Representation Theory of Non-graded, Non-restricted Modular Lie Algebras

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Abstract

We classify in a unified approach the simple restricted modules for the minimal p-envelope of the non-graded, non-restricted Hamiltonian Lie algebra $H(2; (1, 1); \Phi(1))$ over an algebraically closed field k of characteristic $p \geq 5$. We also give the restrictions of these modules to a subalgebra isomorphic to the first Witt Algebra, a result stated in [S. Herpel and D. Stewart, *Selecta Mathematica* 22:2 (2016) 765–799] with an incomplete proof. We end by completing the classification of the simple restricted modules over fields of all characteristics by considering the characteristic 3 case separately.

Contents

Notation and conventions

We let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_{>0} = \mathbb{N} \setminus \{0\}$ and let $\mathbb{C}$ be the field of complex numbers. We let p denote an arbitrary prime number and we denote by $\mathbb{F}_p$ the field of p elements. We denote the Kronecker delta by δ_{ij}. We denote the $n \times n$ identity matrix by I_n. The transpose of a matrix A will be denoted by A^T.

Throughout, k is the ground field, $\mathfrak{g}$ and $\mathfrak{h}$ Lie algebras over k, and A a (not-necessarily associative) k-algebra. If $\mathfrak{g}$ is defined over a field k of positive characteristic p, then we say that $\mathfrak{g}$ is a *modular Lie algebra*.

We let $kv = k\langle v \rangle$ denote the one-dimensional k-vector space with basis v. More generally, we let $k\langle v_1, \ldots, v_n \rangle$ denote the n-dimensional k-vector space with basis $\{v_1, \ldots, v_n\}$, or if one is inside a vector space V, we let it denote the span of the n vectors. Furthermore, if the vectors are inside $\mathfrak{g}$, we let $\mathfrak{g}\langle v_1, \ldots, v_n \rangle$ denote the Lie subalgebra generated by $\{v_1, \ldots, v_n\}$. We denote the centre of $\mathfrak{g}$ by $C(\mathfrak{g})$, see Definition 3.1.18. We adopt the notation $[x, \mathfrak{g}] = 0$ to mean: for all $y \in \mathfrak{g}$, we have $[x, y] = 0$; likewise for $[\mathfrak{g}, x] = 0$.

By analogy with the situation in vector spaces, if M is an A-module, we denote by $A\langle m_1, \ldots, m_n \rangle$ the A-submodule of M generated by the m_i.[1] In an algebra or ring, we let $(a_1, a_2, \ldots, a_t)$ denote the two-sided ideal generated by the elements $a_1, \ldots, a_t$; ideals are understood to be two-sided unless otherwise stated. If A has an identity or unit element, we denote it by 1.

For $m \in \mathbb{N}_{>0}$ and $\underline{n} \in \mathbb{N}^m_{>0}$, we denote the algebras of divided powers by $\mathcal{O}(m; \underline{n})$ and $\mathcal{O}(m)$, see Definition 4.1.2. The symbol W_n denotes the Witt algebra $W(n; (1, \ldots, 1)) = W(n; \underline{1})$, see Definition 4.1.4. The symbols H and $\widehat{H}$ denote the Hamiltonian Lie algebra $H(2; (1, 1); \Phi(1)) = H(2; \underline{1}; \Phi(1))$ and its minimal p-envelope, respectively, see Definition 5.2.7 and the remark after it and Definition 3.2.32 and the discussion after Proposition 3.2.35, respectively. We denote the polynomial ring over k in n indeterminates by $k[X_1, \ldots, X_n]$. For a set X, we use id_X to denote the identity function on X. The

[1] Caution: if M is a $\mathfrak{g}$-module, the notation $\mathfrak{g}\langle x \rangle$ is potentially ambiguous. If $x \in M$ it means the $\mathfrak{g}$-submodule generated by x, while if $x \in \mathfrak{g}$, it means the Lie subalgebra generated by x.

restricted universal enveloping algebra of a restricted Lie algebra $(\mathfrak{g}, [p])$ is denoted by $\mathfrak{u}(\mathfrak{g})$, Definition 3.2.29; more generally for a linear form $S \in \mathfrak{g}^*$, the S-reduced universal enveloping algebra of $\mathfrak{g}$ is denoted by $u(\mathfrak{g}, S)$, see Definition 3.4.5.

Inclusions are denoted by $\subseteq$ and are not assumed to be proper unless stated. Throughout, subspace means vector subspace. Direct sums are vector space direct sums, unless it is clear from context or stated otherwise. We use $\leq$ to denote the suitable notion of substructure given the context, so that $\mathfrak{h} \leq \mathfrak{g}$ denotes that the Lie algebra $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, while $W \leq V$ denotes that W is a vector subspace of V, and so on.

Chapter 1

Introduction and results

In the broad light of day
mathematicians check their equations
and their proofs, leaving no stone
unturned in their search for rigour.
But, at night, under the full moon, they
dream, they float among the stars and
wonder at the miracle of the heavens.
They are inspired. Without dreams
there is no art, no mathematics, no life.

Atiyah, *NAMS Jan 2010*, p. 8

A modular Lie algebra $\mathfrak{g}$ is simply a Lie algebra defined over a field k of positive characteristic p. Much work has gone into classifying the irreducible representations of modular Lie algebras and working out their dimensions, for example by Chang, Holmes, Koreshkov, Shen, Feldvoss, Siciliano and Weigel (Feldvoss et al., 2016; Holmes, 2001, 1998; Koreshkov, 1978; Chang, 1941; Shen, 1988a,b). However, almost all this work has been concentrated on those of restricted type, i.e., those which admit a mapping $[p] : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $\operatorname{ad} x^{[p]} = (\operatorname{ad} x)^p$ for all $x \in \mathfrak{g}$. Nonetheless, most Cartan-type modular Lie algebras are in fact non-restricted. Hence there is much left to do.

In some more detail, consider the Lie algebra $W_1 = W(1;1) = \operatorname{Der}_k(k[X]/(X^p))$, called the first Witt Algebra. It consists of all derivations (endomorphisms satisfying the product rule) of the truncated polynomial ring $k[X]/(X^p)$. Chang (1941) determined the irreducible representations with arbitrary characters of W_1. After some time of relatively little research on these algebras and their representations, Strade (1977) gave new life to their study by giving a new proof of the results of Chang via new methods.

Afterwards Shen, working with graded Lie algebras of Cartan type in particular, was able to generalise the results of Chang to the nth restricted Jacobson-Witt algebra $W_n = W(n;\underline{1})$, the Lie algebra consisting of all derivations of the truncated polynomial ring

$$k[X_1,\dots,X_n]/\left(X_1^p,\dots,X_n^p\right).$$

This algebra has basis (in divided power notation)

$$\left\{x_1^{(a_1)}x_2^{(a_2)}\cdots x_n^{(a_n)}\partial_{x_i} : 0 \le a_i \le p-1, i = 1,\dots,n\right\},$$

where ∂_{x_i} is the (special) derivation uniquely determined by the property

$$\partial_{x_i}x_1^{(a_1)}x_2^{(a_2)}\cdots x_m^{(a_m)} = x_1^{(a_1)}\cdots x_{i-1}^{(a_{i-1})}x_i^{(a_i-1)}x_{i+1}^{(a_{i+1})}\cdots x_m^{(a_m)},$$

see §4.1 for more details. We have $k\left\langle x_i\partial_{x_j} : i,j = 1,\dots,n\right\rangle \cong \mathfrak{gl}_n(k)$, see Proposition 4.1.8. For the following theorem, see Shen (1988a):

Theorem 1.0.1 (Simple restricted W_n-modules). *Let $\lambda \in \mathbb{F}_p^n$. Let $L_0(\lambda)$ be the restricted simple $\mathfrak{gl}_n$-module of highest weight λ. Then*

1. *There are p^n distinct (up to isomorphism) simple restricted W_n-modules, represented by $\left\{L(\lambda) : \lambda \in \mathbb{F}_p^n\right\}$*

2. *If λ is* not *exceptional, then $L(\lambda)$ is the induced module from $L_0(\lambda)$.*

3. *If λ is not exceptional, then $\dim_k L(\lambda) = p^n \dim_k L_0(\lambda)$, and if λ is exceptional, it is either the trivial one-dimensional module, or has dimension $\binom{n-1}{j}(p^n-1)$ for some $0 \le j < n$.*

Here, a restricted $\mathfrak{g}$-module (V,ρ) is just a $\mathfrak{g}$-module that respects the restricted structure of $\mathfrak{g}$, i.e, $\rho(x^{[p]}) = \rho(x)^p$ for all $x \in \mathfrak{g}$. As we will see, Shen's theorem classifying all the simple restricted modules is not too dissimilar from the theorem we will prove concerning restricted modules for $\widehat{H} = H(2;(1,1);\Phi(1))_{[p]}$, the minimal p-envelope of the Hamiltonian Lie algebra of Cartan type $H(2;(1,1);\Phi(1))$. There, too, the representation theory of $\mathfrak{gl}_n$ will play a foundational role (with $n = 2$ as we are concerned with a subalgebra of W_2). Also foundational will be to study induced modules and determine which weights will be exceptional and which will not be. Finally, we will give similar dimension formulas for all restricted simples over $\widehat{H}$.

Continuing our overview of the situation, Holmes (2001) went on to generalise the work of Shen and Chang and determined the simple modules of character height at most one (this includes those of character height -1, which corresponds to the restricted case) for the restricted Witt algebras W_n using a uniform approach.

Then, Holmes and Zhang (2002) undertook work to generalise the classification of simple modules with such character heights to the other three families of restricted Cartan-type Lie algebras, namely the special algebras, the Hamiltonian algebras, and the contact algebras. This work was completed by Zhang in Zhang (2002), where he dealt with the simple modules of exceptional weight.

Holmes and Zhang (2006) studied modules for the restricted Witt algebra W_n with character height greater than one, and also proved several results concerning the simplicity of induced modules of character height greater than one in restricted Cartan-type Lie algebras.

A great overview of the situation can be found in Benkart and Feldvoss (2015). Also see the work of Nakano in his monograph Nakano (1992), where he describes a fairly general setup for studying representations of Cartan-type Lie algebras. Of particular interest for us is his work on finding decompositions for Verma modules.

This thesis will focus on calculating dimensions of irreducible representations of a non-restricted Hamiltonian-type Lie algebra. We classify, then, the simple restricted modules for the Hamiltonian-type Lie algebra $H(2; (1, 1); \Phi(1))$, more precisely for its minimal p-envelope $\widehat{H}$, and give dimension formulas for all of them. Moreover, we calculate the composition factors of all restricted induced modules. This completes the rank one and rank two picture[1]; the other non-restricted Hamiltonian algebra was only recently dealt with by Feldvoss, Siciliano and Weigel in Feldvoss et al. (2016).

Apart from the intrinsic motivation to expand the understanding of the representation theory of modular Lie algebras to non-restricted Cartan-type Lie algebras, it turns out that such an understanding has played an important role in the study of maximal subalgebras of exceptional classical Lie algebras $\mathfrak{g}$ over an algebraically closed field of good characteristic, for instance, in Herpel and Stewart (2016a); Premet and Stewart (2019). In Herpel and Stewart (2016a) the authors show that for such a Lie algebra $\mathfrak{g}$, if it is simple, then any simple subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is either isomorphic to the first Witt algebra W_1 or of classical type. This result relied (among many other things) on knowledge of the restrictions of the simple modules we classify to a subalgebra isomorphic to W_1, but the argument was

[1] In the sense that it completes the description of the restricted modules for Hamiltonian algebras of absolute toral rank 1 and 2, see §6.1 for more details.

incomplete because the representation theory for $H(2;(1,1);\Phi(1))$ turned out to be more complicated than expected; for more details see Lemma 2.7, Lemma 2.9, and the proof of Theorem 1.3 at the end of §4 in Herpel and Stewart (2016a).

Our main result is Theorem 6.1.7, which gives a full description of the $p^2 - p + 1$ isomorphism classes of simple restricted $\widehat{H}$-modules.

1.1 Outline

The structure of the thesis is as follows: Chapter 2 reviews the notions from the theory of associative algebras and commutative algebra that will be needed throughout, including some important results such as the Jordan-Hölder theorem, which tells us important information concerning composition factors and composition series.

In Chapter 3 we cover the general theory of Lie algebras, defining subalgebras, ideals, quotients, homomorphisms, modules and module operations, and giving important results, as well as introducing restricted Lie algebras, restricted universal enveloping algebras, restricted representations, induced representations, filtrations, and gradations.

In Chapter 4, we look at the algebras of divided powers $O(m;\underline{n})$, for m a positive integer and $\underline{n} \in \mathbb{N}^m_{>0}$. We study their Lie algebras of derivations and certain important Lie subalgebras of these algebras of derivations, such as the generalised Jacobson-Witt algebra $W(m;\underline{n})$. These algebras will be important since the Hamiltonian algebras we look at in later chapters will be defined in relation to these algebras.

Chapter 5 continues the study of Lie algebras of derivations and defines the general family of Hamiltonians $H(2r)$ for $r \in \mathbb{N}_{>0}$ by considering certain differential forms. We then go on to define the family of graded Hamiltonian Lie algebras of Cartan type. This allows us to then define certain filtered deformations of these algebras, such as $H(2;\underline{n};\Phi(l))$ and $H(2;\underline{n};\Phi(\tau))$. We conclude by citing some classification theorems and isomorphisms between these algebras, as well as giving some explicit descriptions of them.

Chapter 6 goes on to work out the restricted simple modules in characteristic $p \geq 5$ for the Hamiltonian algebra $H(2;(1,1);\Phi(1))$, as well as determining the module structure of all the restricted induced modules.

Chapter 7 then looks at the question of restricting the simple modules found in Chapter 6 to a subalgebra of $H(2;(1,1);\Phi(1))$ isomorphic to W_1 and decomposing them in terms of the simple modules for W_1. We conclude the chapter by giving an application of this result.

In Chapter 8, we extend the work started in Chapter 6 to all characteristics (considering that the Hamiltonian Lie algebras are not defined in characteristic $p = 2$) and perform the classification of the restricted simples for $\widehat{H}$ over algebraically closed fields of characteristic $p = 3$, giving the module structure of all the restricted induced modules as well.

Chapter 2

Preliminaries

> In the elder days of art
> Builders wrought with greatest care
> Each minute and unseen part,
> For the Gods are everywhere.
>
> Longfellow, 'The Builders'

The material in this chapter concerns the basic commutative algebra and the basic theory of associative algebras and their representations underpinning the theory of modular Lie algebras. It touches only briefly at the end on some notions from homological algebra.

2.1 General algebra and commutative algebra

We refer the reader to (Etingof et al., 2011, §2.1–§2.6) for a quick introduction to the main ideas and results concerning associative algebras and their representations.

A good source for general representation theory is the classic Fulton and Harris (1991).

Definition 2.1.1. The field k is said to be *algebraically closed* if every non-constant polynomial in $k[X]$ has a root in k.

Definition 2.1.2. An *algebra* A over a field k is a vector space A over k equipped with a bilinear product $\cdot : A \times A \longrightarrow A$.

If the product is associative, that is, if

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

for all triples $a, b, c \in A$, then we say that A is an *associative algebra.*

An element $1 \in A$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in A$ is called a *unit.* Algebras having a unit element are called *algebras with unit* or *algebras with unity.*

From now on, if A is an associative algebra we will write the product $a \cdot b$ multiplicatively as ab for all $a, b \in A$.

For the rest of this section we assume that A is an associative algebra.

Example 2.1.3. The following are examples of associative algebras over k:

1. the vector space k over itself with multiplication given by the usual field multiplication;
2. the polynomial ring $k[X_1, \ldots, X_n]$ in indeterminates $X_1, X_2, \ldots, X_n$;
3. the algebra of all k-linear morphisms of any vector space V to itself, denoted $\mathrm{End}_k(V)$, with multiplication given by $fg := f \circ g$ for all $f, g \in \mathrm{End}_k(V)$;
4. the algebra of functions from a set X to k, where the multiplication is defined pointwise: $(fg)(x) := f(x)g(x)$ for all $x \in X$ and $f, g : X \longrightarrow k$.

Definition 2.1.4. We say that a subspace $B \subseteq A$ is a *subalgebra* if $BB = \{bb' : b, b' \in B\} \subseteq B$.

Definition 2.1.5. A *representation* of an associative algebra A is a vector space V equipped with an algebra homomorphism

$$\rho : A \longrightarrow \mathrm{End}_k(V).$$

The vector space V is said to be an A-module if there is a k-bilinear map:

$$\begin{aligned} \cdot : A \times V &\longrightarrow V \\ (a, v) &\mapsto a \cdot v, \end{aligned}$$

such that

$$(ab) \cdot v = a \cdot (b \cdot v)$$

for all $a, b \in A$ and all $v \in V$.

Remark. If A is an algebra with unit, then we require ρ to preserve the unit, i.e., $\rho(1) = \mathrm{id}_V$ and for modules, we require

$$1 \cdot v = v$$

for all $v \in V$.

Remark. An associative representation (V, ρ) defines an A-module, via $x \cdot v := \rho(x)(v)$. Similarly, an A-module V defines a representation $\rho : A :\longrightarrow \mathrm{End}_k(V)$ via $\rho(x)(v) := x \cdot v$. Thus, the two concepts are equivalent, and we use them interchangeably.

Remark. We will consider, unless stated otherwise, left-modules, i. e., we write $a \cdot v = \rho(a)(v)$ for the action of $a \in A$ on elements of the A-module V, but clearly one can define right-modules.

Example 2.1.6. Let V be any vector space. Define the representation ρ by letting A act trivially:

$$a \cdot v = 0$$

for all $a \in A, v \in V$.

If $V = k$ then we say that V is the *trivial representation.*

Example 2.1.7. Let $V = A$ then A acts on itself via

$$a \cdot b := ab,$$

for all $a, b \in A$, i.e. via left-multiplication. This representation is called the *regular representation* of A.

Example 2.1.8. Let $\mathfrak{S}_n$ be the symmetric group on n points. Let $V = k^n$ and let $\{e_i : 1 \leq i \leq n\}$ be the standard basis. Then V becomes a module for the group algebra $k\mathfrak{S}_n$ via $\sigma \cdot e_i := e_{\sigma(i)}$ for all $\sigma \in \mathfrak{S}_n$ and all e_i. In coordinate form:

$$\sigma \cdot (z_1, \dots, z_n) = \left(z_{\sigma^{-1}(1)}, \dots, z_{\sigma^{-1}(n)}\right).$$

This is the *natural permutation representation* of $k\mathfrak{S}_n$.

Definition 2.1.9. A subspace $W \leq V$ of a A-module V is said to be a *submodule* or *sub-representation* if $A \cdot W := \{x \cdot w : x \in A, w \in W\} \subseteq W$. The quotient space V/W acquires the structure of an A-module via

$$x \cdot (v + W) := x \cdot v + W$$

for all $x \in A, v \in V$.

A module $V \neq \{0\}$ is *simple* or *irreducible* if its only submodules are $\{0\}$ and V.

Remark. The subspace W being a submodule guarantees that the action defined above is well-defined.

Definition 2.1.10. Let V, W be A-modules. Then $\varphi : V \longrightarrow W$ is a *map of A-modules* or an *A-homomorphism* if it is linear and preserves the action of A, i.e.:

$$\varphi(x \cdot v) = x \cdot \varphi(v),$$

for all $x \in A$ and all $v \in V$.

Every associative algebra A can be regarded as a (possibly non-commutative) ring. It turns out, there is a corresponding notion of module for rings. Essentially, M is a module over the ring R if it can be regarded as a vector space over R, i.e. a vector space where one allows the scalars to come from R instead from a field.

Definition 2.1.11. Let R be a possibly non-commutative ring with unit. An additive abelian group M is said to be a *left R-module* if one has a multiplication $R \times M \longrightarrow M$ denoted by $(r, m) \mapsto r \cdot m$ satisfying for all $r, s \in R$ and all $m, n \in M$:

1. $r \cdot (m + n) = r \cdot m + r \cdot n$;
2. $(r + s) \cdot m = r \cdot m + s \cdot m$;
3. $r \cdot (s \cdot m) = (rs) \cdot m$;
4. $1 \cdot m = m$.

Remark. Clearly, we can also define right R-modules.

In other words, M is an abelian group equipped with an action of R that is linear in a suitable sense; in particular, $\mathbb{Z}$-linear.

Definition 2.1.12. Let M be an R-module, for R a possibly non-commutative ring. A set of elements $S \subseteq M$ is said to be a *basis* for M if $R\langle S \rangle = M$, i.e. if S generates M as an R-module, and if it is linearly independent.

A module M is said to be *free* if it admits a basis or if it is the zero module.

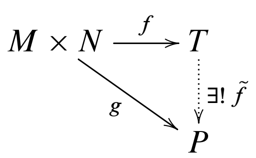
See (Lang, 2000, §3.4, pp. 135–137) for more details or see (Rotman, 2002, §7.4) for a different approach. One thing to note is that free modules can be characterised by a universal property, which allows us to see that every module is a quotient of a free module. Also see (Rotman, 2002, §7.1) for a general and detailed reference on modules.

For the notions of generation and linear independence in the context of modules, see (Lang, 2000, §3.3). It is clear that the notion of free modules only becomes of interest when dealing with modules not over fields. This is because all modules over fields, that is, all vector spaces, are free modules, given that every vector space admits a basis (assuming that the Axiom of Choice holds). Conversely, not all modules are free. For example, the factor ring $\mathbb{Z}/m\mathbb{Z}$ for $m > 1$ seen as a $\mathbb{Z}$-module is not free.

As in the case of A-modules, one can define suitable notions of submodules, quotients and homomorphisms. Having defined homomorphism, submodules, and quotients, all the usual isomorphism theorems apply to R-modules, for more details see, for instance, (Rotman, 2002, §7.1, pp. 429–431)

Definition 2.1.13. Let S be a k-algebra (ring). Let M be a right S-module and N be a left S-module. Let T be a non-empty set. A map $f : M \times N \longrightarrow T$ is *S-balanced* if $f(m \cdot s, n) = f(m, s \cdot n)$ for all $s \in S$ and $(m, n) \in M \times N$.

Definition 2.1.14. Let S be a k-algebra (ring). Let M be a right S-module and N be a left S-module. Suppose that $f : M \times N \longrightarrow T$ is an S-balanced k-bilinear map (S-bilinear map), for T a k-vector space (S-module). The pair (T, f) is a *tensor product of M and N over S* if for all k-vector spaces P (S-modules P) and all S-balanced k-bilinear maps (S-bilinear maps) $g : M \times N \longrightarrow P$ there is a unique linear map $\tilde{f} : T \longrightarrow P$ such that the following diagram



commutes.

Remark. Thanks to the universal property of the tensor product, any two tensor products are isomorphic. We write $M \otimes_S N$ for T and $m \otimes n$ for $f(m, n)$ for all $(m, n) \in M \times N$. Furthermore, $M \otimes_S N$ has basis $\{m \otimes n : m \in M, n \in N\}$.

An important special case of this definition arises when $S = k$ and M and N are simply k-vector spaces. In that case, we see that $M \otimes_k N$ is a k-vector space with basis $\{m \otimes n : m \in M, n \in N\}$ such that $\lambda(m \otimes n) = \lambda m \otimes n = m \otimes \lambda n$, $(m + m') \otimes n = m \otimes n + m' \otimes n$, $m \otimes (n + n') = m \otimes n + m \otimes n'$ for all $\lambda \in k, m, m' \in M, n, n' \in N$.

Definition 2.1.15. Let S and R be k-algebras (rings). Let M be a left R-module and a right S-module. We say that M is an *(R, S)-bimodule* if $r \cdot (m \cdot s) = (r \cdot m) \cdot s$ for all $r \in R, m \in M, s \in S$.

For a proof of the following see (Strade and Farnsteiner, 1988, §5.6, Lemma 6.1, p. 226).

Lemma 2.1.16. *Let S and R be k-algebras (rings). Let M be an (R,S)-bimodule and N be a left S-module. There is an R-module structure on $M \otimes_S N$ given by $r \cdot (m \otimes n) = r \cdot m \otimes n$ for all $r \in R$ and $(m,n) \in M \times N$.*

Definition 2.1.17. Let V be a k-vector space. The *tensor algebra* of V (seen as a k-vector space) is

$$T(V) = \bigoplus_{i=0}^{\infty} V^{\otimes i},$$

where $V^{\otimes 0} = k$, $V^{\otimes 1} = V$ and

$$V^{\otimes i} = \underbrace{V \otimes_k V \otimes_k \cdots \otimes_k V}_{i \text{ times}} = \bigotimes^{i} V.$$

This becomes an algebra by defining multiplication on the subspaces $V^{\otimes i} \times V^{\otimes l} \longrightarrow V^{\otimes i+l}$ thus

$$(v_1 \otimes \cdots \otimes v_i)(w_1 \otimes \cdots \otimes w_l) = v_1 \otimes \cdots \otimes v_i \otimes w_1 \otimes \cdots \otimes w_l,$$

and extending linearly. We refer the reader to (Lang, 2000, §16.7) or (Eisenbud, 1995, A2.2 –A2.3) for more details.

Definition 2.1.18. The *exterior algebra* of V, denoted $\bigwedge(V)$, is defined by

$$\bigwedge(V) = T(V)/\mathfrak{a},$$

where $\mathfrak{a} := (x \otimes x : x \in V)$.

Remark. There are several other ways to define this algebra, see (Lang, 2000, §19.1) for one such way. Amongst others, one can also define it as a special subspace of $T(V)$ consisting of what are called *antisymmetric tensors*. In positive characteristic, however, one must be careful. If p divides $n!$, then one cannot identify the antisymmetric tensors in $V^{\otimes n}$ with the quotient $\bigwedge^n(V)$. The exterior algebra is also sometimes called the *alternating algebra*, or the *Grassmann algebra*.

The previous constructions can be also made where one takes V to be an R-module, where R is some commutative ring. The resulting exterior algebra has the structure of an R-module:

$$r \cdot (v_1 \wedge v_2 \wedge \cdots \wedge v_n) = r \cdot v_1 \wedge v_2 \wedge \cdots \wedge v_n = v_1 \wedge r \cdot v_2 \wedge \cdots \wedge v_n = \cdots,$$

and so on.

Definition 2.1.19. Let M be an R-module for a possibly non-commutative ring R. A *composition series* for M is a sequence of submodules

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

where the inclusions are proper, and the M_{i+1}/M_i are all simple.

The *length* of the composition series is then n. The factor modules M_{i+1}/M_i are called the *composition factors* of the series.

Remark. Recall that one can give a module structure to the quotient M/N (seen here as the set of N-cosets) as long as N is a submodule of M.

The well-known Jordan-Hölder theorem tells us in the case of modules that (see (Curtis and Reiner, 1981, 3.11)):

Theorem 2.1.20 (Jordan-Hölder). *If a module A has a composition series, then any two composition series are equivalent, that is, the number of occurrences of each isomorphism type of simple A-module as a composition factor does not depend on the choice of composition series.*

In particular, since everything we work with in this thesis will be finite-dimensional, we have:

Theorem 2.1.21. *Over any field k, any finite-dimensional module M for a finite-dimensional k-algebra A has a composition series, unique up to equivalence.*

We write $[M_1, M_2, \ldots, M_n] = [V]$ for the list of composition factors of V a module.

2.2 Homological algebra

Since this thesis does not deal with notions from homological algebra directly, we shall only recall some of the notions used. For a more detailed exposition, we refer the reader to Rotman's book: Rotman (2009); also see (Rotman, 2002, §10) and Hilton and Stammbach (1997).

DEFINITION 2.2.1. A module P is *projective* if for every surjective module homomorphism $f : N \longrightarrow M$ and every module homomorphism $g : P \longrightarrow M$, there exists a homomorphism $h : P \longrightarrow N$ such that $f \circ h = g$, i.e., the following diagram commutes:

$$\begin{array}{ccc} P & \overset{\exists h}{\dashrightarrow} & N \\ & {\scriptstyle g}\searrow & \downarrow {\scriptstyle f} \\ & & M \end{array}$$

In other words, *every morphism from P to M factors through every epimorphism to M*.

Remark. An alternative but equivalent definition can be made as follows: A module P is projective if every short exact sequence of modules of the form

$$0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} P \longrightarrow 0$$

in fact splits. In other words, every surjective module homomorphism $f : B \longrightarrow P$ admits a section map, that is, a module homomorphism $h : P \longrightarrow B$ such that $f \circ h = \mathrm{id}_P$. In this case, we have $B \cong A \oplus P$.

Remark. If one dualises the definitions, then one obtains the notion of *injective* modules.

We will briefly treat the Ext functor, but we will state the definitions for the case we will be concerned with only. Let $\mathfrak{g}$ be a Lie algebra and let $u(\mathfrak{g}, S)$ be its S-reduced universal enveloping algebra, an associative k-algebra with unit (see Definition 3.4.5). Let M be a $u(\mathfrak{g}, S)$-module. Then we define ith right-derived functor

$$\mathrm{Ext}^i_{u(\mathfrak{g},S)}(M, -) = R^i \operatorname{Hom}_{u(\mathfrak{g},S)}(M, -).$$

In the case $\mathrm{Ext}^1_{u(\mathfrak{g},S)}(M, N) = \mathrm{Ext}^1(M, N)$, we have (see (Hilton and Stammbach, 1997, §3.2, Thm. 2.4)):

THEOREM 2.2.2. *Let M and N be $u(\mathfrak{g}, S)$-modules. Then $\mathrm{Ext}^1(M, N)$ can be interpreted as the group of extensions with elements short exact sequences:*

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0\,.$$

For the following proposition, see Exercise 2.5.1 and Exercise 2.5.2 in (Weibel, 1994, §2.5, p. 50):

Proposition 2.2.3. *We have that if the $u(\mathfrak{g}, S)$-module M is projective, then $\mathrm{Ext}^i_{u(\mathfrak{g},S)}(M, N) = 0$ for all $i > 0$. Similarly, if N is injective, then $\mathrm{Ext}^i_{u(\mathfrak{g},S)}(M, N) = 0$ for all $i > 0$.*

Chapter 3

General theory

> … la mathématique est l'art de donner
> le même nom à des choses différentes.
>
> ---
> Henri Poincaré, *Science et méthode*

In this chapter we recall the basic notions from the classical theory of Lie algebras and introduce in some detail the rudiments of the modular theory. We cover important results we will use later, both in terms of the structure theory of modular Lie algebras and their representations. The crucial notion of a p-mapping is introduced, setting the stage to study restricted Lie algebras and their restricted representations (those representations that respect the restricted structure). We include a brief discussion of graded and filtered algebras. Finally, the notion of induced representations for Lie algebras is introduced, together with the principle of Frobenius reciprocity.

3.1 Classical theory and basic operations

We refer the reader to Humphreys (1980) for a good introduction to the classical theory of Lie algebras as well as to Chapter 1 of Strade and Farnsteiner (1988). Other good references at a higher level of sophistication are Milne (2013), see in particular §I.1 for most of the material covered in this section, (Serre, 2006, §I.I, §I.III, §I.V, §I.VII) and (Bourbaki, 1975, Chap. 1). See (Rotman, 2002, §9.10) for a very quick overview.

Definition 3.1.1. Let k be a field. A *Lie algebra* $\mathfrak{g}$ is a k-vector space equipped with a k-bilinear bracket $[,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, such that for all $x, y, z \in \mathfrak{g}$:

1. $[x, x] = 0$;

2. $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$ or, equivalently

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Remark. Using (1) and expanding $[x + y, x + y]$ we obtain that $[x, y] = -[y, x]$. Equation (2) is called the *Jacobi identity*. The *dimension* of a Lie algebra is simply its dimension as a k-vector space.

Example 3.1.2. On any vector space V over k define all the Lie brackets to be zero: $[x, x] = 0$ for all $x \in V$. Then V is a Lie algebra.

Example 3.1.3. On a three-dimensional vector space $\mathfrak{g} := kx \oplus ky \oplus kz$ define the Lie bracket $[x, y] = z$, $[x, z] = 0 = [y, z]$. Then $\mathfrak{g}$ is a Lie algebra. It is often referred to as the *Heisenberg Lie algebra*.

Example 3.1.4. On the k-vector space $ke \oplus kh \oplus kf$ define the Lie bracket

$$[h, e] = 2e, \;\; [h, f] = -2f, \;\; [e, f] = h.$$

This is the Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(k)$.

Example 3.1.5 (Diamond algebra). Let $\mathfrak{g} = kt \oplus kx \oplus ky \oplus kz$. Define $[z, \mathfrak{g}] = 0$ and

$$[t, x] = x, \;\; [t, y] = -y, \;\; [x, y] = z.$$

Then $\mathfrak{g}$ is a Lie algebra.

Definition 3.1.6. Let $\mathfrak{h}, \mathfrak{g}$ be Lie algebras. A k-linear map $f : \mathfrak{h} \longrightarrow \mathfrak{g}$ is a *Lie algebra homomorphism* if

$$f([x, y]) = [f(x), f(y)]$$

for all $x, y \in \mathfrak{h}$.

If f is invertible, then f is an *isomorphism* and we say that $\mathfrak{h}$ and $\mathfrak{g}$ are *isomorphic*, which we denote by $\mathfrak{h} \cong \mathfrak{g}$.

Example 3.1.7. On a two-dimensional vector space $\mathfrak{g} := kx \oplus ky$ define the Lie bracket $[x, y] = x$. Then $\mathfrak{g}$ is a Lie algebra, and it is up to isomorphism the only noncommutative Lie algebra of dimension 2 (see Definition 3.1.18 for the definition of commutative Lie algebras).

Example 3.1.8. Note that if k is a field of characteristic 2, then $\mathfrak{sl}_2(k)$ is isomorphic to the Lie algebra in Example 3.1.3.

Definition 3.1.9. Any triple (e, h, f) in $\mathfrak{g}$ such that $k\langle e, h, f\rangle \cong \mathfrak{sl}_2(k)$ and

$$[h, e] = 2e, \ [h, f] = -2f, \ [e, f] = h$$

will be called an $\mathfrak{sl}_2$*-triple.*

Given an associative k-algebra A, we can define a Lie product or bracket

$$[a, b] = ab - ba,$$

which turns A into a Lie algebra. We denote this Lie algebra by A^-.

An important such algebra is the *general linear Lie algebra* $\mathfrak{gl}(V) := \mathrm{End}_k(V)^-$, where $\mathrm{End}_k(V)$ is the algebra of all k-linear endomorphisms of V a vector space over k.

Example 3.1.10. If A is the associative matrix algebra $M_n(k)$ of all $n \times n$ matrices over k, then A^- is the Lie algebra $\mathfrak{gl}_n = \mathfrak{gl}_n(k)$. It is isomorphic to $\mathfrak{gl}(V)$ if $\dim_k V = n$. It has dimension n^2.

Let $E_{i,j}$ be the $n \times n$ matrix with entries

$$\left(E_{i,j}\right)_{kl} = \delta_{ik}\delta_{jl},$$

where δ_{ij} is the Kronecker delta, which is 0 if $i \neq j$ and 1 if $i = j$ (so $E_{i,j}$ is the matrix with a 1 in the (ij)th entry and zeroes everywhere else). The following, then, is a basis for $\mathfrak{gl}_n$:

$$\left\{E_{i,j} : i, j \in \{1, \ldots, n\}\right\}.$$

It's easy to check that the basis elements satisfy:

$$[E_{i,j}, E_{r,s}] = \delta_{jr} E_{i,s} - \delta_{si} E_{r,j}.$$

Subalgebras of the Lie algebra A^- of an associative algebra A are called *commutator algebras.*

Subalgebras of $\mathfrak{gl}(V)$ are often called *linear Lie algebras* or *Lie algebras of linear transformations.*

We know that every Lie algebra is a commutator algebra, for example it embeds into its universal enveloping algebra, see Theorem 3.1.58. However, that does not mean that every Lie algebra is isomorphic to an associative algebra equipped with the commutator bracket. The following proposition gives an example of a Lie algebra that fails to have this property.

Proposition 3.1.11. *Let k be a field of characteristic 0 or $p \geq 3$. There does not exist an associative algebra A such that $A^- \cong \mathfrak{sl}_2(k)$ as Lie algebras.*

Proof. Indeed, suppose that $\mathfrak{sl}_2$ is isomorphic to an associative algebra A equipped with the commutator bracket. Then we can pick an $\mathfrak{sl}_2$-triple (E, H, F) in A. The elements EH, HE, HF, FH, EF, FE of A can all be written as k-linear combinations $\alpha E + \beta H + \gamma F$. Using the associativity of A and the Lie bracket, we can show, with some work, that in fact $EH, HE \in k\langle E\rangle$, $FH, HF \in k\langle F\rangle$, and $EF, FE \in k\langle H\rangle$. Write $EH = \alpha_1 E, HE = \alpha_3 E, HF = \gamma_2 F, FH = \gamma_4 F$, and $EF = \beta_5 H, FE = \beta_6 H$. By considering the equation $(FE)F = F(EF)$, we can show that $\alpha_1 = -2\beta_5$. By considering other such equations, we can show that $\alpha_3 = -2\beta_6$ and that $\alpha_1 = \gamma_2$ and $\alpha_3 = \gamma_4$.

We have $(EF)(HF) = (E(FH))F$, so

$$\beta_5 \gamma_2^2 F = \gamma_4 \beta_5 \gamma_2 F.$$

Hence, $\beta_5(\gamma_2^2 - \gamma_4\gamma_2) = 0$. Since $HF - FH = -2F$, we have $\gamma_2 - \gamma_4 = -2$, and so $\beta_5(\gamma_2^2 - (\gamma_2 + 2)\gamma_2) = 0$. Thus, $\beta_5(-2\gamma_2) = 0$. Therefore $\beta_5 = 0$ or $\gamma_2 = 0$.

Both possibilities lead to $\gamma_2 = 0, \gamma_4 = 2$ and $\beta_5 = 0, \beta_6 = -1$. Hence, $HF = 0, FH = 2F$ and $EF = 0, FE = -H$. Thus, $0 = F(HF)E = (FH)(FE) = -2FH = -4F$. Therefore, $4 = 0$, a contradiction. Thus, $\mathfrak{sl}_2$ is *not* isomorphic to any associative algebra equipped with the commutator bracket. ∎

Definition 3.1.12. The set of all traceless matrices in $\mathfrak{gl}_n(k)$ is closed under the Lie bracket. This Lie algebra is called $\mathfrak{sl}_n(k) = \mathfrak{sl}_n$. It is of dimension $n^2 - 1$.

Define

$$H_{i,i+1} = E_{i,i} - E_{i+1,i+1}.$$

Then Lie algebra $\mathfrak{sl}_n$ admits the following nice basis:

$$\left\{H_{i,i+1}, E_{i,j} : i \neq j\right\}.$$

Example 3.1.13. The Lie algebra $\mathfrak{sl}_2$ has basis $\{H_{1,2}, E_{1,2}, E_{2,1}\}$, where

$$H_{1,2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} E_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is easy to check that $[H_{1,2}, E_{1,2}] = 2E_{1,2}, [H_{1,2}, E_{2,1}] = -2E_{2,1}, [E_{1,2}, E_{2,1}] = H_{1,2}$. Thus $(E_{1,2}, H_{1,2}, E_{2,1})$ is an $\mathfrak{sl}_2$-triple, and the two definitions of $\mathfrak{sl}_2$ do coincide.

Definition 3.1.14. Consider the $2r \times 2r$ matrix

$$S := \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}.$$

Then

$$\mathfrak{sp}_{2r}(k) := \left\{A \in \mathfrak{gl}_{2r}(k) : SA = -A^T S\right\}$$

is a Lie subalgebra of $\mathfrak{sl}_{2r}(k)$. It is called the *symplectic Lie algebra.*

The following proposition can be found in (Strade and Farnsteiner, 1988, §4.4, Exercise 2, p. 168).

Proposition 3.1.15. *The Lie algebra $\mathfrak{sp}_{2r}(k)$ has dimension $2r^2 + r$ and it is simple.*

Definition 3.1.16. For any Lie algebra $\mathfrak{g}$ define its *derived subalgebra* to be

$$[\mathfrak{g}, \mathfrak{g}] = k\langle\{[x, y] : x, y \in \mathfrak{g}\}\rangle.$$

Remark. In general if $\mathfrak{s}, \mathfrak{t} \subseteq \mathfrak{g}$, we denote by

$$[\mathfrak{s}, \mathfrak{t}]$$

the subspace of $\mathfrak{g}$ spanned by the brackets $[x, y]$ with $x \in \mathfrak{s}$ and $y \in \mathfrak{t}$.

Remark. We will often drop the set brackets in notation such as the above and simply write $k\langle[x, y] : x, y \in \mathfrak{g}\rangle$, for instance.

Example 3.1.17. We have $[\mathfrak{gl}_n(k), \mathfrak{gl}_n(k)] \cong \mathfrak{sl}_n(k)$.

Definition 3.1.18. We say that a subspace $\mathfrak{j} \subseteq \mathfrak{g}$ is a *subalgebra* if $[\mathfrak{j}, \mathfrak{j}] \subseteq \mathfrak{j}$.

We say a subspace $\mathfrak{j}$ is an *ideal* if $[\mathfrak{j}, \mathfrak{g}] \subseteq \mathfrak{j}$. A *simple* Lie algebra has no non-trivial ideals (that is, the only ideals are $\{0\}$ and $\mathfrak{g}$).

If $\mathfrak{j} \triangleleft \mathfrak{g}$ is an ideal, the vector space $\mathfrak{g}/\mathfrak{j}$ has the structure of a Lie algebra with Lie bracket

$$[x + \mathfrak{j}, y + \mathfrak{j}] := [x, y] + \mathfrak{j}.$$

The *centre* $C(\mathfrak{g})$ of a Lie algebra is the set $\{x \in \mathfrak{g} : [x, \mathfrak{g}] = 0\}$. It is an ideal of $\mathfrak{g}$. A Lie algebra is called *abelian* or *commutative* if $C(\mathfrak{g}) = \mathfrak{g}$. Generalising, we define the centraliser of a subset $\mathfrak{s} \subseteq \mathfrak{g}$ to be

$$C_{\mathfrak{g}}(\mathfrak{s}) = \{x \in \mathfrak{g} : [x, \mathfrak{s}] = 0\}.$$

Remark. If $\mathfrak{i}, \mathfrak{j}$ are ideals of $\mathfrak{g}$, then $[\mathfrak{i}, \mathfrak{j}]$ is an ideal of $\mathfrak{g}$.

Example 3.1.19.

1. The Lie algebras in Example 3.1.2 are all abelian.
2. The Lie algebra $\mathfrak{sl}_2(k)$ is simple if k has characteristic $p \geq 3$. If $p = 2$, then one has $[h, e] = 0 = [h, f]$, so kh is a non-trivial ideal of $\mathfrak{sl}_2(k)$.
3. In characteristic $p \geq 3$, $C(\mathfrak{sl}_2(k)) = \{0\}$, while if $p = 2$, $C(\mathfrak{sl}_2(k)) = kh$.

As in the case of modules, having defined homomorphisms, ideals, and quotients, we remark that all the usual isomorphism theorems apply to Lie algebras, for more details see, for instance, (Milne, 2013, §I.1, p. 15).

Definition 3.1.20. A *representation* of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V),$$

where V is a k-vector space.

The vector space V is said to be a $\mathfrak{g}$-*module* if there is a k-bilinear map

$$\begin{aligned} \cdot : \mathfrak{g} \times V &\longrightarrow V \\ (x, v) &\mapsto x \cdot v, \end{aligned}$$

such that

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$$

for all $x, y \in \mathfrak{g}$ and all $v \in V$.

Remark. As in the case of associative algebras, representations and modules are equivalent, and we use the language of $\mathfrak{g}$-modules interchangeably with that of representations throughout. Indeed, a representation defines a $\mathfrak{g}$-module, via $x \cdot v := \rho(x)(v)$. Similarly, a $\mathfrak{g}$-module defines a representation $\rho : \mathfrak{g} :\longrightarrow \mathfrak{gl}(V)$ via $\rho(x)(v) := x \cdot v$.

Example 3.1.21 (Adjoint Map). For all $x \in \mathfrak{g}$ define the map

$$\begin{aligned} \operatorname{ad} x : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ y &\mapsto [x, y]. \end{aligned}$$

We call $\operatorname{ad} x$ the *adjoint map* of x. It is straightforward to verify that $(\mathfrak{g}, \operatorname{ad})$ is a $\mathfrak{g}$-module. We call this representation the *adjoint representation.*

Remark. Note that the image $\operatorname{ad}(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$.

Example 3.1.22 (Trivial Representation). On the one-dimensional vector space k define the action $x \cdot v = 0$ for all $x \in \mathfrak{g}, v \in k$. This turns k into a $\mathfrak{g}$-module, the *trivial module* or *trivial representation.*

Example 3.1.23. Consider the action of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathbb{C}[x, y]$ given by:

$$\begin{aligned} \rho(e) &= x \frac{\partial}{\partial_y} \\ \rho(h) &= x \frac{\partial}{\partial_x} - y \frac{\partial}{\partial_y} \\ \rho(f) &= y \frac{\partial}{\partial_x}. \end{aligned}$$

This turns $\mathbb{C}[x, y]$ into an $\mathfrak{sl}_2(\mathbb{C})$-module. Similarly, we can make the polynomial algebra $k[x, y]$ into an $\mathfrak{sl}_2(k)$-module.

Example 3.1.24. Recall the Lie algebra $\mathfrak{g}$ in Example 3.1.3, the Heisenberg Lie algebra. The map $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}_3(k)$ defined by

$$\begin{aligned} \rho(x) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \rho(y) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ \rho(z) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

is a representation of $\mathfrak{g}$, since one can directly compute that $[\rho(x), \rho(y)] = \rho(z)$ and $[\rho(x), \rho(z)] = [\rho(y), \rho(z)] = 0$, the zero matrix.

