore the function $x = \left(a, \dfrac{E}{D}, \dfrac{D^{\sim n}}{D}\right)$ is a rational $\overline{\mathcal{P}}$-inner function and conditions $x(0) = (0,0,0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$ are satisfied.

(3) $\Rightarrow$ (1) Suppose (3) holds, that is,

$$|s_0| < 2, \qquad \frac{2|s_0 - p_0\overline{s}_0| + |s_0^2 - 4p_0|}{4 - |s_0|^2} \le |\lambda_0|$$

and

$$|a_0| \le |\lambda_0| \sqrt{1 - \frac{1}{4}|s_0|^2}.$$

By Theorem 5.2.1, there exists an analytic function $h_1 = (s_1, p_1) : \mathbb{D} \to \Gamma$ such that $h_1(0) = (0,0)$ and $h_1(\lambda_0) = (s_0, p_0)$. By [25, Theorem 4], there is a rational Γ-inner function $h : \mathbb{D} \to \Gamma$ such that $h(0) = (0,0)$ and $h(\lambda_0) = (s_0, p_0)$. Let $n = \deg(h)$. By Proposition 2.1.12, there are polynomials E, D satisfying $\deg(E), \deg(D) \le n$, $E^{\sim n} = E$, $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$ and $|E(\lambda)| \le 2|D(\lambda)|$ on $\overline{\mathbb{D}}$ such that $h = \left(\dfrac{E}{D}, \dfrac{D^{\sim n}}{D}\right)$. By Theorem 4.5.5, we can construct a rational $\overline{\mathcal{P}}$-inner function

$$x = \left(cB\frac{A}{D}, \frac{E}{D}, \frac{D^{\sim n}}{D}\right), \text{ for some finite Blaschke product } B \text{ and some } c \in \mathbb{C} \text{ such that } |c| = 1,$$

where A is an outer polynomial such that

$$|A(\lambda)|^2 = |D(\lambda)|^2 - \frac{1}{4}|E(\lambda)|^2, \quad \text{for all } \lambda \in \mathbb{T}.$$

To fulfill the conditions $x(0) = (0,0,0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$, we define a function

$a : \mathbb{D} \to \mathbb{C}$ such that $a(0) = 0$ and $a(\lambda_0) = a_0$ as in (2) $\Rightarrow$ (1). Recall when (i) $|a_0| = |\lambda_0|\sqrt{1 - \frac{1}{4}|s_0|^2}$, then we define a by, for $\lambda \in \mathbb{D}$,

$$a(\lambda) = \gamma\lambda\frac{A}{D}(\lambda),$$

where $|\gamma| = 1$ such that $a_0 = \gamma\lambda_0\sqrt{1 - \frac{1}{4}|s_0|^2}$. When (ii) $|a_0| < |\lambda_0|\sqrt{1 - \frac{1}{4}|s_0|^2}$, we define a by

$$a(\lambda) = \lambda\tilde{B}(\lambda)\frac{A(\lambda)}{D(\lambda)}, \text{ for } \lambda \in \mathbb{D},$$

where $\tilde{B}(\lambda) = \dfrac{\lambda - \lambda_0 + \mu_0(1 - \overline{\lambda_0}\lambda)}{1 - \overline{\lambda_0}\lambda + \overline{\mu_0}(\lambda - \lambda_0)}$. It is easy to see that in both cases $a(0) = 0$ and $a(\lambda_0) = a_0$.

Note that when

$$|s_0| < 2, \qquad \frac{2|s_0 - p_0\bar{s}_0| + |s_0^2 - 4p_0|}{4 - |s_0|^2} = |\lambda_0|$$

and

$$|a_0| \le |\lambda_0|\sqrt{1 - \frac{1}{4}|s_0|^2},$$

an explicit construction of a rational $\overline{\mathcal{P}}$-inner function $x : \mathbb{D} \to \overline{\mathcal{P}}$ such that $x(0) = (0, 0, 0)$ and $x(\lambda_0) = (a_0, s_0, p_0)$ is given in Theorem 5.3.2. □

Chapter 6. Possible further investigations

We end with some open questions.

