# Auslander-Reiten theory, derived categories, and higher dimensional homological algebra

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*A Giocs, sempre con me.*

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**Abstract**

Auslander-Reiten theory plays an important role in the study of abelian and triangulated categories (in classic homological algebra) and in their higher analogues (in the more recent higher homological algebra). The classic setup studies module categories of the form $\operatorname{mod}\Lambda$ and their bounded derived categories $\mathcal{D}^b(\operatorname{mod}\Lambda)$, where $\Lambda$ is a finite dimensional algebra over a field $k$ and $\operatorname{mod}\Lambda$ is the category of finitely generated (right) $\Lambda$-modules.

If $\operatorname{gldim}\Lambda \le 1$, Brüning proved there is a bijection between the wide subcategories of the abelian category $\operatorname{mod}\Lambda$ and those of the triangulated category $\mathcal{D}^b(\operatorname{mod}\Lambda)$. When $\mathcal{T}$ is a suitable triangulated category, Jørgensen described Auslander-Reiten triangles in the extension closed subcategories of $\mathcal{T}$. If $\mathcal{X} \subseteq \operatorname{mod}\Lambda$ is a precovering extension closed subcategory, Kleiner proved that any indecomposable not Ext-projective $X \in \mathcal{X}$ appears as the end term of an Auslander-Reiten sequence in $\mathcal{X}$ and he further described the case when $\operatorname{End}_\Lambda(X)$ modulo the morphisms factoring through a projective is a division ring.

Letting $d$ be a positive integer, we study higher homological algebra and higher Auslander-Reiten theory. Geiss, Keller and Oppermann generalised triangulated categories to $(d+2)$-angulated categories and Jasso likewise generalised abelian categories to $d$-abelian categories. Note that the case $d = 1$ recovers classic homological algebra. Assuming there is a $d$-cluster tilting subcategory $\mathcal{F} \subseteq \operatorname{mod}\,\Lambda$ , consider

$$\overline{\mathcal{F}} := \operatorname{add}\{\Sigma^{id}\mathcal{F} \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}^b(\operatorname{mod}\,\Lambda).$$

Then $\mathcal{F}$ is $d$-abelian and plays the role of a higher $\operatorname{mod}\Lambda$ having for higher derived category the $(d+2)$-angulated category $\overline{\mathcal{F}}$. With this in mind, we generalise Brüning, Jørgensen and Kleiner's results for higher values of $d$.

We also use higher Auslander-Reiten theory to generalise results on Grothendieck groups of a suitable triangulated category $\mathcal{T}$. We present "higher cluster tilting" versions of results by Xiao and Zhu and by Palu and a "higher angulated" version of Palu's result. Our results express $K_0(\mathcal{T})$ as a quotient of the split Grothendieck group of higher-cluster tilting subcategories of $\mathcal{T}$.

We prove analogues of results by Kleiner on subcategories of $\operatorname{mod}\Lambda$ in the corresponding setup of subcategories of a suitable triangulated category $\mathcal{T}$ with a precovering extension closed subcategory $\mathcal{C}$. In particular, we introduce indecomposable Ext-projective objects $C$ in $\mathcal{C}$, show that such a $C$ appears in what we call a *left-weak Auslander-Reiten triangle in* $\mathcal{C}$ and prove how these objects are related to the concept of Iyama and Yoshino's mutation.

# Contents

# Chapter 1

# Introduction

Homological algebra, that is the study of abelian and triangulated categories, arose from algebraic topology in the early twentieth century and it has since found many applications in different areas of mathematics, such as combinatorics and representation theory. Fixing a field $k$ and a finite dimensional $k$-algebra $\Lambda$, the classic setup studies the abelian category of finitely generated (right) $\Lambda$-modules, denoted by $\operatorname{mod}\Lambda$, and its bounded derived category, denoted by $\mathcal{D}^b(\operatorname{mod}\Lambda)$, which is a triangulated category. Auslander-Reiten theory is a widely used tool to study homological algebra. Auslander-Reiten sequences in $\operatorname{mod}\Lambda$ are non-splitting short exact sequences of the form $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ that are "as close as possible" to split exact sequences. They collect important information about both the building blocks of the objects of $\operatorname{mod}\Lambda$, that is indecomposable modules, and the building blocks of the morphisms of $\operatorname{mod}\Lambda$, that is irreducible morphisms. Auslander and Reiten later extended this theory to general abelian categories and Auslander and Smalø to the study of Auslander-Reiten sequences in their subcategories in [7]. Moreover, Happel developed the corresponding theory of Auslander-Reiten triangles in triangulated categories in [20] and then Jørgensen defined and studied Auslander-Reiten triangles in their non-triangulated subcategories in [37].

