A flat extension theorem for truncated matrix-valued multisequences

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Abstract

Given a truncated multisequence of $p \times p$ Hermitian matrices $S := (S_{\gamma_1,\ldots,\gamma_d})_{\substack{(\gamma_1,\ldots,\gamma_d)\in\mathbb{N}_0^d \\ 0\leq\gamma_1+\cdots+\gamma_d\leq m}}$, *the truncated matrix-valued moment problem on* $\mathbb{R}^d$ asks whether or not there exists a $p \times p$ positive semidefinite matrix-valued measure T, with convergent moments of all orders, such that

$$S_{\gamma_1,\ldots,\gamma_d} = \int \cdots \int_{\mathbb{R}^d} x_1^{\gamma_1} \cdots x_d^{\gamma_d} dT(x_1, \ldots, x_d)$$

for all $(\gamma_1, \ldots, \gamma_d) \in \mathbb{N}_0^d$ which satisfy $0 \leq \sum_{j=1}^d \gamma_j \leq m$. When such a measure exists we say that T is a representing measure for S. We shall see that if m is even, then S has a minimal representing measure (that is, $\sum\limits_{a=1}^{\kappa} \operatorname{rank} Q_a$ is as small as possible) if and only if a block matrix determined entirely by S has a rank-preserving positive extension. In this case, the support of the representing measure has a connection with zeros (suitably interpreted) of a system of matrix-valued polynomials which describe the rank-preserving extension. The proof of this result relies on operator theory and certain results for ideals of multivariate matrix-valued polynomials. Our result subsumes the celebrated flat extension theorem of Curto and Fialkow.

We shall pay particularly close attention to the bivariate quadratic matrix-valued moment problem (that is, where $d = 2$ and $m = 2$).

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Contents

Chapter 1

Introduction

1.1 The flat extension theorem for matricial moments

We will first introduce frequently used definitions and notation. Commonly used sets are $\mathbb{N}_0, \mathbb{R}, \mathbb{C}$ denoting the sets of nonnegative integers, real numbers and complex numbers respectively. Given a nonempty set E, we let

$$E^d = \{(x_1, \dots, x_d) : x_j \in E \ \text{ for } \ j = 1, \dots, d\}.$$

Next, we let $\mathbb{C}^{p\times p}$ denote the set of $p \times p$ matrices with entries in $\mathbb{C}$ and $\mathcal{H}_p \subseteq \mathbb{C}^{p\times p}$ denote the set of $p \times p$ Hermitian matrices with entries in $\mathbb{C}$. Moreover, let $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{N}_0^d$. We define

$$\Gamma_{m,d} := \{\gamma \in \mathbb{N}_0^d : 0 \leq |\gamma| \leq m\}.$$

Given $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{N}_0^d$, we define

$$x^\lambda = \prod_{j=1}^{d} x_j^{\lambda_j} \quad \text{and} \quad |\lambda| = \lambda_1 + \cdots + \lambda_d.$$

We will be considering *the truncated matrix-valued moment problem on* $\mathbb{R}^d$. Given a truncated multisequence $S := (S_\gamma)_{\gamma\in\Gamma_{m,d}}$, we wish to find a $p \times p$ positive semidefinite matrix-valued measure T on $\mathbb{R}^d$ such that

$$S_\gamma = \int_{\mathbb{R}^d} x^\gamma dT(x) \quad \text{for } \gamma \in \Gamma_{m,d}$$

and T has convergent moments of all possible orders. When such a measure exists we say that T is a *representing measure* for S. We will be interested in the case when the representing measure T is minimal, that is, T is of the form $T = \sum_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$ and $\sum_{a=1}^{\kappa} \operatorname{rank} Q_a = \operatorname{rank} M(n)$, or equivalently, $\sum_{a=1}^{\kappa} \operatorname{rank} Q_a$ is as small as possible (see Definition 1.4.31 for the definition of $M(n)$).

In order to communicate our solution to the truncated matrix-valued moment problem on

$\mathbb{R}^d$ we require the notion of flatness for a positive moment matrix. We refer to [16] where this approach has its origin in the truncated moment problem on $\mathbb{R}^d$.

Definition 1.1.1. Let $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$ be a given truncated $\mathcal{H}_p$-valued multisequence and $M(n) \succeq 0$ be the corresponding moment matrix (see Definition 1.4.31). Then $M(n)$ has a *flat extension* if there exist $(S_\gamma)_{\gamma\in\Gamma_{2n+2,d}\setminus\Gamma_{2n,d}}$, where $S_\gamma \in \mathcal{H}_p$ for $\gamma \in \Gamma_{2n+2,d} \setminus \Gamma_{2n,d}$ such that $M(n+1) \succeq 0$ and

$$\operatorname{rank} M(n) = \operatorname{rank} M(n+1).$$

The main purpose of this thesis is to formulate and prove a flat extension theorem which features the minimality of a representing measure for the given data and establishes connections among the flat extension, zeros of the determinants of certain matrix-valued polynomials and the support of a representing measure.

1.2 Motivation and aims

In [64], Mourrain and Schmüdgen studied extensions and representations for Hermitian functionals $L : \mathscr{A} \to \mathbb{C}$, where $\mathscr{A}$ is a unital $*$-algebra. Let $\mathscr{C}$ be a $*$-invariant subspace of a unital $*$-algebra $\mathscr{A}$ and $\mathscr{C}^2 := \operatorname{span}\{ab : a, b \in \mathscr{C}\}$. Suppose $\mathscr{B} \subseteq \mathscr{C}$ is a $*$-invariant subspace of $\mathscr{A}$ such that $1 \in \mathscr{B}$. Mourrain and Schmüdgen say that a Hermitian linear functional $L : \mathscr{C}^2 \to \mathbb{C}$ has a flat extension with respect to $\mathscr{B}$ if

$$\mathscr{C} = \mathscr{B} + K_L(\mathscr{C}),$$

where $K_L(\mathscr{C}) := \{a \in \mathscr{C} : L(b^*a) = 0\}$. In [64], Mourrain and Schmüdgen showed that every positive flat linear functional $L : \mathscr{C} \to \mathbb{C}$ has a unique extension $\tilde{L} : \mathscr{A} \to \mathbb{C}$. Mourrain and Schmüdgen also showed that if $\mathscr{A} = \mathbb{C}^{d\times d}[x_1, \ldots, x_d]$ (see Definition 2.0.1), $\mathscr{B} = \mathbb{C}^{d\times d}_n[x_1, \ldots, x_d]$ (see Definition 2.0.2), $\mathscr{C} = \mathbb{C}^{d\times d}_{n+1}[x_1, \ldots, x_d]$ and $L : \mathscr{C}^2 \to \mathbb{C}$ is a positive linear functional which has a flat extension with respect to $\mathscr{B}$, then

$$L\big((p_{jk})_{j,k=1}^d\big) = \sum_{j,k=1}^{d} \sum_{i=1}^{r} p_{jk}(t_i)u_{ki}\bar{u}_{ji} \quad \text{for} \quad (p_{jk}) \in \mathbb{C}^{d\times d}[x_1, \ldots, x_d] \tag{1.1}$$

for some choice of $t_1, \ldots, t_r \in \mathbb{R}^d$ and $u_1, \ldots, u_r \in \mathbb{C}^d$ with $u_i = \operatorname{col}(u_{ki})_{k=1}^d$ for $i = 1, \ldots, r$, and in particular,

$$L(x^\gamma I_d) = \sum_{j=1}^{d} \sum_{i=1}^{r} t_i^\gamma |u_{ji}|^2 \quad \text{for} \quad 0 \le |\gamma| \le 2n + 2. \tag{1.2}$$

The aim of this thesis is to formulate and prove a flat extension theorem for matricial moments $(S_\gamma)_{\substack{\gamma\in\mathbb{N}_0^d \\ 0\le|\gamma|\le 2n}}$, where S_γ is a $p \times p$ Hermitian matrix for all $\gamma \in \mathbb{N}_0^d$ satisfying $0 \le |\gamma| \le 2n$, that has an integral representation, which is closer in analogy to Curto and

Fialkow's flat extension theorem (which we have reformulated in Theorem 1.4.27 for the convenience of the reader) compared to formula (1.1) that has the additional constraint $d = p$.

Let us further elaborate on the truncated moment problem on $\mathbb{R}^d$ in the scalar setting. In [16] Curto and Fialkow describe a recursive model for singular positive Hankel matrices and show that when the truncated moment problem is of flat data type, a solution exists and it can be constructed from the simultaneous zeros of a collection of polynomials which describe the linear dependence of the extension of the moment matrix. Curto and Fialkow have used the flat extension approach to discover a number of truncated moment problems which have a concrete solution (see, e.g., [17, 18, 19, 20, 21, 22]).

We observe that the flatness condition admits a natural analogue in the setting where the given finite multisequence is Hermitian matrix-valued. However, it is not immediately obvious what the role of the variety of a moment matrix should be. With concepts from noncommutative algebraic geometry for matrix-valued polynomials such as the variety of a right ideal in the set of matrix-valued polynomials, we can extract information concerning the representing measure and its support.

Furthermore, in [16], Curto and Fialkow investigate the bivariate quadratic moment problem in an equivalent setting in $\mathbb{C}$. It is shown that given $s_{00}, s_{10}, s_{01}, s_{20}, s_{11}, s_{02}$, with $s_{00} > 0$, the corresponding moment matrix being positive semidefinite is enough to guarantee the existence of a minimal (that is, $\operatorname{rank} M(1)$-atomic) representing measure. It is natural to wonder if a similar result holds for matrix-valued moments. We investigate the bivariate quadratic matrix-valued moment problem and present a series of necessary and sufficient conditions for a minimal solution with the use of the flat extension theorem for matricial moments. We shall see that the matricial bivariate quadratic moment problem is more technically demanding than its scalar-valued counterpart considered in [16].

1.3 Background

The moment problem on $\mathbb{R}^d$ is a well-known problem in classical analysis and has been studied by mathematicians and engineers since the late 19th century, beginning with Stieltjes [77], Hamburger [42, 43], Hausdorff [44] and Riesz [67]. The full moment problem on $\mathbb{R}$ has a concrete solution discovered by Hamburger [42, 43] which can be communicated solely in terms of the positivity of Hankel matrices built from the given sequence. It is natural to wonder about a multidimensional analogue of the full moment problem on $\mathbb{R}$, that is, the full moment problem on $\mathbb{R}^d$, where the given sequence is a multisequence indexed by d-tuples of nonnegative integers. It is well known that a natural analogue of Hamburger's theorem fails (see, e.g., Schmüdgen [72]), particularly, there exist multisequences such that the corresponding multivariable Hankel matrices are positive semidefinite yet the multisequences do not have a representing measure. It turns out that the Hamburger moment problem on $\mathbb{R}^d$ is a special case of the full K-moment problem on $\mathbb{R}^d$ (where we wish to find a positive measure which is supported on a given closed set $K \subseteq \mathbb{R}^d$). We refer the reader to Riesz [67] (solution on $\mathbb{R}$), Haviland [45, 46] (generalisation for $d > 1$) and Schmüdgen [70] (when K is a compact semialgebraic set). For a solution to the truncated K-moment problem on $\mathbb{R}^d$ based on commutativity conditions of certain matrices see Kimsey [52], where an application to the subnormal completion problem is considered. Moment problems on $\mathbb{R}^d$ intertwine many different areas of mathematics such as matrix and operator theory, probability theory, optimisation theory, and the theory of orthogonal polynomials. Various applications for moment problems on $\mathbb{R}^d$ can be found in control theory, polynomial optimisation and mathematical finance (see, e.g., Lasserre [60] and Laurent [61]). For approaches to the multidimensional moment problem which utilise techniques from real algebra see Marshall [62] and Prestel and Delzell [66]. For a treatment of the abstract multidimensional moment problem see Berg, Christensen and Ressel [7] and Sasvári [68], which, in addition, treats indefinite analogues of multidimensional moment problems.

The truncated moment problem on $\mathbb{R}$, that is, where one is given a truncated sequence $(s_j)_{j=0}^m$ with $s_j \in \mathbb{R}$ for $j = 0, \dots, m$, has a concrete solution which can be communicated in terms of positivity of a Hankel matrix and checking a range inclusion. Moreover, a minimal representing measure can be constructed from the zeros of the polynomial describing a rank-preserving positive extension. We refer the reader to the classical works of Akhiezer [1], Akhiezer and Krein [2], Krein and Nudel'man [59], Shohat and Tamarkin [73] and the fairly recent work of Curto and Fialkow [15]. An area of active interest concerns the truncated moment problem on $\mathbb{N}_0$ where one seeks a measure whose support is contained in a given closed subset $K \subseteq \mathbb{N}_0$ (see, e.g., Infusino, Kuna, Lebowitz and Speer [49]).

Curto and Fialkow in a series of papers studied scalar truncated moment problems on $\mathbb{R}^d$ and $\mathbb{C}^d$ (which is equivalent to the truncated moment problem on $\mathbb{R}^{2d}$). We refer the reader to [16, 17, 18, 19, 20, 21, 22] where concrete conditions for a solution to various moment problems are investigated. For connections between bivariate moment matrices and flat extensions see Fialkow and Nie [37, 38], Fialkow [35] and Curto and Yoo [25]. For the bivariate cubic moment problem we refer the reader to Curto, Lee and Yoon [24], Kimsey [50], and Curto and Yoo [26].

We next wish to mention alternative approaches to the flat extension theorem for the truncated moment problem on $\mathbb{R}^d$. The core variety approach to the truncated moment problem began with the study of Fialkow [36]. Subsequently, Blekherman and Fialkow in [8] strengthened the core variety approach to feature a necessary and sufficient condition for a solution. For additional results related to the core variety approach see Schmüdgen [72] and di Dio and Schmüdgen [30]. Recently, in [23], Curto, Ghasemi, Infusino and Kuhlmann investigated the theory of positive extensions of linear functionals showing the existence of an integral representation for the linear functional.

We now wish to bring the matrix-valued and operator-valued moment problem into focus. The matrix-valued moment problem on $\mathbb{R}$ was initially investigated by Krein [57, 58]. See [65] for a thorough review on Krein's work on moment problems. Andô in [4] was the first to study the truncated moment problem in the operator-valued case. Kovalishina studied the nondegenerate case in [55, 56]. Bolotnikov considered the degenerate truncated matrix-valued Hamburger and Stieltjes moment problems in terms of a linear fractional transformation, see

[10, 11, 12]. Dym [31] considered the truncated matrix-valued Hamburger moment problem associating it with parametrised solutions of a matrix interpolation problem. Alpay and Loubaton in [3] treated the partial trigonometric moment problem on an interval in the matrix case, where Toeplitz matrices built from the moments are associated to orthogonal polynomials. For connections between matrix-valued orthogonal polynomials and CMV matrices we refer the reader to Dym and Kimsey [32].

Simonov studied the strong matrix-valued Hamburger moment problem in [74, 75]. The truncated matrix-valued moment problem on a finite closed interval was studied by Choque Rivero, Dyukarev, Fritzsche and Kirstein [13, 14]. Using Potapov's method of Fundamental Matrix Inequalities they characterised the solutions by nonnegative Hermitian block Hankel matrices and they investigated further the case of an odd number of prescribed moments. Dyukarev, Fritzsche, Kirstein, Mädler and Thiele [34] studied the truncated matrix-valued Hamburger moment problem with an algebraic approach based on matrix-valued polynomials built from a nonnegative Hermitian block Hankel matrix. Dyukarev, Fritzsche, Kirstein and Mädler [33] studied the truncated matrix-valued Stieltjes moment problem via a similar approach.

Bakonyi and Woerdeman in [5] studied the univariate truncated matrix-valued Hamburger moment problem and the odd case of the bivariate truncated matrix-valued moment problem. Kimsey and Woerdeman in [54] investigated the odd case of the truncated matrix-valued K-moment problem on $\mathbb{R}^d$, $\mathbb{C}^d$ and $\mathbb{T}^d$, where they discovered easily checked commutativity conditions for the existence of a minimal representing measure.

Applications on matrix-valued moment problems with related topics have been studied extensively in recent years. Geronimo [39] studied scattering theory and matrix orthogonal polynomials with the construction of a matrix-valued distribution function built from matrix-valued moments. Dette and Studden in [27] investigated matrix orthogonal polynomials and matrix-valued measures associated with certain matricial moments from a numerical analysis point of view. In [28], Dette and Studden considered optimal design problems in linear models as a statistical application of the problem of maximising matrix-valued Hankel determinants built from matricial moments. Moreover, Dette and Tomecki in [29] studied the distribution of random Hankel block matrices and random Hankel determinant processes with respect to certain matricial moments.

1.4 Known results

1.4.1 The truncated moment problem on $\mathbb{R}^d$

We present basic notation and definitions from matrix analysis (see, e.g., [47, 48] for further details).

Definition 1.4.1. We denote by $\mathbb{C}^p$ the *p-dimensional complex vector space*. We consider $\mathbb{C}^p$ equipped with the standard inner product $\langle \xi, \eta \rangle = \eta^* \xi$, where $\xi, \eta \in \mathbb{C}^p$.

Definition 1.4.2. We denote by $\mathbb{C}^{p\times p}$ the *set of* $p \times p$ *matrices over the complex numbers* $\mathbb{C}$ and by $\mathbb{R}^{p\times p}$ the *set of* $p \times p$ *matrices over the real numbers* $\mathbb{R}$. The $p \times p$ matrix of zeros is denoted by $0_{p\times p}$ and the $p \times p$ identity matrix is denoted by I_p.

Definition 1.4.3. A matrix $A \in \mathbb{C}^{p\times p}$ is called *Hermitian* if $A = A^*$.

Definition 1.4.4. We denote by $\mathcal{H}_p \subseteq \mathbb{C}^{p\times p}$ the *set of* $p \times p$ *Hermitian matrices over* $\mathbb{C}$.

Definition 1.4.5. A matrix $A \in \mathbb{C}^{p\times p}$ is called *positive semidefinite* if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^p$. We write $A \succeq 0$.

Definition 1.4.6. Let $A, B \in \mathbb{C}^{p\times p}$. We write $A \succeq B$ if $A - B$ is positive semidefinite.

Definition 1.4.7. A matrix $A \in \mathbb{C}^{p\times p}$ is called *positive definite* if $x^*Ax > 0$ for all $x \in \mathbb{C}^p \setminus \{0\}$.

Definition 1.4.8. Let $A, B \in \mathbb{C}^{p\times p}$. We write $A \succ B$ if $A - B$ is positive definite.

Definition 1.4.9. Let $A \in \mathbb{C}^{p\times p}$. If there exists $c \in \mathbb{C}$ such that $Ax = cx$ for some nonzero vector $x \in \mathbb{C}^p$, then c is called an *eigenvalue of* A corresponding to the eigenvector x. The set of all eigenvalues of A is called the *spectrum of* A and is denoted by $\sigma(A)$.

Let us provide more definitions used to formulate the truncated moment problem on $\mathbb{R}^d$.

Definition 1.4.10. Let $\mathbb{N}_0$ denote the *nonnegative integers*. Let E be a nonempty set and

$$E^d = \{(x_1, \ldots, x_d) : x_j \in E \ \text{ for } \ j = 1, \ldots, d\}.$$

If $E = \mathbb{N}_0$, we let $\varepsilon_j \in \mathbb{N}_0^d$ denote a d-tuple of zeros with 1 in the j-th entry.

Definition 1.4.11. ([71, p. 400]) The *Borel* σ*-algebra* $\mathcal{B}(\mathbb{R}^d)$ is the σ-algebra on $\mathbb{R}^d$ generated by the open subsets of $\mathbb{R}^d$. A Borel set in $\mathcal{B}(\mathbb{R}^d)$ is an element of $\mathcal{B}(\mathbb{R}^d)$.

Definition 1.4.12. Let $w \in \mathbb{R}^d$. We denote by δ_w the *Dirac measure* with respect to w, that is,

$$\delta_w(A) = \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{if } w \notin A, \end{cases}$$

where $A \in \mathcal{B}(\mathbb{R}^d)$.

Definition 1.4.13. Given $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{N}_0^d$, we define the *length of* λ

$$|\lambda| = \lambda_1 + \cdots + \lambda_d$$

and the product

$$x^\lambda = \prod_{j=1}^{d} x_j^{\lambda_j}.$$

Definition 1.4.14. Define by $\mathbb{R}[x_1, \ldots, x_d]$ the *ring of real multivariate polynomials with real indeterminates* $x_1, \ldots, x_d$, that is, the ring of polynomials of the form

$$p(x) = \sum_{\lambda \in \Gamma_{n,d}} p_\lambda x^\lambda,$$

where $p_\lambda \in \mathbb{R}$ for $\lambda \in \Gamma_{n,d}$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

Definition 1.4.15. Let $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and let $\lambda = (\lambda_1, \ldots, \lambda_d),\ \lambda' = (\lambda'_1, \ldots, \lambda'_d) \in \mathbb{N}_0^d$. We define the *lexicographic order* $\prec_{\rm lex}$ on $\mathbb{R}[x_1, \ldots, x_d]$ as follows:

$$x_1^{\lambda_1} \cdots x_d^{\lambda_d} \prec_{\rm lex} x_1^{\lambda'_1} \cdots x_d^{\lambda'_d} \quad \text{for} \;\; \lambda_i > \lambda'_i,$$

where i is the smallest integer $i \in \{1, \ldots, d\}$ for which $\lambda_i \neq \lambda'_i$.

Definition 1.4.16. Let $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}_0^d$. We define

$$\Gamma_{m,d} := \{\gamma \in \mathbb{N}_0^d : 0 \leq |\gamma| \leq m\}.$$

Let $\tilde{\gamma} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_d) \in \mathbb{N}_0^d$. We order $\mathbb{N}_0^d$ by the *graded lexicographic order* $\prec_{\rm grlex}$, that is, $\gamma \prec_{\rm grlex} \tilde{\gamma}$ if $|\gamma| < |\tilde{\gamma}|$, or, if $|\gamma| = |\tilde{\gamma}|$ then $x^\gamma \prec_{\rm lex} x^{\tilde{\gamma}}$. We note that $\Gamma_{m,d}$ inherits the ordering of $\mathbb{N}_0^d$ and is such that

$$\operatorname{card} \Gamma_{m,d} = \binom{m+d}{d} := \frac{(m+d)!}{m!d!}.$$

We now formally state the truncated moment problem on $\mathbb{R}^d$.

Problem 1.4.17 (The truncated moment problem on $\mathbb{R}^d$)**.** Let $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Given a finite multisequence of real numbers $s := (s_\gamma)_{\gamma \in \Gamma_{m,d}}$ with $s_{0_d} > 0$, the *truncated moment problem* with data s entails finding a positive Borel measure μ on $\mathbb{R}^d$ such that

$$s_\gamma = \int_{\mathbb{R}^d} x^\gamma d\mu(x) := \int \cdots \int_{\mathbb{R}^d} x_1^{\gamma_1} \cdots x_d^{\gamma_d} d\mu(x_1, \ldots, x_d) \quad \text{for} \;\; \gamma \in \Gamma_{m,d},$$

and

$$\int_{\mathbb{R}^d} |x^\gamma| d\mu(x) < \infty \quad \text{for} \;\; \gamma \in \mathbb{N}_0^d.$$

Definition 1.4.18. Let $(v_\gamma)_{\gamma \in \Gamma_{m,d}}$, where $v_\gamma \in \mathbb{C}^p$ for $\gamma \in \Gamma_{m,d}$. We denote $\operatorname{col}(v_\gamma)_{\gamma \in \Gamma_{m,d}}$ as

$$\operatorname{col}(v_\gamma)_{\gamma \in \Gamma_{m,d}} := \begin{pmatrix} v_{0,0,\ldots,0} \\ \vdots \\ v_{m,0,\ldots,0} \\ \vdots \\ v_{0,\ldots,0,m} \end{pmatrix}.$$

Definition 1.4.19. Let $s := (s_\gamma)_{\gamma\in\Gamma_{2n,d}}$ be a finite multisequence of real numbers and $M(n)$ the corresponding *moment matrix* based on s and defined as follows. We label the rows and columns by a family of monomials $(x^\gamma)_{\gamma\in\Gamma_{n,d}}$ ordered by $\prec_{\text{grlex}}$ (see Definition 1.4.16). We let the entry in the row indexed by x^γ and in the column indexed by $x^{\tilde{\gamma}}$ be given by

$$s_{\gamma+\tilde{\gamma}}.$$

Let $X^\lambda := \operatorname{col}(s_{\lambda+\gamma})_{\gamma\in\Gamma_{n,d}}$ for $\lambda \in \Gamma_{n,d}$ and $C_{M(n)}$ be the *column space of* $M(n)$. Note that $X^\lambda \in C_{M(n)}$.

Definition 1.4.20. Let $\mathbb{R}_n[x_1,\dots,x_d]$ be the *set of all polynomials in* $\mathbb{R}[x_1,\dots,x_d]$ *with total degree* n, that is, $p(x)$ can be written as $p(x) = \sum\limits_{\lambda\in\Gamma_{n,d}} p_\lambda x^\lambda$. We define

$$p(X) := \sum_{\lambda\in\Gamma_{n,d}} p_\lambda X^\lambda \in C_{M(n)}$$

and

$$\mathcal{Z}(p) := \{(x_1,\dots,x_d) \in \mathbb{R}^d : p(x) = 0\}.$$

Definition 1.4.21. ([17, p. 12]) Let $s := (s_\gamma)_{\gamma\in\Gamma_{2n,d}}$ be a finite multisequence of real numbers and let $M(n)$ the corresponding moment matrix. The *variety of* $M(n)$, denoted by $\mathcal{V}(M(n))$, is given by

$$\mathcal{V}(M(n)) := \bigcap_{\substack{p(X)=0 \\ p\in\mathbb{R}_n[x_1,\dots,x_d]}} \mathcal{Z}(p).$$

An answer to Problem 1.4.17, when $p = 1$, $d > 1$ and $\operatorname{card}\operatorname{supp}\mu$ is as small as possible, can be found in [15]. In [15], it is shown that the Problem 1.4.17 has a minimal solution if and only if there exists a solution which is r-atomic where $r := \operatorname{rank} M(n)$, that is, the solution can be expressed as the measure $\mu = \sum\limits_{a=1}^{r} \varrho_a \delta_{u^{(a)}}$ with $\varrho_a > 0$ for every $a = 1,\dots,r$ and $\delta_{u^{(a)}}$ is as in Definition 1.4.12. We say that such a measure is an r-atomic representing measure for s.

The following theorem characterises r-atomic solutions of Problem 1.4.17 when $d = 1$ and $m = 2n$ (see [15, Theorem 3.9]).

Theorem 1.4.22. *Let $s := (s_0,\dots,s_{2n})$ be a finite multisequence of real numbers and let*

$$M(n) = \begin{pmatrix} s_0 & \dots & s_n \\ \vdots & \iddots & \vdots \\ s_n & \dots & s_{2n} \end{pmatrix} \succeq 0$$

be the corresponding moment matrix with $r := \operatorname{rank} M(n)$. Then s has an r-atomic representing measure $\mu = \sum\limits_{a=1}^{r} \varrho_a \delta_{u^{(a)}}$ if and only if the matrix $M(n)$ has an extension $M(n+1) \succeq 0$ such that $\operatorname{rank} M(n) = \operatorname{rank} M(n+1)$. In this case, $\operatorname{supp}\mu = \mathcal{Z}(p)$ and the scalars $\varrho_1,\dots,\varrho_r$ are *given by the Vandermonde equation*

$$\begin{pmatrix} s_0 \\ \vdots \\ \vdots \\ s_{r-1} \end{pmatrix} = \begin{pmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_r \\ \vdots & & \vdots \\ x_1^{n-1} & \dots & x_r^{n-1} \end{pmatrix} \begin{pmatrix} \varrho_0 \\ \vdots \\ \vdots \\ \varrho_{r-1} \end{pmatrix}.$$

The following example illustrates a way to obtain an r-atomic representing measure for a given finite multisequence of real numbers. We will make use of the algorithm described in [15, p. 621].

Example 1.4.23. Let $s := (s_0, \dots, s_4)$ be a given finite multisequence of real numbers with corresponding moment matrix

$$M(2) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \succ 0.$$

We need first to show that $M(2)$ has an extension $M(3) \succeq 0$ with $\operatorname{rank} M(3) = \operatorname{rank} M(2)$. Since each minor determinant of $M(2)$ is positive, we denote c to be the unique scalar such that for $\alpha, \beta \in \mathbb{R}$,

$$\alpha \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} + \beta \begin{pmatrix} s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} s_3 \\ s_4 \\ c \end{pmatrix}.$$

We then get $\alpha = \frac{-3}{10}$, $\beta = \frac{6}{5}$ and $c = \frac{165}{1000}$. Next we extend the original multisequence to $\tilde{s} := (s_0, s_1, s_2, s_3, s_4, c)$. It suffices to show that $\tilde{s}$ has a representing measure. Let $r = 3$ and denote $\Phi := \Phi(\tilde{s}) = (\phi_0, \phi_1, \phi_2) \in \mathbb{R}^3$. Then $\tilde{s}$ has a representing measure if and only if

$$s_j = \phi_0 s_{j-3} + \phi_1 s_{j-2} + \phi_2 s_{j-1}, \quad j = 3, 4.$$

By

$$\begin{pmatrix} s_3 \\ s_4 \\ c \end{pmatrix} = \phi_0 \begin{pmatrix} s_0 \\ s_1 \\ s_2 \end{pmatrix} + \phi_1 \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} + \phi_2 \begin{pmatrix} s_2 \\ s_3 \\ s_4 \end{pmatrix},$$

we derive $\phi_0 = 0$, $\phi_1 = \frac{-3}{10}$ and $\phi_2 = \frac{6}{5}$. The generating function built from Φ, denoted by $g_{\tilde{s}}(x)$, will give rise to the representing measure μ as follows. Notice that $g_{\tilde{s}}(x)$ is given by

$$\begin{aligned} g_{\tilde{s}}(x) &= x^r - (\phi_0 + \dots + \phi_{r-1} x^{r-1}) \\ &= x^3 - (\phi_0 + \phi_1 x + \phi_2 x^2) \\ &= x^3 - \tfrac{6}{5} x^2 + \tfrac{3}{10} x \end{aligned}$$

with zeros

$$x_1 = 0,\ x_2 = \frac{6+\sqrt{6}}{10} \quad \text{and} \quad x_3 = \frac{6-\sqrt{6}}{10}.$$

Since x_1, x_2, x_3 are distinct, the Vandermonde matrix is invertible and $\varrho = V_x^{-1}\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$ is well-defined. Thus we obtain

$$\varrho_1 = \frac{1}{3},\ \varrho_2 = \frac{16+\sqrt{6}}{36} \quad \text{and} \quad \varrho_3 = \frac{16-\sqrt{6}}{36}.$$

Finally $\mu = \sum_{a=1}^{3} \varrho_a \delta_{u^{(a)}}$ is a 3-atomic representing measure for $\tilde{s}$.

We continue with a characterisation for positive extensions given by Smul'jan [76] via the following result.

Lemma 1.4.24 ([76]). *Let $A \in \mathbb{C}^{n\times n}, A \succeq 0, B \in \mathbb{C}^{n\times m}, C \in \mathbb{C}^{m\times m}$ and let*

$$\tilde{A} := \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}.$$

Then:
(i) $\tilde{A}$ is positive semidefinite if and only if $B = AW$ for some $W \in \mathbb{C}^{n\times m}$ and $C \succeq W^*AW$.
(ii) $\tilde{A}$ is positive semidefinite and $\operatorname{rank}\tilde{A} = \operatorname{rank} A$ *if and only if $B = AW$ for some $W \in \mathbb{C}^{n\times m}$ and $C = W^*AW$.*

Definition 1.4.25. Given distinct points $w^{(1)}, \dots, w^{(k)} \in \mathbb{R}^d$ and a subset $\Lambda = \{\lambda^{(1)}, \dots, \lambda^{(k)}\}$ of $\mathbb{N}_0^d$, we define the *multivariable Vandermonde matrix* by

$$V(w^{(1)}, \dots, w^{(k)}; \Lambda) := \begin{pmatrix} \{w^{(1)}\}^{\lambda^{(1)}} & \dots & \{w^{(1)}\}^{\lambda^{(k)}} \\ \vdots & & \vdots \\ \{w^{(k)}\}^{\lambda^{(1)}} & \dots & \{w^{(k)}\}^{\lambda^{(k)}} \end{pmatrix}.$$

We now present [54, Theorem 2.13] which is based on [69, Algorithm 1] and provides a useful machinery throughout this thesis when the invertibility of a multivariable Vandermonde matrix is needed.

Theorem 1.4.26. *Given distinct points $w^{(1)}, \dots, w^{(\kappa)} \in \mathbb{R}^d$, there exists $\Lambda \subseteq \mathbb{N}_0^d$ such that* $\operatorname{card}\Lambda = \kappa$ *and $V(w^{(1)}, \dots, w^{(\kappa)}; \Lambda)$ is invertible.*

In one or several variables one can derive solutions for Problem 1.4.17 based on matrix positivity and extension, see [16]. The next theorem, due to Curto and Fialkow (see [16, Theorem 5.13]) provides necessary and sufficient conditions for a minimal solution to the Problem 1.4.17 when $m = 2n$.

Theorem 1.4.27. *Let $s := (s_\gamma)_{\gamma\in\Gamma_{2n,d}}$ be a given finite multisequence of real numbers and let $M(n) \succeq 0$ be the corresponding moment matrix with $r := \operatorname{rank} M(n)$. Then s has an r-atomic representing measure $\mu = \sum_{a=1}^{r} \varrho_a \delta_{w^{(a)}}$ if and only if the matrix $M(n)$ admits an extension $M(n+1) \succeq 0$ such that $\operatorname{rank} M(n) = \operatorname{rank} M(n+1)$.*

In this case, $\operatorname{supp}\mu = \mathcal{V}(M(n+1))$, and there exists $\Lambda = \{\lambda^{(1)}, \ldots, \lambda^{(r)}\} \subseteq \mathbb{N}_0^d$ with $\operatorname{card}\Lambda = r$ such that $V(w^{(1)}, \ldots, w^{(r)}; \Lambda)$ is invertible. Then the scalars $\varrho_1, \ldots, \varrho_r$ are given by the Vandermonde equation

$$\operatorname{col}(\varrho_a)_{a=1}^r = V(w^{(1)}, \ldots, w^{(r)}; \Lambda)^{-1} \operatorname{col}(s_\lambda)_{\lambda\in\Lambda},$$

where $V(w^{(1)}, \ldots, w^{(r)}; \Lambda) \in \mathbb{C}^{r\times r}$.

In the following example we illustrate Theorem 1.4.27 when $d = 2$ and $n = 1$. We note that throughout the thesis when $d = 2$, we shall use X, Y in place of X_1, X_2, respectively.

Example 1.4.28. Let $(s_\gamma)_{\gamma\in\Gamma_{2,2}}$ be a finite bisequence of real numbers and let $M(1)$ the corresponding moment matrix given by

$$M(1) = \begin{array}{c c} & \begin{array}{ccc} 1 & X & Y \end{array} \\ \begin{array}{c} 1 \\ X \\ Y \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \end{array} \succ 0.$$

We have $M(1)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$. There exists $W \in \mathbb{C}^{3\times 3}$ such that $W = M(1)^{-1}B$. Choose $B = \begin{pmatrix} 2 & 0 & 3 \\ 0 & \sqrt{12} & 0 \\ \sqrt{12} & 0 & 0 \end{pmatrix}$. Then $W = \begin{pmatrix} 2 & 0 & 3 \\ 0 & \sqrt{3} & 0 \\ \frac{2\sqrt{3}}{3} & 0 & 0 \end{pmatrix}$ and

$$C = W^* M(1) W = W^* B = \begin{pmatrix} 8 & 0 & 6 \\ 0 & 6 & 0 \\ 6 & 0 & 9 \end{pmatrix}.$$

By Lemma 1.4.24,

$$M(2) = \begin{pmatrix} M(1) & B \\ B^* & C \end{pmatrix} = \begin{array}{c c} & \begin{array}{cccccc} 1 & X & Y & X^2 & XY & Y^2 \end{array} \\ \begin{array}{c} 1 \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 2 & 0 & 3 \\ 0 & 2 & 0 & 0 & \sqrt{12} & 0 \\ 0 & 0 & 3 & \sqrt{12} & 0 & 0 \\ 2 & 0 & \sqrt{12} & 8 & 0 & 6 \\ 0 & \sqrt{12} & 0 & 0 & 6 & 0 \\ 3 & 0 & 0 & 6 & 0 & 9 \end{pmatrix} \end{array} \succeq 0$$

and

$$\operatorname{rank} M(1) = \operatorname{rank} M(2) = 3.$$

Hence, Theorem 1.4.27 asserts that there exists an r-atomic representing measure for s, which we compute explicitly as follows. We observe that the columns X^2, XY, Y^2 are the linear combinations of the columns 1, X, Y, that is,

$$X^2 = 2 \cdot 1 + \frac{2\sqrt{3}}{3} \cdot Y, \quad XY = \sqrt{3} \cdot X \ \text{ and } \ Y^2 = 3 \cdot 1.$$

We then have the polynomials in $\mathbb{R}_2[x, y]$

$$p_1(x, y) = x^2 - (2 + \frac{2\sqrt{3}}{3} y), \quad p_2(x, y) = xy - \sqrt{3}x, \ \text{ and } \ p_3(x, y) = y^2 - 3$$

and thus $\mathcal{V}(M(2)) = \{(0, \sqrt{3}), (2, \sqrt{3}), (-2, \sqrt{3})\}$. Theorem 1.4.27 yields

$$\begin{pmatrix} \varrho_1 \\ \varrho_2 \\ \varrho_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{8} \\ 0 & -\frac{1}{4} & \frac{1}{8} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

and hence

$$\varrho_1 = \frac{1}{2}, \quad \varrho_2 = \frac{1}{4}, \quad \varrho_3 = \frac{1}{4}.$$

Finally a 3-atomic representing measure for s is $\mu = \sum_{a=1}^{3} \varrho_a \delta_{(u^{(a)}, y_a)}$.

We continue with a result on the cardinality of the support of the representing measure given in [6, Theorem 2].

Theorem 1.4.29. *Let $s := (s_\gamma)_{\gamma \in \Gamma_{m,d}}$ be a given finite multisequence of real numbers with a representing measure ν. Then s has a finitely atomic representing measure μ with*

$$\operatorname{supp} \mu \subseteq \operatorname{supp} \nu \quad \text{and} \quad \operatorname{card} \operatorname{supp} \mu \leq \binom{m+d}{d}.$$

In view of Theorem 1.4.29, Theorem 1.4.27 can be amplified to the following.

Theorem 1.4.30 ([22, p. 180])**.** *Let $s := (s_\gamma)_{\gamma \in \Gamma_{2n,d}}$ be a given finite multisequence of real numbers. s has a finitely atomic representing measure if and only if the corresponding moment matrix $M(n)$ has a positive extension $M(n+k)$ which in turn admits a rank-preserving moment matrix extension for $k \geq 0$.*

1.4.2 The truncated matrix-valued moment problem on $\mathbb{R}^d$

In the current subsection we provide preliminary definitions concerning the matricial case and we pose the truncated matrix-valued moment problem on $\mathbb{R}^d$, see Problem 1.4.37. We refer the reader to the foundational work of Krein in the matricial setting, see [57].

Definition 1.4.31. Let $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$ be a given truncated $\mathcal{H}_p$-valued multisequence and $M(n)$ the corresponding *moment matrix* based on S and defined as follows. We label the block rows and block columns by a family of monomials $(x^\gamma)_{\gamma\in\Gamma_{n,d}}$ ordered by $\prec_{\text{grlex}}$ (see Definition 1.4.16). We let the entry in the block row indexed by x^γ and in the block column indexed by $x^{\tilde{\gamma}}$ be given by

$$S_{\gamma+\tilde{\gamma}}.$$

Definition 1.4.32. A function $T : \mathcal{B}(\mathbb{R}^d) \to \mathcal{H}_p$ is called a *positive $\mathcal{H}_p$-valued Borel measure on* $\mathbb{R}^d$, if for each $u \in \mathbb{C}^p$, $\langle T(\sigma)u, u\rangle$ defines a positive Borel measure on $\mathbb{R}^d$ for all sets $\sigma \in \mathcal{B}(\mathbb{R}^d)$, or, equivalently, if for T_{ab}, $1 \le a, b \le p$, finite complex-valued Borel measures on $\mathbb{R}^d$, we have

$$T(\sigma) := \big(T_{ab}(\sigma)\big)_{a,b=1}^p = \begin{pmatrix} T_{11}(\sigma) & \dots & T_{1p}(\sigma) \\ \vdots & \ddots & \vdots \\ \overline{T_{1p}(\sigma)} & \dots & T_{pp}(\sigma) \end{pmatrix} \succeq 0$$

for all $\sigma \in \mathcal{B}(\mathbb{R}^d)$.