(1) Is it true that the Carathéodory distance $C_{\mathcal{P}}$ and the Kobayashi distance $K_{\mathcal{P}}$ in $\mathcal{P}$ are equal? A positive answer was given for the symmetrized bidisc $\mathbb{G}$ in [11]. For definitions of $C_{\mathcal{P}}$ and $K_{\mathcal{P}}$, see Chapter C.

(2) Find necessary and sufficient conditions for the solvability of a finite interpolation problem for analytic functions $f : \mathbb{D} \to \mathcal{P}$ satisfying finitely many interpolation conditions $\lambda_j \mapsto W_j \in \mathcal{P}$?

(3) Can we find a geometric classification of all complex geodesics in $\mathcal{P}$, as what was done for the symmetrized bidisc $\mathbb{G}$, see [7].

(4) A more general question: are there other special linear subspaces E of $\mathbb{C}^{n\times m}$ which give rise to interesting domains Ω in $\mathbb{C}^d$ closely related to the μ_E-synthesis problem for functions in $\mathrm{Hol}(\mathbb{D}, \mathbb{C}^{m\times n})$?

Chapter A. Basic definitions

Definition A.0.1. *$H^\infty(\mathbb{D})$ is the Banach space of bounded analytic functions on the open unit disc $\mathbb{D}$ with supremum norm $||f||_\infty = \sup_{\lambda\in\mathbb{D}} |f(\lambda)|$.*

Definition A.0.2. *Let Ω be an open set in $\mathbb{C}$ and $(X, ||.||_X)$ be a Banach space. Then we say a map $f : \Omega \to X$ is analytic on Ω if, for every $z_0 \in \Omega$, there is $f'(z_0) \in X$ such that*

$$\lim_{z\to z_0} \left\| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right\|_X = 0.$$

Remark A.0.3. (Operator norm of a matrix) *Let $A \in \mathbb{C}^{m\times n}$,*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad a_{ij} \in \mathbb{C}.$$

Then A defines a bounded linear operator

$$\begin{aligned} A \ &: \ \mathbb{C}^n \to \mathbb{C}^m \\ &: \ x \longmapsto Ax, \ \ \textit{where} \end{aligned}$$

$$Ax \ = \ \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n a_{1j}x_j \\ \sum_{j=1}^n a_{2j}x_j \\ \vdots \\ \sum_{j=1}^n a_{mj}x_j \end{bmatrix}.$$

The operator norm *of A is given by*

$$\|A\| = \sup_{\|x\|_{\mathbb{C}^n} \leq 1} \|Ax\|_{\mathbb{C}^m}.$$

Definition A.0.4. *A matrix $A \in \mathbb{C}^{n\times n}$ is said to be* positive semi-definite *if $\langle x, Ax\rangle \geq 0$ for all $x \in \mathbb{C}^n$, and* positive definite *if $\langle x, Ax\rangle > 0$ for all vectors $x \neq 0, \ x \in \mathbb{C}^n$.*

Note: A positive semi-definite matrix is positive definite if and only if it is invertible.

Theorem A.0.5. [20, Pages 1, 2] *There are some conditions that characterize positive matrices.*

(i) *A is positive if and only if it is Hermitian and all its eigenvalues are nonnegative. A is strictly positive if and only if it is Hermitian and all its eigenvalues are positive.*

(ii) *A is positive if and only if it is Hermitian and all its principal minors are nonnegative. A is strictly positive if and only if it is Hermitian and all its principal minors are positive.*

(iii) *A is positive if and only if $A = T^*T$ for some upper triangular matrix T. Further, T can be chosen to have nonnegative diagonal entries. If A is strictly positive, then T is unique. A is positive if and only if T is nonsingular.*

Proposition A.0.6. *Let H and G be Hilbert spaces. Define $\mathcal{B}(H, G)$ to be the Banach space of all bounded linear operators $T : H \to G$ with norm*

$$||T|| = \sup\{||Tx||_G : ||x||_H \leq 1\}.$$

Then

$$||T|| \leq 1 \iff I - T^*T \geq 0.$$

Theorem A.0.7. (Schwarz lemma) [15, Theorem 13] *If $f(z)$ is analytic for $|z| < 1$ and satisfies the conditions $|f(z)| \leq 1$, $f(0) = 0$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$, then $f(z) = cz$ for some c such that $|c| = 1$.*