Definition 3.1.25. A subspace $W \leq V$ of a $\mathfrak{g}$-module V is said to be a *submodule* or *sub-representation* if $\mathfrak{g} \cdot W := \{x \cdot w : x \in \mathfrak{g}, w \in W\} \subseteq W$. The quotient space V/W acquires the structure of a $\mathfrak{g}$-module via

$$x \cdot (v + W) := x \cdot v + W.$$

A module $V \neq \{0\}$ is *simple* or *irreducible* if its only submodules are $\{0\}$ and V.

A representation is said to be *faithful* if it is injective.

Example 3.1.26. Consider the span Γ_l of the monomials of degree l in $k[x, y]$. Since the action of $\mathfrak{sl}_2$ given in Example 3.1.23 preserves Γ_l, we see that the Γ_l are finite-dimensional $\mathfrak{sl}_2$-submodules (of dimension $l + 1$) of $k[x, y]$. When $k = \mathbb{C}$, more is true. In fact, one can prove that if V is a finite-dimensional complex simple $\mathfrak{sl}_2$-module, then $V \cong \Gamma_l$, where $l = \dim_k V - 1$ (see, for example, (Fulton and Harris, 1991, §11.1, pp. 146–150)).

Example 3.1.27. The adjoint representation is faithful if and only if $\mathfrak{g}$ is centreless.

Example 3.1.28. The representation in Example 3.1.24 of the Heisenberg Lie algebra is clearly faithful.

Definition 3.1.29. Let V, W be $\mathfrak{g}$-modules. Then $\varphi : V \longrightarrow W$ is a *map of* $\mathfrak{g}$*-modules* or $\mathfrak{g}$*-homomorphism* if it is linear and preserves the action of $\mathfrak{g}$, i.e.:

$$\varphi(x \cdot v) = x \cdot \varphi(v),$$

for all $x \in \mathfrak{g}$ and all $v \in V$.

Definition 3.1.30. A *derivation* of an algebra A over k is a k-linear map $D : A \longrightarrow A$ such that

$$D(ab) = D(a)b + aD(b)$$

for all $a, b \in A$. The set of all derivations of a algebra, denoted $\mathrm{Der}_k(A)$, is a Lie subalgebra of $\mathfrak{gl}(A)$. We call $\mathrm{Der}_k(A)$ the *algebra of derivations of* A.

An ideal $\mathfrak{i} \lhd \mathfrak{g}$ stable under all derivations of $\mathfrak{g}$ is said to be a *characteristic ideal.*

Remark. It is a direct computation to verify that $[D, E] = D \circ E - E \circ D$ is a derivation for all $D, E \in \mathrm{Der}_k(A)$.

Example 3.1.21 together with the Jacobi identity shows that $\operatorname{ad}\mathfrak{g}$ is in fact a Lie subalgebra of $\operatorname{Der}_k(\mathfrak{g})$, often called the subalgebra of *inner derivations*. In fact, the fact that $\operatorname{ad} x$ is a derivation for all $x \in \mathfrak{g}$ provides an easy way to memorise the Jacobi identity:

$$[x,[y,z]] = \operatorname{ad}(x)([y,z]) = [\operatorname{ad}(x)(y),z] + [y,\operatorname{ad}(x)(z)] = [[x,y],z] + [y,[x,z]].$$

Example 3.1.31. The algebra A can be seen as a $\operatorname{Der}_k(A)$-module, by setting $D \cdot a = D(a)$ for all $D \in \operatorname{Der}_k(A), a \in A$, since $\operatorname{Der}_k(A) \leq \mathfrak{gl}(A)$.

Definition 3.1.32. Let V, W be $\mathfrak{g}$-modules. Then the tensor product $V \otimes_k W$ obtains the structure of a $\mathfrak{g}$-module via

$$x \cdot (v \otimes w) = x \cdot v \otimes w + v \otimes x \cdot w.$$

Example 3.1.33. Note that $\mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{gl}_2(\mathbb{C}) \hookrightarrow \mathfrak{gl}(V)$, with V a two-dimensional complex vector space. The module V is the natural $\mathfrak{sl}_2(\mathbb{C})$-module. We will now describe $V \otimes_k V$. Take $V = \mathbb{C}^2$ and pick the standard basis $\{v_1, v_2\}$, where $v_1 = \binom{1}{0}$ and $v_1 = \binom{0}{1}$, so that $\mathfrak{sl}_2(\mathbb{C})$ acts on $\mathbb{C}^2$ via the matrices $E_{1,2}, E_{2,1}, H_{1,2}$, representing the action of e, f, and h, respectively.

Now, $V \otimes_k V$ has basis $\{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\}$. We calculate

$$\begin{aligned}
e \cdot (v_1 \otimes v_1) &= 0 \\
e \cdot (v_1 \otimes v_2) &= v_1 \otimes v_1 \\
e \cdot (v_2 \otimes v_1) &= v_1 \otimes v_1 \\
e \cdot (v_2 \otimes v_2) &= v_1 \otimes v_2 + v_2 \otimes v_1.
\end{aligned}$$

Likewise,

$$\begin{aligned}
f \cdot (v_1 \otimes v_1) &= v_1 \otimes v_2 + v_2 \otimes v_1 \\
f \cdot (v_1 \otimes v_2) &= v_2 \otimes v_2 \\
f \cdot (v_2 \otimes v_1) &= v_2 \otimes v_2 \\
f \cdot (v_2 \otimes v_2) &= 0.
\end{aligned}$$

Finally,

$$h \cdot (v_1 \otimes v_1) = 2v_1 \otimes v_1$$
$$h \cdot (v_1 \otimes v_2) = 0$$
$$h \cdot (v_2 \otimes v_1) = 0$$
$$h \cdot (v_2 \otimes v_2) = -2v_2 \otimes v_2.$$

Note that $v_1 \otimes v_1$ is a weight vector for h of weight 2 and is killed by e. The representation theory of $\mathfrak{sl}_2(\mathbb{C})$ tells us that the vector $v_1 \otimes v_1$ is a maximal vector and thus generates under f a simple submodule of dimension 3 isomorphic to Γ_2. We calculate this submodule to be $W := \mathbb{C}\langle v_1 \otimes v_1, v_1 \otimes v_2 + v_2 \otimes v_1, v_2 \otimes v_2\rangle$. Finally, note that $W' := \mathbb{C}\langle v_1 \otimes v_2 - v_2 \otimes v_1\rangle$ is killed by $\mathfrak{sl}_2(\mathbb{C})$ and is a vector space complement to W. It is the one-dimensional trivial $\mathfrak{sl}_2(\mathbb{C})$-module, isomorphic to Γ_0. Thus $V \otimes_k V = W \oplus W' \cong \Gamma_2 \oplus \Gamma_0$.

For the following proposition see (Strade and Farnsteiner, 1988, §1.2, Proposition 2.3, p. 12):

Proposition 3.1.34. *Let V, W be $\mathfrak{g}$-modules. Then the vector space $\mathrm{Hom}_k(V, W)$ obtains the structure of a $\mathfrak{g}$-module via*

$$(x \cdot f)(v) = x \cdot f(v) - f(x \cdot v)$$

Remark. We observe that $f \in \mathrm{Hom}_k(V, W)$ is a homomorphism of $\mathfrak{g}$-modules if, and only if, $x \cdot f = 0$. So, $\mathrm{Hom}_{\mathfrak{g}}(V, W)$, the space of $\mathfrak{g}$-homomorphisms between V and W, is precisely the space of $\mathfrak{g}$-invariants of $\mathrm{Hom}_k(V, W)$, i.e. $\mathrm{Hom}_{\mathfrak{g}}(V, W) = \{f \in \mathrm{Hom}_k(V, W) : \mathfrak{g} \cdot f = 0\}$.

Example 3.1.35. If one takes $W = k$ the trivial representation, then $\mathrm{Hom}_k(V, W) \cong V^*$. Thus, we see that the dual vector space is a $\mathfrak{g}$-module with module structure defined by $(x \cdot \varphi)(v) = -\varphi(x \cdot v)$ for all $\varphi \in V^*, x \in \mathfrak{g}, v \in V$.

See (Bourbaki, 1975, Chap. I, §3) for a more in-depth treatment on tensor products of representations and homomorphism modules.

For more details on the material that follows, see, for example, (Erdmann and Wildon, 2006, §4.1–4.2, §6) or (Bourbaki, 1975, Chap. I, §4–5).

Definition 3.1.36. Let $\mathfrak{g}$ be a Lie algebra. Define the sequence of subspaces

$$\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}], \ldots, \mathfrak{g}_i = [\mathfrak{g}_{i-1}, \mathfrak{g}], \ldots$$

Note that we have the following inclusions:

$$\mathfrak{g} \supseteq \mathfrak{g}_1 \supseteq \cdots \supseteq \mathfrak{g}_i \supseteq \cdots$$

We say that $\mathfrak{g}$ is *nilpotent* if there is an $i \in \mathbb{N}$ such that $\mathfrak{g}_i = \{0\}$.

Example 3.1.37. Consider $\mathfrak{g} = \mathfrak{sl}_2(k)$ in characteristic 2. Then $\mathfrak{g}_1 = kh$ and $\mathfrak{g}_2 = \{0\}$, so $\mathfrak{sl}_2$ is nilpotent.

Similarly we have:

Definition 3.1.38. Let $\mathfrak{g}$ be a Lie algebra. Define the sequence of subspaces

$$\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}], \dots, \mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}], \dots$$

Note that we have the following inclusions:

$$\mathfrak{g} \supseteq \mathfrak{g}^{(1)} \supseteq \cdots \supseteq \mathfrak{g}^{(i)} \supseteq \cdots$$

We say that $\mathfrak{g}$ is *solvable* if there is an $i \in \mathbb{N}$ such that $\mathfrak{g}^{(i)} = \{0\}$.

Remark. Any nilpotent Lie algebra $\mathfrak{g}$ is solvable, since we have $\mathfrak{g}^{(i)} \subseteq \mathfrak{g}_i$.

Remark. The algebras $\mathfrak{g}^{(i)}$ and $\mathfrak{g}_i$ are ideals of $\mathfrak{g}$. The algebras $\mathfrak{g}^{(i)}$ are ideals in $\mathfrak{g}^{(i-1)}$. Also, note that $\mathfrak{g}^{(i)}$ is the i-th derived subalgebra of $\mathfrak{g}$.

Example 3.1.39. Consider the space of upper triangular matrices in $\mathfrak{gl}_2(k)$. It has basis $\{E_{1,1}, E_{1,2}, E_{2,2}\}$. The Lie bracket satisfies $[E_{1,1}, E_{2,2}] = 0, [E_{1,2}, E_{1,1}] = -E_{1,2}, [E_{1,2}, E_{2,2}] = E_{1,2}$. Thus, $\mathfrak{g} = k\langle E_{1,1}, E_{1,2}, E_{2,2}\rangle$ is a Lie algebra. Clearly, $\mathfrak{g}_1 = \mathfrak{g}^{(1)} = k\langle E_{1,2}\rangle$. Therefore, $\mathfrak{g}^{(2)} = \{0\}$, so $\mathfrak{g}$ is solvable. However, $\mathfrak{g}_2 = [k\langle E_{1,2}\rangle, \mathfrak{g}] = k\langle E_{1,2}\rangle$. Thus, $\mathfrak{g}$ is not nilpotent.

Proposition 3.1.40. *The ideals $\mathfrak{g}^{(i)}$ and $\mathfrak{g}_i$ are in fact characteristic ideals of $\mathfrak{g}$.*

Proof. We proceed by induction. Clearly $\mathfrak{g}^{(1)} = \mathfrak{g}_1$ is stable under all derivations of $\mathfrak{g}$. Suppose that $\mathfrak{g}^{(i)}$ is stable under all derivations for some $i > 0$. Let $D \in \mathrm{Der}_k(\mathfrak{g})$. We have $D(\mathfrak{g}^{(i+1)}) = D([\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]) = [D(\mathfrak{g}^{(i)}), \mathfrak{g}^{(i)}] + [\mathfrak{g}^{(i)}, D(\mathfrak{g}^{(i)})] \subseteq [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] + [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] = \mathfrak{g}^{(i+1')} + \mathfrak{g}^{(i+1)} \subseteq \mathfrak{g}^{(i+1)}$, as required. Similarly, suppose $\mathfrak{g}_j$ is stable under all derivations for some $j > 0$. We have $D(\mathfrak{g}_{j+1}) = D([\mathfrak{g}_j, \mathfrak{g}]) = [D(\mathfrak{g}_j), \mathfrak{g}] + [\mathfrak{g}_j, D(\mathfrak{g})] \subseteq [\mathfrak{g}_j, \mathfrak{g}] + [\mathfrak{g}_j, \mathfrak{g}] = \mathfrak{g}_{j+1} + \mathfrak{g}_{j+1} \subseteq \mathfrak{g}_{j+1}$, as required. ■

For the following theorem see (Bourbaki, 1975, Chap. I, §4.2, p. 39):

Theorem 3.1.41 (Engel's Theorem). *Let k be an arbitrary field. Let $\mathfrak{g} \subseteq \mathfrak{gl}(V) = \mathrm{End}_k(V)^-$ be a finite-dimensional Lie algebra such that for all $X \in \mathfrak{g}$ there is an $N \in \mathbb{N}$ such that $X^N = 0$. Then there is a non-zero vector $v \in V$ such that $Yv = 0$ for all $Y \in \mathfrak{g}$.*

For the following theorem see (Humphreys, 1980, §4.1, p. 15):

Theorem 3.1.42 (Lie's Theorem). *Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a solvable Lie algebra, where V is over $\mathbb{C}$. Then there is a vector $v \in V$ such that v is a common eigenvector for all $X \in \mathfrak{g}$.*

Definition 3.1.43. Let ρ be a representation of a Lie algebra $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. The *Killing form with respect to V* is

$$B_V(x, y) = \operatorname{tr} \rho(x)\rho(y).$$

If $\mathfrak{gl}(V) = \mathfrak{gl}(\mathfrak{g})$ and ρ is the adjoint representation, then we will simply write $B(x, y)$, and call it "the Killing form".

For the following four results, consider k to be a field of characteristic 0 and V a finite-dimensional vector space.

For the following theorem see (Milne, 2013, §I.3, Theorem 3.17, p. 39):

Theorem 3.1.44 (Cartan's Criterion). *Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie algebra such that*

$$B_V(X, Y) = 0$$

for all $X, Y \in \mathfrak{g}$. Then $\mathfrak{g}$ is solvable.

For the following corollary see (Strade and Farnsteiner, 1988, §1.7, Corollary 7.6, p. 42):

Corollary 3.1.45. *A Lie algebra $\mathfrak{g}$ is solvable if and only if $B(\mathfrak{g}, \mathfrak{g}_1) = 0$.*

For the following lemma see (Strade and Farnsteiner, 1988, §1.7, Lemma 7.1, p. 38):

Lemma 3.1.46. *Let $\mathfrak{g}$ be finite-dimensional. Then $\mathfrak{g}$ contains a unique maximal solvable ideal, denoted by* $\mathrm{rad}(\mathfrak{g})$.

Definition 3.1.47. Let $\mathfrak{g}$ be finite-dimensional. The ideal $\mathrm{rad}(\mathfrak{g})$ is called the *solvable radical* of $\mathfrak{g}$. We say that $\mathfrak{g}$ is *semisimple* if $\mathrm{rad}(\mathfrak{g}) = 0$.

For the following corollary see (Milne, 2013, §I.4, Corollary 4.19, p. 45):

Corollary 3.1.48. *If $\mathfrak{g}$ is a semisimple Lie algebra, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.*

For the following theorem see (Erdmann and Wildon, 2006, §9.4, Lemma 9.12, p. 84):

Theorem 3.1.49. *If $\mathfrak{g}$ is semisimple and $\rho : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a Lie algebra homomorphism, then $\rho(\mathfrak{g})$ is also semisimple.*

For the following lemma see (Erdmann and Wildon, 2006, §7.7, Lemma 7.13, p. 62):

Lemma 3.1.50 (Schur). *Let k be an algebraically closed field. Let $\mathfrak{g}$ be a Lie algebra and V be an irreducible $\mathfrak{g}$-module. Then*

$$\dim \operatorname{Hom}_{\mathfrak{g}}(V, V) = 1,$$

where $\operatorname{Hom}_{\mathfrak{g}}(V, V)$ is the set of all maps of $\mathfrak{g}$-modules from V to itself.

Here are four important and useful commutation formulas, see (Strade and Farnsteiner, 1988, §1.1, Proposition 1.3, p. 9):

Proposition 3.1.51. *Let A be an associative k-algebra. Then, for all $a, b, c \in A$, we have in A^-:*

$$(\operatorname{ad} c)^m(a) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} c^j a c^{m-j}$$

$$[ab, c] = [a, c]b + a[b, c]$$

$$[a^m, c] = \sum_{j=0}^{m-1} a^j [a, c] a^{m-j-1}$$

$$ca^m = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} a^j (\operatorname{ad} a)^{m-j} c.$$

Definition 3.1.52. Let $V \subseteq \mathfrak{g}$ be a subspace. The *normaliser* of V in $\mathfrak{g}$ is

$$\mathfrak{n}_{\mathfrak{g}}(V) = \{x \in \mathfrak{g} : [x, V] \subseteq V\}.$$

Definition 3.1.53. A nilpotent subalgebra $\mathfrak{h}$ of a finite-dimensional Lie algebra $\mathfrak{g}$ is a *Cartan subalgebra* if it is self-normalising, i.e.

$$\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}.$$

It is not obvious whether or not Cartan subalgebras exist. In fact, in some cases it is unknown whether they exist. Theorem 4.7 in (Strade and Farnsteiner, 1988, §1.4, p. 26) guarantees that they exist provided that the ground field k is algebraically closed.

Example 3.1.54. Let k be a field of characteristic $p \neq 2$. Let $\mathfrak{d} = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) : a, b \in k, a + b = 0\right\} \subseteq \mathfrak{sl}_2(k)$. We claim that $\mathfrak{d}$ is a Cartan subalgebra of $\mathfrak{sl}_2(k)$. Indeed, observe that $\mathfrak{d}_1 = [\mathfrak{d}, \mathfrak{d}] = \{0\}$, since the Lie bracket of any two diagonal matrices is the zero matrix. Let us now compute the normaliser $\mathfrak{n}_{\mathfrak{sl}_2}(\mathfrak{d})$. Let $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a + d = 0$, i.e. $x \in \mathfrak{sl}_2$. Let $d = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) \in \mathfrak{d}$. We calculate $[x, d] = \left(\begin{smallmatrix} 0 & b(\mu-\lambda) \\ c(\lambda-\mu) & 0 \end{smallmatrix}\right)$. If x lies in the normaliser of $\mathfrak{d}$, we must have thus $b(\mu - \lambda) = 0 = c(\lambda - \mu)$. Therefore, $\mu = \lambda$ or $b = c = 0$. Since this must hold for all choices of μ and λ with $\mu + \lambda = 0$, we conclude that $b = c = 0$ and so $x = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ with $a + d = 0$. Thus, $x \in \mathfrak{d}$, and so $\mathfrak{n}_{\mathfrak{sl}_2}(\mathfrak{d}) = \mathfrak{d}$. Hence, $\mathfrak{d}$ is a Cartan subalgebra of $\mathfrak{sl}_2(k)$.

If $\mathfrak{g}$ is defined over an algebraically closed field, we know that it has a Cartan subalgebra $\mathfrak{h}$. Corollary 4.4 in (Strade and Farnsteiner, 1988, §1.4, p. 23) tells us that with respect to the adjoint representation $\mathrm{ad} : \mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ we have a decomposition $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$, where

$$\mathfrak{g}_\alpha := \left\{x \in \mathfrak{g} : \forall h \in \mathfrak{h}\ \exists n \in \mathbb{N} : (\mathrm{ad}\, h - \alpha(h)\, \mathrm{id}_{\mathfrak{g}})^n(x) = 0\right\}.$$

Moreover, $\mathfrak{g}_0 = \mathfrak{h}$, so that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha$.

Definition 3.1.55. Let B be the Killing form on $\mathfrak{g}$. Then

$$\mathrm{rad}(B) = \{x \in \mathfrak{g} : B(x, y) = 0\ \forall y \in \mathfrak{g}\}.$$

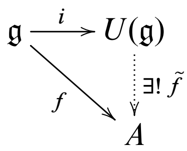
The same definition applies to any symmetric bilinear form on $\mathfrak{g}$.

For the following theorem see (Strade and Farnsteiner, 1988, §1.7, Theorem 7.9, p. 43):

Theorem 3.1.56. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of arbitrary characteristic. Suppose that* $\mathrm{rad}(B) = 0$. *Then every derivation of $\mathfrak{g}$ is inner, i.e.*

$$\mathrm{Der}_k(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g}).$$

Definition 3.1.57. Let $\mathfrak{g}$ be a Lie algebra over k. Suppose that $i : \mathfrak{g} \longrightarrow U(\mathfrak{g})^-$ is a homomorphism of Lie algebras, for $U(\mathfrak{g})^-$ the associated Lie algebra of some associative k-algebra $U(\mathfrak{g})$. The pair $(U(\mathfrak{g}), i)$ is a *universal enveloping algebra* of $\mathfrak{g}$ if for every associative k-algebra A and every Lie algebra homomorphism $f : \mathfrak{g} \longrightarrow A^-$, there is a unique associative homomorphism $\tilde{f} : U(\mathfrak{g}) \longrightarrow A$ such that the following diagram



commutes.

For the following theorem see (Strade and Farnsteiner, 1988, §1.8, Theorem 8.3, p. 49) and (Strade and Farnsteiner, 1988, §1.8, Corollary 8.5, p. 50):

Theorem 3.1.58. *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ possesses a universal enveloping algebra $U(\mathfrak{g})$. Furthermore, it embeds into it.*

The universal enveloping algebra can be realised as the quotient of the tensor algebra $T(\mathfrak{g})$ of $\mathfrak{g}$ by an appropriate ideal. More specifically, consider the ideal $I = (\{a \otimes b - b \otimes a - [a, b] : a, b \in \mathfrak{g}\})$. Then $U(\mathfrak{g}) \cong T(\mathfrak{g})/I$.

Thanks to the universal property of the universal enveloping algebra, any two universal algebras of a given lie algebra, $U(\mathfrak{g})$ and $V(\mathfrak{g})$, say, are isomorphic up to unique isomorphism, see (Strade and Farnsteiner, 1988, §1.8, Theorem 8.1).

The following famous theorem, found in (Strade and Farnsteiner, 1988, §1.8, Theorem 8.4, p. 50), due to Poincaré, Birkhoff, and Witt, the so-called PBW theorem, can be used to see that the map of Lie algebras $i : \mathfrak{g} \longrightarrow U(\mathfrak{g})$ is in fact injective.

Theorem 3.1.59 (P-B-W). *Let $\mathfrak{g}$ be a Lie algebra over k with ordered basis $(x_j)_{j\in\Lambda}$. Let $(U(\mathfrak{g}), i)$ be a universal enveloping algebra of $\mathfrak{g}$. Then the set*

$$\left\{i(x_{j_1}) \cdots i(x_{j_n}) : n \in \mathbb{N}, j_1 \leq j_2 \leq \cdots \leq j_n \in \Lambda\right\}$$

is a k-basis for $U(\mathfrak{g})$. One can also write the basis as

$$\left\{i(x_{j_1})^{s_1} \cdots i(x_{j_n})^{s_n} : n \in \mathbb{N}, j_1 < j_2 < \cdots < j_n \in \Lambda, s_i \geq 0\right\}.$$

Since i is an embedding, we will often suppress it, and regard $\mathfrak{g}$ as a subalgebra of $U(\mathfrak{g})^-$.

Example 3.1.60. Let (e, f, h) be an ordered basis for $\mathfrak{sl}_2$. Then the basis elements for $U(\mathfrak{sl}_2)$ look like $e^{a_1} f^{a_2} h^{a_3}$ for $a_i \in \mathbb{N}$, so one has elements such as

$$e, e^2, e^3, \ldots, ef, ef^2, ef^3, \ldots, efh, efh^2, \ldots, eh, eh^2, \ldots$$

and so on. Meanwhile, elements such as hf, he, hef, f^2e are not valid basis elements in the PBW basis.

For the following theorem see (Strade and Farnsteiner, 1988, §1.8, Corollary 8.2, p. 48):

Theorem 3.1.61. *Let $\mathfrak{g}$ be a Lie algebra and let V be a $\mathfrak{g}$-module, and $(U(\mathfrak{g}), i)$ a universal enveloping algebra of $\mathfrak{g}$. Then there is a unique $U(\mathfrak{g})$-module structure on V such that*

$$a \cdot v = i(a) \cdot v,$$

for all $a \in \mathfrak{g}, v \in V$.

Furthermore, a subspace $W \subseteq V$ is a $\mathfrak{g}$-submodule if and only if it is a $U(\mathfrak{g})$-submodule.

In light of the previous theorem, one might ask: Why study representations of Lie algebras at all? Why not just study modules of associative algebras? At least one point in favour of Lie algebras lies in the fact that $U(\mathfrak{g})$ is infinite-dimensional even if $\mathfrak{g}$ is not.

The following two results are useful in studying $\mathfrak{g}$-modules.

This is Corollary 3.8 from (Strade and Farnsteiner, 1988, p. 19):

Theorem 3.1.62. *Let $\mathfrak{i}$ be a finite-dimensional ideal of a Lie algebra $\mathfrak{g}$ (over an arbitrary field), let V be a simple $\mathfrak{g}$-module. If x acts nilpotently on V for all $x \in \mathfrak{i}$, then x acts trivially for all $x \in \mathfrak{i}$, i.e. $\mathfrak{i} \cdot V = 0$.*

This is Lemma 5.6 from (Strade and Farnsteiner, 1988, p. 31):

Theorem 3.1.63. *Let $\mathfrak{g}$ be an abelian Lie algebra over an algebraically closed field of arbitrary characteristic. Every finite-dimensional irreducible representation V of $\mathfrak{g}$ is one-dimensional.*

3.2 Restrictable Lie algebras

For most of the material in this section, we refer the reader to (Strade and Farnsteiner, 1988, §2). Throughout, we assume that the base field k has characteristic p unless otherwise stated.

Often when calculating, we will appeal to the following without reference, see (Nagell, 1951, §30, p. 99):

Theorem 3.2.1 (Wilson's Theorem). *Let t be an integer greater than one. Then t is prime if and only if $(t-1)! = -1$ modulo t.*

That is, over k one has $(p-1)! = -1 = p-1$.

To introduce restricted structures, we make the following definition.

Definition 3.2.2. Let $\mathfrak{g}$ be a Lie algebra over k. A mapping $[p] : \mathfrak{g} \longrightarrow \mathfrak{g}, a \mapsto a^{[p]}$ is called a *p-mapping* if

1. $\operatorname{ad} a^{[p]} = (\operatorname{ad} a)^p$, for all $a \in \mathfrak{g}$;
2. $(\lambda a)^{[p]} = \lambda^p a^{[p]}$, for all $a \in \mathfrak{g}, \lambda \in k$;
3. $(a+b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a,b)$,

where

$$(\operatorname{ad}(a \otimes X + b \otimes 1))^{p-1}(a \otimes 1) = \sum_{i=1}^{p-1} i s_i(a,b) \otimes X^{i-1}$$

in $\mathfrak{g} \otimes_k k[X]$, for all $a, b \in \mathfrak{g}$ (where $\mathfrak{g} \otimes_k k[X]$ obtains the structure of a Lie algebra via $[l \otimes x, j \otimes y] := [l, j] \otimes xy$ for all $l, j \in \mathfrak{g}, x, y \in k[X]$).

The pair $(\mathfrak{g}, [p])$ is referred to as a *restricted Lie algebra.*

Remark. Calculating the $s_i(a,b)$ correction terms can be in general rather cumbersome. Sometimes it suffices to compute $s_{p-2}(a,b)$ and $s_{p-1}(a,b)$. We refer the reader to (Strade, 2004, p. 18, Eq. (1.1.1)) and (Strade, 2009, p. 18, Rem. 10.2.4) for more details.

Example 3.2.3. The Lie algebra $\mathfrak{sl}_2(k)$ is restricted with $[p]$-mapping given by: $h^{[p]} = h, e^{[p]} = f^{[p]} = 0$.

The following example is universal, as we will see later, see Definition 3.2.29 and Theorem 3.2.30. Take an associative algebra A. The Lie algebra A^- becomes a restricted Lie algebra if we set $x^{[p]} = x^p$ for all $x \in A$. In particular, $(\mathfrak{gl}(V), p)$ is a restricted Lie algebra. Furthermore, any Lie subalgebra $\mathfrak{g} \leq A^-$ with the property that $x^p \in \mathfrak{g}$ for all $x \in \mathfrak{g}$ is a restricted Lie algebra. A particularly important example is the Lie algebra of derivations $\operatorname{Der}_k(B) \leq \mathfrak{gl}(B)$ of any k-algebra B, since $D^p \in \operatorname{Der}_k(B)$ for all $D \in \operatorname{Der}_k(B)$. Thus $\operatorname{Der}_k(B)$ is a restricted Lie algebra with $[p]$-mapping given by p-fold composition of functions. Setting

$$B = k[X_1, \ldots, X_n] / \left(X_1^p, \ldots, X_n^p\right)$$

yields an important family of examples, as we will see later.

The suitable notions of substructures are the following:

Definition 3.2.4. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra over k. A subalgebra $\mathfrak{h} \leq \mathfrak{g}$ (ideal $\mathfrak{i} \lhd \mathfrak{g}$) is called a *p-subalgebra* or *restricted subalgebra* (*p-ideal* or *restricted ideal*) if $x^{[p]} \in \mathfrak{h}$ for all $x \in \mathfrak{h}$ ($x^{[p]} \in \mathfrak{i}$ for all $x \in \mathfrak{i}$).

For the following lemma see (Strade and Farnsteiner, 1988, §2.1, Lemma 1.2, p. 64):

Lemma 3.2.5. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra over k. Let $a, b \in \mathfrak{g}$ and $\mathfrak{h} = \mathfrak{g}\langle a, b\rangle$ be the Lie algebra generated by a and b. Then*

$$s_i(a, b) \in \mathfrak{h}_{p-1}.$$

Remark. Here we mean $\mathfrak{h}_{p-1}$ in the sense of Definition 3.1.36.

Let S be a subset of a restricted Lie algebra $\mathfrak{g}$. Then put

$$S_{[p]} = \bigcap_{\substack{S \subseteq \mathfrak{h} \\ \mathfrak{h} \text{ a } p\text{-subalgebra}}} \mathfrak{h}$$

Definition 3.2.6. We call $S_{[p]}$ the *p-subalgebra generated* by S in $\mathfrak{g}$.

Denote by $S^{[p]^i}$ the image of S under i applications of the map $[p]$. The next proposition gives an explicit description of the p-subalgebra $S_{[p]}$ in the special case $S \leq \mathfrak{g}$.

For the following proposition see (Strade and Farnsteiner, 1988, §2.1, Proposition 1.3, part (1), p. 66):

Proposition 3.2.7. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. If $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra, then we have the following characterisation:*

$$\mathfrak{h}_{[p]} = \sum_{i \geq 0} k\langle \mathfrak{h}^{[p]^i} \rangle.$$

Furthermore, if $(e_j)_{j \in J}$ is a basis for $\mathfrak{h}$, we have

$$\mathfrak{h}_{[p]} = \sum_{\substack{i \geq 0 \\ j \in J}} k e_j^{[p]^i}.$$

Definition 3.2.8. Let $(\mathfrak{g}_1, [p]_1), (\mathfrak{g}_2, [p]_2)$ be restricted Lie algebras. A map $f : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ is called *restricted* or a *p-homomorphism* if

1. f is a Lie algebra homomorphism;

2. for all $x \in \mathfrak{g}_1$, $f(x^{[p]_1}) = f(x)^{[p]_2}$.

Furthermore a *p-representation* or a *restricted representation* is just a restricted homomorphism $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ for some vector space V, where one takes the p-mapping on $\mathfrak{gl}(V)$ to be the one coming from the associative operation (composition of functions).

Now, if I is a p-ideal of $(\mathfrak{g}, [p])$, the quotient $\mathfrak{g}/I$ does carry a natural p-mapping, given by (see (Strade and Farnsteiner, 1988, §2.2, Proposition 1.4, p. 67)):

$$(x + I)^{[p]'} = x^{[p]} + I.$$

Having defined suitable notions of homomorphism and quotient, the usual isomorphism theorems apply.

Given that we can embed $\mathfrak{g}$ into $U(\mathfrak{g})$, one can ask what the relationship between $x^{[p]}$ and x^p is. Indeed, we have:

Proposition 3.2.9. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Then for all $x \in \mathfrak{g}$, the element $x^p - x^{[p]}$ is central in $U(\mathfrak{g})$.*

Proof. Since $\mathfrak{g}$ is identified with its copy in $U(\mathfrak{g})$ and for any associative algebra A the map $a \mapsto a^p$ is a p-mapping on A^-, we have

$$\operatorname{ad} x^p(y) = (\operatorname{ad} x)^p(y) = \operatorname{ad} x^{[p]}(y)$$

for all $y \in \mathfrak{g}$. Thus, $[x^p - x^{[p]}, \mathfrak{g}] = 0$, i.e., $(x^p - x^{[p]})(y) = y(x^p - x^{[p]})$. Note that $U(\mathfrak{g})$ has as k-basis (associative) products of elements of $\mathfrak{g}$. So $x^p - x^{[p]}$ commutes with any product of elements of $\mathfrak{g}$ and so commutes with any sum of such products by distributivity. ∎

Definition 3.2.10 (p-nilpotency). Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra over k. A p-ideal $I \lhd_p \mathfrak{g}$ is *p-nilpotent* if there is $n \in \mathbb{N}$ such that $I^{[p]^n} = 0$. An element x is called *p-nilpotent* if there is an $n \in \mathbb{N}$ such that $x^{[p]^n} = 0$. Finally the p-ideal I is called *p-nil* if every element $x \in I$ is p-nilpotent.

For the following theorem see (Strade and Farnsteiner, 1988, §2.1, Corollary 1.6, p. 68):

Theorem 3.2.11. *Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie algebra. Then there is a unique p-ideal*

$$\operatorname{rad}_p(\mathfrak{g})$$

such that

1. $\mathrm{rad}_p(\mathfrak{g})$ *is p-nilpotent;*

2. *if $I \lhd_p \mathfrak{g}$ is p-nilpotent, then $I \subseteq \mathrm{rad}_p(\mathfrak{g})$.*

Definition 3.2.12. The p-ideal $\mathrm{rad}_p(\mathfrak{g})$ is called the *p-radical* of $\mathfrak{g}$.

For the following theorem see (Strade and Farnsteiner, 1988, §2.1, Corollary 1.7, p. 68):

Theorem 3.2.13. *Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie algebra. Then*

1. $\mathrm{rad}_p(\mathfrak{g}) \subseteq N(\mathfrak{g})$*;*

2. $\mathrm{rad}_p(\mathfrak{g}/\mathrm{rad}_p(\mathfrak{g})) = 0$*,*

where $N(\mathfrak{g})$ is the sum of all nilpotent ideals of $\mathfrak{g}$.

Before looking more closely at p-mappings, we make the following definition:

Definition 3.2.14. Let $f : V \longrightarrow V$ be a map. We say that f is p-semilinear if

1. $f(x+y) = f(x) + f(y)$ for all $x, y \in V$;

2. $f(\alpha x) = \alpha^p f(x)$ for all $x \in V, \alpha \in k$.

For the following proposition see (Strade and Farnsteiner, 1988, §2.2, Proposition 2.1, p. 70):

Proposition 3.2.15. *Let $\mathfrak{g}$ be a subalgebra of a restricted Lie algebra $(\mathfrak{a}, [p])$ and let $[p]_1 : \mathfrak{g} \longrightarrow \mathfrak{g}$ be a mapping. The following are equivalent:*

1. *$[p]_1$ is a p-mapping on $\mathfrak{g}$;*

2. *there is a p-semilinear mapping $f : \mathfrak{g} \longrightarrow C_{\mathfrak{a}}(\mathfrak{g})$ such that $[p]_1 = [p] + f$.*

Remark. Recall that we have $C_{\mathfrak{a}}(\mathfrak{g}) = \{x \in \mathfrak{a} : [x, \mathfrak{g}] = 0\}$.

Remark. If $\mathfrak{g} = \mathfrak{a}$, then the proposition says that if $[p]_1$ is a p-mapping, then the difference $[p]_1 - [p]$ is a p-semilinear mapping $\mathfrak{a} \longrightarrow C(\mathfrak{a})$. Conversely, if f is a p-semilinear mapping $\mathfrak{a} \longrightarrow C(\mathfrak{a})$, then $[p] + f$ is a p-mapping on $\mathfrak{a}$.

For the following corollary see (Strade and Farnsteiner, 1988, §2.2, Corollary 2.2, p. 71):

Corollary 3.2.16. *Let $\mathfrak{g}$ be a Lie algebra. We have the following:*

1. *if $C(\mathfrak{g}) = 0$, then $\mathfrak{g}$ admits at most one p-mapping;*
2. *if two p-mappings coincide on a basis, then they are equal;*
3. *if $(\mathfrak{g}, [p])$ is restricted, there is a p-mapping $[p]'$ of $\mathfrak{g}$ such that*

$$x^{[p]'} = 0$$

for all $x \in C(\mathfrak{g})$.

The foregoing tells us something about the uniqueness of restricted structures. Now we turn to the question of existence.

For the following theorem see (Strade and Farnsteiner, 1988, §2.2, Theorem 2.3, p. 71):

Theorem 3.2.17 (N. Jacobson). *Let $(e_j)_{j \in J}$ be a basis of a Lie algebra $\mathfrak{g}$ such that there exist $y_j \in \mathfrak{g}$ with $(\operatorname{ad} e_j)^p = \operatorname{ad} y_j$. Then there is a unique p-mapping $[p] : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that*

$$e_j^{[p]} = y_j$$

for all $j \in J$.

Definition 3.2.18. A Lie algebra $\mathfrak{g}$ is called *restrictable* if $\operatorname{ad}(\mathfrak{g})$ is a p-subalgebra of $\operatorname{Der}_k(\mathfrak{g})$, that is $(\operatorname{ad} x)^p \in \operatorname{ad}(\mathfrak{g})$ for all $x \in \mathfrak{g}$. In other words, there is a mapping $[p] : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $(\operatorname{ad} x)^p = \operatorname{ad} x^{[p]}$ for all $x \in \mathfrak{g}$.

Remark. Thanks to Jacobson's theorem, we see that a Lie algebra is restrictable if and only if it admits at least one p-mapping, that is, if and only if there is a mapping that makes it a restricted Lie algebra.

For the following theorem see (Strade and Farnsteiner, 1988, §2.2, Theorem 2.4, p. 73):

Theorem 3.2.19. *Let $f : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ be a surjective homomorphism of Lie algebras. If $\mathfrak{g}_1$ is restrictable, so is $\mathfrak{g}_2$.*

Remark. This theorem has the interesting consequence that the Lie algebra $\mathfrak{g}/I$, where $I \lhd \mathfrak{g}$, is restrictable if $\mathfrak{g}$ is restrictable.

Definition 3.2.20. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. A derivation $D \in \operatorname{Der}_k(\mathfrak{g})$ is called a *restricted derivation* if

$$D(a^{[p]}) = (\operatorname{ad} a)^{p-1}(D(a))$$

for all $a \in \mathfrak{g}$. Let $\mathrm{Der}_{[p]}(\mathfrak{g})$ denote the subspace of restricted derivations. Every inner derivation $\operatorname{ad} x$ for all $x \in \mathfrak{g}$ is a restricted derivation.

For the following proposition see (Strade and Farnsteiner, 1988, §2.2, Exercise 2, p. 76):

Proposition 3.2.21. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. For every derivation D, we have*

$$D(x^{[p]}) - (\operatorname{ad} x)^{p-1}(D(x)) \in C(\mathfrak{g}),$$

for all $x \in \mathfrak{g}$.

Definition 3.2.22. Let A, B be Lie algebras and $\varphi : A \longrightarrow \mathrm{Der}_k(B)$ be a homomorphism. On the vector space $A \oplus B$ define a Lie bracket by

$$[(a, b), (a', b')] = ([a, a'], \varphi(a)(b') - \varphi(a')(b) + [b, b']).$$

This algebra, denoted by $A \oplus_\varphi B$, is called the *semidirect product* of A and B.

Remark. Note that $A \oplus_\varphi B$ is a Lie algebra. If $\varphi = 0$ it is the usual direct sum of Lie algebras. If $[B, B] = 0$, then B is simply an A-module and φ its representation.

For the following theorem see (Strade and Farnsteiner, 1988, §2.2, Theorem 2.5, p. 73):

Theorem 3.2.23. *Let $(A, [p])$ and $(B, [p]')$ be restricted Lie algebras and let $\varphi : A \longrightarrow \mathrm{Der}_k(B)$ be a restricted homomorphism such that $\varphi(x)$ is restricted for all $x \in A$. Then $A \oplus_\varphi B$ is restrictable.*

For the following corollary see (Strade and Farnsteiner, 1988, §2.2, Corollary 2.6, p. 74):

Corollary 3.2.24. *Let A, B be ideals of a Lie algebra $\mathfrak{g}$ such that $\mathfrak{g} = A \oplus B$. Then $\mathfrak{g}$ is restrictable if and only if A and B are.*

In the following all vector spaces are assumed to be finite-dimensional.

Definition 3.2.25. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra of k. An element $x \in \mathfrak{g}$ is *semisimple* if $x \in (kx^{[p]})_{[p]}$ and *toral* if $x^{[p]} = x$.

Remark. A restricted subalgebra that is abelian and admits a basis consisting of toral elements is called a *toral subalgebra* or often just a *torus*. Maximal tori play an important role in the classification of restricted modular Lie algebras. Indeed, if M is a finite-dimensional $u(\mathfrak{g}, S)$-module (see Definition 3.4.5), then it has a decomposition

$$M = \bigoplus_{\lambda \in \mathfrak{t}^*} M_\lambda$$

into weight spaces $M_\lambda := \{m \in M : t \cdot m = \lambda(t)m \text{ for all } t \in \mathfrak{t}\}$ relative to a maximal torus $\mathfrak{t}$.

The significance of semisimple elements rests on the following result, which can be found in (Strade and Farnsteiner, 1988, §2.3, Theorem 3.4, p. 80):

Theorem 3.2.26. *Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie algebra over k. For all $x \in \mathfrak{g}$ there is a $j \in \mathbb{N}$ such that $x^{[p]^j}$ is semisimple.*

Definition 3.2.27. A p-mapping $[p]$ on $\mathfrak{g}$ is called *nonsingular* if $x^{[p]} \neq 0$ for all $x \in \mathfrak{g} \setminus \{0\}$.

For the following theorem see (Strade and Farnsteiner, 1988, §2.3, Theorem 3.10, p. 84):

Theorem 3.2.28. *Let k be algebraically closed and let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie algebra with nonsingular p-mapping. Then $\mathfrak{g}$ is abelian.*

Definition 3.2.29. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra over k. Suppose that $i : \mathfrak{g} \longrightarrow \mathfrak{u}(\mathfrak{g})^-$ is a restricted homomorphism of Lie algebras, for $\mathfrak{u}(\mathfrak{g})^-$ the associated Lie algebra of some associative k-algebra with unity $\mathfrak{u}(\mathfrak{g})$. The pair $(\mathfrak{u}(\mathfrak{g}), i)$ is a *restricted universal enveloping algebra* of $\mathfrak{g}$ if for every associative k-algebra with unity A and every restricted Lie algebra homomorphism $f : \mathfrak{g} \longrightarrow A^-$, there is a unique associative homomorphism $\tilde{f} : \mathfrak{u}(\mathfrak{g}) \longrightarrow A$ such that the following diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\;i\;} & \mathfrak{u}(\mathfrak{g}) \\ & {\scriptstyle f}\searrow & \vdots\; {\scriptstyle \exists!\, \tilde{f}} \\ & & A \end{array}$$

commutes.

For the following theorem see (Strade and Farnsteiner, 1988, §2.5, Theorem 5.1, p. 90):

Theorem 3.2.30. *Let $(\mathfrak{g}, [p])$ be restricted and let $(x_j)_{j\in\Lambda}$ be an ordered basis for $\mathfrak{g}$ over k. Then the restricted universal enveloping algebra exists and*

$$i : \mathfrak{g} \longrightarrow \mathfrak{u}(\mathfrak{g})^-$$

is injective and $\dim_k \mathfrak{u}(\mathfrak{g}) = p^n$ *if* $\dim_k \mathfrak{g} = n$*. Furthermore,* $\mathfrak{u}(\mathfrak{g})$ *possesses a PBW-type k-basis given by*

$$\left\{ i(x_{j_1})^{s_1} \cdots i(x_{j_n})^{s_n} : n \in \mathbb{N}, j_1 < j_2 < \cdots < j_n \in \Lambda, 0 \leq s_i \leq p-1 \right\}.$$

Remark. The restricted universal enveloping algebra $\mathfrak{u}(\mathfrak{g})$ can be obtained as the quotient of $U(\mathfrak{g})$ by the ideal

$$\left(x^p - x^{[p]} : x \in \mathfrak{g} \right).$$

Remark. As in the case of $U(\mathfrak{g})$, we identify $\mathfrak{g}$ with its image $i(\mathfrak{g})$ in $\mathfrak{u}(\mathfrak{g})$.