Definition 1.4.33. The *support* of an $\mathcal{H}_p$-valued measure T, denoted by $\operatorname{supp} T$, is defined as the smallest closed subset $\mathcal{G} \subseteq \mathcal{B}(\mathbb{R}^d)$ such that $T(\mathbb{R}^d \setminus \mathcal{G}) = 0_{p\times p}$.

Definition 1.4.34. For a measurable function $f : \mathbb{R}^d \to \mathbb{C}$, we let its integral

$$\int_{\mathbb{R}^d} f(x)\ dT(x) \in \mathcal{H}_p$$

be given by

$$\langle \int_{\mathbb{R}^d} f(x)\ dT(x)u, v\rangle = \int_{\mathbb{R}^d} f(x)\ d\langle T(x)u, v\rangle$$

for all $u, v \in \mathbb{C}^p$, provided all integrals on the right-hand side converge, that is,

$$\int_{\mathbb{R}^d} |f(x)|\ d|\langle T(x)u, v\rangle| < \infty,$$

or, equivalently,

$$\int_{\mathbb{R}^d} f(x)\ dT(x) = \left(\int_{\mathbb{R}^d} f(x)\ dT_{ab}(x)\right)_{a,b=1}^p,$$

where T_{ab} is as in Definition 1.4.32.

Remark 1.4.35. If an $\mathcal{H}_p$-valued measure T is of the form $T = \sum\limits_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$, then

$$\int_{\mathbb{R}^d} f(x)\ dT(x) = \sum_{a=1}^{\kappa} Q_a f(w^{(a)}).$$

Definition 1.4.36. The *power moments* of a positive $\mathcal{H}_p$-valued measure T on $\mathbb{R}^d$ are given by

$$\int\limits_{\mathbb{R}^d} x^\lambda dT(x) \quad \text{for } \lambda \in \mathbb{N}_0^d,$$

provided

$$\int\limits_{\mathbb{R}^d} |x^\lambda| \, d|T_{ab}(x)| < \infty \quad \text{for} \;\; \lambda \in \mathbb{N}_0^d \;\; \text{and} \;\; a, b = 1, \ldots, p.$$

We now present the truncated matrix-valued moment problem on $\mathbb{R}^d$.

Problem 1.4.37 (The truncated matrix-valued moment problem on $\mathbb{R}^d$)**.** Let $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Given a truncated $\mathcal{H}_p$-valued multisequence $S := (S_\gamma)_{\gamma \in \Gamma_{m,d}}$, *the truncated matrix-valued moment problem* with data S entails finding a positive $\mathcal{H}_p$-valued measure T on $\mathbb{R}^d$ such that

$$S_\gamma = \int_{\mathbb{R}^d} x^\gamma dT(x) \quad \text{for} \;\; \gamma \in \Gamma_{m,d},$$

provided

$$\int_{\mathbb{R}^d} |x^\gamma| \, d|T_{ab}(x)| < \infty \quad \text{for} \;\; \gamma \in \mathbb{N}_0^d \quad \text{and} \;\; a, b = 1, \ldots, p.$$

When such a measure exists we say that T is a *representing measure* for S.

Definition 1.4.38. Let $S := (S_\gamma)_{\gamma \in \Gamma_{2n,d}}$ be a given truncated $\mathcal{H}_p$-valued multisequence with a representing measure $T = \sum_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$. We will say that T is *minimal*, if $\sum_{a=1}^{\kappa} \operatorname{rank} Q_a$ is as small as possible. It turns out that the corresponding moment matrix $M(n)$ of S has the property that $\operatorname{rank} M(n) \le \sum_{a=1}^{\kappa} \operatorname{rank} Q_a$ for any representing measure of S (see Lemma 3.3.8) and hence, any minimal representing measure T satisfies

$$\operatorname{rank} M(n) = \sum_{a=1}^{\kappa} \operatorname{rank} Q_a.$$

Throughout the thesis the assumption $S_{0_d} = I_p$ is being used, as stated and explained in the next remark.

Remark 1.4.39. Let $S := (S_\gamma)_{\gamma \in \Gamma_{2n,d}}$ be a given truncated $\mathcal{H}_p$-valued multisequence. We will assume without loss of generality throughout the thesis that S has the property

$$S_{0_d} = I_p. \tag{A1}$$

We note that if S_{0_d} is invertible, then we may consider $\tilde{S} = (\tilde{S}_\gamma)_{\gamma \in \Gamma_{2n,d}}$ in place of S, where

$$\tilde{S}_\gamma = S_{0_d}^{-\frac{1}{2}} S_\gamma S_{0_d}^{-\frac{1}{2}} \in \mathcal{H}_p \quad \text{for } \gamma \in \Gamma_{2n,d}.$$

Notice that S has a representing measure if and only if $\tilde{S}$ has a representing measure. Moreover, notice that if $S_{0_d} = 0_{p \times p}$, then S has the trivial measure. We shall see that if S has a representing measure, then $M(n) \succeq 0$ and there exists an extension $M(n+1) \succeq 0$ (see Lemmas 3.3.1 and 1.4.24). Hence, if S_{0_d} is not invertible and $S_{0_d} \neq 0_{p \times p}$, then for any $\gamma \in \Gamma_{2n,d} \setminus \{0_d\}$, we have

$$\begin{pmatrix} S_{0_d} & S_\gamma \\ S_\gamma & S_{2\gamma} \end{pmatrix} \succeq 0.$$

Thus, by Lemma 1.4.24,

$$\operatorname{Ran} S_\gamma \subseteq \operatorname{Ran} S_{0_d} \tag{1.3}$$

and so

$$(\operatorname{Ran} S_\gamma)^\perp \supseteq (\operatorname{Ran} S_{0_d})^\perp.$$

Therefore, since $S_\gamma, S_{0_d} \in \mathcal{H}_p$,

$$\ker S_\gamma \supseteq \ker S_{0_d}. \tag{1.4}$$

Let $k = \operatorname{rank} S_{0_d}$. Since $S_{0_d} \in \mathcal{H}_p$, we may order the eigenvalues in decreasing order, say

$$\lambda_1 \geq \cdots \geq \lambda_k > \lambda_{k+1} = \cdots = \lambda_p = 0.$$

There exists a set of orthonormal eigenvectors, say $x^{(1)}, \ldots, x^{(p)} \in \mathbb{C}^p$, corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_p$, respectively. Let unitary $U := \left(x^{(1)}\middle| \ \ldots \ \middle| x^{(p)}\right) \in \mathbb{C}^{p\times p}$. Using inclusion (1.4), we see that there exists $\tilde{S}_\gamma \in \mathcal{H}_k$ such that

$$U^* S_\gamma U = \begin{pmatrix} \tilde{S}_\gamma & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for } \gamma \in \Gamma_{2n,d}.$$

Notice that $\tilde{S}_{0_d}$ is invertible. Thus we can proceed as above.

The following example shows that if S_{00} is positive semidefinite and singular, then $M(1) \succeq 0$ does not guarantee the existence of a representing measure.

Example 1.4.40. Let $S := (S_\gamma)_{\gamma\in\Gamma_{2,2}}$ be a given truncated $\mathcal{H}_2$-valued bisequence and let $M(1)$ be the corresponding moment matrix given by

$$M(1) = \begin{array}{c} \\ 1 \\ X \\ Y \end{array} \!\!\begin{array}{c} \begin{array}{ccc} 1 & X & Y \end{array} \\ \begin{pmatrix} S_{00} & S_{10} & S_{01} \\ S_{10} & S_{20} & S_{11} \\ S_{01} & S_{11} & S_{02} \end{pmatrix} \end{array},$$

where $S_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $S_{20} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $S_{10} = S_{01} = S_{11} = S_{02} = 0_{2\times 2}$. We shall see that there are no $S_{30}, S_{21}, S_{12}, S_{03} \in \mathcal{H}_2$ such that

$$\operatorname{Ran} \begin{pmatrix} S_{20} & S_{11} & S_{02} \\ S_{30} & S_{21} & S_{12} \\ S_{21} & S_{12} & S_{03} \end{pmatrix} \subseteq \operatorname{Ran} M(1).$$

Write $M(2) := \begin{pmatrix} M(1) & B \\ B^* & C \end{pmatrix}$, where $B,\ C \in \mathbb{C}^{6\times 6}$. By Lemma 1.4.24, $M(2) \succeq 0$ if and only if there exists $W \in \mathbb{C}^{6\times 6}$ such that

$$M(1)W = B \quad \text{and} \quad C \succeq W^* B.$$

Thus $M(2) \succeq 0$ if and only if

$$\operatorname{Ran} \begin{pmatrix} S_{20} & S_{11} & S_{02} \\ S_{30} & S_{21} & S_{12} \\ S_{21} & S_{12} & S_{03} \end{pmatrix} \subseteq \operatorname{Ran} M(1) \quad \text{and} \quad C \succeq W^* M(1) W,$$

where $W \in \mathbb{C}^{6\times 6}$ is such that $M(1)W = B = \begin{pmatrix} S_{20} & S_{11} & S_{02} \\ S_{30} & S_{21} & S_{12} \\ S_{21} & S_{12} & S_{03} \end{pmatrix}$. However, this would imply that

$$\operatorname{Ran} \begin{pmatrix} S_{20} & S_{11} & S_{02} \end{pmatrix} \subseteq \operatorname{Ran} \begin{pmatrix} S_{00} & S_{10} & S_{01} \end{pmatrix},$$

which cannot hold (see range inclusion (1.3) in Remark 1.4.39) and hence the claim is proved.

For an example where a truncated $\mathcal{H}_2$-valued bisequence does not have a minimal representing measure we refer the reader to [53].

1.5 Results by chapters

In Chapter 2, we establish several algebraic results involving matrix-valued polynomials with real indeterminates. We define the notions of a right ideal and the variety of the ideal in our matricial setting. Furthermore, we are particularly interested in studying the notion of a real radical of the set of the matrix-valued polynomials.

In Chapter 3, we present a series of results on infinite moment matrices with finite rank. We establish necessary conditions for the existence of a solution of the matrix-valued moment problem on $\mathbb{R}^d$ for a full or a truncated $\mathcal{H}_p$-valued multisequence. We abstract the notion of the variety of a moment matrix introduced by Curto and Fialkow in [16] to our matricial setting and obtain a number of algebraic results such as the notion of a right ideal in the set of matrix-valued polynomials with real indeterminates.

In the same chapter, through a series of results on the variety of the moment matrix and its connection with the support of the representing measure, we arrive at the following integral representation for a full $\mathcal{H}_p$-valued multisequence with corresponding moment matrix of finite rank.

Theorem 1.5.1. *Let $S^{(\infty)} := (S_\gamma)_{\gamma\in\mathbb{N}_0^d}$ be a given $\mathcal{H}_p$-valued multisequence. If $S^{(\infty)}$ gives rise to $M(\infty) \succeq 0$ and $r := \operatorname{rank} M(\infty) < \infty$, then $S^{(\infty)}$ has a unique representing measure T. In this case,*

$$\operatorname{supp} T = \mathcal{V}(\mathcal{I}),$$

and moreover,

$$\operatorname{card} \mathcal{V}(\mathcal{I}) = r.$$

In Chapter 3, positive extension results for truncated moment matrices are also provided.

In Chapter 4, we obtain the main result of this thesis, namely the flat extension theorem for matricial moments. We establish necessary and sufficient conditions for the existence of a minimal solution to the truncated matrix-valued moment problem on $\mathbb{R}^d$.

Theorem 1.5.2 (flat extension theorem for matricial moments)**.** *Let $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$ be a given truncated $\mathcal{H}_p$-valued multisequence, $M(n) \succeq 0$ be the corresponding moment matrix and $r := \operatorname{rank} M(n)$. S has a representing measure*

$$T = \sum_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$$

with

$$\sum_{a=1}^{\kappa} \operatorname{rank} Q_a = r$$

if and only if the matrix $M(n)$ admits an extension $M(n+1) \succeq 0$ such that

$$\operatorname{rank} M(n) = \operatorname{rank} M(n+1).$$

Moreover,

$$\operatorname{supp} T = \mathcal{V}(M(n+1)),$$

and there exists $\Lambda = \{\lambda^{(1)}, \dots, \lambda^{(\kappa)}\} \subseteq \mathbb{N}_0^d$ with $\operatorname{card} \Lambda = \kappa$ such that the multivariable Vandermonde matrix $V^{p\times p}(w^{(1)}, \dots, w^{(\kappa)}; \Lambda) \in \mathbb{C}^{\kappa p\times\kappa p}$ is invertible. Then the positive semidefinite matrices $Q_1, \dots, Q_\kappa \in \mathbb{C}^{p\times p}$ are given by the Vandermonde equation

$$\operatorname{col}(Q_a)_{a=1}^{\kappa} = V^{p\times p}(w^{(1)}, \dots, w^{(\kappa)}; \Lambda)^{-1} \operatorname{col}(S_\lambda)_{\lambda\in\Lambda}.$$

Proof. See Theorem 4.0.2. □

In Chapter 5, we study the bivariate quadratic matrix-valued problem where the given matricial truncated bisequence is $\mathcal{H}_p$-valued. We investigate a series of necessary and sufficient conditions for a minimal solution to the bivariate quadratic matrix-valued moment problem. We observe that the matricial setting is more demanding than the scalar-valued considered in [16]. For $p = 1$, Curto and Fialkow [16] showed that every $S := (S_\gamma)_{\gamma\in\Gamma_{2,2}} = (S_{00}, S_{10}, S_{01}, S_{20}, S_{11}, S_{02})$, with $S_{00} > 0$ and $M(1) \succeq 0$ has a minimal representing measure. Curto and Fialkow's proof is divided in three cases according to the value of $\operatorname{rank} M(1)$. The technical challenges in the matrix-valued setting ($p \geq 1$) can been seen in the following theorem where we treat matrix equations to obtain a minimal solution to the bivariate quadratic matrix-valued moment problem.

Theorem 1.5.3. *Let $S := (S_\gamma)_{\gamma\in\Gamma_{2,2}}$ be a given truncated $\mathcal{H}_p$-valued bisequence and*

$$M(1) = \begin{array}{c} \\ 1 \\ X \\ Y \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & X & Y \end{array} \\ \begin{pmatrix} I_p & S_{10} & S_{01} \\ S_{10} & S_{20} & S_{11} \\ S_{01} & S_{11} & S_{02} \end{pmatrix} \end{array}$$

be the corresponding moment matrix. S has a minimal repre-

senting measure if and only if the following conditions hold:
(i) $M(1) \succeq 0$.
(ii) There exist $S_{30}, S_{21}, S_{12}, S_{03} \in \mathcal{H}_p$ *such that*

$$\operatorname{Ran}\begin{pmatrix} S_{20} & S_{11} & S_{02} \\ S_{30} & S_{21} & S_{12} \\ S_{21} & S_{12} & S_{03} \end{pmatrix} \subseteq \operatorname{Ran} M(1)$$

(hence, there exists $W = (W_{ab})_{a,b=1}^{3} \in \mathbb{C}^{3p\times 3p}$ *such that* $M(1)W = B$, *where*

$$B = \begin{pmatrix} S_{20} & S_{11} & S_{02} \\ S_{30} & S_{21} & S_{12} \\ S_{21} & S_{12} & S_{03} \end{pmatrix})$$

and moreover, the following matrix equations hold:

$$W_{11}^* S_{11} + W_{21}^* S_{21} + W_{31}^* S_{12} = S_{11} W_{11} + S_{21} W_{21} + S_{12} W_{31}, \tag{1.5}$$

$$W_{13}^* S_{20} + W_{23}^* S_{30} + W_{33}^* S_{21} = W_{12}^* S_{11} + W_{22}^* S_{21} + W_{32}^* S_{12} \tag{1.6}$$

and

$$W_{12}^* S_{02} + W_{22}^* S_{12} + W_{32}^* S_{03} = S_{02} W_{12} + S_{12} W_{22} + S_{03} W_{32}. \tag{1.7}$$

In the same chapter, we also consider special cases where $M(1)$ is positive semidefinite and singular. When $p \geq 2$, we observe that a straightforward analogue of Curto and Fialkow's result does not hold. However, through a series of theorems we shall see that $S := (S_\gamma)_{\gamma\in\Gamma_{2,2}}$, $S_{00} \succ 0$, with $M(1) \succeq 0$ having certain block columns, has a minimal representing measure.

Chapter 2

Matrix-valued polynomials

In this chapter, we introduce important definitions and notation while establishing several algebraic results involving matrix-valued polynomials with real indeterminates. We study the notions of a right ideal and the variety of the ideal. We also introduce the notion of an ideal of matrix-valued polynomials being real radical.

Definition 2.0.1. Let $\mathbb{C}^{p\times p}[x_1,\dots,x_d]$ denote the *set of $p\times p$ matrix-valued polynomials with real indeterminates* $x_1,\dots,x_d$, that is, $\mathbb{C}^{p\times p}[x_1,\dots,x_d]$ consists of matrix-valued polynomials of the form

$$P(x)=\sum_{\lambda\in\Gamma_{n,d}} x^\lambda P_\lambda,$$

where $P_\lambda\in\mathbb{C}^{p\times p}$, $x^\lambda=\prod_{j=1}^{d} x_j^{\lambda_j}$ for $\lambda\in\Gamma_{n,d}$ and $n\in\mathbb{N}_0$ is arbitrary.

Definition 2.0.2. Let $\mathbb{C}_n^{p\times p}[x_1,\dots,x_d]$ denote the *set of $p\times p$ matrix-valued polynomials with degree at most n with real indeterminates* $x_1,\dots,x_d$, that is, $\mathbb{C}_n^{p\times p}[x_1,\dots,x_d]$ consists of matrix-valued polynomials of the form

$$P(x)=\sum_{\lambda\in\Gamma_{n,d}} x^\lambda P_\lambda,$$

where $P_\lambda\in\mathbb{C}^{p\times p}$, $x^\lambda=\prod_{j=1}^{d} x_j^{\lambda_j}$ for $\lambda\in\Gamma_{n,d}$.

Definition 2.0.3. A set $\mathscr{I}\subseteq\mathbb{C}^{p\times p}[x_1,\dots,x_d]$ is a *right ideal* if it satisfies the following conditions:
(i) $P+Q\in\mathscr{I}$ whenever $P,Q\in\mathscr{I}$.
(ii) $PQ\in\mathscr{I}$ whenever $P\in\mathscr{I}$ and $Q\in\mathbb{C}^{p\times p}[x_1,\dots,x_d]$.

Definition 2.0.4. Let $\mathscr{I}\subseteq\mathbb{C}^{p\times p}[x_1,\dots,x_d]$ be a right ideal. We shall let

$$\mathcal{V}(\mathscr{I}):=\{x\in\mathbb{R}^d:\det P(x)=0\quad\text{for all}\quad P\in\mathscr{I}\}$$

be the *variety associated with the ideal $\mathscr{I}$*.

Definition 2.0.5. A right ideal $\mathscr{I} \subseteq \mathbb{C}^{p\times p}[x_1,\ldots,x_d]$ is *real radical* if

$$\sum_{a=1}^{\kappa} P^{(a)}(x)\{P^{(a)}(x)\}^* \in \mathscr{I} \Longrightarrow P^{(a)}(x) \in \mathscr{I} \quad \text{for } \ a=1,\ldots,\kappa.$$

Remark 2.0.6. We wish to justify the usage of the moniker real radical of Definition 2.0.5 when $p=1$. We note that one usually says that a real ideal $\mathscr{K} \subseteq \mathbb{R}[x_1,\ldots,x_d]$ is real radical if

$$\sum_{a=1}^{\kappa}(f^{(a)}(x))^2 \in \mathscr{K} \Longrightarrow f^{(a)} \in \mathscr{K} \quad \text{for } \ a=1,\ldots,\kappa$$

(see, e.g., Laurent [61]). Suppose $\mathscr{I} = \mathscr{I}_1 + \mathscr{I}_2\mathrm{i}$, where

$$\mathscr{I}_1 = \{\mathrm{Re}(f(x)) : f \in \mathscr{I}\} \quad \text{and} \quad \mathscr{I}_2 = \{\mathrm{Im}(f(x)) : f \in \mathscr{I}\}$$

and let $f^{(a)} = q^{(a)} + r^{(a)}\mathrm{i}$, where

$$q^{(a)}(x) = \mathrm{Re}(f(x)) \quad \text{and} \quad r^{(a)}(x) = \mathrm{Im}(f(x)).$$

We claim that

$$\sum_{a=1}^{\kappa}((q^{(a)}(x))^2 + (r^{(a)}(x))^2) \in \mathscr{I}_1 \Longrightarrow q^{(a)} \in \mathscr{I}_1,\ r^{(a)} \in \mathscr{I}_2 \quad \text{for } \ a=1,\ldots,\kappa \tag{2.1}$$

holds. We wish to demonstrate a connection between the notion of a real ideal $\mathscr{K} \subseteq \mathbb{R}[x_1,\ldots,x_d]$ being real radical and our notion of a complex ideal $\mathscr{I} \subseteq \mathbb{C}[x_1,\ldots,x_d]$ being real radical, that is,

$$\sum_{a=1}^{\kappa}|f^{(a)}(x)|^2 \in \mathscr{I} \Longrightarrow f^{(a)} \in \mathscr{I} \quad \text{for } \ a=1,\ldots,\kappa.$$

Then

$$\sum_{a=1}^{\kappa}|f^{(a)}(x)|^2 = \sum_{a=1}^{\kappa}((q^{(a)}(x))^2 + (r^{(a)}(x))^2) \in \mathscr{I} \Longrightarrow q^{(a)}(x) + r^{(a)}(x) \in \mathscr{I} \quad \text{for } \ a=1,\ldots,\kappa.$$

Notice that $\mathscr{I}_1, \mathscr{I}_2$ are closed under scalar addition and multiplication and so they are ideals in $\mathbb{R}[x_1,\ldots,x_d]$. If $\sum_{a=1}^{\kappa}|f^{(a)}(x)|^2 \in \mathscr{I}$, then $q^{(a)} + r^{(a)}\mathrm{i} \in \mathscr{I}$ for all $a=1,\ldots,\kappa$. But then

$$q^{(a)} \in \mathscr{I}_1 \quad \text{and} \quad r^{(a)} \in \mathscr{I}_2 \quad \text{for } \ a=1,\ldots,\kappa,$$

since $\mathscr{I} = \mathscr{I}_1 + \mathscr{I}_2\mathrm{i}$. However $|f^{(a)}(x)|^2 = (q^{(a)}(x))^2 + (r^{(a)}(x))^2$ and so

$$\sum_{a=1}^{\kappa}|f^{(a)}(x)|^2 \in \mathscr{I} \Longrightarrow q^{(a)} \in \mathscr{I}_1,\ r^{(a)} \in \mathscr{I}_2 \quad \text{for } \ a=1,\ldots,\kappa$$

can be written as

$$\sum_{a=1}^{\kappa}((q^{(a)}(x))^2 + (r^{(a)}(x))^2) \in \mathscr{I} \Longrightarrow q^{(a)} \in \mathscr{I}_1,\ r^{(a)} \in \mathscr{I}_2 \quad \text{for } \ a = 1, \dots, \kappa.$$

Notice that $\sum\limits_{a=1}^{\kappa}((q^{(a)}(x))^2 + (r^{(a)}(x))^2) \in \mathscr{I}_1$ from which we conclude that the claim (2.1) holds.

In the following remark we will introduce an additional assumption on $\mathscr{I} \subseteq \mathbb{C}[x_1, \dots, x_d]$ which appears in Remark 2.0.6. As we noted in Remark 2.0.6, $\mathscr{I} = \mathscr{I}_1 + \mathscr{I}_2\mathrm{i}$, where $\mathscr{I}_1, \mathscr{I}_2$ are real ideals in $\mathbb{R}[x_1, \dots, x_d]$. Thus, it is clear that $f \in \mathscr{I}$ vanishes on a set $V \subseteq \mathbb{R}^d$ if and only if $\mathrm{Re}(f(x))$ and $\pm\,\mathrm{Im}(f(x))$ vanish on V. In view of the Real Nullstellensatz (see, e.g., [9]), any real radical ideal must agree with its vanishing ideal (that is, the set of polynomials which vanish on the variety). Therefore, if $\mathscr{I} \subseteq \mathbb{C}[x_1, \dots, x_d]$ is real radical, then $f \in \mathscr{I}$ implies that $\bar{f} \in \mathscr{I}$.

Remark 2.0.7. Let $\mathscr{I} \subseteq \mathbb{C}[x_1, \dots, x_d]$ and $\mathscr{I}_1, \mathscr{I}_2 \subseteq \mathbb{R}[x_1, \dots, x_d]$ be as in Remark 2.0.6. Suppose $\mathscr{I}$ has the additional property that $f \in \mathscr{I}$ implies $\bar{f} \in \mathscr{I}$. Then
(i) $\mathscr{I}_1 \subseteq \mathbb{R}[x_1, \dots, x_d]$ is real radical.
(ii) $\mathscr{I}_2 \subseteq \mathbb{R}[x_1, \dots, x_d]$ is real radical.

Since $\mathscr{I}$ is an ideal in $\mathbb{C}[x_1, \dots, x_d]$ which is closed under complex conjugation, we have that $\mathscr{I}_1$ and $\mathscr{I}_2$ are subideals of $\mathscr{I}$ over $\mathbb{R}[x_1, \dots, x_d]$. Hence, we may use the fact that $\mathscr{I}$ is real radical to deduce (i) and (ii) .

Lemma 2.0.8. *Fix $\gamma \in \mathbb{N}_0^d$ with $|\gamma| > n$ and let $P(x) = x^\gamma I_p + \sum\limits_{\lambda \in \Gamma_{n,d}} x^\lambda P_\lambda \in \mathbb{C}^{p \times p}[x_1, \dots, x_d]$. Then*

$$\det P(x) = x^{\gamma p} + \sum_{\lambda \in \Gamma_{m,d}} x^\lambda h_\lambda,$$

where $\gamma p := (\gamma_1 p, \dots, \gamma_d p) \in \mathbb{N}_0^d$ and $m < |\gamma| p$.

Proof. We proceed by induction on p. For $p = 2$,

$$P(x) = \begin{pmatrix} x^\gamma + \beta_{11}(x) & \beta_{12}(x) \\ \beta_{21}(x) & x^\gamma + \beta_{22}(x) \end{pmatrix},$$

where $\beta_{ab}(x) = \sum\limits_{\lambda \in \Gamma_{n,d}} x^\lambda P_\lambda^{(a,b)} \in \mathbb{C}[x_1, \dots, x_d]$ with $P_\lambda^{(a,b)}$ the (a, b)-th entry of P_λ, $1 \le a, b \le 2$. We also have

$$\begin{aligned} \det P(x) &= (x^\gamma + \beta_{11}(x))(x^\gamma + \beta_{22}(x)) - \beta_{12}(x)\beta_{21}(x) \\ &= x^{2\gamma} + x^\gamma\beta_{22}(x) + x^\gamma\beta_{11}(x) + \beta_{11}(x)\beta_{22}(x) - \beta_{12}(x)\beta_{21}(x) \\ &= x^{2\gamma} + L(x) + C(x), \end{aligned}$$

where $L(x) = x^\gamma\beta_{22}(x) + +x^\gamma\beta_{11}(x)$, $C(x) = \beta_{11}(x)\beta_{22}(x) - \beta_{12}(x)\beta_{21}(x) \in \mathbb{C}[x_1, \dots, x_d]$.

Suppose the claim holds for $p > 2$. We have

$$P(x) = \begin{pmatrix} x^\gamma + \beta_{11}(x) & \dots & \beta_{1p}(x) \\ \vdots & \ddots & \vdots \\ \beta_{p1}(x) & \dots & x^\gamma + \beta_{pp}(x) \end{pmatrix}$$

and so

$$\begin{aligned} \det P(x) &= (x^\gamma + \beta_{11}(x)) \det \begin{pmatrix} x^\gamma + \beta_{22}(x) & \dots & \beta_{2p}(x) \\ \vdots & \ddots & \vdots \\ \beta_{p2}(x) & \dots & x^\gamma + \beta_{pp}(x) \end{pmatrix} + \dots + \\ &+(-1)^{1+p}\beta_{1p}(x) \det \begin{pmatrix} \beta_{21}(x) & \dots & \beta_{2,p-1}(x) \\ \vdots & \ddots & \vdots \\ \beta_{p1}(x) & \dots & \beta_{p,p-1}(x) \end{pmatrix} \\ &= (x^\gamma + \beta_{11}(x)) \Bigg[(x^\gamma + \beta_{22}(x)) \begin{pmatrix} x^\gamma + \beta_{33}(x) & \dots & \beta_{3p}(x) \\ \vdots & \ddots & \vdots \\ \beta_{p3}(x) & \dots & x^\gamma + \beta_{pp}(x) \end{pmatrix} + \dots + \\ &+(-1)^{1+(2+p-1)}\beta_{2,p-1}(x) \det \begin{pmatrix} \beta_{31}(x) & \dots & \beta_{3,p-2}(x) \\ \vdots & \ddots & \vdots \\ \beta_{p1}(x) & \dots & \beta_{p-1,p-1}(x) \end{pmatrix} \Bigg]. \end{aligned}$$

Let $\widetilde{L}(x)$ be the sum of the terms of $\det P(x)$ of degree up to $\gamma(p-1)$ with $|\gamma| > 0$ and $\widetilde{C}(x)$ the sum of the terms of $\det P(x)$ of degree up to γp with $|\gamma| = 0$. Then

$$\widetilde{L}(x) + \widetilde{C}(x) = \sum_{\lambda \in \Gamma_{m,d}} x^\lambda h_\lambda,$$

where $m < |\gamma| p$. Thus

$$\det P(x) = x^{\gamma p} + \sum_{\lambda \in \Gamma_{m,d}} x^\lambda h_\lambda. \qquad \square$$

We order the monomials in $\mathbb{C}[x_1, \dots, x_d]$ by the graded lexicographic order $\prec_{\text{grlex}}$ (see Definition 1.4.16).

Remark 2.0.9. Fix $\gamma \in \mathbb{N}_0^d$ with $|\gamma| > n$ and let $P(x) = x^\gamma I_p + \sum\limits_{\lambda \in \Gamma_{n,d}} x^\lambda P_\lambda \in \mathbb{C}^{p \times p}[x_1, \dots, x_d]$. For a polynomial $\varphi(x) \in \mathbb{C}[x_1, \dots, x_d]$ given by

$$\varphi(x) := \det P(x) = x^{\gamma p} + \sum_{\lambda \in \Gamma_{m,d}} x^\lambda h_\lambda,$$

where $\gamma p := (\gamma_1 p, \dots, \gamma_d p) \in \mathbb{N}_0^d$ and $m < |\gamma| p$, the leading term of $\varphi(x)$ is

$$\mathrm{LT}(\varphi(x)) = x^{\gamma p}.$$

Definition 2.0.10. We define the basis of $\mathbb{C}^{p\times p}$ viewed as a vector space over $\mathbb{C}$

$$\mathcal{A}^{p\times p} := \{E_{11}, E_{12}, \ldots, E_{1p}, E_{21}, \ldots, E_{2p}, \ldots, E_{p1}, \ldots, E_{pp}\},$$

where $E_{jk} \in \mathbb{C}^{p\times p}$ is the matrix with 1 in the (j,k)-th entry and 0 in the rest of the entries, $j,k = 1,\ldots,p$.

Definition 2.0.11. Given a right ideal $\mathscr{I} \subseteq \mathbb{C}^{p\times p}[x_1,\ldots,x_d]$, we define

$$\mathscr{I}_{jk} := \{f \in \mathbb{C}[x_1,\ldots,x_d] : \text{there exists } F \in \mathscr{I} \text{ such that } F(x)E_{jk} = f(x)E_{jk}\} \subseteq \mathbb{C}[x_1,\ldots,x_d],$$

where $E_{jk} \in \mathbb{C}^{p\times p}$ is as in Definition 2.0.10 for all $j,k = 1,\ldots,p$.

Lemma 2.0.12. *Suppose $\mathscr{I} \subseteq \mathbb{C}^{p\times p}[x_1,\ldots,x_d]$ is a right ideal. Then $\mathscr{I}_{jk} \subseteq \mathbb{C}[x_1,\ldots,x_d]$ is an ideal for all $j,k = 1,\ldots,p$.*

Proof. If $f,g \in \mathscr{I}_{jk}$, then

$$f(x)E_{jk} = F(x)E_{jk} \text{ for } F \in \mathscr{I}$$

and

$$g(x)E_{jk} = G(x)E_{jk} \text{ for } G \in \mathscr{I}.$$

Since $(f+g)(x)E_{jk} = (F+G)(x)E_{jk}$, we have

$$f + g \in \mathscr{I}_{jk}.$$

If $f \in \mathscr{I}_{jk}$ and $h \in \mathbb{C}[x_1,\ldots,x_d]$, then

$$(fh)(x)E_{jk} = (Fh)(x)E_{jk}$$

and thus $fh \in \mathscr{I}_{jk}$. ☐

Lemma 2.0.13. *Suppose $\mathscr{I} \subseteq \mathbb{C}^{p\times p}[x_1,\ldots,x_d]$ is a right ideal. If $\mathscr{I}$ is real radical, then $\mathscr{I}_{jj}$ is real radical for all $j = 1,\ldots,p$.*

Proof. We need to show

$$\sum_{a=1}^{\kappa} |f^{(a)}(x)|^2 \in \mathscr{I}_{jj} \Longrightarrow f^{(a)}(x) \in \mathscr{I}_{jj} \quad \text{for } a = 1,\ldots,\kappa.$$

Let $f(x) = \sum_{a=1}^{\kappa} |f^{(a)}(x)|^2 \in \mathscr{I}_{jj}$. Then there exists $F \in \mathscr{I}$ such that

$$f(x)E_{jj} = F(x)E_{jj}.$$

Without loss of generality, we may assume that $F(x) = f(x)E_{jj}$. If we let $F^{(a)}(x) = f^{(a)}(x)E_{jj}$, then

$$\sum_{a=1}^{\kappa} F^{(a)}(x)\{F^{(a)}(x)\}^* = \sum_{a=1}^{\kappa} |f^{(a)}(x)|^2 E_{jj} = f(x)E_{jj}.$$

Thus

$$\sum_{a=1}^{\kappa} F^{(a)}(x)\{F^{(a)}(x)\}^* = F(x)$$

and hence

$$\sum_{a=1}^{\kappa} F^{(a)}(x)\{F^{(a)}(x)\}^* \in \mathscr{I},$$

which implies that $F^{(a)}(x) \in \mathscr{I}$ for all $a = 1, \ldots, \kappa$, since $\mathscr{I}$ is real radical. Consequently,

$$f^{(a)}(x) \in \mathscr{I}_{jj} \quad \text{for} \quad a = 1, \ldots, \kappa$$

and $\mathscr{I}_{jj}$ is real radical. $\square$

Chapter 3

Infinite moment matrices with finite rank

The main aim of this chapter is to study infinite moment matrices with finite rank, necessary conditions for a full or a truncated $\mathcal{H}_p$-valued multisequence to have a representing measure and extension results for moment matrices.

3.1 Infinite moment matrices and matrix-valued polynomials

In this section, we define moment matrices associated with an $\mathcal{H}_p$-valued multisequence. Our aim is to investigate infinite moment matrices and their relation with matrix-valued polynomials. We define the variety of a moment matrix in our matricial setting using zeros of determinants of matrix-valued polynomials. We also present a series of results which connect positivity of an infinite moment matrix and an associated right ideal of matrix-valued polynomials being real radical.

In analogy to Definition 1.4.18, we denote the following block column.

Definition 3.1.1. Let $(V_\lambda)_{\lambda\in\mathbb{N}_0^d}$, where $V_\lambda \in \mathbb{C}^{p\times p}$ for $\lambda \in \mathbb{N}_0^d$. We let

$$\operatorname{col}(V_\lambda)_{\lambda\in\mathbb{N}_0^d} := \begin{pmatrix} V_{0,0,\dots,0} \\ \vdots \\ V_{m,0,\dots,0} \\ \vdots \\ V_{0,\dots,0,m} \\ \vdots \end{pmatrix}.$$

Definition 3.1.2. A *right module $\mathcal{E}$ over* $\mathbb{C}^{p\times p}$ is a set under the operation of addition $+ : \mathcal{E}\times\mathcal{E} \to \mathcal{E}$ together with the right multiplication $\cdot : \mathcal{E}\times\mathbb{C}^{p\times p} \to \mathcal{E}$, which satisfies the following axioms:

For all $V, U \in \mathcal{E}$ and $\Upsilon, \Xi \in \mathbb{C}^{p\times p}$, we have
(i) $(V+U)\Upsilon = V\Upsilon + U\Upsilon$.
(ii) $V(\Upsilon + \Xi) = V\Upsilon + V\Xi$.
(iii) $V(\Xi\Upsilon) = (V\Xi)\Upsilon$.
(iv) $VI_p = V$.

Definition 3.1.3. Let

$$(\mathbb{C}^{p\times p})_0^\omega := \{V = \operatorname{col}(V_\lambda)_{\lambda\in\mathbb{N}_0^d} : V_\lambda \in \mathbb{C}^{p\times p} \text{ and } V_\lambda = 0_{p\times p} \text{ for all but finitely many } \lambda \in \mathbb{N}_0^d\}.$$

Lemma 3.1.4. $(\mathbb{C}^{p\times p})_0^\omega$ *is a right module over* $\mathbb{C}^{p\times p}$*, under the operation of addition given by*

$$A + B = \operatorname{col}(A_\lambda + B_\lambda)_{\lambda\in\mathbb{N}_0^d} \in (\mathbb{C}^{p\times p})_0^\omega$$

for $A = \operatorname{col}(A_\lambda)_{\lambda\in\mathbb{N}_0^d}, B = \operatorname{col}(B_\lambda)_{\lambda\in\mathbb{N}_0^d} \in (\mathbb{C}^{p\times p})_0^\omega$*, together with the right multiplication given by*

$$A \cdot C := \operatorname{col}(A_\lambda C)_{\lambda\in\mathbb{N}_0^d} \in (\mathbb{C}^{p\times p})_0^\omega$$

for $A = \operatorname{col}(A_\lambda)_{\lambda\in\mathbb{N}_0^d} \in (\mathbb{C}^{p\times p})_0^\omega$ *and* $C \in \mathbb{C}^{p\times p}$.

Proof. The axioms of Definition 3.1.2 can be easily verified. □

We now give the definition of an infinite moment matrix based on $S^{(\infty)} := (S_\gamma)_{\gamma\in\mathbb{N}_0^d}$, where $S_\gamma \in \mathcal{H}_p$ for all $\gamma \in \mathbb{N}_0^d$.

Definition 3.1.5. Let $S^{(\infty)} := (S_\gamma)_{\gamma\in\mathbb{N}_0^d}$ be a given $\mathcal{H}_p$-valued multisequence. We define $M(\infty)$ to be the corresponding *moment matrix* based on $S^{(\infty)}$ as follows. We label the block rows and block columns by a family of monomials $(x^\gamma)_{\gamma\in\mathbb{N}_0^d}$ ordered by $\prec_{\text{grlex}}$ (see Definition 1.4.16). We let the entry in the block row indexed by x^γ and in the block column indexed by $x^{\tilde\gamma}$ be given by

$$S_{\gamma+\tilde\gamma}.$$

Let $X^\lambda := \operatorname{col}(S_{\lambda+\gamma})_{\gamma\in\mathbb{N}_0^d}$, $\lambda \in \Gamma_{n,d}$ and $C_{M(\infty)} = \{M(\infty)V : V \in (\mathbb{C}^{p\times p})_0^\omega\}$. We notice that $X^\lambda \in C_{M(\infty)}$.

For $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$ a given truncated $\mathcal{H}_p$-valued multisequence and $M(n)$ the corresponding moment matrix, we let $X^\lambda := \operatorname{col}(S_{\lambda+\gamma})_{\gamma\in\Gamma_{n,d}}$ for $\lambda \in \Gamma_{n,d}$ and $C_{M(n)}$ be the *column space of* $M(n)$.