Proof. Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic map such that $f(0) = 0$. Let $g(z) = \dfrac{f(z)}{z}$ for $z \neq 0$, then g is analytic on $\mathbb{D} \setminus 0$. The Taylor expansion of $f(z)$ at $z = 0$ is

$$f(z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \cdots = z\, h(z)$$

where $h(z)$ is analytic on $\mathbb{D}$. Since $h(z) = g(z)$ for $z \neq 0$, $g(z)$ is analytic on $\mathbb{D}$. On the circle $|z| = r < 1,\ r > 0$,

$$|g(z)| = \left|\frac{f(z)}{z}\right| \leq \frac{1}{r}.$$

By the maximum principle, on $|z| \leq r$, $|g(z)|$ has maxima on $|z| = r$, and hence for $|z| \leq r$, $|g(z)| \leq \dfrac{1}{r}$. Letting r tend to 1 we find that $|g(z)| \leq 1$ for all $z \in \mathbb{D}$, and so $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.
Further, let z_0 be such that $|z_0| < 1, z_0 \neq 0$ and $|f(z_0)| = |z_0|$. Then $|g(z_0)| = 1$. By the maximum principle, g is a constant with modulus 1, that means $g(z) = e^{i\theta}$, and so $f(z) = e^{i\theta}z$ is a rotation. $\square$

A.0.1 The Nevanlinna class

Definition A.0.8. [43, Section 15.22] *For any real number* t*, define* $\log^+ t = \log t$ *if* $t \geq 1$ *and* $\log^+ t = 0$ *if* $t < 1$*. The Nevanlinna class on the unit disc* $\mathbb{D}$ *is the class of all* $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ *for which*

$$\sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta < \infty. \tag{A.1}$$

We denote this class by N*.*

Remark A.0.9. *It is clear that* $H^\infty(\mathbb{D}) \subset N$*. Note that equation* (A.1) *imposes a restriction on the rate of growth of* $|f(z)|$ *as* $|z| \to 1$*, whereas the boundedness of the integrals*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})| d\theta \tag{A.2}$$

imposes no such restriction. For instance, equation (A.2) *is independent of* r *if* $f = e^g$ *for any* $g \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$*. The point is that equation* (A.2) *can stay small because* $\log |f|$ *assumes large negative values as well as large positive ones, whereas* $\log^+ |f| \geq 0$*.*

Chapter B. The fundamental group of a topological space

Definition B.0.1. [34, Definition, page 150] *Let $f, g : X \to Y$ be two mappings where X and Y are topological spaces and let $I = [0, 1]$ be the unit interval. Then f and g are said to be* homotopic*, denoted by $f \simeq g$, if there exists a continuous mapping $h : X \times I \to Y$ such that for each x in X*

$$h(x, 0) = f(x) \textit{ and } h(x, 1) = g(x).$$

such a map h is called a homotopy between f and g.

Definition B.0.2. [34, Definition, page 157] *Two topological spaces X and Y are said to be* homotopy equivalent (or of the same homotopy type)*, if there is a pair of continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g : Y \to Y$ is homotopic to the identity map id_Y and $g \circ f : X \to X$ is homotopic to id_X.*

Let y_0 be in a topological space Y and let $C(Y, y_0)$ be the collection of all continuous maps $f : I \to Y$ such that

$$f(0) = y_0 = f(1).$$

Definition B.0.3. [34, Definition, page 159] *If f and g are two maps in $C(Y, y_0)$. Then f is* homotopic to g modulo y_0*, denoted by $f \underset{y_0}{\simeq} g$ if there exists a continuous map $h : I \times I \to Y$ such that*

$$h(x, 0) = f(x) \textit{ and } h(x, 1) = g(x) \text{ for every } x \in I,$$

$$\textit{and } h(0, t) = y_0 = h(1, t) \text{ for every } t \in I.$$

Lemma B.0.4. [34, Lemma 4-16] *Homotopy modulo y_0 is an equivalence relation on $C(Y, y_0)$.*