Example 3.2.31. Let (e, f, h) be an ordered basis for $\mathfrak{sl}_2$. Then $\mathfrak{u}(\mathfrak{g})$ is a p^3-dimensional associative algebra with basis

$$\{e^{a_1} f^{a_2} h^{a_3} : 0 \leq a_i \leq p-1\}.$$

As in the case of universal enveloping algebras, the universal property of restricted universal enveloping algebras guarantees that any two restricted universal enveloping algebras of $\mathfrak{g}$ are isomorphic.

It is extremely useful to have the extra tools and structure of restricted Lie algebras. Indeed, to engage in the Classification Theory one needs to consider the root space decomposition of a Lie algebra with respect to a maximal toral subalgebra. In this context, thus, embedding a non-restricted Lie algebra into a restricted Lie algebra is a necessity.

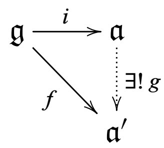
We therefore consider the following ways of embedding an arbitrary Lie algebra into restricted Lie algebras.

Definition 3.2.32. Let $\mathfrak{g}$ be a Lie algebra.

1. A triple $(\mathfrak{a}, [p], i)$ consisting of a restricted Lie algebra $(\mathfrak{a}, [p])$ and a Lie algebra homomorphism $i : \mathfrak{g} \longrightarrow \mathfrak{a}$ is a *p-envelope* of $\mathfrak{g}$ if $i : \mathfrak{g} \longrightarrow \mathfrak{a}$ is injective and the p-subalgebra $i(\mathfrak{g})_{[p]} = \mathfrak{a}$.

2. A p-envelope is called *universal* if for every restricted Lie algebra $(\mathfrak{a}', [p]')$ and every homomorphism $f : \mathfrak{g} \longrightarrow \mathfrak{a}'$, there is a unique restricted homomorphism

$g : \mathfrak{a} \longrightarrow \mathfrak{a}'$ such that the diagram



commutes.

Theorem 3.2.33 (Mil'ner (1975)). *Every Lie algebra $\mathfrak{g}$ has a universal p-envelope, denoted $\hat{\mathfrak{g}}$.*

Proof. See (Strade and Farnsteiner, 1988, Theorem 2.5.2, p. 92). ■

Remark. Firstly, the universal p-envelope is unique in the same fashion as the universal enveloping algebras. Secondly, as the proof referred above mentions, one may identify $\hat{\mathfrak{g}}$ with $i(\mathfrak{g})_{[p]} \leq U(\mathfrak{g})$, where $i : \mathfrak{g} \longrightarrow U(\mathfrak{g})$ is the canonical embedding.

Example 3.2.34. Let $\mathfrak{g} = kh \oplus kx$. Setting $[h, x] = 0$ turns $\mathfrak{g}$ into a Lie algebra. As we saw in the previous remark, the universal p-envelope of $\mathfrak{g}$ can be identified with $i(\mathfrak{g})_{[p]} \leq U(\mathfrak{g})$. By Proposition 3.2.7, we have $\hat{\mathfrak{g}} = i(\mathfrak{g})_{[p]} = \sum_{i \geq 0} kh^{p^i} + \sum_{i \geq 0} kx^{p^i}$.

Exercise 4 in (Strade and Farnsteiner, 1988, §2.5) tells us that:

Proposition 3.2.35. *Let $(\mathfrak{a}, [p], i)$ and $(\mathfrak{a}', [p]', i')$ be two p-envelopes of $\mathfrak{g}$. We have*

$$\mathrm{ad}(\mathfrak{a}) \cong \mathrm{ad}(\mathfrak{a}').$$

There is a notion of a *minimal p-envelope*, which indeed is minimal in the sense of inclusion. A p-envelope $(\mathfrak{a}, [p], i)$ is called minimal if $C(\mathfrak{a}) \subseteq i(\mathfrak{g})$. For details see (Strade, 2004, pp. 19–22) or (Strade and Farnsteiner, 1988, §2.5, pp. 94–97). The existence of minimal p-envelopes is guaranteed by Theorem 1.1.6 (2) in (Strade, 2004, §1.1, p. 20).

Example 3.2.36. Working out minimal p-envelopes of Lie algebras is not trivial. Theorem 7.2.7 in (Strade, 2004, §7.2 , p. 368–372) determines these for all the simple Lie algebras. In Chapter 6 we work out explicitly the minimal p-envelope of one of these algebras.

The minimal p-envelope should not be confused with the universal p-envelope of a Lie algebra. For one, if $\mathfrak{g}$ is finite-dimensional, we know that the minimal p-envelope of $\mathfrak{g}$ will be finite dimensional (see Theorem 1.1.6 (3) in (Strade, 2004, §1.1, p. 20)), while as we saw in the above remark, universal p-envelopes are infinite-dimensional.

3.3 Filtered and graded Lie algebras

Definition 3.3.1. A *descending filtration* of a Lie algebra $\mathfrak{g}$ is a family $(\mathfrak{g}_{(i)})_{i\in\mathbb{Z}}$ of subspaces such that

1. $\mathfrak{g}_{(l)} \subseteq \mathfrak{g}_{(i)}$ if $i \leq l$;
2. $[\mathfrak{g}_{(i)}, \mathfrak{g}_{(l)}] \subseteq \mathfrak{g}_{(i+l)}$ for all $i, l \in \mathbb{Z}$.

A Lie algebra $\mathfrak{g}$ admitting a filtration is said to be a *filtered Lie algebra*.
If $\bigcup_{n\in\mathbb{Z}} \mathfrak{g}_{(n)} = \mathfrak{g}$, the filtration is said to be *exhaustive*. If

$$\bigcap_{n\in\mathbb{Z}} \mathfrak{g}_{(n)} = \{0\}$$

it is said to be *separating*.

Example 3.3.2. Here's a way of building a descending filtration on $\mathfrak{g}$. Let $\mathfrak{j} \leq \mathfrak{g}$. Define $\mathfrak{g}_{(-1)} = \mathfrak{g}$ and $\mathfrak{g}_{(0)} = \mathfrak{j}$ and define recursively

$$\mathfrak{g}_{(n+1)} = \left\{x \in \mathfrak{g}_{(n)} : [x, \mathfrak{g}] \subseteq \mathfrak{g}_{(n)}\right\}.$$

Then it is clear that the descending property holds. The multiplication property can be checked by induction.

Definition 3.3.3. Let $\mathfrak{g}$ be a Lie algebra. A family of subspaces $(\mathfrak{g}_i)_{i\in\mathbb{Z}}$ such that $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ and $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ is called a $\mathbb{Z}$-gradation of $\mathfrak{g}$. We call the algebra $\mathfrak{g}$ a *graded algebra* or a *$\mathbb{Z}$-graded algebra*. The elements inside $\mathfrak{g}_i$ are called *homogeneous elements of degree i*.

Remark. The notion of a $\mathbb{Z}$-gradation can be naturally generalised to abelian groups. In this thesis, however, we limit ourselves to the study of $\mathbb{Z}$-gradations. We will often simply say that a Lie algebra is graded to mean that it has a $\mathbb{Z}$-gradation.

There exist a functor from the category of graded algebras to the category of filtered algebras and vice-versa. One way around, a graded Lie algebra $(\mathfrak{g}, (\mathfrak{g}_i)_{i\in\mathbb{Z}})$ gives rise to a filtered Lie algebra $(\mathfrak{g}, (\mathfrak{g}_{(i)})_{i\in\mathbb{Z}})$ by defining

$$\mathfrak{g}_{(i)} = \bigoplus_{j\geq i} \mathfrak{g}_j.$$

Indeed, it is clear that $\mathfrak{g}_{(l)} \subseteq \mathfrak{g}_{(i)}$ if $i \leq l$. Let $a_i \in \mathfrak{g}_{(i)}, a_l \in \mathfrak{g}_{(l)}$. Then the smallest subspace in the gradation that any element in the Lie bracket of a_i with a_l can be is when one takes the lowest graded component of a_i and the lowest graded component of a_l, $a^{i'} \in \mathfrak{g}_{i'}$ and $a^{l'} \in \mathfrak{g}_{l'}$, say. Their Lie bracket $[a^{i'}, a^{l'}] \in \mathfrak{g}_{i'+l'}$. But, $i' \geq i, l' \geq l$, so $\mathfrak{g}_{i'+l'} \subseteq \mathfrak{g}_{(i+l)}$.

Going from a filtration to gradation is not as straightforward, however.

Definition 3.3.4. Let $\mathfrak{g}$ be Lie algebra and $(\mathfrak{g}_{(n)})_{n\in\mathbb{Z}}$ a descending filtration. Define

$$\mathfrak{g}_i = \mathfrak{g}_{(i)}/\mathfrak{g}_{(i+1)}$$

and endow the vector space $\mathrm{gr}(\mathfrak{g}) = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ with the structure of a Lie algebra by setting

$$[x + \mathfrak{g}_{(i+1)}, y + \mathfrak{g}_{(j+1)}] = [x, y] + \mathfrak{g}_{(i+j+1)}$$

for all $x \in \mathfrak{g}_{(i)}$ and $y \in \mathfrak{g}_{(j)}$. The $\mathbb{Z}$-graded Lie algebra $(\mathrm{gr}(\mathfrak{g}), (\mathfrak{g}_i)_{i\in\mathbb{Z}})$ is called the *graded Lie algebra associated with* $(\mathfrak{g}, (\mathfrak{g}_{(i)})_{i\in\mathbb{Z}})$.

Remark. One can carry out the above construction for ascending filtrations as well.

Analogous definitions and constructions work for algebras in general where one replaces the Lie bracket with the algebra multiplication.

Definition 3.3.5. A filtration $(\mathfrak{g}_{(n)})_{n\in\mathbb{Z}}$ of a restricted Lie algebra $(\mathfrak{g}, [p])$ is *restricted* if

$$\mathfrak{g}_{(n)}^{[p]} \subseteq \mathfrak{g}_{(pn)}$$

for all $n \in \mathbb{Z}$.

Definition 3.3.6. A $\mathbb{Z}$-gradation $(\mathfrak{g}_i)_{i\in\mathbb{Z}}$ of a restricted Lie algebra $(\mathfrak{g}, [p])$ is *restricted* if

$$\mathfrak{g}_i^{[p]} \subseteq \mathfrak{g}_{pi}$$

for all $i \in \mathbb{Z}$.

Remark. Note that the zero-graded piece $\mathfrak{g}_0$ for any $\mathbb{Z}$-graded Lie algebra is a Lie subalgebra of $\mathfrak{g}$ (since $[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_0$). Likewise, every subspace $\mathfrak{g}_g$, for $g \in \mathbb{Z}$ is a $\mathfrak{g}_0$-module via the adjoint representation.

Example 3.3.7. Consider the Lie algebra $\mathfrak{g} = \mathrm{Der}_k(k[X])$. If D is the derivation given by $D(X^n) = nX^{n-1}$, then $\mathfrak{g} = k\left\langle X^i D : i \in \mathbb{N}\right\rangle$. Then $\mathfrak{g}$ can be $\mathbb{Z}$-graded by putting

$\mathfrak{g}_n := k\left\langle X^{n+1}D\right\rangle$, since $[X^n D, X^m D] = (m-n)\, X^{n+m-1} D \in \mathfrak{g}_{n+m-2}$, as a direct computation shows.

Before proceeding to the last section, we give a generalisation of the notion of a $\mathbb{Z}$-gradation.

Definition 3.3.8. Let $\mathfrak{g}$ be $\mathbb{Z}$-graded. Let V be a $\mathfrak{g}$-module. We say that V is *$\mathbb{Z}$-graded* if there are subspaces V_g such that

1. $V = \bigoplus_{g\in\mathbb{Z}} V_g$;
2. $\mathfrak{g}_g \cdot V_h \subseteq V_{g+h}$ for all $g, h \in \mathbb{Z}$.

The elements $v \in V_g$ are said to be *homogeneous of degree* g.

One can then show that (see (Strade and Farnsteiner, 1988, §3.4, Prop 4.4) for a proof of this result):

Proposition 3.3.9. *Let V, W be finite-dimensional $\mathbb{Z}$-graded $\mathfrak{g}$-modules. Then* $\mathrm{Hom}_k(V, W)$ *is $\mathbb{Z}$-graded via*

$$\mathrm{Hom}_k(V, W)_g := \{f \in \mathrm{Hom}_k(V, W) : f(V_h) \subseteq W_{h+g} \text{ for all } h \in \mathbb{Z}\}.$$

3.4 Representations of modular Lie algebras

For an excellent exposition of the following, and proofs of the main results, see (Strade and Farnsteiner, 1988, §5.2–§5.9).

The following is Theorem 2.4 in (Strade and Farnsteiner, 1988, p. 207).

Theorem 3.4.1 (N. Jacobson). *Let k be a field of positive characteristic and let $\mathfrak{g}$ be a finite-dimensional Lie algebra over k. Then every simple $\mathfrak{g}$-module V is finite-dimensional.*

For the following theorem see (Strade and Farnsteiner, 1988, §5.2, Theorem 2.5, p. 207):

Theorem 3.4.2. *Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie algebra over an algebraically closed field and let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be an irreducible representation. Then there is a linear form $S \in \mathfrak{g}^*$ such that*

$$\rho(x)^p - \rho(x^{[p]}) = S(x)^p \,\mathrm{id}_V$$

for all $x \in \mathfrak{g}$.

Definition 3.4.3. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra over k and let $S \in \mathfrak{g}^*$ be a linear form. A representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is called an S-*representation* if

$$\rho(x)^p - \rho(x^{[p]}) = S(x)^p \operatorname{id}_V$$

for all $x \in \mathfrak{g}$. The form S is called the *character* of the representation. Putting $S = 0$ gives us the whole theory of p-representations.

Therefore, every simple module over a finite-dimensional restricted Lie algebra over an algebraically closed field has a character.

Definition 3.4.4. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra and supposed it is graded, so $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, and let $(\mathfrak{g}, (\mathfrak{g}_{(i)})_{i \in \mathbb{Z}})$ be the associated filtration. For a character χ, we define the useful notion of the *height* of a character:

$$\operatorname{ht} \chi = \min \left\{ i \geq -1 : \chi(\mathfrak{g}_{(i)}) = 0 \right\}.$$

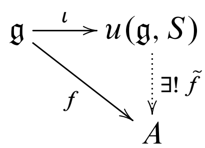
Remark. Note that simple restricted modules correspond to modules of character height -1.

Now we introduce a generalisation of the restricted universal enveloping algebra:

Definition 3.4.5. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra over k and let $S \in \mathfrak{g}^*$. A pair $(u(\mathfrak{g}, S), i)$ consisting of a homomorphism of Lie algebras $\iota : \mathfrak{g} \longrightarrow u(\mathfrak{g}, S)^-$, for $u(\mathfrak{g}, S)$ some associative k-algebra with unity such that

$$\iota(x)^p - \iota(x^{[p]}) = S(x)^p 1$$

for all $x \in \mathfrak{g}$ is an S-*reduced universal enveloping algebra* of $\mathfrak{g}$ if for every associative k-algebra with unity A and every Lie algebra homomorphism $f : \mathfrak{g} \longrightarrow A^-$ such that $f(x)^p - f(x^{[p]}) = S(x)^p 1$ for all $x \in \mathfrak{g}$, there is a unique associative homomorphism $\tilde{f} : u(\mathfrak{g}, S) \longrightarrow A$ such that the following diagram



commutes.

Remark. The S-reduced universal enveloping algebra exists, and any two S-reduced universal enveloping algebras are isomorphic, by the usual categorical argument. We have $u(\mathfrak{g}, 0) = \mathfrak{u}(\mathfrak{g})$. Also, $u(\mathfrak{g}, S)$ is obtained as the quotient of $U(\mathfrak{g})$ by the ideal generated by

$$\left\{x^p - x^{[p]} - S(x)^p 1 : x \in \mathfrak{g}\right\}.$$

If $(x_i)_{i\in\Lambda}$ is an ordered basis for $\mathfrak{g}$, $u(\mathfrak{g}, S)$ has a PBW-type k-basis given by

$$\left\{\iota(x_{j_1})^{s_1} \cdots \iota(x_{j_n})^{s_n} : n \in \mathbb{N}, j_1 < j_2 < \cdots < j_n \in \Lambda, 0 \leq s_i \leq p-1\right\}.$$

The map ι is then injective and so, as in the case of $U(\mathfrak{g})$, we identify $\mathfrak{g}$ with its image $\iota(\mathfrak{g})$. For more details see Theorem 3.1 and its proof in §5.3 in Strade and Farnsteiner (1988). As in the restricted case, we have $\dim_k u(\mathfrak{g}, S) = p^n$ if $\dim_k \mathfrak{g} = n$.

Remark. As in the case of modules over the universal enveloping algebra $U(\mathfrak{g})$ and $\mathfrak{g}$-modules, there is a natural equivalence between modules over the S-reduced universal enveloping algebra $u(\mathfrak{g}, S)$ and $\mathfrak{g}$-representations with character S.

Remark. If $\mathfrak{h}$ is a restricted subalgebra of $\mathfrak{g}$, then $u(\mathfrak{h}, S)$ can be naturally identified with a subalgebra of $u(\mathfrak{g}, S)$.

For the following theorem see (Strade and Farnsteiner, 1988, §5.3, Corollary 3.2, p. 214):

Theorem 3.4.6. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Then for all $S \in \mathfrak{g}^*$ there is an irreducible S-representation of $\mathfrak{g}$, i.e., every linear form S is the character of some irreducible representation.*

For the following theorem see (Strade and Farnsteiner, 1988, §5.2, Theorem 2.7, p. 211):

Theorem 3.4.7. *Let V_i be $\mathfrak{g}$-modules with characters S_i for $i = 1, 2$. Then*

1. $\mathrm{Hom}_k(V_1, V_2)$ *is a $\mathfrak{g}$-module with character $S_2 - S_1$;*
2. V_1^* *is a $\mathfrak{g}$-module with character $-S_1$;*
3. $V_1 \otimes_k V_2$ *is a $\mathfrak{g}$-module with character $S_1 + S_2$.*

In particular, we see that the restrictedness of restricted modules (so when $S = 0$) is preserved by taking Hom-spaces, tensor products, and duals, i.e. if V_1 and V_2 are restricted $\mathfrak{g}$-modules, then $\mathrm{Hom}_k(V_1, V_2)$ and $V_1 \otimes_k V_2$ are both restricted $\mathfrak{g}$-modules; furthermore V_1^* and V_2^* are both restricted $\mathfrak{g}$-modules.

Definition 3.4.8. A representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is called *completely reducible* if there is a family of irreducible submodules $(V_i)_{i \in I}$ such that $V = \bigoplus_{i \in I} V_i$. Alternatively, it is completely reducible if for every submodule W there exists a submodule U such that $V = W \oplus U$.

Weyl's complete reducibility tells us that every finite-dimensional representation of a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic 0 is completely reducible. This, however, is not the case in the modular world.

Example 3.4.9. Consider the $\mathfrak{sl}_2$-submodule $\Gamma_p \leq k[x, y]$ consisting of all the homogeneous polynomials of degree p (see Example 3.1.23 and Example 3.1.26). We claim that Γ_p is not completely reducible. Indeed, consider the subspace $k \langle x^p, y^p \rangle$. It is an $\mathfrak{sl}_2$-submodule, since all the elements act trivially on it. Suppose that there exists a submodule $V \leq \Gamma_p$ such that $V \oplus k \langle x^p, y^p \rangle = \Gamma_p$. Let $v \in V \setminus \{0\}$. We can write

$$v = \sum_{i=j}^{p-1} \lambda_i x^i y^{p-i},$$

where $j > 0$ and $\lambda_j \neq 0$. Now, $(\rho(e))^{p-j}(v) = c\lambda_j x^p$, where $c \neq 0$. Therefore, $x^p \in V$, a contradiction. Thus, Γ_p is not completely reducible.

The following theorem (contrast with Weyl's theorem) is due to Jacobson (see (Strade and Farnsteiner, 1988, §5.5, p. 220)):

Theorem 3.4.10. *Let $\mathfrak{g} \neq \{0\}$ be a finite-dimensional Lie algebra over a field of positive characteristic. Then $\mathfrak{g}$ possesses a finite-dimensional faithful completely reducible representation.*

Now we define the important notion of an induced representation (see (Strade and Farnsteiner, 1988, §5.6) for more details):

Definition 3.4.11 (Induced Representation). Let $\mathfrak{h}$ be a p-subalgebra of a restricted Lie algebra $(\mathfrak{g}, [p])$. Let $S \in \mathfrak{g}^*$. Let M be a left $\mathfrak{h}$-module with character S. Then

$$\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(M, S) := u(\mathfrak{g}, S) \otimes_{u(\mathfrak{h}, S)} M$$

is a left $\mathfrak{g}$-module with character S. It is referred to as the $\mathfrak{g}$-*module induced by the* $\mathfrak{h}$-*module* M.

Remark. The tensor product over $u(\mathfrak{h}, S)$ is taken seeing $u(\mathfrak{g}, S)$ as a right $u(\mathfrak{h}, S)$-module and M as a left $u(\mathfrak{h}, S)$-module. See Definition 2.1.14.

Remark. Lemma 2.1.16 says that the action of $\mathfrak{g}$ on $\operatorname{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(M, S)$ is given by

$$y \cdot (x \otimes m) = y \cdot x \otimes m = yx \otimes m,$$

for all $y \in \mathfrak{g}$, $x \in u(\mathfrak{g}, S)$, $m \in M$.

Induced modules will be very important in classifying the simple modules for the Hamiltonian algebra $H(2; (1, 1); \Phi(1))$, as we will later see. The following well-known theorem will be of much import too.

Theorem 3.4.12 (Frobenius Reciprocity). *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra over k, and $S \in \mathfrak{g}^*$. Let V be a $\mathfrak{g}$-module with character S and let $\mathfrak{h} \leq \mathfrak{g}$ be a p-subalgebra. Let M be an $\mathfrak{h}$-module with character $S_{\mathfrak{h}} \in \mathfrak{h}^*$. Then:*

$$\operatorname{Hom}_{\mathfrak{h}}(M, V_{Res}) \cong \operatorname{Hom}_{\mathfrak{g}}(\operatorname{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(M, S), V),$$

where V_{Res} is simply the restriction of the $\mathfrak{g}$-module V to $\mathfrak{h}$.

For a proof for Frobenius reciprocity, we refer the interested reader to (Strade and Farnsteiner, 1988, §5.6, Theorem 6.3). From it, we see that if $f \in \operatorname{Hom}_{\mathfrak{h}}(M, V_{\mathrm{Res}})$, then the map

$$x \otimes m \mapsto x \cdot f(m)$$

is a $\mathfrak{g}$-homomorphism $\operatorname{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(M, S) \longrightarrow V$.

Concerning dimensions, we have (assuming that $\mathfrak{g}$ and M are finite-dimensional):

$$\dim_k \operatorname{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(M, S) = p^{\dim_k \mathfrak{g}/\mathfrak{h}} \dim_k M.$$

A proof of this can be found in (Strade and Farnsteiner, 1988, §5.6, Prop. 6.2).

The restricted representation theory of $\mathfrak{sl}_2$ will be of importance to us when we classify the simple restricted modules of $H(2; (1, 1); \Phi(1))$. For more details, see, for example, (Strade and Farnsteiner, 1988, §5.2, pp. 207–209).

Theorem 3.4.13. *There are p isomorphism classes of restricted irreducible representations of $\mathfrak{sl}_2$. The classes have representatives $L(z)$ for $z \in \{0, 1, \ldots, p-1\}$, where $L(z)$ has dimension $z + 1$.*

Remark. These representations have already been introduced as $\Gamma_z = L(z)$ in Example 3.1.26.

Remark. Richard E. Block (see Block (1979, 1981)) has classified all the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$, including all the infinite-dimensional ones. The simple modules fall into three types: highest weight modules, Whittaker modules, and those belonging to a family of pairwise nonisomorphic and mostly new modules that the author constructs in Block (1981).

Chapter 4

Derivation algebras and Lie algebras of Cartan type

In this chapter we will study the Lie algebras of derivations of divided power algebras as well as certain important Lie subalgebras of these algebras of derivations, such as the generalised Jacobson-Witt algebra $W(m;\underline{n})$. These algebras turn out to be important since the Hamiltonian algebras we look at in later chapters will be defined in relation to these algebras.

Recall that a *derivation* of an algebra A over k is a k-linear map $D : A \longrightarrow A$ such that

$$D(ab) = D(a)b + aD(b)$$

for all $a, b \in A$. In the context of Lie algebras, a derivation D of a Lie algebra $\mathfrak{g}$ thus satisfies $D([a, b]) = [D(a), b] + [a, D(b)]$ for all $a, b \in \mathfrak{g}$. Recall that $\mathrm{Der}_k(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

Any such derivation D satisfies *Leibniz's rule*:

$$D^n(xy) = \sum_{i=0}^{n} \binom{n}{i} D^i(x)D^{n-i}(y).$$

In particular, if k is a field of characteristic $p > 0$, we see that $D^p(xy) = xD^p(y) + D^p(x)y$ for all $x, y \in A$, since the binomial coefficients $\binom{p}{i}$ for all $1 \leq i \leq p-1$ are divisible by p. Therefore, in the modular case, $D^p \in \mathrm{Der}_k(A)$, proving indeed that $\mathrm{Der}_k(A)$ is a restricted Lie algebra.

It turns out that several important families of Cartan-type modular Lie algebras, not just the Hamiltonian algebras, are closely related to certain types of derivation algebras.

4.1 Graded Lie algebras of Cartan type

Rather than going into the general theory of graded Lie algebras of Cartan type, in this section we will study in some detail the simplest family of these, the W, or Witt, family.

Let k be a field of characteristic $p > 0$. Consider the truncated polynomial ring

$$k[X]/(X^p).$$

We consider the Lie algebra $\mathrm{Der}_k(k[X]/(X^p))$. It turns out that this Lie algebra admits the following k-basis: $\{\partial, x\partial, \ldots, x^{p-1}\partial\}$, where x denotes the image of X in the quotient, i.e. $x = X + (X^p)$, and ∂ is the derivation of $k[X]/(X^p)$ defined by $\partial(x^a) = ax^{a-1}$.

It turns out that the Lie bracket on basis elements satisfies:

$$[x^i\partial, x^j\partial] = (j-i)x^{i+j-1}\partial.$$

This Lie algebra is called the *first Witt algebra*. It is denoted $W(1;1) = W_1$. It is restricted with p-mapping given by p-fold composition: $D^{[p]} = D^p$ for all $D \in W_1$. If $p \geq 3$, it is simple. If $p = 3$, in fact $W_1 \cong \mathfrak{sl}_2(k)$, but for higher characteristic, it is not of classical type (i.e. not a Lie algebra that is a modular analogue of a simple Lie algebra in characteristic 0). It is graded as follows.

Definition 4.1.1. Define the r-th graded piece to be:

$$W(1;1)_r = k\left\langle x^{r+1}\partial\right\rangle.$$

Then $W(1;1) = \bigoplus_{r=-1}^{p-2} W(1;1)_r$.

The first Witt algebra is the simplest example of a modular Lie algebra of type W, and indeed it is the simplest example of a graded Lie algebra of Cartan type, for whose definition see Definition 5.1.9 and the second remark following it.

We want to generalise this somewhat. Suppose one was to consider a truncated polynomial ring

$$k[X_1, \ldots, X_m]/\left(X_1^{p^{n_1}}, \ldots, X_m^{p^{n_m}}\right).$$

This would present problems once any of the $n_i > 1$. This is principally because once one considers elements such as $x_1^p \neq 0$ they will admit no non-zero derivations, since if D is a derivation, an inductive argument shows $D(x_1^p) = px_1^{p-1}D(x_1) = 0$.

To get around this fact, it is necessary to introduce *algebras of divided powers*. From now on, let m be a positive integer and let $\underline{n} \in \mathbb{N}_{>0}^m$.

Definition 4.1.2. The algebra $O(m)$ is the associative and commutative algebra (with unit element) with k-basis

$$\left\{x^{(a)} : a \in \mathbb{N}^m, 0 \leq a_i\right\}$$

satisfying

$$x^{(a)}x^{(b)} = \binom{a+b}{a} x^{(a+b)},$$

where $a+b = (a_1+b_1, \ldots, a_m+b_m)$ and $\binom{a}{b} = \prod_{i=1}^{m} \binom{a_i}{b_i}$, adopting the convention that $\binom{a_i}{b_i} = 0$ if $b_i > a_i$.

The *divided power algebra* $O(m;\underline{n})$ is the associative and commutative subalgebra (with unit element) of $O(m)$ with k-basis

$$\left\{x^{(a)} : a \in \mathbb{N}^m, 0 \leq a_i \leq p^{n_i} - 1\right\}.$$

Remark. Note that if $a_i + b_i \geq p^{n_i}$, we have that p^{n_i}, and therefore p, divides the binomial coefficient $\binom{a_i+b_i}{a_i}$. Thus the binomial coefficient is zero in k and $x^{(a)}x^{(b)} = 0$. Therefore, $O(m;\underline{n})$ is indeed a subalgebra.

For a more detailed description of this algebra in terms of generators and relations, see for instance (Strade, 2004, §2.1, pp. 59–60) or (Strade, 2004, p. 3) for a quick overview. For a description in terms of "divided" formal power series, see (Strade and Farnsteiner, 1988, §3.5, p. 132), where the algebra is denoted $A(m;\underline{n})$.

We now give a construction of $O(m)$. Consider the polynomial ring $\mathbb{C}[X_1, \ldots, X_m]$. Define $X_i^{(r)} = \frac{1}{r!}X_i^r$ for all $i \in \{1, \ldots, m\}$ and $r \geq 0$. Then

$$\mathcal{P}_{\mathbb{Z}} := \sum_{a \in \mathbb{N}^m} \left(\mathbb{Z} \prod_{i=1}^{m} X_i^{(a_i)}\right)$$

is a $\mathbb{Z}$-subalgebra of $\mathbb{C}[X_1, \ldots, X_m]$. It turns out that $O(m) \cong k \otimes_{\mathbb{Z}} \mathcal{P}_{\mathbb{Z}}$.

The algebra $O(m;\underline{n})$ is graded via

$$O(m;\underline{n})_i := k\left\langle x^{(a)} : |a| = i\right\rangle,$$

setting $|a| := a_1 + a_2 + \ldots + a_m$. This gradation gives rise to the filtration (see §3.3 for more details)

$$O(m;\underline{n})_{(i)} := k\left\langle x^{(a)} : |a| \geq i\right\rangle.$$

To see that divided powers occur naturally, observe that if we apply the partial derivative $\frac{\partial}{\partial X_1}$ to the polynomial $X_1^a X_2^b \in \mathbb{C}[X_1, X_2]$, say, we obtain $aX_1^{a-1}X_2^b$. This can grow unwieldy as coefficients grow in number as one applies partial derivatives repeatedly. To circumvent this problem, put $X_i^{(a_i)} = \frac{1}{a_i!}X_i^{a_i}$ for all i. Now applying the partial derivative to $X_1^{(a)}X_2^{(b)}$ yields

$$\frac{a}{a!}X_1^{a-1}X_2^{(b)} = X_1^{(a-1)}X_2^{(b)}.$$

More generally, in the commutative polynomial ring in n variables $\mathbb{C}[X_1, \ldots, X_n]$, applying the i-th partial derivative $\partial_{x_i} = \frac{\partial}{\partial x_i}$ to a polynomial $X_1^{(a_1)}X_2^{(a_2)} \cdots X_n^{(a_n)}$ yields

$$X_1^{(a_1)} \cdots X_{i-1}^{(a_{i-1})} X_i^{(a_i - 1)} X_{i+1}^{(a_{i+1})} \cdots X_n^{(a_n)}.$$

We will sometimes write $X^{(a)} = (X_1 \cdots X_n)^{(a_1, \ldots, a_n)}$ for $X_1^{(a_1)}X_2^{(a_2)} \cdots X_n^{(a_n)} = \prod_{i=1}^n X_i^{(a_i)}$. Here, again, we refer the reader to (Strade, 2004, §2.1, pp. 59–62). In the quotient, we denote the image of the monomial X_i by x_i, for all $1 \leq i \leq m$. It is easy to see that

$$X^{(r)}X^{(s)} = \binom{r+s}{r} X^{(r+s)}.$$

Furthermore, they arise in the algebra of distributions for a connected reductive algebraic group, see (Jantzen, 1987, Part II, §1.12, pp. 184–185) for more details. They also occur naturally in Kostant $\mathbb{Z}$-form of $U(\mathfrak{g})$ for $\mathfrak{g}$ a complex semisimple Lie algebra, see (Kostant, 1966, §2.3–2.5) for more details.

Now, going back to W_1 and the truncated polynomial algebra $k[X]/(X^p)$, we do have the following isomorphism more generally:

$$k[X_1, \ldots, X_m]/\left(X_1^p, \ldots, X_m^p\right) \cong O(m; \underline{1})$$

where $\underline{1} := (1, 1, \ldots, 1) \in \mathbb{N}^m$.

Another interesting isomorphism is (see (Strade, 2004, §7.6, p. 423)):

$$O(m; \underline{n}) \cong O\left(\sum_{i=1}^m n_i; \underline{1}\right) = O(|\underline{n}|; \underline{1}).$$

In particular,

$$O(m; \underline{n}) \cong k[X_1, \ldots, X_{|\underline{n}|}]/\left(X_1^p, \ldots, X_{|\underline{n}|}^p\right).$$

Definition 4.1.3. A derivation $D : O(m;\underline{n}) \longrightarrow O(m;\underline{n})$ is said to be *special* if

$$D(x^{(a)}) = \sum_{i=1}^{m} x^{(a-\varepsilon_i)} D(x_i),$$

where ε_i is the m-tuple with j-th entry given by δ_{ij} and $x_i := x^{(\varepsilon_i)}$.

Definition 4.1.4. The Lie algebra $W(m;\underline{n})$, where $\underline{n} \in \mathbb{N}^m_{>0}$, called the *generalised Jacobson-Witt algebra*, is the Lie algebra consisting of the special derivations of $O(m;\underline{n})$.

For the following proposition see (Strade and Farnsteiner, 1988, §4.2, Theorem 2.4, p. 149):

Proposition 4.1.5. *The algebra $W(m;\underline{n})$ is simple and has dimension $mp^{|\underline{n}|}$.*

Remark. Observe that $\dim_k O(m;\underline{n}) = p^{|\underline{n}|}$.

It has (in divided power notation) k-basis

$$\left\{x_1^{(a_1)} x_2^{(a_2)} \cdots x_m^{(a_m)} \partial_{x_i} : 0 \leq a_i \leq p^{n_i} - 1, i = 1, \ldots, m\right\},$$

where ∂_{x_i} is the (special) derivation uniquely determined by the property

$$\partial_{x_i} x_1^{(a_1)} x_2^{(a_2)} \cdots x_m^{(a_m)} = x_1^{(a_1)} \cdots x_{i-1}^{(a_i-1)} x_i^{(a_i-1)} x_{i+1}^{(a_i+1)} \cdots x_m^{(a_m)},$$

that is,

$$\partial_{x_i} x^{(a)} = x^{(a-\varepsilon_i)},$$

where we adopt the convention that $x^{(b)} = 0$ if any of the $b_i < 0$. The derivation ∂_{x_i} is called the *i-th partial derivative* of $O(m;\underline{n})$.

Definition 4.1.6. We set $W(m) = \sum_{i=1}^{m} O(m)\partial_{x_i}$.

Clearly, $W(m)$ has k-basis

$$\left\{x^{(a)} \partial_{x_i} : 0 \leq a_i, i = 1, \ldots, m\right\}.$$

The Lie bracket on the basis elements of $W(m;\underline{n})$ and $W(m)$ is given by the formula (see the proof of Proposition 5.9 in Chapter 3 of Strade and Farnsteiner (1988) or (Holmes, 2001, p. 448))

$$[x^{(a)}\partial_{x_i}, x^{(b)}\partial_{x_j}] = \binom{a+b-\varepsilon_i}{a} x^{(a+b-\varepsilon_i)}\partial_{x_j} - \binom{a+b-\varepsilon_j}{b} x^{(a+b-\varepsilon_j)}\partial_{x_i}, \quad (4.1)$$

for $a, b \in \mathbb{N}^m, i, j \in \{1, \ldots, m\}$.

Moreover, $W(m;\underline{n})$ is restricted if and only if $\underline{n} = \underline{1} = (1, 1, \ldots, 1)$, and in such a case, it is isomorphic to the full derivation algebra of the truncated polynomial ring $k[X_1, \ldots, X_m]/\left(X_1^p, \ldots, X_m^p\right)$:

$$\mathrm{Der}_k\left(k[X_1, \ldots, X_m]/\left(X_1^p, \ldots, X_m^p\right)\right) \cong W(m;\underline{1}).$$

Additionally, it is the full derivation algebra $\mathrm{Der}_k(O(m;\underline{1}))$. For a proof that $\mathrm{Der}_k(O(m;\underline{1})) = W(m;\underline{1})$, we refer the reader to Proposition 5.9, part (3) in (Strade and Farnsteiner, 1988, p. 132). In general it is only true that $W(m;\underline{n}) \leq \mathrm{Der}_k(O(m;\underline{n}))$, see Proposition 5.9, part (2) in the previous reference. Furthermore, Theorem 2.4 in (Strade and Farnsteiner, 1988, p. 149) tells us that $W(m;\underline{n})$ is restrictable if and only if $\underline{n} = \underline{1}$, and in that case $D^{[p]} = D^p$ for all $D \in W(m;\underline{n})$.

More generally, $W(m;\underline{n})$ is in fact a free $O(m;\underline{n})$-module with basis $\{\partial_{x_i} : i = 1, \ldots, m\}$. For a proof of this fact, see Proposition 5.9, part (1) in (Strade and Farnsteiner, 1988, p. 133). Alternatively, (Strade, 2004, §2.1, p. 60) defines $W(m;\underline{n})$ as this free module: $W(m;\underline{n}) = \sum_{i=1}^m O(m;\underline{n})\partial_{x_i}$.

The algebra $W(m;\underline{n})$ acts naturally on $O(m;\underline{n})$ via $D \cdot f = D(f)$. This is called the *canonical representation of* $W(m;\underline{n})$, see Example 3.1.31 for more details.

The algebra $W(m;\underline{n})$ is graded as follows.

Definition 4.1.7. Define r-th graded piece to be

$$W(m;\underline{n})_r = k\left\langle x^{(a)}\partial_{x_i} : |a| = r + 1, i = 1, \ldots, m\right\rangle,$$

so that the "degree" function on monomial derivations $x^{(a)}\partial_{x_i}$ of the Witt algebra is computed by taking the degree of the monomial $x^{(a)}$ and subtracting one.

One has

$$W(m;\underline{n}) = \bigoplus_{r=-1}^{s} W(m;\underline{n})_r,$$

where $s = \sum_{i=1}^m (p^{n_i} - 1) - 1 = \sum_{i=1}^m (p^{n_i}) - m - 1$. Furthermore, when $\underline{n} = \underline{1}$ the gradation is restricted (that is, $W(m;\underline{1})_i^{[p]} \subseteq W(m;\underline{1})_{pi}$ for all i).

Proposition 4.1.8. *One has the isomorphism:*

$$W(m;\underline{n})_0 = k\left\langle x_i\partial_{x_j} : i, j = 1, \ldots, m\right\rangle \cong \mathfrak{gl}_m(k).$$

Furthermore, this is a restricted representation of $W(m;\underline{n})_0$.

Proof. We map $x_i\partial_{x_j} \mapsto E_{i,j} \in M_n(k)$, see Example 3.1.10 for more details. Notice that $W(m;\underline{n})_0$ can be seen as endomorphisms of the m-dimensional k-vector space $k\langle x_1, x_2, \dots, x_m\rangle$. Since $x_i\partial_{x_j}(x_t) = \delta_{jt}x_i$ for all $1 \le t \le m$, we see that the matrix of $x_i\partial_{x_j}$ with respect to the basis $\{x_1, \dots, x_m\}$ is just $E_{i,j}$.

To see that this is a restricted representation, observe that $\mathfrak{gl}_m(k)$ is a restricted Lie algebra with p-mapping given by p-fold composition of functions, and that the p-mapping on $W(m;\underline{n})_0$ is also given by p-fold composition of functions. ■

An important subalgebra of $W(m;\underline{n})$ is

$$\mathfrak{t} := k\left\langle x^{(\varepsilon_i)}\partial_{x_i} : i = 1, \dots, m\right\rangle = k\left\langle x_i\partial_{x_i} : i = 1, \dots, m\right\rangle$$

Clearly $\mathfrak{t}$ is an abelian subalgebra. If $\underline{n} = \underline{1}$, Theorem 2.5 in (Strade and Farnsteiner, 1988, §4.2, pp. 150–151) tells us that $\mathfrak{t}$ is a torus. In particular, $\left(x^{(\varepsilon_i)}\partial_{x_i}\right)^{[p]} = x^{(\varepsilon_i)}\partial_{x_i}$ for all $i = 1, \dots, m$.

As we saw in Chapter 1, Shen in Shen (1988a) classified the restricted simple $W_n = W(n;\underline{1})$-modules. Holmes in Holmes (2001) expanded this work to all simple W_n-modules of character height at most one, see Theorem 4.3 and Theorem 4.4 in Holmes (2001) for more details.

Chapter 5

Hamiltonian Lie algebras

The family of Hamiltonian Lie algebras is the centre-piece of this thesis. We give an introduction here to set the stage for the later chapters, focusing on the Hamiltonian algebras most relevant to our discussion.

Throughout this chapter, let m be a positive integer and let $\underline{n} \in \mathbb{N}_{>0}^m$. We take our base field to be an algebraically closed field k of characteristic $p \geq 3$.

Let $\mathcal{O}((m))$ and $\mathcal{O}((m; \underline{n}))$ denote the topological completions of $\mathcal{O}(m)$ and $\mathcal{O}(m; \underline{n})$, respectively, with respect to the topologies induced from the descending chains of ideals $(\mathcal{O}(m)_{(j)})_{j\geq 0}$ and $(\mathcal{O}(m; \underline{n})_{(j)})_{j\geq 0}$, respectively.

One can then go on to define $W((m))$ and $W((m; \underline{n}))$ as well as $H((m))$ and $H((m; \underline{n}))$. It turns out, however, that $H(m; \underline{n}) = H((m; \underline{n}))$, see (Strade, 2004, §4.2, p. 186) for more details. Thus, we restrict in this chapter our attention to $H(m)$ and $H(m; \underline{n})$.

5.1 Hamiltonian forms and graded Hamiltonians

We shall start by looking at the general family of Hamiltonians.

For a more in-depth treatment of the topics covered in this section, we refer the interested reader to (Strade, 2004, §4.2). Let the space of $\mathcal{O}(m)$-homomorphisms $\mathrm{Hom}_{\mathcal{O}(m)}(W(m), \mathcal{O}(m))$ be denoted by $\Omega^1(m)$[1]. Define $\mathrm{d} : \mathcal{O}(m) \longrightarrow \Omega^1(m)$ by

$$\mathrm{d}(f)(D) = D(f),$$

for all $f \in \mathcal{O}(m)$, $D \in W(m)$.

[1]This becomes an $\mathcal{O}(m)$-module via $(f \cdot \lambda)(D) := f\lambda(D)$ for all $f \in \mathcal{O}(m)$, $\lambda \in \Omega^1(m)$, $D \in W(m)$.

Now, $\Omega^1(m)$ can be given the structure of a $W(m)$-module. First, we define an action of $W(m)$ on $\mathrm{Hom}_k(W(m), O(m))$. To do so we apply Definition 3.1.34, where we take $W(m)$ to be a $W(m)$-module via the adjoint representation, i.e., $D \cdot D' = [D, D']$ and the action of $W(m)$ on $O(m)$ is simply $D \cdot f = D(f)$. Thus, we have $D(\alpha)(D') = D(\alpha(D')) - \alpha([D, D'])$ for all $D, D' \in W(m)$ and all $\alpha \in \Omega^1(m)$. It remains to show that $D(\alpha) \in \Omega^1(m)$ for all $D \in W(m), \alpha \in \Omega^1(m)$. Let $D, D' \in W(m)$, $f \in O(m)$. We wish to show

$$D(\alpha)(fD') = f(D(\alpha)(D')).$$

We have $D(\alpha)(fD') = D(\alpha(fD')) - \alpha([D, fD'])$. Since α is an $O(m)$-homomorphism, we have $D(\alpha(fD')) = D(f\alpha(D'))$. By the product rule, we have $D(f\alpha(D')) = D(f)\alpha(D') + fD(\alpha(D'))$. One calculates the following identity in $W(m)$: $[D, fD] = f[D, D'] + D(f)D'$. Therefore, using again the fact that α is an $O(m)$-homomorphism,

$$\alpha([D, fD']) = \alpha(f[D, D']) + \alpha(D(f)D') = f\alpha([D, D']) + D(f)\alpha(D').$$

Hence, $D(\alpha)(fD') = D(f)\alpha(D') + fD(\alpha(D')) - (f\alpha([D, D']) + D(f)\alpha(D')) = fD(\alpha(D')) - f\alpha([D, D']) = f(D(\alpha)(D'))$, as required.

It turns out that d is a $W(m)$-homomorphism between $O(m)$ and $\Omega^1(m)$. Indeed, it is a direct computation to verify that $\mathrm{d}(D(f)) = D \cdot \mathrm{d}(f)$ for all $D \in W(m)$, $f \in O(m)$.