Remark 3.1.6. Let $S^{(\infty)} := (S_\gamma)_{\gamma\in\mathbb{N}_0^d}$ be a given $\mathcal{H}_p$-valued multisequence. Then we can view $M(\infty) : (\mathbb{C}^{p\times p})_0^\omega \to C_{M(\infty)}$ as a right linear operator, that is,

$$M(\infty)(VQ + V) = M(\infty)VQ + M(\infty)V,$$

for $V = \operatorname{col}(V_\lambda)_{\lambda\in\mathbb{N}_0^d} \in (\mathbb{C}^{p\times p})_0^\omega$ and $Q = \operatorname{col}(Q_\lambda)_{\lambda\in\mathbb{N}_0^d} \in (\mathbb{C}^{p\times p})_0^\omega$.

Definition 3.1.7. Let $S^{(\infty)} := (S_\gamma)_{\gamma\in\mathbb{N}_0^d}$ be a given $\mathcal{H}_p$-valued multisequence and let $M(\infty)$ be the corresponding moment matrix. We define

$$\operatorname{rank} M(\infty) := \sup_{n\in\mathbb{N}} \operatorname{rank} M(n),$$

where $M(n)$ the corresponding moment matrix based on $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$.

Definition 3.1.8. We define the *right linear map*

$$\Phi : \mathbb{C}^{p\times p}[x_1,\dots,x_d] \to C_{M(\infty)}$$

to be given by

$$\Phi(P) = \sum_{\lambda\in\Gamma_{n,d}} X^\lambda P_\lambda,$$

where $P(x) = \sum\limits_{\lambda\in\Gamma_{n,d}} x^\lambda P_\lambda \in \mathbb{C}^{p\times p}[x_1,\dots,x_d]$.

Definition 3.1.9. Given $P(x) = \sum\limits_{\lambda\in\Gamma_{n,d}} x^\lambda P_\lambda \in \mathbb{C}^{p\times p}[x_1,\dots,x_d]$, we let

$$P(X) := \sum_{\lambda\in\Gamma_{n,d}} X^\lambda P_\lambda$$

and

$$\widehat{P} := \operatorname{col}(P_\lambda)_{\lambda\in\Gamma_{n,d}} \oplus \operatorname{col}(0_{p\times p})_{\gamma\in\mathbb{N}_0^d} \in (\mathbb{C}^{p\times p})_0^\omega.$$

Remark 3.1.10. Given $P(x) = \sum\limits_{\lambda\in\Gamma_{n,d}} x^\lambda P_\lambda \in \mathbb{C}^{p\times p}[x_1,\dots,x_d]$, we observe that

$$\Phi(P) = M(\infty)\widehat{P}.$$

Indeed, notice that

$$M(\infty)\widehat{P} = \operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} S_{\gamma+\lambda} P_\lambda\right)_{\gamma\in\mathbb{N}_0^d} = \sum_{\lambda\in\Gamma_{n,d}} X^\lambda P_\lambda = P(X) = \Phi(P).$$

Definition 3.1.11. Let $S^{(\infty)} := (S_\gamma)_{\gamma\in\mathbb{N}_0^d}$ be a given $\mathcal{H}_p$-valued multisequence and let $M(\infty)$ be the corresponding moment matrix. Suppose

$$P(x) = \sum_{\lambda\in\Gamma_{n,d}} x^\lambda P_\lambda \in \mathbb{C}^{p\times p}[x_1,\dots,x_d].$$

We will write $M(\infty) \succeq 0$ if

$$\widehat{P}^* M(\infty)\widehat{P} \succeq 0_{p\times p} \quad \text{for} \;\; P \in \mathbb{C}^{p\times p}[x_1,\dots,x_d],$$

or, equivalently, $M(n) \succeq 0$ for all $n \in \mathbb{N}_0^d$.

Definition 3.1.12. Let $\mathbb{C}^p[x_1,\dots,x_d]$ be the set of *vector-valued polynomials*, that is,

$$q(x)=\sum_{\lambda\in\Gamma_{n,d}} q_\lambda x^\lambda,$$

where $q_\lambda\in\mathbb{C}^p$, $x^\lambda=\prod_{j=1}^{d} x_j^{\lambda_j}$ for $\lambda\in\Gamma_{n,d}$ and n is arbitrary.

Lemma 3.1.13. *Let $S^{(\infty)}:=(S_\gamma)_{\gamma\in\mathbb{N}_0^d}$ be a given $\mathcal{H}_p$-valued multisequence and let $M(\infty)$ be the corresponding moment matrix. Suppose*

$$q(x)=\sum_{\lambda\in\Gamma_{n,d}} q_\lambda x^\lambda\in\mathbb{C}^p[x_1,\dots,x_d].$$

If $M(\infty)\succeq 0$, then

$$\hat{q}^*M(\infty)\hat{q}\geq 0 \quad \textit{for } q\in\mathbb{C}^p[x_1,\dots,x_d].$$

Proof. Let $P(x)=\sum_{\lambda\in\Gamma_{n,d}} x^\lambda P_\lambda\in\mathbb{C}^{p\times p}[x_1,\dots,x_d]$. Then by Definition 3.1.11, $M(\infty)\succeq 0$ if

$$\widehat{P}^*M(\infty)\widehat{P}\succeq 0_{p\times p} \quad \text{for } P\in\mathbb{C}^{p\times p}[x_1,\dots,x_d].$$

If e_1 is a standard basis vector in $\mathbb{C}^p$, then $e_1^*\widehat{P}^*M(\infty)\widehat{P}e_1\geq 0$. Let $q(x):=P(x)e_1$. Notice that $q\in\mathbb{C}^p[x_1,\dots,x_d]$ and

$$\hat{q}^*M(\infty)\hat{q}\geq 0.$$

Since $P\in\mathbb{C}^{p\times p}[x_1,\dots,x_d]$ is arbitrary, so is $q\in\mathbb{C}^p[x_1,\dots,x_d]$. Thus

$$\hat{q}^*M(\infty)\hat{q}\geq 0 \quad \text{for } q\in\mathbb{C}^p[x_1,\dots,x_d].$$

□

Definition 3.1.14. Suppose $M(\infty)\succeq 0$. Let $P\in\mathbb{C}^{p\times p}[x_1,\dots,x_d]$. We define the set

$$\mathcal{I}:=\{P\in\mathbb{C}^{p\times p}[x_1,\dots,x_d]:\widehat{P}^*M(\infty)\widehat{P}=0_{p\times p}\}\subseteq\mathbb{C}^{p\times p}[x_1,\dots,x_d]$$

and the kernel of the map $\Phi:\mathbb{C}^{p\times p}[x_1,\dots,x_d]\to C_{M(\infty)}$ by

$$\ker\Phi:=\{P\in\mathbb{C}^{p\times p}[x_1,\dots,x_d]:M(\infty)\widehat{P}=\operatorname{col}(0_{p\times p})_{\lambda\in\mathbb{N}_0^d}\}.$$

Lemma 3.1.15. *Suppose $M(\infty)\succeq 0$. Then*

$$\mathcal{I}=\ker\Phi,$$

where $\mathcal{I}$ and $\ker\Phi$ are as in Definition 3.1.14.

Proof. By Definition 3.1.11, $M(\infty)\succeq 0$ if $\widehat{P}^*M(\infty)\widehat{P}\succeq 0_{p\times p}$ for $P\in\mathbb{C}^{p\times p}[x_1,\dots,x_d]$ and thus by Lemma 3.1.13, the corresponding moment matrix $M(m)$ based on $S:=(S_\gamma)_{\gamma\in\Gamma_{2m,d}}$ is positive semidefinite for all $m \in \mathbb{N}$. Hence $M(m)^{\frac{1}{2}}$ exists and we let $A := M(m)^{\frac{1}{2}} \operatorname{col}(P_\lambda)_{\lambda \in \Gamma_{m,d}}$, for $P(x) = \sum\limits_{\lambda \in \Gamma_{m,d}} x^\lambda P_\lambda$. Since $P \in \mathcal{I}$,

$$\widehat{P}^* M(\infty) \widehat{P} = 0_{p \times p}.$$

But $\widehat{P}^* M(\infty) \widehat{P} = A^* A$ and hence $A^* A = 0_{p \times p}$. Thus, all singular values of A are 0 and so $\operatorname{rank} A = 0$, which forces

$$A = \operatorname{col}(0_{p \times p})_{\lambda \in \Gamma_{m,d}}.$$

Therefore

$$M(m)^{\frac{1}{2}} \operatorname{col}(P_\lambda)_{\lambda \in \Gamma_{m,d}} = \operatorname{col}(0_{p \times p})_{\lambda \in \Gamma_{m,d}}$$

and

$$M(m) \operatorname{col}(P_\lambda)_{\lambda \in \Gamma_{m,d}} = \operatorname{col}(0_{p \times p})_{\lambda \in \Gamma_{m,d}}. \tag{3.1}$$

We have to show

$$M(\infty) \widehat{P} = \operatorname{col}(0_{p \times p})_{\gamma \in \mathbb{N}_0^d}.$$

We will show that for all $\ell \geq m$,

$$M(\ell) \{ \operatorname{col}(P_\lambda)_{\lambda \in \Gamma_{m,d}} \oplus \operatorname{col}(0_{p \times p})_{\gamma \in \Gamma_{l,d} \setminus \Gamma_{m,d}} \} = \operatorname{col}(0_{p \times p})_{\gamma \in \Gamma_{\ell,d}}. \tag{3.2}$$

First notice

$$\operatorname{card} \Gamma_{m,d} = \binom{m+d}{d}, \quad \operatorname{card} \Gamma_{\ell,d} = \binom{\ell+d}{d}$$

and

$$\operatorname{card}(\Gamma_{\ell,d} \setminus \Gamma_{m,d}) = \binom{\ell+d}{d} - \binom{m+d}{d}.$$

We write

$$M(\ell) = \begin{pmatrix} M(m) & B \\ B^* & C \end{pmatrix} \succeq 0,$$

where

$$M(m) \in \mathbb{C}^{(\operatorname{card} \Gamma_{m,d})p \times (\operatorname{card} \Gamma_{m,d})p}, \quad B \in \mathbb{C}^{(\operatorname{card} \Gamma_{m,d})p \times (\operatorname{card}(\Gamma_{\ell,d} \setminus \Gamma_{m,d}))p}$$

and $C \in \mathbb{C}^{(\operatorname{card} \Gamma_{\ell,d} \setminus \Gamma_{m,d})p \times (\operatorname{card}(\Gamma_{\ell,d} \setminus \Gamma_{m,d}))p}$. Since $M(\ell) \succeq 0$, by Lemma 1.4.24, there exists $W \in \mathbb{C}^{(\operatorname{card} \Gamma_{m,d})p \times (\operatorname{card}(\Gamma_{\ell,d} \setminus \Gamma_{m,d}))p}$ such that $M(m)W = B \quad \text{and} \quad C \succeq W^* M(m) W$. Then

$$\begin{aligned} M(\ell) \{ \operatorname{col}(P_\lambda)_{\lambda \in \Gamma_{m,d}} \oplus \operatorname{col}(0_{p \times p})_{\gamma \in \Gamma_{\ell,d} \setminus \Gamma_{m,d}} \} &= \begin{pmatrix} M(m) \operatorname{col}(P_\lambda)_{\lambda \in \Gamma_{m,d}} \\ B^* \operatorname{col}(P_\lambda)_{\lambda \in \Gamma_{m,d}} \end{pmatrix} \\ &= \begin{pmatrix} \operatorname{col}(0_{p \times p})_{\gamma \in \Gamma_{m,d}} \\ W^* M(m) \operatorname{col}(P_\lambda)_{\lambda \in \Gamma_{m,d}} \end{pmatrix} \\ &= \operatorname{col}(0_{p \times p})_{\gamma \in \Gamma_{\ell,d}}, \end{aligned}$$

by equation (3.1).

Thus, equation (3.2) holds for all $\ell \geq m$ and we obtain

$$M(\infty)\widehat{P} = \mathrm{col}(0_{p\times p})_{\gamma\in\mathbb{N}_0^d},$$

which implies $P \in \ker \Phi$.

Conversely, if $P \in \ker \Phi$ then

$$M(\infty)\widehat{P} = \mathrm{col}(0_{p\times p})_{\lambda\in\mathbb{N}_0^d}$$

and so $\widehat{P}^* M(\infty)\widehat{P} = 0_{p\times p}$, that is, $P \in \mathcal{I}$. □

Lemma 3.1.16. *Suppose $M(\infty) \succeq 0$. Then $\mathcal{I} = \ker \Phi$ is a right ideal.*

Proof. Let $P, Q \in \mathbb{C}^{p\times p}[x_1, \ldots, x_d]$. We have to show the following:
(i) If $P \in \ker \Phi$ and $Q \in \ker \Phi$, then $P + Q \in \ker \Phi$.
(ii) If $P \in \ker \Phi$ and $Q \in \mathbb{C}^{p\times p}[x_1, \ldots, x_d]$, then $PQ \in \ker \Phi$.
To prove (i) notice that since $P \in \ker \Phi$,

$$M(\infty)\widehat{P} = \mathrm{col}(0_{p\times p})_{\lambda\in\mathbb{N}_0^d}$$

and similarly, since $Q \in \ker \Phi$,

$$M(\infty)\widehat{Q} = \mathrm{col}(0_{p\times p})_{\lambda\in\mathbb{N}_0^d}.$$

We then have

$$M(\infty)\widehat{Q} + M(\infty)\widehat{P} = M(\infty)(\widehat{P+Q}) = \mathrm{col}(0_{p\times p})_{\lambda\in\mathbb{N}_0^d},$$

that is, $P + Q \in \ker \Phi$.
To prove (ii) we need to show that if $P \in \ker \Phi$ and $Q \in \mathbb{C}^{p\times p}[x_1, \ldots, x_d]$, then

$$M(\infty)(\widehat{PQ}) = \mathrm{col}(0_{p\times p})_{\lambda\in\mathbb{N}_0^d}.$$

For

$$P(x) = \sum_{\lambda\in\Gamma_{n,d}} x^\lambda P_\lambda \quad \text{and} \quad Q(x) = \sum_{\lambda\in\Gamma_{n,d}} x^\lambda Q_\lambda,$$

we let

$$R(x) = P(x)Q(x) = \sum_{\lambda'\in\Gamma_{n,d}} P(x)x^{\lambda'}Q_{\lambda'}.$$

We will show

$$M(\infty)(\widehat{x^{\lambda'}P}) = \mathrm{col}(0_{p\times p})_{\gamma\in\mathbb{N}_0^d}. \tag{3.3}$$

We have

$$M(\infty)(\widehat{x^{\lambda'}P}) = \mathrm{col}\left(\sum_{\lambda\in\Gamma_{n,d}} S_{\gamma+\lambda'+\lambda}P_\lambda\right)_{\gamma\in\mathbb{N}_0^d}.$$

But since $P \in \ker \Phi$,

$$M(\infty)\widehat{P} = \operatorname{col}(0_{p\times p})_{\tilde{\gamma}\in\mathbb{N}_0^d},$$

which means that

$$\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} S_{\lambda+\tilde{\gamma}}P_\lambda\right)_{\tilde{\gamma}\in\mathbb{N}_0^d} = \operatorname{col}(0_{p\times p})_{\tilde{\gamma}\in\mathbb{N}_0^d}.$$

For $\tilde{\gamma} = \gamma + \lambda'$, we have

$$\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} S_{\gamma+\lambda'+\lambda}P_\lambda\right)_{\gamma\in\mathbb{N}_0^d} = M(\infty)(\widehat{x^{\lambda'}P}) = \operatorname{col}(0_{p\times p})_{\gamma\in\mathbb{N}_0^d}$$

and equation (3.3) holds. For any fixed $\lambda' \in \Gamma_{n,d}$, by equation (3.3),

$$M(\infty)(\widehat{x^{\lambda'}P}) \cdot Q_{\lambda'} = \operatorname{col}(0_{p\times p})_{\gamma\in\mathbb{N}_0^d} \cdot Q_{\lambda'}$$

and so

$$M(\infty)(\widehat{x^{\lambda'}P}) \cdot Q_{\lambda'} = \operatorname{col}(0_{p\times p})_{\gamma\in\mathbb{N}_0^d}.$$

Hence

$$\sum_{\lambda'\in\Gamma_{n,d}} M(\infty)(\widehat{x^{\lambda'}P}) \cdot Q_{\lambda'} = \sum_{\lambda'\in\Gamma_{n,d}} \operatorname{col}(0_{p\times p})_{\gamma\in\mathbb{N}_0^d} = \operatorname{col}(0_{p\times p})_{\gamma\in\mathbb{N}_0^d}.$$

Finally, since

$$\widehat{R} = \sum_{\lambda'\in\Gamma_{n,d}} \widehat{x^{\lambda'}P}Q_{\lambda'},$$

we have

$$M(\infty)\sum_{\lambda'\in\Gamma_{n,d}} \widehat{x^{\lambda'}P}Q_{\lambda'} = M(\infty)\widehat{R} = \operatorname{col}(0_{p\times p})_{\gamma\in\mathbb{N}_0^d},$$

as desired and we derive that $\ker \Phi$ is a right ideal. By Lemma 3.1.15, $\mathcal{I} = \ker \Phi$ and so $\mathcal{I}$ is a right ideal as well. $\square$

Definition 3.1.17. Let $M(\infty) \succeq 0$ and $\mathcal{I}$ be as in Definition 3.1.14. We define the *right quotient module*

$$\mathbb{C}^{p\times p}[x_1,\dots,x_d]/\mathcal{I} := \{P + \mathcal{I} : P \in \mathbb{C}^{p\times p}[x_1,\dots,x_d]\}$$

of equivalence classes modulo $\mathcal{I}$, that is, we will write

$$P + \mathcal{I} = P' + \mathcal{I},$$

whenever

$$P - P' \in \mathcal{I} \quad \text{for} \;\; P, P' \in \mathbb{C}^{p\times p}[x_1,\dots,x_d].$$

Lemma 3.1.18. *$\mathbb{C}^{p\times p}[x_1,\dots,x_d]/\mathcal{I}$ is a right module over $\mathbb{C}^{p\times p}$, under the operation of addition $(+)$ given by*

$$(P + \mathcal{I}) + (P' + \mathcal{I}) := (P + P') + \mathcal{I}$$

for $P, P' \in \mathbb{C}^{p\times p}[x_1, \dots, x_d]$, together with the right multiplication $(\cdot)$ given by

$$(P + \mathcal{I}) \cdot R := PR + \mathcal{I}$$

for $P \in \mathbb{C}^{p\times p}[x_1, \dots, x_d]$ and $R \in \mathbb{C}^{p\times p}$.

Proof. Let $P, Q \in \mathbb{C}^{p\times p}[x_1, \dots, x_d]$. The following properties can be easily checked:
(i) $((P + \mathcal{I}) + (Q + \mathcal{I}))R = (P + \mathcal{I})R + (Q + \mathcal{I})R$ for all $R \in \mathbb{C}^{p\times p}$.
(ii) $(P + \mathcal{I})(R + S) = (P + \mathcal{I})R + (P + \mathcal{I})S$ for all $R, S \in \mathbb{C}^{p\times p}$.
(ii) $(P + \mathcal{I})(SR) = ((P + \mathcal{I})S)R$ for all $R, S \in \mathbb{C}^{p\times p}$.
(iv) $(P + \mathcal{I})I_p = P + \mathcal{I}$. □

Definition 3.1.19. For every $P, Q \in \mathbb{C}^{p\times p}[x_1, \dots, x_d]$, we define the form

$$[\cdot, \cdot] : \mathbb{C}^{p\times p}[x_1, \dots, x_d]/\mathcal{I} \times \mathbb{C}^{p\times p}[x_1, \dots, x_d]/\mathcal{I} \to \mathbb{C}^{p\times p}$$

given by

$$[P + \mathcal{I}, Q + \mathcal{I}] := \widehat{Q}^* M(\infty)\widehat{P}.$$

The following lemma shows that the form in Definition 3.1.19 is a well-defined positive semidefinite sesquilinear form.

Lemma 3.1.20. *Suppose $M(\infty) \succeq 0$ and let $P, Q \in \mathbb{C}^{p\times p}[x_1, \dots, x_d]$. Then $[P + \mathcal{I}, Q + \mathcal{I}]$ is well-defined, sesquilinear and positive semidefinite.*

Proof. We first show that the form $[P + \mathcal{I}, Q + \mathcal{I}]$ is well-defined. We need to prove that if $P + \mathcal{I} = P' + \mathcal{I}$ and $Q + \mathcal{I} = Q' + \mathcal{I}$, then

$$[P + \mathcal{I}, Q + \mathcal{I}] = [P' + \mathcal{I}, Q' + \mathcal{I}],$$

where $P, P', Q, Q' \in \mathbb{C}^{p\times p}[x_1, \dots, x_d]$. We have

$$[P + \mathcal{I}, Q + \mathcal{I}] = \widehat{Q}^* M(\infty)\widehat{P} \quad \text{and} \quad [P' + \mathcal{I}, Q' + \mathcal{I}] = \widehat{Q'}^* M(\infty)\widehat{P'}.$$

Since $P - P' \in \mathcal{I}$,

$$\widehat{Q}^* M(\infty)(\widehat{P - P'}) = 0_{p\times p}$$

and since $Q - Q' \in \mathcal{I}$,

$$(\widehat{Q - Q'})^* M(\infty)\widehat{P'} = 0_{p\times p}.$$

We write

$$\widehat{Q}^* M(\infty)(\widehat{P - P'}) = \widehat{Q}^* M(\infty)\widehat{P} - \widehat{Q}^* M(\infty)\widehat{P'} = 0_{p\times p} \tag{3.4}$$

and

$$(\widehat{Q - Q'})^* M(\infty)\widehat{P'} = \widehat{Q}^* M(\infty)\widehat{P'} - \widehat{Q'}^* M(\infty)\widehat{P'} = 0_{p\times p}. \tag{3.5}$$

We sum both hand sides of equations (3.4) and (3.5) and we obtain

$$\widehat{(Q-Q')}^{*} M(\infty)\widehat{(P-P')} = 0_{p\times p},$$

that is,

$$\widehat{Q}^{*} M(\infty)\widehat{P} = \widehat{Q'}^{*} M(\infty)\widehat{P'}.$$

Therefore

$$[P+\mathcal{I}, Q+\mathcal{I}] = [P'+\mathcal{I}, Q'+\mathcal{I}].$$

We now show that $[P+\mathcal{I}, Q+\mathcal{I}]$ is sesquilinear. Let $A, \tilde{A} \in \mathbb{C}^{p\times p}$. If

$$P(x) = \sum_{\lambda\in\Gamma_{m,d}} x^{\lambda}P_{\lambda} \quad \text{and} \quad Q(x) = \sum_{\lambda\in\Gamma_{n,d}} x^{\lambda}Q_{\lambda},$$

then

$$P(x)A = \sum_{\lambda\in\Gamma_{m,d}} x^{\lambda}P_{\lambda}A \quad \text{and} \quad Q(x)A = \sum_{\lambda\in\Gamma_{n,d}} x^{\lambda}Q_{\lambda}A.$$

Let $\tilde{m} := \max(m,n)$. Without loss of generality suppose $\tilde{m} = m$. For $\lambda \in \Gamma_{m,d} \setminus \Gamma_{n,d}$, let $Q_{\lambda} := 0_{p\times p}$. We may view Q as $Q(x) = \sum\limits_{\lambda\in\Gamma_{m,d}} x^{\lambda}Q_{\lambda}$. We have

$$\begin{aligned}
[(P+\mathcal{I})A + (\tilde{P}+\mathcal{I})\tilde{A}, Q+\mathcal{I}] &= \widehat{Q}^{*} M(\infty)(\widehat{P}A + \widehat{\tilde{P}}\tilde{A}) \\
&= (\widehat{Q}^{*} M(\infty)\widehat{P})A + (\widehat{Q}^{*} M(\infty)\widehat{\tilde{P}})\tilde{A} \\
&= [P+\mathcal{I}, Q+\mathcal{I}]A + [\tilde{P}+\mathcal{I}, Q+\mathcal{I}]\tilde{A}
\end{aligned}$$

and

$$\begin{aligned}
[Q+\mathcal{I}, (P+\mathcal{I})A + (\tilde{P}+\mathcal{I})\tilde{A}] &= (\widehat{P}A + \widehat{\tilde{P}}\tilde{A})^{*} M(\infty)\widehat{Q} \\
&= A^{*}(\widehat{P}^{*} M(\infty)\widehat{Q}) + \tilde{A}^{*}(\widehat{\tilde{P}}^{*} M(\infty)\widehat{Q}) \\
&= A^{*}[Q+\mathcal{I}, P+\mathcal{I}] + \tilde{A}^{*}[Q+\mathcal{I}, \tilde{P}+\mathcal{I}]
\end{aligned}$$

and so $[P+\mathcal{I}, Q+\mathcal{I}]$ is sesquilinear. Finally, we show that $[P+\mathcal{I}, Q+\mathcal{I}]$ is positive semidefinite. By definition,

$$[P+\mathcal{I}, P+\mathcal{I}] = 0_{p\times p} \quad \text{if and only if} \quad P \in \mathcal{I}.$$

Moreover, it follows from the definition of $M(\infty) \succeq 0$ (see Definition 3.1.11) that

$$[P+\mathcal{I}, P+\mathcal{I}] = \widehat{P}^{*} M(\infty)\widehat{P} \succeq 0_{p\times p}.$$

Thus $[P+\mathcal{I}, Q+\mathcal{I}]$ is positive semidefinite. □

We next define the variety of a moment matrix in our matrix-valued setting. We introduce zeros of determinants of matrix-valued polynomials abstracting that way the notion of the variety of a moment matrix introduced by Curto and Fialkow in [16].

Definition 3.1.21. Let $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$ be a truncated $\mathcal{H}_p$-valued multisequence and let $M(n)$ be the corresponding moment matrix. Let $P(x) = \sum_{\lambda\in\Gamma_{n,d}} x^\lambda P_\lambda \in \mathbb{C}_n^{p\times p}[x_1,\dots,x_d]$ such that $P(X) \in C_{M(n)}$. The *variety of* $M(n)$, denoted by $\mathcal{V}(M(n))$, is given by

$$\mathcal{V}(M(n)) := \bigcap_{\substack{P\in\mathbb{C}_n^{p\times p}[x_1,\dots,x_d] \\ P(X)=\operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}}}} \mathcal{Z}(\det(P(x))).$$

In analogy to Definition 3.1.21, we define the variety associated with the right ideal $\mathcal{I}$.

Definition 3.1.22. Let $\mathcal{I}$ be the right ideal as in Definition 3.1.14 and let the matrix-valued polynomial $P(x) = \sum\limits_{\lambda\in\Gamma_{n,d}} x^\lambda P_\lambda \in \mathbb{C}^{p\times p}[x_1,\dots,x_d]$. We define the *variety associated with* $\mathcal{I}$ by

$$\mathcal{V}(\mathcal{I}) := \bigcap_{P\in\mathcal{I}} \mathcal{Z}(\det P(x)).$$

Lemma 3.1.23. *Let $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$ be a given truncated $\mathcal{H}_p$-valued multisequence and $M(n)$ the corresponding moment matrix. Suppose $M(n)\succeq 0$ has an extension $M(n+1)\succeq 0$. If there exists $P(x) = \sum\limits_{\lambda\in\Gamma_{n,d}} x^\lambda P_\lambda \in \mathbb{C}^{p\times p}[x_1,\dots,x_d]$ such that $P(X) = \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}} \in C_{M(n)}$, then*

$$P(X) = \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n+1,d}} \in C_{M(n+1)}.$$

Proof. If there exists $P \in \mathbb{C}^{p\times p}[x_1,\dots,x_d]$ such that $P(X) = \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}} \in C_{M(n)}$, then since $M(n)\succeq 0$, we have

$$M(n)\operatorname{col}(P_\lambda)_{\lambda\in\Gamma_{n,d}} = \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}}. \tag{3.6}$$

We will show

$$M(n+1)\{\operatorname{col}(P_\lambda)_{\lambda\in\Gamma_{n,d}} \oplus \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n+1,d}\setminus\Gamma_{n,d}}\} = \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n+1,d}}. \tag{3.7}$$

Notice that

$$\operatorname{card}\Gamma_{n,d} = \binom{n+d}{d}, \quad \operatorname{card}\Gamma_{n+1,d} = \binom{n+1+d}{d}$$

and

$$\operatorname{card}(\Gamma_{n+1,d}\setminus\Gamma_{n,d}) = \binom{n+1+d}{d} - \binom{n+d}{d}.$$

We write

$$M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix} \succeq 0,$$

where

$$M(n) \in \mathbb{C}^{(\operatorname{card}\Gamma_{n,d})p\times(\operatorname{card}\Gamma_{n,d})p},$$

$$B \in \mathbb{C}^{(\operatorname{card}\Gamma_{n,d})p\times(\operatorname{card}(\Gamma_{n+1,d}\setminus\Gamma_{n,d}))p}$$

and

$$C \in \mathbb{C}^{(\operatorname{card}\Gamma_{n+1,d}\setminus\Gamma_{n,d})p\times(\operatorname{card}(\Gamma_{n+1,d}\setminus\Gamma_{n,d}))p}.$$

Since $M(n+1) \succeq 0$, by Lemma 1.4.24, there exists $W \in \mathbb{C}^{(\operatorname{card}\Gamma_{n,d})p\times(\operatorname{card}(\Gamma_{n+1,d}\setminus\Gamma_{n,d}))p}$ such that

$$M(n)W = B \quad \text{and} \quad C \succeq W^*M(n)W.$$

Then

$$\begin{aligned}
M(n+1)\{\operatorname{col}(P_\lambda)_{\lambda\in\Gamma_{n,d}} \oplus \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n+1,d}\setminus\Gamma_{n,d}}\} &= \begin{pmatrix} M(n)\operatorname{col}(P_\lambda)_{\lambda\in\Gamma_{n,d}} \\ B^*\operatorname{col}(P_\lambda)_{\lambda\in\Gamma_{n,d}} \end{pmatrix} \\
&= \begin{pmatrix} \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}} \\ W^*M(n)\operatorname{col}(P_\lambda)_{\lambda\in\Gamma_{n,d}} \end{pmatrix} \\
&= \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n+1,d}},
\end{aligned}$$

by equation (3.6). Thus, equation (3.7) holds and the proof is complete. □

Definition 3.1.24. The *Kronecker product* of $A = (a_{ij})_{\substack{i=1,\dots,p\\ j=1,\dots,q}}$ and $B = (b_{ij})_{\substack{i=1,\dots,p\\ j=1,\dots,q}} \in \mathbb{C}^{p\times q}$ is denoted by $A\otimes B$ and is defined to be the block matrix

$$A\otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \in \mathbb{C}^{mp\times nq}.$$

We refer the reader to [47] for more details on the Kronecker product.

Definition 3.1.25. Given distinct points $w^{(1)},\dots,w^{(k)} \in \mathbb{R}^d$ and a a subset $\Lambda = \{\lambda^{(1)},\dots,\lambda^{(k)}\}$ of $\mathbb{N}_0^d$, we define the *multivariable Vandermonde matrix* for $p\times p$ matrix-valued polynomials by

$$\begin{aligned}
V^{p\times p}(w^{(1)},\dots,w^{(k)};\Lambda) &:= \begin{pmatrix} \{w^{(1)}\}^{\lambda^{(1)}}I_p & \dots & \{w^{(1)}\}^{\lambda^{(k)}}I_p \\ \vdots & & \vdots \\ \{w^{(k)}\}^{\lambda^{(1)}}I_p & \dots & \{w^{(k)}\}^{\lambda^{(k)}}I_p \end{pmatrix} \\
&= V(w^{(1)},\dots,w^{(k)};\Lambda)\otimes I_p \in \mathbb{C}^{kp\times kp}.
\end{aligned}$$

The following lemma is well-known. However for completeness, we will provide a proof.

Lemma 3.1.26. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{m\times m}$ be given. Then*

$$(\det A)^m(\det B)^n = \det(A\otimes B) = \det(B\otimes A).$$

$A\otimes B$ and $B\otimes A$ are invertible if and only if A and B are both invertible.

Proof. Let ν_i be the eigenvalues of $A \otimes B$, $i = 1, \dots, mn$. Then

$$\det(A \otimes B) = \prod_{i=1}^{mn} \nu_i.$$

If $\alpha \in \sigma(A)$ and $x \in \mathbb{C}^n$ is a corresponding eigenvector of A, and if $\beta \in \sigma(A)$ and $y \in \mathbb{C}^m$ is a corresponding eigenvector of B, then $\alpha\beta \in \sigma(A \otimes B)$ and $x \otimes y \in \mathbb{C}^{nm}$ is a corresponding eigenvector of $A \otimes B$. If $\sigma(A) = \{\alpha_1, \dots, \alpha_n\}$ and $\sigma(B) = \{\beta_1, \dots, \beta_m\}$, then

$$\sigma(A \otimes B) = \{\alpha_j \beta_k : j = 1, \dots, n,\ k = 1, \dots, m\}$$

(including algebraic multiplicities). In particular $\sigma(A \otimes B) = \sigma(B \otimes A)$ (see [47, Theorem 4.2.12]). Therefore, each eigenvalue of $A \otimes B$ is given by $\nu_i = \alpha_j \beta_k$, where α_j is an eigenvalue of A and β_k is an eigenvalue of B for every $i = 1, \dots, mn$. Hence

$$\begin{aligned} \det(A \otimes B) &= \prod_{i=1}^{mn} \nu_i \\ &= \prod_{j=1}^{n} \prod_{k=1}^{m} \alpha_j \beta_k \\ &= (\prod_{j=1}^{n} {\alpha_j}^m)(\prod_{k=1}^{m} {\beta_k}^n) \\ &= (\det A)^m (\det B)^n. \end{aligned}$$

Since the eigenvalues of $A \otimes B$ and $B \otimes A$ are the same,

$$\det(A \otimes B) = \det(B \otimes A).$$

So $A \otimes B$ is invertible if and only if $(\det A)^m (\det B)^n \neq 0$, which is in turn true if and only if $\det A \neq 0$ and $\det B \neq 0$. Finally $\det A \neq 0$ and $\det B \neq 0$ if and only if A and B are both invertible. □

Remark 3.1.27. By Lemma 3.1.26, $V^{p \times p}(w^{(1)}, \dots, w^{(k)}; \Lambda)$ is invertible if and only if $V(w^{(1)}, \dots, w^{(k)}; \Lambda)$ is invertible and I_p is invertible. However, since I_p is obviously invertible, we have $V^{p \times p}(w^{(1)}, \dots, w^{(k)}; \Lambda)$ is invertible if and only if $V(w^{(1)}, \dots, w^{(k)}; \Lambda)$ is invertible.

In the following example we highlight the importance of the variety of a moment matrix for computing a representing measure of a truncated $\mathcal{H}_2$-valued bisequence. We shall see how the block column relations of the initial moment matrix $M(1)$ give rise to the respective variety. As a result, we will construct a positive extension $M(2)$ which is rank-preserving and compute the variety of $M(2)$. We observe that the variety of the extension is smaller than the variety of the initial matrix. Finally, via calculations with multivariable Vandermonde matrices for 2×2 matrix-valued polynomials we shall obtain a representing measure for the truncated $\mathcal{H}_2$-valued bisequence.

Example 3.1.28. Let $S := (S_\gamma)_{\gamma\in\Gamma_{2,2}}$ be a truncated $\mathcal{H}_2$-valued bisequence given by

$$M(1) = \begin{array}{c c} & \begin{array}{ccc} 1 & X & Y \end{array} \\ \begin{array}{c} 1 \\ X \\ Y \end{array} & \begin{pmatrix} S_{00} & S_{10} & S_{01} \\ S_{10} & S_{20} & S_{11} \\ S_{01} & S_{11} & S_{02} \end{pmatrix} \end{array} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 1 & -1 & 1 & -1 \\ 0 & 2 & -1 & 1 & -1 & 1 \\ 1 & -1 & 4 & 0 & 1 & -1 \\ -1 & 1 & 0 & 4 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix} \succeq 0.$$

$M(1)$ is described by the block column relation $Y = 1 \cdot P_{00} \in C_{M(1)}$, where

$$P_{00} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Thus $P_1(X,Y) = \operatorname{col}(0_{2\times 2})_{\gamma\in\Gamma_{1,2}}$, where

$$P_1(x,y) = yI_2 - P_{00}. \tag{3.8}$$

Since $\det(P_1(x,y)) = y(y-1)$, we obtain

$$\mathcal{V}(M(1)) = \mathcal{Z}(\det(P(x,y))) = \{(x,0) : x \in \mathbb{R}\} \cup \{(x,1) : x \in \mathbb{R}\}.$$

Since $P_1(X,Y) = \operatorname{col}(0_{2\times 2})_{\gamma\in\Gamma_{1,2}} \in C_{M(1)}$, where P_1 as described in formula (3.8), Lemma 3.1.23 implies that any positive extension

$$M(2) = \begin{array}{c c} & \begin{array}{cccccc} 1 & X & Y & X^2 & XY & Y^2 \end{array} \\ \begin{array}{c} 1 \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \end{array} & \begin{pmatrix} S_{00} & S_{10} & S_{01} & S_{20} & S_{11} & S_{02} \\ S_{10} & S_{20} & S_{11} & S_{30} & S_{21} & S_{12} \\ S_{01} & S_{11} & S_{02} & S_{21} & S_{12} & S_{03} \\ S_{20} & S_{30} & S_{21} & S_{40} & S_{31} & S_{22} \\ S_{11} & S_{21} & S_{12} & S_{31} & S_{22} & S_{13} \\ S_{02} & S_{12} & S_{03} & S_{22} & S_{13} & S_{04} \end{pmatrix} \end{array} \succeq 0$$

must have the block column relation

$$P_1(X,Y) = \operatorname{col}(0_{2\times 2})_{\gamma\in\Gamma_{2,2}} \in C_{M(2)}.$$

Thus

$$\begin{pmatrix} S_{21} \\ S_{12} \\ S_{03} \end{pmatrix} = \begin{pmatrix} S_{20} \\ S_{11} \\ S_{02} \end{pmatrix} P_{00}.$$

If we let

$$\begin{pmatrix} S_{22} \\ S_{13} \\ S_{04} \end{pmatrix} = \begin{pmatrix} S_{21} \\ S_{12} \\ S_{03} \end{pmatrix} P_{00} \quad \text{and} \quad S_{40} = 2S_{20},$$

then one can check that

$$X^2 = 1 \cdot (2I_2) \in C_{M(2)},$$
$$XY = X \cdot P_{00} \in C_{M(2)}$$

and

$$Y^2 = Y \in C_{M(2)}.$$

Let $W = \begin{pmatrix} 2I_2 & 0 & 0 \\ 0 & P_{00} & 0 \\ 0 & 0 & I_2 \end{pmatrix} \in \mathbb{C}^{6\times 6}$. Then

$$\begin{pmatrix} S_{20} & S_{11} & S_{02} \\ S_{30} & S_{21} & S_{12} \\ S_{21} & S_{12} & S_{03} \end{pmatrix} = M(1)W \quad \text{and} \quad \begin{pmatrix} S_{40} & S_{31} & S_{22} \\ S_{31} & S_{22} & S_{13} \\ S_{22} & S_{13} & S_{04} \end{pmatrix} = W^* M(1)W.$$

Lemma 1.4.24 asserts that $M(2) \succeq 0$ and

$$\operatorname{rank} M(1) = \operatorname{rank} M(2).$$

We have the following matrix-valued polynomials in $\mathbb{C}^{2\times 2}[x,y]$:

$$P_1(x,y) = yI_2 - P_{00}, \quad P_2(x,y) = x^2 I_2 - 2I_2,$$

$$P_3(x,y) = xyI_2 - xP_{00}, \quad P_4(x,y) = y^2 I_2 - yI_2,$$

with

$$\det(P_1(x,y)) = y(y-1), \quad \det(P_2(x,y)) = (x^2-2)^2,$$
$$\det(P_3(x,y)) = x^2 y(y-1), \quad \det(P_4(x,y)) = y^2(y-1)^2.$$

We obtain

$$\mathcal{V}(M(2)) = \{(\sqrt{2},0), (-\sqrt{2},0), (\sqrt{2},1), (-\sqrt{2},1)\}.$$

We wish now to compute a representing measure for S. Remark 3.1.27 asserts that for a subset $\Lambda = \{(0,0),(1,0),(0,1),(1,1)\} \subseteq \mathbb{N}_0^2$, the matrix

$$V^{2\times 2}((\sqrt{2},0),(-\sqrt{2},0),(\sqrt{2},1),(-\sqrt{2},1);\Lambda) = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \sqrt{2} & 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 & -\sqrt{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & -\sqrt{2} \end{pmatrix}$$

is invertible. The positive semidefinite matrices $Q_1, Q_2, Q_3, Q_4 \in \mathbb{C}^{2\times 2}$ are given by the Vandermonde equation

$$\operatorname{col}(Q_a)_{a=1}^4 = V^{2\times 2}((\sqrt{2},0),(-\sqrt{2},0),(\sqrt{2},1),(-\sqrt{2},1);\Lambda)^{-1}\operatorname{col}(S_\lambda)_{\lambda\in\Lambda}. \tag{3.9}$$

We then get

$$V^{2\times 2}((\sqrt{2},0),(-\sqrt{2},0),(\sqrt{2},1),(-\sqrt{2},1);\Lambda)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{\sqrt{2}}{4} & 0 & -\frac{1}{2} & 0 & -\frac{\sqrt{2}}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{\sqrt{2}}{4} & 0 & -\frac{1}{2} & 0 & -\frac{\sqrt{2}}{4} \\ \frac{1}{2} & 0 & -\frac{\sqrt{2}}{4} & 0 & \frac{1}{2} & 0 & \frac{\sqrt{2}}{4} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{\sqrt{2}}{4} & 0 & \frac{1}{2} & 0 & \frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{\sqrt{2}}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{\sqrt{2}}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{\sqrt{2}}{4} \end{pmatrix}$$

and so, equation (3.9) yields

$$Q_1 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} = Q_2, \quad Q_3 = \begin{pmatrix} \frac{1}{4}+\frac{\sqrt{2}}{8} & -\frac{1}{4}-\frac{\sqrt{2}}{8} \\ -\frac{1}{4}-\frac{\sqrt{2}}{8} & \frac{1}{4}+\frac{\sqrt{2}}{8} \end{pmatrix} \quad \text{and} \quad Q_4 = \begin{pmatrix} \frac{1}{4}-\frac{\sqrt{2}}{8} & -\frac{1}{4}+\frac{\sqrt{2}}{8} \\ -\frac{1}{4}+\frac{\sqrt{2}}{8} & \frac{1}{4}-\frac{\sqrt{2}}{8} \end{pmatrix},$$

where $\operatorname{rank} Q_a = 1$ and $Q_a \succeq 0$ for $a = 1, \dots, 4$. We note that

$$\sum_{a=1}^{4} \operatorname{rank} Q_a = \operatorname{rank} M(1) = 4.$$

Finally, a representing measure T for S with $\sum\limits_{a=1}^{4} \operatorname{rank} Q_a = 4$ is $T = \sum\limits_{a=1}^{4} Q_a \delta_{w^{(a)}}$.