$C(Y, y_0)$ can be decomposed by the relation homotopy modulo y_0 into disjoint equivalence classes, specifically the arcwise-connected components of $C(Y, y_0)$. Let $\pi_1(Y, y_0)$ denote the collection of these equivalence classes. Let $[f]$ denote the homotopy class such that f is an element of $C(Y, y_0)$. This means $[f]$ denotes the collection of all g in $C(Y, y_0)$

such that $f \underset{y_0}{\simeq} g$. Define the *juxtaposition* $f * g$ of f and g in $\pi_1(Y, y_0)$ by

$$(f * g)(x) = \begin{cases} f(2x) & 0 \leq x \leq \frac{1}{2} \\ g(2x-1) & \frac{1}{2} \leq x \leq 1 \end{cases}$$

Then $f * g$ is also an element of $C(Y, y_0)$, since $(f * g)(\frac{1}{2}) = f(1) = g(0) = y_0$. If $[f]$ and $[g]$ are two elements of $\pi_1(Y, y_0)$, we define their product by

$$[f] \circ [g] = [f * g].$$

The set $\pi_1(Y, y_0)$ is called the *fundamental group* and it is a group under the $\circ$ operation.

Theorem B.0.5. [34, Theorem 4-20] *A mapping $h : (X, x_0) \to (Y, y_0)$ induces a homomorphism $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.*

Proof. Define a mapping

$$\begin{aligned} h_\# \quad &: \quad C(X, x_0) \to C(Y, y_0) \\ &: \quad f \longmapsto h_\# f \end{aligned}$$

given by $(h_\# f)(t) = h(f(t))$. To show that $h_\#$ is continuous, let f_0 belongs to $C(X, x_0)$, and let U be any basis element in the compact open topology of $C(Y, y_0)$ where $h_\# f_0$ is contained in U. Now by definition, U is the collection of all continuous functions in $C(Y, y_0)$ that take a compact set K into an open set O. Consider the basis element U^{-1} of $C(X, x_0)$ comprising all continuous functions taking K into $h^{-1}(O)$. Since $[h_\# f](K)$ lies in O, then $h(f(K))$ lies in O and $f(K)$ lies in $h^{-1}(O)$, and consequently f_0 lies in U^{-1}. Suppose g lies in U^{-1}, then $g(K)$ lies in $h^{-1}(O)$ and $[h_\# g](K) = h(g(K))$ lies in O, and so $h_\# g$ belongs to U. Thus $h_\#$ is continuous. Define h_* by $h_*([f]) = [h_\# f]$. Since $h_\#$ is continuous, it maps $C(X, x_0)$ into $C(Y, y_0)$, so h_* is well-defined. To prove that h_* is a homomorphism, which means,

$$h_*([f] \circ [g]) = h_*([f]) \circ h_*([g]).$$

It suffices to show that

$$h_\#(f * g) = h_\# f * h_\# g$$

which is immediate.

$$[h_\#(f * g)](x) = \begin{cases} h(f(2x)) = [h_\# f](2x) & \text{for } 0 \leq x \leq \frac{1}{2} \\ h(g(2x-1)) = [h_\# g](2x-1) & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$= [h_\# f * h_\# g](x).$$

□

Theorem B.0.6. [34, Theorem 4-21] *Let the mappings f and g from (X, x_0) to (Y, y_0) be homotopic. Then the induced homomorphisms coincide. If $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$, then $(gf)_* = g_* f_*$.*

Theorem B.0.7. [34, Theorem 4-3] *Let Y^X denote the space of all continuous functions of X into Y. Then the homotopy classes of Y^X are precisely the arcwise-connected components of Y^X.*

Lemma B.0.8. [16, Lemma 4.13] *Let B be a finite Blaschke product. Then the degree of B is equal to $B_*(1)$.*

Chapter C. Kobayashi and Carathéodory distances

In this chapter we recall definitions of Kobayashi and Carathéodory distances from [12]. One can find the notions below in a book by Jarnicki and Pflug [36].