Proposition 5.1.1. *The space $\Omega^1(m)$ is a free $O(m)$-module with basis $\{\mathrm{d}x_1, \dots, \mathrm{d}x_m\}$, so $\Omega^1(m) = \sum_{i=1}^m O(m)\mathrm{d}x_i$, recalling that $O(m)$ acts on $\Omega^1(m)$ via $(f \cdot \lambda)(D) = f\lambda(D)$ for all $f \in O(m), \lambda \in \Omega^1(m), D \in W(m)$.*

Proof. Since $W(m)$ is a free $O(m)$-module with basis $\{\partial_{x_1}, \dots, \partial_{x_m}\}$, if $D \in W(m)$, we can write $D = \sum_{i=1}^m f_i \partial_{x_i}$, where $f_i \in O(m)$ for all i. Hence,

$$\lambda(D) = \sum_{i=1}^m f_i \lambda(\partial_{x_i}).$$

Therefore, the action of $\lambda \in \Omega^1(m)$ is determined by how λ acts on the partial derivatives $\partial_{x_1}, \dots, \partial_{x_m}$.

Consider the map $\alpha = \sum_{j=1}^m \lambda(\partial_{x_j})\mathrm{d}x_j \in \Omega^1(m)$. Observe that $\alpha(\partial_{x_i}) = \lambda(\partial_{x_i})$ for all i. Thus, α acts the same way λ acts on the partial derivatives, so $\alpha = \sum_{j=1}^m \lambda(\partial_{x_j})\mathrm{d}x_j = \lambda$. ■

Remark. Taking $\lambda = \mathrm{d}(f)$ for $f \in O(m)$, we see that the action of $\mathrm{d}(f)$ is determined by $\mathrm{d}(f)(\partial_{x_i}) = \partial_{x_i}(f)$. Again, thanks to the fact we that we have $(\mathrm{d}x_i)(\partial_{x_j}) = \delta_{ij}$, we see that the map $\sum_{i=1}^m \partial_{x_i}(f)\mathrm{d}x_i$ acts the same way as $\mathrm{d}(f)$ on the partial derivatives. Therefore, $\mathrm{d}(f) = \sum_{i=1}^m \partial_{x_i}(f)\mathrm{d}x_i$.

Example 5.1.2. Suppose $m \geq 5$. We have $\mathrm{d}(x_1 x_3^{(2)}) \in \Omega^1(m)$. We calculate that $\mathrm{d}(x_1 x_3^{(2)})(\partial_{x_1}) = x_3^{(2)}$, $\mathrm{d}(x_1 x_3^{(2)})(\partial_{x_3}) = x_1 x_3$, and $\mathrm{d}(x_1 x_3^{(2)})(\partial_{x_i}) = 0$ otherwise. Thus,

$$\mathrm{d}(x_1 x_3^{(2)}) = x_3^{(2)}\mathrm{d}x_1 + x_1 x_3 \mathrm{d}x_3.$$

Similarly,

$$\mathrm{d}(x_2^{(p)} x_5^{(p-1)}) = x_2^{(p-1)} x_5^{(p-1)}\mathrm{d}x_2 + x_2^{(p)} x_5^{(p-2)}\mathrm{d}x_5.$$

Definition 5.1.3. The elements of the exterior algebra over $O(m)$ of $\Omega^1(m)$, denoted $\Omega(m)$, are called *differential forms* on $O(m)$. We sometimes also use the r-fold exterior power: $\Omega^r(m) := \bigwedge^r \Omega^1(m)$ (also taken over $O(m)$).

Remark. We have $\Omega^0(m) = O(m)$.

Remark. Replace the algebra $O(m)$ with the algebra $O(m;\underline{n})$ and $W(m)$ with $W(m;\underline{n})$ in all the previous constructions. This gives us $\Omega^1(m;\underline{n})$ and $\Omega(m;\underline{n})$, and so on.

The algebra $\Omega(m)$ is graded via $\Omega(m) = \bigoplus_{r\in\mathbb{N}} \Omega^r(m)$, since clearly for all $r, l \in \mathbb{N}$ we have $\Omega^r(m)\Omega^l(m) \subseteq \Omega^{r+l}(m)$.

Similarly, the algebra $\Omega(m;\underline{n})$ is graded via $\Omega(m;\underline{n}) = \bigoplus_{r\in\mathbb{N}} \Omega^r(m;\underline{n})$.

The elements of $W(m)$ can be extended to be derivations of $\Omega(m)$ by defining:

$$D(\omega_1 \wedge \omega_2) = D(\omega_1) \wedge \omega_2 + \omega_1 \wedge D(\omega_2)$$

for two-fold wedge products, and defining higher-order wedge products inductively, i.e. $D(\omega_1 \wedge \omega_2 \wedge \omega_3) = D(\omega_1 \wedge \omega_2) \wedge \omega_3 + \omega_1 \wedge \omega_2 \wedge D(\omega_3)$ and so on.

The map d can be extended to a square-zero (i.e. $\mathrm{d}^2 = 0$) linear operator on $\Omega(m)$. Indeed, set $\mathrm{d}(f\mathrm{d}g) = \mathrm{d}f \wedge \mathrm{d}g$ for all $f, g \in O(m)$. Extend this inductively to $\Omega(m)$ by defining

$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}(\omega_1) \wedge \omega_2 + (-1)^{\deg(\omega_1)} \omega_1 \wedge \mathrm{d}(\omega_2)$$

for all $\omega_1, \omega_2 \in \Omega(m)$.

The following proposition can be found in (Strade, 2004, §4.2, p. 185).

Proposition 5.1.4. *The map* d *satisfies*

$$\mathrm{d}(f\omega) = (\mathrm{d}f) \wedge \omega + f\mathrm{d}\omega$$

for all $f \in O(m), \omega \in \Omega(m)$.

Definition 5.1.5. The following differential form will be of particular interest:

$$\omega_H := \sum_{i=1}^{r} \mathrm{d}x_i \wedge \mathrm{d}x_{i+r},$$

where $m = 2r \geq 2$. The form ω_H is often called a *Hamiltonian form* (though not the *only* Hamiltonian form, see Definition 5.2.2).

Indeed, this differential form gives rise to the family of *Hamiltonian Lie algebras:*

$$H(2r) := \{D \in W(2r) : D(\omega_H) = 0\}.$$

For $\underline{n} \in \mathbb{N}_{>0}^{2r}$, define

$$H(2r;\underline{n}) = H(2r) \cap W(2r;\underline{n}).$$

What do the elements looks like? How does one compute with them? Now we will attempt to gain a more concrete understanding of what these algebras look like. We will begin by fixing some notation.

Given $\underline{n} \in \mathbb{N}_{>0}^{m}$, we put $\tau = \tau(\underline{n}) = (p^{n_1} - 1, p^{n_2} - 1, \dots, p^{n_m} - 1)$. Put also

$$\sigma(i) := \begin{cases} 1 & \text{if } 1 \leq i \leq r \\ -1 & \text{if } r+1 \leq i \leq 2r \end{cases}$$

and

$$i' := \begin{cases} i + r & \text{if } 1 \leq i \leq r \\ i - r & \text{if } r+1 \leq i \leq 2r. \end{cases}$$

Note that $(i')' = i$ and $\sigma(i') = -\sigma(i)$ for all $1 \leq i \leq 2r$.

Definition 5.1.6. Define a linear operator $D_H : O(2r;\underline{n}) \longrightarrow W(2r;\underline{n})$ by

$$x^{(a)} \mapsto \sum_{i=1}^{2r} \sigma(i)\partial_{x_i}(x^{(a)})\partial_{x_{i'}}.$$

So for instance $D_H(x_i) = \sigma(i)\partial_{x_{i'}}$, $D(x_1) = \partial_{x_{1+r}}$ and (if $r = 2$)

$$D_H(x_2^{(2)}x_3^{(4)}) = x_2 x_3^{(4)}\partial_{x_4} - x_2^{(2)}x_3^{(3)}\partial_{x_1}.$$

The Lie bracket in $W(2r;\underline{n})$ satisfies

$$[D_H(x^{(a)}), D_H(x^{(b)})] = D_H\left(D_H(x^{(a)})(x^{(b)})\right).$$ [2]

Then the graded simple Hamiltonian Lie algebra $H(2r;\underline{n})^{(2)}$ is the k-span

$$k\left\langle D_H(x^{(a)}) : 0 \leq a < \tau(\underline{n})\right\rangle,$$

recalling from Definition 3.1.38 that $H(2r;\underline{n})^{(2)}$ is the second derived subalgebra of $H(2r;\underline{n})$. For an explicit description of $H(2r;\underline{n})$ see (Strade, 2004, §4.2, p. 188). We also have $H(2r;\underline{n})^{(1)} = D_H(O(2r;\underline{n}))$, see (Strade and Farnsteiner, 1988, §4.4, p. 163), bearing in mind that $H(2r;\underline{n})^{(1)}$ is denoted by $H(2r;\underline{n})'$.

The grading of $H(2r;\underline{n})^{(2)}$ is inherited from that of $W(2r;\underline{n})$. That is, we have $H(2r;\underline{n})^{(2)}_l := H(2r;\underline{n})^{(2)} \cap W(2r;\underline{n})_l$ and

$$H(2r;\underline{n})^{(2)} = \bigoplus_{l=-1}^{s} H(2r;\underline{n})^{(2)}_l,$$

where $s = \sum_{i=1}^{2r}(p^{n_i}-1) - 3 = |\tau(\underline{n})| - 3$. We observe that D_H is a linear mapping of degree -2 in the sense of Proposition 3.3.9, i.e.,

$$D_H(O(2r;\underline{n})_i) \subseteq W(2r;\underline{n})_{i-2}$$

for all i.

For the following see (Strade and Farnsteiner, 1988, §4.4, Prop. 4.4), bearing in mind that $H(2r;\underline{n})^{(2)}$ is denoted by $H(2r;\underline{n})$.

Proposition 5.1.7. *We have*

$$H(2r;\underline{n})^{(2)}_0 \cong \mathfrak{sp}_{2r}(k).$$

In (Strade and Farnsteiner, 1988, Thm. 4.5, p. 166) we find the following:

Theorem 5.1.8. *Let $r \geq 1$ be an integer. The Hamiltonian Lie algebra $H(2r;\underline{n})^{(2)}$ is simple and has dimension $p^{\sum_{i=1}^{2r} n_i} - 2 = p^{|\underline{n}|} - 2$. It is restrictable if and only if $n_i = 1$ for*

[2]This identity allows us to define a Lie multiplication on $O(2r;\underline{n})$ via $\{f, g\} := D_H(f)(g)$. The Jacobi identity is satisfied thanks to this identity and that $\{f, f\} = 0$ can be computed explicitly. The mapping $\{,\}$ is usually referred to as the *Poisson bracket.*

all $1 \leq i \leq 2r$, and in that case $H(2r;\underline{n})^{(2)}$ is a p-subalgebra of $W(2r;\underline{n})$ with restricted gradation.

By Exercise 9 in (Strade and Farnsteiner, 1988, §4.4, p. 169) we have that

$$H(2r;\underline{n}) = H(2r;\underline{n}+\underline{1})^{(2)} \cap W(2r;\underline{n}).$$

Proof. The inclusion $H(2r;\underline{n}+\underline{1})^{(2)} \cap W(2r;\underline{n}) \subseteq H(2r;\underline{n})$ is immediate from the definitions. We will now prove $H(2r;\underline{n}) \subseteq H(2r;\underline{n}+\underline{1})^{(2)} \cap W(2r;\underline{n})$. Clearly $H(2r;\underline{n}) \subseteq W(2r;\underline{n})$. We note that the remarks in (Strade and Farnsteiner, 1988, §4.4, p. 163) preceding Lemma 4.1 imply that for all $D \in H(2r;\underline{n})$ there exists an $f \in O(2r)$ such that $D_H(f) = D$. Let $U = \{f \in O(2r) : D_H(f) \in H(2r;\underline{n})\}$. We seek a basis for U. Observe that $U = \{f \in O(2r) : D_H(f) \in W(2r;\underline{n})\}$. The condition $D_H(f) = \sum_{i=1}^{2r} \sigma(i)\partial_{x_i}(f)\partial_{x_{i'}} \in W(2r;\underline{n})$ yields that $\partial_{x_i}(f) \in O(2r;\underline{n})$ for all $1 \leq i \leq 2r$. Thus, a basis for all such f is given by $\left\{x^{(a)} : 0 \leq a_i \leq p^{n_i}-1\right\} \cup \left\{x^{(p^{n_i}\varepsilon_i)} : 1 \leq i \leq 2r\right\}$. For all a with $0 \leq a_i \leq p^{n_i}-1$, $D_H(x^{(a)}) \in H(2r;\underline{n}+\underline{1})^{(2)}$, since $a < \tau(\underline{n}+\underline{1})$. Lastly, $D_H(x^{(p^{n_i}\varepsilon_i)}) \in H(2r;\underline{n}+\underline{1})^{(2)}$, since $p^{n_i}\varepsilon_i < \tau(\underline{n}+\underline{1})$. This concludes the proof. ■

The representation theory of these algebras is relatively well understood. See for examples the papers by Holmes (1998) and Yao and Shu (2011).

Definition 5.1.9. Any graded Lie subalgebra of $H(2r;\underline{n})$ containing $H(2r;\underline{n})^{(\infty)}$ is called a *graded Hamiltonian Lie algebra of Cartan type.*

Remark. By $\mathfrak{g}^{(\infty)}$ we mean $\bigcap \mathfrak{g}^{(i)}$.

Remark. The general definition of *graded Lie algebras of Cartan type* is made similarly, where one replaces the type H with all the types that exist (W, S, CS, H, CH, and K).

5.2 Filtered deformations

Starting from graded Cartan-type Lie algebras one goes to general Cartan-type Lie algebras by considering filtered deformations of these algebras. Let us understand in some more detail what this means.

The following definition of Cartan type Lie algebras can be found in (Strade, 2004, §4.2, Def 4.2.4). We only state the case we are concerned with.

Definition 5.2.1. Let $\mathfrak{g} = \mathfrak{g}_{(-s')} \supseteq \cdots \supseteq \mathfrak{g}_{(s)} \supseteq \{0\}$ be a separating filtration[3] of $\mathfrak{g}$. If there is $m = 2r \in \mathbb{N}_{>0}$ and $\underline{n} \in \mathbb{N}^m_{>0}$ and an embedding $\psi : \operatorname{gr}\mathfrak{g} \longrightarrow H(m;\underline{n})$ of graded Lie algebras such that

$$H(m;\underline{n})^{(\infty)} \subseteq \psi(\operatorname{gr}\mathfrak{g}) \subseteq H(m;\underline{n}),$$

then $\mathfrak{g}$ is called a *Hamiltonian Lie algebra of Cartan type.*

In other words, we say that $\mathfrak{g}$ is a Hamiltonian Lie algebra of Cartan type if $\mathfrak{g}$ admits a separating filtration such that the graded of $\mathfrak{g}$ is isomorphic to a graded Hamiltonian Lie algebra of Cartan type.

For more structural information concerning Hamiltonian Lie algebras of Cartan type, we refer the interested reader to Theorem 4.2.6 and Theorem 4.2.7 in (Strade, 2004, §4.2). We do mention that if $\mathfrak{g}$ is a Hamiltonian Lie algebra of Cartan type, then $\mathfrak{g}^{(\infty)}$ is a simple Hamiltonian Lie algebra of Cartan type.

Definition 5.2.2. A *Hamiltonian form* is a form

$$\omega = \sum_{i,j=1}^{2r} f_{i,j}\mathrm{d}x_i \wedge \mathrm{d}x_j \in \Omega^2(2r)$$

such that $f_{i,j} = -f_{j,i}$, $\mathrm{d}\omega = 0$, and $\det(f_{i,j}) \in O(2r)^*$, the set of invertible elements of $O(2r)$.

Definition 5.2.3. A *Hamiltonian form subordinate to* $O(2r;\underline{n})$ is a Hamiltonian form ω such that $\omega \in u\Omega^2(2r;\underline{n})$ for some $u \in \mathscr{U}(2r;\underline{n})$, where

$$\mathscr{U}(2r;\underline{n}) := \left\{u \in O(2r)^* : u^{-1}\mathrm{d}u \in \Omega^1(m;\underline{n}), u(0) = 1\right\}.$$

Now we subdivide such Hamiltonian forms into those of *first type*, which are those where one can take $u = 1$ in the above definition, i.e., those Hamiltonian forms ω with $\omega \in \Omega^2(2r;\underline{n})$, and those of *second type*, which are simply all others.

Before we define an important family of Hamiltonian forms subordinate to $O(2r;\underline{n})$ of second type, we need to make the following definitions.

Definition 5.2.4. Let R be a commutative ring with unit element and M_R a maximal ideal of R. A *system of divided powers* on M_R is a sequence of maps $\gamma_r : M_R \longrightarrow R$, where we denote the image $\gamma_r(f)$ by $f^{(r)}$, such that for all $f, g \in M_R$:

[3] So $\mathfrak{g}_{(s)} = \{0\}$, see Definition 3.3.1.

1. $f^{(0)} = 1$, $f^{(r)} \in M_R$ for all $r > 0$;
2. $f^{(1)} = f$;
3. $f^{(r)} f^{(s)} = \binom{r+s}{r} f^{(r+s)}$ for all $r, s \geq 0$;
4. $(f+g)^{(r)} = \sum_{l=0}^{r} f^{(l)} g^{(r-l)}$ for all $r \geq 0$;
5. $(hf)^{(r)} = h^r f^{(r)}$ for all $h \in R$ and $r \geq 0$;
6. $\left(f^{(s)}\right)^{(r)} = \frac{(rs)!}{r!(s!)^r} f^{(rs)}$ for all $r \geq 0, s > 0$.

Remark. This implies that:

$$\left(f^{(p^s)}\right)^{(p^r)} = f^{(p^{s+r})}.$$

A proposition due to Skryabin, see (Strade, 2004, §2.1, Proposition 2.1.4, p. 63), tells us that there is a unique system of divided powers on $O(m)_{(1)}$ such that $\gamma_r(x_i) = x_i^{(r)}$ for all $r \geq 0$ and all $1 \leq i \leq m$. Note that successive application of (5) and (6) allows us to compute $(x^{(a)})^{(r)}$.

Definition 5.2.5. For $f \in O(m)_{(1)}$ define:

$$\exp(f) = \sum_{i=0}^{\infty} f^{(i)}.$$

Definition 5.2.6. Define for $l \in \{1, \ldots, 2r\}$

$$\omega_{H,l} = \mathrm{d}\left(\exp\left(x_l^{(p^{n_l})}\right) \sum_{j=1}^{2r} \sigma(j) x_j \mathrm{d}x_{j'}\right).$$

Definition 5.2.7. Let ω be a Hamiltonian form subordinate to $O(2r; \underline{n})$. Then (see (Strade, 2004, §6.5, p. 337))

$$H(2r; \underline{n}; \omega) := \{D \in W(2r; \underline{n}) : D(\omega) = 0\}.$$

Remark. In the case of the form $\omega = \omega_{H,l}$ and $r = 1$, we will adopt the notation $H(2; (n_1, n_2); \Phi(l)) = H(2; (n_1, n_2); \omega_{H,l})$.

Thus the Hamiltonian algebra $H(2; (1,1); \Phi(1))$ is

$$\{D \in W(2; (1,1)) : D(\omega_{H,1}) = 0\}.$$

In this case, we can calculate $\omega_{H,1} = 2\exp\left(x_1^{(p)}\right) \mathrm{d}x_1 \wedge \mathrm{d}x_2$. Note, however, that for non-zero $\lambda \in k$, we have $D(\omega) = 0$ if and only if $D(\lambda\omega) = 0$. Thus, we have

$$\begin{aligned} H(2;(1,1);\Phi(1)) &= \{D \in W(2;(1,1)) : D(\omega_{H,1}) = 0\} \\ &= \left\{D \in W(2;(1,1)) : D\left(\exp\left(x_1^{(p)}\right) \mathrm{d}x_1 \wedge \mathrm{d}x_2\right) = 0\right\} \end{aligned}$$

Thanks to Theorem 6.5.8 in (Strade, 2004, §6.5), we have the following description (from now on we adopt the notation $x := x_1$ and $y := x_2$ in $O(2;\underline{1})$ and $W(2;\underline{1})$, noting that this means we write $x^{(a_1,a_2)} = x^{(a_1)}y^{(a_2)}$).

Theorem 5.2.8. $H(2;(1,1);\Phi(1)) = \{D_{H,1}(f) : f \in O(2;(1,1))\}$*, where*

$$\begin{aligned} D_{H,1}(f) &= \partial_x(f)\partial_y - \partial_y(f)\partial_x - x^{p-1} f\partial_y \\ &= \partial_x(f)\partial_y - \partial_y(f)\partial_x + x^{(p-1)} f\partial_y \\ &= D_H(f) + x^{(p-1)} f\partial_y. \end{aligned}$$

Remark. See Definition 5.1.6 for the definition of D_H. Note also that for the second equality we use Wilson's Theorem.

Furthermore, the same theorem tells us that $H(2;(1,1);\Phi(1))$ is simple of dimension p^2 (as long as one is in characteristic $p \geq 3$).

The Hamiltonian algebra $H(2;\underline{n};\Phi(\tau))$ is

$$H(2;\underline{n};\omega) = \{D \in W(2;\underline{n}) : D(\omega) = 0\},$$

where ω is the Hamiltonian form:

$$\left(1 + x^{(\tau(\underline{n}))}\right) \mathrm{d}x_1 \wedge \mathrm{d}x_2 = \left(1 + x^{(p^{n_1}-1)}y^{(p^{n_2}-1)}\right) \mathrm{d}x \wedge \mathrm{d}y.$$

For more details see (Strade, 2004, §6.3, pp. 308–309) and (Strade, 2009, §10.3), noting that $H(2;\underline{n};\sigma) = S(2;\underline{n};\sigma)$, since the special form $\omega_S := \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m$ coincides with the Hamiltonian form ω_H when $m = 2$.

An explicit description can be found in Theorem 6.3.7 of (Strade, 2004, §6.3, p. 309):

Theorem 5.2.9. *The Lie algebra* $H(2;\underline{n};\Phi(\tau))^{(1)}$ *is a simple Cartan type Lie algebra of dimension* $p^{|\underline{n}|} - 1$*. More exactly,*

$$H(2;\underline{n};\Phi(\tau)) = k\left(1 - x^{(\tau(\underline{n}))}\right)\partial_x + k\left(1 - x^{(\tau(\underline{n}))}\right)\partial_y + \sum_{l\geq 0} H(2;\underline{n})_l,$$

and

$$H(2;\underline{n};\Phi(\tau))^{(1)} = k\left(1 - x^{(\tau(\underline{n}))}\right)\partial_x + k\left(1 - x^{(\tau(\underline{n}))}\right)\partial_y + \sum_{l\geq 0} H(2;\underline{n})^{(1)}{}_l.$$

We have the following classification (see Theorem 6.3.10, part (3) in Strade (2004)):

Theorem 5.2.10. *Every simple Lie algebra of Hamiltonian type ($r = 1$) is isomorphic to one of*

1. $H(2;\underline{n})^{(2)}$;
2. $H(2;\underline{n};\Phi(\tau))^{(1)}$;
3. $H(2;\underline{n};\Phi(l))$,

where $l = 1, 2$ and $n_1 \leq n_2$, with the condition that either $l = 1$ or $l = 2$, $n_1 < n_2$.

Furthermore, with these restrictions on $\underline{n}$, the exposed algebras are pairwise non-isomorphic.

Corollary 5.2.11. *We have $H(2;(1,1);\Phi(1)) \cong H(2;(1,1);\Phi(2))$.*

For a more general classification theorem that covers $r > 1$, we refer the reader to Theorem 6.5.1 in (Strade, 2004, §6.5, p. 329).

Finally, we refer the reader to recently translated work of Skryabin Skryabin (2019), noting that in it the term "Hamiltonian form" has been replaced by "symplectic form". Some of the classification had already been published in English in Skryabin (1991), see Skryabin (1990) for the original Russian publication.

In Skryabin (2019), we find a classification of the Hamiltonian forms of second type, see Theorem 5.1, and a classification of the Hamiltonian forms of first type, see Theorem 7.2 and its new formulation in Theorem 7.3. We note that Skryabin works over perfect fields, a more general setting than the one we are concerned with, that of algebraically closed fields.

Chapter 6

Simple restricted modules for the non-graded Hamiltonian $H(2; (1, 1); \Phi(1))$

In this chapter we will be calculating the dimensions of the restricted irreducible representations of $\widehat{H}$. We shall give dimension formulas for all of them and give a full classification, giving a list of representatives for the isomorphism classes of restricted simples. Moreover, we shall calculate the composition factors of all restricted induced modules.

We begin by finding a generating set for the important subalgebra consisting of p-nilpotent elements N; then we calculate formulae for the action of important elements of $\widehat{H}$. Using this, we then classify the maximal vectors in the induced modules $Z(M)$, splitting the classification into (a) modules induced from one-dimensional $\widehat{H}_0$-modules, (b) modules induced from two-dimensional $\widehat{H}_0$-modules, and (c) modules induced from higher-dimensional $\widehat{H}_0$-modules. Finally, we use this knowledge of maximal vectors to determine in full the module structure of the restricted induced modules.

The material in this chapter can be found in an article form in Guerra (2020).

6.1 Preliminaries and notation

Let k be an algebraically closed field of positive characteristic $p \geq 5$.

Put $\mathcal{A} = \{a \in \mathbb{Z}^2 : 0 \leq a_i \leq p-1\}$.

See (Strade, 2009, §10.4) and (Strade, 2004, §4.2) for more on the descriptions of the

Hamiltonian algebras.

Applying Theorem 5.2.8 to basis elements in $O(2;(1,1)) \cong k[X,Y]/(X^p,Y^p)$, we see that the non-graded Hamiltonian algebra $H := H(2;(1,1);\Phi(1))$, of dimension p^2, can be realised as the subalgebra of

$$W(2;(1,1)) = \mathrm{Der}\,(k[X,Y]/(X^p,Y^p))$$

with basis

$$\left\{y^{(j-1)}\partial_x - x^{(p-1)}y^{(j)}\partial_y, x^{(i-1)}y^{(j)}\partial_y - x^{(i)}y^{(j-1)}\partial_x : 1 \le i \le p-1, 0 \le j \le p-1\right\},$$

where $x^{(-1)}$ and $y^{(-1)}$ are understood to be zero, and x and y denote the images of X and Y in the truncated polynomial ring $k[X,Y]/(X^p,Y^p)$, respectively, using divided power notation, see (Strade, 2004, §2). The fact that these elements are not homogeneous makes calculating with this algebra more subtle, see §6.2.1 for more details.

Given how we obtained this basis for H, we can also describe it as follows. Define for all $(a,b) \in \mathcal{A}$ an element $e_{a,b} := D_{H,1}(x^{(a)}y^{(b)}) \in H(2;(1,1);\Phi(1))$. The set $\{e_{a,b} : (a,b) \in \mathcal{A}\}$ is a basis for H. We extend this notation to all non-negative integers by setting $e_{a,b} = 0$ if $a, b \ge p$.

For a general formula for commutators in $W(n;(1,\dots,1))$, we refer the reader to Equation (4.1).

Recall we have a k-basis for W_2

$$\left\{x^{(a)}y^{(b)}\partial_\alpha : 0 \le a, b \le p-1, \alpha = x, y\right\}.$$

By Equation (4.1), the Lie bracket in W_2, and hence H is given by, for instance,

$$\begin{aligned}[x^{(a)}y^{(b)}\partial_x, x^{(c)}y^{(d)}\partial_y] = &\binom{c+a-1}{a}\binom{d+b}{b}x^{(c+a-1)}y^{(d+b)}\partial_y \\ &- \binom{a+c}{c}\binom{b+d-1}{d}x^{(a+c)}y^{(b+d-1)}\partial_x,\end{aligned}$$

and other commutators are computed similarly. Recall that we adopt the convention that $x^{(b)} = 0$ if any of the $b_i < 0$.

We want to describe the bracket of H in our chosen basis. To this end, we begin by defining a map $^{-} : \mathbb{Z} \to \{0,\dots,p-1\}$ such that $\overline{a} - a \in p\mathbb{Z}$ for all $a \in \mathbb{Z}$. Then a straightforward calculation shows that

$$e_{a,b} = x^{(\overline{a-1})}y^{(b)}\partial_y - x^{(a)}y^{(b-1)}\partial_x$$

for all $(a,b) \in \mathcal{A}$. In particular, if $b > 0$, then $e_{0,b} = x^{(p-1)}y^{(b)}\partial_y - y^{(b-1)}\partial_x$ and $e_{0,0} = x^{(p-1)}\partial_y$.

Lemma 6.1.1. *For any $(a,b),(c,d) \in \mathcal{A}$ we have*

$$\begin{aligned}[e_{a,b}, e_{c,d}] &= D_{H,1}(x^{(\overline{a-1})}y^{(b)}x^{(c)}y^{(d-1)} - x^{(a)}y^{(b-1)}x^{(\overline{c-1})}y^{(d)}) \\ &= \left(\binom{\overline{a-1}+c}{\overline{a-1}}\binom{b+d-1}{b} - \binom{a+\overline{c-1}}{a}\binom{b+d-1}{b-1}\right)e_{a+c-1,b+d-1},\end{aligned}$$

with the caveat that when $a = c = 0$, we take $e_{a+\overline{c-1},b+d-1}$ instead.

Proof. By Equation (10.4.2) of Strade (2009) we have

$$[D_{H,1}(f), D_{H,1}(g)] = D_{H,1}\big(D_H(f)(g) + x^{(p-1)}f\partial_y(g) - x^{(p-1)}g\partial_y(f)\big)$$

for any $f, g \in O(2;(1,1))$.

Taking $f = x^{(a)}y^{(b)}$ and $g = x^{(c)}y^{(d)}$ we establish the first claimed equality. Note that we have $D_H(f) = x^{(a-1)}y^{(b)}\partial_y - x^{(a)}y^{(b-1)}\partial_x$. If $a, c > 0$, then since $x^{(p-1)}x^{(t)} = 0$ if $t > 0$, the bracket is given by $D_{H,1}(D_H(f)(g))$ which gives the formula in this case. Now suppose $a = 0$ and $c > 0$. Then the product above becomes

$$\begin{aligned}[D_{H,1}(f), D_{H,1}(g)] &= D_{H,1}\big(x^{(p-1)}y^{(b)}x^{(c)}y^{(d-1)} - x^{(a)}y^{(b-1)}x^{(c-1)}y^{(d)}\big) \\ &= D_{H,1}\big(x^{(\overline{a-1})}y^{(b)}x^{(c)}y^{(d-1)} - x^{(a)}y^{(b-1)}x^{(\overline{c-1})}y^{(d)}\big)\end{aligned}$$

and the formulas agree. The remaining cases are similar.

From the first equality we get the bracket is

$$\binom{\overline{a-1}+c}{\overline{a-1}}\binom{b+d-1}{b}e_{\overline{a-1}+c,b+d-1} - \binom{a+\overline{c-1}}{a}\binom{b+d-1}{b-1}e_{a+\overline{c-1},b+d-1}.$$

If either $a = c = 0$ or $a, c > 0$ then $e_{\overline{a-1}+c,b+d-1} = e_{a+\overline{c-1},b+d-1}$ and this common term can be factored out. In the case where exactly one of a or c is zero we see that only one term survives and it agrees with the stated formula. ■

Remark. Note that if $b = d = 0$, the undefined expression $e_{a+c-1,-1}$ is involved. Since the coefficients $\binom{b+d-1}{b}$ and $\binom{b+d-1}{b-1}$ are both zero, we take the expression as a whole to be zero, and indeed $[e_{a,0}, e_{c,0}] = 0$.

The Lie algebra H is simple and its minimal p-envelope $\widehat{H} := H_{[p]}$ can be obtained by adding the element $K := x\partial_x + y\partial_y$, see (Strade, 2009, §10.4) for more details. As we noted in Chapter 1, classifying the restricted simple modules for $\widehat{H}$ completes the rank one and rank two picture, in the sense that it completes the description of the restricted simples for Hamiltonian algebras of absolute toral rank 1 and 2. The only Hamiltonian algebra of absolute toral rank 1, $H(2;(1,1))^{(2)}$, was done by Koreshkov (1978); the absolute toral rank 2 Hamiltonian algebras are $H(2;(1,1),\Phi(\tau))^{(1)}$, which was done by Feldvoss et al. (2016)[1], $H(4;(1,1,1,1))^{(2)}$, which was done by Shen (1988a,b), together with certain corrections made in Holmes (1998), $H(2;(1,2))^{(2)}$, which was done by Yao and Shu (2011), and lastly the algebra we are concerned with, $H(2;(1,1);\Phi(1))$. For the classification of the absolute toral rank 1 and 2 simple Hamiltonian Lie algebras, see (Strade, 2009, §10.6, p. 106).

Since $K \in \widehat{H}$ and $e_{1,1} = y\partial_y - x\partial_x \in H \subseteq \widehat{H}$ we get two elements

$$T_1 := \frac{1}{2}(K - e_{1,1}) = x\partial_x \qquad \text{and} \qquad T_2 := \frac{1}{2}(K + e_{1,1}) = y\partial_y$$

of $\widehat{H}$. It is straightforward to check from the bracket in $W(2;(1,1))$ that $[T_1, T_2] = 0$. So the subspace $T := k\langle T_1, T_2\rangle \subseteq \widehat{H}$ is a 2-dimensional abelian subalgebra which is a maximal toral subalgebra, as remarked in the proof of Theorem 10.4.6 in Strade (2009). Hence this is indeed an algebra of absolute toral rank 2.

Note that $\{x\partial_x, y\partial_y, e_{a,b} : (1,1) \neq (a,b) \in \mathcal{A}\}$ is a basis for $\widehat{H}$. We have

Lemma 6.1.2. *For any* $(a,b) \in \mathcal{A}$ *we have*

$$[T_1, e_{a,b}] = (a-1)e_{a,b}$$
$$[T_2, e_{a,b}] = (b-1)e_{a,b}.$$

In particular, $[K, e_{a,b}] = (a+b-2)e_{a,b}$.

Proof. Direct calculation using the bracket in $W(2;(1,1))$. Note this is correct even when $a = 0$. ■

[1] See the remark after Proposition 6.1.6 for why of the two non-graded Hamiltonians $H(2;(1,1);\Phi(1))$ rather than $H(2;(1,1);\Phi(\tau))^{(1)}$ turns out to be trickier to handle.

We will induce representations from a suitable subalgebra to all of $\widehat{H}$, which we will now define.

To this end, we must first define a restricted descending filtration[2] $\left(\widehat{H}_{(n)}\right)_{n\in\mathbb{Z}}$ on $\widehat{H}$ from the natural grading, see Definition 4.1.7,

$$W(2;(1,1)) = \bigoplus_{d=-1}^{2p-3} W(2;(1,1))_d,$$

namely $\widehat{H}_{(n)} := \widehat{H} \cap W(2;(1,1))_{(n)}$, where $W(2;(1,1))_{(n)} := \bigoplus_{d\geq n} W(2;(1,1))_d$.

Then $\widehat{H}_{(0)}$ is a codimension 2 subalgebra of $\widehat{H}$ having $\widehat{H}_{(1)}$ as an ideal. Indeed, we have a linear map $\widehat{H} \hookrightarrow W(2;(1,1)) \longrightarrow W(2;(1,1))/W(2;(1,1))_{(0)} \cong W(2;(1,1))_{-1}$ with kernel $\widehat{H}_{(0)}$ (by definition). Now, $W(2;(1,1))_{-1}$ is two-dimensional with basis $\{\partial_x, \partial_y\}$ and the linear map surjects onto this, since $-e_{0,1} = \partial_x - x^{(p-1)}y\partial_y \in \widehat{H}$ is mapped onto ∂_x and $e_{1,0} = \partial_y \in \widehat{H}$ is mapped onto ∂_y. Hence, $\widehat{H}_{(0)}$ has codimension 2.

We lift representations from $\widehat{H}_0 := \widehat{H}_{(0)}/\widehat{H}_{(1)}$ to $\widehat{H}_{(0)}$ via the canonical map, i.e., we take the pullback via the quotient map: if ρ is a representation and π is the canonical projection

$$\widehat{H}_{(0)} \xrightarrow{\pi} \widehat{H}_0 \xrightarrow{\rho} \mathfrak{gl}(V)$$

then $\rho\circ\pi$ is the desired representation.

We have $\widehat{H}_{(0)}/\widehat{H}_{(1)} \cong \mathfrak{gl}_2$[3] because we have the following map of Lie algebras

$$\widehat{H}_{(0)} \hookrightarrow W(2;(1,1))_{(0)} \longrightarrow W(2;(1,1))_{(0)}/W(2;(1,1))_{(1)} \longrightarrow W(2;(1,1))_0 \longrightarrow \mathfrak{gl}_2(k)$$

with kernel $\widehat{H}_{(1)}$ (by definition). We know from Proposition 4.1.8 that the last map is an isomorphism. Thus, the map is surjective, and $\widehat{H}_0 \cong \mathfrak{gl}_2$. We see that we have elements $x\partial_x + \widehat{H}_{(1)}, y\partial_y + \widehat{H}_{(1)}, x\partial_y + \widehat{H}_{(1)}, \left(y\partial_x - x^{(p-1)}y^{(2)}\partial_y\right) + \widehat{H}_{(1)}$ in the quotient, and they go under our fixed isomorphism to the matrices $E_{1,1}, E_{2,2}, E_{1,2}$, and $E_{2,1}$, respectively.

In this thesis we will be considering only restricted representations, also known as p-representations, i.e., those for which

$$\rho(x^{[p]}) = \rho(x)^p,$$

for all $x \in \widehat{H}$, see (Strade and Farnsteiner, 1988, §2.1) for more details.

[2] See Definition 3.3.1 and Definition 3.3.5.

[3] In Herpel and Stewart (2016a), the authors claim that $\widehat{H}_0 \cong \mathfrak{sl}_2$.

Recall that we write $\mathfrak{u}(\widehat{H})$ for the restricted universal enveloping algebra of $\widehat{H}$.

Given a restricted module M for $\widehat{H}_{(0)}$ we will study the induced $\mathfrak{u}(\widehat{H})$-module, i.e., the restricted $\widehat{H}$-module,

$$Z(M) := \operatorname{Ind}_{\widehat{H}_{(0)}}^{\widehat{H}}(M,0) := \mathfrak{u}(\widehat{H}) \otimes_{\mathfrak{u}(\widehat{H}_{(0)})} M,$$

where $\widehat{H}$ acts on $Z(M)$ as usual.

Concerning the restricted structure, according to Strade in (Strade, 2009, §10.4), one has $D^{[p]} = D^p$ if $D \in \widehat{H}_{(0)}$. For such D, we have $D^p = D$ when $D = x\partial_x$ or $D = y\partial_y$. Otherwise $D^p = 0$ for single terms $x^{(a)}y^{(b)}\partial_x$ and $x^{(a)}y^{(b)}\partial_y$. For basis elements $D \notin \widehat{H}_{(0)}$, we have

$$\partial_y^{[p]} = 0$$

$$\left(-\partial_x + x^{(p-1)}y\partial_y\right)^{[p]} = y\partial_y.$$

Let M be a restricted $\widehat{H}_0$-module, and hence a restricted $\widehat{H}_{(0)}$-module, with $\widehat{H}_{(1)} \cdot M = 0$.

We seek a way to express elements of $Z(M)$ uniquely. Observe that

$$\partial'_x := \partial_x - x^{(p-1)}y\partial_y \notin \widehat{H}_{(0)}.$$

Also $\partial_y \notin \widehat{H}_{(0)}$. These are linearly independent and in $\widehat{H}$. Hence, $k\langle \partial'_x, \partial_y\rangle$ is a vector space complement of $\widehat{H}_{(0)}$ in $\widehat{H}$, i.e., $\widehat{H} = \widehat{H}_{(0)} \oplus k\langle \partial'_x, \partial_y\rangle$. Thus, by the PBW theorem for $\mathfrak{u}(\widehat{H})$, any $v \in Z(M)$ can be expressed uniquely in the form

$$v = \sum_{a\in\mathcal{A}} \left(\partial'_x\partial_y\right)^a \otimes m_a, \tag{6.1}$$

where $m_a \in M$ and $\left(\partial'_x\partial_y\right)^a := \partial'^{a_1}_x \partial^{a_2}_y$.

Set $N = \widehat{H}_{(1)} \oplus k\left\langle x\partial_y\right\rangle$. We call it N because it is a subalgebra of $\widehat{H}$ consisting of p-nilpotent elements (recall Definition 3.2.10). Let $f : \widehat{H}_{(0)} \longrightarrow \mathfrak{gl}_2(k)$ be the map of Lie algebras we have fixed above. Consider the usual Borel subalgebra

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a,b,c \in k \right\} \subseteq \mathfrak{gl}_2(k).$$

Define $\mathcal{B} = f^{-1}(B)$, which is clearly a subalgebra of $\widehat{H}_{(0)}$. It is easy to see that $\mathcal{B} = N \oplus T$

(as vector spaces), where we recall that $T = k\langle x\partial_x, y\partial_y\rangle$ is a 2-dimensional abelian subalgebra which is a maximal toral subalgebra. Note, furthermore, that $\mathcal{B}$ is a split extension of N by T (in other words, $\mathcal{B}$ is a semidirect product of Lie algebras).

Definition 6.1.3. Let M be a $\mathcal{B}$-module. Let $\lambda \in k^2$. Set

$$M(\lambda) = \left\{m \in M : x\partial_x \cdot m = \lambda_1 m, y\partial_y \cdot m = \lambda_2 m\right\}.$$

We call elements of $M(\lambda)$ *weight vectors (of weight λ)*. If in addition $v \in M(\lambda)$ is nonzero and $N \cdot v = 0$, then we say that v is a *maximal vector (of weight λ)*, following Holmes (2001).

Remark. Every $\widehat{H}_0$-module M is a $\mathcal{B}$-module, by inflation to $\widehat{H}_{(0)}$ and then restriction to $\mathcal{B}$. Thus, it makes sense to talk about maximal vectors v for M. In this setting, such maximal vectors are equivalent to maximal vectors for M in the classical sense, recalling that $\widehat{H}_0 \cong \mathfrak{gl}_2$, where v is a maximal vector for $\mathfrak{gl}_2$ if it is an eigenvector for $x\partial_x$ and $y\partial_y$ and is annihilated by $x\partial_y$. This is because the algebra $\mathcal{B}$ in the quotient by $\widehat{H}_{(1)}$ becomes

$$\mathcal{B}/\widehat{H}_{(1)} \cong k\left\langle x\partial_x, y\partial_y, x\partial_y\right\rangle.$$

Remark. Since we are looking at restricted modules, we have that if a restricted $\mathcal{B}$-module M has a maximal vector of weight λ, then necessarily $\lambda \in \mathbb{F}_p^2$, where $\mathbb{F}_p$ is the prime subfield of our field k, see (Holmes, 2001, §2) for details. (Briefly, this is because in the restricted case one gets $\lambda_i^p = \lambda_i$.)

The following results show the importance of maximal vectors and of induced modules. See (Holmes, 2001, Lem. 2.1), for the proof of Lemma 6.1.4.

Lemma 6.1.4. *Let M be a finite-dimensional restricted $\widehat{H}$-module. The following are equivalent:*

1. *M is non-zero and is generated (as an $\widehat{H}$-module) by each of its maximal vectors;*

2. *M is simple.*

Proposition 6.1.5. *Let M be a finite-dimensional restricted $\widehat{H}$-module. Then M has a maximal vector.*

Proof. Note that M is a restricted $\widehat{H}_{(0)}$-module. It has a simple restricted $\widehat{H}_{(0)}$-submodule S. Now, since $\widehat{H}_{(1)} \subseteq N$, the proof of Lemma 6.1.4 in (Holmes, 2001, Lem. 2.1) shows that $\widehat{H}_{(1)}$ acts trivially on S. Thus, we see that S is a simple restricted $\widehat{H}_0 = \widehat{H}_{(0)}/\widehat{H}_{(1)}$-module. Thus, S has a maximal vector v of weight λ as a restricted $\widehat{H}_0 \cong \mathfrak{gl}_2$-module. We now claim that v is a maximal vector for $\widehat{H}$. Indeed, it is non-zero, and it is a weight vector. Finally, we see that $\widehat{H}_{(1)} \cdot v = 0$, and that $x\partial_y \cdot v = 0$, the latter because v is a maximal vector for $\widehat{H}_0$. Thus, $N = \widehat{H}_{(1)} \oplus k\langle x\partial_y\rangle$ annihilates v, as required. ■

Proposition 6.1.6. *Let M be a simple restricted $\widehat{H}$-module. Then M is a homomorphic image of $Z(S)$ for some simple restricted $\widehat{H}_0$-module S, i.e., every simple restricted $\widehat{H}$-module M is a quotient of some induced module $Z(S)$.*

Proof. Since M is finite-dimensional, we let $v \in M$ be a maximal vector of weight λ. Apply Frobenius reciprocity, where one takes S to be a simple restricted $\mathfrak{gl}_2$-submodule of weight λ, so that

$$\operatorname{Hom}_{\widehat{H}}(Z(S), M) \neq 0,$$

noting that any non-zero map must be surjective due to the simplicity of M. ■

Remark. Now we can explain why classifying the restricted simples for $\widehat{H}$ turns out to be harder than for L the minimal p-envelope of $H(2;(1,1);\Phi(\tau))^{(1)}$. Thanks to Theorem 4.3 in (Feldvoss et al., 2016, p. 387), the authors are able to reduce the problem of classifying the simples for L into that of classifying the simples for the graded of L, which turns out to be $H(2;(1,1))$, which in turn depends (see the proof of Theorem 5.3 in (Feldvoss et al., 2016, p. 391) on classifying the simples for $H(2;(1,1))^{(2)}$, which is done in Holmes (1998). The whole procedure works because the space $L_{(0)}/L_{(1)} \cong \mathfrak{sl}_2(k)$ coincides with the zero-graded piece $H(2;(1,1))_0$. In our case, the space $\widehat{H}_{(0)}/\widehat{H}_{(1)} \cong \mathfrak{gl}_2(k)$ does not coincide with the zero-graded piece of its graded algebra (still $H(2;(1,1))$, see (Herpel and Stewart, 2016a, p. 776)), so this strategy would only allow limited information on the induced modules $Z(M)$ based on inducing from simple restricted $\mathfrak{sl}_2$-modules $L_0(r)$, but as we just proved, a full classification of the simples of $\widehat{H}$ requires inducing from simple restricted $\widehat{H}_0 \cong \mathfrak{gl}_2(k)$-modules.