3.2 Existence of a representing measure for an infinite moment matrix with finite rank

In this section we shall see that if $M(\infty) \succeq 0$ and $\operatorname{rank} M(\infty) < \infty$, then the associated $\mathcal{H}_p$-valued multisequence has a representing measure T.

To this end, throughout this section we shall define and use vector-valued polynomials. We shift our perspective from the previous setting of matrix-valued polynomials and we observe that vector-valued polynomials shall serve as a tool for defining commuting self-adjoint multiplication operators on a quotient space; the desired existence of the representing measure for an infinite moment matrix will then arise. Both settings are equally important to state and prove the flat extension theorem for matricial moments (see Theorem 4.0.2).

Definition 3.2.1. We define the vector space

$$(\mathbb{C}^p)_0^\omega := \{v = \operatorname{col}(v_\lambda)_{\lambda\in\mathbb{N}_0^d} : v_\lambda \in \mathbb{C}^p \quad \text{and } v_\lambda = 0_p \text{ for all but finitely many } \lambda \in \mathbb{N}_0^d\}.$$

Definition 3.2.2. We let $\tilde{C}_{M(\infty)}$ be the complex vector space

$$\tilde{C}_{M(\infty)} = \{M(\infty)v : v \in (\mathbb{C}^p)_0^\omega\}.$$

Remark 3.2.3. We note that

$$\tilde{C}_{M(\infty)} = \{M(\infty)v : v \in (\mathbb{C}^p)_0^\omega\} = \{\sum_{\lambda\in\Gamma_{n,d}} X^\lambda v^{(\lambda)} : v \in (\mathbb{C}^p)_0^\omega, \quad \lambda \in \Gamma_{n,d}\}.$$

Definition 3.2.4. Given $q(x) = \sum_{\lambda\in\Gamma_{n,d}} q_\lambda x^\lambda \in \mathbb{C}^p[x_1, \dots, x_d]$, we let

$$\hat{q} := \mathrm{col}(q_\lambda)_{\lambda\in\Gamma_{n,d}} \oplus \mathrm{col}(0_p)_{\gamma\in\mathbb{N}_0^d} \in (\mathbb{C}^p)_0^\omega.$$

Lemma 3.2.5. *Suppose $M(\infty) \succeq 0$ and $r = \operatorname{rank} M(\infty) < \infty$. Then $r = \dim \tilde{C}_{M(\infty)}$.*

Proof. If $\dim \tilde{C}_{M(\infty)} = m$ and $m \neq r$, then there exists a basis

$$\mathcal{B} := \{X^{\lambda^{(1)}} e_{k_1}, \dots, X^{\lambda^{(m)}} e_{k_m}\}$$

of $\tilde{C}_{M(\infty)}$ for $1 \le k_a \le p$, where e_{k_a} is a standard basis vector in $\mathbb{C}^p$ and $a = 1, \dots, m$. We will show that

$$\tilde{\mathcal{B}} := \{\tilde{X}^{\lambda^{(1)}} e_{k_1}, \dots, \tilde{X}^{\lambda^{(m)}} e_{k_m}\}$$

is a basis of $\tilde{C}_{M(\kappa)}$, where

$$\tilde{X}^{\lambda^{(a)}} = \mathrm{col}(S_{\lambda^{(a)}+\gamma})_{\gamma\in\Gamma_{\kappa,d}} \quad \text{and} \quad \kappa \ge \max_{a=1,\dots,m} \left(|\lambda^{(a)}|\right).$$

First we need to show that $\tilde{\mathcal{B}}$ is linearly independent in $\tilde{C}_{M(\kappa)}$. For this, suppose that there exist $c_1, \dots, c_m \in \mathbb{C}$ not all zero such that

$$\sum_{a=1}^{m} c_a \tilde{X}^{\lambda^{(a)}} e_{k_a} = \mathrm{col}(0_p)_{\gamma\in\Gamma_{\kappa,d}} \in \tilde{C}_{M(\kappa)}. \tag{3.10}$$

Let $v = \mathrm{col}(v_\lambda)_{\lambda\in\Gamma_{\kappa,d}}$ be a vector in $\mathbb{C}^{(\mathrm{card}\,\Gamma_{\kappa,d})p}$ with

$$v_\lambda = \begin{cases} 0_p, & \text{when } \lambda \in \Gamma_{\kappa,d} \setminus \lambda^{(a)} \\ c_a, & \text{when } \lambda = \lambda^{(a)} \end{cases} \quad \text{for } \ a = 1, \dots, m.$$

Then by equation (3.10), $M(\kappa)v = \mathrm{col}(0_p)_{\gamma\in\Gamma_{\kappa,d}} \in \tilde{C}_{M(\kappa)}$. Since $M(\kappa+\ell) \succeq 0$ for all $\ell = 1, 2, \dots$, we have

$$M(\kappa + \ell)\{v \oplus \mathrm{col}(0_p)_{\gamma\in\Gamma_{\kappa+\ell,d}\setminus\Gamma_{\kappa,d}}\} = \mathrm{col}(0_p)_{\gamma\in\Gamma_{\kappa+\ell,d}} \in \tilde{C}_{M(\kappa)}.$$

For $\eta \in \mathbb{C}^p[x_1, \dots, x_d]$ with $\hat{\eta} := v \oplus \mathrm{col}(0_p)_{\gamma\in\mathbb{N}_0^d\setminus\Gamma_{\kappa,d}}$, we have

$$M(\infty)\hat{\eta} = \mathrm{col}(0_p)_{\gamma\in\mathbb{N}_0^d} \in \tilde{C}_{M(\infty)},$$

that is, there exist $c_1, \dots, c_m \in \mathbb{C}$ not all zero such that

$$\sum_{a=1}^{m} c_a X^{\lambda^{(a)}} e_{k_a} = \text{col}(0_p)_{\gamma \in \mathbb{N}_0^d} \in \tilde{C}_{M(\infty)}.$$

However, this contradicts the fact that $\mathcal{B}$ is linear independent. Hence $\tilde{\mathcal{B}}$ is linearly independent in $\tilde{C}_{M(\kappa)}$. It remains to show that $\tilde{\mathcal{B}}$ spans $\tilde{C}_{M(\kappa)}$. Since $\mathcal{B}$ is a basis of $\tilde{C}_{M(\infty)}$, for any $\text{col}(d_\gamma)_{\gamma \in \mathbb{N}_0^d} \in \tilde{C}_{M(\infty)}$ with $d_\gamma \in \mathbb{C}^p$, there exists $c_1, \dots, c_m \in \mathbb{C}$ such that

$$\sum_{a=1}^{m} c_a X^{\lambda^{(a)}} e_{k_a} = \text{col}(d_\gamma)_{\gamma \in \mathbb{N}_0^d}.$$

We next let $\mathcal{X}^{\lambda^{(a)}} = \text{col}(S_{\lambda^{(a)}+\gamma})_{\gamma \in \mathbb{N}_0^d \setminus \Gamma_{\kappa,d}}$. We have

$$\sum_{a=1}^{m} c_a \{\tilde{X}^{\lambda^{(a)}} \oplus \mathcal{X}^{\lambda^{(a)}}\} e_{k_a} = \text{col}(d_\gamma)_{\gamma \in \Gamma_{\kappa,d}} \oplus \text{col}(d_\gamma)_{\gamma \in \mathbb{N}_0^d \setminus \Gamma_{\kappa,d}}$$

and so

$$\sum_{a=1}^{m} c_a \tilde{X}^{\lambda^{(a)}} e_{k_a} = \text{col}(d_\gamma)_{\gamma \in \Gamma_{\kappa,d}}.$$

Hence $\tilde{\mathcal{B}}$ spans $\tilde{C}_{M(\kappa)}$. Therefore $\tilde{\mathcal{B}}$ is a basis of $\tilde{C}_{M(\kappa)}$, which forces $\text{rank}\, M(\kappa) = m$ for all κ. Thus $\sup_\kappa \text{rank}\, M(\kappa) = \sup_\kappa m$ for all κ, that is, $r = m$, a contradiction. Consequently $\dim \tilde{C}_{M(\infty)} = r$. $\square$

Remark 3.2.6. Presently, we shall view $M(\infty)$ as a linear operator

$$M(\infty) : (\mathbb{C}^p)_0^\omega \to \tilde{C}_{M(\infty)}$$

and not as a linear operator

$$M(\infty) : (\mathbb{C}^{p \times p})_0^\omega \to C_{M(\infty)}$$

as in Section 3.1.

Remark 3.2.7. Assume $r = \text{rank}\, M(\infty)$ (or, equivalently, $\dim \tilde{C}_{M(\infty)} < \infty$). Suppose

$$\mathcal{B} := \{X^{\lambda^{(1)}} e_{k_1}, \dots, X^{\lambda^{(r)}} e_{k_r}\} \quad \text{for} \quad 1 \le k_a \le p,$$

is a basis for $\tilde{C}_{M(\infty)}$, where e_{k_a} is a standard basis vector in $\mathbb{C}^p$ and $a = 1, \dots, r$. Then there exist $c_1, \dots, c_r \in \mathbb{C}$ such that any $w \in \tilde{C}_{M(\infty)}$ can be written as

$$w = \sum_{a=1}^{r} c_a X^{\lambda^{(a)}} e_{k_a} \in \tilde{C}_{M(\infty)}.$$

We shall proceed with a result on positivity when $M(\infty)$ is treated as a linear operator $M(\infty) : (\mathbb{C}^p)_0^\omega \to \tilde{C}_{M(\infty)}$.

In analogy to results from Section 3.1, we move on to the following.

Definition 3.2.8. We define the map $\phi : \mathbb{C}^p[x_1, \dots, x_d] \to \tilde{C}_{M(\infty)}$ given by

$$\phi(v) = \sum_{a=1}^{r} c_a X^{\lambda^{(a)}} e_{k_a},$$

where $v(x) = \sum\limits_{\lambda \in \Gamma_{n,d}} v_\lambda x^\lambda \in \mathbb{C}^p[x_1, \dots, x_d]$.

Definition 3.2.9. Suppose $M(\infty) \succeq 0$. Let $q \in \mathbb{C}^p[x_1, \dots, x_d]$. We define the subspace of $\mathbb{C}^p[x_1, \dots, x_d]$

$$\mathcal{J} := \{q \in \mathbb{C}^p[x_1, \dots, x_d] : \langle M(\infty)\hat{q}, \hat{q}\rangle = 0\}$$

and the kernel of the map ϕ

$$\ker \phi := \{q \in \mathbb{C}^p[x_1, \dots, x_d] : \phi(q) = \mathrm{col}(0_p)_{\gamma \in \mathbb{N}_0^d}\},$$

where ϕ is as in Definition 3.2.8.

Lemma 3.2.10. *Suppose $M(\infty) \succeq 0$. Then*

$$\mathcal{J} = \ker \phi,$$

where $\mathcal{J}$ and $\ker \phi$ are as in Definition 3.2.9.

Proof. If $q(x) = \sum\limits_{\lambda \in \Gamma_{n,d}} q_\lambda x^\lambda \in \ker \phi$, then

$$\phi(q) = \sum_{a=1}^{r} c_a X^{\lambda^{(a)}} e_{k_a} = \mathrm{col}(0_p)_{\gamma \in \mathbb{N}_0^d},$$

that is, $M(\infty)\hat{q} = \mathrm{col}(0_p)_{\gamma \in \mathbb{N}_0^d}$, where $\hat{q} \in (\mathbb{C}^p)_0^\omega$. Thus $\langle M(\infty)\hat{q}, \hat{q}\rangle = 0$ and so $q \in \mathcal{J}$.

Conversely, let $q(x) = \sum_{\lambda \in \Gamma_{n,d}} q_\lambda x^\lambda \in \mathcal{J}$. Then $\langle M(\infty)\hat{q}, \hat{q}\rangle = 0$. It suffices to show that for every $\eta(x) = \sum_{\lambda \in \Gamma_{m,d}} \eta_\lambda x^\lambda \in \mathbb{C}^p[x_1, \dots, x_d]$,

$$\langle M(\infty)\hat{q}, \hat{\eta}\rangle = 0.$$

Let $\tilde{m} = \max(n, m)$. Without loss of generality suppose $\tilde{m} = m$. Let $\eta_\lambda = 0_p$ for $\lambda \in \Gamma_{n,d} \setminus \Gamma_{m,d}$. We may view η as $\eta(x) = \sum\limits_{\lambda \in \Gamma_{n,d}} \eta_\lambda x^\lambda$. Since $\langle M(\infty)\hat{q}, \hat{q}\rangle = 0$, we have $\hat{q}^* M(\infty)\hat{q} = 0$ and so

$$\mathrm{col}(q_\lambda)^*_{\lambda \in \Gamma_{m,d}} M(m)\, \mathrm{col}(q_\lambda)_{\lambda \in \Gamma_{m,d}} = 0.$$

Moreover, since $M(\infty) \succeq 0$, $M(m) \succeq 0$ and hence, the square root of $M(m)$ exists. Next, $\langle M(m)^{\frac{1}{2}}\hat{q}, \hat{q}\rangle = 0$ implies $\langle M(m)^{\frac{1}{2}}\hat{q}, M(m)^{\frac{1}{2}}\hat{q}\rangle = 0$, that is,

$$\| M(m)^{\frac{1}{2}}\hat{q} \| = 0.$$

Then $M(m)^{\frac{1}{2}}\hat{q} = \mathrm{col}(0_p)_{\lambda\in\Gamma_{m,d}}$ and

$$M(m)^{\frac{1}{2}}M(m)^{\frac{1}{2}}\hat{q} = M(m)^{\frac{1}{2}}\,\mathrm{col}(0_p)_{\lambda\in\Gamma_{m,d}},$$

which implies

$$M(m)\hat{q} = \mathrm{col}(0_p)_{\lambda\in\Gamma_{m,d}}. \tag{3.11}$$

If $q(x) = \sum\limits_{\lambda\in\Gamma_{n,d}} q_\lambda x^\lambda \in \mathcal{J}$ and $\eta(x) = \sum\limits_{\lambda\in\Gamma_{n,d}} \eta_\lambda x^\lambda \in \mathcal{J}$ with $\hat{q}, \hat{\eta} \in (\mathbb{C}^p)_0^\omega$, then

$$\begin{aligned} \langle M(\infty)\hat{q}, \hat{\eta}\rangle &= \hat{\eta}^* M(\infty)\hat{q} \\ &= \mathrm{col}(\eta_\lambda)^*_{\lambda\in\Gamma_{m,d}} M(m)\,\mathrm{col}(q_\lambda)_{\lambda\in\Gamma_{m,d}} \\ &= \langle M(m)\hat{q}, \hat{\eta}\rangle \\ &= 0, \end{aligned}$$

by equation (3.11). □

Definition 3.2.11. Let $M(\infty) \succeq 0$ and $\mathcal{J}$ be as in Definition 3.2.9. We define the quotient space

$$\mathbb{C}^p[x_1,\ldots,x_d]/\mathcal{J} = \{q + \mathcal{J} : q \in \mathbb{C}^p[x_1,\ldots,x_d]\}$$

of equivalence classes modulo $\mathcal{J}$, that is, if

$$q + \mathcal{J} = q' + \mathcal{J},$$

then

$$q - q' \in \mathcal{J} \quad \text{for } q, q' \in \mathbb{C}^p[x_1,\ldots,x_d].$$

Definition 3.2.12. For every $h, q \in \mathbb{C}^p[x_1,\ldots,x_d]$, we define the inner product

$$\langle\cdot,\cdot\rangle : \mathbb{C}^p[x_1,\ldots,x_d]/\mathcal{J} \times \mathbb{C}^p[x_1,\ldots,x_d]/\mathcal{J} \to \mathbb{C}$$

given by

$$\langle h + \mathcal{J}, q + \mathcal{J}\rangle = \hat{q}^* M(\infty)\hat{h}.$$

Lemma 3.2.13. *Suppose $M(\infty) \succeq 0$ and let $h, q \in \mathbb{C}^p[x_1,\ldots,x_d]$. Then $\langle h + \mathcal{J}, q + \mathcal{J}\rangle$ is well-defined, linear and positive semidefinite.*

Proof. We first show that the inner product $\langle h + \mathcal{J}, q + \mathcal{J}\rangle$ is well-defined. We need to prove that if $h + \mathcal{J} = h' + \mathcal{J}$ and $q + \mathcal{J} = q' + \mathcal{J}$, then

$$\langle h + \mathcal{J}, q + \mathcal{J}\rangle = \langle h' + \mathcal{J}, q' + \mathcal{J}\rangle,$$

where $h, h', q, q' \in \mathbb{C}^p[x_1,\ldots,x_d]$. We write

$$\langle h + \mathcal{J}, q + \mathcal{J}\rangle = \hat{q}^* M(\infty)\hat{h} \quad \text{and} \quad \langle h' + \mathcal{J}, q' + \mathcal{J}\rangle = \hat{q'}^* M(\infty)\hat{h'}.$$

Since $h - h' \in \mathcal{J}$,

$$\hat{q}^* M(\infty)(\widehat{h - h'}) = 0$$

and since $q - q' \in \mathcal{J}$,

$$(\widehat{q - q'})^* M(\infty)\hat{h}' = 0.$$

We write

$$\hat{q}^* M(\infty)(\widehat{h - h'}) = \hat{q}^* M(\infty)\hat{h} - \hat{q}^* M(\infty)\hat{h}' = 0 \tag{3.12}$$

and

$$(\widehat{q - q'})^* M(\infty)\hat{h}' = \hat{q}^* M(\infty)\hat{h}' - \hat{q'}^* M(\infty)\hat{h}' = 0. \tag{3.13}$$

We sum both hand sides of equations (3.12) and (3.13) and we obtain

$$(\widehat{q - q'})^* M(\infty)(\widehat{h - h'}) = 0,$$

that is,

$$\hat{q}^* M(\infty)\hat{h} = \hat{q'}^* M(\infty)\hat{h}',$$

and hence

$$\langle h + \mathcal{J}, q + \mathcal{J}\rangle = \langle h' + \mathcal{J}, q' + \mathcal{J}\rangle.$$

We now show that the inner product $\langle h + \mathcal{J}, q + \mathcal{J}\rangle$ is linear. We must prove that for every $h, \tilde{h}, q \in \mathbb{C}^p[x_1, \ldots, x_d]$ and $a, \tilde{a} \in \mathbb{C}$,

$$\langle a(h + \mathcal{J}) + \tilde{a}(\tilde{h} + \mathcal{J}), q + \mathcal{J}\rangle = a\langle h + \mathcal{J}, q + \mathcal{J}\rangle + \tilde{a}\langle \tilde{h} + \mathcal{J}, q + \mathcal{J}\rangle.$$

Let

$$h(x) = \sum_{\lambda \in \Gamma_{n,d}} h_\lambda x^\lambda \quad \text{and} \quad q(x) = \sum_{\lambda \in \Gamma_{m,d}} q_\lambda x^\lambda.$$

Then

$$ah(x) = \sum_{\lambda \in \Gamma_{n,d}} a h_\lambda x^\lambda.$$

Let $\tilde{m} = \max(n, m)$. Without loss of generality suppose $\tilde{m} = m$. Let $q_\lambda = 0_h$ for $\lambda \in \Gamma_{n,d} \setminus \Gamma_{m,d}$. We may view q as $q(x) = \sum_{\lambda \in \Gamma_{n,d}} q_\lambda x^\lambda$ and we have

$$\begin{aligned}
\langle a(h + \mathcal{J}) + \tilde{a}(\tilde{h} + \mathcal{J}), q + \mathcal{J}\rangle &= \hat{q}^* M(\infty)(\widehat{ah + \tilde{a}\tilde{h}}) \\
&= \hat{q}^* M(\infty)(\widehat{ah}) + \hat{q}^* M(\infty)(\widehat{\tilde{a}\tilde{h}}) \\
&= a\hat{q}^* M(\infty)\hat{h} + \tilde{a}\hat{q}^* M(\infty)\hat{\tilde{h}} \\
&= a\langle h + \mathcal{J}, q + \mathcal{J}\rangle + \tilde{a}\langle \tilde{h} + \mathcal{J}, q + \mathcal{J}\rangle.
\end{aligned}$$

Finally, we show $\langle h + \mathcal{J}, q + \mathcal{J}\rangle$ is positive semidefinite. By definition,

$$\langle h + \mathcal{J}, h + \mathcal{J}\rangle = 0 \quad \text{if and only if} \quad h \in \mathcal{J}.$$

Since $M(\infty) \succeq 0$, by Lemma 3.1.13,

$$\langle h + \mathcal{J}, h + \mathcal{J} \rangle = \hat{h}^* M(\infty) \hat{h} \geq 0.$$

Hence $\langle h + \mathcal{J}, h + \mathcal{J} \rangle$ is positive semidefinite. $\square$

Definition 3.2.14. We define the map $\Psi : \tilde{C}_{M(\infty)} \to \mathbb{C}^p[x_1, \ldots, x_d]/\mathcal{J}$ given by

$$\Psi(w) = \sum_{a=1}^{r} c_a x^{\lambda(a)} e_{k_a} + \mathcal{J},$$

where

$$w = \sum_{a=1}^{r} c_a X^{\lambda(a)} e_{k_a} \in \tilde{C}_{M(\infty)}.$$

Lemma 3.2.15. *Ψ as in Definition 3.2.14 is an isomorphism.*

Proof. We consider the map $\phi : \mathbb{C}^p[x_1, \ldots, x_d] \to \tilde{C}_{M(\infty)}$ is as in Definition 3.2.8 and we first show that ϕ is an homomorphism. For $\sum\limits_{a=1}^{r} d_a x^{\lambda(a)} e_{k_a} \in \mathbb{C}^p[x_1, \ldots, x_d]$, where $d_1, \ldots, d_r \in \mathbb{C}$, we have

$$\begin{aligned} \phi\left(\sum_{a=1}^{r} d_a x^{\lambda(a)} e_{k_a} + \sum_{a=1}^{r} c_a x^{\lambda(a)} e_{k_a} \right) &= \phi\left(\sum_{a=1}^{r} d_a x^{\lambda(a)} e_{k_a} \right) + \phi\left(\sum_{a=1}^{r} c_a x^{\lambda(a)} e_{k_a} \right) \\ &= \sum_{a=1}^{r} d_a X^{\lambda(a)} e_{k_a} + \sum_{a=1}^{r} c_a X^{\lambda(a)} e_{k_a}. \end{aligned}$$

Moreover, we shall see that ϕ is surjective. Indeed, for every $\sum\limits_{a=1}^{r} c_a X^{\lambda(a)} e_{k_a} \in \tilde{C}_{M(\infty)}$, there exists $\sum\limits_{a=1}^{r} c_a x^{\lambda(a)} e_{k_a} \in \mathbb{C}^p[x_1, \ldots, x_d]$ such that

$$\phi\left(\sum_{a=1}^{r} c_a x^{\lambda(a)} e_{k_a} \right) = \sum_{a=1}^{r} c_a X^{\lambda(a)} e_{k_a}.$$

By the Fundamental homomorphism theorem (see, e.g., [41, Theorem 1.11]), $\tilde{C}_{M(\infty)}$ is isomorphic to $\mathbb{C}^p[x_1, \ldots, x_d]/\ker \phi$ and thus to $\mathbb{C}^p[x_1, \ldots, x_d]/\mathcal{J}$, by Lemma 3.2.10. Hence, the map Ψ is an isomorphism. $\square$

Remark 3.2.16. By Lemma 3.2.5, $r = \operatorname{rank} M(\infty) = \dim \tilde{C}_{M(\infty)} < \infty$. Since Ψ is an isomorphism, we derive that $r = \dim(\mathbb{C}^p[x_1, \ldots, x_d]/\mathcal{J})$.

In this setting, we present the multiplication operators M_{x_j}, $j = 1, \ldots, d$, as defined below.

Definition 3.2.17. Let $q \in \mathbb{C}^p[x_1, \ldots, x_d]$. We define the *multiplication operators*

$$M_{x_j} : \mathbb{C}^p[x_1, \ldots, x_d]/\mathcal{J} \to \mathbb{C}^p[x_1, \ldots, x_d]/\mathcal{J} \quad \text{for} \ \ j = 1, \ldots, d$$

given by

$$M_{x_j}(q+\mathcal{J}) := \sum_{k=1}^{p} M_{x_j}^{(k)}\Big(\sum_{\lambda\in\Gamma_{m,d}} q_\lambda^{(k)} x^\lambda e_k + \mathcal{J}\Big),$$

where

$$M_{x_j}^{(k)} : \mathbb{C}^p[x_1,\dots,x_d]/\mathcal{J} \to \mathbb{C}^p[x_1,\dots,x_d]/\mathcal{J}$$

is the multiplication operator defined by

$$M_{x_j}^{(k)}\Big(\sum_{\lambda\in\Gamma_{m,d}} q_\lambda^{(k)} x^\lambda e_k + \mathcal{J}\Big) := \sum_{\lambda\in\Gamma_{m,d}} q_\lambda^{(k)} x^{\lambda+\varepsilon_j} e_k + \mathcal{J}$$

for all $j = 1,\dots,d$ and $\varepsilon_j \in \mathbb{N}_0^d$, $j = 1,\dots,d$.

Let us now continue with lemmas on properties of the multiplication operators M_{x_j}.

Lemma 3.2.18. *Let M_{x_j}, $j = 1,\dots,d$, be the multiplication operators as in Definition 3.2.17. Then M_{x_j} is well-defined for all $j = 1,\dots,d$.*

Proof. Let $q(x) = \sum_{\lambda\in\Gamma_{m,d}} q_\lambda x^\lambda$ and $h(x) = \sum_{\lambda\in\Gamma_{m,d}} h_\lambda x^\lambda$. If $q+\mathcal{J} = h+\mathcal{J}$, then

$$M_{x_j}(q+\mathcal{J}) = M_{x_j}(h+\mathcal{J}),$$

that is,

$$\sum_{k=1}^{p} M_{x_j}^{(k)}\Big(\sum_{\lambda\in\Gamma_{m,d}} q_\lambda^{(k)} x^\lambda e_k + \mathcal{J}\Big) = \sum_{k=1}^{p} M_{x_j}^{(k)}\Big(\sum_{\lambda\in\Gamma_{m,d}} h_\lambda^{(k)} x^\lambda e_k + \mathcal{J}\Big),$$

or equivalently,

$$\sum_{\lambda\in\Gamma_{m,d}} \sum_{k=1}^{p} q_\lambda^{(k)} x^\lambda x^{\varepsilon_j} e_k + \mathcal{J} = \sum_{\lambda\in\Gamma_{m,d}} \sum_{k=1}^{p} h_\lambda^{(k)} x^\lambda x^{\varepsilon_j} e_k + \mathcal{J},$$

which is equivalent to

$$x^{\varepsilon_j} \sum_{\lambda\in\Gamma_{m,d}} q_\lambda x^\lambda + \mathcal{J} = x^{\varepsilon_j} \sum_{\lambda\in\Gamma_{m,d}} h_\lambda x^\lambda + \mathcal{J},$$

that is,

$$x^{\varepsilon_j} q + \mathcal{J} = x^{\varepsilon_j} h + \mathcal{J},$$

and hence

$$x_j(q-h) \in \mathcal{J}$$

as required. $\square$

Lemma 3.2.19. *Let M_{x_j}, $j = 1,\dots,d$, be as in Definition 3.2.17. Then $M_{x_j}(q+\mathcal{J}) = x^{\varepsilon_j} q+\mathcal{J}$ for all $j = 1,\dots,d$.*

Proof. For all $j = 1, \dots, d$,

$$\begin{aligned}
M_{x_j}(q + \mathcal{J}) &= \sum_{k=1}^{p} M_{x_j}^{(k)} \Big(\sum_{\lambda \in \Gamma_{m,d}} q_\lambda^{(k)} x^\lambda e_k + \mathcal{J} \Big) \\
&= \sum_{k=1}^{p} \sum_{\lambda \in \Gamma_{m,d}} q_\lambda^{(k)} x^{\lambda + \varepsilon_j} e_k + \mathcal{J} \\
&= \sum_{\lambda \in \Gamma_{m,d}} \sum_{k=1}^{p} q_\lambda^{(k)} x^\lambda x^{\varepsilon_j} e_k + \mathcal{J} \\
&= x^{\varepsilon_j} \sum_{\lambda \in \Gamma_{m,d}} q_\lambda x^\lambda + \mathcal{J} \\
&= x^{\varepsilon_j} q + \mathcal{J} \\
&= x_j q + \mathcal{J}
\end{aligned}$$

as required. □

Lemma 3.2.20. *Let M_{x_j}, $j = 1, \dots, d$, be as in Definition 3.2.17. Then $M_{x_j} M_{x_\ell} = M_{x_\ell} M_{x_j}$ for all $j, \ell = 1, \dots, d$.*

Proof. We need to show that for every $q, f \in \mathbb{C}^p[_1, \dots, x_d]$,

$$\langle M_{x_j} M_{x_\ell}(q + \mathcal{J}), f + \mathcal{J} \rangle = \langle M_{x_\ell} M_{x_j}(q + \mathcal{J}), f + \mathcal{J} \rangle,$$

that is, $\langle x^{\varepsilon_j} x^{\varepsilon_\ell}(q + \mathcal{J}), f + \mathcal{J} \rangle = \langle x^{\varepsilon_\ell} x^{\varepsilon_j}(q + \mathcal{J}), f + \mathcal{J} \rangle$. We have

$$\begin{aligned}
\langle x^{\varepsilon_j} x^{\varepsilon_\ell}(q + \mathcal{J}), f + \mathcal{J} \rangle &= \langle x_j x_\ell (q + \mathcal{J}), f + \mathcal{J} \rangle \\
&= \hat{f}^* M(\infty) \widehat{(x_j x_\ell q)} \\
&= \hat{f}^* M(\infty) \widehat{(x_\ell x_j q)} \\
&= \langle x_\ell x_j (q + \mathcal{J}), f + \mathcal{J} \rangle \\
&= \langle x^{\varepsilon_\ell} x^{\varepsilon_j}(q + \mathcal{J}), f + \mathcal{J} \rangle.
\end{aligned}$$

Thus $M_{x_j} M_{x_\ell} = M_{x_\ell} M_{x_j}$ for all $j, \ell = 1, \dots, d$. □

Lemma 3.2.21. *Let M_{x_j}, $j = 1, \dots, d$, be as in Definition 3.2.17. Then M_{x_j} is self-adjoint for all $j = 1, \dots, d$.*

Proof. We need to show that

$$\begin{aligned}
&\Big\langle \sum_{k=1}^{p} M_{x_j}^{(k)} \Big(\sum_{\lambda \in \Gamma_{m,d}} q_\lambda^{(k)} x^\lambda e_k + \mathcal{J} \Big), \sum_{\ell=1}^{p} \sum_{\lambda \in \Gamma_{m,d}} f_\lambda^{(\ell)} x^\lambda e_\ell + \mathcal{J} \Big\rangle \\
&= \Big\langle \sum_{k=1}^{p} \sum_{\lambda \in \Gamma_{m,d}} q_\lambda^{(k)} x^\lambda e_k + \mathcal{J}, \sum_{\ell=1}^{p} M_{x_j}^{(\ell)} \Big(\sum_{\lambda \in \Gamma_{m,d}} f_\lambda^{(\ell)} x^\lambda e_\ell + \mathcal{J} \Big) \Big\rangle,
\end{aligned}$$

that is,

$$\langle M_{x_j}(q + \mathcal{J}), f + \mathcal{J} \rangle = \langle q + \mathcal{J}, M_{x_j}(f + \mathcal{J}) \rangle.$$

We have

$$\begin{aligned}
&\left\langle \sum_{k=1}^{p} M_{x_j}^{(k)}\Big(\sum_{\lambda\in\Gamma_{m,d}} q_\lambda^{(k)} x^\lambda e_k + \mathcal{J}\Big), \sum_{\ell=1}^{p} \sum_{\lambda\in\Gamma_{m,d}} f_\lambda^{(\ell)} x^\lambda e_\ell + \mathcal{J} \right\rangle \\
&= \left\langle \sum_{k=1}^{p} \sum_{\lambda\in\Gamma_{m,d}} q_\lambda^{(k)} x^{\lambda+\varepsilon_j} e_k + \mathcal{J}, \sum_{\ell=1}^{p} \sum_{\lambda\in\Gamma_{m,d}} f_\lambda^{(\ell)} x^\lambda e_\ell + \mathcal{J} \right\rangle
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
&\left\langle \sum_{k=1}^{p} \sum_{\lambda\in\Gamma_{m,d}} q_\lambda^{(k)} x^\lambda e_k + \mathcal{J}, \sum_{\ell=1}^{p} M_{x_j}^{(\ell)}\Big(\sum_{\lambda\in\Gamma_{m,d}} f_\lambda^{(\ell)} x^\lambda e_\ell + \mathcal{J}\Big) \right\rangle \\
&= \left\langle \sum_{k=1}^{p} \sum_{\lambda\in\Gamma_{m,d}} q_\lambda^{(k)} x^{\lambda} e_k + \mathcal{J}, \sum_{\ell=1}^{p} \sum_{\lambda\in\Gamma_{m,d}} f_\lambda^{(\ell)} x^{\lambda+\varepsilon_j} e_\ell + \mathcal{J} \right\rangle.
\end{aligned} \tag{3.15}$$

Equation (3.14) is equal to

$$\sum_{k,\ell=1}^{p} \hat{f}^* M(\infty)\widehat{(x_j q)},$$

where $\hat{f} \in (\mathbb{C}^p)_0^\omega$ and $\widehat{(x_j q)} \in (\mathbb{C}^p)_0^\omega$ and equation (3.15) is equal to

$$\sum_{\ell,k=1}^{p} \widehat{(x_j f)}^* M(\infty)\hat{q},$$

where $\widehat{(x_j f)} \in (\mathbb{C}^p)_0^\omega$ and $\hat{q} \in (\mathbb{C}^p)_0^\omega$. It remains to show that $\hat{f}^* M(\infty)\widehat{(x_j q)} = \widehat{(x_j f)}^* M(\infty)\hat{q}$. We have

$$\begin{aligned}
\hat{f}^* M(\infty)\widehat{(x_j q)} &= \hat{f}^* \operatorname{col}\big(\sum_{\lambda\in\Gamma_{m,d}} S_{\gamma+\lambda}\widehat{(x_j q)}\big)_{\gamma\in\mathbb{N}_0^d} \\
&= \hat{f}^* \operatorname{col}\big(\sum_{\lambda\in\Gamma_{m,d}} S_{\gamma+\lambda+\varepsilon_j}\hat{q}\big)_{\gamma\in\mathbb{N}_0^d} \\
&= \widehat{(x_j f)}^* M(\infty)\hat{q}
\end{aligned}$$

and the proof is now complete. □

Next, we shall use spectral theory involving the preceding multiplication operators. First, we denote by $\mathcal{P}$ the set of the orthogonal projections on $\mathbb{C}^p[x_1,\dots,x_d]/\mathcal{J}$ as in the next definition.

Definition 3.2.22. Let $\mathcal{H}$ be a complex Hilbert space. A bounded linear map $Q : \mathcal{H} \to \mathcal{H}$ is called an orthogonal projection if Q is self-adjoint and $Q^2 = Q$.

We also define $\mathcal{P}$ to be the set of the orthogonal projections on $\mathbb{C}^p[x_1,\dots,x_d]/\mathcal{J}$.

Remark 3.2.23. M_{x_j} is self-adjoint for all $j = 1,\dots,d$ and so by the spectral theorem for bounded self-adjoint operators on a Hilbert space (see, e.g., [71, Theorem 5.1]), there exists a unique spectral measure $E_j : \mathcal{B}(\sigma(E_j)) \to \mathcal{P}$, $\sigma(E_j) \subseteq \mathcal{B}(\mathbb{R}^d)$, such that

$$M_{x_j}(q+\mathcal{J}) = \int_{\sigma(E_j)} x dE_j(x)(q+\mathcal{J}) \quad \text{for } j = 1,\dots,d.$$

E_j is unique, in the sense that if $F_j : \mathcal{B}(\mathbb{R}) \to \mathcal{P}$ is another spectral measure such that

$$M_{x_j}(q+\mathcal{J}) = \int_{\sigma(E_j)} x dF_j(x)(q+\mathcal{J}) \quad \text{for } \; j = 1, \dots, d,$$

then we have

$$E_j\big(\alpha \cap \sigma(E_j)\big) = F_j(\alpha) \quad \text{for } \; \alpha \in \mathcal{B}(\mathbb{R}), \; j = 1, \dots, d.$$

By [71, Lemma 4.3], $E_j(\alpha)E_j(\beta) = E_j(\alpha \cap \beta)$ for $\alpha, \beta \in \mathcal{B}(\sigma(E_j))$, which implies that

$$E_j(\alpha)E_k(\beta) = E_k(\beta)E_j(\alpha) \;\text{ for } \alpha \in \mathcal{B}(\sigma(E_j)), \; \beta \in \mathcal{B}(\sigma(E_k)), \; j, k = 1, \dots, d.$$

Since M_{x_j} is self-adjoint and pairwise commute, that is, $M_{x_j}M_{x_k} = M_{x_k}M_{x_j}$ for all $j, k = 1, \dots, d$ (see Lemma 3.2.20), we have that for all Borel sets $\alpha, \beta \in \mathcal{B}(\mathbb{R}^d)$,

$$M_{x_j}(\alpha)M_{x_k}(\beta) = M_{x_k}(\beta)M_{x_j}(\alpha) \quad \text{for } \; j, k = 1, \dots, d.$$

Thus, by [71, Theorem 4.10], there exists a unique spectral measure E on the Borel algebra $\mathcal{B}(\Omega)$ of the product space $\Omega = \sigma(E_1) \times \cdots \times \sigma(E_d)$ such that

$$E(\alpha_1 \times \cdots \times \alpha_d) = E_{x_1}(\alpha_1) \cdots E_{x_d}(\alpha_d) \quad \text{for } \; \alpha_j \in \mathcal{B}(\Omega), \; j = 1, \dots, d.$$

Remark 3.2.24 ([71, Theorem 5.23])**.** For M_{x_j}, $j = 1, \dots, d$, commuting self-adjoint operators on the quotient space $\mathbb{C}^p[x_1, \dots, x_d]/\mathcal{J}$, there exists a joint spectral measure $E : \mathcal{B}(\mathbb{R}^d) \to \mathcal{P}$ such that for every $q, f \in \mathbb{C}^p$,

$$\langle M_{x_1}^{\gamma_1} \cdots M_{x_d}^{\gamma_d}(q+\mathcal{J}), f+\mathcal{J}\rangle = \int_{\mathbb{R}^d} x_1^{\gamma_1} \cdots x_d^{\gamma_d} d\langle E(x_1, \dots, x_d)(q+\mathcal{J}), f+\mathcal{J}\rangle, \quad j = 1, \dots, d.$$

Definition 3.2.25 ([71, Definition 5.3])**.** The support of the spectral measure E is called the *joint spectrum of* $M_{x_1}, \dots, M_{x_d}$ and is denoted by $\sigma(M_x) = \sigma(M_{x_1}, \dots, M_{x_d})$.