The *pseudohyperbolic* distance on $\mathbb{D}$ is defined as

$$d(z,w) = \left| \frac{z-w}{1-\overline{w}z} \right|, \quad z,w \in \mathbb{D}.$$

The *Carathéodary extremal problem* for a domain Ω and for a given pair of points $z, w \in \Omega$ is to find

$$C_\Omega(z,w) := \sup\{d(F(z),F(w)) : F \text{ maps } \Omega \text{ analytically into } \mathbb{D}\}.$$

A function F for which the supremum on the right-hand side is attained is called a *Carathéodary extremal function* for Ω and the points z, w, and C_Ω is called the *Carathéodary pseudodistance* on Ω.

The *Kobayashi extremal problem* for a pair of points $z, w \in \Omega$ is to find the quantity

$$\delta_\Omega(z,w) = \inf d(\lambda_1, \lambda_2),$$

over all pairs $\lambda_1, \lambda_2 \in \mathbb{D}$ such that there exists an analytic function $h : \mathbb{D} \to \Omega$ such that $h(\lambda_1) = z$ and $h(\lambda_2) = w$. Any such function h for which the infimum on the right hand side is attained is called a *Kobayashi extremal function* for Ω and the points z, w. The *Kobayashi distance* K_Ω on Ω is defined to be the largest pseudodistance on Ω dominated by δ_Ω [26].

It is standard that

$$C_\Omega \le K_\Omega \le \delta_\Omega$$

for any domain Ω [26]. If a domain D is bounded and convex *Lempert's theorem* asserts that the Carathéodory and Kobayashi distance coincide on D, that is, $C_D = K_D$, [39].

Chapter D. Some results on the function theory of $\mathbb{G}$

D.1 Royal variety of $\mathbb{G}$ and Aut $\mathbb{G}$

A *flat geodesic* of $\mathbb{G}$ is a complex geodesic of $\mathbb{G}$ which is the intersection of $\mathbb{G}$ with a complex line.

The royal variety $\mathcal{R}_\Gamma$ of the symmetrized bidisc is

$$\mathcal{R}_\Gamma = \{(s,p) \in \mathbb{C}^2 : s^2 = 4p\}.$$

For $w \in \mathbb{T}$, define the function $\Phi_w : \Gamma \to \mathbb{C}$ by

$$\Phi_w(s,p) = \Phi(w,s,p) = \frac{2wp - s}{2 - ws} \text{ for } (s,p) \in \Gamma \text{ such that } ws \neq 2. \tag{D.1}$$

Theorem D.1.1. (1) *Each point of $\mathbb{G}$ lies on a unique flat geodesic.*

(2) *Each flat geodesic meets the royal variety exactly once.*

(3) *For $z_1, z_2 \in \mathbb{G}$, the following are equivalent:*

(i) *Φ_w is an extremal function for the Carathéodory extremal problem for z_1, z_2 for every $w \in \mathbb{T}$;*

(ii) *z_1, z_2 lie either on the royal variety or on a flat geodesic.*

Statement 3(i) means: for all $w \in \mathbb{T}$,

$$C_G(z_1, z_2) = d\big(\Phi_w(z_1), \Phi_w(z_2)\big).$$

Lemma D.1.2. [14] *Every automorphism of $\mathbb{G}$ maps the royal variety $\mathcal{R}_\Gamma \cap \mathbb{G}$ to itself and maps every flat geodesic to a flat geodesic.*

Proof. Let $\alpha \in$ Aut $\mathbb{G}$ and let φ be either the royal or a flat geodesic. Consider any pair of points on the complex geodesic $\alpha \circ \varphi$ of $\mathbb{G}$, say $z_1 = \alpha \circ \varphi(\lambda_1), z_2 = \alpha \circ \varphi(\lambda_2)$ where $\lambda_1 \neq \lambda_2$ in $\mathbb{D}$. Observe that, by Theorem D.1.1, statement (3)(i),

$$C_G\big(\varphi(\lambda_1), \varphi(\lambda_2)\big) = d\big(\Phi_\zeta \circ \varphi(\lambda_1), \Phi_\zeta \circ \varphi(\lambda_2)\big).$$

for all $\zeta \in \mathbb{T}$. We claim that every $\Phi_w, w \in \mathbb{T}$, is a Carathéodory extremal function for z_1, z_2. Indeed, for $w \in \mathbb{T}$, by virtue of Corollary , there exist $m \in \operatorname{Aut} \mathbb{D}, \zeta \in \mathbb{T}$ such that