Lastly, even if this technique worked, it would not give us all the information we shall find, including a complete description of the module structure of all the induced modules, their composition series, and a complete description of all the composition factors and all the isomorphisms between them.

Certain weights will be important for us. They are the following: $\omega_0 = (-1,-1)$, $\omega_1 = (0,-1)$, $\omega_2 = (0,0)$, and all $\lambda \in \mathbb{F}_p^2$ with $\lambda_1 - \lambda_2 = 1$. These weights we call the *exceptional weights*.

We will prove:

Theorem 6.1.7. *Let $p \geq 5$ be a prime, k be an algebraically closed field of characteristic p, $\lambda \in \mathbb{F}_p^2$ a weight, $L_0(\lambda)$ be the simple restricted $\mathfrak{gl}_2(k)$-module of highest weight λ, $Z(\lambda) = Z(L_0(\lambda))$ the corresponding induced $\widehat{H}$-module, and $L(\lambda)$ its simple head.*

1. *The full list of simple pairwise nonisomorphic restricted $\widehat{H}$-modules is given by $\left\{L(\lambda) : \lambda \in \mathbb{F}_p^2, \lambda_1 - \lambda_2 \neq 1 \text{ or } \lambda = \omega_1\right\}$. There are $p^2 - p + 1$ of them.*

2. *If λ is not exceptional, then $L(\lambda) = Z(\lambda)$, and its dimension is $p^2 \dim_k L_0(\lambda) = p^2(\lambda_1 - \lambda_2 + 1)$.*

3. *For exceptional λ, the modules $L(\lambda)$ in the list are as follows:*

   (a) if $\lambda = \omega_0 = (-1,-1)$, $L(\lambda) \cong O(2;(1,1))/(k \cdot 1)$, with dimension $p^2 - 1$;

   (b) if $\lambda = \omega_1 = (0,-1)$, $L(\lambda) \cong \widehat{H}\left\langle \partial_y \otimes m \right\rangle \leq Z(0,0)$, with dimension $p^2 - 1$;

   (c) if $\lambda = \omega_2 = (0,0)$, $L(\lambda) \cong k$, with dimension 1 (this is the trivial module).

Remark. The condition $\lambda_1 - \lambda_2 \neq 1$ comes from the fact that all the simple heads $L(\lambda)$ of modules induced from two-dimensional $\mathfrak{gl}_2$-modules (with the exception of $\lambda = \omega_1$) are isomorphic to some other simple restricted $\widehat{H}$-module. See the remark after the proof of Theorem 6.3.13 for more details.

Definition 6.1.8. Let $\lambda \in \mathbb{F}_p^2$, $\mathcal{A} = \left\{a \in \mathbb{Z}^2 : 0 \leq a_i \leq p-1\right\}$, $a \in \mathcal{A}$. For brevity we define the following

$$\begin{aligned}
\lambda(a)_i &= \lambda_i + a_i, \\
r_a &= a_1(\lambda(a)_1 - \lambda(a)_2) + a_1 a_2 - \binom{a_1}{2} \\
s_a &= a_2(\lambda(a)_1 - \lambda(a)_2) - a_1 a_2 + \binom{a_2}{2} \\
t_a &= \binom{a_1}{2}(\lambda(a)_2 - \lambda(a)_1) - \binom{a_1}{2} a_2 + \binom{a_1}{3},
\end{aligned}$$

Furthermore, $x\partial_y$ will also be referred to as X, especially when it is acting on M.

6.1.1 Generating the subalgebra N

To facilitate the arguments concerning maximal vectors in what follows, we will find a generating set for our subalgebra N. First, however, we describe $N = \widehat{H}_{(1)} \oplus k\left\langle x\partial_y\right\rangle$ in more detail.

Proposition 6.1.9. *The Lie subalgebra $N = \widehat{H}_{(1)} \oplus k\left\langle x\partial_y\right\rangle$ has dimension p^2-4.*

Proof. Recall that $\widehat{H}_{(0)}$ has codimension 2 in $\widehat{H}$. Thus, $\dim_k \widehat{H}_{(0)} = \left(p^2+1\right)-2 = p^2-1$. We also saw that the quotient $\widehat{H}_{(0)}/\widehat{H}_{(1)}$ is 4-dimensional. Thus $\dim_k \widehat{H}_{(1)} = \left(p^2-1\right)-4 = p^2-5$. Therefore, we have $\dim_k N = p^2-4$. ∎

We have the following:

Proposition 6.1.10. *Let k be an algebraically closed field of characteristic $p \geq 5$. We have*

$$N = \widehat{H}\left\langle x\partial_y, x^{(p-1)}\partial_y, e_{1,2}, e_{0,3}\right\rangle$$

(as a Lie subalgebra) if $p \neq 5$. If $p=5$

$$N = \widehat{H}\left\langle x\partial_y, x^{(p-1)}\partial_y, e_{1,2}, e_{0,3}, e_{4,4}\right\rangle,$$

where $e_{4,4} = x^{(3)}y^{(4)}\partial_y - x^{(4)}y^{(3)}\partial_x$.

Proof. Put $S = \widehat{H}\left\langle x\partial_y, x^{(p-1)}\partial_y, e_{1,2}, e_{0,3}\right\rangle \leq N$.

First we will obtain all $y^{(j-1)}\partial_x - x^{(p-1)}y^{(j)}\partial_y$ for $j=3,\ldots,p-1$. This will show that $\dim_k S \geq p-3$. For $j=3$, we observe that this is just the element $-e_{0,3}$, which is already in S. We proceed by induction on j. The base case is clear.

Now, we have

$$[y^{(j-1)}\partial_x - x^{(p-1)}y^{(j)}\partial_y, e_{1,2}] = -\binom{j+1}{2}\left(y^{(j)}\partial_x - x^{(p-1)}y^{(j+1)}\partial_y\right),$$

which is never zero since $j \neq p-1$. So we obtain all the desired elements by induction.

Now we claim that $\left\{x^{(i)}y\partial_y - x^{(i+1)}\partial_x, x^{(i)}\partial_y\right\} \subseteq S$ for $i=1,\ldots,p-2$. Since $x^{(p-1)}\partial_y$ is in our set of generators, this will show that $\dim_k S \geq (p-3)+(2p-3) = 3p-6$.

Proceed by induction on i. For $i=1$, we already have $x\partial_y \in S$ and we have $e_{2,1} = xy\partial_y - x^{(2)}\partial_x \in S$, which we obtain from $[x\partial_y, e_{1,2}] = 2e_{2,1}$. For the inductive step, we have

$$[x\partial_y, x^{(i)}y\partial_y - x^{(i+1)}\partial_x] = (i+2)\left(x^{(i+1)}\partial_y\right),$$

and

$$[x^{(i+1)}\partial_y, e_{1,2}] = (i+2)\left(x^{(i+1)}y\partial_y - x^{(i+2)}\partial_x\right).$$

Hence, in step-wise fashion we get the terms we want up to the point we obtain the terms $x^{(p-3)}y\partial_y - x^{(p-2)}\partial_x$ and $x^{(p-3)}\partial_y$. Taking the Lie bracket of the former with $x\partial_y$, we obtain the term $(p-1)x^{(p-2)}\partial_y$. By taking the Lie bracket of this term with $e_{1,2}$, we obtain $x^{(p-2)}y\partial_y - x^{(p-1)}\partial_x$. Thus, we have proved our claim.

We have

$$[x^{(i)}\partial_y, y^{(j-1)}\partial_x - x^{(p-1)}y^{(j)}\partial_y] = -\left(x^{(i-1)}y^{(j-1)}\partial_y - x^{(i)}y^{(j-2)}\partial_x\right),$$

so $x^{(i-1)}y^{(j-1)}\partial_y - x^{(i)}y^{(j-2)}\partial_x \in S$ for $i = 1, \dots, p-1$, $j = 3, \dots, p-1$. This shows that $\dim_k S \geq (3p-6) + (p-1)(p-3) = p^2 - p - 3$.

Since $\dim_k N = p^2 - 4$, we are only missing $p-1$ elements. Note that the following $p-1$ are both in N and have not yet been shown to lie in S:

$$x^{(i-1)}y^{(p-1)}\partial_y - x^{(i)}y^{(p-2)}\partial_x,$$

$1 \leq i \leq p-1$. We calculate

$$[e_{1,2}, x^{(i-1)}y^{(j)}\partial_y - x^{(i)}y^{(j-1)}\partial_x] = \gamma_{i,j}\left(x^{(i-1)}y^{(j+1)}\partial_y - x^{(i)}y^{(j)}\partial_x\right),$$

where $\gamma_{i,j} = \binom{j+1}{2} - i(j+1)$. Taking $j = p-2$ in the above gives us the elements we need as i runs from 1 to $p-1$, as long as the coefficient $\gamma_{i,p-2} \neq 0$. However, $\gamma_{i,p-2} = 1 + i = 0$ when $i = p-1$. So we still need to find the last term

$$x^{(p-2)}y^{(p-1)}\partial_y - x^{(p-1)}y^{(p-2)}\partial_x.$$

We calculate

$$[y^{(p-4)}\partial_x - x^{(p-1)}y^{(p-5)}\partial_y, -e_{0,3}] = 2\left(x^{(p-2)}y^{(p-1)}\partial_y - x^{(p-1)}y^{(p-2)}\partial_x\right).$$

Finally, we note that if $p = 5$, $y^{(p-4)}\partial_x - x^{(p-1)}y^{(p-5)}\partial_y \notin N$, so we add the element $e_{4,4} = x^{(p-2)}y^{(p-1)}\partial_y - x^{(p-1)}y^{(p-2)}\partial_x$ in characteristic 5. By dimensions, we are done. ∎

Remark. Computer verification confirms that N is not generated by S alone when $p = 5$.

From the previous result we see that the Lie algebra $\widehat{H}$ is in fact generated by

$$\mathcal{G} := \left\{x\partial_y, x^{(p-1)}\partial_y, e_{1,2}, e_{0,3}, e_{0,2}, \partial'_x, \partial_y, x\partial_x - y\partial_y, x\partial_x + y\partial_y\right\},$$

if $p > 5$ and by $\mathcal{G} \cup \{e_{4,4}\}$ if $p = 5$, since the set $\left\{x\partial_y, x^{(p-1)}\partial_y, e_{1,2}, e_{0,3}\right\}$ ($\left\{x\partial_y, x^{(p-1)}\partial_y, e_{1,2}, e_{0,3}, e_{4,4}\right\}$ if $p = 5$) generates N, which is of dimension $p^2 - 4$, so the Lie subalgebra generated by $\mathcal{G}$ ($\mathcal{G} \cup \{e_{4,4}\}$ if $p = 5$) has dimension $p^2 - 4 + 5 = p^2 + 1$, and thus must be all of $\widehat{H}$. Having this generating set is a good thing because it allows us to make certain arguments easier.

For instance, it gives us an effective way of proving that a particular set of elements obtained from a maximal vector v in fact forms *the whole submodule* generated by it. Assume $\mathcal{U} \subseteq \widehat{H}\langle v\rangle$ is a k-linearly independent set. Then it is easy to prove, using the properties of bases, linearity, and vector subspaces, that if $D \cdot u \in k\langle\mathcal{U}\rangle$ for all $D \in \mathcal{G}$ and $u \in \mathcal{U}$, then $k\langle\mathcal{U}\rangle$ is an $\widehat{H}$-module.

6.2 The action of $\widehat{H}$ on induced modules

6.2.1 Calculating the actions

Throughout, let $v \in Z(M)$ be a maximal vector of weight λ, for M a simple restricted $\widehat{H}_0$-module as above. We are now interested in the action of $\widehat{H}$ on $Z(M)$.

A useful lemma used throughout this chapter is the following:

Lemma 6.2.1. *Let $\mathscr{A}$ be an associative k-algebra. Suppose $D, A_0, \ldots, A_N \in \mathscr{A}$ and that for all $t \in \{0, \ldots, N-1\}$*

$$A_t D = DA_t + A_{t+1}$$

Then we have for $0 \leq n \leq N$

$$A_0 D^n = \sum_{t=0}^{n} \binom{n}{t} D^{n-t} A_t.$$

For the proof, it is a direct application of (Strade and Farnsteiner, 1988, Chap. 1, Prop. 1.3 (4)), noting that $\operatorname{ad}(D)^t(A_0) = (-1)^t A_t$.

We can now explain in more detail why calculating in $\widehat{H}$ is subtler. Since $v = \sum_{c\in\mathcal{A}} \left(\partial'_x\partial_y\right)^c \otimes m_c$, we will want to work in $\mathfrak{u}(\widehat{H})$ with expressions of the form $e_{a,b}(\partial'_x)^n$.

Note that $\partial'_x = -e_{0,1}$. It is straightforward to verify using the formula for the bracket in $\widehat{H}$ that for all $(a,b) \in \mathcal{A}$, we have

$$[e_{a,b}, -e_{0,1}] = \begin{cases} -e_{a-1,b} & a > 0, \\ (b-1)e_{p-1,b} & a = 0. \end{cases}$$

Hence, the existence of the non-homogeneous (in the W_2 grading) elements $e_{0,b} = x^{(p-1)}y^{(b)}\partial_y - y^{(b-1)}\partial_x$ complicates calculating $e_{a,b}(\partial'_x)^n$. To commute $e_{a,b}$ past the ∂'_x terms in the expression $e_{a,b}(\partial'_x)^n = e_{a,b}(-e_{0,1})^n$, we need to calculate $A_1 := [e_{a,b}, -e_{0,1}]$, $A_2 := [A_1, -e_{0,1}]$, $A_3 := [A_2, -e_{0,1}], \ldots, A_n := [A_{n-1}, -e_{0,1}]$.

The bracket behaves differently depending on whether a is zero or not, and taking the bracket with $-e_{0,1}$ decreases a by one as long as $a > 0$. Thus, when computing the A_t terms we will often need to use the second equation. When and whether this flip occurs depends on the value of a and the value of n, and so there will not in general be a simple formula for the A_t, such as $A_t = (-1)^t e_{a-t,b}$, for instance. Furthermore, since we will be seeking a formula for $e_{a,b} \cdot v$, we will need to consider $0 \le n \le p-1$. Therefore, formulas for $e_{a,b} \cdot v$ will often involve multiple cases and will not be easily stated.

Lemma 6.2.2. *We have the following identities in* $\mathfrak{u}(\widehat{H})$*:*

1. $x\partial_y\partial'_x = \partial'_x x\partial_y - \partial_y$
2. $-\partial_y\partial'_x = -\partial'_x\partial_y + x^{(p-1)}\partial_y$
3. $\partial_y^i\partial'_x = \partial'_x\partial_y^i - i\,\partial_y^{i-1}x^{(p-1)}\partial_y$
4. $y\partial_y\partial_y^i = \partial_y^i y\partial_y - i\,\partial_y^i$.

Proof. We use the identity $ab - ba = [a,b]$ in $\mathfrak{u}(\widehat{H})$. Since $\partial'_x = \partial_x - x^{(p-1)}\partial_y = -e_{0,1}$ it is easy to see that $[x\partial_y, \partial'_x] = -\partial_y$ and $[-\partial_y, \partial'_x] = x^{(p-1)}\partial_y$. Setting $a = x\partial_y$ and $b = \partial'_x$, and $a = -\partial_y$ and $b = \partial'_x$ gives the first two identities, respectively.

For the third identity, we proceed by induction. The base case $i = 1$ is given by the second identity. Assume inductively that the identity holds for some i, we calculate

$$\begin{aligned} \partial_y^{i+1}\partial'_x &= \partial_y\left(\partial'_x\partial_y^i - i\,\partial_y^{i-1}x^{(p-1)}\partial_y\right) \\ &= \partial'_x\partial_y^{i+1} - x^{(p-1)}\partial_y\partial_y^i - i\,\partial_y^i x^{(p-1)}\partial_y \\ &= \partial'_x\partial_y^{i+1} - (i+1)\partial_y^i x^{(p-1)}\partial_y, \end{aligned}$$

as required. The last identity holds since $[x^{(j)}\partial_y, \partial_y] = 0$ so that $x^{(j)}\partial_y\partial_y^{a_2} = \partial_y^{a_2}x^{(j)}\partial_y$.

Lastly, we proceed by induction again. The base case holds since we calculate that $[y\partial_y, \partial_y] = -\partial_y$, so that $y\partial_y\partial_y = \partial_y y\partial_y - \partial_y$. Assume inductively that the identity holds for some i, we calculate

$$\begin{aligned} y\partial_y\partial_y^{i+1} &= \left(\partial_y^i y\partial_y - i\,\partial_y^i\right)\partial_y \\ &= \partial_y^{i+1}y\partial_y - \partial_y^{i+1} - i\,\partial_y^{i+1} \\ &= \partial_y^{i+1}y\partial_y - (i+1)\partial_y^{i+1}, \end{aligned}$$

as required. ■

We will now give the calculation for the action of one of the elements of $\widehat{H}$ on $Z(M)$, and the rest is done similarly.

Since $x\partial_y \in N$, observe $x\partial_y \cdot v = 0$ because v is a maximal vector. Recall that we write $x\partial_y = X$.

Lemma 6.2.3. *In fact we have:*

$$\begin{aligned} 0 = x\partial_y \cdot v &= \sum_{a\in\mathcal{A}} \left(x\partial_y\partial_x'^{a_1}\right)\partial_y^{a_2} \otimes m_a \\ &= \sum_{a\in\mathcal{A}} \left(\partial_x'\partial_y\right)^a \otimes X \cdot m_a - \sum_{a\in\mathcal{A}} a_1\partial_x'^{a_1-1}\partial_y^{a_2+1} \otimes m_a. \end{aligned}$$

Proof. Apply $x\partial_y$ to Equation (6.1). We proceed by commuting the $x\partial_y$ past the ∂_x' terms. By Lemma 6.2.2 we have

$$x\partial_y\partial_x' = \partial_x' x\partial_y - \partial_y$$

$$-\partial_y\partial_x' = -\partial_x'\partial_y + x^{(p-1)}\partial_y.$$

In general for $a > 1$ we calculate that

$$x^{(a)}\partial_y\partial_x' = \partial_x' x^{(a)}\partial_y - x^{(a-1)}\partial_y.$$

Put $D = \partial_x'$, $A_0 = x\partial_y$, $A_1 = -\partial_y$ and

$$A_j = (-1)^j x^{(p-j+1)}\partial_y$$

for $j \geq 2$.

One can verify that $[A_j, D] = A_{j+1}$, and thus that the above satisfy the conditions of Lemma 6.2.1. Consequently, we have :

$$x\partial_y \partial_x'^{a_1} = A_0 D^{a_1} = \sum_{t=0}^{a_1} \binom{a_1}{t} \partial_x'^{a_1-t} A_t.$$

Recall that $x^{(j)}\partial_y \partial_y^{a_2} = \partial_y^{a_2} x^{(j)}\partial_y$. Hence, we have:

$$A_0 \partial_x'^{a_1} \partial_y^{a_2} = \partial_x'^{a_1} \partial_y^{a_2} x\partial_y - a_1 \partial_x'^{a_1-1} \partial_y^{a_2+1} + \sum_{t=2}^{a_1} \binom{a_1}{t} \partial_x'^{a_1-t} \partial_y^{a_2} A_t.$$

Looking at the A_t terms above, we see that $2 \le t \le a_1 \le p-1$, so they all have degree greater than or equal to 1. Thus they act trivially on M, as they lie inside our subalgebra N.

Thus, tensoring with m_a, we conclude,

$$x\partial_y \partial_x'^{a_1} \partial_y^{a_2} \otimes m_a = \partial_x'^{a_1} \partial_y^{a_2} \otimes X \cdot m_a - a_1 \partial_x'^{a_1-1} \partial_y^{a_2+1} \otimes m_a.$$

Summing over all indices we obtain the result, as required. ■

Now, from this alone we can obtain the following information: if $a_1 = p-1$, we see that the term $\left(\partial_x' \partial_y\right)^a \otimes X \cdot m_a$ cannot cancel with any other term, so

$$X \cdot m_a = 0$$

for all a with $a_1 = p-1$. Likewise, if $a_2 = 0$ we see

$$X \cdot m_a = 0$$

for all a with $a_2 = 0$.

We continue studying the action of $\widehat{H}$ on $Z(M)$. We calculate:

$$\begin{aligned} \lambda_1 v = x\partial_x \cdot v &= \sum_{a \in \mathcal{A}} \left(\partial_x' \partial_y\right)^a \otimes (x\partial_x \cdot m_a - a_1 m_a) \\ &= \sum_{a \in \mathcal{A}} \left(\partial_x' \partial_y\right)^a \otimes \lambda_1 m_a. \end{aligned}$$

Since $[y\partial_y, \partial_x'] = 0$, we have $y\partial_y \partial_x'^{a_1} = \partial_x'^{a_1} y\partial_y$, so using the fourth identity in

Lemma 6.2.2, we calculate:

$$\begin{aligned}\lambda_2 v = y\partial_y \cdot v &= \sum_{a\in\mathcal{A}} \left(\partial'_x\partial_y\right)^a \otimes \left(y\partial_y \cdot m_a - a_2 m_a\right)\\ &= \sum_{a\in\mathcal{A}} \left(\partial'_x\partial_y\right)^a \otimes \lambda_2 m_a.\end{aligned}$$

In light of this, we have for $a \in \mathcal{A}$:

$$x\partial_x \cdot m_a = \lambda(a)_1 m_a$$

and

$$y\partial_y \cdot m_a = \lambda(a)_2 m_a,$$

where $\lambda(a)_i = \lambda_i + a_i$, see Definition 6.1.8.

Since $T = k\left\langle x\partial_x, y\partial_y\right\rangle$ is a maximal torus and $Z(M)$ is a T-module, we have a decomposition

$$Z(M) = \bigoplus_{\alpha\in T^*} Z(M)_\alpha.$$

For $\alpha \in T^*$ write $\lambda_1 = \alpha(x\partial_x), \lambda_2 = \alpha(y\partial_y)$ and $Z(M)_\alpha = Z(M)_{(\lambda_1,\lambda_2)}$ as well as $M_\alpha = M_{(\lambda_1,\lambda_2)}$. The previous two calculations show that for all $w = \sum_{a\in\mathcal{A}} \left(\partial'_x\partial_y\right)^a \otimes m_a \in Z(M)$, we have $w \in Z(M)_{(\lambda_1,\lambda_2)}$ if and only if $m_a \in M_{(\lambda_1+a_1,\lambda_2+a_2)}$ for all $a \in \mathcal{A}$. Therefore,

$$Z(M) = \bigoplus_{(\lambda_1,\lambda_2)\in\mathbb{F}_p^2} Z(M)_{(\lambda_1,\lambda_2)} = \bigoplus_{(\lambda_1,\lambda_2)\in\mathbb{F}_p^2} \left(\bigoplus_{a\in\mathcal{A}} (\partial'_x\partial_y)^a \otimes M_{(\lambda_1+a_1,\lambda_2+a_2)}\right).$$

The element $x\partial_x + y\partial_y$ acts on any simple $\mathfrak{gl}_2$-module by a constant. Let $c \in k$ be the constant for M. We thus have for all $w = \sum_{a\in\mathcal{A}} \left(\partial'_x\partial_y\right)^a \otimes m_a \in Z(M)$:

$$\begin{aligned}(x\partial_x + y\partial_y)\cdot w &= \sum_{a\in\mathcal{A}} \left(\partial'_x\partial_y\right)^a \otimes \left((x\partial_x + y\partial_y)\cdot m_a - (a_1+a_2)m_a\right)\\ &= \sum_{a\in\mathcal{A}} \left(\partial'_x\partial_y\right)^a \otimes (c-(a_1+a_2))m_a.\end{aligned}$$

Now, we have

$$0 = x^{(2)}\partial_y \cdot v = \sum_{a\in\mathcal{A}} \binom{a_1}{2} \partial_x'^{a_1-2}\partial_y^{a_2+1} \otimes m_a - \sum_{a\in\mathcal{A}} a_1 \partial_x'^{a_1-1}\partial_y^{a_2} \otimes X\cdot m_a.$$

We have

$$0 = e_{2,1} \cdot v = \sum_{a \in \mathcal{A}} r_a \partial_x'^{a_1 - 1} \partial_y^{a_2} \otimes m_a - \sum_{a \in \mathcal{A}} a_2 \partial_x'^{a_1} \partial_y^{a_2 - 1} \otimes X \cdot m_a.$$

From this we can immediately obtain that if $a_2 = p - 1$, then the term

$$r_a \partial_x'^{a_1 - 1} \partial_y^{a_2} \otimes m_a$$

cannot cancel with any other term, forcing either $m_a = 0$ or $r_a = 0$.

Now we study the action of the element $A := e_{1,2} = y^{(2)}\partial_y - xy\partial_x$, where we set $Y := -e_{0,2} = y\partial_x - x^{(p-1)}y^{(2)}\partial_y$. We have

$$\begin{aligned} 0 = A \cdot v = & \sum_{\substack{a \in \mathcal{A} \\ a_1 \neq p-1}} s_a \partial_x'^{a_1} \partial_y^{a_2 - 1} \otimes m_a + \sum_{\substack{a \in \mathcal{A} \\ a_1 \neq p-1}} a_1 \partial_x'^{a_1 - 1} \partial_y^{a_2} \otimes Y \cdot m_a \\ & + \sum_{\substack{0 \leq a_2 \leq p-1 \\ a_1 = p-1}} s_a \partial_x'^{p-1} \partial_y^{a_2 - 1} \otimes m_a \\ & - \sum_{\substack{0 \leq a_2 \leq p-1 \\ a_1 = p-1}} \partial_x'^{p-2} \partial_y^{a_2} \otimes Y \cdot m_a - \sum_{\substack{0 \leq a_2 \leq p-1 \\ a_1 = p-1}} \binom{a_2}{2} \partial_y^{a_2 - 2} \otimes X \cdot m_a. \end{aligned}$$

Using that when $a_1 = p - 1$, $X \cdot m_a = 0$, we can simplify the above, since the terms $\binom{a_2}{2}\partial_y^{a_2-1} \otimes X \cdot m_a = 0$ for $a_1 = p - 1$, to simply:

$$\begin{aligned} 0 = A \cdot v = & \sum_{a \in \mathcal{A}} s_a \partial_x'^{a_1} \partial_y^{a_2 - 1} \otimes m_a \\ & + \sum_{a \in \mathcal{A}} a_1 \partial_x'^{a_1 - 1} \partial_y^{a_2} \otimes Y \cdot m_a. \end{aligned}$$

From this we can see that if $a_1 = p - 1$, then the term

$$s_a \partial_x'^{a_1} \partial_y^{a_2 - 1} \otimes m_a$$

cannot cancel so either $m_a = 0$ or $s_a = 0$.

Similarly, if $a_2 = p - 1$, then the term $a_1 \partial_x'^{a_1 - 1} \partial_y^{a_2} \otimes Y \cdot m_a$ cannot cancel, forcing either $Y \cdot m_a = 0$ or $a_1 = 0$.

Now, we study the action of the element $C := -e_{0,3} = y^{(2)}\partial_x - x^{(p-1)}y^{(3)}\partial_y$.

$$\begin{aligned}
0 = C\cdot v = &\sum_{\substack{a\in\mathcal{A}\\ a_1\neq p-1,p-2}} \binom{a_2}{2}\partial_x'^{a_1+1}\partial_y^{a_2-2}\otimes m_a - \sum_{\substack{a\in\mathcal{A}\\ a_1\neq p-1,p-2}} a_2\partial_x'^{a_1}\partial_y^{a_2-1}\otimes Y\cdot m_a\\
&\sum_{\substack{0\leq a_2\leq p-1\\ a_1=p-2}} \binom{a_2}{2}\partial_x'^{p-1}\partial_y^{a_2-2}\otimes m_a - \sum_{\substack{0\leq a_2\leq p-1\\ a_1=p-2}} a_2\partial_x'^{p-2}\partial_y^{a_2-1}\otimes Y\cdot m_a\\
&+\sum_{\substack{0\leq a_2\leq p-1\\ a_1=p-2}} 2\binom{a_2}{3}\partial_y^{a_2-3}\otimes X\cdot m_a\\
&-\sum_{\substack{0\leq a_2\leq p-1\\ a_1=p-1}} a_2\partial_x'^{p-1}\partial_y^{a_2-1}\otimes Y\cdot m_a - \sum_{\substack{0\leq a_2\leq p-1\\ a_1=p-1}} 2\binom{a_2}{3}\partial_x'\partial_y^{a_2-3}\otimes X\cdot m_a\\
&+\sum_{\substack{0\leq a_2\leq p-1\\ a_1=p-1}} \left(\binom{a_2}{2}(\lambda(a)_2 - 2\lambda(a)_1 + a_2 - 2) - 2\binom{a_2}{3}\right)\partial_y^{a_2-2}\otimes m_a.
\end{aligned}$$

Using again that for $a\in\mathcal{A}$ with $a_1 = p-1$, $X\cdot m_a = 0$, we can simplify the above to:

$$\begin{aligned}
0 = C\cdot v = &\sum_{\substack{a\in\mathcal{A}\\ a_1\neq p-1,p-2}} \binom{a_2}{2}\partial_x'^{a_1+1}\partial_y^{a_2-2}\otimes m_a - \sum_{\substack{a\in\mathcal{A}\\ a_1\neq p-1,p-2}} a_2\partial_x'^{a_1}\partial_y^{a_2-1}\otimes Y\cdot m_a\\
&+\sum_{\substack{0\leq a_2\leq p-1\\ a_1=p-2}} \binom{a_2}{2}\partial_x'^{p-1}\partial_y^{a_2-2}\otimes m_a - \sum_{\substack{0\leq a_2\leq p-1\\ a_1=p-2}} a_2\partial_x'^{p-2}\partial_y^{a_2-1}\otimes Y\cdot m_a\\
&+\sum_{\substack{0\leq a_2\leq p-1\\ a_1=p-2}} 2\binom{a_2}{3}\partial_y^{a_2-3}\otimes X\cdot m_a - \sum_{\substack{0\leq a_2\leq p-1\\ a_1=p-1}} a_2\partial_x'^{p-1}\partial_y^{a_2-1}\otimes Y\cdot m_a\\
&+\sum_{\substack{0\leq a_2\leq p-1\\ a_1=p-1}} \left(\binom{a_2}{2}(\lambda(a)_2 - 2\lambda(a)_1 + a_2 - 2) - 2\binom{a_2}{3}\right)\partial_y^{a_2-2}\otimes m_a.
\end{aligned}$$

Consider the term

$$-a_2\partial_x'^{p-1}\partial_y^{a_2-1}\otimes Y\cdot m_a$$

If $a_2 = p-1$, we see that this cannot cancel with any other term. Thus, we deduce that

$$Y\cdot m_{(p-1,p-1)} = 0.$$

Likewise, consider the term

$$-a_2\partial_x'^{p-2}\partial_y^{a_2-1}\otimes Y\cdot m_a.$$

If $a_2 = p-1$, we see that this cannot cancel with any other term. Thus, we deduce that

$$Y\cdot m_{(p-2,p-1)} = 0.$$

Now, consider the term in the second sum

$$-a_2\partial_x'^{a_1}\partial_y^{a_2-1}\otimes Y\cdot m_a,$$

where $a_1 = 0$. If $a_2 = p-1$, then no cancellation can occur with any other term, so we deduce that

$$Y\cdot m_{(0,p-1)} = 0.$$

We also have

$$\begin{aligned}0 = e_{3,1}\cdot v = &\sum_{a\in\mathcal{A}} t_a\partial_x'^{a_1-2}\partial_y^{a_2}\otimes m_a\\ &+\sum_{a\in\mathcal{A}} a_1a_2\partial_x'^{a_1-1}\partial_y^{a_2-1}\otimes X\cdot m_a.\end{aligned}$$

Here, we also see that if $a_2 = p-1$, then no cancellation can occur with any other terms, so either $m_a = 0$ or $t_a = 0$.

Finally we calculate the action of $e_{2,p-1} = xy^{(p-1)}\partial_y - x^{(2)}y^{(p-2)}\partial_x$:

$$\begin{aligned}0 = e_{2,p-1}\cdot v = &-\sum_{\substack{0\leq a_1\leq p-1\\ a_2=p-3}}\binom{a_1}{2}\partial_x'^{a_1-2}\otimes Y\cdot m_a + \sum_{\substack{0\leq a_1\leq p-1\\ a_2=p-2}} 2\binom{a_1}{2}\partial_x'^{a_1-2}\partial_y\otimes Y\cdot m_a\\ &+\sum_{\substack{0\leq a_1\leq p-1\\ a_2=p-2}}\left(a_1\left(\lambda(a)_2-\lambda(a)_1\right)+\binom{a_1}{2}\right)\partial_x'^{a_1-1}\otimes m_a\\ &+\sum_{\substack{0\leq a_1\leq p-1\\ a_2=p-1}}\partial_x'^{a_1}\otimes X\cdot m_a - \sum_{\substack{0\leq a_1\leq p-1\\ a_2=p-1}}\binom{a_1}{2}\partial_x'^{a_1-2}\partial_y^2\otimes Y\cdot m_a\\ &+\sum_{\substack{0\leq a_1\leq p-1\\ a_2=p-1}}\left(a_1\left(\lambda(a)_1-\lambda(a)_2-1\right)-\binom{a_1}{2}\right)\partial_x'^{a_1-1}\partial_y\otimes m_a\end{aligned}$$

From this we can see that if $m_{\omega_0} \neq 0$, then $\lambda(a)_1 - \lambda(a)_2 = (a_1+1)/2 = 0$.

We also have:

$$0 = x^{(p-1)}\partial_y \cdot v = -\sum_{\substack{0\le a_2\le p-1\\ a_1=p-2}} \partial_y^{a_2} \otimes X\cdot m_a + \sum_{\substack{0\le a_2\le p-1\\ a_1=p-1}} \partial'_x\partial_y^{a_2} \otimes X\cdot m_a + \sum_{\substack{0\le a_2\le p-1\\ a_1=p-1}} \partial_y^{a_2+1} \otimes m_a$$

From this we can also confirm that if $m_{\omega_0} \neq 0$, then $\lambda(a)_1 - \lambda(a)_2 = 0$.

To handle the $p=5$ case with more ease, we have computed the action of $e_{4,4}$ on vectors in $Z(M)$.

By applying $e_{4,4}$ to both sides of the identity (6.1), we have for all $v \in Z(M)$:

$$\begin{aligned} e_{4,4}\cdot v &= 1\otimes X\cdot m_{(2,4)} + 1\otimes(\lambda((3,3))_2 - \lambda((3,3)_1)\, m_{(3,3)} - 1\otimes Y\cdot m_{(4,2)} \\ &+ 3\partial'_x \otimes X\cdot m_{(3,4)} + (4\,(\lambda((3,4))_2 - \lambda((3,4)_1) - 1)\,\partial_y \otimes m_{(3,4)} \\ &+ (4\,(\lambda((4,3))_2 - \lambda((4,3)_1) + 1)\,\partial'_x \otimes m_{(4,3)} - 3\partial_y \otimes Y\cdot m_{(4,3)} \\ &+ (\lambda((4,4))_2 - \lambda((4,4)_1)\,\partial'_x\partial_y \otimes m_{(4,4)} + \partial'^2_x \otimes X\cdot m_{(4,4)} - \partial_y^2 \otimes Y\cdot m_{(4,4)}. \end{aligned}$$

Later on we will need to have a formula for the action of $Y = -e_{0,2}$ on arbitrary vectors $v \in Z(M)$. We have

$$\begin{aligned} Y\cdot v = &\sum_{\substack{a\in\mathcal{A}\\ a_1\neq p-1,p-2}} \partial'^{a_1}_x\partial_y^{a_2} \otimes Y\cdot m_a - \sum_{\substack{a\in\mathcal{A}\\ a_1\neq p-1,p-2}} a_2\partial'^{a_1+1}_x\partial_y^{a_2-1} \otimes m_a \\ &+ \sum_{\substack{0\le a_2\le p-1\\ a_1=p-2}} \partial'^{a_1}_x\partial_y^{a_2} \otimes Y\cdot m_a - \sum_{\substack{0\le a_2\le p-1\\ a_1=p-2}} a_2\partial'^{a_1+1}_x\partial_y^{a_2-1} \otimes m_a \\ &- \sum_{\substack{0\le a_2\le p-1\\ a_1=p-2}} \binom{a_2}{2}\partial_y^{a_2-2} \otimes X\cdot m_a \\ &+ \sum_{\substack{0\le a_2\le p-1\\ a_1=p-1}} \partial'^{a_1}_x\partial_y^{a_2} \otimes Y\cdot m_a + \sum_{\substack{0\le a_2\le p-1\\ a_1=p-1}} w_a\partial_y^{a_2-1} \otimes m_a \\ &+ \sum_{\substack{0\le a_2\le p-1\\ a_1=p-1}} \binom{a_2}{2}\partial'_x\partial_y^{a_2-2} \otimes X\cdot m_a, \end{aligned}$$

where
$$w_a := a_2\lambda(a)_1 - \binom{a_2}{2}$$

We lastly state the formula for the action of ∂_y on vectors in $Z(M)$. This will become useful when checking that a set of k-linearly independent vectors does form an $\widehat{H}$-submodule.

We have for $v \in Z(M)$:

$$\partial_y \cdot v = \sum_{\substack{0\le a_2\le p-1 \\ a_1\neq p-1}} \partial_x'^{a_1}\partial_y^{a_2+1} \otimes m_a + \sum_{\substack{0\le a_2\le p-1 \\ a_1= p-1}} \partial_x'^{a_1}\partial_y^{a_2+1} \otimes m_a + \sum_{\substack{0\le a_2\le p-1 \\ a_1= p-1}} \partial_y^{a_2} \otimes X \cdot m_a. \tag{6.2}$$

Before we move on, we summarise the information we extracted throughout this section for ease of reference.

We proved the following:

Proposition 6.2.4. *Let M be a simple restricted $\widehat{H}_0$-module and let $Z(M)$ be the induced $\mathfrak{u}(\widehat{H})$-module. Let $v = \sum_{a\in\mathcal{A}} \left(\partial_x'\partial_y\right)^a \otimes m_a \in Z(M)$ be a maximal vector, where $m_a \in M$ for all $a \in \mathcal{A} = \{0,1,\dots,p-1\}^2$. Let $Y = -e_{0,2}$. Then we have, recalling the notation in Definition 6.1.8,*

1. *$X \cdot m_a = 0$ for all a with $a_1 = p-1$ or $a_2 = 0$;*
2. *$m_a = 0$ or $r_a = 0$ for all a with $a_2 = p-1$;*
3. *$m_a = 0$ or $s_a = 0$ for all a with $a_1 = p-1$;*
4. *$Y \cdot m_a = 0$ for all a with $a_2 = p-1$;*
5. *$m_a = 0$ or $t_a = 0$ for all a with $a_2 = p-1$.*

6.2.2 Using the $\mathfrak{sl}_2$-module structure

We begin by proving that the extension of $\widehat{H}_{(1)}$ by $\widehat{H}_{(0)}/\widehat{H}_{(1)}$ in fact splits. Indeed, we show there is a subalgebra $\mathfrak{u} \le \widehat{H}_{(0)}$ such that $\mathfrak{u} \oplus \widehat{H}_{(1)} = \widehat{H}_{(0)}$, i.e. such that $\mathfrak{u} \cong \widehat{H}_0 \cong \mathfrak{gl}_2(k)$. Consider the subspace of $\widehat{H}_{(0)}$ spanned by the elements $x\partial_y$, $x\partial_x$, $y\partial_y$, and $-e_{0,2} = y\partial_x - x^{(p-1)}y^{(2)}\partial_y$. It is easy to see using the formula for the bracket in $\widehat{H}$

that this subspace is in fact a subalgebra of $\widehat{H}_{(0)}$, call it $\mathfrak{u}$. The map $f : \widehat{H}_{(0)} \longrightarrow \mathfrak{gl}_2(k)$ fixed in §6.1 gives us a bijective map of Lie algebras $f \circ \iota : \mathfrak{u} \longrightarrow \mathfrak{gl}_2(k)$. Thus, $\mathfrak{u} \cong \mathfrak{gl}_2(k)$. The elements $x\partial_y, x\partial_x, y\partial_y$, and $-e_{0,2}$ are mapped to the matrices $E_{1,2}, E_{1,1}, E_{2,2}$, and $E_{2,1}$, respectively.

It is easy to check that $\mathfrak{u}$ contains the $\mathfrak{sl}_2$-triple $(X = x\partial_y, H := x\partial_x - y\partial_y, Y = -e_{0,2})$ as one can verify that

$$[H,X] = 2X,\ [H,Y] = -2Y,\ \text{and}\ [X,Y] = H.$$

Under our fixed isomorphism X goes to $E_{1,2}$, H goes to $E_{1,1} - E_{2,2} = H_{1,2}$ and Y goes to $E_{2,1}$.

Recall that M is a simple restricted $\widehat{H}_0 \cong \mathfrak{gl}_2$-module. Thus, we can view M as a restricted $\mathfrak{sl}_2$-module, by restriction. In fact in the quotient $\widehat{H}_{(0)}/\widehat{H}_{(1)}$ we have the $\mathfrak{sl}_2$-triple $(X + \widehat{H}_{(1)}, H + \widehat{H}_{(1)}, Y + \widehat{H}_{(1)})$, where our fixed isomorphism $\widehat{H}_0 \cong \mathfrak{gl}_2$ tells us that $X + \widehat{H}_{(1)}$, $H + \widehat{H}_{(1)}$, and $Y + \widehat{H}_{(1)}$ are mapped to the matrices $E_{1,2}$, $H_{1,2}$, and $E_{2,1}$, respectively.

First recall some of the basic results concerning $\mathfrak{sl}_2$-modules.

Proposition 6.2.5. *Let V be an $\mathfrak{sl}_2$-module and let $m \in V_\alpha$, where*

$$V_\alpha := \{m \in V : H \cdot m = \alpha m\},$$

noting that this is non-zero for some scalar α, as k is algebraically closed. Then we have

1. *$X \cdot m \in V_{\alpha+2}$;*

2. *$H \cdot m \in V_\alpha$;*

3. *$Y \cdot m \in V_{\alpha-2}$.*

Also, using an inductive argument, we obtain the following well-known lemma:

Lemma 6.2.6. *For $m \in V_\alpha$ such that $Y \cdot m = 0$, we have*

$$(YX^i) \cdot m = i(-\alpha - i + 1)X^{i-1} \cdot m.$$

Now, we know that simple restricted $\mathfrak{gl}_2$-modules are always simple after restriction to $\mathfrak{sl}_2$. Thus we have a decomposition of our simple restricted $\mathfrak{sl}_2$-module M into its

H-eigenspaces, with each eigenspace one-dimensional:

$$M = M_{-n} \oplus M_{-n+2} \oplus \cdots \oplus M_n,$$

where $n+1$ is the dimension of M.

Therefore, pick an eigenbasis $\{v_{-n}, v_{-n+2}, \ldots, v_n\}$ for M such that

$$X \cdot v_\alpha = v_{\alpha+2},$$

for all eigenvalues α not equal to n.

Using our lemma and this basis we have that

$$Y \cdot v_{-n+2i} = i(n-i+1)v_{-n+2i-2}$$

for all $i \in \{0, \ldots, n\}$.

We restate the information we already had in Proposition 6.2.4 in these new terms:

Proposition 6.2.7. *Let M be a simple restricted $\widehat{H}_0$-module and let $Z(M)$ be the induced $\mathfrak{u}(\widehat{H})$-module. Let $v = \sum_{a \in \mathcal{A}} \left(\partial'_x \partial_y\right)^a \otimes m_a \in Z(M)$ be a maximal vector, where $m_a \in M$ for all $a \in \mathcal{A} = \{0, 1, \ldots, p-1\}^2$. Then we have, recalling the notation in Definition 6.1.8,*

1. *$m_a = 0$ or $m_a \in k\langle v_n \rangle$ for all a with $a_1 = p-1$ or $a_2 = 0$;*
2. *$m_a = 0$ or $r_a = 0$ for all a with $a_2 = p-1$;*
3. *$m_a = 0$ or $s_a = 0$ for all a with $a_1 = p-1$;*
4. *$m_a = 0$ or $m_a \in k\langle v_{-n} \rangle$ for all a with $a_2 = p-1$;*
5. *$m_a = 0$ or $t_a = 0$ for all a with $a_2 = p-1$.*

From this we can see that if $m_{\omega_0} = m_{(p-1,p-1)} \neq 0$, then it lies in the highest $\mathfrak{sl}_2$-weight space *and* in the lowest $\mathfrak{sl}_2$-weight space. This tells us that the only case when $m_{\omega_0} \neq 0$ is when we are inducing from a one-dimensional $\mathfrak{sl}_2$-module $L_0(a, a)$.