Lemma 3.2.26. *If* $r = \operatorname{rank} M(\infty) = \dim(\mathbb{C}^p[x_1, \dots, x_d]/\mathcal{J}) < \infty$, *then*

$$\operatorname{card} \sigma(M_x) \le r,$$

where $\sigma(M_x)$ *is as defined in Definition 3.2.25.*

Proof. Since M_{x_j} $j = 1, \dots, d$ are self-adjoint operators on the finite dimensional Hilbert space $\mathbb{C}^p[x_1, \dots, x_d]/\mathcal{J}$, we have $\sigma(M_{x_j}) \subseteq \mathbb{R}$ with

$$\operatorname{card} \sigma(M_{x_j}) \le \dim(\mathbb{C}^p[x_1, \dots, x_d]/\mathcal{J}) = r < \infty.$$

We next fix a basis $\mathcal{D}$ of $\mathbb{C}^p[x_1, \dots, x_d]/\mathcal{J}$ and let $A_j \in \mathbb{C}^{r \times r}$ be the matrix representation of M_{x_j} with respect to $\mathcal{D}$. Then since M_{x_j} are commuting self-adjoint operators we get

$$A_j^* = A_j \;\text{ for } \; j = 1, \dots, d.$$

By [47, Theorem 2.5.5], there exists unitary $U \in \mathbb{C}^{r\times r}$ such that

$$A_j = U\mathrm{diag}(\nu_1^{(j)}, \dots, \nu_r^{(j)})U^* \text{ for } j = 1, \dots, d$$

and $\mathrm{diag}(\nu_1^{(j)}, \dots, \nu_r^{(j)}) \in \mathbb{C}^{r\times r}$ with $\nu_1^{(j)}, \dots, \nu_r^{(j)}$ the eigenvalues of A_j. Therefore

$$\sigma(M_x) = \{(\nu_1^{(1)}, \dots, \nu_1^{(d)}), \dots, (\nu_r^{(1)}, \dots, \nu_r^{(d)})\},$$

from which we derive $\mathrm{card}\,\sigma(M_x) \le r$. □

The following proposition is a significant tool in our construction, since it proves the existence of a representing measure T for a given $\mathcal{H}_p$-valued multisequence $S^{(\infty)} := (S_\gamma)_{\gamma\in\mathbb{N}_0^d}$ which gives rise to an infinite moment matrix with finite rank.

Proposition 3.2.27. *Let $S^{(\infty)} := (S_\gamma)_{\gamma\in\mathbb{N}_0^d}$ be a given $\mathcal{H}_p$-valued multisequence with corresponding moment matrix $M(\infty) \succeq 0$. Suppose $r := \mathrm{rank}\, M(\infty) < \infty$. Then $S^{(\infty)}$ has a representing measure T.*

Proof. First we show

$$v^* S_\gamma v = \langle M_{x_1}^{\gamma_1} \cdots M_{x_d}^{\gamma_d}(v + \mathcal{J}), v + \mathcal{J}\rangle = \int_{\mathbb{R}^d} x_1^{\gamma_1} \cdots x_d^{\gamma_d} d\langle E(x_1, \dots, x_d)(v + \mathcal{J}), v + \mathcal{J}\rangle, \quad (3.16)$$

that is,

$$v^* S_\gamma v = \langle M_{x_1}^{\gamma_1} \cdots M_{x_d}^{\gamma_d}(v + \mathcal{J}), v + \mathcal{J}\rangle = \int_{\mathbb{R}^d} x^\gamma d\langle E(x_1, \dots, x_d)(v + \mathcal{J}), v + \mathcal{J}\rangle \quad (3.17)$$

for all $v \in \mathbb{C}^p$ and $\gamma \in \mathbb{N}_0^d$. For all $v \in \mathbb{C}^p$, we have

$$\begin{aligned} \langle M_{x_1}^{\gamma_1} \cdots M_{x_d}^{\gamma_d}(v + \mathcal{J}), v + \mathcal{J}\rangle &= \langle x_1^{\gamma_1} \cdots x_d^{\gamma_d} v + \mathcal{J}, v + \mathcal{J}\rangle \\ &= \hat{v}^* M(\infty)\widehat{(x^\gamma v)} \\ &= \hat{v}^* \,\mathrm{col}(S_{\tilde{\gamma}+\gamma}\hat{v})_{\tilde{\gamma}\in\mathbb{N}_0^d} \\ &= v^* S_\gamma v \end{aligned}$$

Therefore, we have obtained the left hand side of the equation (3.16). The right hand side is implied by Remark 3.2.24. Indeed we have

$$\begin{aligned} \langle \int_{\mathbb{R}^d} x_1^{\gamma_1} \cdots x_d^{\gamma_d} dE(x_1, \dots, x_d)(v + \mathcal{J}), (v + \mathcal{J})\rangle &= \int_{\mathbb{R}^d} x_1^{\gamma_1} \cdots x_d^{\gamma_d} d\langle E(x_1, \dots, x_d)(v + \mathcal{J}), v + \mathcal{J}\rangle \\ &= \langle M_{x_1}^{\gamma_1} \cdots M_{x_d}^{\gamma_d}(v + \mathcal{J}), v + \mathcal{J}\rangle, \end{aligned}$$

for $\gamma \in \mathbb{N}_0^d$ and equation (3.17) holds.

Let $v^* T(\alpha) v := \langle E(\alpha)(v + \mathcal{J}), v + \mathcal{J}\rangle$ for every $\alpha \in \mathcal{B}(\mathbb{R}^d)$. We rewrite equation (3.17) as

$$v^* S_\gamma v = \langle M_{x_1}^{\gamma_1} \cdots M_{x_d}^{\gamma_d}(v + \mathcal{J}), v + \mathcal{J}\rangle = \int_{\mathbb{R}^d} x^\gamma d\langle T(x) v, v\rangle$$

and let $T_{v,v}(\alpha) := v^*T(\alpha)v$, where $\alpha \in \mathcal{B}(\mathbb{R}^d)$. Notice that $T_{v,v}(\alpha) \succeq 0$. We need to show

$$v^*S_\gamma w = \int_{\mathbb{R}^d} x^\gamma dT_{w,v}(x) \quad \text{for} \;\; \gamma \in \mathbb{N}_0^d.$$

Fix $\alpha \in \mathcal{B}(\mathbb{R}^d)$ and define

$$T_{w,v}(\alpha) := \frac{1}{4}(T_{w+v}(\alpha) - T_{w-v}(\alpha) + iT_{w+iv}(\alpha) - iT_{w-iv}(\alpha)) \quad \text{for } v, w \in \mathbb{C}^p. \tag{3.18}$$

We observe

$$\begin{aligned}
4\int_{\mathbb{R}^d} x^\gamma dT_{w,v}(x) &= \int_{\mathbb{R}^d} x^\gamma dT_{w+v}(x) - \int_{\mathbb{R}^d} x^\gamma dT_{w-v}(x) \\
&+i\int_{\mathbb{R}^d} x^\gamma dT_{w+iv}(x) - i\int_{\mathbb{R}^d} x^\gamma dT_{w-iv}(x) \\
&= (w+v)^*S_\gamma(w+v) - (w-v)^*S_\gamma(w-v) \\
&+i(w+iv)^*S_\gamma(w+iv) - i(w-iv)^*S_\gamma(w-iv) \\
&= 4v^*S_\gamma w
\end{aligned}$$

for all $\gamma \in \mathbb{N}_0^d$ and $v, w \in \mathbb{C}^p$. Thus

$$v^*S_\gamma w = \int_{\mathbb{R}^d} x^\gamma dT_{w,v}(x) \quad \text{for} \;\; v, w \in \mathbb{C}^p \;\; \text{and } \gamma \in \mathbb{N}_0^d. \tag{3.19}$$

Let $\beta(w,v) : \mathbb{C}^p \times \mathbb{C}^p \to \mathbb{C}$ be given by $\beta(w,v) := T_{w,v}(\alpha)$ where $\alpha \in \mathcal{B}(\mathbb{R}^d)$ is fixed. Using assumption (A1) we have

$$T_{v,v}(\alpha) \le T_{v,v}(\mathbb{R}^d) = v^*S_{0_d}v = v^*I_pv \le \max \frac{v^*I_pv}{v^*v}v^*v = ||v||^2,$$

by the Rayleigh-Ritz Theorem (see, e.g., [47, Theorem 4.2.2]), where $\max \frac{v^*I_pv}{v^*v}$, $v \neq 0_p$, is the maximum eigenvalue of the matrix I_p. For all $w, v \in \mathbb{C}^p$, formula (3.18) yields

$$\begin{aligned}
|\beta(w,v)| &= |T_{w,v}(\alpha)| \\
&\le \left|\tfrac{1}{4}\Big(||w+v||^2 - ||w-v||^2 + i||w+iv||^2 - i||w-iv||^2\Big)\right| \\
&= \Big|\tfrac{1}{4}\Big(||w||^2 + ||v||^2 + 2\mathrm{Re}(\langle w,v\rangle) - (||w||^2 + ||v||^2 - 2\mathrm{Re}(\langle w,v\rangle)) \\
&+i(||w||^2 + ||v||^2 - 2i\mathrm{Re}(\langle w,v\rangle)) - i(||w||^2 + ||v||^2 + 2i\mathrm{Re}(\langle w,v\rangle))\Big)\Big| \\
&= |2\mathrm{Re}(\langle w,v\rangle)| \\
&\le 2||w||\,||v||,
\end{aligned}$$

by the Cauchy-Schwarz inequality. Hence β is a bounded sesquilinear form. For every $v \in \mathbb{C}^p$, the linear functional $L_v : \mathbb{C}^p \to \mathbb{C}$ given by $L_v(w) = \beta(w,v)$ is such that

$$|L_v(w)| = |\beta(w,v)| \le ||\beta||\,||w||\,||v||.$$

By the Riesz Representation Theorem for Hilbert spaces (see, e.g., [63, Theorem 4, Section 6.3]), there exists a unique $\varphi \in \mathbb{C}^p$ such that

$$L_v(w) = \langle \varphi, v \rangle \text{ for all } v \in \mathbb{C}^p.$$

Let $T : \mathcal{B}(\mathbb{R}^d) \to \mathcal{H}_p$ be given by

$$v^* T(\alpha) w = \beta(w, v) = T_{w,v}(\alpha) \quad \text{for} \ \ w, v \in \mathbb{C}^p,$$

for which $T(\alpha)w = \varphi$, $\alpha \in \mathcal{B}(\mathbb{R}^d)$. Since

$$w^* T(\alpha) w = T_{w,w}(\alpha) \geq 0 \ \text{ for } w \in \mathbb{C}^p,$$

we have $T(\alpha) \succeq 0$ for $\alpha \in \mathcal{B}(\mathbb{R}^d)$. Therefore, formula (3.19) implies

$$S_\gamma = \int_{\mathbb{R}^d} x^\gamma dT(x) \ \text{ for } \gamma \in \mathbb{N}_0^d$$

and so, $S^{(\infty)}$ has a representing measure T. □

3.3 Necessary conditions for the existence of a representing measure

Throughout the section a series of lemmas are shown on the variety of the moment matrix and its connection with the support of the representing measure. We study necessary conditions for a solution to the matrix-valued moment problem and our aim is to state and prove the main result of this chapter (see Theorem 3.3.15). We show that if $M(\infty) \succeq 0$ and $\operatorname{rank} M(\infty) < \infty$, then the associated $\mathcal{H}_p$-valued multisequence has a representing measure T with $\sum\limits_{a=1}^{\kappa} \operatorname{rank} Q_a = \operatorname{rank} M(\infty)$ and $\operatorname{supp} T = \mathcal{V}(\mathcal{I})$, where $\mathcal{I} \subseteq \mathbb{C}^{p\times p}[x_1, \ldots, x_d]$ is the right ideal associated with $M(\infty)$.

This in turn yields to an analogous result (see Corollary 3.3.16) where a truncated $\mathcal{H}_p$-valued multisequence has a unique representing measure T and is later being used in the proof of the flat extension theorem for matricial moments (see Theorem 4.0.2).

Lemma 3.3.1. *Let $S := (S_\gamma)_{\gamma \in \Gamma_{2n,d}}$ be a given truncated $\mathcal{H}_p$-valued multisequence and $M(n)$ the corresponding moment matrix. If S has a representing measure T, then $M(n) \succeq 0$.*

Proof. For $\eta = \operatorname{col}(\eta_\lambda)_{\lambda \in \Gamma_{n,d}}$, we have

$$\eta^* M(n) \eta = \int_{\mathbb{R}^d} \zeta(x)^* dT(x) \zeta(x) \geq 0,$$

where $\zeta(x) = \sum\limits_{\lambda \in \Gamma_{n,d}} x^\lambda \eta_\lambda$. □

Definition 3.3.2. Let T be a representing measure for $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$, where $S_\gamma \in \mathcal{H}_p$ for $\gamma \in \Gamma_{2n,d}$ and $P(x) = \sum\limits_{\lambda\in\Gamma_{n,d}} x^\lambda P_\lambda \in \mathbb{C}_n^{p\times p}[x_1,\dots,x_d]$. We define

$$\int_{\mathbb{R}^d} P(x)^* dT(x) P(x) := \sum_{\lambda,\gamma\in\Gamma_{n,d}} P_\lambda^* S_{\gamma+\lambda} P_\gamma.$$

Remark 3.3.3. In view of [51, Theorem 2], if S has a representing measure T, then we can always find a representing measure $\tilde{T}$ for S of the form $\tilde{T} = \sum\limits_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$ with $\kappa \le \binom{2n+d}{d} p$. Then we may let

$$\int_{\mathbb{R}^d} P(x)^* d\tilde{T}(x) P(x) := \sum_{a=1}^{\kappa} P(w^{(a)})^* Q_a P(w^{(a)}).$$

The following lemma is very important for connecting the support of a representing measure of an $\mathcal{H}_p$-valued truncated multisequence and the variety of a moment matrix.

Lemma 3.3.4. *Let $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$ be a given truncated $\mathcal{H}_p$-valued multisequence with a representing measure T. Suppose $M(n)$ is the corresponding moment matrix. If*

$$\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} S_{\gamma+\lambda} P_\lambda\right)_{\gamma\in\Gamma_{n,d}} = \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}},$$

then

$$\operatorname{supp} T \subseteq \mathcal{Z}(\det P(x)),$$

where $P(x) = \sum\limits_{\lambda\in\Gamma_{n,d}} x^\lambda P_\lambda \in \mathbb{C}_n^{p\times p}[x_1,\dots,x_d]$.

Proof. If $\operatorname{col}\left(\sum\limits_{\lambda\in\Gamma_{n,d}} S_{\gamma+\lambda} P_\lambda\right)_{\gamma\in\Gamma_{n,d}} = \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}}$, then

$$\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} S_{\gamma+\lambda} P_\lambda\right)^*_{\gamma\in\Gamma_{n,d}} \operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} S_{\gamma+\lambda} P_\lambda\right)_{\gamma\in\Gamma_{n,d}} = 0_{p\times p},$$

that is, $\sum\limits_{\lambda,\gamma\in\Gamma_{n,d}} P_\lambda^* S_{\gamma+\lambda} P_\gamma = 0_{p\times p}$, which is equivalent to $\int_{\mathbb{R}^d} P(x)^* dT(x) P(x) = 0_{p\times p}$. Indeed

$$\operatorname{col}(P_\lambda)^*_{\lambda\in\Gamma_{n,d}} \operatorname{col}\left(\sum_{\gamma\in\Gamma_{n,d}} S_{\gamma+\lambda} P_\gamma\right)_{\lambda\in\Gamma_{n,d}} = 0_{p\times p}$$

and so

$$\begin{aligned}\operatorname{col}(P_\lambda)^*_{\lambda\in\Gamma_{n,d}} M(n) \operatorname{col}(P_\lambda)_{\lambda\in\Gamma_{n,d}} &= \sum_{\lambda,\gamma\in\Gamma_{n,d}} P_\lambda^* S_{\gamma+\lambda} P_\gamma \\ &= \int_{\mathbb{R}^d} P(x)^* dT(x) P(x) \\ &= 0_{p\times p}.\end{aligned}$$

Suppose to the contrary that

$$\operatorname{supp} T \nsubseteq \mathcal{Z}(\det P(x)).$$

Then there exists a point $u^{(0)} \in \operatorname{supp} T$ such that $u^{(0)} \notin \mathcal{Z}(\det P(x))$ and

$$B_\varepsilon(u^{(0)}) = \{x \in \mathbb{R}^d : ||x - u^{(0)}|| < \varepsilon\} \quad \text{for } \varepsilon > 0 \text{ small enough,}$$

has the property $T(\overline{B_\varepsilon(u^{(0)})}) \neq 0_{p\times p}$ and $\overline{B_\varepsilon(u^{(0)})} \cap \mathcal{Z}(\det P(x)) = \emptyset$. We write

$$\int_{\mathbb{R}^d} P(x)^* dT(x) P(x) = \int_{\overline{B_\varepsilon(u^{(0)})}} P(x)^* dT(x) P(x) + \int_{\mathbb{R}^d \setminus \overline{B_\varepsilon(u^{(0)})}} P(x)^* dT(x) P(x)$$

and we note that both terms on the right hand side are positive semidefinite.
Let $Y := T|_{\overline{B_\varepsilon(u^{(0)})}} = T(\sigma \cap \overline{B_\varepsilon(u^{(0)})})$ for $\sigma \in \mathcal{B}(\mathbb{R}^d)$. Consider $\tilde{S} := (\tilde{S}_\gamma)_{\gamma\in\Gamma_{2n,d}}$, where

$$\tilde{S}_\gamma = \int_{\mathbb{R}^d} x^\gamma dY(x) \quad \text{for } \gamma \in \Gamma_{2n,d}$$

and note that $\tilde{S}_{0_d} = \int_{\mathbb{R}^d} dY(x) = Y(\overline{B_\varepsilon(u^{(0)})}) \neq 0_{p\times p}$. Applying [51, Theorem 2] we obtain a representing measure for $\tilde{S}$ of the form $\widetilde{Y} = \sum_{a=1}^{\kappa} Q_a \delta_{u^{(a)}}$, with nonzero $Q_a \succeq 0$, $\kappa \leq \binom{2n+d}{d} p$ and $u^{(1)}, \dots, u^{(\kappa)} \in \overline{B_\varepsilon(u^{(0)})}$. But then

$$\begin{aligned} 0_{p\times p} = \int_{\overline{B_\varepsilon(u^{(0)})}} P(x)^* dT(x) P(x) &= \int_{\mathbb{R}^d} P(x)^* dY(x) P(x) \\ &= \int_{\mathbb{R}^d} P(x)^* d\widetilde{Y}(x) P(x) \\ &= \sum_{a=1}^{\kappa} P(u^{(a)})^* Q_a P(u^{(a)}), \end{aligned}$$

by Remark 3.3.3. Since $P(u^{(a)})^* Q_a P(u^{(a)}) \succeq 0_{p\times p}$ for $a = 1, \dots \kappa$, we derive

$$P(u^{(a)})^* Q_a P(u^{(a)}) = 0_{p\times p} \text{ for } a = 1, \dots \kappa. \tag{3.20}$$

But $P(u^{(a)})$ is invertible and therefore formula (3.20) implies $Q_a = 0_{p\times p}$ for $a = 1, \dots \kappa$, a contradiction. □

The next example illustrates that the converse of Lemma 3.3.4 does not hold when $p > 1$. Note that if $p = 1$, then the assertion in Lemma 3.3.4 is necessary and sufficient, see [16, Proposition 3.1].

Example 3.3.5. Let $S := (S_\gamma)_{\gamma\in\Gamma_{2,2}}$ be a truncated $\mathcal{H}_2$-valued bisequence with $S_{00} = I_2$, $S_{10} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = S_{20}$, $S_{01} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = S_{02}$ and $S_{11} = 0_{2\times 2}$. Then S has a representing measure T given by

$$T = \frac{1}{2}\left(I_2 \delta_{(0,0)} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \delta_{(1,0)} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \delta_{(0,1)} \right).$$

Choose the matrix-valued polynomial in $\mathbb{C}_1^{2\times 2}[x,y]$

$$\begin{aligned} P(x,y) &= \begin{pmatrix} x & 1 \\ 0 & y \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + x \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

and notice that $\det P(x,y) = xy$ and

$$\det P(x,y)|_{\operatorname{supp} T} = 0.$$

We have

$$P(X,Y) = \begin{pmatrix} S_{00} \\ S_{10} \\ S_{01} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} S_{10} \\ S_{20} \\ S_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} S_{01} \\ S_{11} \\ S_{02} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq \operatorname{col}(0_{2\times 2})_{\gamma \in \Gamma_{1,2}} ,$$

which asserts that the converse of Lemma 3.3.4 does not hold.

We continue with results on the variety of a moment matrix and its connection with the support of a representing measure T.

Lemma 3.3.6. *Suppose $S := (S_\gamma)_{\gamma \in \Gamma_{2n,d}}$ is a given truncated $\mathcal{H}_p$-valued multisequence with a representing measure T. Let $M(n)$ be the corresponding moment matrix and let $\mathcal{V}(M(n))$ be the variety of $M(n)$ (see Definition 3.1.21). Let $P(x) = \sum\limits_{\lambda \in \Gamma_{n,d}} x^\lambda P_\lambda \in \mathbb{C}_n^{p\times p}[x_1, \dots, x_d]$. If*

$$\operatorname{col}\left(\sum_{\lambda \in \Gamma_{n,d}} S_{\gamma+\lambda} P_\lambda \right)_{\gamma \in \Gamma_{n,d}} = \operatorname{col}(0_{p\times p})_{\gamma \in \Gamma_{n,d}},$$

then

$$\operatorname{supp} T \subseteq \mathcal{V}(M(n)).$$

Proof. By Lemma 3.3.4, for any $P(x) \in \mathbb{C}_n^{p\times p}[x_1, \dots, x_d]$ with

$$P(X) = \operatorname{col}\left(\sum_{\lambda \in \Gamma_{n,d}} S_{\gamma+\lambda} P_\lambda \right)_{\gamma \in \Gamma_{n,d}} = \operatorname{col}(0_{p\times p})_{\gamma \in \Gamma_{n,d}},$$

we have $\operatorname{supp} T \subseteq \mathcal{Z}(\det P(x))$. Thus

$$\bigcap_{\substack{P(X) = \operatorname{col}(0_{p\times p})_{\gamma \in \Gamma_{n,d}} \\ P \in \mathbb{C}_n^{p\times p}[x_1,\dots,x_d]}} \operatorname{supp} T \subseteq \bigcap_{\substack{P(X) = \operatorname{col}(0_{p\times p})_{\gamma \in \Gamma_{n,d}} \\ P \in \mathbb{C}_n^{p\times p}[x_1,\dots,x_d]}} \mathcal{Z}(\det P(x)),$$

which implies that

$$\operatorname{supp} T \subseteq \mathcal{V}(M(n)).$$

□

Lemma 3.3.7. *Let $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$ be a given truncated $\mathcal{H}_p$-valued multisequence and let $M(n)$ be the corresponding moment matrix. If S has a representing measure T and $w^{(1)}, \dots, w^{(\kappa)} \in \mathbb{R}^d$ are given such that*

$$\operatorname{supp} T = \{w^{(1)}, \dots, w^{(\kappa)}\},$$

then there exists $P(x) \in \mathbb{C}_n^{p\times p}[x_1, \dots, x_d]$ such that

$$\mathcal{Z}(\det P(x)) = \{w^{(1)}, \dots, w^{(\kappa)}\}.$$

Moreover

$$\operatorname{col}\Bigg(\sum_{\lambda\in\Gamma_{n,d}} S_{\gamma+\lambda}P_\lambda\Bigg)_{\gamma\in\Gamma_{n,d}} = \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}},$$

and

$$\mathcal{V}(M(n)) \subseteq \operatorname{supp} T,$$

where $\mathcal{V}(M(n))$ the variety of $M(n)$ (see Definition 3.1.21).

Proof. If we let $P(x) := \prod_{a=1}^{\kappa}\prod_{j=1}^{d}(x_j - w_j^{(a)})I_p$, then $\det P(x) = \prod_{a=1}^{\kappa}\prod_{j=1}^{d}(x_j - w_j^{(a)})^p$ and so

$$\det P(w^{(a)}) = \prod_{a=1}^{\kappa}\prod_{j=1}^{d}(w_j^{(a)} - w_j^{(a)})^p = 0.$$

Thus

$$\{w^{(1)}, \dots, w^{(\kappa)}\} \subseteq \mathcal{Z}(\det P(x)). \tag{3.21}$$

If we let $P(x) := \prod_{a=1}^{\kappa}\Big(\sum_{j=1}^{d}(x_j - w_j^{(a)})^2\Big)I_p$, then $\det P(x) = \prod_{a=1}^{\kappa}\Big(\sum_{j=1}^{d}(x_j - w_j^{(a)})^2\Big)^p$ and hence

$$\det P(w^{(a)}) = \prod_{a=1}^{\kappa}\Bigg(\sum_{j=1}^{d}(w_j^{(a)} - w_j^{(a)})^2\Bigg)^p = 0,$$

which yields

$$\mathcal{Z}(\det P(x)) \subseteq \{w^{(1)}, \dots, w^{(\kappa)}\}. \tag{3.22}$$

Then by inclusions (3.21) and (3.22), we obtain

$$\mathcal{Z}(\det P(x)) = \{w^{(1)}, \dots, w^{(\kappa)}\}.$$

Hence $\mathcal{Z}(\det P(x)) = \{w^{(1)}, \dots, w^{(\kappa)}\} = \operatorname{supp} T$, where $T = \sum\limits_{a=1}^{\kappa} Q_a\delta_{w^{(a)}}$. We will next show that for both choices of $P \in \mathbb{C}_n^{p\times p}[x_1, \dots, x_d]$, one obtains

$$\operatorname{col}\Bigg(\sum_{\lambda\in\Gamma_{n,d}} S_{\gamma+\lambda}P_\lambda\Bigg)_{\gamma\in\Gamma_{n,d}} = \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}}$$

and thus $\mathcal{V}(M(n)) \subseteq \operatorname{supp} T$. For the choice of $P(x) := \prod_{a=1}^{\kappa}\prod_{j=1}^{d}(x_j - w_j^{(a)})I_p \in \mathbb{C}_n^{p\times p}[x_1, \dots, x_d]$, we have

$$\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} S_{\gamma+\lambda}P_\lambda\right)_{\gamma\in\Gamma_{n,d}} = \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}}.$$

Consider $P(X)\in C_{M(n)}$. We have $\mathcal{Z}(\det P(x)) = \operatorname{supp} T$ and we shall see $P(X) = \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}}$. We notice

$$\begin{aligned} P(X) = \operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} X^{\gamma+\lambda}P_\lambda\right)_{\gamma\in\Gamma_{n,d}} &= \operatorname{col}\left(\int_{\mathbb{R}^d} x^\gamma dT(x)P(x)\right)_{\gamma\in\Gamma_{n,d}} \\ &= \operatorname{col}\left(\int_{\mathbb{R}^d} x^\gamma dT(x)\prod_{a=1}^{\kappa}\prod_{j=1}^{d}(x_j - w_j^{(a)})I_p\right)_{\gamma\in\Gamma_{n,d}} \\ &= \operatorname{col}\left(\int_{\mathbb{R}^d} x^\gamma \varphi(x)dT(x)\right)_{\gamma\in\Gamma_{n,d}}, \end{aligned}$$

where $\varphi(x) = \prod_{a=1}^{\kappa}\prod_{j=1}^{d}(x_j - w_j^{(a)}) \in \mathbb{R}[x_1,\dots,x_d]$. Since $T = \sum\limits_{a=1}^{\kappa} Q_a\delta_{w^{(a)}}$, $P(X)$ becomes

$$\begin{aligned} \operatorname{col}\left(\sum_{a=1}^{\kappa}\{w^{(a)}\}^\gamma\varphi(w^{(a)})Q_a\right)_{\gamma\in\Gamma_{n,d}} &= \operatorname{col}\left(\sum_{a=1}^{\kappa}\{w^{(a)}\}^\gamma\prod_{a=1}^{\kappa}\prod_{j=1}^{d}(w_j^{(a)} - w_j^{(a)})Q_a\right)_{\gamma\in\Gamma_{n,d}} \\ &= \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}} \end{aligned}$$

and hence $P(X) = \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}}$. Since there exists matrix-valued polynomial $P(x)$ such that $P(X) = \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}}$ and $\mathcal{Z}(\det P(x)) = \operatorname{supp} T$, we then have

$$\bigcap_{\substack{P(X)=\operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}} \\ P\in\mathbb{C}_n^{p\times p}[x_1,\dots,x_d]}} \mathcal{Z}(\det P(x)) \subseteq \bigcap_{\substack{P(X)=\operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}} \\ P\in\mathbb{C}_n^{p\times p}[x_1,\dots,x_d]}} \operatorname{supp} T,$$

which implies $\mathcal{V}(M(n)) \subseteq \operatorname{supp} T$. Next, for the choice of $P(x) := \prod_{a=1}^{\kappa}\left(\sum_{j=1}^{d}(x_j - w_j^{(a)})^2\right)I_p$, we will show that

$$\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} S_{\gamma+\lambda}P_\lambda\right)_{\gamma\in\Gamma_{n,d}} = \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}}.$$

We have $\mathcal{Z}(\det P(x)) = \operatorname{supp} T$ and we consider $P(X)\in C_{M(n)}$. We will show that for this choice of $P(x)$, $P(X) = \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}}$. Notice that

$$\begin{aligned} P(X) &= \operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} X^{\gamma+\lambda}P_\lambda\right)_{\gamma\in\Gamma_{n,d}} \\ &= \operatorname{col}\left(\int_{\mathbb{R}^d} x^\gamma dT(x)P(x)\right)_{\gamma\in\Gamma_{n,d}} \\ &= \operatorname{col}\left(\int_{\mathbb{R}^d} x^\gamma dT(x)\prod_{a=1}^{\kappa}\left(\sum_{j=1}^{d}(x_j - w_j^{(a)})^2\right)I_p\right)_{\gamma\in\Gamma_{n,d}} \\ &= \operatorname{col}\left(\int_{\mathbb{R}^d} x^\gamma\tilde{\varphi}(x)dT(x)\right)_{\gamma\in\Gamma_{n,d}}, \end{aligned}$$

where $\tilde{\varphi}(x) = \prod_{a=1}^{\kappa} \sum_{j=1}^{d} (x_j - w_j^{(a)})^2 \in \mathbb{R}[x_1, \dots, x_d]$. Since $T = \sum\limits_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$, $P(X)$ becomes

$$\begin{aligned} \operatorname{col}\left(\sum_{a=1}^{\kappa} \{w^{(a)}\}^{\gamma} \tilde{\varphi}(w^{(a)}) Q_a \right)_{\gamma \in \Gamma_{n,d}} &= \operatorname{col}\left(\sum_{a=1}^{\kappa} \{w^{(a)}\}^{\gamma} \prod_{a=1}^{\kappa} \left(\sum_{j=1}^{d} (w_j^{(a)} - w_j^{(a)})^2 \right) Q_a \right)_{\gamma \in \Gamma_{n,d}} \\ &= \operatorname{col}(0_{p \times p})_{\gamma \in \Gamma_{n,d}} \end{aligned}$$

and so $P(X) = \operatorname{col}(0_{p \times p})_{\gamma \in \Gamma_{n,d}}$. We thus conclude that there exists a matrix-valued polynomial $P(x)$ such that $P(X) = \operatorname{col}(0_{p \times p})_{\gamma \in \Gamma_{n,d}}$ and $\mathcal{Z}(\det P(x)) = \operatorname{supp} T$ and thus we obtain

$$\bigcap_{\substack{P(X) = \operatorname{col}(0_{p \times p})_{\gamma \in \Gamma_{n,d}} \\ P \in \mathbb{C}_n^{p \times p}[x_1, \dots, x_d]}} \mathcal{Z}(\det P(x)) \subseteq \bigcap_{\substack{P(X) = \operatorname{col}(0_{p \times p})_{\gamma \in \Gamma_{n,d}} \\ P \in \mathbb{C}_n^{p \times p}[x_1, \dots, x_d]}} \operatorname{supp} T,$$

which asserts

$$\mathcal{V}(M(n)) \subseteq \operatorname{supp} T. \qquad \square$$

Lemma 3.3.8. *Let $S := (S_\gamma)_{\gamma \in \Gamma_{2n,d}}$ be truncated $\mathcal{H}_p$-valued multisequence and let $M(n)$ be the corresponding moment matrix. If T is a representing measure for S, then*

$$\operatorname{rank} M(n) \leq \sum_{a=1}^{\kappa} \operatorname{rank} Q_a.$$

Proof. If $\operatorname{supp} T$ is infinite, then $\operatorname{rank} M(n) \leq \sum\limits_{a=1}^{\kappa} \operatorname{rank} Q_a$ holds trivially. If $\operatorname{supp} T$ is finite, that is, T is of the form $T = \sum\limits_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$, then

$$M(n) = V^T R\, V,$$

where $V := V^{p \times p}(w^{(1)}, \dots, w^{(\kappa)}; \Lambda) \in \mathbb{C}^{\kappa p \times \kappa p}$ with $\Lambda \subseteq \mathbb{N}_0^d$ and $\operatorname{card} \Lambda = \kappa$ and

$$R := Q_1 \oplus \cdots \oplus Q_\kappa = \begin{pmatrix} Q_1 & & 0 \\ & \ddots & \\ 0 & & Q_\kappa \end{pmatrix} \in \mathbb{C}^{\kappa p \times \kappa p}.$$

Hence

$$\begin{aligned} \operatorname{rank} M(n) &\leq \min(\operatorname{rank} V^T, \operatorname{rank} RV) \\ &\leq \min(\operatorname{rank} V^T, \operatorname{rank} R, \operatorname{rank} V) \\ &\leq \min(\operatorname{rank} V, \operatorname{rank} R) \\ &\leq \operatorname{rank} R \\ &= \sum_{a=1}^{\kappa} \operatorname{rank} Q_a \end{aligned}$$

and the proof is complete. $\square$

Proposition 3.3.9. *Let $S^{(\infty)} := (S_\gamma)_{\gamma\in\mathbb{N}_0^d}$ be a given $\mathcal{H}_p$-valued multisequence with a representing measure T which has $\sum\limits_{a=1}^{\kappa} \operatorname{rank} Q_a < \infty$ and $M(\infty)$ be the corresponding moment matrix. Then*

$$r := \operatorname{rank} M(\infty) = \sum_{a=1}^{\kappa} \operatorname{rank} Q_a.$$

Proof. By Theorem 1.4.26, there exists $\Lambda \subseteq \mathbb{N}_0^d$ such that $\operatorname{card}\Lambda = \kappa$ and $V(w^{(1)}, \dots, w^{(\kappa)}; \Lambda)$ is invertible. If $S^{(\infty)}$ has a representing measure $T = \sum\limits_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$, then

$$\operatorname{rank} M_\Lambda(\infty) \leq \operatorname{rank} M(\infty) \leq \sum_{a=1}^{\kappa} \operatorname{rank} Q_a, \tag{3.23}$$

where $M_\Lambda(\infty)$ is a principal submatrix of $M(\infty)$ with block rows and block columns indexed by Λ. Notice that since $V(w^{(1)}, \dots, w^{(\kappa)}; \Lambda)$ is invertible, by Remark 3.1.27 we deduce that $V := V^{p\times p}(w^{(1)}, \dots, w^{(\kappa)}; \Lambda) \in \mathbb{C}^{\kappa p\times\kappa p}$ is invertible. Moreover, since $V \in \mathbb{R}^{\kappa p\times\kappa p}$, $M_\Lambda(\infty)$ can be written as

$$M_\Lambda(\infty) = V^T R V = V^* R V,$$

where

$$R := Q_1 \oplus \cdots \oplus Q_\kappa = \begin{pmatrix} Q_1 & & 0 \\ & \ddots & \\ 0 & & Q_\kappa \end{pmatrix} \in \mathbb{C}^{\kappa p\times\kappa p}.$$

By Sylvester's law of inertia (see, e.g., [48, Theorem 4.5.8]), we have $i_+(M_\Lambda(\infty)) = i_+(R)$, where i_+ indicates the number of positive eigenvalues. So $\operatorname{rank} M_\Lambda(\infty) = \operatorname{rank} R$. However $\operatorname{rank} R = \sum\limits_{a=1}^{\kappa} \operatorname{rank} Q_a$. By inequality (3.23),

$$\sum_{a=1}^{\kappa} \operatorname{rank} Q_a \leq \operatorname{rank} M(\infty) \leq \sum_{a=1}^{\kappa} \operatorname{rank} Q_a,$$

which implies

$$\sum_{a=1}^{\kappa} \operatorname{rank} Q_a = \operatorname{rank} M(\infty) = r. \qquad \square$$

Lemma 3.3.10. *Suppose $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$ is a truncated $\mathcal{H}_p$-valued multisequence with a representing measure T. Let $M(n)$ be the corresponding moment matrix and $\mathcal{V}(M(n))$ be the variety of $M(n)$ (see Definition 3.1.21). Then*

$$\operatorname{rank} M(n) \leq \operatorname{card} \mathcal{V}(M(n)).$$

Proof. Lemma 3.3.8 asserts that $\operatorname{rank} M(n) \leq \sum\limits_{a=1}^{\kappa} \operatorname{rank} Q_a$ and by Lemma 3.3.6,

$$\operatorname{supp} T \subseteq \mathcal{V}(M(n)),$$

which implies $\sum_{a=1}^{\kappa} \operatorname{rank} Q_a \leq \operatorname{card} \mathcal{V}(M(n))$. Hence

$$\operatorname{rank} M(n) \leq \operatorname{card} \mathcal{V}(M(n)). \qquad \square$$

In analogy to Lemma 3.3.7, we proceed to Lemmas 3.3.11 and 3.3.12 for $P \in \mathbb{C}^{p\times p}[x_1, \ldots, x_d]$.

Lemma 3.3.11. *Let $S^{(\infty)} := (S_\gamma)_{\gamma\in\mathbb{N}_0^d}$ be a given $\mathcal{H}_p$-valued multisequence. If $S^{(\infty)}$ has a representing measure T and $w^{(1)}, \ldots, w^{(\kappa)} \in \mathbb{R}^d$ are given, then there exists $P \in \mathbb{C}^{p\times p}[x_1, \ldots, x_d]$ such that*

$$\mathcal{Z}(\det P(x)) = \{w^{(1)}, \ldots, w^{(\kappa)}\}.$$

Moreover, $P \in \mathcal{I}$ and

$$\mathcal{V}(\mathcal{I}) \subseteq \operatorname{supp} T$$

where $\mathcal{I}$ is as in Definition 3.1.14 and $\mathcal{V}(\mathcal{I})$ the variety of $\mathcal{I}$ (see Definition 3.1.22).