$$\Phi_w \circ \alpha = m \circ \Phi_\zeta.$$

Then

$$\begin{aligned} d\big(\Phi_w(z_1), \Phi_w(z_2)\big) &= d\Big(\Phi_w\big(\alpha \circ \varphi(\lambda_1)\big), \Phi_w\big(\alpha \circ \varphi(\lambda_2)\big)\Big) \\ &= d\big(m \circ \Phi_\zeta \circ \varphi(\lambda_1), m \circ \Phi_\zeta \circ \varphi(\lambda_2)\big) \\ &= d\big(\Phi_\zeta \circ \varphi(\lambda_1), \Phi_\zeta \circ \varphi(\lambda_2)\big) \\ &= C_G\big(\varphi(\lambda_1), \varphi(\lambda_2)\big) \\ &= C_G\big(\alpha \circ \varphi(\lambda_1), \alpha \circ \varphi(\lambda_2)\big) \\ &= C_G(z_1, z_2), \end{aligned}$$

and Φ_w is a Carathéodory extremal as claimed. Hence, by Theorem D.1.1, statement (3)(ii), the geodesic $\alpha \circ \varphi$ is either royal or flat. Among the class of royal or flat geodesics, the royal variety is the unique one that meets more than one other geodesic in the class, and this property is preserved by automorphisms. Hence if φ is the royal variety then so is $\alpha \circ \varphi$, and $\alpha \circ \varphi_\beta$ is a flat geodesic for every $\beta \in \mathbb{D}$. □

D.2 Solutions to interpolation problems from $\mathbb{D}$ to $\mathbb{G}$

Definition D.2.1. [44] *Let $(\mathbb{C}^n, \|.\|_{\mathbb{C}^n})$ be a Hilbert space. Then:*

(1) *$L^2(\mathbb{T}, \mathbb{C}^n)$ is defined to be the space of square-integrable $\mathbb{C}^n$-valued functions on the unit circle with its natural inner product and norm*

$$\|f\|_{L^2} = \left(\frac{1}{2\pi}\int_0^{2\pi} \|f(e^{i\theta})\|^2_{\mathbb{C}^n}\, d\theta\right)^{\frac{1}{2}}.$$

(2) *$H^2(\mathbb{D}, \mathbb{C}^n)$ is defined to be the space of analytic $\mathbb{C}^n$-valued functions on the unit disc such that*

$$\lim_{r\to 1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \|f(re^{i\theta})\|^2_{\mathbb{C}^n}\, d\theta\right)^{\frac{1}{2}} < \infty.$$

The space $H^2(\mathbb{D}, \mathbb{C}^n)$ can be identified with a closed subspace of $L^2(\mathbb{T}, \mathbb{C}^n)$.

To prove Theorem D.2.3, Costara in [25] relied on the following theorem by Agler and Young.

Theorem D.2.2. [13, Theorem 0.1] *Let $\lambda_1, \dots, \lambda_m \in \mathbb{D}$ be distinct points and $W_1, \dots, W_m \in \Omega$. Put $s_j = \operatorname{tr}(W_j)$ and $p_j = \det(W_j)$ for $j = 1, \dots, m$, and suppose that $s_j^2/4 - p_j \neq 0$ for $j \le k$, and $s_j^2/4 - p_j = 0$ for $j > k$, where $k \ge 1$ (that is, at least one matrix W_j has two distinct eigenvalues). The following assertions are equivalent.*

(1) *There exists an analytic 2×2 matrix-valued function $F : \mathbb{D} \to \Omega$ such that $F(\lambda_j) = W_j$ for $j = 1, \ldots, m$.*

(2) *There exist constants $b_1, \ldots, b_m, c_1, \ldots, c_m \in \mathbb{C}$ such that $b_j c_j = s_j^2/4 - p_j$ for $j = 1, \ldots, k, b_j = c_j = 0$ if $j > k$ and W_j is a scalar matrix, $b_j \neq 0$ and $c_j = 0$ if $j > k$ and W_j is not a scalar matrix, and such that we can find $G : \mathbb{D} \to \mathbb{C}^{2\times 2}$ analytic with $G(\lambda_j) = Y_j$ for $j = 1, \ldots, m$ and $\|G(\lambda)\| \leq 1$ on $\mathbb{D}$, where*

$$Y_j := \begin{bmatrix} s_j/2 & b_j \\ c_j & s_j/2 \end{bmatrix} \qquad (j = 1, \ldots, m).$$