6.3 Finding maximal vectors and determining induced modules and their composition factors

6.3.1 General considerations

Recall that we have the following result:

Theorem 6.3.1. *There are p isomorphism classes of irreducible restricted representations of $\mathfrak{sl}_2$, with representatives $L_0(z)$ for $z \in \{0,1,\ldots,p-1\}$, where $L_0(z)$ has dimension $z+1$.*

Theorem 6.3.2. *There are p^2 isomorphism classes of irreducible restricted representations of $\mathfrak{gl}_2$, with representatives $L_0(\lambda)$ for $\lambda \in \mathbb{F}_p^2$, where $L_0(\lambda)$ has dimension $\lambda_1 - \lambda_2 + 1$.*

In what follows, let $L_0(\lambda)$ be the $\mathfrak{gl}_2 \cong \widehat{H}_0$-module of highest weight $\lambda = (\lambda_1, \lambda_2)$, which we often view as the $\mathfrak{sl}_2$-module $L_0(\lambda_1 - \lambda_2)$ by restriction.

We adopt the following setup for our restricted $\widehat{H}_0$-modules M (see §6.2.2):

We pick an eigenbasis $\{m_1, m_2, \ldots, m_{n+1}\}$. With this eigenbasis we have

$$X \cdot m_i = m_{i+1},$$

where $X \cdot m_{n+1} = 0$.

From this and by using the results in §6.2.2, we get the following formula for the action of $Y := -e_{0,2}$ on our chosen basis:

$$Y \cdot m_i = (i-1)\,(n-i+2)\,m_{i-1},$$

noting again that $Y \cdot m_1 = 0$.

Consider a maximal vector $v = \sum_{a\in\mathcal{A}} \left(\partial'_x \partial_y\right)^a \otimes m_a \in Z(L_0(a,b))$, where $m_a \in L_0(a,b)$ for all $a \in \mathcal{A}$. If v has weight (λ_1, λ_2), we saw in §6.2.1 that $x\partial_x \cdot m_a = \lambda(a)_1 m_a$ and $y\partial_y \cdot m_a = \lambda(a)_2 m_a$ for all $a \in \mathcal{A}$. Thus, we see that each m_a is a weight vector for $x\partial_x - y\partial_y$, and thus lies in exactly one weight space for the $\mathfrak{sl}_2$ action, i.e. in one H-eigenspace.

Throughout, we write $Z(a,b)$ for $Z(L_0(a,b))$ and $L(a,b)$ for the unique maximal simple quotient of $Z(a,b)$.

Recall that we write $[M_1, M_2, \ldots, M_n] = [V]$ for the list of composition factors of V a module.

6.3.2 Modules induced from one-dimensional modules

We start by looking at inducing to $\widehat{H}$ from one-dimensional modules $M \cong L_0(a, a)$, where $a \in \mathbb{F}_p$. Here we have an eigenbasis $\{m\}$ for M with $X \cdot m = 0 = Y \cdot m$.

We have the following:

Proposition 6.3.3. *Let $M \cong L_0(a, a)$, where $k \langle m \rangle = M$. Every maximal vector v for $Z(M)$ is contained in the subspace*

$$k \langle 1 \otimes m \rangle \oplus k \langle \partial_y \otimes m \rangle \oplus k \langle \partial_x'^{p-1} \partial_y^{p-1} \otimes m \rangle .$$

Proof. Let v be a maximal vector, so we write $v = \sum_{a \in \mathcal{A}} \left(\partial_x' \partial_y\right)^a \otimes m_a$, where $m_a \in M$ for all $a \in \mathcal{A}$. For each m_a write in fact $m_a = k_a m$, where $k_a \in k$. From $x\partial_y \cdot v = 0$, we obtain the following (see Lemma 6.2.3):

$$0 = -\sum_{a \in \mathcal{A}} a_1 \partial_x'^{a_1 - 1} \partial_y^{a_2 + 1} \otimes k_a m.$$

The terms $a_1 \partial_x'^{a_1-1} \partial_y^{a_2+1} \otimes k_a m$ are linearly independent. Thus, if $k_a \neq 0$, then $a_1 = 0$ or $a_2 = p - 1$.

The rest of the following are done similarly, see §6.2.1 for the formulae.

By considering the linearly independent terms in $e_{2,1} \cdot v = 0$, we obtain the following:

$$\text{if } k_a \neq 0, \text{ then } a_1 = 0 \text{ or } r_a = a_1(\lambda(a)_1 - \lambda(a)_2) + a_1 a_2 - \tbinom{a_1}{2} = 0.$$

Likewise, from $e_{1,2} \cdot v = 0$, we obtain the following:

$$\text{if } k_a \neq 0, \text{ then } a_2 = 0 \text{ or } s_a = a_2(\lambda(a)_1 - \lambda(a)_2) - a_1 a_2 + \tbinom{a_2}{2} = 0.$$

Suppose now that there is a $k_a \neq 0$, with $a = (a_1, a_2) \in \mathcal{A}$. If $a_1 \neq 0$, then by the action of $x\partial_y$ we conclude that $a_2 = p - 1$. From the action of $e_{2,1}$, we have $r_a = 0$. Since $p - 1 = a_2 \neq 0$, from the action of $e_{1,2}$ we conclude that $s_a = 0$.

The condition $s_a = 0 = r_a$ yields:

$$a_1 a_2 - \binom{a_1}{2} = -a_1 a_2 + \binom{a_2}{2} = 0.$$

This implies

$$-a_1 - \binom{a_1}{2} = a_1 + 1 = 0,$$

which in turn implies $a_1 = p-1$. Thus $(a_1, a_2) = (p-1, p-1)$.

Summarising, we showed:

$$\text{if } k_a \neq 0 \text{ and } a_1 \neq 0, \text{ then } a = (p-1, p-1).$$

Hence our maximal vector is of the form:

$$v = \partial_x'^{p-1}\partial_y^{p-1} \otimes k_{(p-1,p-1)}m + \sum_{0\leq a_2\leq p-1} \partial_y^{a_2} \otimes k_{(0,a_2)}m.$$

By considering the terms in $e_{1,2} \cdot v = 0$, we conclude:

$$\text{if } a = (0, a_2) \text{ and } k_a \neq 0, \text{ then } a_2 = 0 \text{ or } s_a = 0.$$

Suppose $a_2 \neq 0$, then

$$s_a = -a_1a_2 + \binom{a_2}{2} = 0.$$

Since $a_1 = 0$, we must have $\binom{a_2}{2} = 0$, and so $a_2 = 0, 1$. Summarising, if $a = (0, a_2)$ and $k_a \neq 0$, then $a_2 = 0$ or $a_2 = 1$.

Thus our maximal vector must be contained in the subspace

$$k\left\langle 1 \otimes m\right\rangle \oplus k\left\langle \partial_y \otimes m\right\rangle \oplus k\left\langle \partial_x'^{p-1}\partial_y^{p-1} \otimes m\right\rangle,$$

as claimed. ■

We refine the previous proposition into:

Proposition 6.3.4. *Let $M \cong L_0(a,a)$, where $k\langle m\rangle = M$. If v is a maximal vector for $Z(M)$, then $v = \mu_1(1 \otimes m)$ or $v = \mu_2\left(\partial_y \otimes m\right)$ or $v = \mu_3\left(\partial_x'^{p-1}\partial_y^{p-1} \otimes m\right)$, where $\mu_i \in k$ for all i.*

Proof. Let v be a maximal vector for $Z(M)$ of weight $\lambda = (\lambda_1, \lambda_2)$, so we write

$$v = \mu_1(1 \otimes m) + \mu_2\left(\partial_y \otimes m\right) + \mu_3\left(\partial_x'^{p-1}\partial_y^{p-1} \otimes m\right).$$

Now, each of the terms is a weight vector for $x\partial_x$ and $y\partial_y$. We calculate:

$$x\partial_x \cdot v = \mu_1 a(1 \otimes m) + \mu_2 a\left(\partial_y \otimes m\right) + \mu_3(a+1)\left(\partial_x'^{p-1}\partial_y^{p-1} \otimes m\right) = \lambda_1 v.$$

Thus, by comparing coefficients, we have $\mu_1 a = \lambda_1\mu_1$, $\mu_2 a = \lambda_1\mu_2$, and $\mu_3(a+1) = \lambda_1\mu_3$. We conclude that either $\lambda_1 = a$ and $\mu_3 = 0$ or $\lambda_1 \neq a$ and $\mu_1 = \mu_2 = 0$. Therefore, either $v = \mu_1(1\otimes m) + \mu_2(\partial_y \otimes m)$ or $v = \mu_3\left(\partial_x'^{p-1}\partial_y^{p-1}\otimes m\right)$.

Suppose the former is the case. We calculate:

$$y\partial_y \cdot v = \mu_1 a(1\otimes m) + \mu_2(a-1)(\partial_y\otimes m) = \lambda_2 v.$$

So, by comparing coefficients, we have $\mu_1 a = \lambda_2\mu_1$, $\mu_2(a-1) = \lambda_2\mu_2$. Hence, either $\lambda_2 = a$ and $\mu_2 = 0$ or $\lambda_2 \neq a$ and $\mu_1 = 0$, as required. ∎

Lemma 6.3.5. *Let $a \in \mathbb{F}_p$. Consider $\widehat{H}\langle\partial_y\otimes m\rangle \leq Z(a,a)$.*

1. *If $a \neq 0$, then $\widehat{H}\langle\partial_y\otimes m\rangle = Z(a,a)$, with dimension p^2.*
2. *If $a = 0$, then*

$$\widehat{H}\langle\partial_y\otimes m\rangle = k\left\langle\partial_x'^i\partial_y^{j+1}\otimes m : 0\leq i\leq p-1, 0\leq j\leq p-2\right\rangle \oplus k\left\langle\partial_x'^j\otimes m : 1\leq j\leq p-1\right\rangle,$$

as vector spaces, with dimension p^2-1.

Proof. We must check that the basis elements are stable under the generators of $\widehat{H}$.

Consider $v := \partial_y\otimes m$. Then using $\partial_x^i\partial_y^j \in \mathfrak{u}(\widehat{H})$ we see $\widehat{H}\langle v\rangle$ contains

$$\left\{\partial_x'^i\partial_y^{j+1}\otimes m : 0\leq i\leq p-1, 0\leq j\leq p-2\right\}.$$

Now, $[Y,\partial_y] = -\partial_x'$, so

$$Y\cdot v = \partial_y\otimes Y\cdot m - \partial_x'\otimes m = -\partial_x'\otimes m.$$

Hence, $\widehat{H}\langle v\rangle$ also contains the elements

$$\left\{\partial_x'^j\otimes m : 1\leq j\leq p-1\right\}.$$

Now $\partial_x'\cdot\partial_x'^{p-1} = -y\partial_y\otimes m = -a\cdot 1\otimes m$.

If $a\neq 0$, then $-a\cdot 1\otimes m\neq 0$, and $\widehat{H}\langle v\rangle = Z(a,a)$. Thus in this case, $Z(a,a)$ is simple.

If $a = 0$, then $-a \cdot 1 \otimes m = 0$, and this is all we get, since we can use our basis for $\widehat{H}$ to check that the above k-basis is closed under the action of $\widehat{H}$. Thus, $\dim_k \widehat{H} \langle v \rangle = p^2 - 1$. ■

Lemma 6.3.6. *In $Z(-1, -1)$, we have*

$$\widehat{H} \left\langle \partial_x'^{p-1} \partial_y^{p-1} \otimes m \right\rangle = k \left\langle \partial_x'^{p-1} \partial_y^{p-1} \otimes m \right\rangle.$$

Proof. Clearly $k \langle \partial_x'^{p-1} \partial_y^{p-1} \otimes m \rangle \subseteq \widehat{H} \langle \partial_x'^{p-1} \partial_y^{p-1} \otimes m \rangle$. We leave it to the reader to use the basis for $\widehat{H}$ to check that $k \langle \partial_x'^{p-1} \partial_y^{p-1} \otimes m \rangle$ is closed under the action of $\widehat{H}$. ■

We will need the following lemma to prove the main result of this subsection.

Lemma 6.3.7. *The restricted $\widehat{H}$-module $\mathcal{O}(2; (1, 1))/(k \cdot 1)$ is simple.*

Proof. Recall that $\widehat{H} \leq W(2; (1, 1))$ acts on $\mathcal{O}(2; (1, 1))$ via $D \cdot f = D(f)$ for all $D \in \widehat{H}$, $f \in \mathcal{O}(2; (1, 1))$ (see Example 3.1.31). By Lemma 6.1.4 it suffices to show that all the maximal vectors generate the whole module.

Let $v \in \mathcal{O}(2; (1, 1))/(k \cdot 1)$ be a maximal vector. Then we can write

$$v = \sum_{0 \leq a, b \leq p-1} k_{a,b} x^{(a)} y^{(b)},$$

as its representative in $\mathcal{O}(2; (1, 1))$, so that in the quotient, we identify the term $k_{0,0} 1$ with 0. We calculate

$$0 = x \partial_y \cdot v = \sum_{0 \leq a, b \leq p-1} (a + 1) \, k_{a,b} x^{(a+1)} y^{(b-1)}.$$

Therefore,

$$\text{if } k_{a,b} \neq 0, \text{ then } a = p - 1 \text{ or } b = 0.$$

Hence our maximal vector is of the form:

$$v = \sum_{1 \leq b \leq p-1} k_{p-1,b} x^{(p-1)} y^{(b)} + \sum_{1 \leq a \leq p-1} k_{a,0} x^{(a)}.$$

We calculate

$$0 = e_{1,2} \cdot v = \sum_{1 \leq b \leq p-1} \left(\binom{b+1}{2} + b + 1 \right) k_{p-1,b} x^{(p-1)} y^{(b+1)} - \sum_{1 \leq a \leq p-1} a k_{a,0} x^{(a)} y.$$

Hence,

$$k_{a,0} = 0 \text{ for all } 1 \le a \le p-1.$$

We also get that

$$\text{if } k_{p-1,b} \neq 0, \text{ then } b = p-1, p-2.$$

We conclude that v must be of the form

$$v = k_{p-1,p-2}x^{(p-1)}y^{(p-2)} + k_{p-1,p-1}x^{(p-1)}y^{(p-1)}.$$

Since v is a weight vector, we argue as before to conclude that in fact $v = \mu_1 x^{(p-1)}y^{(p-1)}$ or $v = \mu_2 x^{(p-1)}y^{(p-2)}$, noting that indeed $N \cdot v = 0$, that is, that $x^{(p-1)}\partial_y$ and $C := -e_{0,3}$ kill v. Note that in characteristic $p = 5$, one must also check that $e_{4,4} \cdot v = 0$, which is clear.

Suppose now $v = \mu_1 x^{(p-1)}y^{(p-1)} \neq 0$. We calculate

$$\begin{aligned} \partial'_x \cdot x^{(a)}y^{(b)} &= x^{(a-1)}y^{(b)} \\ \partial_y \cdot x^{(a)}y^{(b)} &= x^{(a)}y^{(b-1)}, \end{aligned}$$

the first identity being valid only for $1 \le a \le p-1$. Consequently, by applying powers of ∂'_x and ∂_y consecutively, we see we can obtain all of $O(2;(1,1))/(k \cdot 1)$.

Suppose now that $v = \mu_2 x^{(p-1)}y^{(p-2)} \neq 0$. By using the above identities, we see that $v_1 := y^{(p-3)} \in \widehat{H}\langle v \rangle$. Then we calculate

$$C \cdot v_1 = x^{(p-1)}y^{(p-1)},$$

and so $\widehat{H}\langle v \rangle = O(2;(1,1))/(k \cdot 1)$, and we are done. ■

Theorem 6.3.8. *The induced module $Z(M) \cong Z(a,a)$ is simple unless $a = 0$ or $a = p-1$, in which case it has composition factors of dimension 1 and $p^2 - 1$.*

Proof. Consider the vector $v = \partial_x'^{p-1}\partial_y^{p-1} \otimes m$, which given Proposition 6.3.4 is a good candidate for a maximal vector. Because $C = -e_{0,3}$ must annihilate maximal vectors, and

$$C \cdot v = (p-1-\lambda(a)_2)\,\partial_y^{p-3} \otimes m = (p-1-a)\,\partial_y^{p-3} \otimes m,$$

we conclude that v is maximal only when $a = p-1$.

Now, $\widehat{H}\langle v\rangle = k\langle v\rangle$ is one-dimensional, so in the $a = p-1$ case, we conclude that $Z(a,a)$ is not simple. Furthermore, this is the only proper submodule, as $\widehat{H}\langle \partial_y \otimes m\rangle$ here generates all of $Z(-1,-1)$.

We calculate that the vector v has weight $\lambda = (a+1, a+1) = (0,0)$. It remains to show that the quotient $Z(-1,-1)/\widehat{H}\langle v\rangle$ is simple.

We have by Frobenius reciprocity that, given a simple $\widehat{H}$-module M:

$$\mathrm{Hom}_{\widehat{H}_{(0)}}(L_0(-1,-1), M) \cong \mathrm{Hom}_{\widehat{H}}(Z(-1,-1), M).$$

This tells us that there is a simple $\widehat{H}_0$-submodule of M isomorphic to $L_0(-1,-1)$ if and only if $Z(-1,-1)$ surjects to M. That is, M has a maximal vector of highest weight $(-1,-1)$ if and only if $Z(-1,-1)$ surjects to M.

But $O(2;(1,1))/(k\cdot 1)$ is simple by Lemma 6.3.7 and it has a $(-1,-1)$ weight maximal vector. Hence, $Z(-1,-1)$ surjects to it. Hence, $Z(-1,-1)$ has a (p^2-1)-dimensional simple quotient. By consideration of dimensions, the quotient $Z(-1,-1)/\widehat{H}\langle v\rangle$ is this simple quotient, so it is $L(-1,-1)$.

Now, $\widehat{H}\langle v\rangle$ is a one-dimensional simple $\widehat{H}$-module of highest weight $(0,0)$, which must be trivial and is isomorphic to $L(0,0)$. Thus, we have composition factors

$$[L(-1,-1), L(0,0)]$$

of dimension $p^2-1, 1$.

Now let $a \neq p-1$. So v above is not maximal. Clearly $1\otimes m$ always generates all of $Z(M)$, so we now look at $v = \partial_y \otimes m$.

If $a \neq 0$, then $\widehat{H}\langle v\rangle = Z(a,a)$ by Lemma 6.3.5, and so $Z(a,a)$ is simple.

On the other hand, if $a = 0$, then $\widehat{H}\langle v\rangle$ is a non-trivial simple submodule of dimension p^2-1, as it is generated by each of its maximal vectors, namely the vectors of the form $\partial_y \otimes \mu m$ for non-zero μ. We also calculate that the vector v has weight $\lambda = (a, a-1)$. Hence, v here is maximal vector of weight $(0,-1)$, which means that $\widehat{H}\langle v\rangle \cong L(0,-1)$. The quotient by $\widehat{H}\langle v\rangle$ is one-dimensional and hence simple, and therefore is $L(0,0)$. Thus, $Z(0,0)$ has composition factors

$$[L(0,-1), L(0,0)]$$

of dimension p^2-1 and 1. ■

6.3.3 Modules induced from two-dimensional modules

Let $M \cong L_0(a, a-1)$, with $a \in \mathbb{F}_p$. Pick an eigenbasis $\{m_1, m_2\}$ for M with $X \cdot m_1 = m_2$ and $Y \cdot m_2 = m_1$. We refer the reader to §6.2.2 for more details.

Proposition 6.3.9. *Let $M \cong L_0(a, a-1)$, with $a \in \mathbb{F}_p$. Every maximal vector v for $Z(M)$ is contained in the subspace*

$$k\langle 1 \otimes m_2 \rangle \oplus k\left\langle \partial'_x \otimes m_2 + \partial_y \otimes m_1 \right\rangle \oplus k\left\langle \partial'_x\partial_y \otimes m_2 + \partial_y^2 \otimes m_1 \right\rangle.$$

Proof. Let v be a maximal vector, so we write $v = \sum_{a\in\mathcal{A}} \left(\partial'_x\partial_y\right)^a \otimes m_a$, where $m_a \in M$ for all $a \in \mathcal{A}$ (see Equation (6.1)). Since m_a can only be in one H-eigenspace (see §6.3.1), we have for all $a \in \mathcal{A}$, $m_a = \mu_a m_1$ or $m_a = \mu_a m_2$, where $\mu_a \in k$ (we write $m_a = \mu_a m_j$, with $\mu_a \in k$, generally). As with the one-dimensional case, we refer the reader to §6.2.1 for the formulae for the actions we will consider here. We do the first one in detail. The others are done similarly.

From $e_{2,1} \cdot v = 0$, we see

$$0 = \sum_{a\in\mathcal{A}} r_a \partial'^{a_1-1}_x \partial^{a_2}_y \otimes m_a - \sum_{a\in\mathcal{A}} a_2 \partial'^{a_1}_x \partial^{a_2-1}_y \otimes X \cdot m_a.$$

Let $a \in \mathcal{A}$. If $m_a \in k\langle m_1 \rangle$ and $m_a \neq 0$, we see that the term $r_a \partial'^{a_1-1}_x \partial^{a_2}_y \otimes m_a = r_a \partial'^{a_1-1}_x \partial^{a_2}_y \otimes \mu_a m_1$ cannot cancel with any term in the right sum, since $X \cdot m_c \in k\langle m_2 \rangle$ for all $c \in \mathcal{A}$. Due to exponents, it cannot cancel with another term $r_b \partial'^{b_1-1}_x \partial^{b_2}_y \otimes m_b$ in the same left sum.

Therefore, we conclude that if $m_a \in k\langle m_1 \rangle$ and $m_a \neq 0$, then $r_a \partial'^{a_1-1}_x \partial^{a_2}_y \otimes m_a = 0$. Consequently, either $a_1 = 0$ or $r_a = 0$. Suppose $a_1 \neq 0$, and so $r_a = 0$. This implies

$$0 = a_1(\lambda(a)_1 - \lambda(a)_2) + a_1 a_2 - \binom{a_1}{2}.$$

Since m_a is in the lowest $\mathfrak{sl}_2$-weight space, we have $\lambda(a)_1 - \lambda(a)_2 = -1$ Thus,

$$0 = -a_1 + a_1 a_2 - \binom{a_1}{2}.$$

Because $a_1 \neq 0$, we deduce that $-1 + a_2 - (a_1 - 1)/2 = 0$, i.e., that $a_2 = (a_1 + 1)/2$. Summarising:

$$\text{if } m_a \in k\langle m_1\rangle, m_a \neq 0, \text{ and } a_1 \neq 0, \text{ then } a_2 = (a_1+1)/2.$$

By considering the terms in $e_{1,2} \cdot v = 0$ we see that:

$$\text{if } m_a \in k\langle m_2\rangle \text{ and } m_a \neq 0, \text{ then either } s_a = 0 \text{ or } a_2 = 0.$$

We use this to conclude that:

$$\text{if } m_a \in k\langle m_2\rangle, m_a \neq 0, \text{ and } a_2 \neq 0, \text{ then } a_1 = (a_2+1)/2.$$

From the action of $x\partial_y$ we derive that:

$$\text{if } m_a \in k\langle m_1\rangle \text{ and } m_a \neq 0, \text{ then either } a_1 = 0 \text{ or } a_2 = p-1.$$

Therefore, if $m_a \in k\langle m_1\rangle, m_a \neq 0$ and $a_1 \neq 0$, we must have $a_2 = p-1$. Furthermore, the conclusion from $e_{2,1}$'s action yields $a_2 = (a_1+1)/2$. Hence, $p-1 = (a_1+1)/2$, and so $a_1 = p-3$.

Summarising:

$$\text{if } m_a \in k\langle m_1\rangle, m_a \neq 0 \text{ and } a_1 \neq 0, \text{ then } a = (p-3, p-1).$$

From the action of $e_{3,1}$ we see that:

$$\text{if } m_a \in k\langle m_1\rangle \text{ and } m_a \neq 0, \text{ then } a_1 = 0, 1 \text{ or } t_a = 0.$$

Suppose $m_a \in k\langle m_1\rangle, m_a \neq 0$, and $a_1 \neq 0$. From the conclusion from $x\partial_y$ and $e_{2,1}$ we see that $(a_1, a_2) = (p-3, p-1)$. Thus, since $a_1 \neq 0, 1$, from $e_{3,1}$'s action we see that

$$t_a = \binom{a_1}{2}(\lambda(a)_2 - \lambda(a)_1) - \binom{a_1}{2}a_2 + \binom{a_1}{3} = 0.$$

This implies that $a_1 = 0, 1$ or $a_2 = (a_1+1)/3$. Since $a_1 = p-3 \neq 0, 1$, we conclude that $a_2 = (a_1+1)/3$. Hence, $a_2 = (p-2)/3$, but we have $a_2 = p-1$. Thus $p-1 = (p-2)/3$, which implies $p-2 = 3p-3 = -3 = p-3$, a contradiction.

Summarising:

$$\text{for all } a \in \mathcal{A}, \text{ if } m_a \in k\langle m_1\rangle \text{ and } m_a \neq 0, \text{ then } a_1 = 0.$$

Therefore, for all $a \in \mathcal{A}$ with $a_1 \neq 0$ we write $m_a = \mu_a m_2$.

Recall that if $m_a \in k\langle m_2\rangle$ and $m_a \neq 0$, then $a_2 = 0$ or $a_1 = (a_2+1)/2$. Therefore, if $a_2 \neq 0$ and $a_1 = 0$, then $a_2 = p-1$. This implies that when $a_1 = 0$, if $a_2 \neq p-1$, then $a_2 = 0$. Thus, for $a \in \mathcal{A}$ with $a = (0, a_2)$, if $m_a \in k\langle m_2\rangle$ and $1 \leq a_2 \leq p-2$, it must be the case that $m_a = 0$.

Thus we write $m_{(0,a_2)} = \mu_{(0,a_2)}m_1$ for all $1 \leq a_2 \leq p-2$.

Putting all of this together, we have that our maximal vector is of the form:

$$\begin{aligned} v =& 1 \otimes m_{(0,0)} + \partial_y^{p-1} \otimes m_{(0,p-1)} \\ &+ \sum_{1 \leq a_2 \leq p-2} \partial_y^{a_2} \otimes \mu_{(0,a_2)}m_1 + \sum_{\substack{a_1 \neq 0 \\ 0 \leq a_2 \leq p-1}} \partial_x'^{a_1}\partial_y^{a_2} \otimes \mu_a m_2. \end{aligned}$$

Set $C = -e_{0,3}$. From $C \cdot v = 0$, we see that if $a = (p-1, a_2) \in \mathcal{A}, m_a \in k\langle m_2\rangle$, and $m_a \neq 0$, then $a_2 = 0$.

We also deduce from $C \cdot v = 0$ that $m_{(0,p-1)} = \mu_{(0,p-1)}m_1$.

Thus our maximal vector is of the form:

$$\begin{aligned} v =& 1 \otimes m_{(0,0)} + \partial_x'^{p-1} \otimes \mu_{(p-1,0)}m_2 \\ &+ \sum_{1 \leq a_2 \leq p-1} \partial_y^{a_2} \otimes \mu_{(0,a_2)}m_1 + \sum_{\substack{a_1 \neq 0, p-1 \\ 0 \leq a_2 \leq p-1}} \partial_x'^{a_1}\partial_y^{a_2} \otimes \mu_a m_2. \end{aligned}$$

Applying $e_{2,1}$ to v yields either $\mu_{(p-1,0)} = 0$ or $r_{(p-1,0)} = 0$. It's straightforward to compute that $r_{(p-1,0)} \neq 0$. Thus, $\mu_{(p-1,0)} = 0$.

Hence, our maximal vector is of the form:

$$\begin{aligned} v =& 1 \otimes m_{(0,0)} \\ &+ \sum_{1 \leq a_2 \leq p-1} \partial_y^{a_2} \otimes \mu_{(0,a_2)}m_1 + \sum_{\substack{a_1 \neq 0, p-1 \\ 0 \leq a_2 \leq p-1}} \partial_x'^{a_1}\partial_y^{a_2} \otimes \mu_a m_2. \end{aligned}$$

We want to know when the $\mu_a m_2$ in the rightmost sum are nonzero, so we are considering $a \in \mathcal{A}$ such that $a_1 \neq 0, p-1$ and $0 \leq a_2 \leq p-1$.

By considering the terms in $e_{3,1} \cdot v = 0$, we see that

$$\text{if } \mu_a m_2 \neq 0, \text{ then } a_1 = 0, 1 \text{ or } a_2 = (a_1 - 5)/3.$$

On the other hand, by considering the terms in $e_{2,1} \cdot v = 0$, we see that

$$\text{if } \mu_a m_2 \neq 0, \text{ then } a_1 = 0, 1, p-1 \text{ or } a_2 = (a_1 - 3)/2.$$

Hence, assume $\mu_a m_2 \neq 0$ and $a_1 \neq 0, 1$. Since we have already seen that $a_1 \neq p-1$, we have $a_2 = (a_1 - 5)/3 = (a_1 - 3)/2$. This implies $a_1 = p-1$, which is not possible. Thus, we conclude that if $\mu_a m_2 \neq 0$, then $a_1 = 0, 1$. Since $a_1 \neq 0$, we conclude that if $\mu_a m_2 \neq 0$, then $a_1 = 1$.

Therefore, our maximal vector is of the form:

$$\begin{aligned} v =& 1 \otimes m_{(0,0)} \\ &+ \sum_{1 \le a_2 \le p-1} \partial_y^{a_2} \otimes \mu_{(0,a_2)} m_1 + \sum_{0 \le a_2 \le p-1} \partial'_x \partial_y^{a_2} \otimes \mu_{(1,a_2)} m_2. \end{aligned}$$

Applying C again, we see that if $\mu_{(1,a_2)} \neq 0$, then $a_2 = 0, 1$. Thus, we have

$$\begin{aligned} v =& 1 \otimes m_{(0,0)} \\ &+ \sum_{1 \le a_2 \le p-1} \partial_y^{a_2} \otimes \mu_{(0,a_2)} m_1 + \partial'_x \otimes \mu_{(1,0)} m_2 + \partial'_x \partial_y \otimes \mu_{(1,1)} m_2. \end{aligned}$$

We want to know when the $\mu_{(0,a_2)} m_1$ are nonzero. By considering the terms in $e_{2,1} \cdot v = 0$, we get that if $a_2 \geq 3$, then $\mu_{(0,a_2)} = 0$.

Thus, our maximal vector is of the form:

$$\begin{aligned} v =& 1 \otimes m_{(0,0)} \\ &+ \partial_y \otimes \mu_{(0,1)} m_1 + \partial_y^2 \otimes \mu_{(0,2)} m_1 + \partial'_x \otimes \mu_{(1,0)} m_2 + \partial'_x \partial_y \otimes \mu_{(1,1)} m_2. \end{aligned}$$

From $X \cdot v = 0$ it is easy to see that $m_{(0,0)} = \mu_{(0,0)} m_2$.

Finally, we see from $e_{1,2} \cdot v = 0$ that

$$\begin{aligned} \mu_{(1,0)} &= \mu_{(0,1)} \\ \mu_{(1,1)} &= \mu_{(0,2)}. \end{aligned}$$

Thus, we conclude that

$$v \in k \langle 1 \otimes m_2 \rangle \oplus k \left\langle \partial'_x \otimes m_2 + \partial_y \otimes m_1 \right\rangle \oplus k \left\langle \partial'_x \partial_y \otimes m_2 + \partial_y^2 \otimes m_1 \right\rangle,$$

which completes the proof. ■

We will break up the proof of our determination of the modules induced from two-dimensional modules and their composition factors into several lemmas, as depending on the weight one obtains wildly different structures.

In what follows, we adopt the following shorthand:

$$w := \partial'_x \partial_y \otimes m_2 + \partial_y^2 \otimes m_1$$

$$v := \partial'_x \otimes m_2 + \partial_y \otimes m_1.$$

We refine the previous proposition into the following:

Proposition 6.3.10. *Let* $M \cong L_0(a, a-1)$*, with* $a \in \mathbb{F}_p$*. If* u *is a maximal vector for* $Z(M)$*, then* $u = \mu_1 (1 \otimes m_2)$ *or* $u = \mu_2 v$ *or* $u = \mu_3 w$*, where* $\mu_i \in k$ *for all* i*.*

Proof. Let u be a maximal vector for $Z(M)$ of weight $\lambda = (\lambda_1, \lambda_2)$, so we write

$$u = \mu_1 (1 \otimes m) + \mu_2 v + \mu_3 w$$

Now, each of the terms is a weight vector for $x\partial_x$ and $y\partial_y$. We calculate:

$$x\partial_x \cdot u = \mu_1 a (1 \otimes m) + \mu_2 (a-1) v + \mu_3 (a-1) w = \lambda_1 u.$$

Thus, by comparing coefficients, we have $\mu_1 a = \lambda_1 \mu_1$, $\mu_2 (a-1) = \lambda_1 \mu_2$, and $\mu_3 (a-1) = \lambda_1 \mu_3$. We conclude that either $\lambda_1 = a$ and $\mu_2 = 0 = \mu_3$ or $\lambda_1 \neq a$ and $\mu_1 = 0$. Therefore, either $u = \mu_1 (1 \otimes m)$ or $u = \mu_2 v + \mu_3 w$.

Suppose the latter is the case and let $b = a - 1$. We calculate:

$$y\partial_y \cdot u = \mu_2 b v + \mu_3 (b-1) w = \lambda_2 u.$$

So, by comparing coefficients, we have $\mu_2 b = \lambda_2 \mu_2$, $\mu_3 (b-1) = \lambda_2 \mu_3$. Hence, either $\lambda_2 = b$ and $\mu_3 = 0$ or $\lambda_2 \neq b$ and $\mu_2 = 0$, as required. ■

Lemma 6.3.11. *Let* $a \in \mathbb{F}_p$*. In* $Z(a, a-1)$*, if* $a \neq 1$*, then* $\widehat{H}\langle v \rangle = \widehat{H}\langle w \rangle$*. In* $Z(1,0)$*,* $\widehat{H}\langle v \rangle \neq \widehat{H}\langle w \rangle$*.*

Proof. Acting on w by powers of ∂_y and ∂'_x gives that $\widehat{H}\langle w \rangle$ contains at least the following:

$$\left\{\partial'^i_x \partial^{j+2}_y \otimes m_1 + \partial'^{i+1}_x \partial^{j+1}_y \otimes m_2 : 0 \leq i \leq p-1, 0 \leq j \leq p-2\right\},$$

which gives distinct elements as long as $(i,j) \neq (p-1, p-2)$. If $(i,j) = (p-1, p-2)$, we obtain the element

$$\partial'^p_x \partial^{p-1}_y \otimes m_2 = \partial^{p-1}_y \otimes -a m_2,$$

so if $a - 1 \neq -1$, we have $\dim_k \widehat{H}\langle v \rangle \geq p^2 - p$.

Now, we calculate:

$$Y \cdot w = -\partial'_x \partial_y \otimes m_1 - \partial'^2_x \otimes m_2.$$

Hence,

$$\left\{\partial_x'^{i+1}\partial_y \otimes m_1 + \partial_x'^{i+2} \otimes m_2 : i \in \{0, 1 \ldots, p-1\}\right\}$$

is contained in $\widehat{H}\langle w\rangle$.

More specifically, when $i = p-2$, this gives the element

$$\partial_x'^{p-1}\partial_y \otimes m_1 + (-a+1)\cdot 1 \otimes m_2$$

and when $i = p-1$ the element

$$\partial_y \otimes (-a+1)\, m_1 + \partial_x' \otimes (-a+1)\, m_2,$$

noting that $x\partial_x$ and $y\partial_y$ have weights of $a-1$ and a on the lower $\mathfrak{sl}_2$-weight space $k\langle m_1\rangle$, respectively.

Then if $a-1 \neq 0$, then we see that $\widehat{H}\langle w\rangle$ contains v.

Hence, if $a-1 \neq 0$, $\widehat{H}\langle w\rangle = \widehat{H}\langle v\rangle$. ■

Lemma 6.3.12. *Let* $a \in \mathbb{F}_p$. *We have in* $Z(a, a-1)$ *that*

$$\begin{aligned}\widehat{H}\langle w\rangle = k&\left\langle \partial_x'^{i}\partial_y^{j+2} \otimes m_1 + \partial_x'^{i+1}\partial_y^{j+1} \otimes m_2 : 0 \le i \le p-1, 0 \le j \le p-2\right\rangle\\ &\oplus k\left\langle \partial_x'^{i+1}\partial_y \otimes m_1 + \partial_x'^{i+2} \otimes m_2 : i \in \{0, 1 \ldots, p-1\}\right\rangle\end{aligned}$$

and

$$\widehat{H}\langle v\rangle = k\left\langle \partial_x'^{i}\partial_y^{j+1} \otimes m_1 + \partial_x'^{i+1}\partial_y^{j} \otimes m_2 : 0 \le i \le p-1, 0 \le j \le p-1\right\rangle,$$

as vector spaces.

Therefore, if $a \neq 0$, *then* $\dim_k \widehat{H}\langle v\rangle = p^2$. *If* $a = 0$, *then* $\dim_k \widehat{H}\langle v\rangle = p^2 - 1$. *Concerning the dimension of* $\widehat{H}\langle w\rangle$, *if* $a = 1$, *then* $\dim_k \widehat{H}\langle w\rangle = p^2 - 1$. *The other cases are covered by* $\dim_k \widehat{H}\langle v\rangle$, *thanks to Lemma 6.3.11.*

Proof. Let $b = a-1$. We only study $\widehat{H}\langle v\rangle$ and leave the other case to the interested reader, noting that one must only check $\widehat{H}\langle w\rangle$ when $b \neq 0$, i.e., in $Z(1,0)$. Now, $\widehat{H}\langle v\rangle$ certainly contains

$$\left\{\partial_y^{j+1} \otimes m_1 + \partial_x'\partial_y^{j} \otimes m_2 : 0 \le j \le p-1\right\},$$

using Lemma 6.2.2.

By letting ∂'_x act on each the elements of the previous set we obtain:

$$\left\{\partial'^{i}_x \partial^{j+1}_y \otimes m_1 + \partial'^{i+1}_x \partial^{j}_y \otimes m_2 : 0 \le i \le p-1, 0 \le j \le p-1\right\},$$

which gives distinct elements as long as $(i,j) \neq (p-1,p-1)$. If $(i,j) = (p-1,p-1)$, we obtain again the element

$$\partial'^{p}_x \partial^{p-1}_y \otimes m_2 = \partial^{p-1}_y \otimes (-b-1)\, m_2,$$

so if $b \neq -1$, we have $\dim_k \widehat{H}\,\langle v\rangle \ge p^2$. We leave it to the reader to use the basis for $\widehat{H}$ to check that the above k-basis is indeed closed under the action of $\widehat{H}$. ■

Theorem 6.3.13. *The induced module* $Z(M) \cong Z(a,a-1)$*, with* $a \in \mathbb{F}_p$*, is not simple. If* $(a,a-1) = (p-1,p-2)$ *or* $(1,0)$*, then* $Z(a,a-1)$ *has composition factors of dimension* 1, p^2-1 *and* p^2*. If* $(a,a-1) = (0,-1)$*, then* $Z(a,a-1)$ *has two one-dimensional composition factors and two composition factors of dimension* p^2-1*. In the remaining cases* $Z(a,a-1)$ *has two composition factors of dimension* p^2*.*

Proof. Write $b = a-1$. First, we calculate that the vector v has weight $\lambda = (a-1,b)$. The vector w has weight, $\lambda = (a-1,b-1)$.

We start by outlining a basic Frobenius reciprocity argument that takes care of lots of cases.

We have by Frobenius reciprocity that

$$\mathrm{Hom}_{\widehat{H}_{(0)}}(L_0(a,b), Z(a,a)) \cong \mathrm{Hom}_{\widehat{H}}(Z(a,b), Z(a,a)).$$

The left side is non-zero as $Z(a,a)$ has a maximal vector of highest weight $(a,a-1) = (a,b)$, as we saw previously. Thus there is a non-zero $\widehat{H}$-homomorphism

$$f : Z(a,b) \longrightarrow Z(a,a).$$

Now, if $a \neq 0,-1$, we know that $Z(a,a)$ is simple, of dimension p^2, and thus that f must be surjective.

Hence, $Z(a,b)$ has a p^2-dimensional simple quotient isomorphic to $Z(a,a) = L(a,a)$ if $(a,b) \neq (0,-1),(-1,-2)$.

We start with the general case $Z(a,b)$, where $(a,b) \neq (1,0),(0,-1),(-1,-2)$. Here we have $\widehat{H}\,\langle w\rangle = \widehat{H}\,\langle v\rangle \le Z(a,b)$ of dimension p^2, and simple, as the submodule is generated by its maximal vectors v and w. It is isomorphic to $L(a-1,b) = Z(a-1,b)$.

By the above, and by consideration of dimensions, the quotient $Z(a,b)/\widehat{H}\,\langle v\rangle$ is simple and we call it $L(a,b)$. Thus, we have found all the composition factors:

$$[L(a-1,b),L(a,b)],$$

both of dimension p^2.

Note: since w is also a maximal vector of weight $(a-1,b-1)$, $\widehat{H}\,\langle v\rangle = \widehat{H}\,\langle w\rangle$ can be viewed as a simple p^2-dimensional $\widehat{H}$-module, and we have then $L(a-1,b)\cong L(a-1,b-1)$, noting that $(a-1,b-1)\neq(0,-1),(-1,-2),(-2,-3)$, so this isomorphism is not a problem as if $(a-1,b-1)\neq(1,0)$, we are guaranteed that $L(a-1,b-1)$ is the p^2-dimensional quotient of $Z(a-1,b-1)$, and if $(a-1,b-1)=(1,0)$, we are in the case $(2,1)$, and the statement says, $L(1,1)\cong L(1,0)$, where $L(1,1)=Z(1,1)$ is a p^2-dimensional simple module, and $L(1,0)$ is the p^2-dimensional quotient of $Z(1,0)$ we find below.

Consider now the induced module $Z(1,0)$. It has the submodule $\widehat{H}\,\langle v\rangle$ of dimension p^2 inside it. The quotient $Z(1,0)/\widehat{H}\,\langle v\rangle$ must be simple, by the above argument and by consideration of dimensions. We call this quotient $L(1,0)$. Now, $\widehat{H}\,\langle v\rangle$ has the (p^2-1)-dimensional submodule $\widehat{H}\,\langle w\rangle$, which is simple, and of weight $(0,-1)$, so by Frobenius reciprocity, we see that $\widehat{H}\,\langle w\rangle\cong L(0,-1)$. The quotient $\widehat{H}\,\langle v\rangle/L(0,-1)$ is one-dimensional, and so simple and isomorphic to $L(0,0)$. Thus we have all the composition factors:

$$[L(0,0),L(0,-1),L(1,0)],$$

of dimensions 1, p^2-1, and p^2, respectively. Note that $\widehat{H}\,\langle v\rangle$ has a maximal vector of highest weight $(0,0)$, and from the above, $\widehat{H}\,\langle v\rangle\cong Z(0,0)$.

Now we study $Z(0,-1)$. Here we have $\widehat{H}\,\langle w\rangle = \widehat{H}\,\langle v\rangle \leq Z(a,b)$ of dimension p^2-1, and simple, as the submodule is generated by its maximal vectors v and w, so we have $\widehat{H}\,\langle v\rangle\cong L(-1,-1)\cong L(-1,-2)$.

Note: The previous is not a problem, as we will see that $L(-1,-2)$ is the (p^2-1)-dimensional simple quotient of $Z(-1,-2)$, and $L(-1,-1)$ is the (p^2-1)-dimensional simple quotient of $Z(-1,-1)$.

We turn our attention to the quotient $Z(0,-1)/\widehat{H}\,\langle v\rangle$. There are two vectors not in $\widehat{H}\,\langle v\rangle$,

$$\theta := \partial_y^{p-1}\otimes m_2$$

$$\varphi := \partial_x'^{p-1}\otimes m_1$$

with the following property: $\widehat{H}\cdot\eta \in \widehat{H}\langle v\rangle$ (so in particular, $x\partial_x$ and $y\partial_y$ have weight $(0,0)$ on them in the quotient). Note here one *must* calculate $e_{4,4}\cdot\eta$ to handle the characteristic $p=5$ case. Thus there is a two-dimensional submodule $k\langle\theta,\varphi\rangle \leq Z(0,-1)/\widehat{H}\langle v\rangle$. The quotient here is (p^2-1)-dimensional. By Frobenius reciprocity, we have that $Z(0,-1)$ must have a (p^2-1)-dimensional simple quotient isomorphic to $L(0,-1)\subseteq Z(0,0)$, where $L(0,-1) = \widehat{H}\left\langle\partial_y\otimes m\right\rangle$. By consideration of dimensions, the above quotient has to be this one. It remains to decompose the module $k\langle\theta,\varphi\rangle$, but this has just a one-dimensional simple submodule with a one-dimensional simple quotient. Thus the compositions factors are:

$$[L(-1,-1), L(0,-1), L(0,0), L(0,0)],$$

the first two of dimension p^2-1 and the last two one-dimensional.

Finally, we have $Z(-1,-2)$. As above, we have $\widehat{H}\langle w\rangle = \widehat{H}\langle v\rangle \leq Z(a,b)$ of dimension p^2, and simple, as the submodule is generated by its maximal vectors v and w. Here we have $\widehat{H}\langle v\rangle \cong L(-2,-2)\cong L(-2,-3)$.

Note: Again, the above isomorphism is not a problem, as $L(-2,-2)=Z(-2,-2)$ is a p^2-dimensional simple $\widehat{H}$-module and $L(-2,-3)$ is the p^2-dimensional simple quotient of $Z(-2,-3)$.