Higher homological algebra was first introduced by Iyama in 2007, see [28], as a higher dimensional generalisation of the above theory, and it is currently a very active area of research. Let $d \geq 1$ be an integer. Jasso generalised abelian categories to $d$-abelian categories in [34], where kernels and cokernels are replaced by complexes of $d+1$ objects and short exact sequences by $d$-exact sequences, which are complexes of $d+2$ objects. In the same fashion, Geiss, Keller and Oppermann generalised triangulated categories to $(d+2)$-angulated categories in [19], where triangles are replaced by complexes of $d+2$ objects. Note that the base case $d = 1$ recovers classic homological algebra. As for homological algebra, higher Auslander-Reiten theory plays a crucial role in the study of higher homological algebra. Higher Auslander-Reiten sequences were first introduced by Iyama in [28] and Auslander-Reiten $(d+2)$-angles by Iyama and Yoshino in [32]. If we further assume that the global dimension of $\operatorname{mod}\Lambda$ is at most $d$ and that $\operatorname{mod}\Lambda$ has a $d$-cluster tilting subcategory $\mathcal{F}$, we can introduce the higher case corresponding to the classic setup of $\operatorname{mod}\Lambda$ and $\mathcal{D}^b(\operatorname{mod}\Lambda)$. In this setup, $\mathcal{F}$ is $d$-abelian,

$$\overline{\mathcal{F}} \coloneqq \operatorname{add}\{\Sigma^{id}\mathcal{F} \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}^b(\operatorname{mod}\Lambda)$$

is $(d+2)$-angulated and $\mathcal{F}$ and $\overline{\mathcal{F}}$ play the roles of a higher $\operatorname{mod}\Lambda$ and its higher bounded derived category respectively. Note that the classic case is recovered when $d=1$ as $\operatorname{mod}\Lambda$ is the only 1-cluster tilting subcategory of itself.

### Outline of thesis

The structure of this thesis is as follows. Chapter 2 introduces the background material that will be used in the rest of the thesis. We start by giving an overview on categories and subcategories. Then, we recall some definitions and results on homological algebra, including the functor Ext, Auslander-Reiten theory and the classic case of $\operatorname{mod}\Lambda$ and $\mathcal{D}^b(\operatorname{mod}\Lambda)$ mentioned above. The third and last part of the chapter consists of an introduction to higher homological algebra. We define $d$-abelian and $(d+2)$-angulated categories and present a series of results that will be widely used to prove the main results in this thesis. Moreover, in Remark 2.3.44, we describe the generalised higher $\operatorname{mod}\Lambda$ and its higher derived category mentioned above. When introducing higher Auslander-Reiten theory, we give an alternative proof for the $d$-Auslander-Reiten duality, first proved by Iyama in [28], see Theorem 2.3.26.

Chapter 3 presents three examples of the categories defined in Chapter 2. These are presented at this stage to give examples of the categories defined in Chapter 2, but they will be also used in later chapters to give applications of the main results of this thesis. The first is a classic example of a triangulated category: the cluster category of Dynkin type $A_n$, which we denote by $\mathcal{C}_{A_n}$. The second one is the triangulated $q$-cluster category of Dynkin type $A_n$, denoted by $\mathcal{C}_q(A_n)$ and it is a generalisation of $\mathcal{C}_{A_n}$. The third is a class of examples first defined by Vaso, see [55], and it is an example of the higher $\operatorname{mod}\Lambda$ and its higher derived category.