Theorem D.2.3. [25, Theorem 4] *Let $\lambda_1, \ldots, \lambda_m \in \mathbb{D}$ be distincts points, and $(s_1, p_1), \ldots, (s_m, p_m) \in \mathbb{G}$. There exists an analytic function $f : \mathbb{D} \to \mathbb{G}$ with $f(\lambda_j) = (s_j, p_j)$ for $j = 1, \ldots, m$ if and only if there exists $N \leq 4m$ and an analytic function $g = (s, p) : \mathbb{D} \to \mathbb{G}$, where p and s are of the form given by*

$$p(\lambda) = \zeta \prod_{j=1}^{N} \frac{\lambda - \alpha_j}{1 - \overline{\alpha_j}\lambda}, \qquad \text{for } \lambda \in \mathbb{D},\ \zeta \in \mathbb{T} \text{ and } \alpha_1, \ldots, \alpha_N \in \mathbb{D}. \tag{D.2}$$

$$s(\lambda) = r\xi \frac{(\prod_{j=1}^{t}(\lambda + \xi_j))(\prod_{j=1}^{q}(1 - \overline{\beta}_j\lambda)(\lambda - \beta_j))}{\prod_{j=1}^{N}(1 - \overline{\alpha}_j\lambda)}, \tag{D.3}$$

for $\lambda \in \mathbb{D}$, $t, q \in \mathbb{N} \cup \{0\}$ with $t + 2q = N$, $\beta_1, \ldots, \beta_q \in \mathbb{D}, r \in \mathbb{R}_+$ and $\xi, \xi_1, \ldots, \xi_t \in \mathbb{T}$ with $\xi^2 \prod_{j=1}^{t} \xi_j = \zeta$ such that $g(\lambda_j) = (s_j, p_j)$ for $j = 1, \ldots, m$.

Proof. Let $f = (\tilde{s}, \tilde{p})$, and define, for $\lambda \in \mathbb{D}$,

$$F(\lambda) = \begin{bmatrix} 0 & 1 \\ -\tilde{p}(\lambda) & \tilde{s}(\lambda) \end{bmatrix}.$$

Using the fact that $f(\mathbb{D}) \subseteq \mathbb{G}$ we obtain that $F(\mathbb{D}) \subseteq \Omega$, and, by applying Theorem D.2.2, we can find matrices $W_1, \ldots, W_m \in \Omega$ such that W_j is cojugate to

$$\begin{bmatrix} 0 & 1 \\ -p_j & s_j \end{bmatrix} \quad \text{for } j = 1, \ldots, m,$$

and such that there exists an analytic function $\tilde{F} : \mathbb{D} \to \mathbb{C}^{2\times 2}$ with $\tilde{F}(\lambda_j) = W_j$ for $j = 1, \ldots, m$ and $\|\tilde{F}(\lambda)\| \leq 1$ on $\mathbb{D}$. Then [8, p. 181] we can find a rational analytic function $G : \overline{\mathbb{D}} \to \mathbb{C}^{2\times 2}$ of order at most $2m$, which is also inner (that is, its values on $\mathbb{T}$ are unitary matrices), such that $G(\lambda_j) = W_j$ for $j = 1, \ldots, m$. Put $g = (\mathrm{tr}\,(G), \det(G))$ on $\mathbb{D}$. Then g is a rational function of order at most $4m$, we have $g(\lambda_j) = (s_j, p_j)$ for $j = 1, \ldots, m$, and since $r(.) \leq \|.\|$ on $\mathbb{C}^{2\times 2}$ we obtain that $g(\mathbb{D}) \subseteq \mathbb{G}$. Also, since the spectrum of a unitary matrix is always a subset of $\mathbb{T}$, by the definition of the distinguished boundary of Γ we obtain that g is Γ-inner.