By Frobenius reciprocity,

$$\mathrm{Hom}_{\widehat{H}_{(0)}}(L_0(-1,-2), M) \cong \mathrm{Hom}_{\widehat{H}}(Z(-1,-2), M).$$

If we take M to be the (p^2-1)-dimensional simple submodule of $Z(0,-1)$, we see that the left side is non-zero because M has a maximal vector v of weight $(-1,-2)$. Thus the right hand is non-zero, and so $Z(-1,-2)$ surjects onto M, as M is simple. Hence, we have shown that $Z(-1,-2)$ has a (p^2-1)-dimensional simple quotient. Indeed, we can argue that $Z(-1,-2)/\widehat{H}\langle v\rangle$ has a one-dimensional submodule. The vector $\gamma := \partial_x'^{p-1}\partial_y^{p-2}\otimes m_2 \notin \widehat{H}\langle v\rangle$ is such that $\widehat{H}\cdot\gamma\subseteq\widehat{H}\langle v\rangle$. The quotient of $Z(-1,-2)/\widehat{H}\langle v\rangle$ by this one-dimensional submodule $k\langle\gamma\rangle$ must then be the (p^2-1)-dimensional simple quotient above, so it must be $L(-1,-2)$. Thus, we have the composition factors:

$$[L(-2,-2), L(0,0), L(-1,-2)],$$

of dimensions, p^2, 1, and p^2-1, respectively. ■

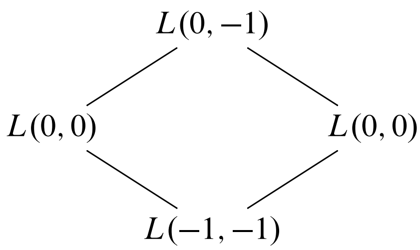
Remark. All the composition factors of modules induced from two-dimensional modules are isomorphic to simple quotients of modules induced from one-dimensional induced modules except for $L(0,-1)$. More precisely, we have for all pairs $(a, a-1)$, with $a \in \mathbb{F}_p$:

$$L(a, a-1) \cong L(a, a),$$

except when $(a, a-1) = (0, -1)$, in which case $L(0,-1)$ is still isomorphic to a composition factor of a module induced from a one-dimensional induced module, more precisely $L(0,-1) \cong \widehat{H} \langle \partial_y \otimes m \rangle \leq Z(0,0)$.

We will later see that $L(0,-1)$ is not isomorphic to $L(-1,-1)$.

Furthermore, the proof of Theorem 6.3.13 in fact shows that the Alperin diagram (see Alperin (1980)) of $Z(0,-1)$ is



Hence, we have

$$\dim_k \mathrm{Ext}^1(k, L(0,-1)), \dim_k \mathrm{Ext}^1(k, L(-1,-1)) \geq 2.$$

Thus, we see that the trivial module is not projective and that $L(0,-1)$ and $L(-1,-1)$ are not injective.

6.3.4 Higher-dimensional induced modules

Proposition 6.3.14. *Let $M \cong L_0(a,b)$, with $p-1 \geq a-b = n \geq 2$, where $k\langle m_1, m_2, \ldots, m_{n+1} \rangle = M$ and $X \cdot m_{n+1} = 0$. Every maximal vector v for $Z(M)$ is contained in the subspace*

$$k\langle 1 \otimes m_{n+1} \rangle .$$

Proof. We recall here the general setup for restricted $\widehat{H}_0$-modules M:

We pick an eigenbasis $\{m_1, m_2, \ldots, m_{n+1}\}$. With this eigenbasis we have

$$X \cdot m_i = m_{i+1},$$

where $X \cdot m_{n+1} = 0$, and

$$Y \cdot m_i = (i-1)(n-i+2)\, m_{i-1},$$

noting again that $Y \cdot m_1 = 0$.

Let $v = \sum_{a \in \mathcal{A}} \left(\partial'_x \partial_y\right)^a \otimes m_a$ be a maximal vector, where $m_a \in M$ for all $a \in \mathcal{A}$. As with the lower-dimensional cases, each m_a can only be in one H-eigenspace (see §6.3.1), so one has, for all $a \in \mathcal{A}$:

$$m_a = \mu_a m_j,$$

with $j \in \{1, \ldots, n+1\}$ and $\mu_a \in k$.

Arguing as before, from $e_{1,2} \cdot v = 0$ one gets that:

$$\text{if } m_a = \mu_a m_{n+1} \neq 0, \text{ then either } a_2 = 0 \text{ or } a_1 = \tfrac{a_2+2n-1}{2}.$$

From $e_{2,1} \cdot v = 0$ one gets that:

$$\text{if } m_a = \mu_a m_1 \neq 0, \text{ then either } a_1 = 0 \text{ or } a_2 = \tfrac{a_1+2n-1}{2}.$$

From $x\partial_y \cdot v = 0$, we see that:

$$\text{if } m_a = \mu_a m_1 \neq 0, \text{ then either } a_1 = 0 \text{ or } a_2 = p-1.$$

Suppose $m_a = \mu_a m_1 \neq 0$ and $a_1 \neq 0$. Then

$$a_2 = p - 1 = \frac{a_1 + 2n - 1}{2}.$$

This gives that $a_1 = -1 - 2n$.

From the action of $e_{3,1}$ together with the previous, we see that:

$$\text{if } m_a = \mu_a m_1 \neq 0, \text{ then } a_1 = 0, 1.$$

If $a_1 \neq 0$, this case also implies that:

$$\text{if } m_a = \mu_a m_1 \neq 0 \text{ then } a_1 = 1 \text{ and } a_2 = p-1 \text{ and } n = p-1.$$

We also deduce that:

$$\text{if } m_a = \mu_a m_2 \neq 0, \text{ then } a_1 = 0, 1 \text{ or } a_2 = \tfrac{a_1-2}{3} + n - 2,$$

provided one is not in the $n = p-1$ case. But, in fact we can improve this by considering the action of $x^{(2)}\partial_y$ too, which gives:

$$\text{if } m_a = \mu_a m_2 \neq 0 \text{ we have either } a_1 = 0, 1 \text{ or } a_2 = p-1.$$

So, if one is in the $a_1 \neq 0, 1$ case we have $p - 1 = \frac{a_1-2}{3} + n - 2$, which implies $a_1 = 5 - 3n$, again, provided one is not in the $n = p - 1$ case. We now consider what happens in the $n = p - 1$ in the above when we consider the non-zero $m_a = \mu_a m_2$. For that case we see that we are not allowed to conclude what we have if $a = (2, a_2)$.

Summarising:

$$\text{If } m_a = \mu_a m_2 \neq 0, \text{ then } a_1 = 0, 1 \text{ or } a_1 = 2 \text{ or } a = (5-3n, p-1).$$

Write $\tau = (5 - 3n, p - 1)$.

We write our maximal vector

$$\begin{aligned} v = &\sum_{0\le a_2\le p-1} \partial_y^{a_2} \otimes m_{(0,a_2)} + \sum_{0\le a_2\le p-2} \partial'_x \partial_y^{a_2} \otimes \underbrace{m_{(1,a_2)}}_{\geq \mu_a m_2} + \partial'_x \partial_y^{p-1} \otimes m_{(1,p-1)} \\ &+ \sum_{0\le a_2\le p-1} \partial_x'^2 \partial_y^{a_2} \otimes \underbrace{m_{(2,a_2)}}_{\geq \mu_a m_2} + \sum_{\substack{3\le a_1\le p-1\\ 0\le a_2\le p-1\\ a\neq\tau}} \partial_x'^{a_1} \partial_y^{a_2} \otimes \underbrace{m_a}_{\geq \mu_a m_3} + \left(\partial'_x \partial_y\right)^{\tau} \otimes m_\tau. \end{aligned}$$

By Proposition 6.2.4, we know that $Y \cdot m_a = 0$ if $a_2 = p - 1$. Thus, $m_\tau = \mu_\tau m_1$ and $m_{(1,p-1)} = \mu_{(1,p-1)} m_1$.

Acting on our maximal vector by $x^{(2)}\partial_y$ again, we see that the $\partial_y^{p-1} \otimes \mu_{(1,p-1)} m_2$ term can only cancel with the term $\partial_y^{p-1} \otimes m_{(2,p-1)}$. But in fact, $m_{(2,p-1)} = \mu_{(2,p-1)} m_1$, and so no cancellation can occur, and we conclude

$$m_{(1,p-1)} = 0 = m_{(2,p-1)}.$$

Now, since $m_\tau = \mu_\tau m_1$, we see from the previous that we must have $\tau = (0, p - 1)$ or $\tau = (1, p - 1)$. Thus, we can write

$$\begin{aligned} v = &\sum_{0\le a_2\le p-1} \partial_y^{a_2} \otimes m_{(0,a_2)} + \sum_{0\le a_2\le p-2} \partial'_x \partial_y^{a_2} \otimes \underbrace{m_{(1,a_2)}}_{\geq \mu_a m_2} \\ &+ \sum_{0\le a_2\le p-2} \partial_x'^2 \partial_y^{a_2} \otimes \underbrace{m_{(2,a_2)}}_{\geq \mu_a m_2} + \sum_{\substack{3\le a_1\le p-1\\ 0\le a_2\le p-2}} \partial_x'^{a_1} \partial_y^{a_2} \otimes \underbrace{m_a}_{\geq \mu_a m_3}. \end{aligned}$$

Looking at $x^{(2)}\partial_y \cdot v = 0$ again, we gather that $m_{(2,a_2)} \neq 0$ implies that $m_{(2,a_2)} = \mu_{(2,a_2)} m_j$ for some $j \geq 3$. Secondly, we also see that if $X \cdot m_{(1,a_2)}$ and $m_{(2,a_2-1)}$ are in the same weight space, then $\mu_{(1,a_2)} = \mu_{(2,a_2-1)}$ for $0 \leq a_2 \leq p-2$. Otherwise $m_{(1,a_2)} = \mu_{(1,a_2)} m_{n+1}$ and $\mu_{(2,a_2-1)} = 0$. In particular, $m_{(1,0)} = \mu_{(1,0)} m_{n+1}$ and $m_{(2,p-2)} = 0$.

We also see that if $m_a = \mu_a m_3$, then the associated terms cannot cancel with anything and we conclude $\mu_a m_3 = 0$. Thus, we have:

$$v = \sum_{0 \leq a_2 \leq p-1} \partial_y^{a_2} \otimes m_{(0,a_2)} + \partial'_x \otimes \mu_{(1,0)} m_{n+1} + \sum_{1 \leq a_2 \leq p-2} \partial'_x \partial_y^{a_2} \otimes \underbrace{m_{(1,a_2)}}_{\geq \mu_a m_2}$$
$$+ \sum_{0 \leq a_2 \leq p-3} \partial_x'^2 \partial_y^{a_2} \otimes \underbrace{m_{(2,a_2)}}_{\geq \mu_a m_3} + \sum_{\substack{3 \leq a_1 \leq p-1 \\ 0 \leq a_2 \leq p-2}} \partial_x'^{a_1} \partial_y^{a_2} \otimes \underbrace{m_a}_{\geq \mu_a m_4} .$$

By looking at the action of $x^{(p-1)} \partial_y$ on v we see that we have:

$$m_a = \mu_a m_n \text{ or } m_a = \mu_a m_{n+1} \text{ for } a_1 = p-2.$$

Furthermore

$$\mu_{(p-1,a_2-1)} = \mu_{(p-2,a_2)}$$

for $1 \leq a_2 \leq p-2$, when $m_a = \mu_a m_n$. When such $m_a = \mu_a m_{n+1}$, then $\mu_{(p-1,a_2-1)} = 0$. Finally, $\mu_{(p-1,p-2)} = 0$.

We also see that $m_a = \mu_a m_4 \neq 0$ implies $a_1 = 3$, again by looking at the action of $x^{(2)} \partial_y$.

Let's study the $m_{(p-2,a_2)}$ and $m_{(p-1,a_2)}$. We gather from $x\partial_y \cdot v = 0$ that if $m_{(p-2,a_2)} = \mu_{(p-2,a_2)} m_{n+1}$, then

$$\mu_{(p-1,a_2-1)} = 0$$

for $1 \leq a_2 \leq p-2$, as above. On the other hand, if $m_{(p-2,a_2)} = \mu_{(p-2,a_2)} m_n$, then

$$\mu_{(p-1,a_2-1)} = -\mu_{(p-2,a_2)},$$

again for $1 \leq a_2 \leq p-2$. Therefore, putting it all together we see that if $m_{(p-2,a_2)} = \mu_{(p-2,a_2)} m_n$, then

$$\mu_{(p-1,a_2-1)} = -\mu_{(p-1,a_2-1)},$$

so they are all zero. On the other hand, if $m_{(p-2,a_2)} = \mu_{(p-2,a_2)} m_{n+1}$, then the $\mu_{(p-1,a_2-1)}$ are all zero. Either way

$$\mu_{(p-1,a_2)} = 0$$

for all $0 \leq a_2 \leq p-2$. And we have $m_{(p-2,a_2)} = \mu_{(p-2,a_2)} m_{n+1}$.

We have

$$\begin{aligned} v = &\sum_{0\le a_2\le p-1} \partial_y^{a_2}\otimes m_{(0,a_2)} + \partial_x'\otimes \mu_{(1,0)}m_{n+1} + \sum_{1\le a_2\le p-2}\partial_x'\partial_y^{a_2}\otimes \underbrace{m_{(1,a_2)}}_{\ge\mu_a m_2} \\ &+ \sum_{0\le a_2\le p-3}\partial_x'^2\partial_y^{a_2}\otimes \underbrace{m_{(2,a_2)}}_{\ge\mu_a m_3} + \sum_{0\le a_2\le p-2}\partial_x'^3\partial_y^{a_2}\otimes\underbrace{m_{(3,a_2)}}_{\ge\mu_a m_4} \\ &+ \sum_{\substack{4\le a_1\le p-2\\0\le a_2\le p-2}}\partial_x'^{a_1}\partial_y^{a_2}\otimes\underbrace{m_a}_{\ge\mu_a m_5}. \end{aligned}$$

By considering the action of C, we see that the terms

$$-a_2\partial_y^{a_2-1}\otimes Y\cdot m_{(0,a_2)}$$

cannot cancel with anything and thus, either $a_2=0$ or $m_{(0,a_2)}=\mu_{(0,a_2)}m_1$.

We write thus,

$$\begin{aligned} v = &\sum_{1\le a_2\le p-1} \partial_y^{a_2}\otimes \mu_{(0,a_2)}m_1 + \partial_x'\otimes \mu_{(1,0)}m_{n+1} + \sum_{1\le a_2\le p-2}\partial_x'\partial_y^{a_2}\otimes \underbrace{m_{(1,a_2)}}_{\ge\mu_a m_2} \\ &+ \sum_{0\le a_2\le p-3}\partial_x'^2\partial_y^{a_2}\otimes \underbrace{m_{(2,a_2)}}_{\ge\mu_a m_3} + \sum_{0\le a_2\le p-2}\partial_x'^3\partial_y^{a_2}\otimes\underbrace{m_{(3,a_2)}}_{\ge\mu_a m_4} \\ &+ \sum_{\substack{4\le a_1\le p-2\\0\le a_2\le p-2}}\partial_x'^{a_1}\partial_y^{a_2}\otimes\underbrace{m_a}_{\ge\mu_a m_5} + 1\otimes m_{(0,0)}. \end{aligned}$$

Now we let $x\partial_y$ act on our maximal vector. We see that the term $1\otimes X\cdot m_{(0,0)}$ cannot cancel with anything, so we conclude that $m_{(0,0)}=\mu_{(0,0)}m_{n+1}$.

Furthermore, we see that the $\partial_y^{a_2}\otimes\mu_{(0,a_2)}m_2$ terms can only cancel with the terms $-\partial_y^{a_2}\otimes m_{(1,a_2-1)}$, for $2\le a_2\le p-1$. Thus,

$$\mu_{(0,a_2)}m_2 = m_{(1,a_2-1)} = \mu_{(1,a_2-1)}m_j,$$

and thus either $j=2$, and we have $\mu_{(0,a_2)}=\mu_{(1,a_2-1)}$, or $\mu_{(0,a_2)}=\mu_{(1,a_2-1)}=0$. Consequently, we have

$$\text{if } 0\ne m_{(1,a_2)}, \text{ then } m_{(1,a_2)}=\mu_{(1,a_2)}m_2, \text{ for } 1\le a_2\le p-2.$$

Considering the term $m_{(0,a_2)}$ when $a_2=1$, we see that it can only cancel with $-\partial_y\otimes\mu_{(1,0)}m_{n+1}$, which is not possible, thus we deduce that $\mu_{(0,1)}=0=\mu_{(1,0)}$.

But, in fact, now we can deduce information on all the m_a from this. Looking again at the action of $x\partial_y$, we see that the $\partial'_x\partial_y^{a_2}\otimes X\cdot m_{(1,a_2)}=\partial'_x\partial_y^{a_2}\otimes\mu_{(1,a_2)}m_3$ terms can only cancel with the terms $-2\partial'_x\partial_y^{a_2}\otimes m_{(2,a_2-1)}$, for $1\leq a_2\leq p-2$. So, as above, we see that either they lie in the same $\mathfrak{sl}_2$-weight space, and we have

$$\mu_{(1,a_2)}=2\mu_{(2,a_2-1)},$$

or they are both zero. Thus, we have $0\neq m_{(2,a_2)}=\mu_{(2,a_2)}m_3$.

Continuing likewise, for higher values of a_1 up to and including $p-2$, we see that

$$\mu_{(a_1,a_2)}=(a_1+1)\,\mu_{(a_1+1,a_2-1)},$$

if $m_{(a_1+1,a_2-1)}$ is in the same $\mathfrak{sl}_2$-weight space as $X\cdot m_{(a_1,a_2)}$, and they are zero otherwise, where $0\leq a_2\leq p-2$ if $a_1\geq 3$, meaning in such cases we can immediately see that $m_{(a_1,0)}=0=m_{(a_1+1,p-2)}$. In the $a_1=2$ case we can say

$$m_{(2,0)}=0=m_{(3,p-2)}=m_{(3,p-3)}.$$

We summarise what we have:

$$\begin{aligned} v=&1\otimes\mu_{(0,0)}m_{n+1}+\sum_{2\leq a_2\leq p-1}\partial_y^{a_2}\otimes\mu_{(0,a_2)}m_1+\sum_{1\leq a_2\leq p-2}\partial'_x\partial_y^{a_2}\otimes\underbrace{\mu_{(1,a_2)}}_{=\mu_{(0,a_2+1)}}m_2\\ &+\sum_{1\leq a_2\leq p-3}\partial_x'^2\partial_y^{a_2}\otimes\underbrace{\mu_{(2,a_2)}}_{=\mu_{(1,a_2+1)}/2}m_3+\sum_{1\leq a_2\leq p-4}\partial_x'^3\partial_y^{a_2}\otimes\underbrace{\mu_{(3,a_2)}}_{=\mu_{(2,a_2+1)}/3}m_4\\ &+\sum_{\substack{4\leq a_1\leq n\\1\leq a_2\leq p-3}}\partial_x'^{a_1}\partial_y^{a_2}\otimes\underbrace{\mu_{(a_1,a_2)}}_{=\mu_{(a_1-1,a_2+1)}/a_1}m_{a_1+1}.\end{aligned}$$

We now apply C to v. Comparing the terms with exponent 1 in the ∂'_x component, we see that $\mu_{(0,a_2)}\neq 0$ implies that $a_2=1,2n$, for $2\leq a_2\leq p-1$. Also, since

$$\mu_{(0,a_2)}=\mu_{(1,a_2-1)}=2\mu_{(2,a_2-2)}=\ldots=n\mu_{(n,a_2-n)},$$

we see that if $\mu_{(a_1,a_2)}\neq 0$, then $a_2=2n-a_1$.

We write, then

$$v = 1 \otimes \mu_{(0,0)} m_{n+1} + \sum_{0 \le a_1 \le n} \partial_x'^{a_1} \partial_y^{2n-a_1} \otimes \underbrace{\mu_{(a_1,2n-a_1)}}_{=\mu_{(a_1-1,2n-a_1+1)}/a_1} m_{a_1+1}.$$

We apply the action of $e_{2,1}$ to conclude. From it we see that we get the term

$$s_{(0,2n)} \partial_y^{2n-1} \otimes \mu_{(0,2n)} m_1,$$

which can only cancel with

$$\partial_y^{2n-1} \otimes n\mu_{(1,2n-1)} m_1,$$

noting that $Y \cdot m_2 = nm_1$. Now, we compute that $s_{(0,2n)} = 4n^2 - n$. Thus we have either $\mu_{(0,2n)} = \mu_{(1,2n-1)} = 0$ or $4n^2 - n + n = 0$. The latter cannot happen, as this implies that $4n^2 = pt$, for some $t \in \mathbb{N}$, but since $p \geq 5$, p doesn't not divide 4, so it must divide n^2, and thus must divide n itself, which is not possible.

We conclude, hence,

$$0 = \mu_{(0,2n)} = \mu_{(1,2n-1)} = \mu_{(2,2n-2)} = \ldots = \mu_{(n,2n-n)}.$$

Thus, $v = 1 \otimes \mu_{(0,0)} m_{n+1}$, as required. ■

From this it follows that

Theorem 6.3.15. *The induced module $Z(M) \cong Z(a,b)$, where $p-1 \geq a-b \geq 2$, is simple.*

Lastly, we prove the following:

Proposition 6.3.16. *There are two isomorphism classes of (p^2-1)-dimensional restricted simple $\widehat{H}$-modules, one represented by $L(-1,-1)$, the other by $L(0,-1)$.*

Proof. The only (p^2-1)-dimensional restricted simple modules arise as composition factors of modules induced from one-dimensional or two-dimensional modules. All of these are isomorphic to either $L(0,-1)$ or $L(-1,-1)$, as we have seen. It remains to show that these two are not isomorphic.

Now, if they *were* isomorphic, this would tell us that $Z(0,-1)$ has a simple quotient isomorphic to $L(-1,-1)$, i.e.,

$$0 \neq \mathrm{Hom}_{\widehat{H}}(Z(0,-1), L(-1,-1)) \cong \mathrm{Hom}_{\widehat{H}_{(0)}}(L_0(0,-1), L(-1,-1)).$$

Thus, $L(-1,-1)$ would need to have a maximal vector of weight $(0,-1)$. Recall that

$$L(-1,-1) = Z(-1,-1)/k\left\langle \left(\partial'_x\partial_y\right)^{\omega_0} \otimes m\right\rangle.$$

If $0 \neq \delta \in L(-1,-1)$ is a vector of weight $(0,-1)$, then working in the quotient we deduce that $\delta = \partial_x'^{p-1} \otimes m$. This is a problem, as $X \cdot \delta = \partial_x'^{p-2}\partial_y \otimes m \neq 0$, so that δ is not maximal. Thus no maximal vector of such a weight exists, and we are done. ■

Lemma 6.3.17. *Let $\lambda, \mu \in \mathbb{F}_p^2$ with $\lambda_1 - \lambda_2 = \mu_1 - \mu_2$ and $\lambda \neq \mu$. If $Z(\lambda)$ and $Z(\mu)$ are both simple, then they are not isomorphic as $\widehat{H}$-modules.*

Proof. If $Z(\lambda) \cong Z(\mu)$, then $Z(\lambda)$ has a maximal vector v of weight μ. We consider the two cases where $Z(\lambda)$ is simple. The first is when $\lambda_1 - \lambda_2 \geq 2$. Then Proposition 6.3.14 tells us that $v \in k\langle 1 \otimes m_{n+1}\rangle$. Thus, v i s a maximal vector of weight λ, since $1 \otimes m_{n+1}$ is a maximal vector of weight λ. This is a contradiction. Therefore $Z(\lambda) \ncong Z(\mu)$. The second case arises when $\lambda = (a,a)$, with $a \neq 0,-1$. Then Proposition 6.3.4 tells us that $v \in k\langle 1 \otimes m\rangle$ or $v \in k\left\langle \partial_y \otimes m\right\rangle$ or $v \in \left\langle \partial_x'^{p-1}\partial_y^{p-1} \otimes m\right\rangle$. We observe that $C = -e_{0,3} \in N$ only kills $\partial_x'^{p-1}\partial_y^{p-1} \otimes m$ when $a = -1$. Thus, $v \in k\langle 1 \otimes m\rangle$ or $v \in k\left\langle \partial_y \otimes m\right\rangle$. If $v \in k\langle 1 \otimes m\rangle$, we have a contradiction, since $1 \otimes m$ is a maximal vector of weight λ. Finally if $v \in k\left\langle \partial_y \otimes m\right\rangle$, then v has weight $(a, a-1)$, a contradiction. Therefore, $Z(\lambda) \ncong Z(\mu)$. ■

We can now prove our main result, Theorem 6.1.7, restated here for convenience.

Theorem. *Let $p \geq 5$ be a prime, k be an algebraically closed field of characteristic p, $\lambda \in \mathbb{F}_p^2$ a weight, $L_0(\lambda)$ be the simple restricted $\mathfrak{gl}_2(k)$-module of highest weight λ, $Z(\lambda) = Z(L_0(\lambda))$ the corresponding induced $\widehat{H}$-module, and $L(\lambda)$ its simple head.*

1. *The full list of simple pairwise nonisomorphic restricted $\widehat{H}$-modules is given by $\left\{L(\lambda) : \lambda \in \mathbb{F}_p^2, \lambda_1 - \lambda_2 \neq 1 \text{or } \lambda = \omega_1\right\}$. There are $p^2 - p + 1$ of them.*

2. *If λ is not exceptional, then $L(\lambda) = Z(\lambda)$, and its dimension is $p^2 \dim_k L_0(\lambda) = p^2(\lambda_1 - \lambda_2 + 1)$.*

3. *For exceptional* λ*, the modules* $L(\lambda)$ *in the list are as follows:*

   (a) *if* $\lambda = \omega_0 = (-1,-1)$*,* $L(\lambda) \cong O(2;(1,1))/(k \cdot 1)$*, with dimension* $p^2 - 1$*;*

   (b) *if* $\lambda = \omega_1 = (0,-1)$*,* $L(\lambda) \cong \widehat{H}\langle \partial_y \otimes m \rangle \leq Z(0,0)$*, with dimension* $p^2 - 1$*;*

   (c) *if* $\lambda = \omega_2 = (0,0)$*,* $L(\lambda) \cong k$*, with dimension* 1 *(this is the trivial module).*

Proof of Theorem 6.1.7. That $Z(\lambda)$ has dimension $p^2 \dim_k L_0(\lambda) = p^2(\lambda_1 - \lambda_2 + 1)$ follows immediately from the definition of $Z(\lambda)$ and from the fact that $\dim_k L_0(\lambda) = \lambda_1 - \lambda_2 + 1$, stated in Theorem 6.3.2.

By Proposition 6.1.6, every simple restricted $\widehat{H}$-module is a simple quotient of some induced $\widehat{H}$-module $Z(\lambda)$. The full list of these simple quotients is given by $\{L(\lambda) : \lambda \in \mathbb{F}_p^2\}$. These are not all pairwise nonisomorphic, however. If $\lambda = (a,a)$, the proof of Theorem 6.3.8 tells us that $L(0,0)$ is one-dimensional and $L(-1,-1)$ is (p^2-1)-dimensional, so they are not isomorphic, by dimensions. It also tells us that if $a \neq 0,-1$, then $L(\lambda) = Z(\lambda)$, and thus has dimension p^2. By Lemma 6.3.17, they are all pairwise nonisomorphic, and by dimensions they are all pairwise nonisomorphic to $L(0,0)$ and $L(-1,-1)$. Thus, the modules $\{L(a,a) : a \in \mathbb{F}_p\}$ are all pairwise nonisomorphic.

The remark after Theorem 6.3.13 tells us that for all $a \in \mathbb{F}_p : L(a,a-1) \cong L(a,a)$, except when $a = 0$. Therefore, the only $L(a,a-1)$ possibly not isomorphic to one of the nonisomorphic simple restricted $\widehat{H}$-modules already described is $L(0,-1) = L(\omega_1)$. Proposition 6.3.16 guarantees that it is indeed not isomorphic to any of them. Hence, the modules $\{L(a,a) : a \in \mathbb{F}_p\} \cup \{L(0,-1)\}$ are all pairwise nonisomorphic.

It remains to consider the modules $L(\lambda)$ for all $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 - \lambda_2 \geq 2$. Theorem 6.3.15 implies that $L(\lambda) = Z(\lambda)$ for all such λ. By Lemma 6.3.17 and dimensions they are all pairwise nonisomorphic, and by dimensions they are all pairwise nonisomorphic to the other $L(\lambda)$. Thus, the modules in the list $\{L(a,a) : a \in \mathbb{F}_p\} \cup \{L(0,-1)\} \cup \{L(\lambda) : \lambda_1 - \lambda_2 \geq 2\}$ are all pairwise nonisomorphic and these exhaust all the possibilities for isomorphism types of simple restricted $\widehat{H}$-modules. Thus, the full list of simples is indeed $\{L(\lambda) : \lambda \in \mathbb{F}_p^2, \lambda_1 - \lambda_2 \neq 1 \text{or } \lambda = \omega_1\}$. Computing how many modules there are in this list is straightforward. This proves (1).

That $L(\lambda) = Z(\lambda)$ for all λ not exceptional is immediate from the previous paragraphs. Thus we have proved (2). Statements 3 (a) and 3 (c) are proved in the proof of Theorem 6.3.8. Statement 3 (b) is proved in the proof of Theorem 6.3.13. ∎

Chapter 7

Restrictions to W_1-subalgebras and balanced toral elements

We will now be giving a characterisation of how the simple restricted $\widehat{H}$-modules we classified in the previous chapter restrict to a subalgebra of H isomorphic to the first Witt algebra.

7.1 Preliminaries

We have from Lemma 2.8 in Herpel and Stewart (2016a):

Lemma 7.1.1. *The subalgebra H of $W(2;(1,1))$ contains a p-subalgebra $W := W(1;1)$ with basis*

$$\left\{\partial_y, y\partial_y - x\partial_x, y^{(2)}\partial_y - xy\partial_x, \ldots, y^{(p-1)}\partial_y - xy^{(p-2)}\partial_x\right\},$$

with these elements playing the roles of $\partial, x\partial, x^{(2)}\partial, \ldots, x^{(p-1)}\partial$, respectively, where x is the image of X in the truncated polynomial ring $k[X]/(X^p)$.

Briefly, we recall the restricted representation theory for W, see Chang in Chang (1941).

Consider the standard subalgebras $W_{(1)}$ and $W_{(0)}$ of W from the filtration obtained from the natural grading on W, see §4.1 for more details. Then $W_{(1)}$ is an ideal of $W_{(0)}$. Consequently, we can define for $\lambda \in k$ a one-dimensional $W_{(0)}$-module k_λ, much as we did in §6.1, where $W_{(1)}$ acts trivially and $x\partial$ acts via $x\partial \cdot v = \lambda v$. From this we define the

Verma module

$$Z^+(\lambda) = \mathfrak{u}(W) \otimes_{\mathfrak{u}(W_{(0)})} k_\lambda.$$

Then we have the following description of all the isomorphism classes of simple restricted W-modules, see (Chang, 1941, Hauptsatz 2', p. 176).

Theorem 7.1.2. *There are p isomorphism classes of irreducible restricted representations of W, with representatives $L_W(r)$ for $r \in \{0, 1, \ldots, p-1\}$. $L_W(r)$ is obtained from the induced representation $Z^+(r)$, the Verma module, and is equal to it if $r \neq 0, -1$, with dimension p. If $r = 0$, then $Z^+(0)$ has a trivial simple quotient, which is $L_W(0)$, and $Z^+(p-1)$ has a $(p-1)$-dimensional simple quotient, denoted $L_W(p-1)$.*

Now, in Herpel and Stewart (Herpel and Stewart, 2016a, Lem. 2.1, Prop. 2.2), the authors also provide two key results, one an algorithm, to work out the composition factors of a graded W-module. They are as follows:

Lemma 7.1.3. *Suppose V is a W-module admitting a grading $V = \bigoplus_{i\in\mathbb{Z}} V(i)$ such that $\partial \cdot V(i) \subset V(i+2)$ and such that each $V(i)$ is stable under $x\partial$. Then there exists a unique semisimple W-module $V_s = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ with $V_s = \bigoplus_{i\in\mathbb{Z}} V_s(i)$ with $V_s(i) = V(i)$ as $x\partial$-modules and each V_i is a a graded irreducible W-module.*

For this module V_s, the set of composition factors $[V|W]$ and $[V_s|W]$ coincide.

Proposition 7.1.4. *Let V be as in Lemma 7.1.3. For $i \in \mathbb{Z}$ with $V(i) \neq 0$, let ℓ_i be a list (with multiplicities) of the $x\partial$-weights on $V(i)$. Then the following algorithm determines the composition factors (with multiplicities) of V as a W-module:*

1. *Let $r \in \mathbb{Z}$ be maximal such that ℓ_r is nonempty. Pick $\mu \in \ell_r$.*

2. *Record a composition factor $U = L(\lambda)$ for $\lambda = \mu - 1$ if $\mu \neq 0, 1$ and $U = L(p-1)$, $L(0)$ if $\mu = 1, 0$ respectively. Form a new set of lists $\{\ell_r'\}$ by removing weights from $\{\ell_r\}$ in the following way: If $U = L(0)$ remove a 0-weight from ℓ_r, if $U = L(p-1)$ remove one weight $1, 2, \ldots p-1$ from $\ell_r, \ell_{r-2}, \ldots, \ell_{r-2p+4}$ respectively and otherwise remove one weight $\mu, \mu+1, \ldots, \mu+p-1$ from $\ell_r, \ell_{r-2}, \ldots, \ell_{r-2p+2}$.*

3. *If the new lists $\{\ell_r'\}$ are not all empty, repeat from Step (i).*

As an H-subalgebra, W is generated by the elements ∂_y and $L := y^{(p-1)}\partial_y - xy^{(p-2)}\partial_x$.

We calculate the action of the latter as

$$\begin{aligned} 0 = L \cdot v = &- \sum_{\substack{0 \le a_1 \le p-1 \\ a_2 = p-3}} a_1 \partial_x'^{a_1-1} \otimes Y \cdot m_a + \sum_{\substack{0 \le a_1 \le p-1 \\ a_2 = p-2}} 2a_1 \partial_x'^{a_1-1} \partial_y \otimes Y \cdot m_a \\ &+ \sum_{\substack{0 \le a_1 \le p-1 \\ a_2 = p-2}} \left(\lambda(a)_2 - \lambda(a)_1 + a_1\right) \partial_x'^{a_1} \otimes m_a \\ &- 2 \otimes X \cdot m_{\omega_0} - \sum_{\substack{0 \le a_1 \le p-1 \\ a_2 = p-1}} a_1 \partial_x'^{a_1-1} \partial_y^2 \otimes Y \cdot m_a \\ &+ \sum_{\substack{0 \le a_1 \le p-1 \\ a_2 = p-1}} \left(\lambda(a)_1 - \lambda(a)_2 - 1 - a_1\right) \partial_x'^{a_1} \partial_y \otimes m_a. \end{aligned}$$

This will be useful as we will often need to check that a given k-span of vectors is indeed a W-module.

7.2 Restrictions

We are now ready to prove the main theorem of this chapter:

Theorem 7.2.1. *The restrictions of simple restricted modules $L(\lambda)$ to the subalgebra W provided by Lemma 7.1.1 are as follows. We have*

1. $[L(0,0)|W] = L_W(0)$,
2. $[L(-1,-1)|W] = [L(0,-1)|W] = [\bigoplus_{j=0}^{p-2} L_W(j) \oplus L_W(p-1)^2]$,
3. *for λ not exceptional*

$$[L(\lambda)|W] = \left[\left(\bigoplus_{j=1}^{p-2} L_W(j) \oplus L_W(0)^2 \oplus L_W(p-1)^2 \right)^{(r+1)} \right],$$

where $\lambda_1 - \lambda_2 = r$.

In particular every p-representation of $\widehat{H}$ restricted to W contains the same number of composition factors of each $L_W(j)$, where $1 \le j \le p-2$.

Proof. The trivial module's restriction is clear. First we deal with the case when the simple restricted $\widehat{H}$-module is equal to the associated Verma module, i.e., when $L(\lambda) = Z(\lambda)$, i.e. when λ is not exceptional.

We take a basis for $L_0(\lambda)$ as usual, but we label it so that v_i spans the i-th weight space for $h := y\partial_y - x\partial_x$. The strategy will be to perform the algorithm on W-sub-modules of $Z(\lambda)$, pass to quotients, and repeat.

Define in general

$$Z(\lambda)_i = k\left\langle \partial_x'^a \partial_y^b \otimes v_i : 0 \leq a \leq p-2, 0 \leq b \leq p-1 \right\rangle.$$

Take now $i = r$, where $r = \lambda_1 - \lambda_2$. Then $Z(\lambda)_r$ is the first W-sub-module of $Z(\lambda)$ we will consider. We grade it thus

$$Z(\lambda)_r = \bigoplus_{b\in\mathbb{Z}} Z(\lambda)_r(2b),$$

where

$$Z(\lambda)_r(2b) := k\left\langle \partial_x'^a \partial_y^b \otimes v_r : 0 \leq a \leq p-2 \right\rangle.$$

This grading satisfies the conditions in Lemma 7.1.3. That $Z(\lambda)_r$ is indeed a W-module can be checked by using the formula for ∂_y found in Equation (6.2) and that for the action of L found above.

Note that the basis vector $\partial_x'^a \partial_y^b \otimes v_i$ is a weight vector for h with weight $a - b + i$.

As in the algorithm, let ℓ_i be the list of weights with multiplicities of h on $Z(\lambda)_r(i)$. The element h representing $x\partial$ has weight $r + 1 + a$ on the highest graded piece $Z(\lambda)_r(2p-2)$, for $0 \leq a \leq p-2$, so we have weights $\{0, 1, \ldots, p-1\} \setminus \{r\}$, and so obtain composition factors $L_W(0), L_W(1), \ldots, L_W(p-1)$ excluding $L_W(r-1)$ if $r \neq 0$ and $L_W(0)$ if $r = 0$, remembering here that $r \neq 1$. Now remove the relevant h-weights according to part (ii) of the algorithm.

It is convenient at this point to consider the $r = 0$ case separately, i.e., we have $Z(\lambda)$ of dimension p^2. In this case, we have recorded composition factors $L_W(1), L_W(2), \ldots, L_W(p-1)$, so we remove weights $\mu, \mu+1, \ldots, \mu+p-1$ for $\mu = 2, \ldots, p-1$, from $\ell_{2p-2}, \ell_{2p-4}, \ldots, \ell_0$, respectively, and remove weights $1, 2, \ldots, p-1$ from $\ell_{2p-2}, \ell_{2p-4}, \ldots, \ell_2$, respectively. This leaves ℓ_{2p-2} empty. Each of the non-empty ℓ_i had $p-1$ weights to begin with, and we have removed $p-1$ distinct weights for all $\ell_i \neq \ell_0$. Thus, only ℓ_0 is non-empty, containing just the weight 0. Therefore we find a copy of $L_W(0)$ and the algorithm stops. Looking at the quotient $Z(\lambda)/Z(\lambda)_r$, which is p-dimensional, we find it to be a W-submodule

$$k\left\langle \partial_x'^{p-1} \partial_y^b \otimes v_r : 0 \leq b \leq p-1 \right\rangle + Z(\lambda)_r,$$

which we grade similarly by powers of ∂_y. The grading satisfies the conditions in Lemma 7.1.3, since $\partial_y^b \otimes X \cdot v_r = 0$ in the quotient. In the highest graded piece, as above, we have the weight $p-1-(p-1)+r = 0$, so we remove this 0-weight from it, and record a composition factor $L_W(0)$. We see that we have the weight $p-1-(p-2)+r = 1$, so we remove the weight 1 from ℓ_{2p-4}, and the weights $2, 3, \dots, p-1$ as we go down to ℓ_0, leaving all the lists of weights empty, and picking up the composition factor $L_W(p-1)$. So, indeed,

$$[L(\lambda)|W] = \left[\bigoplus_{j=1}^{p-2} L_W(j) \oplus L_W(0)^2 \oplus L_W(p-1)^2\right],$$

where $\lambda_1 - \lambda_2 = r = 0$, λ not exceptional.

We go back to our generic case, $r \neq 0$. Recall that we found composition factors $L_W(0), L_W(1), \dots, L_W(p-1)$ excluding $L_W(r-1)$. So, we remove weights $\mu, \mu + 1, \dots, \mu + p - 1$ for $\mu = 2, \dots, p-1$, $\mu \neq r$, from $\ell_{2p-2}, \ell_{2p-4}, \dots, \ell_0$, respectively, and remove weights $1, 2, \dots, p-1$ from $\ell_{2p-2}, \ell_{2p-4}, \dots, \ell_2$, respectively, and remove a 0-weight from ℓ_{2p-2}. This leaves ℓ_{2p-2} empty.

In the lower graded pieces, each of the non-empty ℓ_i had $p-1$ weights to begin with, and we have removed $p-2$ distinct weights for all $\ell_i \neq \ell_0$, and $p-3$ distinct weights for ℓ_0. We see that ℓ_{2p-4} has only the weight 1 remaining in it.

Thus, we record a composition factor $L_W(p-1)$, and remove weights $1, 2, \dots, p-1$ from $\ell_{2p-4}, \dots, \ell_0$. Therefore, we have removed all the weights up to, but not including, those in ℓ_0. The only weight remaining in it is a 0-weight, so we record a composition factor $L_W(0)$, and the algorithm terminates. So far, we have found composition factors

$$\bigoplus_{j=1}^{p-2} L_W(j) \oplus L_W(0)^2 \oplus L_W(p-1)^2$$

not including $L_W(r-1)$.

Before passing to the quotient we deal with the subquotient that will be left at the end, consisting of the k-span of the vectors

$$\left\langle \partial_x'^{p-1} \partial_y^b \otimes v_i : 0 \leq b \leq p-1, -r \leq i \leq r \right\rangle.$$

It is a W-module (as the interested reader can verify) and we grade it as usual. It gives us all the following composition factors, each with multiplicity 1:

$$L_W(i-1) \text{ for } i \in \{-r, -r+2, \dots, r-2, r\} \setminus \{0, 1\}$$

and if r is even, we also pick up a copy of $L_W(p-1)$ and $L_W(0)$ at the end of the process.

If r is odd, we also obtain a copy of $L_W(p-1)$ and $L_W(0)$ at the end of the process, omitting some of the details, which the reader can verify, noting that we obtain $r+2$ composition factors in both cases.

Looking at the quotient $Z(\lambda)/Z(\lambda)_r$, we find a W-submodule

$$Z(\lambda)_{r-2} := k\left\langle \partial_x'^a \partial_y^b \otimes v_{r-2} : 0 \leq a \leq p-2, 0 \leq b \leq p-1 \right\rangle + Z(\lambda)_r,$$

which we grade similarly by powers of ∂_y. The grading satisfies the conditions in Lemma 7.1.3, so we perform the algorithm on it.

The vectors in the highest graded piece have weights

$$a + 1 + (r-2),$$

so $a + r - 1$ for $0 \leq a \leq p-2$. Thus we have all weights in the range $\{0, 1, \dots, p-1\}$ except for $r-2$. So, we obtain composition factors $L_W(0), \dots, L_W(p-1)$ excluding $L_W(0)$ if $r = 2$, $L_W(p-1)$ if $r = 3$, and $L_W(r-3)$ otherwise. If we are in the latter case, then the argument as above runs, and we obtain composition factors $\bigoplus_{j=1}^{p-2} L_W(j) \oplus L_W(0)^2 \oplus L_W(p-1)^2$ excluding $L_W(r-3)$.

If $r = 2$, then we argue as in the $r = 0$ case, and obtain composition factors $\bigoplus_{j=1}^{p-2} L_W(j) \oplus L_W(0) \oplus L_W(p-1)$.

Now, if $r = 3$, we have composition factors $L_W(0), \dots, L_W(p-2)$. Proceeding as usual, we see that there is a 1-weight remaining in ℓ_{2p-4}, so we record a $L_W(p-1)$ composition factor and remove weights according to the algorithm, leaving all the lists of weights empty. So we obtain composition factors $\bigoplus_{j=1}^{p-2} L_W(j) \oplus L_W(0) \oplus L_W(p-1)$ in this case too.

Proceeding to the submodule $Z(\lambda)_{r-4}$, which is defined analogously, it is easy to see that the vectors in the highest graded piece have weights

$$a + 1 + (r-4),$$

so $a + r - 3$ for $0 \leq a \leq p-2$. Thus, again, we have all weights in the range

$\{0, 1, \ldots, p-1\}$ except for $r-4$. And again, as above, depending on the value of r, one argues three separate cases, obtaining composition factors

$$\bigoplus_{j=1}^{p-2} L_W(j) \oplus L_W(0) \oplus L_W(p-1)$$

if $r = 4, 5$, i.e., when one misses out an $L_W(0)$ or and $L_W(p-1)$ in the first step, and

$$\bigoplus_{j=1}^{p-2} L_W(j) \oplus L_W(0)^2 \oplus L_W(p-1)^2$$

excluding $L_W(r-5)$ in the other cases.

We perform the same task all the way down to $Z(\lambda)_{-r}$, i.e., we perform it $r+1$ times, with the composition factors as outlined above.

Now, we can put everything together. As $r \neq 0$, we in fact have that $r \geq 2$. As we apply the algorithm repeatedly, we obtain the following composition factors. From $Z(\lambda)_r$ we get:

$$\bigoplus_{j=1}^{p-2} L_W(j) \oplus L_W(0)^2 \oplus L_W(p-1)^2,$$

not including $L_W(r-1)$.