In the same setup as above, further assume that $k$ is algebraically closed. When $d=1$, Brüning proved in [10] that there is a bijection between the wide subcategories of $\operatorname{mod}\Lambda$ and those of $\mathcal{D}^b(\operatorname{mod}\Lambda)$. In Chapter 4, assuming $\operatorname{mod}\Lambda$ has a $d$-cluster tilting subcategory $\mathcal{F}$, we prove the higher version of Brüning's result.Moreover, when $\mathcal{T}$ is a suitable triangulated category, Jørgensen described Auslander-Reiten triangles in the extension closed subcategories of $\mathcal{T}$ in [37]. We prove a generalised version of his result. We conclude this chapter by applying the two main results to Vaso's class of examples introduced in Chapter 3.

Chapter 5 presents a higher version of Kleiner's results from [43] on Auslander-Reiten sequences in precovering extension closed subcategories $\mathcal{X}$ of $\operatorname{mod}\Lambda$. In particular, Kleiner proved that any indecomposable not Ext-projective $X \in \mathcal{X}$ appears as the end term of an Auslander-Reiten sequence in $\mathcal{X}$ and he further described the case when $\operatorname{End}_\Lambda(X)$ modulo the morphisms factoring through a projective is a division ring. Assuming $\operatorname{mod}\Lambda$ has a $d$-cluster tilting subcategory $\mathcal{F}$, we prove the higher version of these results and apply our results to an example.

Chapter 6 gives a way to express the Grothendieck group of a suitable triangulated category $\mathcal{C}$ as a quotient of the split Grothendieck group of a higher-cluster tilting subcategory of $\mathcal{C}$. The results proved in this chapter are a "higher cluster tilting" version of Xiao and Zhu's result from [57] and a "higher angulated" version of Palu's theorem from [50]. We illustrate the higher cluster tilting result in the example $\mathcal{C}_q(A_n)$ introduced in Chapter 3 and the higher angulated result in another example.

Finally, Chapter 7 proves analogues of results by Kleiner on $\operatorname{mod}\Lambda$ from [43] in triangulated categories. In particular, we prove that if $\mathcal{T}$ is a suitable triangulated category with a precovering extension closed subcategory $\mathcal{C}$, then any Ext-projective object $C$ in $\mathcal{C}$ appears in something very similar to an Auslander-Reiten triangle in $\mathcal{C}$, that is an essentially unique triangle in $\mathcal{T}$ of the form

$$X \to B \to C \to \Sigma X.$$

Moreover, under some extra assumptions, we show that the process of removing the indecomposable $C$ from $\mathcal{C}$ and replacing it with $X$ coincides with the classic mutation described by Iyama and Yoshino in [32]. Finally, we apply these results to the example $\mathcal{C}_{A_n}$ introduced in Chapter 3.

Work in this thesis has been in part covered by the following papers: Chapter 4 is based on [15]; Chapter 5 is based on [16]; Chapter 6 is based on [17]; Chapter 7 is based on [14].

# Chapter 2

# Background

This chapter presents the background material needed in the thesis.

## 2.1 Categories, functors and subcategories

In this section, we give an overview on categories and subcategories, see [46, Chapter I] for more details. We also introduce some definitions and results on left (and right) almost split morphisms that we will widely use for the study of Auslander-Reiten theory in homological and higher homological algebra in later sections. Some of these definitions are typically presented in the setup of a more specific category, such as an abelian or a triangulated one, but they still make sense for a general category $\mathcal{A}$.

### 2.1.1 Categories

**Definition 2.1.1.** A *category* is a triple $\mathcal{A} = (\operatorname{Ob}\mathcal{A}, \operatorname{Hom}\mathcal{A}, \circ)$, where $\operatorname{Ob}\mathcal{A}$ is called the *class of objects* of $\mathcal{A}$, $\operatorname{Hom}\mathcal{A}$, which is the union of the sets $\operatorname{Hom}_{\mathcal{A}}(A,B)$, is called the *class of morphisms* of $\mathcal{A}$, $\circ$ is a partial binary operation on $\operatorname{Hom}\mathcal{A}$ and the triple satisfies the following.

(a) We associate to each pair $A, B \in \operatorname{Ob}\mathcal{A}$ the *set of morphisms* from $A$ to $B$, denoted by $\operatorname{Hom}_{\mathcal{A}}(A,B)$, such that if $(A,B) \neq (C,D)$, then $\operatorname{Hom}_{\mathcal{A}}(A,B) \cap \operatorname{Hom}_{\mathcal{A}}(C,D) = \varnothing$.