Proof. Let the matrix-valued polynomial $P(x) := \prod_{a=1}^{\kappa}\prod_{j=1}^{d}(x_j - w_j^{(a)})I_p \in \mathbb{C}^{p\times p}[x_1, \ldots, x_d]$. Then $\det P(x) = \prod_{a=1}^{\kappa}\prod_{j=1}^{d}(x_j - w_j^{(a)})^p$ and so

$$\det P(w^{(a)}) = \prod_{a=1}^{\kappa}\prod_{j=1}^{d}(w_j^{(a)} - w_j^{(a)})^p = 0.$$

Thus $\{w^{(1)}, \ldots, w^{(\kappa)}\} \subseteq \mathcal{Z}(\det P(x))$. To show the other inclusion, choose the matrix-valued polynomial $P(x) := \prod_{a=1}^{\kappa}\left(\sum_{j=1}^{d}(x_j - w_j^{(a)})^2\right)I_p \in \mathbb{C}^{p\times p}[x_1, \ldots, x_d]$. Then we shall obtain $\det P(x) = \prod_{a=1}^{\kappa}\left(\sum_{j=1}^{d}(x_j - w_j^{(a)})^2\right)^p$ and so

$$\det P(w^{(a)}) = \prod_{a=1}^{\kappa}\left(\sum_{j=1}^{d}(w_j^{(a)} - w_j^{(a)})^2\right)^p = 0,$$

which implies that $\mathcal{Z}(\det P(x)) \subseteq \{w^{(1)}, \ldots, w^{(\kappa)}\}$. Thus

$$\mathcal{Z}(\det P(x)) = \{w^{(1)}, \ldots, w^{(\kappa)}\}.$$

Let $\operatorname{supp} T = \{w^{(1)}, \ldots, w^{(\kappa)}\}$ where $T = \sum_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$. In the following, we shall see that for both choices of the matrix-valued polynomial $P \in \mathbb{C}^{p\times p}[x_1, \ldots, x_d]$, one obtains $P \in \mathcal{I}$ and this in turn yields the inclusion $\mathcal{V}(\mathcal{I}) \subseteq \operatorname{supp} T$. Consider first the matrix-valued polynomial $P(x) := \prod_{a=1}^{\kappa}\prod_{j=1}^{d}(x_j - w_j^{(a)})I_p$ such that $P(X) \in C_{M(\infty)}$. We have

$$\mathcal{Z}(\det P(x)) = \operatorname{supp} T$$

and we shall show that $P(X) = \operatorname{col}(0_{p\times p})_{\gamma\in\mathbb{N}_0^d}$.

Notice that

$$
\begin{aligned}
P(X) = \operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} X^{\gamma+\lambda} P_\lambda \right)_{\gamma\in\mathbb{N}_0^d} &= \operatorname{col}\left(\int_{\mathbb{R}^d} x^\gamma dT(x) P(x) \right)_{\gamma\in\mathbb{N}_0^d} \\
&= \operatorname{col}\left(\int_{\mathbb{R}^d} x^\gamma dT(x) \prod_{a=1}^{\kappa} \prod_{j=1}^{d} (x_j - w_j^{(a)}) I_p \right)_{\gamma\in\mathbb{N}_0^d} \\
&= \operatorname{col}\left(\int_{\mathbb{R}^d} x^\gamma \varphi(x) dT(x) \right)_{\gamma\in\mathbb{N}_0^d},
\end{aligned}
$$

where $\varphi(x) = \prod_{a=1}^{\kappa} \prod_{j=1}^{d} (x_j - w_j^{(a)}) \in \mathbb{R}[x_1, \dots, x_d]$. Since $T = \sum\limits_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$, $P(X)$ becomes

$$
\begin{aligned}
\operatorname{col}\left(\sum_{a=1}^{\kappa} \{w^{(a)}\}^\gamma \varphi(w^{(a)}) Q_a \right)_{\gamma\in\mathbb{N}_0^d} &= \operatorname{col}\left(\sum_{a=1}^{\kappa} \{w^{(a)}\}^\gamma \prod_{a=1}^{\kappa} \prod_{j=1}^{d} (w_j^{(a)} - w_j^{(a)}) Q_a \right)_{\gamma\in\mathbb{N}_0^d} \\
&= \operatorname{col}(0_{p\times p})_{\gamma\in\mathbb{N}_0^d}
\end{aligned}
$$

and hence $P \in \mathcal{I}$.
Since there exists $P \in \mathcal{I}$ such that $\mathcal{Z}(\det P(x)) = \operatorname{supp} T$,

$$
\mathcal{V}(\mathcal{I}) := \bigcap_{P\in\mathcal{I}} \mathcal{Z}(\det P(x)) \subseteq \bigcap_{P\in\mathcal{I}} \operatorname{supp} T
$$

and thus $\mathcal{V}(\mathcal{I}) \subseteq \operatorname{supp} T$. We continue on showing that for the choice of the matrix-valued polynomial $P(x) := \prod_{a=1}^{\kappa} \left(\sum_{j=1}^{d} (x_j - w_j^{(a)})^2 \right) I_p$, one obtains that $P \in \mathcal{I}$ as well. Consider $P(X) \in C_{M(\infty)}$. We have $\mathcal{Z}(\det P(x)) = \operatorname{supp} T$ and we shall see $P(X) = \operatorname{col}(0_{p\times p})_{\gamma\in\mathbb{N}_0^d}$. Indeed

$$
\begin{aligned}
P(X) &= \operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} X^{\gamma+\lambda} P_\lambda \right)_{\gamma\in\mathbb{N}_0^d} \\
&= \operatorname{col}\left(\int_{\mathbb{R}^d} x^\gamma dT(x) P(x) \right)_{\gamma\in\mathbb{N}_0^d} \\
&= \operatorname{col}\left(\int_{\mathbb{R}^d} x^\gamma dT(x) \prod_{a=1}^{\kappa} \left(\sum_{j=1}^{d} (x_j - w_j^{(a)})^2 \right) I_p \right)_{\gamma\in\mathbb{N}_0^d} \\
&= \operatorname{col}\left(\int_{\mathbb{R}^d} x^\gamma \tilde{\varphi}(x) dT(x) \right)_{\gamma\in\mathbb{N}_0^d},
\end{aligned}
$$

where $\tilde{\varphi}(x) = \prod_{a=1}^{\kappa} \sum_{j=1}^{d} (x_j - w_j^{(a)})^2 \in \mathbb{R}[x_1, \dots, x_d]$. Since $T = \sum\limits_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$, $P(X)$ becomes

$$
\begin{aligned}
\operatorname{col}\left(\sum_{a=1}^{\kappa} \{w^{(a)}\}^\gamma \tilde{\varphi}(w^{(a)}) Q_a \right)_{\gamma\in\mathbb{N}_0^d} &= \operatorname{col}\left(\sum_{a=1}^{\kappa} \{w^{(a)}\}^\gamma \prod_{a=1}^{\kappa} \left(\sum_{j=1}^{d} (w_j^{(a)} - w_j^{(a)})^2 \right) Q_a \right)_{\gamma\in\mathbb{N}_0^d} \\
&= \operatorname{col}(0_{p\times p})_{\gamma\in\mathbb{N}_0^d}
\end{aligned}
$$

and so $P \in \mathcal{I}$. Since there exists $P \in \mathcal{I}$ such that $\mathcal{Z}(\det P(x)) = \operatorname{supp} T$, we again obtain

$$\mathcal{V}(\mathcal{I}) \subseteq \operatorname{supp} T$$

as desired. $\square$

Lemma 3.3.12. *Let T be a representing measure for $S^{(\infty)} := (S_\gamma)_{\gamma \in \mathbb{N}_0^d}$, where $S_\gamma \in \mathcal{H}_p$, $\gamma \in \mathbb{N}_0^d$ and $w^{(1)}, \ldots, w^{(\kappa)} \in \mathbb{R}^d$ be such that*

$$\operatorname{supp} T = \{w^{(1)}, \ldots, w^{(\kappa)}\}.$$

If there exists $P \in \mathbb{C}^{p\times p}[x_1, \ldots, x_d]$ with $P \in \mathcal{I}$, then

$$\operatorname{supp} T \subseteq \mathcal{V}(\mathcal{I}),$$

where $\mathcal{I}$ is as in Definition 3.1.14 and $\mathcal{V}(\mathcal{I})$ the variety of $\mathcal{I}$ (see Definition 3.1.22).

Proof. By Lemma 3.3.11, if we choose $P(x) := \prod_{a=1}^{\kappa} \prod_{j=1}^{d} (x_j - w_j^{(a)}) I_p \in \mathbb{C}^{p\times p}[x_1, \ldots, x_d]$, then $P \in \mathcal{I}$ and

$$\{w^{(1)}, \ldots, w^{(\kappa)}\} \subseteq \mathcal{Z}(\det P(x)),$$

that is,

$$\operatorname{supp} T \subseteq \mathcal{Z}(\det P(x)).$$

Therefore

$$\bigcap_{P\in\mathcal{I}} \operatorname{supp} T \subseteq \bigcap_{P\in\mathcal{I}} \mathcal{Z}(\det P(x))$$

and so

$$\operatorname{supp} T \subseteq \mathcal{V}(\mathcal{I}).$$ $\square$

In the next lemma we treat the multiplication operators of Definition 3.2.17 to provide a connection between the joint spectrum of $M_{x_1}, \ldots, M_{x_d}$ and a representing measure T.

Lemma 3.3.13. *If T is a representing measure for $S^{(\infty)} := (S_\gamma)_{\gamma \in \mathbb{N}_0^d}$, where $S_\gamma \in \mathcal{H}_p$, $\gamma \in \mathbb{N}_0^d$, then*

$$\operatorname{supp} T \subseteq \sigma(M_x),$$

where $\sigma(M_x)$ is as in Definition 3.2.25.

Proof. Since M_{x_j}, $j = 1, \ldots, d$, are commuting self-adjoint operators on $\mathbb{C}^p[x_1, \ldots, x_d]/\mathcal{J}$, by Remark 3.2.24, there exists a joint spectral measure $E : \mathcal{B}(\mathbb{R}^d) \to \mathcal{P}$ such that for every $q, f \in \mathbb{C}^p$,

$$\langle M_{x_1}^{\gamma_1} \ldots M_{x_d}^{\gamma_d}(q + \mathcal{J}), f + \mathcal{J}\rangle = \int_{\mathbb{R}^d} x_1^{\gamma_1} \ldots x_d^{\gamma_d} d\langle E(x_1, \ldots, x_d)(q + \mathcal{J}), f + \mathcal{J}\rangle, \quad j = 1, \ldots, d.$$

Moreover

$$v^* T(\alpha) v = \langle E(\alpha)(v + \mathcal{J}), v + \mathcal{J}\rangle \text{ for every } \alpha \in \mathcal{B}(\mathbb{R}^d).$$

If $\alpha \subseteq \operatorname{supp} T$, then $T(\alpha) \neq 0_{p\times p}$. Thus, there exists $v \in \mathbb{C}^p$ such that $v^*T(\alpha)v > 0$. Hence

$$\langle E(\alpha)(v + \mathcal{J}), v + \mathcal{J}\rangle > 0$$

and so $E(\alpha) \neq 0_{p\times p}$. □

The next lemma describes the block column relations of an infinite moment matrix in terms of the variety of a right ideal built from matrix-valued polynomials.

Lemma 3.3.14. *Let $S^{(\infty)} := (S_\gamma)_{\gamma\in\mathbb{N}_0^d}$ be a given $\mathcal{H}_p$-valued multisequence with a representing measure T. Let $M(\infty)$ be the corresponding moment matrix with $r := \operatorname{rank} M(\infty)$. If there exists $P \in \mathbb{C}^{p\times p}[x_1, \dots, x_d]$ such that $P \in \mathcal{I}$ then*

$$\operatorname{card} \mathcal{V}(\mathcal{I}) = r,$$

where $\mathcal{I}$ is as in Definition 3.1.14 and $\mathcal{V}(\mathcal{I})$ the variety of $\mathcal{I}$ (see Definition 3.1.22).

Proof. By Lemma 3.3.11, there exists $P \in \mathcal{I}$ with $\mathcal{V}(\mathcal{I}) \subseteq \operatorname{supp} T$ such that

$$\mathcal{Z}(\det P(x)) = \operatorname{supp} T$$

and by Lemma 3.3.13, $\operatorname{supp} T \subseteq \sigma(M_x)$. Then

$$\operatorname{supp} T = \mathcal{Z}(\det P(x)) \subseteq \sigma(M_x)$$

and thus

$$\bigcap_{P\in\mathcal{I}} \mathcal{Z}(\det P(x)) \subseteq \sigma(M_x),$$

which is equivalent to $\mathcal{V}(\mathcal{I}) \subseteq \sigma(M_x)$. Therefore

$$\operatorname{card} \mathcal{V}(\mathcal{I}) \leq \operatorname{card} \sigma(M_x) \leq \dim(\mathbb{C}^p[x_1, \dots, x_d]/\mathcal{J}) = r.$$

Moreover, by Remark 3.3.6, $\operatorname{supp} T \subseteq \mathcal{V}(\mathcal{I})$ and so $\sum\limits_{a=1}^{\kappa} \operatorname{rank} Q_a \leq \operatorname{card} \mathcal{V}(\mathcal{I})$. Then Proposition 3.3.9 implies $\operatorname{card} \mathcal{V}(\mathcal{I}) \geq r$. Finally

$$\operatorname{card} \mathcal{V}(\mathcal{I}) = r.$$ □

We next state and prove the main theorem of this chapter. We shall see that if $M(\infty) \succeq 0$ with $\operatorname{rank} M(\infty) < \infty$, then the associated $\mathcal{H}_p$-valued multisequence has a unique representing measure T and one can extract information on the support of the representing measure in terms of the variety of the right ideal associated with $M(\infty)$.

Theorem 3.3.15. *Let $S^{(\infty)} := (S_\gamma)_{\gamma\in\mathbb{N}_0^d}$ be a given $\mathcal{H}_p$-valued multisequence. If $S^{(\infty)}$ gives rise to $M(\infty) \succeq 0$ and $r := \operatorname{rank} M(\infty) < \infty$, then $S^{(\infty)}$ has a unique representing measure T. In* *this case,*

$$\operatorname{supp} T = \mathcal{V}(\mathcal{I}),$$

where $\mathcal{I}$ is as in Definition 3.1.14, and moreover,

$$\operatorname{card} \mathcal{V}(\mathcal{I}) = r.$$

Proof. By Proposition 3.2.27, if $S^{(\infty)}$ gives rise to $M(\infty) \succeq 0$ and $r := \operatorname{rank} M(\infty) < \infty$, then $S^{(\infty)}$ has a representing measure T. Moreover, by Lemma 3.3.12, we have $\operatorname{supp} T \subseteq \mathcal{V}(\mathcal{I})$ and by Lemma 3.3.11, $\mathcal{V}(\mathcal{I}) \subseteq \operatorname{supp} T$. Thus

$$\operatorname{supp} T = \mathcal{V}(\mathcal{I}).$$

Next, Proposition 3.3.9 yields $\sum_{a=1}^{\kappa} \operatorname{rank} Q_a = r = \operatorname{rank} M(\infty)$. Since $\sum_{a=1}^{\kappa} \operatorname{rank} Q_a = r < \infty$, the measure T is of the form $T = \sum_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$, with

$$\sum_{a=1}^{\kappa} \operatorname{rank} Q_a = r \quad \text{and} \quad Q_1, \ldots, Q_\kappa \succeq 0.$$

To prove T is unique, suppose $\widetilde{T}$ is another representing measure for $S^{(\infty)}$. By Remark 3.3.6, we have $\operatorname{supp} \widetilde{T} \subseteq \mathcal{V}(\mathcal{I})$ and by Remark 3.3.11, $\mathcal{V}(\mathcal{I}) \subseteq \operatorname{supp} \widetilde{T}$. As before $\operatorname{supp} \widetilde{T} = \mathcal{V}(\mathcal{I})$, and moreover, $\sum_{b=1}^{\widetilde{\kappa}} \operatorname{rank} \widetilde{Q}_b = r < \infty$, by Proposition 3.3.9. So $\widetilde{T}$ is of the form $\widetilde{T} = \sum_{b=1}^{\widetilde{\kappa}} \widetilde{Q}_b \delta_{\tilde{w}^{(b)}}$ with

$$\sum_{b=1}^{\widetilde{\kappa}} \operatorname{rank} \widetilde{Q}_b = r \quad \text{and} \quad \widetilde{Q}_1, \ldots, \widetilde{Q}_\kappa \succeq 0.$$

Since $\operatorname{supp} T = \mathcal{V}(\mathcal{I}) = \operatorname{supp} T$, we have $\{w^{(a)}\}_{a=1}^{\kappa} = \{\tilde{w}^{(b)}\}_{b=1}^{\widetilde{\kappa}}$. Thus $\kappa = \widetilde{\kappa}$ and $w^{(a)} = \tilde{w}^{(b)} = \tilde{w}^{(a)}$ for all $a = 1, \ldots, \kappa$. By Theorem 1.4.26, there exists $\Lambda = \{\lambda^{(1)}, \ldots, \lambda^{(\kappa)}\} \subseteq \mathbb{N}_0^d$ such that $\operatorname{card} \Lambda = \kappa$ and $V(w^{(1)}, \ldots, w^{(\kappa)}; \Lambda)$ is invertible. Remark 3.1.27 implies then that $V^{p\times p}(w^{(1)}, \ldots, w^{(\kappa)}; \Lambda)$ is invertible. The positive semidefinite matrices $Q_1, \ldots, Q_\kappa \in \mathbb{C}^{p\times p}$ are computed by the Vandermonde equation

$$\operatorname{col}(Q_a)_{a=1}^{\kappa} = V^{p\times p}(w^{(1)}, \ldots, w^{(\kappa)}; \Lambda)^{-1} \operatorname{col}(S_\lambda)_{\lambda\in\Lambda},$$

where $Q_1, \ldots, Q_\kappa \succeq 0$. Moreover, the positive semidefinite matrices $\widetilde{Q}_1, \ldots, \widetilde{Q}_\kappa \in \mathbb{C}^{p\times p}$ are computed by the Vandermonde equation

$$\operatorname{col}(\widetilde{Q}_a)_{a=1}^{\kappa} = V^{p\times p}(w^{(1)}, \ldots, w^{(\kappa)}; \Lambda)^{-1} \operatorname{col}(S_\lambda)_{\lambda\in\Lambda},$$

where $\widetilde{Q}_1, \ldots, \widetilde{Q}_\kappa \succeq 0$. Hence $\operatorname{col}(Q_a)_{a=1}^{\kappa} = \operatorname{col}(\widetilde{Q}_a)_{a=1}^{\kappa}$ and $(Q_a)_{a=1}^{\kappa} = (\widetilde{Q}_a)_{a=1}^{\kappa}$ which asserts that the positive semidefinite matrices $Q_1, \ldots, Q_\kappa$ are uniquely determined for all $a = 1, \ldots, \kappa$. Consequently, the representing measure T is unique and the proof is complete. □

In analogy to Theorem 3.3.15, we formulate the next corollary for a given truncated $\mathcal{H}_p$-valued multisequence $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$.

Corollary 3.3.16. *Let $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$ be a given truncated $\mathcal{H}_p$-valued multisequence. Suppose there exist moments $(S_\gamma)_{\gamma\in\Gamma_{2n+2,d}\setminus\Gamma_{2n,d}}$ such that $M(n+1)\succeq 0$ and*

$$\operatorname{rank} M(n) = \operatorname{rank} M(n+1).$$

Then $(S_\gamma)_{\gamma\in\Gamma_{2n+2,d}}$ has a unique representing measure T. In this case,

$$\operatorname{supp} T = \mathcal{V}(M(n+k)) \quad \textit{for} \;\; k = 1, 2, \ldots,$$

where $\mathcal{V}(M(n+k))$ denotes the variety of $M(n+k)$ for $P(x)\in\mathbb{C}^{p\times p}[x_1,\ldots,x_d]$ such that $M(n+k)\operatorname{col}(P_\lambda)_{\lambda\in\Gamma_{n+k,d}} = \operatorname{col}(0_{p\times p})_{\lambda\in\Gamma_{n+k,d}}$ for all $k=1,2,\ldots,$ and moreover,

$$\operatorname{card}\mathcal{V}(M(n+k)) = r.$$

Proof. By Lemma 3.4.2, there exist moments $(S_\gamma)_{\gamma\in\mathbb{N}_0^d\setminus\Gamma_{2n+2,d}}$ which give rise to a unique sequence of extensions

$$M(n+k)\succeq 0 \quad \text{for} \;\; k = 2, 3, \ldots$$

and thus to $M(\infty)\succeq 0$. Hence, by Proposition 3.2.27, $(S_\gamma)_{\gamma\in\mathbb{N}_0^d}$ has a representing measure T and its uniqueness follows from Theorem 3.3.15. So if S gives rise to $M(n+1)\succeq 0$ and $r := \operatorname{rank} M(n+1) = \operatorname{rank} M(n) < \infty$, then $(S_\gamma)_{\gamma\in\Gamma_{2n+2,d}}$ has a unique representing measure T. Moreover, Lemma 3.3.6 applied for $P(x)\in\mathbb{C}^{p\times p}_{n+1}[x_1,\ldots,x_d]$ with

$$\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} S_{\gamma+\lambda}P_\lambda\right)_{\gamma\in\Gamma_{n+1,d}} = \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n+1,d}}$$

yields

$$\operatorname{supp} T \subseteq \mathcal{V}(M(n+1)). \tag{3.24}$$

Notice that since $(S_\gamma)_{\gamma\in\Gamma_{2n+2,d}}$ has a representing measure T, for $P(x)\in\mathbb{C}^{p\times p}_{n+1}[x_1,\ldots,x_d]$, Lemma 3.3.7 asserts

$$\mathcal{V}(M(n+1)) \subseteq \operatorname{supp} T, \tag{3.25}$$

By inclusions (3.24) and (3.25),

$$\operatorname{supp} T = \mathcal{V}(M(n+1)).$$

We need to show $\operatorname{supp} T = \mathcal{V}(M(n+k))$ for all $k=1,2,\ldots$ We apply Lemma 3.3.12 for $P(x)\in\mathbb{C}^{p\times p}[x_1,\ldots,x_d]$ such that $M(n+k)\operatorname{col}(P_\lambda)_{\lambda\in\Gamma_{n+k,d}} = \operatorname{col}(0_{p\times p})_{\lambda\in\Gamma_{n+k,d}}$. Then

$$\operatorname{supp} T \subseteq \mathcal{V}(M(n+k)) \quad \text{for} \;\; k = 1, 2, \ldots \tag{3.26}$$

Next, since $(S_\gamma)_{\gamma\in\mathbb{N}_0^d}$ has a representing measure T, $(S_\gamma)_{\gamma\in\Gamma_{2n+2k,d}}$ has a representing measure T for all $k=1,2,\dots$, and thus, Lemma 3.3.11 applied for $P(x)\in\mathbb{C}^{p\times p}[x_1,\dots,x_d]$ implies

$$\mathcal{V}(M(n+k))\subseteq \operatorname{supp} T \quad \text{for } \ k=1,2,\dots \tag{3.27}$$

We shall derive $\operatorname{supp} T=\mathcal{V}(M(n+k))$ for all $k=1,2,\dots$, by inclusions (3.26) and (3.27). Furthermore, since T is a representing measure for $(S_\gamma)_{\gamma\in\mathbb{N}_0^d}$, then T is a representing measure for $(S_\gamma)_{\gamma\in\Gamma_{2n+2k,d}}$ and Lemma 3.3.14 implies $\operatorname{card}\mathcal{V}(M(n+k))=r$ for all $k=1,2,\dots$ Hence $\sum\limits_{a=1}^{\kappa}\operatorname{rank} Q_a=r<\infty$ and the measure T is of the form $T=\sum\limits_{a=1}^{\kappa}Q_a\delta_{w^{(a)}}$ with

$$\sum_{a=1}^{\kappa}\operatorname{rank} Q_a=r \ \text{ and } \ Q_1,\dots,Q_\kappa\succeq 0. \qquad \square$$

We next present an algebraic result involving an ideal (see Definition 3.1.14) associated to an infinite positive moment matrix.

Proposition 3.3.17. *If $M(\infty)\succeq 0$ and $\mathcal{I}\subseteq\mathbb{C}^{p\times p}[x_1,\dots,x_d]$ is the associated right ideal (see Definition 3.1.14), then $\mathcal{I}$ is real radical.*

Proof. We need to show that $\sum_{a=1}^{\kappa}P^{(a)}\{P^{(a)}\}^*\in\mathcal{I}\Rightarrow P^{(a)}\in\mathcal{I}$ for all $a=1,\dots,\kappa$. Let

$$\widehat{R}^{(a)}:=\operatorname{col}\left(I_{p,\lambda}^{(a)}\right)_{\lambda=(0,\dots,0)}\oplus\operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n+1,d}\setminus\gamma=(0,\dots,0)}$$

and

$$\widehat{P}^{(a)}:=\operatorname{col}\left(P_\lambda^{(a)}\right)_{\lambda\in\Gamma_{n,d}}\oplus\operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n+1,d}\setminus\Gamma_{n,d}} \quad \text{for } \ a=1,\dots,\kappa.$$

Since $\sum\limits_{a=1}^{\kappa}\{\widehat{R}^{(a)}\}^*M(n+1)\widehat{P}^{(a)}=0_{p\times p}$, we may write

$$\sum_{a=1}^{\kappa}\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}}S_{\gamma+\lambda}P_\lambda^{(a)}\right)_{\gamma\in\Gamma_{n+1,d}}\operatorname{col}\left(\{P_\lambda^{(a)}\}^*\right)_{\lambda\in\Gamma_{n,d}}=0_{p\times p}$$

and so

$$\sum_{a=1}^{\kappa}\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}}S_{\gamma+\lambda}P_\lambda^{(a)}\{P_\lambda^{(a)}\}^*\right)_{\gamma\in\Gamma_{n+1,d}}=0_{p\times p}.$$

We then have

$$\sum_{a=1}^{\kappa}\operatorname{tr}\left(\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}}S_{\gamma+\lambda}P_\lambda^{(a)}\{P_\lambda^{(a)}\}^*\right)\right)_{\gamma\in\Gamma_{n,d}}=\operatorname{tr}(0_{p\times p})_{\gamma\in\Gamma_{n,d}},$$

which by properties of the trace is equivalent to

$$\sum_{a=1}^{\kappa}\operatorname{tr}\left(\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}}\{P_\lambda^{(a)}\}^*S_{\gamma+\lambda}P_\lambda^{(a)}\right)\right)_{\gamma\in\Gamma_{n,d}}=\operatorname{tr}(0_{p\times p})_{\gamma\in\Gamma_{n,d}},$$

that is,

$$\sum_{a=1}^{\kappa} \operatorname{tr}\left(\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} \{P_\lambda^{(a)}\}^* S_{\gamma+\lambda} P_\lambda^{(a)}\right)\right)_{\gamma\in\Gamma_{n,d}} = 0$$

and thus

$$\sum_{a=1}^{\kappa} \operatorname{col}\left(\{P_\lambda^{(a)}\}^*\right)_{\lambda\in\Gamma_{n,d}}^* \operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} S_{\gamma+\lambda} P_\lambda^{(a)}\right)_{\gamma\in\Gamma_{n,d}} = 0_{p\times p}.$$

Hence

$$\sum_{a=1}^{\kappa} \{\widehat{P}^{(a)}\}^* M(n) \widehat{P}^{(a)} = 0_{p\times p},$$

which implies $P^{(a)} \in \mathcal{I}$ for all $a = 1, \dots, \kappa$ as desired. $\square$

3.4 Positive extensions of truncated moment matrices

In this section, we investigate positive extensions of truncated moment matrices based on a truncated $\mathcal{H}_p$-valued multisequence. Both results provided in the following are important for obtaining the flat extension theorem for matricial moments stated and proved in Chapter 4.

The next lemma will be referred to as the extension lemma.

Lemma 3.4.1. *Let $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$ be a given truncated $\mathcal{H}_p$-valued multisequence and let $M(n)$ be the corresponding moment matrix. If $M(n) \succeq 0$ has an extension $M(n+1)$ such that $M(n+1) \succeq 0$ and $\operatorname{rank} M(n+1) = \operatorname{rank} M(n)$, then there exist $(S_\gamma)_{\gamma\in\mathbb{N}_0^d\setminus\Gamma_{2n,d}}$ such that*

$$M(n+k) \succeq 0$$

and

$$\operatorname{rank} M(n+k) = \operatorname{rank} M(n+k-1) \quad \textit{for } k = 2, 3, \dots$$

Proof. See Lemma A.0.1 for a proof. $\square$

Lemma 3.4.2. *Let $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$ be a given truncated $\mathcal{H}_p$-valued multisequence and let $M(n) \succeq 0$ be the corresponding moment matrix. Suppose that $M(n)$ has a positive extension $M(n+1)$ with*

$$\operatorname{rank} M(n+1) = \operatorname{rank} M(n).$$

Then there exists a unique sequence of extensions

$$M(n+k) \succeq 0$$

with

$$\operatorname{rank} M(n+k) = \operatorname{rank} M(n+k-1) \quad \textit{for } k = 2, 3, \dots$$

Proof. See Lemma A.0.2 for a proof. $\square$

Chapter 4

The flat extension theorem for matricial moments

In this chapter we will formulate and prove a flat extension theorem for matricial moments. We shall see that a given truncated $\mathcal{H}_p$-valued multisequence $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$ has a minimal representing measure (see Definition 1.4.38) if and only if the corresponding moment matrix $M(n)$ has a flat extension $M(n+1)$. In this case, one can find a minimal representing measure such that the support of the minimal representing measure is the variety of the moment matrix $M(n+1)$.

The definition that follows is an adaptation of the notion of flatness introduced by Curto and Fialkow in [16] to our matricial setting.

Definition 4.0.1. Let $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$ be a given truncated $\mathcal{H}_p$-valued multisequence and $M(n) \succeq 0$ be the corresponding moment matrix. Then $M(n)$ has a *flat extension* if there exist $(S_\gamma)_{\gamma\in\Gamma_{2n+2,d}\setminus\Gamma_{2n,d}}$, where $S_\gamma \in \mathcal{H}_p$ for $\gamma \in \Gamma_{2n+2,d} \setminus \Gamma_{2n,d}$ such that $M(n+1) \succeq 0$ and

$$\operatorname{rank} M(n) = \operatorname{rank} M(n+1).$$

Theorem 4.0.2 (flat extension theorem for matricial moments)**.** *Let $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$ be a given truncated $\mathcal{H}_p$-valued multisequence, $M(n) \succeq 0$ be the corresponding moment matrix and $r := \operatorname{rank} M(n)$. S has a representing measure*

$$T = \sum_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$$

with

$$\sum_{a=1}^{\kappa} \operatorname{rank} Q_a = r$$

if and only if the matrix $M(n)$ admits an extension $M(n+1) \succeq 0$ such that

$$\operatorname{rank} M(n) = \operatorname{rank} M(n+1).$$

Moreover,

$$\operatorname{supp} T = \mathcal{V}(M(n+1)),$$

and there exists $\Lambda = \{\lambda^{(1)}, \ldots, \lambda^{(\kappa)}\} \subseteq \mathbb{N}_0^d$ *with* $\operatorname{card} \Lambda = \kappa$ *such that the multivariable Vandermonde matrix* $V^{p\times p}(w^{(1)}, \ldots, w^{(\kappa)}; \Lambda) \in \mathbb{C}^{\kappa p\times \kappa p}$ *is invertible. Then the positive semidefinite matrices* $Q_1, \ldots, Q_\kappa \in \mathbb{C}^{p\times p}$ *are given by the Vandermonde equation*

$$\operatorname{col}(Q_a)_{a=1}^{\kappa} = V^{p\times p}(w^{(1)}, \ldots, w^{(\kappa)}; \Lambda)^{-1} \operatorname{col}(S_\lambda)_{\lambda\in\Lambda}.$$

Proof. Suppose the matrix $M(n) \succeq 0$ admits an extension $M(n+1) \succeq 0$ such that

$$\operatorname{rank} M(n+1) = \operatorname{rank} M(n) = r.$$

By Corollary 3.3.16, $(S_\gamma)_{\gamma\in\Gamma_{2n+2,d}}$ has a unique representing measure T such that

$$\operatorname{supp} T = \mathcal{V}(M(n+1)) \quad \text{and} \quad \operatorname{card} \mathcal{V}(M(n+1)) = r,$$

that is,

$$\sum_{a=1}^{\kappa} \operatorname{rank} Q_a = r.$$

Consequently, T is of the form

$$T = \sum_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$$

with $\sum_{a=1}^{\kappa} \operatorname{rank} Q_a = r$.

Conversely, suppose that S has a representing measure $T = \sum_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$ with

$$\sum_{a=1}^{\kappa} \operatorname{rank} Q_a = r.$$

Consider the matrix $M(n+1)$ built from the moments $(S_\gamma)_{\gamma\in\Gamma_{2n+2,d}\setminus\Gamma_{2n,d}}$. T is a representing measure for $M(n+1)$ and so, by Lemma 3.3.8 we obtain

$$\operatorname{rank} M(n+1) \leq \sum_{a=1}^{\kappa} \operatorname{rank} Q_a = \operatorname{rank} M(n).$$

The extension lemma (see Lemma 3.4.1) asserts that $M(n+1)$ is a flat extension of $M(n)$. $\square$

Chapter 5

The bivariate quadratic matrix-valued moment problem

In this chapter we will study the bivariate quadratic matrix-valued moment problem. Given a truncated $\mathcal{H}_p$-valued bisequence $S := (S_\gamma)_{\gamma\in\Gamma_{2,2}} = (S_{00}, S_{10}, S_{01}, S_{20}, S_{11}, S_{02})$, we wish to determine when S has a minimal representing measure. For $p = 1$, Curto and Fialkow [16] showed that every $S = (S_\gamma)_{\gamma\in\Gamma_{2,2}}$ with $S_{00} > 0$ and $M(1) \succeq 0$ has a minimal representing measure.

Notice that a direct analogue of Curto and Fialkow's result on the bivariate quadratic moment problem does not hold when $p \geq 2$ (see Example 1.4.40). However, we shall see that if $M(1)$ is positive semidefinite and certain block column relations hold, then $S = (S_\gamma)_{\gamma\in\Gamma_{2,2}}$, $S_{00} \succ 0$, has a minimal representing measure.

In the following we shall make use of the assumption (A1) of Remark 1.4.39, that is, $S_{0_d} = I_p$ for $d = 2$.

The next theorem illustrates necessary and sufficient conditions for a given quadratic $\mathcal{H}_p$-valued bisequence to have a minimal representing measure. We observe that the positivity and flatness conditions are essential to obtain a minimal solution to the bivariate quadratic matrix-valued moment problem.

Theorem 5.0.1. *Let $S := (S_\gamma)_{\gamma\in\Gamma_{2,2}}$ be a given truncated $\mathcal{H}_p$-valued bisequence and*

$$M(1) = \begin{array}{c c} & \begin{array}{ccc} 1 & X & Y \end{array} \\ \begin{array}{c} 1 \\ X \\ Y \end{array} & \begin{pmatrix} I_p & S_{10} & S_{01} \\ S_{10} & S_{20} & S_{11} \\ S_{01} & S_{11} & S_{02} \end{pmatrix} \end{array}$$

be the corresponding moment matrix. S has a minimal representing measure if and only if the following conditions hold:

(i) $M(1) \succeq 0$.

(ii) There exist $S_{30}, S_{21}, S_{12}, S_{03} \in \mathcal{H}_p$ such that

$$\operatorname{Ran} \begin{pmatrix} S_{20} & S_{11} & S_{02} \\ S_{30} & S_{21} & S_{12} \\ S_{21} & S_{12} & S_{03} \end{pmatrix} \subseteq \operatorname{Ran} M(1)$$

(hence, there exists $W = (W_{ab})_{a,b=1}^{3} \in \mathbb{C}^{3p\times 3p}$ *such that* $M(1)W = B$*, where*

$$B = \begin{pmatrix} S_{20} & S_{11} & S_{02} \\ S_{30} & S_{21} & S_{12} \\ S_{21} & S_{12} & S_{03} \end{pmatrix})$$

and moreover, the following matrix equations hold:

$$W_{11}^{*}S_{11} + W_{21}^{*}S_{21} + W_{31}^{*}S_{12} = S_{11}W_{11} + S_{21}W_{21} + S_{12}W_{31}, \tag{5.1}$$

$$W_{13}^{*}S_{20} + W_{23}^{*}S_{30} + W_{33}^{*}S_{21} = W_{12}^{*}S_{11} + W_{22}^{*}S_{21} + W_{32}^{*}S_{12} \tag{5.2}$$

and

$$W_{12}^{*}S_{02} + W_{22}^{*}S_{12} + W_{32}^{*}S_{03} = S_{02}W_{12} + S_{12}W_{22} + S_{03}W_{32}. \tag{5.3}$$

Proof. Since

$$\operatorname{Ran}\begin{pmatrix} S_{20} & S_{11} & S_{02} \\ S_{30} & S_{21} & S_{12} \\ S_{21} & S_{12} & S_{03} \end{pmatrix} \subseteq \operatorname{Ran} M(1),$$

there exists $W = (W_{ab})_{a,b=1}^{3} \in \mathbb{C}^{3p\times 3p}$ such that

$$B := \begin{pmatrix} S_{20} & S_{11} & S_{02} \\ S_{30} & S_{21} & S_{12} \\ S_{21} & S_{12} & S_{03} \end{pmatrix} = M(1)W.$$

Let $W := \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{pmatrix}$. Then

$$S_{20} = W_{11} + S_{10}W_{21} + S_{01}W_{31}, \tag{5.4}$$

$$S_{30} = S_{10}W_{11} + S_{20}W_{21} + S_{11}W_{31}, \tag{5.5}$$

$$\begin{aligned} S_{21} &= S_{01}W_{11} + S_{11}W_{21} + S_{02}W_{31} \\ &= S_{10}W_{12} + S_{20}W_{22} + S_{11}W_{32}, \end{aligned} \tag{5.6}$$

$$S_{11} = W_{12} + S_{10}W_{22} + S_{01}W_{32}, \tag{5.7}$$

$$\begin{aligned} S_{12} &= S_{01}W_{12} + S_{11}W_{22} + S_{02}W_{32} \\ &= S_{10}W_{13} + S_{20}W_{23} + S_{11}W_{33}, \end{aligned} \tag{5.8}$$

$$S_{02} = W_{13} + S_{10}W_{23} + S_{01}W_{33} \tag{5.9}$$

and

$$S_{03} = S_{01}W_{13} + S_{11}W_{23} + S_{02}W_{33}. \tag{5.10}$$

Let $C := W^*M(1)W = W^*B$ and write $C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$. By formulas (5.6), (5.7) and (5.8), we have

$$\begin{aligned} C_{12} &= W_{11}^*S_{11} + W_{21}^*S_{21} + W_{31}^*S_{12} \\ &= W_{11}^*(W_{12} + S_{10}W_{22} + S_{01}W_{32}) + W_{21}^*(S_{10}W_{12} + S_{20}W_{22} + S_{11}W_{32}) \\ &+ W_{31}^*(S_{01}W_{12} + S_{11}W_{22} + S_{02}W_{32}). \end{aligned}$$

Since the matrix equation (5.1) holds, $C_{12} = C_{12}^* = C_{21}$. Next, by formulas (5.6), (5.7) and (5.8),

$$\begin{aligned} C_{22} &= W_{12}^*S_{11} + W_{22}^*S_{21} + W_{32}^*S_{12} \\ &= W_{12}^*(W_{12} + S_{10}W_{22} + S_{01}W_{32}) + W_{22}^*(S_{10}W_{12} + S_{20}W_{22} + S_{11}W_{32}) \\ &+ W_{32}^*(S_{01}W_{12} + S_{11}W_{22} + S_{02}W_{32}) \end{aligned}$$

and by formulas (5.4), (5.5) and (5.6),

$$\begin{aligned} C_{31} &= W_{13}^*S_{20} + W_{23}^*S_{30} + W_{33}^*S_{21} \\ &= W_{13}^*(W_{11} + S_{10}W_{21} + S_{01}W_{31}) + W_{23}^*(S_{10}W_{11} + S_{20}W_{21} + S_{11}W_{31}) \\ &+ W_{33}^*(S_{01}W_{11} + S_{11}W_{21} + S_{02}W_{31}). \end{aligned}$$

Since the matrix equation (5.2) holds, $C_{22} = C_{31}$. Moreover, by formulas (5.8), (5.9) and (5.10),

$$\begin{aligned} C_{23} &= W_{12}^*S_{02} + W_{22}^*S_{12} + W_{32}^*S_{03} \\ &= W_{12}^*(W_{13} + S_{10}W_{23} + S_{01}W_{33}) + W_{22}^*(S_{10}W_{13} + S_{20}W_{23} + S_{11}W_{33}) \\ &+ W_{32}^*(S_{01}W_{13} + S_{11}W_{23} + S_{02}W_{33}). \end{aligned}$$

Since the matrix equation (5.3) holds, $C_{23} = C_{23}^* = C_{32}$. Thus, by Lemma 1.4.24,

$$M(2) := \begin{pmatrix} M(1) & B \\ B^* & C \end{pmatrix} \succeq 0$$

is a flat extension of $M(1)$. By the flat extension theorem for matricial moments (see Theorem 4.0.2), there exists a minimal representing measure T for S.