From $Z(\lambda)_i$, for $i \in \{-r, -r+2, \ldots, r-2\}$ one gets either

$$\bigoplus_{j=1}^{p-2} L_W(j) \oplus L_W(0) \oplus L_W(p-1),$$

if either $i = 0$ or $i = 1$, or

$$\bigoplus_{j=1}^{p-2} L_W(j) \oplus L_W(0)^2 \oplus L_W(p-1)^2,$$

excluding $L_W(i-1)$, otherwise. Thus, we miss out

$$L_W(i-1) \text{ for } i \in \{-r, -r+2, \ldots, r-2, r\} \setminus \{0, 1\},$$

which we recover as we saw above from the subquotient consisting of the $\partial_x'^{p-1}$ terms. This subquotient gave us in addition a copy of $L_W(0)$ and a copy of $L_W(p-1)$. So, we have shown, as required, that for λ not exceptional

$$[L(\lambda)|W] = \left[\left(\bigoplus_{j=1}^{p-2} L_W(j) \oplus L_W(0)^2 \oplus L_W(p-1)^2 \right)^{(r+1)} \right],$$

where $\lambda_1 - \lambda_2 = r$.

Finally, we will deal with the exceptional modules. First we deal with $L(-1,-1) \cong Z(-1,-1)/k\left\langle \partial_x'^{p-1}\partial_y^{p-1} \otimes m \right\rangle$. We define the first submodule to study as

$$M_1 = k\left\langle \partial_x'^a \partial_y^b \otimes m : 0 \le a \le p-2, 0 \le b \le p-1 \right\rangle + k\left\langle \partial_x'^{p-1}\partial_y^{p-1} \otimes m \right\rangle.$$

Grade this as usual by powers of ∂_y. This *is* a W-submodule, as both ∂_y and L preserve the basis, and the grading is as in the lemma. We note that we have already run the algorithm for the same set of weights when we dealt with $L(a,a)$, for $a \neq 0,-1$. We thus get composition factors $L_W(0), L_W(1), \ldots, L_W(p-1)$.

Now we move on to the quotient $M_2 := L(-1,-1)/M_1$. We find a W-submodule which is in fact the whole quotient, with basis

$$k\left\langle \partial_x'^{p-1}\partial_y^b \otimes m : 0 \le b \le p-2 \right\rangle + M_1.$$

Again, grade this as usual, and everything is as in Lemma 7.1.3. Here, we see that the highest graded piece $M_2(2p-4)$ has a single weight $-1-(p-2) = 1$. Thus, we record a copy of $L_W(p-1)$ and remove weights, removing 1 from ℓ_{2p-4}, 2 from ℓ_{2p-6} and so on down to $p-1$ from ℓ_0, remarking that $\ell_{2b} = \{-1-b\}$. Thus all the lists of weights are now empty, and the algorithm terminates, and we have confirmed that $[L(-1,-1)|W] = [\bigoplus_{r=0}^{p-2} L_W(r) \oplus L_W(p-1)^2]$, as required.

Lastly, we turn to $L(0,-1) \cong \widehat{H}\left\langle \partial_y \otimes m \right\rangle \le Z(0,0)$. Recall that we saw that this has a basis

$$k\left\langle \partial_x'^a \partial_y^b \otimes m : 0 \le a, b \le p-1, (a,b) \neq (0,0) \right\rangle.$$

We take the following W-submodule

$$M_1 := k\left\langle \partial_x'^a \partial_y^b \otimes m : 0 \le a \le p-2, 0 \le b \le p-1, (a,b) \neq (0,0) \right\rangle,$$

and we grade it as usual. This is indeed a W-submodule, as one can check using our formulae. Hence, we can run the algorithm on it. The highest graded piece has weights

$\{a+1 : 0 \leq a \leq p-2\}$. We record composition factors $L_W(1), \ldots, L_W(p-1)$. As in the $r=0$ case we have removed $p-1$ weights from $\ell_{2p-2}, \ldots, \ell_2$ and $p-2$ weights from ℓ_0. In this case, however, as the reader can verify ℓ_0 is left empty.

Now, we look at the quotient

$$L(0,-1)/M_1 = k\left\langle \partial_x'^{p-1}\partial_y^b \otimes m : 0 \leq b \leq p-1 \right\rangle + M_1,$$

and we grade it as usual. Perform the algorithm. In general we have $\ell_{2b} = \{-1-b\}$. We get a 0-weight from the highest graded piece, so we record a copy of $L_W(0)$. Then we pick up a 1-weight from ℓ_{2p-4}, record a copy of $L_W(p-1)$ and remove weights $1, 2, \ldots, p-1$ from $\ell_{2p-4}, \ldots, \ell_0$, terminating the algorithm. Thus, we have verified that $[L(0,-1)|W] = [\bigoplus_{j=0}^{p-2} L_W(j) \oplus L_W(p-1)^2]$, as required. ■

Remark. The proof of Theorem 1.3 in Herpel and Stewart (2016a) relied on knowledge of the restrictions of restricted modules for $\widehat{H}$ to a subalgebra isomorphic to W, in particular on the multiplicities of the composition factors $L_W(j)$ with $1 \leq j \leq p-2$, which we have confirmed and given a proof for above.

Premet in Premet (2017) introduced the notion of a d-balanced toral[1] element. We have:

Definition 7.2.2. Let $\mathfrak{g}$ be a restricted Lie algebra. Let $d > 0$ be an integer. A toral element $h \in \mathfrak{g}$ is *d-balanced* if

$$\dim_k \mathfrak{g}(h,i) = \dim_k \mathfrak{g}(h,j)$$

for all $i, j \in \mathbb{F}_p^\times$ and all eigenspaces have $d \mid \dim_k \mathfrak{g}(h,i)$ for $i \neq 0$, where $\mathfrak{g}(h,i)$ denotes the i-th eigenspace of $\operatorname{ad} h$ acting on $\mathfrak{g}$.

Applying this to our setting, we see that the toral element $h := y\partial_y - x\partial_x$ has eigenspaces when it acts on $\widehat{H}$ by $\operatorname{ad} h$ of equal dimension. This is because in the algorithm we used to work out the composition factors of the restriction of V a restricted $\widehat{H}$-module to W, recording a composition factor $L_W(\mu)$ corresponded to finding a non-zero vector v with $h \cdot v = (\mu+1)v$, if $\mu \neq 0, p-1$ and $h \cdot v = 0$ if $\mu = 0$, $h \cdot v = v$ if $\mu = p-1$.

[1] See Definition 3.2.25

Chapter 8

Characteristic 3

In this chapter we continue the classification of simple restricted modules for the Hamiltonian Lie algebra H to the setting of an algebraically closed field k of characteristic 3, which we were not able to treat systematically in Chapter 6, for several reasons. For one, the crucial element $C = -e_{0,3} = y^{(2)}\partial_x - x^{(p-1)}y^{(3)}\partial_y$ in the subalgebra N of p-nilpotent elements does not exist in characteristic 3. Furthermore, in determining the maximal vectors that arise, we used the action of the element $e_{3,1} = x^{(2)}y\partial_y - x^{(3)}\partial_x$, which is not available to us in characteristic 3.

Throughout the chapter, fix k to be an algebraically closed field of characteristic 3. Thus, $\dim_k \widehat{H} = 3^2 + 1 = 10$. We adopt the same notation and setup as in the general case ($p \geq 5$) in Chapter 6, for which see §6.1. We thus have

$$\widehat{H} = k\left\langle \partial'_x, \partial_y, x\partial_x, y\partial_y, e_{0,2}, x\partial_y, x^{(2)}\partial_y, e_{1,2}, e_{2,1}, e_{2,2} \right\rangle,$$

where $e_{2,2} = xy^{(2)}\partial_y - x^{(2)}y\partial_x$ and we recall that $e_{1,2} = y^{(2)}\partial_y - xy\partial_x, e_{2,1} = xy\partial_y - x^{(2)}\partial_x$, and $Y := -e_{0,2} = y\partial_x - x^{(p-1)}y^{(2)}\partial_y = y\partial_x - x^{(2)}y^{(2)}\partial_y$. Here, $N = k\langle x\partial_y, x^{(2)}\partial_y, e_{1,2}, e_{2,1}, e_{2,2}\rangle$.

We note that the filtration on $\widehat{H}$ gives $\widehat{H}_{(1)} = k\langle x^{(2)}\partial_y, e_{1,2}, e_{2,1}, e_{2,2}\rangle$ and $\widehat{H}_{(0)} = k\langle x\partial_x, y\partial_y, Y, x\partial_y, x^{(2)}\partial_y, e_{1,2}, e_{2,1}, e_{2,2}\rangle$, so $\widehat{H}_0 := \widehat{H}_{(0)}/\widehat{H}_{(1)} \cong \mathfrak{gl}_2$. Note also that $\mathcal{A} = \{0, 1, 2\}^2$.

The action of $e_{2,2}$ is given by

$$e_{2,2} \cdot v = \sum_{a \in \mathcal{A}} v_a \partial_x'^{a_1-1} \partial_y^{a_2-1} \otimes m_a + \sum_{a \in \mathcal{A}} \binom{a_2}{2} \partial_x'^{a_1} \partial_y^{a_2-2} \otimes X \cdot m_a - \sum_{a \in \mathcal{A}} \binom{a_1}{2} \partial_x'^{a_1-2} \partial_y^{a_2} \otimes Y \cdot m_a,$$

where one sets

$$v_a = a_1 \left(a_2 \left(\lambda(a)_2 - \lambda(a)_1 \right) - \binom{a_2}{2} \right) + a_2 \binom{a_1}{2}.$$

The following propositions will be useful in dealing with maximal vectors and in verifying that sets of vectors do form $\widehat{H}$-submodules (see the discussion at the end of §6.1 for more details).

Proposition 8.0.1. *We have $H = H \langle \partial_y, Y \rangle$.*

Proof. We calculate $[Y, \partial_y] = -\partial_x'$. Recall that $[\partial_x', \partial_y] = x^{(2)} \partial_y$ (see Lemma 6.2.2, part (2)). We also have $[Y, \partial_x'] = -e_{2,2}$. We calculate $[e_{2,2}, \partial_x'] = e_{1,2}$ and $[e_{2,2}, \partial_y] = -e_{2,1}$. We have that $[-e_{2,1}, \partial_y] = x\partial_y$. Finally, $[e_{1,2}, \partial_y] = x\partial_x - y\partial_y$. ■

Proposition 8.0.2. *We have $N = H \langle x\partial_y, x^{(2)}\partial_y, e_{1,2}, e_{2,2} \rangle$.*

Proof. We have $[x\partial_y, e_{1,2}] = 2e_{2,1}$. ■

8.1 Modules induced from one-dimensional modules

We start by looking at inducing to $\widehat{H}$ from one-dimensional modules $M \cong L_0(a, a)$, where $a \in \mathbb{F}_p$. Here we have an eigenbasis $\{m\}$ for M with $X \cdot m = 0 = Y \cdot m$.

We have the following:

Proposition 8.1.1. *Let $M \cong L_0(a, a)$, then any maximal vector v for $Z(M)$ is contained in the subspace*

$$k \langle 1 \otimes m \rangle \oplus k \langle \partial_y \otimes m \rangle \oplus k \langle \partial_x'^2 \partial_y^2 \otimes m \rangle,$$

where $k \langle m \rangle = M$.

Proof. Let v be a maximal vector, so we write $v = \sum_{a \in \mathcal{A}} \left(\partial'_x \partial_y\right)^a \otimes m_a$, where $m_a \in M$ for all $a \in \mathcal{A}$. For each m_a write in fact $m_a = k_a m$, where $k_a \in k$. From $x\partial_y \cdot v = 0$, we obtain the following (see Lemma 6.2.3):

$$0 = -\sum_{a \in \mathcal{A}} a_1 \partial_x'^{a_1 - 1} \partial_y^{a_2+1} \otimes k_a m.$$

The terms $a_1 \partial_x'^{a_1-1} \partial_y^{a_2+1} \otimes k_a m$ are linearly independent. Thus, if $k_a \neq 0$, then $a_1 = 0$ or $a_2 = p - 1 = 2$. Therefore, we have $0 = k_{(1,0)} = k_{(1,1)} = k_{(2,0)} = k_{(2,1)}$.

See §6.2.1 for the formulae for the rest of the following calculations.

From $e_{1,2} \cdot v = 0$, we obtain the following:

$$\begin{aligned} 0 =& s_{(0,1)} 1 \otimes k_{(0,1)} m + s_{(0,2)} \partial_y \otimes k_{(0,2)} m \\ &+ s_{(1,2)} \partial'_x \partial_y \otimes k_{(1,2)} m + s_{(2,2)} \partial_x'^2 \partial_y \otimes k_{(2,2)} m, \end{aligned}$$

see Definition 6.1.8 for the definition of s_a. One calculates that $s_{(0,1)} = s_{(2,2)} = 0$ and $s_{(1,2)} = -1, s_{(0,2)} = 1$. Hence, $k_{(0,2)} = k_{(1,2)} = 0$.

Therefore

$$v = 1 \otimes k_{(0,0)} m + \partial_y \otimes k_{(0,1)} m + \partial_x'^2 \partial_y^2 \otimes k_{(2,2)} m.$$

Thus, v is of the claimed form. Furthermore, we have that $x^{(2)}\partial_y, e_{2,1}, e_{2,2}$ do kill this vector. ■

Arguing as in Proposition 6.3.4, we refine the previous proposition into:

Proposition 8.1.2. *Let $M \cong L_0(a,a)$. If v is a maximal vector for $Z(M)$, then $v = \mu_1 (1 \otimes m)$ or $v = \mu_2 \left(\partial_y \otimes m\right)$ or $v = \mu_3 \left(\partial_x'^2 \partial_y^2 \otimes m\right)$, where $k \langle m \rangle = M$ and $\mu_i \in k$ for all i.*

As in the case of characteristic $p \geq 5$, we will need the following lemma to prove the main result of this section.

Lemma 8.1.3. *The restricted $\widehat{H}$-module $\mathcal{O}(2;(1,1))/(k \cdot 1)$ is simple.*

Proof. By Lemma 6.1.4 it suffices to show that all the maximal vectors generate the whole module.

The argument determining the maximal vectors in Lemma 6.3.7 still works in this setting, since we only use the action of $x\partial_y$ and $e_{1,2}$, both of which are available to us in characteristic 3.

Thus, $v = \mu_1 x^{(p-1)} y^{(p-1)}$ or $v = \mu_2 x^{(p-1)} y^{(p-2)}$, that is $v = \mu_1 x^{(2)} y^{(2)}$ or $v = \mu_2 x^{(2)} y$, noting that $x^{(2)}\partial_y$, $e_{2,1}$, and $e_{2,2}$ kill both vectors.

The following identities still hold:

$$\partial'_x \cdot x^{(a)} y^{(b)} = x^{(a-1)} y^{(b)}$$
$$\partial_y \cdot x^{(a)} y^{(b)} = x^{(a)} y^{(b-1)},$$

the first equation being valid only for $a = 1, 2$.

The argument showing that $v = \mu_1 x^{(2)} y^{(2)}$ generates all of $O(2; (1,1))/(k \cdot 1)$ still holds. However, the argument for $w = \mu_2 x^{(2)} y$ is no longer valid, since $C \notin \widehat{H}$. To see that w still generates all of $O(2; (1,1))/(k \cdot 1)$, note that $Y \cdot w = 2xy^{(2)}$, so $y^{(2)} \in \widehat{H}\langle w \rangle$. We calculate that

$$\partial'_x \cdot y^{(2)} = x^{(2)} y^{(2)},$$

and so $\widehat{H}\langle w \rangle = O(2; (1,1))/(k \cdot 1)$, and we are done. ■

Theorem 8.1.4. *The induced module $Z(1,1)$ is simple. The modules $Z(0,0)$ and $Z(2,2) = Z(-1,-1)$ are not simple and have composition factors of dimension* 1 *and* $p^2 - 1 = 8$.

Proof. It is easy to see that the argument in Lemma 6.3.5 still holds, so that $\widehat{H}\langle \partial_y \otimes m \rangle = Z(a,a)$ if $a \neq 0$, and in $Z(0,0)$

$$\widehat{H}\langle \partial_y \otimes m \rangle = k\left\langle \partial'^i_x \partial^j_y \otimes m : (i,j) \neq (0,0) \right\rangle.$$

Put $w = \partial'^2_x \partial^2_y \otimes m$ and $v = \partial_y \otimes m$. We shall not deal with the weights of w and their relation to Frobenius reciprocity in the cases $Z(0,0)$ and $Z(1,1)$ until §8.3.

We calculate $\partial'_x \cdot w = (2-a)\partial^2_y \otimes m$. Hence, as long as $a \neq 2$, $\partial'_x \cdot w \neq 0$. We also have $\partial_y \cdot w = 0$ (see ∂_y formula). We have $Y \cdot w = w_{(2,2)} \partial_y \otimes m = (2a-1)\partial_y \otimes m$, which is non-zero as long as $a \neq 2$.

Therefore, we see that in $Z(2,2)$, $\widehat{H} \cdot w \subseteq k\langle w \rangle$, since $\widehat{H} \cdot w = 0$, so $\widehat{H}\langle w \rangle = k\langle w \rangle$. So, in this case we have a simple one-dimensional submodule, the trivial module, noting that w has a weight of $(a+1, a+1) = (0,0)$, so $\widehat{H}\langle w \rangle \cong L(0,0)$.

We have by Frobenius reciprocity that, given a simple $\widehat{H}$-module M:

$$\mathrm{Hom}_{\widehat{H}_{(0)}}(L_0(2,2), M) \cong \mathrm{Hom}_{\widehat{H}}(Z(2,2), M).$$

This tells us that there is a simple $\widehat{H}_0$-submodule of M isomorphic to $L_0(2,2)$ if, and only if, $Z(2,2)$ surjects to M. That is, M has a maximal vector of highest weight $(2,2)$ if, and only if, $Z(2,2)$ surjects to M.

But $O(2;(1,1))/(k\cdot 1)$ is simple by Lemma 8.1.3 and it has a $(2,2)$ weight maximal vector. Hence, $Z(2,2)$ surjects to it. Hence, $Z(2,2)$ has an 8-dimensional simple quotient. By dimensions, the quotient $Z(2,2)/\widehat{H}\langle v\rangle$ is this simple quotient, call it $L(2,2)$. Thus, we have composition factors

$$[L(2,2), L(0,0)]$$

of dimension 8 and 1.

The previous calculations regarding w show that in $Z(0,0)$ and $Z(1,1)$, $\widehat{H}\langle w\rangle$ contains the maximal vector v, so $\widehat{H}\langle v\rangle \subseteq \widehat{H}\langle w\rangle$. Since $w \in \widehat{H}\langle v\rangle$, we conclude $\widehat{H}\langle w\rangle = \widehat{H}\langle v\rangle$. But we saw $\widehat{H}\langle v\rangle = Z(1,1)$. Thus, $\widehat{H}\langle w\rangle = Z(1,1)$, and hence since all maximal vectors generate the whole module, $Z(1,1)$ is simple.

Finally, in $Z(0,0)$ we have that both w and v generate the same 8-dimensional submodule, which is simple as it is generated by each of its maximal vectors. Since v has weight $(a,a-1)=(0,2)$, we have $\widehat{H}\langle v\rangle \cong L(0,2)$. The quotient by this submodule is one-dimensional, and hence simple. Thus, $Z(0,0)$ has composition factors

$$[L(0,2), L(0,0)]$$

of dimension 8 and 1. ■

8.2 Modules induced from two-dimensional modules

Let $M \cong L_0(a,a-1)$, with $a \in \mathbb{F}_p$. Pick an eigenbasis $\{m_1, m_2\}$ for M with $X\cdot m_1 = m_2$ and $Y\cdot m_2 = m_1$. We refer the reader to §6.2.2 for more details.

Proposition 8.2.1. *Let $M \cong L_0(a,a-1)$, with $a \in \mathbb{F}_p$, then any maximal vector v for $Z(M)$ has the general form*

$$\mu_1(1\otimes m_2)+\mu_2\left(\partial_x'\otimes m_2+\partial_y\otimes m_1\right)+\mu_3\left(\partial_x'\partial_y\otimes m_2+\partial_y^2\otimes m_1\right)+\mu_4\left(\partial_y^2\otimes m_2\right),$$

where the $\mu_i \in k$ and at most one of $\mu_3, \mu_4 \neq 0$.

Proof. Let $v=\sum_{a\in\mathcal{A}}\left(\partial_x'\partial_y\right)^a\otimes m_a$ be a maximal vector, where $m_a \in M$ for all $a\in\mathcal{A}$. Since m_a can only be in one H-eigenspace (see §6.3.1) one has, for all $a\in\mathcal{A}$:

$$m_a = \mu_a m_j,$$

with $j = 1$ or $j = 2$.

As with the one-dimensional case, we refer the reader to Section 6.2.1 for the formulae for the actions we will consider here. We do the first one in detail. The others are done similarly.

From $x\partial_y \cdot v = 0$, we see

$$0 = \sum_{a\in\mathcal{A}} \left(\partial'_x\partial_y\right)^a \otimes X \cdot m_a - \sum_{a\in\mathcal{A}} a_1 \partial'^{a_1-1}_x \partial^{a_2+1}_y \otimes m_a.$$

Thus, if $m_a = \mu_a m_1 \neq 0$, then either $a_1 = 0$ or $a_2 = 2$. Also, $m_{(0,0)} = \mu_{(0,0)} m_2$.

From $x^{(2)}\partial_y \cdot v = 0$, we see that $\mu_{(2,0)} = 0$, $m_{(2,2)} = \mu_{(2,2)} m_2$, and

$$\mu_{(2,1)} m_2 = X \cdot m_{(1,2)}.$$

From the last equation we see that if $m_{(1,2)} \in k\langle m_2\rangle$, then $\mu_{(2,1)} = 0$, while if $m_{(1,2)} \in k\langle m_1\rangle$, then $\mu_{(1,2)} = \mu_{(2,1)}$.

From $e_{1,2} \cdot v = 0$, we see that $\mu_{(2,2)} = 0$. Furthermore, if $m_{(0,1)} \in k\langle m_2\rangle$, then $s_{(0,1)} = 1$ and $\mu_{(0,1)} = \mu_{(1,0)} = 0$ (see Definition 6.1.8 for the definition of s_a); if $m_{(0,1)} \in k\langle m_1\rangle$, then $s_{(0,1)} = -1$ and $\mu_{(0,1)} = \mu_{(1,0)}$. Thus, without loss of generality, we write $m_{(0,1)} = \mu_{(0,1)} m_1$ and insist that $\mu_{(0,1)} = \mu_{(1,0)}$. We also see that if $m_{(0,2)} \in k\langle m_2\rangle$, then $s_{(0,2)} = 0$ and $\mu_{(1,1)} = 0$, while if $m_{(0,2)} \in k\langle m_1\rangle$, then $s_{(0,2)} = -1$ and $\mu_{(1,1)} = \mu_{(0,2)}$. Finally, we see that $Y \cdot m_{(1,2)} = 0$, so $m_{(1,2)} \in k\langle m_1\rangle$ and that $s_{(2,1)} = -1$, so $\mu_{(2,1)} = 0$.

Thus

$$\begin{aligned} v =& 1 \otimes \mu_{(0,0)} m_2 + \mu_{(0,1)} \left(\partial'_x \otimes m_2 + \partial_y \otimes m_1\right) \\ &+ \mu_{(1,1)} \left(\partial'_x\partial_y \otimes m_2 + \partial^2_y \otimes m_1\right) + \partial'_x\partial^2_y \otimes \mu_{(1,2)} m_1 \end{aligned}$$

or

$$\begin{aligned} v =& 1 \otimes \mu_{(0,0)} m_2 + \mu_{(0,1)} \left(\partial'_x \otimes m_2 + \partial_y \otimes m_1\right) \\ &+ \mu_{(0,2)} \left(\partial^2_y \otimes m_2\right) + \partial'_x\partial^2_y \otimes \mu_{(1,2)} m_1. \end{aligned}$$

From $e_{2,1} \cdot v = 0$, we see that $\mu_{(1,2)} = 0$. Hence, v has the claimed form. Furthermore, we note that $e_{2,2} \cdot v = 0$. ■

Again, arguing as in Proposition 6.3.4, we refine the previous proposition into:

Proposition 8.2.2. *Let $M \cong L_0(a, a-1)$, with $a \in \mathbb{F}_p$. If v is a maximal vec-*

tor for $Z(M)$, then $v = \mu_1(1 \otimes m)$ or $v = \mu_2\left(\partial'_x \otimes m_2 + \partial_y \otimes m_1\right)$ or $v = \mu_3\left(\partial'_x\partial_y \otimes m_2 + \partial_y^2 \otimes m_1\right)$ or $v = \mu_4\left(\partial_y^2 \otimes m_2\right)$, where $\mu_i \in k$ for all i.

As in the case of characteristic $p \geq 5$, we will break up the proof of our determination of the modules induced from two-dimensional modules and their composition factors.

In what follows, we adopt the following shorthand:

$$v := \partial'_x \otimes m_2 + \partial_y \otimes m_1$$

$$w := \partial'_x\partial_y \otimes m_2 + \partial_y^2 \otimes m_1$$

$$z := \partial_y^2 \otimes m_2.$$

The weights of the maximal vectors are $(a-1, a-1)$, $(a-1, a-2)$, and $(a, a-3)$, respectively.

Proposition 8.2.3. *We have $\widehat{H}\langle z\rangle = \widehat{H}\langle w\rangle$ in $Z(1,0)$ and $Z(2,1)$. In $Z(0,2)$, we have $\widehat{H}\langle z\rangle = \widehat{H}\langle w\rangle \oplus k\langle z\rangle$.*

Proof. We have $Y \cdot z = -2\partial'_x\partial_y \otimes m_2 + \partial_y^2 \otimes m_1 = w$. Thus $w \in \widehat{H}\langle z\rangle$, and so $\widehat{H}\langle w\rangle \subseteq \widehat{H}\langle z\rangle$. From the k-basis for $\widehat{H}\langle w\rangle$ (see Lemma 6.3.12), we see that $\partial_x'^p\partial_y^{p-1} \otimes m_2 = -\partial_y^2 \otimes am_2 \in \widehat{H}\langle w\rangle$. Therefore we see that in $Z(1,0)$ and $Z(2,1)$, $z \in \widehat{H}\langle w\rangle$. Hence, in $Z(1,0)$ and $Z(2,1)$, $\widehat{H}\langle z\rangle = \widehat{H}\langle w\rangle$, as claimed.

In $Z(0,2)$, we have $k\langle z\rangle \subseteq \widehat{H}\langle z\rangle$, so we see that $\widehat{H}\langle w\rangle \oplus k\langle z\rangle \subseteq \widehat{H}\langle z\rangle$. To show the reverse inclusion, we show that $\widehat{H}\langle w\rangle \oplus k\langle z\rangle$ is an $\widehat{H}$-submodule containing z. Since $\widehat{H}\langle w\rangle$ is an $\widehat{H}$-submodule, it is only necessary to show that $\widehat{H} \cdot z \subseteq \widehat{H}\langle w\rangle \oplus k\langle z\rangle$. Indeed, one has $N \cdot z = 0$, $x\partial_x$ and $y\partial_y$ give scalar multiples of z, $Y \cdot z = w$, $\partial_y \cdot z = 0$ and $\partial'_x \cdot z \in \widehat{H}\langle w\rangle$ (see Lemma 6.3.12). ■

Corollary 8.2.4. *In $Z(1,0)$ we have*

$$\widehat{H}\langle z\rangle = \widehat{H}\langle w\rangle \leq \widehat{H}\langle v\rangle$$

with dimensions 8 and 9.

In $Z(2,1)$ we have

$$\widehat{H}\langle z\rangle = \widehat{H}\langle w\rangle = \widehat{H}\langle v\rangle$$

of dimension 9.

In $Z(0,2)$ we have

$$\widehat{H}\langle v\rangle = \widehat{H}\langle w\rangle \leq \widehat{H}\langle z\rangle$$

with dimensions 8 *and* 9.

Proof. Apply Lemma 6.3.11 and Lemma 6.3.12 to the previous proposition. ■

Theorem 8.2.5. *The induced module $Z(M) \cong Z(a, a-1)$, where $a \in \mathbb{F}_p$, is not simple. If $(a, a-1) = (2, 1)$ or $(1, 0)$, then we get composition factors of dimension 1, $p^2 - 1 = 8$ and $p^2 = 9$. If $(a, a-1) = (0, 2)$, we get composition factors of dimension $1, 1, 8, 8$.*

Proof. We shall not deal with the weights of z and their relation to Frobenius reciprocity until §8.3.

We deal with $Z(1, 0)$ first. We have by Frobenius reciprocity that

$$\mathrm{Hom}_{\widehat{H}_{(0)}}(L_0(1, 0), Z(1, 1)) \cong \mathrm{Hom}_{\widehat{H}}(Z(1, 0), Z(1, 1)).$$

The left side is non-zero as $Z(1, 1)$ has a maximal vector of highest weight $(1, 1-1) = (1, 0)$, as we saw previously. Thus there is a non-zero $\widehat{H}$-homomorphism

$$f : Z(1, 0) \longrightarrow Z(1, 1).$$

Now, by Theorem 8.1.4 we know that $Z(1, 1)$ is simple, of dimension 9. Thus f must be surjective.

Hence, $Z(1, 0)$ has a 9-dimensional simple quotient isomorphic to $Z(1, 1) = L(1, 1)$. Indeed, the quotient $Z(1, 0)/\widehat{H}\langle v \rangle$ is 9-dimensional, so it must be this simple quotient; we denote it by $L(1, 0)$. Finally, we have

$$\widehat{H}\langle z \rangle = \widehat{H}\langle w \rangle \leq \widehat{H}\langle v \rangle,$$

of dimensions 8 and 9. The quotient $\widehat{H}\langle v \rangle / \widehat{H}\langle w \rangle$ is one-dimensional, and hence simple, and $\widehat{H}\langle z \rangle = \widehat{H}\langle w \rangle$ is simple as it is generated by its maximal vectors w and z. We have $\widehat{H}\langle w \rangle \cong L(0, 2)$, by Frobenius reciprocity. Thus, we have found all the compositions factors:

$$[L(0, 0), L(0, 2), L(1, 0)],$$

of dimensions 1, 8, 9.

Now we study $Z(0, 2)$. Here we have $\widehat{H}\langle v \rangle = \widehat{H}\langle w \rangle \leq \widehat{H}\langle z \rangle$ of dimensions 8 and 9. The module $\widehat{H}\langle v \rangle = \widehat{H}\langle w \rangle$ is simple as it generated by its maximal vectors, and by Frobenius reciprocity we have $\widehat{H}\langle v \rangle \cong L(2, 2) \cong L(2, 1)$. The quotient $\widehat{H}\langle z \rangle / \widehat{H}\langle v \rangle$ is one-dimensional and simple.

Note: The previous isomorphisms are not a problem, as $L(2,2)$ is the 8-dimensional simple quotient of $Z(2,2)$ and we will see that $L(2,1)$ is the 8-dimensional simple quotient of $Z(2,1)$.

We turn our attention to the quotient $Z(0,2)/\widehat{H}\langle z\rangle$. There is a vector not in $\widehat{H}\langle z\rangle$,

$$\varphi := \partial_x'^{p-1} \otimes m_1 = \partial_x'^2 \otimes m_1$$

with the following property: $\widehat{H} \cdot \varphi \subseteq \widehat{H}\langle z\rangle$ (so in particular, $x\partial_x$ and $y\partial_y$ have weight $(0,0)$ on φ in the quotient). Thus there is a one-dimensional submodule $k\langle\varphi\rangle \leq Z(0,2)/\widehat{H}\langle z\rangle$. The quotient here is 8-dimensional. By Frobenius reciprocity, we have that $Z(0,2)$ must have an 8-dimensional simple quotient isomorphic to $\widehat{H}\langle\partial_y \otimes m\rangle \leq Z(0,0)$. By dimensions, the above quotient has to be this one, which we call $L(0,2)$. Thus the compositions factors are:

$$[L(2,2), L(0,2), L(0,0), L(0,0)],$$

of dimensions $8, 8, 1, 1$.

Finally, we have $Z(2,1)$. Here, we have $\widehat{H}\langle w\rangle = \widehat{H}\langle v\rangle = \widehat{H}\langle z\rangle \leq Z(a, a-1)$ of dimension 9, and simple, as the submodule is generated by its maximal vectors v, w and z. Here we have $\widehat{H}\langle v\rangle \cong L(1,1) \cong L(1,0)$.

Note: Again, the above isomorphism is not a problem, as $L(1,1) = Z(1,1)$ is a 9-dimensional simple $\widehat{H}$-module and $L(1,0)$ is the 9-dimensional simple quotient of $Z(1,0)$.

By Frobenius reciprocity,

$$\mathrm{Hom}_{\widehat{H}_{(0)}}(L_0(2,1), M) \cong \mathrm{Hom}_{\widehat{H}}(Z(2,1), M).$$

If we take M to be the 8-dimensional simple submodule $\widehat{H}\langle w\rangle = \widehat{H}\langle v\rangle \leq Z(0,2)$, we see that the left side is non-zero because M has a maximal vector w of weight $(2,1)$. Thus the right hand is non-zero, and so $Z(2,1)$ surjects onto M, as $M \cong L(2,2)$ is simple. We have shown hence that $Z(2,1)$ has an 8-dimensional simple quotient. Indeed, we can argue that $Z(2,1)/\widehat{H}\langle v\rangle$ has a one-dimensional submodule. Indeed, the vector

$$\gamma := \partial_x'^{p-1}\partial_y^{p-2} \otimes m_2 = \partial_x'^2\partial_y \otimes m_2 \notin \widehat{H}\langle v\rangle$$

is such that $\widehat{H} \cdot \gamma \subseteq \widehat{H}\langle v\rangle$. The quotient of $Z(2,1)/\widehat{H}\langle v\rangle$ by this one-dimensional submodule $k\langle\gamma\rangle$ must then be the 8-dimensional simple quotient above, which we call $L(2,1)$. Thus, we have the composition factors:

$$[L(1,1), L(0,0), L(2,1)],$$

of dimensions, 9, 1, 8. ■

Remark. All the composition factors of modules induced from two-dimensional modules are isomorphic to simple quotients of modules induced from one-dimensional induced modules except for $L(0,2)$. More precisely, we have $L(1,0) \cong L(1,1)$ and $L(2,1) \cong L(2,2)$. The module $L(0,2)$ is still isomorphic to a composition factor of a module induced from a one-dimensional induced module, more precisely $L(0,2) \cong \widehat{H}\left\langle\partial_y \otimes m\right\rangle \leq Z(0,0)$.

We will later see that $L(0,2)$ is not isomorphic to $L(2,2)$.

8.3 Frobenius reciprocity and maximal vectors

Before finishing the classification, we will address a technicality that arises in characteristic 3 with the maximal vectors $w = \partial_x'^2\partial_y^2 \otimes m \in Z(a,a)$ and $z = \partial_y^2 \otimes m_2 \in Z(a,a-1)$.

Throughout this thesis, both in the general case and in this chapter, we have been assuming in order to apply Frobenius reciprocity that if M was a restricted $\widehat{H}$-module with maximal vector v of weight $\lambda = (\lambda_1, \lambda_2)$, then

$$\mathrm{Hom}_{\widehat{H}_{(0)}}(L_0(\lambda_1,\lambda_2), M) \neq 0.$$

Note that if this is true, it means that M has a simple $\widehat{H}_{(0)}$-submodule isomorphic to $L_0(\lambda_1,\lambda_2)$. If $M = \widehat{H}\langle v\rangle$, then this simple $\widehat{H}_{(0)}$-submodule *has* to be $\widehat{H}_{(0)}\langle v\rangle$.

In characteristic 3, it turns out, however, that $\widehat{H}_{(0)}\langle v\rangle$ is not necessarily simple for v a maximal vector. Indeed, in $Z(0,0)$ and $Z(1,1)$, we have that $\widehat{H}_{(0)}\langle w\rangle = k\left\langle w, \partial_y \otimes m, \partial_x' \otimes m\right\rangle$ (this can be verified using the known formulae) with $k\left\langle\partial_y \otimes m, \partial_x' \otimes m\right\rangle \cong L_0(1)$ a two-dimensional $\widehat{H}_{(0)}$-submodule. Likewise, in $Z(a,a-1)$,

$$\widehat{H}_{(0)}\langle z\rangle = k\left\langle z, \partial_x'\partial_y \otimes m_2 + \partial_y^2 \otimes m_1, \partial_x'^2 \otimes m_2 + \partial_x'\partial_y \otimes m_1\right\rangle,$$

with $k\left\langle\partial_x'\partial_y \otimes m_2 + \partial_y^2 \otimes m_1, \partial_x'^2 \otimes m_2 + \partial_x'\partial_y \otimes m_1\right\rangle \cong L_0(1)$ a two-dimensional

$\widehat{H}_{(0)}$-submodule (again this can be verified with the known formulae).

To see that this issue does not arise elsewhere, we cite Corollary 8.12 of Herpel and Stewart (2016b):

Corollary 8.3.1. *Let G be a semisimple algebraic group and let V be a $\mathfrak{g}$-module with $p > \dim_k V$. Assume that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Then V is semisimple.*

Thus, if we consider $V = \widehat{H}_{(0)} \langle v \rangle$ for v a maximal vector, we see that V is an $\widehat{H}_0$-module, and so an $\mathfrak{sl}_2$-module. Thus, by the corollary above, if $\dim_k V < p$, V is semisimple, which forces V to be a simple $\widehat{H}_0 \cong \mathfrak{gl}_2$-module, and so since $\widehat{H}_{(1)} \cdot V = 0$ a simple $\widehat{H}_{(0)}$-module. Now, with the exception of the modules described above, all the modules $V_i := \widehat{H}_{(0)} \langle v_i \rangle$ for $v_i \neq 1 \otimes m$ a maximal vector are at most two-dimensional. Thus, the condition $\dim_k V_i < p$ is met, and we have

$$\mathrm{Hom}_{\widehat{H}_{(0)}}(L_0(\lambda_1, \lambda_2), V_i) \neq 0,$$

so we are justified in applying Frobenius reciprocity in all other cases.

8.4 Modules induced from three-dimensional modules

Proposition 8.4.1. *Let $M \cong L_0(a, b)$, with $a - b = 2$, then any maximal vector v for $Z(M)$ is contained in the subspace*

$$k \langle 1 \otimes m_3 \rangle ,$$

where $k \langle m_1, m_2, m_3 \rangle = M$ and $X \cdot m_3 = 0$.

Proof. We recall here our general setup for our restricted $\widehat{H}_0$-modules M:

We pick an eigenbasis $\{m_1, m_2, m_3\}$. With this eigenbasis we have

$$X \cdot m_i = m_{i+1},$$

where $X \cdot m_3 = 0$, and

$$Y \cdot m_i = (i - 1)(4 - i) m_{i-1},$$

noting again that $Y \cdot m_1 = 0$.

Let $v = \sum_{a\in\mathcal{A}} \left(\partial'_x \partial_y\right)^a \otimes m_a$ be a maximal vector, where $m_a \in M$ for all $a \in \mathcal{A}$. As with the lower-dimensional cases, each m_a can only be in one H-eigenspace (see §6.3.1), so one has, for all $a \in \mathcal{A}$:

$$m_a = \mu_a m_j,$$

with $j \in \{1, 2, 3\}$.

Arguing as before, from $x\partial_y \cdot v = 0$, we see that $m_{(a_1,0)}, m_{(2,a_2)} \in k\langle m_3\rangle$. We also see that

$$X \cdot m_{(0,1)} = \mu_{(1,0)} m_3 \tag{8.1}$$
$$X \cdot m_{(0,2)} = m_{(1,1)} \tag{8.2}$$
$$X \cdot m_{(1,1)} = 2\mu_{(2,0)} m_3 \tag{8.3}$$
$$X \cdot m_{(1,2)} = 2\mu_{(2,1)} m_3. \tag{8.4}$$

From $x^{(2)}\partial_y \cdot v = 0$, we have that

$$X \cdot m_{(1,1)} = \mu_{(2,0)} m_3 \tag{8.5}$$
$$X \cdot m_{(1,2)} = \mu_{(2,1)} m_3. \tag{8.6}$$

We conclude from Equation (8.3) and Equation (8.5) that if $\mu_{(2,0)} \neq 0$, then $\mu_{(2,0)} m_3 = 2\mu_{(2,0)} m_3$ implies $1 = 2$, a contradiction. Therefore, $\mu_{(2,0)} = 0$, and $m_{(1,1)} \in k\langle m_3\rangle$. Similarly, from Equation (8.4) and Equation (8.6) we conclude that $\mu_{(2,1)} = 0$ and $m_{(1,2)} \in k\langle m_3\rangle$.

Thus, we have

$$\begin{aligned} v =& 1 \otimes \mu_{(0,0)} m_3 + \partial_y \otimes m_{(0,1)} + \partial_y^2 \otimes m_{(0,2)} \\ &+ \partial'_x \otimes \mu_{(1,0)} m_3 + \partial'_x \partial_y \otimes \mu_{(1,1)} m_3 + \partial'_x \partial_y^2 \otimes \mu_{(1,2)} m_3 + \partial'^2_x \partial_y^2 \otimes \mu_{(2,2)} m_3. \end{aligned}$$

From $e_{2,1} \cdot v = 0$ one gets that $\mu_{(2,2)} = 0 = \mu_{(1,2)}$, since $r_{(1,2)}$ and $r_{(2,2)}$ are both non-zero, see Definition 6.1.8 for the definition of r_a. Thus, we have, since $r_{(1,0)} = 2$ and $r_{(1,1)} = 0$, that $X \cdot m_{(0,2)} = 0$. Therefore, we have $m_{(0,2)} \in k\langle m_3\rangle$. Finally, we see that $X \cdot m_{(0,1)} = 2\mu_{(1,0)} m_3$. This, together with Equation (8.1), yields $\mu_{(1,0)} = 0$ and $m_{(0,1)} \in k\langle m_3\rangle$.

Using Equation (8.2), we see that $m_{(1,1)} = 0$. Hence, we have

$$v = 1 \otimes \mu_{(0,0)} m_3 + \partial_y \otimes \mu_{(0,1)} m_3 + \partial_y^2 \otimes \mu_{(0,2)} m_3.$$

We calculate that (see Definition 6.1.8 for the definition of s_a)

$$0 = e_{1,2} \cdot v = s_{(0,1)} 1 \otimes \mu_{(0,1)} m_3 + s_{(0,2)} \partial_y \otimes \mu_{(0,2)} m_3.$$

Now, $s_{(0,1)} = 2 = s_{(0,2)}$, so $\mu_{(0,1)} = 0 = \mu_{(0,2)}$. Thus, $v = 1 \otimes \mu_{(0,0)} m_3$, as required. ∎

From this it follows that

Theorem 8.4.2. *The induced module $Z(M) \cong Z(a, b)$, where $a - b = 2$, is simple.*

Proposition 8.4.3. *There are two isomorphism classes of 8-dimensional restricted simple $\widehat{H}$-modules, one represented by $L(2, 2)$, the other by $L(0, 2)$.*

Proof. Arguing as in the proof of Proposition 6.3.16, if $L(2, 2) \cong L(0, 2)$, then $L(2, 2)$ would need to have a maximal vector of weight $(0, 2)$. If $0 \neq \delta \in L(2, 2)$ is a vector of weight $(0, 2)$, then working in the quotient we deduce that $\delta = \partial_x'^2 \otimes m$. This is a problem, as $X \cdot \delta = \partial_x' \partial_y \otimes m \neq 0$, so that δ is not maximal. Thus no maximal vector of such a weight exists, and we are done. ∎

By arguing in a manner completely analogous to the proof of the main result in characteristic $p \geq 5$, we complete the proof of our main result. To state which, recall that the exceptional weights for us are the following: $\omega_0 = (-1, -1) = (2, 2), \omega_1 = (0, -1) = (0, 2), \omega_2 = (0, 0)$, and all $\lambda \in \mathbb{F}_3^2$ with $\lambda_1 - \lambda_2 = 1$.

Theorem 8.4.4. *Let $p = 3$, k be an algebraically closed field of characteristic p, $\lambda \in \mathbb{F}_3^2$ a weight, $L_0(\lambda)$ be the simple restricted $\mathfrak{gl}_2(k)$-module of highest weight λ, $Z(\lambda) = Z(L_0(\lambda))$ the corresponding induced $\widehat{H}$-module, and $L(\lambda)$ its simple head.*

1. *The full list of simple pairwise nonisomorphic restricted $\widehat{H}$-modules is given by $\{L(\lambda) : \lambda \in \mathbb{F}_3^2, \lambda_1 - \lambda_2 \neq 1 \text{ or } \lambda = \omega_1\}$. There are $p^2 - p + 1 = 7$ of them.*

2. *If λ is not exceptional, then $L(\lambda) = Z(\lambda)$, and its dimension is $9 \dim_k L_0(\lambda) = 9 (\lambda_1 - \lambda_2 + 1)$.*

3. *For exceptional λ, the modules $L(\lambda)$ in the list are as follows:*

   (a) if $\lambda = \omega_0 = (-1, -1)$, $L(\lambda) \cong O(2; (1, 1)) / (k \cdot 1)$, with dimension $p^2 - 1 = 8$;

   (b) if $\lambda = \omega_1 = (0, -1)$, $L(\lambda) \cong \widehat{H} \left\langle \partial_y \otimes m \right\rangle \leq Z(0, 0)$, with dimension 8;

   (c) if $\lambda = \omega_2 = (0, 0)$, $L(\lambda) \cong k$, with dimension 1 (this is the trivial module).

Appendix A

Extra formulae

We include the formula for the action of an element we calculated in the course of proving our results, that ended not being needed, in case it might be of use to anyone studying such things. We have the following general formula:

$$
\begin{aligned}
0 = x^{(n)}\partial_y \cdot v = &\sum_{a\in\mathcal{A}} \binom{a_1}{n} (-1)^n \, \partial_x'^{a_1-n} \partial_y^{a_2+1} \otimes m_a \\
&+ \sum_{a\in\mathcal{A}} \binom{a_1}{n-1} (-1)^{n-1} \partial_x'^{a_1-n+1} \partial_y^{a_2} \otimes X \cdot m_a.
\end{aligned}
$$

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