(b) For each triple $A$, $B$, $C \in \operatorname{Ob}\mathcal{A}$ the operation

$$\circ : \operatorname{Hom}_{\mathcal{A}}(B,C) \times \operatorname{Hom}_{\mathcal{A}}(A,B) \to \operatorname{Hom}_{\mathcal{A}}(A,C), \text{ given by } (\beta,\alpha) \mapsto \beta \circ \alpha$$

is defined (we call it the *composition* of $\alpha$ and $\beta$) and has the following properties.

- Associativity: $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$ for every triple $\alpha \in \mathrm{Hom}_{\mathcal{A}}(A, B)$, $\beta \in \mathrm{Hom}_{\mathcal{A}}(B, C)$, $\gamma \in \mathrm{Hom}_{\mathcal{A}}(C, D)$ and
- Existence of the *identity morphism* $1_A$: for each $A \in \mathrm{Ob}\,\mathcal{A}$, there exists $1_A \in \mathrm{Hom}_{\mathcal{A}}(A, A)$ such that for every $\alpha \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ and $\gamma \in \mathrm{Hom}_{\mathcal{A}}(C, A)$ we have $\alpha \circ 1_A = \alpha$ and $1_A \circ \gamma = \gamma$.

**Notation 2.1.2.** We often write $\alpha : A \to B$ instead of $\alpha \in \mathrm{Hom}_{\mathcal{A}}(A, B)$. Moreover, we often write $A = A$ in diagrams to mean $1_A$.

**Example 2.1.3.** Abelian groups together with group homomorphisms and their usual composition form a category, which we denote by Ab.

#### Some important categories

Most of the categories we will study in this thesis have some extra structure. In particular, we will almost always assume our categories are $k$-linear additive categories for some field $k$. In addition, we will also often assume some of the following: they are skeletally small, they have split idempotents, their Hom spaces are finite dimensional over the given field $k$. We recall here the relevant definitions.

**Definition 2.1.4.** Let $k$ be a field. A category $\mathcal{A}$ is called a *$k$-linear category* if for each pair of objects $A$, $B$ in $\mathcal{A}$, the set $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is equipped with a $k$-vector space structure such that the composition of morphisms in $\mathcal{A}$ is a $k$-bilinear map.

**Definition 2.1.5.** A category $\mathcal{A}$ is an *additive category* if the following conditions are satisfied.

(a) For any finite set of objects $A_1, A_2, \ldots, A_n$ in $\mathcal{A}$, there exists a direct sum $A_1 \oplus A_2 \oplus \cdots \oplus A_n$.

(b) For each pair of objects $A$, $B$ in $\mathcal{A}$, the set $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is equipped with an abelian group structure.

(c) For each triple of objects $A$, $B$, $C$ in $\mathcal{A}$, the composition

$$\circ : \mathrm{Hom}_{\mathcal{A}}(B, C) \times \mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{A}}(A, C)$$

is bilinear.

(d) There exists an object $0$ in $\mathcal{A}$, called the *zero object*, such that $1_0$ is the element zero of the abelian group $\mathrm{Hom}_{\mathcal{A}}(0, 0)$.

**Remark 2.1.6.** If $\mathcal{A}$ is a $k$-linear additive category, then the $k$-linear and additive structures have the same addition of morphisms, denoted by "+".

**Definition 2.1.7** ([46, Chapter VIII.2])**.** Let $\mathcal{A}$ be a category. A *biproduct diagram* for two objects $A$ and $B$ in $\mathcal{A}$ is a diagram of the form

$$\begin{tikzcd} A \arrow[r, bend left, "\iota"] & C \arrow[l, bend left, "\pi"] \arrow[r, bend right, "\pi'"'] & B, \arrow[l, bend right, "\iota'"'] \end{tikzcd} \tag{2.1}$$