Conversely, if S has a minimal representing measure, then by the flat extension theorem for matricial moments (see Theorem 4.0.2), there exists a flat extension $M(2) := \begin{pmatrix} M(1) & B \\ B^* & C \end{pmatrix} \succeq 0$ of $M(1)$ such that $\operatorname{rank} M(1) = \operatorname{rank} M(2)$. By Lemma 1.4.24, $C = W^*M(1)W$ for some $W \in \mathbb{C}^{3p \times 3p}$ such that

$$B := \begin{pmatrix} S_{20} & S_{11} & S_{02} \\ S_{30} & S_{21} & S_{12} \\ S_{21} & S_{12} & S_{03} \end{pmatrix} = M(1)W$$

and consequently, $\operatorname{Ran} B \subseteq \operatorname{Ran} M(1)$. Hence there exists $W := \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{pmatrix}$ satisfying $B = M(1)W$. Since $C = \begin{pmatrix} S_{40} & S_{31} & S_{22} \\ S_{31} & S_{22} & S_{13} \\ S_{22} & S_{13} & S_{04} \end{pmatrix} = W^* M(1) W$, we have

$$\begin{aligned} S_{31} &= W_{11}^* S_{11} + W_{21}^* S_{21} + W_{31}^* S_{12}, \\ S_{22} &= W_{13}^* S_{20} + W_{23}^* S_{30} + W_{33}^* S_{21} \\ &= W_{12}^* S_{11} + W_{22}^* S_{21} + W_{32}^* S_{12} \end{aligned}$$

and

$$S_{13} = W_{12}^* S_{02} + W_{22}^* S_{12} + W_{32}^* S_{03}.$$

We derive the matrix equations (5.1), (5.2) and (5.3), respectively. □

The next corollary is a special case of Theorem 5.0.1 when $M(1) \succ 0$.

Corollary 5.0.2. *Let $S := (S_\gamma)_{\gamma \in \Gamma_{2,2}} \in \mathcal{H}_p$ be a given truncated $\mathcal{H}_p$-valued bisequence. Suppose* $M(1) = \begin{array}{c} \\ 1 \\ X \\ Y \end{array} \begin{array}{c} \begin{array}{ccc} 1 & X & Y \end{array} \\ \begin{pmatrix} I_p & S_{10} & S_{01} \\ S_{10} & S_{20} & S_{11} \\ S_{01} & S_{11} & S_{02} \end{pmatrix} \end{array} \succ 0$ *and write $M(1)^{-1} = (P_{ab})_{a,b=1}^3$. S has a minimal representing measure if and only if there exist $S_{30}, S_{21}, S_{12}, S_{03} \in \mathcal{H}_p$ such that the following matrix equations have a solution:*

$$(S_{20}P_{13} + S_{30}P_{23} + S_{21}P_{33})S_{12} - S_{12}(S_{20}P_{13} + S_{30}P_{23} + S_{21}P_{33})^* = R_1, \tag{5.11}$$

$$(S_{20}P_{13} + S_{30}P_{23} + S_{21}P_{33})S_{03} - S_{03}0_{p\times p} = R_2 \tag{5.12}$$

and

$$(S_{11}P_{13} + S_{21}P_{23} + S_{12}P_{33})S_{03} - S_{03}(S_{11}P_{13} + S_{21}P_{23} + S_{12}P_{33})^* = R_3, \tag{5.13}$$

where

$$\begin{aligned} R_1 = {} & S_{11}P_{11}S_{20} + S_{11}P_{12}S_{30} + S_{11}P_{13}S_{21} + S_{21}P_{12}^*S_{20} \\ & + S_{21}P_{22}S_{30} + S_{21}P_{23}S_{21} - S_{20}P_{11}S_{11} - S_{30}P_{12}^*S_{11} \\ & - S_{21}P_{13}^*S_{11} - S_{20}P_{12}S_{21} - S_{30}P_{22}S_{21} - S_{21}P_{23}^*S_{21}, \end{aligned}$$

$$\begin{aligned} R_2 = {} & S_{11}P_{11}S_{11} + S_{21}P_{12}^*S_{11} + S_{12}P_{13}^*S_{11} + S_{11}P_{12}S_{21} + S_{21}P_{22}S_{21} \\ & + S_{12}P_{23}^*S_{21} + S_{11}P_{13}S_{12} + S_{21}P_{23}S_{12} + S_{12}P_{33}S_{12} - S_{20}P_{11}S_{02} \\ & - S_{30}P_{12}^*S_{02} - S_{21}P_{13}^*S_{02} - S_{20}P_{12}S_{12} - S_{30}P_{22}S_{12} - S_{21}P_{23}^*S_{12} \end{aligned}$$

and

$$\begin{aligned}R_3 &= S_{02}P_{11}S_{11} + S_{02}P_{12}S_{21} + S_{02}P_{13}S_{12} + S_{12}P_{12}^*S_{11}\\ &\quad + S_{12}P_{22}S_{21} + S_{12}P_{23}S_{12} - S_{11}P_{11}S_{02} - S_{21}P_{12}^*S_{02}\\ &\quad - S_{12}P_{13}^*S_{02} - S_{11}P_{12}S_{12} - S_{21}P_{22}S_{12} - S_{12}P_{23}^*S_{12}.\end{aligned}$$

Proof. Write

$$M(1)^{-1} = \begin{pmatrix} P_{11} & P_{12} & P_{13}\\ P_{12}^* & P_{22} & P_{23}\\ P_{13}^* & P_{23}^* & P_{33}\end{pmatrix}$$

and let $W = M(1)^{-1}B$, where $B := \begin{pmatrix} S_{20} & S_{11} & S_{02}\\ S_{30} & S_{21} & S_{12}\\ S_{21} & S_{12} & S_{03}\end{pmatrix}$. Then we get $M(1)W = B$. Write $W := \begin{pmatrix} W_{11} & W_{12} & W_{13}\\ W_{21} & W_{22} & W_{23}\\ W_{31} & W_{32} & W_{33}\end{pmatrix}$ and notice that

$$\begin{aligned}W_{11} &= P_{11}S_{20} + P_{12}S_{30} + P_{13}S_{21},\\ W_{21} &= P_{12}^*S_{20} + P_{22}S_{30} + P_{23}S_{21},\\ W_{31} &= P_{13}^*S_{20} + P_{23}^*S_{30} + P_{33}S_{21},\\ W_{12} &= P_{11}S_{11} + P_{12}S_{21} + P_{13}S_{12},\\ W_{22} &= P_{12}^*S_{11} + P_{22}S_{21} + P_{23}S_{12},\\ W_{32} &= P_{13}^*S_{11} + P_{23}^*S_{21} + P_{33}S_{12},\\ W_{13} &= P_{11}S_{02} + P_{12}S_{12} + P_{13}S_{03},\\ W_{23} &= P_{12}^*S_{02} + P_{22}S_{12} + P_{23}S_{03}\end{aligned}$$

and

$$W_{33} = P_{13}^*S_{02} + P_{23}^*S_{12} + P_{33}S_{03}.$$

We fix the moments $S_{30}, S_{21} \in \mathcal{H}_p$. The matrix equation (5.1) in Theorem 5.0.1 then becomes the Lyapunov equation, namely the matrix equation (5.11)

$$A_1S_{12} - S_{12}A_1^* = R_1,$$

where $A_1 = S_{20}P_{13} + S_{30}P_{23} + S_{21}P_{33}$ and

$$\begin{aligned}R_1 &= S_{11}P_{11}S_{20} + S_{11}P_{12}S_{30} + S_{11}P_{13}S_{21} + S_{21}P_{12}^*S_{20}\\ &\quad + S_{21}P_{22}S_{30} + S_{21}P_{23}S_{21} - S_{20}P_{11}S_{11} - S_{30}P_{12}^*S_{11}\\ &\quad - S_{21}P_{13}^*S_{11} - S_{20}P_{12}S_{21} - S_{30}P_{22}S_{21} - S_{21}P_{23}^*S_{21}.\end{aligned}$$

We next fix the moments S_{30}, S_{21} and the matrix equation (5.2) in Theorem 5.0.1 yields the following Sylvester equation, namely the matrix equation (5.12)

$$A_2 S_{03} - S_{03} 0_{p\times p} = R_2,$$

where $A_2 = S_{20}P_{13} + S_{30}P_{23} + S_{21}P_{33} = A_1$ and

$$\begin{aligned} R_2 = {} & S_{11}P_{11}S_{11} + S_{21}P_{12}^{*}S_{11} + S_{12}P_{13}^{*}S_{11} + S_{11}P_{12}S_{21} + S_{21}P_{22}S_{21} \\ & + S_{12}P_{23}^{*}S_{21} + S_{11}P_{13}S_{12} + S_{21}P_{23}S_{12} + S_{12}P_{33}S_{12} - S_{20}P_{11}S_{02} \\ & - S_{30}P_{12}^{*}S_{02} - S_{21}P_{13}^{*}S_{02} - S_{20}P_{12}S_{12} - S_{30}P_{22}S_{12} - S_{21}P_{23}^{*}S_{12}. \end{aligned}$$

Next, we fix the moments S_{12}, S_{21}. The matrix equation (5.3) in Theorem 5.0.1 yields the following Lyapunov equation, namely the matrix equation (5.13)

$$A_3 S_{03} - S_{03} A_2^{*} = R_3,$$

where $A_3 = S_{11}P_{13} + S_{21}P_{23} + S_{12}P_{33}$ and

$$\begin{aligned} R_3 = {} & S_{02}P_{11}S_{11} + S_{02}P_{12}S_{21} + S_{02}P_{13}S_{12} + S_{12}P_{12}^{*}S_{11} \\ & + S_{12}P_{22}S_{21} + S_{12}P_{23}S_{12} - S_{11}P_{11}S_{02} - S_{21}P_{12}^{*}S_{02} \\ & - S_{12}P_{13}^{*}S_{02} - S_{11}P_{12}S_{12} - S_{21}P_{22}S_{12} - S_{12}P_{23}^{*}S_{12}. \end{aligned}$$

Since there exist $S_{30}, S_{21}, S_{12}, S_{03} \in \mathcal{H}_p$ such that the above matrix equations have a solution, Theorem 5.0.1 asserts that there exists a minimal representing measure T for S.

Conversely, if S has a minimal representing measure, then by Theorem 5.0.1, there exists $W := \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{pmatrix}$ such that $M(1)W = \begin{pmatrix} S_{20} & S_{11} & S_{02} \\ S_{30} & S_{21} & S_{12} \\ S_{21} & S_{12} & S_{03} \end{pmatrix} := B$ and the matrix equations (5.1), (5.2) and (5.3) hold. Since $M(1) \succ 0$, we have $W = M(1)^{-1}B$. The matrix equations (5.11), (5.12) and (5.13) follow from the matrix equations (5.1), (5.2) and (5.3), respectively. $\square$

In the following theorem we obtain a minimal representing measure for a given truncated $\mathcal{H}_p$-valued bisequence when the associated moment matrix has certain block column relations. Moreover, we extract information on the support of the representing measure observing its connection with the aforementioned block column relations. Theorem 5.0.3 can be thought of as an analogue of [16, Proposition 6.2] for $p > 1$.

Theorem 5.0.3. *Let $S := (S_\gamma)_{\gamma\in\Gamma_{2,2}}$ be a given truncated $\mathcal{H}_p$-valued bisequence. Suppose*

$$M(1) = \begin{array}{c c} & \begin{array}{ccc} 1 & X & Y \end{array} \\ \begin{array}{c} 1 \\ X \\ Y \end{array} & \begin{pmatrix} I_p & S_{10} & S_{01} \\ S_{10} & S_{20} & S_{11} \\ S_{01} & S_{11} & S_{02} \end{pmatrix} \end{array} \succeq 0 \text{ and } X = 1\cdot\Phi \text{ and } Y = 1\cdot\Psi \text{ for } \Phi, \Psi \in \mathbb{C}^{p\times p}.$$

Then $\Phi = S_{10}$ *and* $\Psi = S_{01}$ *and there exists a minimal (that is,* $\sum_{a=1}^{\kappa} \operatorname{rank} Q_a = p$*) representing measure* T *for* S *of the form*

$$T = \sum_{a=1}^{\kappa} Q_a \delta_{w^{(a)}},$$

where $1 \leq \kappa \leq p$ *and*

$$\operatorname{supp} T = \{w^{(1)}, \dots, w^{(\kappa)}\} \subseteq \sigma(\Phi) \times \sigma(\Psi).$$

Proof. Since $X = 1 \cdot \Phi$ and $Y = 1 \cdot \Psi$ for $\Phi, \Psi \in \mathbb{C}^{p \times p}$, we have

$$S_{10} = \Phi = \Phi^*, \tag{5.14}$$

$$S_{20} = S_{10}\Phi = \Phi^2, \tag{5.15}$$

$$S_{01} = \Psi = \Psi^*, \tag{5.16}$$

$$S_{02} = S_{01}\Psi = \Psi^2$$

and

$$S_{11} = S_{10}\Psi = \Phi\Psi. \tag{5.17}$$

Let

$$S_{30} := S_{10}\Phi = S_{20}, \tag{5.18}$$

$$S_{21} := S_{01}\Phi = \Psi\Phi,\ S_{12} := S_{10}\Psi = \Phi\Psi = S_{11}$$

and

$$S_{03} := S_{01}\Psi = S_{02}. \tag{5.19}$$

Then $S_{30}, S_{12} \in \mathcal{H}_p$ and $S_{03} \in \mathcal{H}_p$. Moreover, $S_{21}^* = \Phi^*\Psi^* = S_{12}^* = S_{12} = S_{11}$ and so $S_{21} \in \mathcal{H}_p$ and

$$S_{12} = S_{11} = S_{21}, \tag{5.20}$$

that is,

$$\Phi\Psi = \Psi\Phi. \tag{5.21}$$

If we let $W := \begin{pmatrix} 0 & 0 & 0 \\ \Phi & \Psi & 0 \\ 0 & 0 & \Psi \end{pmatrix}$, then $B := \begin{pmatrix} S_{20} & S_{11} & S_{02} \\ S_{30} & S_{21} & S_{12} \\ S_{21} & S_{12} & S_{03} \end{pmatrix} = M(1)W.$

Notice that $B = \begin{pmatrix} S_{20} & S_{11} & S_{02} \\ S_{20} & S_{11} & S_{11} \\ S_{11} & S_{11} & S_{02} \end{pmatrix}$, by formulas (5.18), (5.19) and (5.20). Let

$$C := W^*M(1)W = W^*B = \begin{pmatrix} \Phi^*S_{20} & \Phi^*S_{11} & \Phi^*S_{11} \\ \Psi^*S_{20} & \Psi^*S_{11} & \Psi^*S_{11} \\ \Psi^*S_{11} & \Psi^*S_{11} & \Psi^*S_{02} \end{pmatrix}$$

and write $C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$. In order for C to have the appropriate block Hankel structure we need to show

$$\Psi^* S_{20} = \Phi^* S_{11} \ \text{ and } \ \Psi^* S_{11} = \Phi^* S_{11}.$$

By formulas (5.14), (5.15) and (5.21),

$$\Psi^* S_{20} = \Psi^* \Phi^2 = \Psi \Phi^2 = \Psi \Phi \Phi = \Phi \Psi \Phi.$$

By formulas (5.16), (5.17) and (5.21), we have

$$\Phi^* S_{11} = \Phi^* \Phi \Psi = \Phi^2 \Psi = \Phi \Phi \Psi = \Phi \Psi \Phi = \Psi^* S_{20}$$

as desired. Furthermore, we have $C_{22} = C_{31} = \Psi^* S_{11}$. Thus $C_{31} = \Psi^* S_{11} \in \mathcal{H}_p$. However $C_{31} = C_{13}^*$ forces $C_{13} = C_{22} = C_{31}$. Hence

$$M(2) := \begin{pmatrix} M(1) & B \\ B^* & C \end{pmatrix} \succeq 0$$

is a flat extension of $M(1)$ by Lemma 1.4.24. By the flat extension theorem for matricial moments (see Theorem 4.0.2), there exists a minimal representing measure T for S of the form

$$T = \sum_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$$

such that $\operatorname{supp} T = \mathcal{V}(M(2))$, where

$$\sum_{a=1}^{\kappa} \operatorname{rank} Q_a = \operatorname{card} \mathcal{V}(M(2)) = \operatorname{rank} M(1) = p.$$

Since $X = 1 \cdot \Phi$ and $Y = 1 \cdot \Psi$, the matrix-valued polynomials

$$P_1(x, y) = xI_p - \Phi \quad \text{and} \quad P_2(x, y) = yI_p - \Psi$$

are such that $P_1(X, Y) = P_2(X, Y) = \operatorname{col}(0_{p\times p})_{\gamma \in \Gamma_{1,2}} \in C_{M(2)}$.
Lemma 3.3.6 implies that

$$\begin{aligned} \operatorname{supp} T = \mathcal{V}(M(2)) \ &\subseteq \mathcal{Z}(\det(P_1(x, y))) \bigcap \mathcal{Z}(\det(P_2(x, y))) \\ &= \{(x, y) \in \mathbb{R}^2 : \det(xI_p - \Phi) = 0\} \\ &\bigcap \{(x, y) \in \mathbb{R}^2 : \det(yI_p - \Psi) = 0\} \\ &= \sigma(\Phi) \times \sigma(\Psi). \end{aligned}$$

Thus

$$\operatorname{supp} T = \{w^{(1)}, \dots, w^{(\kappa)}\} \subseteq \mathcal{V}(M(2)) = \sigma(\Phi) \times \sigma(\Psi)$$

and

$$T = \sum_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$$

is a representing measure for S with $\sum\limits_{a=1}^{\kappa} \operatorname{rank} Q_a = p$. Since $1 \leq \operatorname{rank} Q_a \leq p$, we must have $1 \leq \kappa \leq p$. $\square$

The following example showcases Theorem 5.0.3 for an explicit truncated $\mathcal{H}_2$-valued bisequence.

Example 5.0.4. Let $S := (S_\gamma)_{\gamma \in \Gamma_{2,2}}$ be a truncated $\mathcal{H}_2$-valued bisequence given by

$$M(1) = \begin{array}{c} \\ 1 \\ X \\ Y \end{array} \!\!\! \begin{array}{c} \begin{array}{ccc} 1 & X & Y \end{array} \\ \begin{pmatrix} I_2 & S_{10} & S_{01} \\ S_{10} & S_{20} & S_{11} \\ S_{01} & S_{11} & S_{02} \end{pmatrix} \end{array} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \succeq 0$$

and suppose $X = 1 \cdot \Phi$ and $Y = 1 \cdot \Psi$ for $\Phi = S_{10}$ and $\Psi = S_{01}$. The matrix-valued polynomials $P_1(x,y) = xI_2 - \Phi$ and $P_2(x,y) = yI_2 - \Psi$ are such that

$$P_1(X,Y) = P_2(X,Y) = \operatorname{col}(0_{2\times 2})_{\gamma \in \Gamma_{1,2}} \in C_{M(1)}$$

and we have $\det(P_1(x,y)) = x(x-1)$ and $\det(P_2(x,y)) = y(y-1)$. By Theorem 5.0.3, $M(1)$ has a flat extension of the form

$$M(2) := \begin{pmatrix} M(1) & B \\ B^* & C \end{pmatrix} = \begin{array}{c} \\ 1 \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \end{array} \!\!\! \begin{array}{c} \begin{array}{cccccc} 1 & X & Y & X^2 & XY & Y^2 \end{array} \\ \begin{pmatrix} I_2 & S_{10} & S_{01} & S_{20} & S_{11} & S_{02} \\ S_{10} & S_{20} & S_{11} & S_{20} & S_{11} & S_{11} \\ S_{01} & S_{11} & S_{02} & S_{11} & S_{11} & S_{02} \\ S_{20} & S_{20} & S_{11} & S_{40} & S_{31} & S_{22} \\ S_{11} & S_{11} & S_{11} & S_{31} & S_{22} & S_{13} \\ S_{02} & S_{11} & S_{02} & S_{22} & S_{13} & S_{04} \end{pmatrix} \end{array} \succeq 0$$

and there exists a minimal representing measure $T = \sum\limits_{a=1}^{\kappa} Q_a \delta_{w^{(a)}}$, where $1 \leq \kappa \leq 2$ and

$$\begin{aligned} \operatorname{supp} T &\subseteq \sigma(\Phi) \times \sigma(\Psi) \\ &= \mathcal{Z}(\det(P_1(x,y))) \bigcap \mathcal{Z}(\det(P_2(x,y))) \\ &= \{(0,0),(1,0),(0,1),(1,1)\}. \end{aligned}$$

We note that $M(2)$ is also described by the block column relation $X + Y = 1$ and so the matrix-valued polynomial $P_3(x,y) = I_2 - xI_2 - yI_2$ is such that

$$P_3(X,Y) = P_1(X,Y) = P_2(X,Y) = \operatorname{col}(0_{2\times 2})_{\gamma\in\Gamma_{1,2}} \in C_{M(2)}.$$

Then $\det(P_3(x,y)) = (1-x-y)^2$ and hence $\mathcal{V}(M(2)) \subseteq \{(1,0),(0,1)\}$. We will show

$$\mathcal{V}(M(2)) = \{(1,0),(0,1)\}.$$

Indeed, if $\mathcal{V}(M(2)) \neq \{(1,0),(0,1)\}$, then since $1 \le \kappa \le 2$,

$$\mathcal{V}(M(2)) = \{(1,0)\} \text{ or } \mathcal{V}(M(2)) = \{(0,1)\}.$$

If $\mathcal{V}(M(2)) = \{(1,0)\}$, then

$$T = Q_1\delta_{(1,0)}$$

is a representing measure for S, where $\operatorname{rank} Q_1 = 2$. But then $\operatorname{rank} Q_1 = \operatorname{rank} S_{20} = 2$, a contradiction. Similarly, if $\mathcal{V}(M(2)) = \{(0,1)\}$, then

$$T = Q_1\delta_{(0,1)}$$

is a representing measure for S, where $\operatorname{rank} Q_1 = 2$. However $\operatorname{rank} Q_1 = \operatorname{rank} S_{01} = 2$, a contradiction. Hence $\kappa \neq 1$ and

$$\mathcal{V}(M(2)) = \{(1,0),(0,1)\}.$$

We will now compute a representing measure for S. Remark 3.1.27 for $\Lambda = \{(0,0),(1,0)\} \subseteq \mathbb{N}_0^2$ asserts that the multivariable Vandermonde matrix

$$V^{2\times 2}((1,0),(0,1);\Lambda) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

is invertible. By the flat extension theorem for matricial moments (see Theorem 4.0.2), the positive semidefinite matrices $Q_1, Q_2 \in \mathbb{C}^{2\times 2}$ are given by the Vandermonde equation

$$\operatorname{col}(Q_a)_{a=1}^2 = V^{2\times 2}((1,0),(0,1);\Lambda)^{-1} \operatorname{col}(S_\lambda)_{\lambda\in\Lambda}. \tag{5.22}$$

We have

$$V^{2\times 2}((1,0),(0,1);\Lambda)^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

and thus by equation (5.22),

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\operatorname{rank} Q_1 = \operatorname{rank} Q_2 = 1$. Hence $T = \sum_{a=1}^{2} Q_a \delta_{w^{(a)}}$ is a representing measure for S with $\operatorname{rank} Q_1 + \operatorname{rank} Q_2 = 2$.

Next, we shall see that every truncated $\mathcal{H}_p$-valued bisequence $S := (S_\gamma)_{\gamma \in \Gamma_{2,2}}$ with a certain description has a minimal representing measure and we will describe its support.

Theorem 5.0.5. *Let $S := (S_\gamma)_{\gamma \in \Gamma_{2,2}}$ be a given truncated $\mathcal{H}_p$-valued bisequence with moments $S_{10} = S_{01} = S_{11} = S_{20} = 0_{p\times p}$. Suppose*

$$M(1) = \begin{array}{c} \\ 1 \\ X \\ Y \end{array} \begin{array}{c} \begin{array}{ccc} 1 & X & Y \end{array} \\ \begin{pmatrix} I_p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S_{02} \end{pmatrix} \end{array} \succeq 0.$$

Then S has a minimal representing measure T with

$$\operatorname{supp} T \subseteq \{(x, y) \in \mathbb{R}^2 : y^2 \in \sigma(S_{02})\}.$$

Proof. Let $S_{30} = S_{21} = S_{12} = S_{03} = 0_{p\times p}$. Then $W := \begin{pmatrix} 0 & 0 & S_{02} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ will satisfy

$$B := M(1)W = \begin{pmatrix} 0 & 0 & S_{02} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C := W^* M(1) W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S_{02}^2 \end{pmatrix}.$$

Lemma 1.4.24 asserts that $M(2) := \begin{pmatrix} M(1) & B \\ B^* & C \end{pmatrix} \succeq 0$ is a flat extension of $M(1)$. By the flat extension theorem for matricial moments (see Theorem 4.0.2), there exists a minimal representing measure T for S with $\operatorname{supp} T = \mathcal{V}(M(2))$. Let $P^{(0,2)}(x, y) = y^2 I_p - S_{02} \in \mathbb{C}^{p\times p}[x, y]$ and notice that

$$\begin{aligned} \mathcal{V}(M(2)) &\subseteq \mathcal{Z}(\det(P^{(2,0)}(x, y))) \\ &= \{(x, y) \in \mathbb{R}^2 : \det(y^2 I_p - S_{02}) = 0\} \end{aligned}.$$

Since $P^{(0,2)}(y, 0)$ is not invertible, there exists $\eta \in \mathbb{C}^p \setminus \{0\}$ such that $y^2\eta = S_{02}\eta$. Thus $y^2 \in \sigma(S_{02})$ and

$$\operatorname{supp} T = \mathcal{V}(M(2)) \subseteq \{(x, y) \in \mathbb{R}^2 : y^2 \in \sigma(S_{02})\}. \qquad \square$$

Definition 5.0.6. Let $P(x) = \sum_{\lambda\in\Gamma_{n,2}} x^\lambda P_\lambda \in \mathbb{C}_2^{p\times p}[x,y]$ and consider the matrix $J \in \mathbb{C}^{6p\times 6p}$. Suppose the map $\Psi(x,y) : \mathbb{R}^2 \to (\mathbb{C}^{p\times p}, \mathbb{C}^{p\times p})$ is given by $\Psi(x,y) = (\Psi_1(x,y), \Psi_2(x,y))$ with

$$\Psi_1(x,y) = J_{00} + J_{10}x + J_{01}y \quad \text{and} \quad \Psi_2(x,y) = K_{00} + K_{10}x + K_{01}y$$

for some $J_{00}, J_{10}, J_{01}, K_{00}, K_{10}, K_{01} \in \mathbb{C}^{p\times p}$. J is defined as a transformation matrix given by

$$J\widehat{P} = \widehat{P}_{00} + \Psi_1\widehat{P}_{10} + \Psi_2\widehat{P}_{01} + \Psi_1^2\widehat{P}_{20} + \Psi_1\Psi_2\widehat{P}_{11} + \Psi_2^*\Psi_2\widehat{P}_{02}.$$

If J is invertible, then we may view J^{-1} as the matrix given by

$$J^{-1}\widehat{P} = \widehat{P}_{00} + \Psi_1^{-1}\widehat{P}_{10} + \Psi_2^{-1}\widehat{P}_{01} + (\Psi_1^{-1})^2\widehat{P}_{20} + \Psi_1^{-1}\Psi_2^{-1}\widehat{P}_{11} + \Psi_2^{-*}\Psi_2^{-1}\widehat{P}_{02},$$

where

$$\Psi_1^{-1}(x,y) = \tilde{J}_{00} + \tilde{J}_{10}x + \tilde{J}_{01}y \quad \text{and} \quad \Psi_2^{-1}(x,y) = \tilde{K}_{00} + \tilde{K}_{10}x + \tilde{K}_{01}y$$

for some $\tilde{J}_{00}, \tilde{J}_{10}, \tilde{J}_{01}, \tilde{K}_{00}, \tilde{K}_{10}, \tilde{K}_{01} \in \mathbb{C}^{p\times p}$.

In the next theorem we derive a minimal representing measure for a given truncated $\mathcal{H}_p$-valued bisequence when the associated moment matrix has a certain block column relation. Theorem 5.0.7 can be considered as an analogue of [16, Proposition 6.3] for $p > 1$.

Theorem 5.0.7. *Let $S := (S_\gamma)_{\gamma\in\Gamma_{2,2}}$ be a given truncated $\mathcal{H}_p$-valued bisequence and suppose*

$$M(1) = \begin{array}{c} \\ 1 \\ X \\ Y \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & X & Y \end{array} \\ \begin{pmatrix} I_p & S_{10} & S_{01} \\ S_{10} & S_{20} & S_{11} \\ S_{01} & S_{11} & S_{02} \end{pmatrix} \end{array} \succeq 0, \quad \begin{pmatrix} I_p & S_{10} \\ S_{10} & S_{20} \end{pmatrix} \succ 0$$

and $Y = 1\cdot W_1 + X\cdot W_2$ for $W_1, W_2 \in \mathbb{C}^{p\times p}$. Then the following statements hold:
(i) There exist $J_{00}, J_{10}, J_{01}, K_{00}, K_{10}, K_{01} \in \mathbb{C}^{p\times p}$ such that J (as in Definition 5.0.6) is invertible, and if we write $J = \begin{pmatrix} J_{11} & J_{12} \\ J_{12} & J_{22} \end{pmatrix}$, where $J_{11} \in \mathbb{C}^{3p\times 3p}$, then $J_{11}^* M(1) J_{11} = \begin{pmatrix} I_p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tilde{S}_{02} \end{pmatrix}$, where $\tilde{S}_{02} = S_{20} - S_{10}^2 \in \mathcal{H}_p$.
(ii) Let J be as in (i). Let $\tilde{S} = (\tilde{S}_\gamma)_{\gamma\in\Gamma_{2,2}}$ be a truncated $\mathcal{H}_p$-valued bisequence given by $\tilde{S}_{10} = \tilde{S}_{01} = \tilde{S}_{11} = 0_{p\times p} = \tilde{S}_{20}$, and let $\tilde{M}(1) = \begin{pmatrix} I_p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tilde{S}_{02} \end{pmatrix}$ be the corresponding moment matrix. If $\tilde{M}(2)$ is a flat extension of $\tilde{M}(1)$ such that $J^{-*}\tilde{M}(2)J^{-1}$ is of the form $M(2) := \begin{pmatrix} M(1) & B \\ B^* & C \end{pmatrix}$, for some choice of $(S_\gamma)_{\gamma\in\Gamma_{2,4}\setminus\Gamma_{2,2}}$ with $S_\gamma \in \mathcal{H}_p$ for $\gamma \in \Gamma_{2,4} \setminus \Gamma_{2,2}$, then S has a minimal representing measure.

Proof. (i) Let $J \in \mathbb{C}^{6p\times 6p}$ be the transformation matrix given in Definition 5.0.6 where

$$\begin{aligned}
J_{00} &= -S_{10}(S_{20}-S_{10}^2)^{-1}(S_{10}S_{01}-S_{11})-S_{01},\\
J_{10} &= (S_{20}-S_{10}^2)^{-1}(S_{10}S_{01}-S_{11}),\, J_{01}=I_p,\\
K_{00} &= -S_{10}, K_{10}=I_p \ \text{ and } \ K_{01}=0_{p\times p}.
\end{aligned}$$

J is invertible and we write $J=\begin{pmatrix} J_{11} & J_{12}\\ J_{12} & J_{22}\end{pmatrix}$, where $J_{11}\in\mathbb{C}^{3p\times 3p}$. Then $J_{11}^{-1}=\begin{pmatrix} I_p & \tilde{J}_{00} & \tilde{K}_{00}\\ 0 & \tilde{J}_{10} & \tilde{K}_{10}\\ 0 & \tilde{J}_{01} & \tilde{K}_{01}\end{pmatrix}$, where $\tilde{J}_{00}=S_{10}$, $\tilde{J}_{10}=0_{p\times p}$, $\tilde{J}_{01}=I_p$, $\tilde{K}_{00}=S_{01}$, $\tilde{K}_{10}=I_p$, and $\tilde{K}_{01}=-(S_{20}-S_{10}^2)^{-1}(S_{10}S_{01}-S_{11})$ and $J_{11}^*M(1)J_{11}=\begin{pmatrix} I_p & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \tilde{S}_{02}\end{pmatrix}$, where $\tilde{S}_{02}=S_{20}-S_{10}^2\in\mathcal{H}_p$.

(ii) Let J be as in (i). Since $\tilde{M}(2)$ is a flat extension of $\tilde{M}(1)$ and

$$J^{-*}\tilde{M}(2)J^{-1}=M(2):=\begin{pmatrix} M(1) & B\\ B^* & C\end{pmatrix},$$

we have that $M(2)\succeq 0$ and

$$2p=\operatorname{rank} M(1)\le \operatorname{rank} M(2)=\operatorname{rank}\tilde{M}(2)=\operatorname{rank}\tilde{M}(1)=2p.$$

Thus $M(2)$ is a flat extension of $M(1)$ and so by the flat extension theorem for matricial moments (see Theorem 4.0.2), there exists a minimal representing measure T for S. $\square$

Definition 5.0.8. ([40, p. 405]) If $A, B\in\mathbb{C}^{n\times n}$, then the set of all matrices of the form $A-\lambda B$, $\lambda\in\mathbb{C}$ is a *pencil*. The *generalised eigenvalues* of $A-\lambda B$ are elements of the set $\sigma(A,B)$ defined by

$$\sigma(A,B):=\{\lambda\in\mathbb{C}:\det(A-\lambda B)=0\}.$$

In the next theorem we shall see that every truncated $\mathcal{H}_p$-valued bisequence $S:=(S_\gamma)_{\gamma\in\Gamma_{2,2}}$ with $M(1)\succ 0$ being block diagonal has a minimal representing measure.

Theorem 5.0.9. *Let $S:=(S_\gamma)_{\gamma\in\Gamma_{2,2}}$ be a given truncated $\mathcal{H}_p$-valued bisequence with moments $S_{10}=S_{01}=S_{11}=0_{p\times p}$. Suppose* $M(1)=\begin{array}{c|c} & \begin{matrix} \ \ 1 & \ \ X & \ \ Y \end{matrix} \\ \begin{matrix} 1\\ X\\ Y\end{matrix} & \begin{pmatrix} I_p & 0 & 0\\ 0 & S_{20} & 0\\ 0 & 0 & S_{02}\end{pmatrix}\end{array}\succ 0$. *Then S has a minimal representing measure T with*

$$\operatorname{supp} T=\{(x,0): x\in\sigma(S_{20}^{-1}S_{02},-S_{02})\}\cup\{(1,y): y^2\in\sigma(S_{02}+S_{20}^{-1}S_{02})\},$$

where $\sigma(S_{20}^{-1}S_{02},-S_{02})$ is the set of generalised eigenvalues of $\{S_{20}^{-1}S_{02},-S_{02}.\}$

Proof. Let $B := \begin{pmatrix} S_{20} & 0 & S_{02} \\ S_{30} & S_{21} & S_{12} \\ S_{21} & S_{12} & S_{03} \end{pmatrix}$. We have $W := M(1)^{-1}B = \begin{pmatrix} S_{20} & 0 & S_{02} \\ S_{20}^{-1}S_{30} & S_{20}^{-1}S_{21} & S_{20}^{-1}S_{12} \\ S_{02}^{-1}S_{21} & S_{02}^{-1}S_{12} & S_{02}^{-1}S_{03} \end{pmatrix}$.

We then let $C := W^*M(1)W = W^*B$ and we write $\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$.

Notice that

$$\begin{aligned} C_{12} &= S_{30}S_{20}^{-1}S_{21} + S_{21}S_{02}^{-1}S_{21}, \\ C_{13} &= S_{20}S_{02} + S_{30}S_{20}^{-1}S_{12} + S_{21}S_{02}^{-1}S_{03}, \\ C_{22} &= S_{21}S_{20}^{-1}S_{21} + S_{12}S_{02}^{-1}S_{12} \end{aligned}$$

and

$$C_{23} = S_{21}S_{20}^{-1}S_{12} + S_{12}S_{02}^{-1}S_{03}.$$

Let $S_{21} := 0_{p\times p}$ and $S_{03} := 0_{p\times p}$. Then $C_{12} = 0_{p\times p} = C_{23} \in \mathcal{H}_p$ and

$$C_{22} = C_{13} \tag{5.23}$$

if and only if

$$S_{12}S_{02}^{-1}S_{12} = S_{20}S_{02} + S_{30}S_{20}^{-1}S_{12}. \tag{5.24}$$

We assume S_{12} is invertible and we solve equation (5.24) for S_{30}. We obtain

$$S_{30} = (S_{12}S_{02}^{-1}S_{12} - S_{20}S_{02})S_{12}^{-1}S_{20}.$$

Hence

$$S_{30} = S_{12}S_{02}^{-1}S_{20} - S_{20}S_{02}S_{12}^{-1}S_{20}.$$

Let $S_{12} := S_{02} \succ 0$ and $S_{30} := S_{20} - S_{20}^2$. Then $S_{30} \in \mathcal{H}_p$ and equation (5.23) holds. Hence, by Lemma 1.4.24 $M(2) := \begin{pmatrix} M(1) & B \\ B^* & C \end{pmatrix} \succeq 0$ is a flat extension of $M(1)$. By the flat extension theorem for matricial moments (see Theorem 4.0.2), there exists a minimal representing measure T for S. We now write

$$B = \begin{pmatrix} S_{20} & 0 & S_{02} \\ S_{30} & 0 & S_{02} \\ 0 & S_{02} & 0 \end{pmatrix}$$

and W becomes

$$W = M(1)^{-1}B = \begin{pmatrix} S_{20} & 0 & S_{02} \\ I_p - S_{20} & 0 & S_{20}^{-1}S_{02} \\ 0 & I_p & 0 \end{pmatrix}.$$

We let the following matrix-valued polynomials in $\mathbb{C}^{p\times p}[x,y]$:

$$P^{(2,0)}(x,y) = x^2 I_p - S_{20} - x(I_p - S_{20}),$$
$$P^{(1,1)}(x,y) = xyI_p - yI_p$$

and

$$P^{(0,2)}(x,y) = y^2 I_p - S_{02} - x(S_{20}^{-1}S_{02}).$$

Let $\mathcal{Z}_{20} := \mathcal{Z}(\det(P^{(2,0)}(x,y)))$, $\mathcal{Z}_{11} := \mathcal{Z}(\det(P^{(1,1)}(x,y)))$ and $\mathcal{Z}_{02} := \mathcal{Z}(\det(P^{(0,2)}(x,y)))$. Then

$$\operatorname{supp} T = \mathcal{Z}_{20} \bigcap \mathcal{Z}_{11} \bigcap \mathcal{Z}_{02}.$$

We observe that $(x,y) \in Z_{20}$ if and only if $P^{(2,0)}(x,y)$ is singular, i.e., there exists a nonzero vector $\xi \in \mathbb{C}^p \setminus \{0\}$ such that

$$\{x^2 I_p - S_{20} - x(I_p - S_{20})\}\xi = 0,$$

that is,

$$x(x-1)\xi = -(x-1)S_{20}\xi. \tag{5.25}$$

We have $\mathcal{Z}_{11} = \{(1,y) : y \in \mathbb{R}\} \cup \{(x,0) : x \in \mathbb{R}\}$ and, in view of equation (5.25), we get

$$\mathcal{Z}_{11} \cap \mathcal{Z}_{20} = \{(x,0) : -x \in \sigma(S_{20})\} \cup \{(1,y) : y \in \mathbb{R}\}.$$

Notice that $P^{(0,2)}(x,y)$ is singular if and only if

$$y^2\xi = (S_{02} + S_{20}^{-1}S_{02}x)\xi \qquad for \quad \xi \in \mathbb{C}^p \setminus \{0\}. \tag{5.26}$$

By equations (5.25) and (5.26) we see that

$$\begin{aligned} Z_{11} \cap Z_{20} \cap Z_{02} = \{(x,0) : \xi \in \mathbb{C}^p \setminus \{0\} \ \text{such that} \ xS_{20}^{-1}S_{02}\xi = -S_{02}\xi\} \\ \cup \{(1,y) : \xi \in \mathbb{C}^p \setminus \{0\} \ \text{such that} \ y^2\xi = (S_{02} + S_{20}^{-1}S_{02})\xi\}, \end{aligned}$$

that is,

$$Z_{11} \cap Z_{20} \cap Z_{02} = \{(x,0) : x \in \sigma(S_{20}^{-1}S_{02}, -S_{02})\} \cup \{(1,y) : y^2 \in \sigma(S_{02} + S_{20}^{-1}S_{02})\},$$

where $\sigma(S_{20}^{-1}S_{02}, -S_{02})$ is the set of generalised eigenvalues of $\{S_{20}^{-1}S_{02}, -S_{02}.\}$ $\square$

Remark 5.0.10. We note that the set $\{(x,0) : x \in \sigma(S_{20}^{-1}S_{02}, -S_{02})\}$ describing the support of the representing measure in Theorem 5.0.9 is finite. Notice that both $S_{20}^{-1}S_{02}$ and $-S_{02}$ are invertible and thus the upper triangular matrices appearing in the respective Generalised Schur Decomposition are invertible. Hence the set of generalised eigenvalues of $\{S_{20}^{-1}S_{02}, -S_{02}\}$ $\sigma(S_{20}^{-1}S_{02}, -S_{02})$ is finite. We refer the reader to [40, Theorem 7.7.1] for further details.