Write $g = (s, p)$. Then p is a finite Blaschke product, and therefore we can find $\zeta \in \mathbb{T}$ and $\alpha_1, \ldots, \alpha_N \in \mathbb{D}$ such that p is given by equation (D.2). For the function s, using the characterization of the distinguished boundary of Γ we obtain that $s = \bar{s}p$ on $\mathbb{T}$. If we consider the Hardy space H^2 (on $\mathbb{T}$), this implies that $s\bar{p} \in \overline{H}^2(\mathbb{D})$, where

$$\begin{aligned}\overline{H}^2(\mathbb{D}) &= \{h \in L^2 : \bar{h} \in H^2(\mathbb{D})\}\\ &= \{h \in L^2 : \widehat{h}(k) = 0,\ \forall\, k \geq 1\}.\end{aligned}$$

If we denote $H^2_-(\mathbb{D}) = \{h \in L^2 : \widehat{h}(k) = 0,\ \forall\, k \geq 0\}$, then $\overline{H}^2(\mathbb{D}) = \lambda H^2_-(\mathbb{D})$, where λ is the identity from $\mathbb{D}$ to $\mathbb{D}$. Therefore $s\bar{p} \in \lambda H^2_-(\mathbb{D})$, and since $|p| = 1$ on $\mathbb{T}$ we obtain that $s \in (\lambda p)H^2_-(\mathbb{D})$. Therefore $s \in H^2(\mathbb{D}) \cap (\lambda p)H^2_-(\mathbb{D})$, and now observe that $H^2(\mathbb{D}) \cap (\lambda p)H^2_-(\mathbb{D})$ is the model space [40, p. 228] for the inner function λp. We have

$$H^2(\mathbb{D}) \bigcap (\lambda p)\ H^2_-(\mathbb{D}) = H^2(\mathbb{D}) \ominus (\lambda p)\ H^2(\mathbb{D}),$$

and this space is finite dimensional, since λp is a finite Blaschke product. If we put $\alpha_0 = 0$ and $b_k = (\lambda - \alpha_k)/(1 - \bar{\alpha}_k\lambda)$ for $k = 0, \ldots, N$, then according to [31, p. 279]

$$\varphi_0 = \frac{d_0}{1 - \bar{\alpha}_0\lambda}$$

and

$$\varphi_j = d_j \frac{1}{1 - \bar{\alpha}_j\lambda} b_{j-1} \ldots b_0 \qquad (1 \leq j \leq N),$$

where $d_j = (1 - |\alpha_j|^2)^{\frac{1}{2}}$ for all j, form a basis of $H^2(\mathbb{D}) \ominus (\lambda p)H^2(\mathbb{D})$. Therefore, for $s \in H^2(\mathbb{D}) \ominus (\lambda p)H^2(\mathbb{D})$ we can find a polynomial Q of degree at most N such that $s = Q/P$, where $P(\lambda) = \prod_{j=1}^N (1 - \bar{\alpha}_j\lambda)$. Since $s = \bar{s}p$ on $\mathbb{T}$, this implies that $Q/\overline{Q} = \zeta\lambda^N$ on $\mathbb{T}$, and then Q must be of the form

$$Q(\lambda) = r\xi \left(\prod_{j=1}^{t} (\lambda + \xi_j)\right) \left(\prod_{j=1}^{q} (1 - \bar{\beta}_j\lambda)(\lambda - \beta_j)\right),$$

where $t, q \in \mathbb{N} \cup \{0\}$ with $t + 2q = N$, $\beta_1, \ldots, \beta_q \in \mathbb{D}$, $r \in \mathbb{R}_+$ and $\xi, \xi_1, \ldots, \xi_t \in \mathbb{T}$ are such that $\xi^2 \prod_{j=1}^t \xi_j = \zeta$. Therefore, s is of the form (D.3). Since $|s| < 2$ on $\mathbb{D}$, then $r \geq 0$ is such that $\|s\|_\infty \leq 2$.

Let us also observe that if p and s are given respectively by equations (D.2) and (D.3), then by putting $g = (s, p)$ it follows that g is a rational analytic function on $\mathbb{D}$, with $g(\mathbb{T}) \subseteq b\Gamma$. If, in addition, we also have $g(0) \in \mathbb{G}$, then g is Γ-inner. $\square$

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