In the next theorem we shall see that every truncated $\mathcal{H}_p$-valued bisequence $S := (S_\gamma)_{\gamma\in\Gamma_{2,2}}$ with $M(1) \succ 0$ satisfying a certain matrix transformation has a minimal representing measure.

We note that if $p = 1$, then Theorem 5.0.11 can be viewed as an analogue of [16, Proposition 6.5], albeit $M(1) \succ 0$ is sufficient to prove that there exists a minimal representing measure.

Theorem 5.0.11. *Let $S := (S_\gamma)_{\gamma\in\Gamma_{2,2}}$ be a given truncated $\mathcal{H}_p$-valued bisequence and suppose*

$$M(1) = \begin{array}{c|c} & \begin{array}{ccc} 1 & X & Y \end{array} \\ \begin{array}{c} 1 \\ X \\ Y \end{array} & \begin{pmatrix} I_p & S_{10} & S_{01} \\ S_{10} & S_{20} & S_{11} \\ S_{01} & S_{11} & S_{02} \end{pmatrix} \end{array} \succ 0.$$

(i) There exist $J_{00}, J_{10}, J_{01}, K_{00}, K_{10}, K_{01} \in \mathbb{C}^{p\times p}$ such that J (as in Definition 5.0.6) is invertible, and if we write $J = \begin{pmatrix} J_{11} & J_{12} \\ J_{12} & J_{22} \end{pmatrix}$, where $J_{11} \in \mathbb{C}^{3p\times 3p}$, then $J_{11}^* M(1) J_{11} = \begin{pmatrix} I_p & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{pmatrix}$.

(ii) Let J be as in (i). *Let $\tilde{S} = (\tilde{S}_\gamma)_{\gamma\in\Gamma_{2,2}}$ be a truncated $\mathcal{H}_p$-valued bisequence given by $\tilde{S}_{10} = \tilde{S}_{01} = \tilde{S}_{11} = 0_{p\times p}$ and let $\tilde{M}(1) = \begin{pmatrix} I_p & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{pmatrix}$ be the corresponding moment matrix. If $\tilde{M}(2)$ is a flat extension of $\tilde{M}(1)$ such that $J^{-*}\tilde{M}(2)J^{-1}$ is of the form $M(2) := \begin{pmatrix} M(1) & B \\ B^* & C \end{pmatrix}$, for some choice of $(S_\gamma)_{\gamma\in\Gamma_{2,4}\setminus\Gamma_{2,2}}$ with $S_\gamma \in \mathcal{H}_p$ for $\gamma \in \Gamma_{2,4} \setminus \Gamma_{2,2}$, then S has a minimal representing measure.*

Proof. (i) Suppose $\Theta = S_{20} - S_{10}^2$. $\Theta \succ 0$, by a Schur complement argument applied to $M(1) \succ 0$. Let

$$\Omega = -(S_{10}S_{01} - S_{11})^*(S_{20} - S_{10}^2)^{-1}(S_{10}S_{01} - S_{11}) + S_{02} - S_{01}^2$$

and $\mathcal{J} \in \mathbb{C}^{3p\times 3p}$ be as in Definition 5.0.6 with

$$\begin{aligned} \mathcal{J}_{00} &= -S_{10}(S_{20} - S_{10}^2)^{-1}(S_{10}S_{01} - S_{11}) - S_{01}, \\ \mathcal{J}_{10} &= (S_{20} - S_{10}^2)^{-1}(S_{10}S_{01} - S_{11}), \mathcal{J}_{01} = I_p, \\ \mathcal{K}_{00} &= -S_{10}, \mathcal{K}_{10} = I_p \text{ and } \mathcal{K}_{01} = 0_{p\times p}. \end{aligned}$$

Then $\mathcal{J}$ is invertible and if we write $\mathcal{J} = \begin{pmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} \\ \mathcal{J}_{12} & \mathcal{J}_{22} \end{pmatrix}$, where $\mathcal{J}_{11} \in \mathbb{C}^{3p\times 3p}$, then $\mathcal{J}_{11}$ is invertible and the $(2,2)$ block of $\mathcal{J}_{11}^* M(1)\mathcal{J}_{11} \succ 0$ is given by Ω, and hence $\Omega \succ 0$. Next we let $J \in \mathbb{C}^{6p\times 6p}$ be the transformation matrix given in Definition 5.0.6 where

$$
\begin{aligned}
J_{00} &= \{-S_{10}(S_{20} - S_{10}^2)^{-1}(S_{10}S_{01} - S_{11}) - S_{01}\}\Omega^{-1/2},\\
J_{10} &= (S_{20} - S_{10}^2)^{-1}(S_{10}S_{01} - S_{11})\Omega^{-1/2}, J_{01} = \Omega^{-1/2},\\
K_{00} &= -S_{10}\Theta^{-1/2}, K_{10} = \Theta^{-1/2} \text{ and } K_{01} = 0_{p\times p}.
\end{aligned}
$$

Then J is invertible and we write $J = \begin{pmatrix} J_{11} & J_{12} \\ J_{12} & J_{22} \end{pmatrix}$, where $J_{11} \in \mathbb{C}^{3p\times 3p}$. We then have $J_{11}^{-1} = \begin{pmatrix} I_p & \tilde{J}_{00} & \tilde{K}_{00} \\ 0 & \tilde{J}_{10} & \tilde{K}_{10} \\ 0 & \tilde{J}_{01} & \tilde{K}_{01} \end{pmatrix}$, where $\tilde{J}_{00} = S_{10}, \tilde{J}_{10} = 0_{p\times p}, \tilde{J}_{01} = \Theta^{1/2}$, $\tilde{K}_{00} = S_{01}, \tilde{K}_{10} = \Omega^{1/2}, \tilde{K}_{01} = -\Theta^{-1/2}(S_{10}S_{01} - S_{11})$ and $J_{11}^* M(1) J_{11} = \begin{pmatrix} I_p & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{pmatrix}$.

(ii) Let J be as in (i) . Since $\tilde{M}(2)$ is a flat extension of $\tilde{M}(1)$ and

$$
J^{-*}\tilde{M}(2)J^{-1} = M(2) := \begin{pmatrix} M(1) & B \\ B^* & C \end{pmatrix},
$$

we have that $M(2) \succeq 0$ and

$$
3p = \operatorname{rank} M(1) \le \operatorname{rank} M(2) = \operatorname{rank} \tilde{M}(2) = \operatorname{rank} \tilde{M}(1) = 3p.
$$

Hence $M(2)$ is a flat extension of $M(1)$ and so by the flat extension theorem for matricial moments (see Theorem 4.0.2), there exists a minimal representing measure T for S. $\square$

Appendix A

Proof of the extension lemma

The goal of this Appendix is to provide proofs for Lemma 3.4.1 and Lemma 3.4.2, which deal with extensions of moment matrices based on matricial moments.

Lemma A.0.1 (extension lemma). *Let $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$ be a given truncated $\mathcal{H}_p$-valued multisequence and let $M(n)$ be the corresponding moment matrix. If $M(n) \succeq 0$ has an extension $M(n{+}1)$ such that $M(n{+}1) \succeq 0$ and* $\operatorname{rank} M(n{+}1) = \operatorname{rank} M(n)$, *then there exist $(S_\gamma)_{\gamma\in\mathbb{N}_0^d\setminus\Gamma_{2n,d}}$ such that*

$$M(n+k) \succeq 0$$

and

$$\operatorname{rank} M(n+k) = \operatorname{rank} M(n+k-1) \quad \textit{for} \;\; k = 2, 3, \dots$$

Proof. We first consider the case when $d = 2$, while the general case $d > 2$ can be proved similarly. We have

$$M(n) \succeq 0 \quad \text{and} \quad \operatorname{rank} M(n) = \operatorname{rank} M(n-1).$$

We wish to choose moments $(S_\gamma)_{\gamma\in\mathbb{N}_0^2\setminus\Gamma_{2n,2}}$ such that $M(n+1) \succeq 0$ and

$$\operatorname{rank} M(n+1) = \operatorname{rank} M(n).$$

There exist matrix-valued polynomials in $\mathbb{C}_n^{p\times p}[x, y]$

$$P^{(a,b)}(x, y) = x^a y^b I_p - \sum_{(j,k)\in\Gamma_{n-1,2}} x^j y^k P_{jk}^{(a,b)},$$

with $(a, b) \in \Gamma_{n,2}$

$$P^{(n,0)}(x, y) = x^n I_p - \sum_{(j,k)\in\Gamma_{n-1,2}} x^j y^k P_{jk}^{(n,0)}$$

and

$$P^{(0,n)}(x, y) = y^n I_p - \sum_{(j,k)\in\Gamma_{n-1,2}} x^j y^k P_{jk}^{(0,n)}$$

such that

$$P^{(a,b)}(X, Y) = P^{(n,0)}(X, Y) = P^{(0,n)}(X, Y) = \operatorname{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}}. \tag{A.1}$$

Let the new moments in $M(n+1)$ be defined by

$$X^{n+1} = (xP^{(n,0)})(X,Y) \quad \text{and} \quad Y^{n+1} = (yP^{(0,n)})(X,Y).$$

Write $X^{n+1} \in C_{M(n+1)}$ as

$$\begin{aligned} X^{n+1} &= (xP^{(n,0)})(X,Y) \\ &= \operatorname{col}\left(S_{n+1+c,d}\right)_{(c,d)\in\Gamma_{n+1,2}} \\ &= \operatorname{col}\left(\sum_{(j,k)\in\Gamma_{n-1,2}} S_{c+j+1,d+k}P_{jk}^{(n,0)}\right)_{(c,d)\in\Gamma_{n+1,2}} \end{aligned}$$

and $Y^{n+1} \in C_{M(n+1)}$ as

$$\begin{aligned} Y^{n+1} &= (yP^{(0,n)})(X,Y) \\ &= \operatorname{col}\left(S_{c,d+n+1}\right)_{(c,d)\in\Gamma_{n+1,2}} \\ &= \operatorname{col}\left(\sum_{(j,k)\in\Gamma_{n-1,2}} S_{c+j,d+k+1}P_{jk}^{(0,n)}\right)_{(c,d)\in\Gamma_{n+1,2}}. \end{aligned}$$

We shall proceed by following certain steps to check that there exists a rank preserving extension $M(n+1)$. We write

$$M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix},$$

where

$$M(n) \in \mathbb{C}^{(\operatorname{card}\Gamma_{n,2})p\times(\operatorname{card}\Gamma_{n,2})p},$$

$$B \in \mathbb{C}^{(\operatorname{card}\Gamma_{n,2})p\times(\operatorname{card}(\Gamma_{n+1,2}\setminus\Gamma_{n,2}))p}$$

and

$$C \in \mathbb{C}^{(\operatorname{card}\Gamma_{n+1,2}\setminus\Gamma_{n,2})p\times(\operatorname{card}(\Gamma_{n+1,2}\setminus\Gamma_{n,2}))p}.$$

Step 1: We need to show that

$$S_{n+1+c,d} = \sum_{(j,k)\in\Gamma_{n-1,2}} S_{c+j+1,d+k}P_{jk}^{(n,0)} \tag{A.2}$$

for all $(n+1+c,d) \in \Gamma_{2n,2}$, that is, $(c,d) \in \Gamma_{n-1,2}$. Similarly, we must prove

$$S_{c,d+n+1} = \sum_{(j,k)\in\Gamma_{n-1,2}} S_{c+j,d+k+1}P_{jk}^{(0,n)} \tag{A.3}$$

for all $(c,d+n+1) \in \Gamma_{2n,2}$, that is, $(c,d) \in \Gamma_{n-1,2}$.

We have $X^n = P^{(n,0)}(X,Y) \in C_{M(n)}$, which is equivalent to

$$\operatorname{col}\Big(S_{n+\ell,m}\Big)_{(\ell,m)\in\Gamma_{n,2}} = \operatorname{col}\Big(\sum_{(j,k)\in\Gamma_{n-1,2}} S_{\ell+j,m+k}P_{jk}^{(n,0)}\Big)_{(\ell,m)\in\Gamma_{n,2}}.$$

Thus $S_{n+\ell,m} = \sum_{(j,k)\in\Gamma_{n-1,2}} S_{\ell+j,m+k}P_{jk}^{(n,0)}$ for all $(\ell,m)\in\Gamma_{n,2}$. Let $\ell = c+1$ and $m = d$. We then have $(\ell,m)\in\Gamma_{n,2}$ and so the equation (A.2) holds for all $(c,d)\in\Gamma_{n-1,2}$. Similarly, since

$$Y^n = P^{(0,n)}(X,Y) \in C_{M(n)},$$

that is,

$$\operatorname{col}\Big(S_{\ell,m+n}\Big)_{(\ell,m)\in\Gamma_{n,2}} = \operatorname{col}\Big(\sum_{(j,k)\in\Gamma_{n-1,2}} S_{\ell+j,m+k}P_{jk}^{(0,n)}\Big)_{(\ell,m)\in\Gamma_{n,2}},$$

we obtain $S_{\ell,m+n} = \sum_{(j,k)\in\Gamma_{n-1,2}} S_{\ell+j,m+k}P_{jk}^{(0,n)}$ for all $(\ell,m)\in\Gamma_{n,2}$. Let $\ell = c$ and $m = d+1$. Then $(\ell,m)\in\Gamma_{n,2}$ and thus the equation (A.3) holds for all $(c,d)\in\Gamma_{n-1,2}$. Notice that the moments

$$\operatorname{col}\Big(S_{c+n+1,d}\Big)_{(c,d)\in\Gamma_{n+1,2}\setminus\Gamma_{n,2}} = \operatorname{col}\Big(\sum_{(j,k)\in\Gamma_{n-1,2}} S_{c+j+1,d+k}P_{jk}^{(n,0)}\Big)_{(c,d)\in\Gamma_{n+1,2}\setminus\Gamma_{n,2}}$$

are already defined. We wish to show that for all $(c,d)\in\Gamma_{n,2}$,

$$S_{n+1+c,d} := \sum_{(j,k)\in\Gamma_{n-1,2}} S_{c+j+1,d+k}P_{jk}^{(n,0)} \in \mathcal{H}_p,$$

which are new moments. For this, consider $\mathcal{M}$ as the submatrix of $M(n+1)$ with block columns indexed by $\{(j+c+1,d+k):(j,k)\in\Gamma_{n-1,2}\}$ and block rows indexed by $(j,k)\in\Gamma_{n-1,2}$. Notice that $\mathcal{M}$ is Hermitian. We have

$$\begin{aligned}
\sum_{(j,k)\in\Gamma_{n-1,2}} S_{c+j+1,d+k}P_{jk}^{(n,0)} &= \operatorname{col}\Big(S_{c+j+1,d+k}\Big)^*_{(j,k)\in\Gamma_{n-1,2}} \operatorname{col}\Big(P_{jk}^{(n,0)}\Big)_{(j,k)\in\Gamma_{n-1,2}} \\
&= \operatorname{col}\Big(P_{jk}^{(n,0)}\Big)^*_{(j,k)\in\Gamma_{n-1,2}} \mathcal{M}\operatorname{col}\Big(P_{jk}^{(n,0)}\Big)_{(j,k)\in\Gamma_{n-1,2}} \\
&= \operatorname{col}\Big(P_{jk}^{(n,0)}\Big)^*_{(j,k)\in\Gamma_{n-1,2}} \operatorname{col}\Big(S_{c+j+1,d+k}\Big)_{(j,k)\in\Gamma_{n-1,2}} \\
&= \sum_{(j,k)\in\Gamma_{n-1,2}} P_{jk}^{(n,0)*}S_{c+j+1,d+k} \\
&= S^*_{n+1+c,d}.
\end{aligned}$$

Similarly, we shall note that

$$\operatorname{col}\Big(S_{c,n+1+d}\Big)_{(c,d)\in\Gamma_{n+1,2}\setminus\Gamma_{n,2}} = \operatorname{col}\Big(\sum_{(j,k)\in\Gamma_{n-1,2}} S_{c+j,d+k+1}P_{jk}^{(0,n)}\Big)_{(c,d)\in\Gamma_{n+1,2}\setminus\Gamma_{n,2}}$$

are already defined and we need to show that the new moments

$$S_{c,n+1+d} := \sum_{(j,k)\in\Gamma_{n-1,2}} S_{c+j,d+k+1} P^{(0,n)}_{jk} \in \mathcal{H}_p$$

for all $(c,d) \in \Gamma_{n,2}$. As before

$$\begin{aligned}
\sum_{(j,k)\in\Gamma_{n-1,2}} S_{c+j,d+k+1} P^{(0,n)}_{jk} &= \operatorname{col}\left(S_{c+j,d+k+1}\right)^*_{(j,k)\in\Gamma_{n-1,2}} \operatorname{col}\left(P^{(0,n)}_{jk}\right)_{(j,k)\in\Gamma_{n-1,2}} \\
&= \operatorname{col}\left(P^{(0,n)}_{jk}\right)^*_{(j,k)\in\Gamma_{n-1,2}} \mathcal{M} \operatorname{col}\left(P^{(0,n)}_{jk}\right)_{(j,k)\in\Gamma_{n-1,2}} \\
&= \operatorname{col}\left(P^{(0,n)}_{jk}\right)^*_{(j,k)\in\Gamma_{n-1,2}} \operatorname{col}\left(S_{c+j,d+k+1}\right)_{(j,k)\in\Gamma_{n-1,2}} \\
&= \sum_{(j,k)\in\Gamma_{n-1,2}} P^{(0,n)*}_{jk} S_{c+j,d+k+1} \\
&= S^*_{c,n+1+d}.
\end{aligned}$$

Step 2: In this step, we need to show

$$X^{a+1}Y^b = (xP^{(n,0)})(X,Y) \in C_{M(n+1)}$$

for all $a+b=n$ with $a \neq n$ and $b \neq 0$, and moreover,

$$X^aY^{b+1} = (yP^{(0,n)})(X,Y) \in C_{M(n+1)}$$

for all $a+b=n$ with $a \neq 0$ and $b \neq n$, where $X^{a+1}Y^b$ and X^aY^{b+1} are block columns of the B block.
We first consider the case when $(c,d) \in \Gamma_{n,2}$. We have

$$X^{a+1}Y^b = \operatorname{col}\left(S_{a+1+c,b+d}\right)_{(c,d)\in\Gamma_{n-1,2}}$$

and

$$(xP^{(n,0)})(X,Y) = \operatorname{col}\left(\sum_{(j,k)\in\Gamma_{n-1,2}} S_{c+j+1,d+k} P^{(n,0)}_{jk}\right)_{(c,d)\in\Gamma_{n-1,2}}.$$

By condition (A.1), $X^n = P^{(n,0)}(X,Y) \in C_{M(n)}$, that is,

$$\operatorname{col}\left(S_{n+\ell,m}\right)_{(\ell,m)\in\Gamma_{n,2}} = \operatorname{col}\left(\sum_{(j,k)\in\Gamma_{n-1,2}} S_{\ell+j,m+k} P^{(n,0)}_{jk}\right)_{(\ell,m)\in\Gamma_{n,2}}$$

and thus $S_{n+\ell,m} = \sum_{(j,k)\in\Gamma_{n-1,2}} S_{\ell+j,m+k} P^{(n,0)}_{jk}$ for all $(\ell,m) \in \Gamma_{n,2}$. If we let $\ell = c+1$ and $m = d$, then $(a+1+c, b+d) \in \Gamma_{2n,2}$.

Similarly, we have
$$X^aY^{b+1} = \operatorname{col}\left(S_{a+c,b+d+1}\right)_{(c,d)\in\Gamma_{n-1,2}}$$
and
$$(yP^{(0,n)})(X,Y) = \operatorname{col}\left(\sum_{(j,k)\in\Gamma_{n-1,2}} S_{c+j,d+k+1}P^{(0,n)}_{jk}\right)_{(c,d)\in\Gamma_{n-1,2}}.$$
Furthermore, by condition (A.1), $Y^n = P^{(0,n)}(X,Y) \in C_{M(n)}$, that is,
$$\operatorname{col}\left(S_{\ell,n+m}\right)_{(\ell,m)\in\Gamma_{n,2}} = \operatorname{col}\left(\sum_{(j,k)\in\Gamma_{n-1,2}} S_{\ell+j,m+k}P^{(0,n)}_{jk}\right)_{(\ell,m)\in\Gamma_{n,2}}$$
and hence
$$S_{\ell,n+m} = \sum_{(j,k)\in\Gamma_{n-1,2}} S_{\ell+j,m+k}P^{(0,n)}_{jk}$$
for all $(\ell,m)\in\Gamma_{n,2}$. If we let $\ell = c$ and $m = d+1$, then $(a+c,b+d+1)\in\Gamma_{2n,2}$. We continue with the case when $(c,d)\in\Gamma_{n+1,2}\setminus\Gamma_{n,2}$. We have defined
$$\operatorname{col}\left(S_{n+\ell+1,m}\right)_{(\ell,m)\in\Gamma_{n+1,2}\setminus\Gamma_{n,2}} = \operatorname{col}\left(\sum_{(j,k)\in\Gamma_{n-1,2}} S_{\ell+j+1,m+k}P^{(n,0)}_{jk}\right)_{(\ell,m)\in\Gamma_{n+1,2}\setminus\Gamma_{n,2}}.$$
We have to show
$$\operatorname{col}\left(S_{a+c+1,b+d}\right)_{(c,d)\in\Gamma_{n+1,2}\setminus\Gamma_{n,2}} = \operatorname{col}\left(\sum_{(j,k)\in\Gamma_{n-1,2}} S_{c+j+1,d+k}P^{(n,0)}_{jk}\right)_{(c,d)\in\Gamma_{n+1,2}\setminus\Gamma_{n,2}}.$$
For $\ell = c$ and $m = d$, we obtain $(a+c+1,b+d)\in\Gamma_{2n+2,2}$. We have also defined
$$\operatorname{col}\left(S_{\ell,m+n+1}\right)_{(\ell,m)\in\Gamma_{n+1,2}\setminus\Gamma_{n,2}} = \operatorname{col}\left(\sum_{(j,k)\in\Gamma_{n-1,2}} S_{\ell+j,m+k+1}P^{(0,n)}_{jk}\right)_{(\ell,m)\in\Gamma_{n+1,2}\setminus\Gamma_{n,2}}.$$
As before, we need to prove
$$\operatorname{col}\left(S_{a+c,b+d+1}\right)_{(c,d)\in\Gamma_{n+1,2}\setminus\Gamma_{n,2}} = \operatorname{col}\left(\sum_{(j,k)\in\Gamma_{n-1,2}} S_{c+j,d+k+1}P^{(0,n)}_{jk}\right)_{(c,d)\in\Gamma_{n+1,2}\setminus\Gamma_{n,2}}.$$
For $\ell = c$ and $m = d$, we derive $(a+c,b+d+1)\in\Gamma_{2n+2,2}$.

Step 3: Let the following moment of the C block $S_{n+1,n+1} := \sum_{(j,k)\in\Gamma_{n-1,2}} S_{j+1,k+n+1}P^{(n,0)}_{jk}$. We must show
$$S_{n+1,n+1} = \sum_{(j,k)\in\Gamma_{n-1,2}} S_{j+n+1,k+1}P^{(0,n)}_{jk}.$$
Consider the submatrix $\mathcal{M}$ of $M(n+1)$ as described in Step 1.

We compute

$$\begin{aligned}
\sum_{(j,k)\in\Gamma_{n-1,2}} S_{j+n+1,k+1}P_{jk}^{(0,n)} &= \mathrm{col}\left(S_{j+n+1,k+1}\right)^*_{(j,k)\in\Gamma_{n-1,2}} \mathrm{col}\left(P_{jk}^{(0,n)}\right)_{(j,k)\in\Gamma_{n-1,2}} \\
&= \mathrm{col}\left(P_{jk}^{(0,n)}\right)^*_{(j,k)\in\Gamma_{n-1,2}} \mathcal{M}\,\mathrm{col}\left(P_{jk}^{(0,n)}\right)_{(j,k)\in\Gamma_{n-1,2}} \\
&= \mathrm{col}\left(P_{jk}^{(0,n)}\right)^*_{(j,k)\in\Gamma_{n-1,2}} \mathrm{col}\left(S_{j+1,k+n+1}\right)_{(j,k)\in\Gamma_{n-1,2}} \\
&= S^*_{n+1,n+1}
\end{aligned}$$

by the definition of $S_{n+\ell,m}$ in Step 2 for $\ell = 1$ and $m = n+1$.

Step 4: In the final step of this proof, we shall consider the case when $d > 2$. If $M(n) \succeq 0$ and $\mathrm{rank}\, M(n) = \mathrm{rank}\, M(n-1)$, then we must choose moments $(S_\gamma)_{\gamma\in\mathbb{N}_0^d\setminus\Gamma_{2n,d}}$ such that

$$M(n+1) \succeq 0 \quad \text{and} \quad \mathrm{rank}\, M(n+1) = \mathrm{rank}\, M(n).$$

There exist matrix-valued polynomials of the form

$$P(x) = x^\gamma I_p - \sum_{\lambda\in\Gamma_{n,d}} x^\lambda P_\lambda \in \mathbb{C}_n^{p\times p}[x_1,\ldots,x_d]$$

with $|\gamma| > 0$ such that

$$P(X) = \mathrm{col}(0_{p\times p})_{\gamma\in\Gamma_{n,d}}.$$

Application of Steps 1-3 (for $d > 2$) will yield the existence of moments $(S_\gamma)_{\gamma\in\mathbb{N}_0^d\setminus\Gamma_{2n,d}}$ such that

$$M(n+k) \succeq 0$$

and

$$\mathrm{rank}\, M(n+k) = \mathrm{rank}\, M(n+k-1) \quad \text{for } \ k = 2, 3, \ldots$$

as desired. $\square$

Lemma A.0.2. *Let $S := (S_\gamma)_{\gamma\in\Gamma_{2n,d}}$ be a given truncated $\mathcal{H}_p$-valued multisequence and let $M(n) \succeq 0$ be the corresponding moment matrix. Suppose that $M(n)$ has a positive extension $M(n+1)$ with*

$$\mathrm{rank}\, M(n+1) = \mathrm{rank}\, M(n).$$

Then there exists a unique sequence of extensions

$$M(n+k) \succeq 0$$

with

$$\mathrm{rank}\, M(n+k) = \mathrm{rank}\, M(n+k-1) \quad \textit{for } \ k = 2, 3, \ldots$$

Proof. Suppose there exists a choice of moments $(S_\gamma)_{\gamma\in\mathbb{N}_0^d\setminus\Gamma_{2n+2,d}}$ which gives rise to a sequence of extensions $M(n+k)\succeq 0$ for all $k=2,3,\dots$ and thus to $M(\infty)\succeq 0$. Consider a matrix-valued polynomial $P\in\mathbb{C}^{p\times p}[x_1,\dots,x_d]$ and let $\mathcal{I}$ be the right ideal associated with $M(\infty)$ (see Definition 3.1.14 for the precise definition of $\mathcal{I}$). Suppose there exists another choice of moments $(\tilde S_\gamma)_{\gamma\in\mathbb{N}_0^d\setminus\Gamma_{2n+2,d}}$ which gives rise to $\tilde M(\infty)\succeq 0$ and is such that $\tilde S_\gamma = S_\gamma$ for all $\gamma\in\Gamma_{2n+2,d}$. Let $\tilde{\mathcal{I}}$ be the right ideal associated with $\tilde M(\infty)$. Since $M(n)$ has a positive extension $M(n+1)$ with

$$\operatorname{rank} M(n+1) = \operatorname{rank} M(n),$$

there exist matrix-valued polynomials of the form

$$P^{(\gamma)}(x) = x^\gamma I_p + \sum_{\lambda\in\Gamma_{n,d}} x^\lambda P_\lambda^{(\gamma)} \in \mathbb{C}^{p\times p}[x_1,\dots,x_d]$$

with $\gamma\in\Gamma_{n+1,d}\setminus\Gamma_{n,d}$ such that $P^{(\gamma)}(X)=\operatorname{col}(0_{p\times p})_{\tilde\gamma\in\mathbb{N}_0^d}\in C_{M(\infty)}$. Thus for $\varepsilon_j\in\mathbb{N}_0^d$,

$$\operatorname{col}\left(S_{(n+1)\varepsilon_j+\tilde\gamma}\right)_{\tilde\gamma\in\mathbb{N}_0^d} = -\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} S_{\lambda+\tilde\gamma}P_\lambda^{(\gamma)}\right)_{\tilde\gamma\in\mathbb{N}_0^d} \quad \text{for } \ j=1,\dots,d. \tag{A.4}$$

We need to show first that

$$\{P^{(\gamma)}\}_{\gamma\in\Gamma_{n+1,d}\setminus\Gamma_{n,d}} \subseteq \tilde{\mathcal{I}}.$$

Since $\tilde S_\gamma = S_\gamma$ for all $\gamma\in\Gamma_{2n+2,d}$, we have

$$\tilde S_{(n+1)\varepsilon_j+\tilde\gamma} = S_{(n+1)\varepsilon_j+\tilde\gamma} \ \text{ for } \ \tilde\gamma\in\Gamma_{n+1,d}.$$

By equation (A.4),

$$\operatorname{col}\left(\tilde S_{(n+1)\varepsilon_j+\tilde\gamma}\right)_{\tilde\gamma\in\Gamma_{n+1,d}} = -\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} \tilde S_{\lambda+\tilde\gamma}P_\lambda^{(\gamma)}\right)_{\tilde\gamma\in\Gamma_{n+1,d}}$$

and

$$\begin{aligned}\tilde M(n+1)\operatorname{col}(P_\lambda^{(\gamma)})_{\lambda\in\Gamma_{n+1,d}} &= M(n+1)\operatorname{col}(P_\lambda^{(\gamma)})_{\lambda\in\Gamma_{n+1,d}}\\ &= \operatorname{col}(0_{p\times p})_{\tilde\gamma\in\Gamma_{n+1,d}}.\end{aligned} \tag{A.5}$$

To show $\{P^{(\gamma)}\}_{\gamma\in\Gamma_{n+1,d}\setminus\Gamma_{n,d}}\subseteq\tilde{\mathcal{I}}$, we need to prove

$$\tilde M(n+k)\{\operatorname{col}(P_\lambda^{(\gamma)})_{\lambda\in\Gamma_{n+1,d}}\oplus\operatorname{col}(0_{p\times p})_{\tilde\gamma\in\Gamma_{n+k,d}\setminus\Gamma_{n+1,d}}\} = \operatorname{col}(0_{p\times p})_{\tilde\gamma\in\Gamma_{n+k,d}} \text{ for } \ k=2,3,\dots \tag{A.6}$$

Write

$$\tilde M(n+k) = \begin{pmatrix}\tilde M(n+1) & B\\ B^* & C\end{pmatrix}\succeq 0,$$

where

$$\tilde M(n+1)\in\mathbb{C}^{(\operatorname{card}\Gamma_{n+1,d})p\times(\operatorname{card}\Gamma_{n+1,d})p},$$

$$B \in \mathbb{C}^{(\operatorname{card}\Gamma_{n+1,d})p \times (\operatorname{card}(\Gamma_{n+k,d} \setminus \Gamma_{n+1,d}))p}$$

and

$$C \in \mathbb{C}^{(\operatorname{card}\Gamma_{n+k,d} \setminus \Gamma_{n+1,d})p \times (\operatorname{card}(\Gamma_{n+k,d} \setminus \Gamma_{n+1,d}))p}.$$

Since $\tilde{M}(n+k) \succeq 0$, by Lemma 1.4.24, there exists $W \in \mathbb{C}^{(\operatorname{card}\Gamma_{n+1,d})p \times (\operatorname{card}(\Gamma_{n+k,d} \setminus \Gamma_{n+1,d}))p}$ such that

$$\tilde{M}(n+1)W = B \quad \text{and} \quad C \succeq W^* \tilde{M}(n+1)W.$$

Then

$$\begin{aligned}
\tilde{M}(n+k)\{\operatorname{col}(P_\lambda^{(\gamma)})_{\lambda\in\Gamma_{n+1,d}} \oplus \operatorname{col}(0_{p\times p})_{\tilde{\gamma}\in\Gamma_{n+k,d}\setminus\Gamma_{n+1,d}}\} &= \begin{pmatrix} \tilde{M}(n+1)\operatorname{col}(P_\lambda^{(\gamma)})_{\lambda\in\Gamma_{n+1,d}} \\ B^* \operatorname{col}(P_\lambda^{(\gamma)})_{\lambda\in\Gamma_{n+1,d}} \end{pmatrix} \\
&= \begin{pmatrix} M(n+1)\operatorname{col}(P_\lambda^{(\gamma)})_{\lambda\in\Gamma_{n+1,d}} \\ B^* \operatorname{col}(P_\lambda^{(\gamma)})_{\lambda\in\Gamma_{n+1,d}} \end{pmatrix} \\
&= \begin{pmatrix} \operatorname{col}(0_{p\times p})_{\tilde{\gamma}\in\Gamma_{n+1,d}} \\ W^* M(n+1)\operatorname{col}(P_\lambda)_{\lambda\in\Gamma_{n+1,d}} \end{pmatrix} \\
&= \operatorname{col}(0_{p\times p})_{\tilde{\gamma}\in\Gamma_{n+k,d}}
\end{aligned}$$

for all $k = 2, 3, \ldots,$ by equation (A.5). Thus, equation (A.6) holds as desired and so

$$\{P^{(\gamma)}\}_{\gamma\in\Gamma_{n+1,d}\setminus\Gamma_{n,d}} \subseteq \tilde{\mathcal{I}}.$$

This in turn will yield $x^{\varepsilon_j} P^{(\gamma)} \in \tilde{\mathcal{I}}$. Indeed

$$\tilde{M}(\infty)\left(x^{\varepsilon_j}\operatorname{col}(P_\lambda^{(\gamma)})_{\lambda\in\Gamma_{n,d}}\right) = \operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} \tilde{S}_{\lambda'+\varepsilon_j+\lambda} P_\lambda^{(a)}\right)_{\lambda'\in\mathbb{N}_0^d}.$$

But since $P^{(\gamma)} \in \tilde{\mathcal{I}}$,

$$\tilde{M}(\infty)\operatorname{col}(P_\lambda^{(\gamma)})_{\lambda\in\Gamma_{n,d}} = \operatorname{col}(0_{p\times p})_{\tilde{\gamma}\in\mathbb{N}_0^d},$$

that is,

$$\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} \tilde{S}_{\lambda+\tilde{\gamma}} P_\lambda^{(\gamma)}\right)_{\tilde{\gamma}\in\mathbb{N}_0^d} = \operatorname{col}(0_{p\times p})_{\tilde{\gamma}\in\mathbb{N}_0^d},$$

For $\tilde{\gamma} = \lambda' + \varepsilon_j,\ j = 1, \ldots, d,$

$$\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} \tilde{S}_{\lambda'+\varepsilon_j+\lambda} P_\lambda^{(\gamma)}\right)_{\lambda'\in\mathbb{N}_0^d} = \operatorname{col}(0_{p\times p})_{\lambda'\in\mathbb{N}_0^d}$$

and so

$$\tilde{M}(\infty)\left(x^{\varepsilon_j}\operatorname{col}(P_\lambda^{(\gamma)})_{\lambda\in\Gamma_{n,d}}\right) = \operatorname{col}(0_{p\times p})_{\tilde{\gamma}\in\mathbb{N}_0^d}.$$

Since $x^{\varepsilon_j} P^{(\gamma)} \in \tilde{\mathcal{I}}$, we have

$$\operatorname{col}\left(S_{(n+2)\varepsilon_j+\tilde{\gamma}}\right)_{\tilde{\gamma}\in\Gamma_{n,d}} = -\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} S_{\lambda+\tilde{\gamma}+\varepsilon_j} P_\lambda^{(\gamma)}\right)_{\tilde{\gamma}\in\Gamma_{n,d}} \quad \text{for } \ j=1,\dots,d.$$

Moreover

$$\operatorname{col}\left(\tilde{S}_{(n+2)\varepsilon_j+\tilde{\gamma}}\right)_{\tilde{\gamma}\in\Gamma_{n+1,d}\setminus\Gamma_{n,d}} = -\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} S_{\lambda+\tilde{\gamma}+\varepsilon_j} P_\lambda^{(\gamma)}\right)_{\tilde{\gamma}\in\Gamma_{n+1,d}\setminus\Gamma_{n,d}} \tag{A.7}$$

and

$$\operatorname{col}\left(\tilde{S}_{(n+2)\varepsilon_j+\tilde{\gamma}}\right)_{\tilde{\gamma}\in\mathbb{N}_0^d\setminus\Gamma_{n+1,d}} = -\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} \tilde{S}_{\lambda+\tilde{\gamma}+\varepsilon_j} P_\lambda^{(\gamma)}\right)_{\tilde{\gamma}\in\mathbb{N}_0^d\setminus\Gamma_{n+1,d}} \tag{A.8}$$

for all $j=1,\dots,d$. In view of equation (A.4),

$$\tilde{S}_{(n+2)\varepsilon_j+\tilde{\gamma}} = S_{(n+2)\varepsilon_j+\tilde{\gamma}} \ \text{ for } \ \tilde{\gamma}\in\Gamma_{n+1,d}\setminus\Gamma_{n,d} \quad \text{and} \quad j=1,\dots,d.$$

Hence $\tilde{S}_{\tilde{\lambda}} = S_{\tilde{\lambda}}$ for $\tilde{\lambda}\in\Gamma_{2n+3,d}$. We next rewrite the equations (A.7) and (A.8) as

$$\operatorname{col}\left(\tilde{S}_{(n+2)\varepsilon_j+\tilde{\gamma}}\right)_{\tilde{\gamma}\in\Gamma_{n+2,d}\setminus\Gamma_{n+1,d}} = -\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} S_{\lambda+\tilde{\gamma}+\varepsilon_j} P_\lambda^{(\gamma)}\right)_{\tilde{\gamma}\in\Gamma_{n+2,d}\setminus\Gamma_{n+1,d}}$$

and

$$\operatorname{col}\left(\tilde{S}_{(n+2)\varepsilon_j+\tilde{\gamma}}\right)_{\tilde{\gamma}\in\mathbb{N}_0^d\setminus\Gamma_{n+2,d}} = -\operatorname{col}\left(\sum_{\lambda\in\Gamma_{n,d}} \tilde{S}_{\lambda+\tilde{\gamma}+\varepsilon_j} P_\lambda^{(\gamma)}\right)_{\tilde{\gamma}\in\mathbb{N}_0^d\setminus\Gamma_{n+2,d}}$$

for all $j=1,\dots,d$. Thus

$$\tilde{S}_{(n+2)\varepsilon_j+\tilde{\gamma}} = S_{(n+2)\varepsilon_j+\tilde{\gamma}} \ \text{ for } \ \tilde{\gamma}\in\Gamma_{n+2,d}\setminus\Gamma_{n+1,d} \quad \text{and} \quad j=1,\dots,d.$$

Hence $\tilde{S}_{\tilde{\lambda}} = S_{\tilde{\lambda}}$ for all $\tilde{\lambda}\in\Gamma_{2n+4,d}$. Continuing inductively we conclude that

$$\tilde{S}_{\tilde{\lambda}} = S_{\tilde{\lambda}} \ \text{ for } \ \tilde{\lambda}\in\mathbb{N}_0^d,$$

from which we derive uniqueness for the sequence of extensions

$$M(n+k)\succeq 0$$

with

$$\operatorname{rank} M(n+k) = \operatorname{rank} M(n+k-1) \quad \text{for } \ k=2,3,\dots \qquad \square$$

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