where $\pi \circ \iota = 1_A$, $\pi' \circ \iota' = 1_B$ and $\iota \circ \pi + \iota' \circ \pi' = 1_C$. By [46, Theorem VIII.2], two objects $A$, $B$ in $\mathcal{A}$ have a product if and only if they have a biproduct diagram (2.1) if and only if they have a coproduct, or a direct sum, and in this case $C = A \oplus B$. Specifically, given a biproduct diagram (2.1), the object $C$ together with $\iota$ and $\iota'$ is a coproduct of $A$ and $B$, while $C$ together with $\pi$ and $\pi'$ is a product of $A$ and $B$. Conversely, each coproduct $C$ of $A$ and $B$ with inclusions $\iota$ and $\iota'$ can be augmented to a biproduct diagram and so can each product of $A$ and $B$ with projections $\pi$ and $\pi'$. Note that if $\mathcal{A}$ is additive, then any two objects have a product, a coproduct and a biproduct diagram.

We recall the definition of (Jacobson) radical of an additive category, see [42, Lemma 6].

**Definition 2.1.8.** Let $\mathcal{A}$ be an additive category. The *(Jacobson) radical of* $\mathcal{A}$ is the two sided ideal $\mathrm{rad}_{\mathcal{A}}$ in $\mathcal{A}$ defined by the formula

$$\mathrm{rad}_{\mathcal{A}}(A, B) = \{\alpha : A \to B \mid 1_A - \beta \circ \alpha \text{ is invertible for any } \beta : B \to A\},$$

for all objects $A$ and $B$ in $\mathcal{A}$.

**Definition 2.1.9.** A category $\mathcal{A}$ is said to be *skeletally small* if the collection of isomorphism classes of objects is a set.

**Definition 2.1.10.** Let $\mathcal{A}$ be a category and $A$ be an object in $\mathcal{A}$. A morphism $e \in \mathrm{Hom}_{\mathcal{A}}(A, A)$ is called an *idempotent* if $e^2 = e \circ e = e$. We say that the category $\mathcal{A}$ *has split idempotents* if for any object $A$ in $\mathcal{A}$ and any idempotent $e \in \mathrm{Hom}_{\mathcal{A}}(A, A)$, there is an object $B$ in $\mathcal{A}$ and morphisms $\pi : A \to B$ and $\iota : B \to A$ such that $\iota \circ \pi = e$ and $\pi \circ \iota = 1_B$.

**Definition 2.1.11.** Let $\mathcal{A}$ be an additive category. We say that an object $A \in \mathcal{A}$ is *indecomposable* if when written as a direct sum of the form $A = A_1 \oplus A_2$, we have that either $A_1 = 0$ or $A_2 = 0$.

**Remark 2.1.12.** Let $k$ be a field and $\mathcal{A}$ be a skeletally small, $k$-linear additive category with split idempotents and finite dimensional Hom spaces over $k$. By [45, Corollary 4.4], we have that $\mathcal{A}$ is a *Krull-Schmidt category*, in the sense that for each object $A$ in $\mathcal{A}$, there is a finite direct sum decomposition of the form $A = A_1 \oplus \cdots \oplus A_n$, where $A_i$ is indecomposable for each $i = 1, \dots, n$. Note that this is equivalent to $\mathrm{Hom}_{\mathcal{A}}(A_i, A_i)$ being a local ring for each $i$. Moreover, by [45, Theorem 4.2] the objects $A_1, \dots, A_n$ are determined uniquely up to isomorphism.

### 2.1.2 Functors

We recall the definition of functors, which are maps between categories that send objects to objects and morphisms to morphisms.

**Definition 2.1.13.** A *covariant functor* $F : \mathcal{A} \to \mathcal{B}$ between two categories $\mathcal{A}$ and $\mathcal{B}$ is defined by assigning to each $A \in \mathrm{Ob}\,\mathcal{A}$ some $F(A) \in \mathrm{Ob}\,\mathcal{B}$ and to each $\alpha : A \to A'$ in $\mathrm{Hom}\,\mathcal{A}$ some $F(\alpha) : F(A) \to F(A')$ in $\mathrm{Hom}\,\mathcal{B}$ such that the following conditions hold.

(a) For every $A \in \mathrm{Ob}\,\mathcal{A}$, $F(1_A) = 1_{F(A)}$.

(b) For each pair of morphisms $\alpha : A \to A'$ and $\alpha' : A' \to A''$ in $\mathrm{Hom}\,\mathcal{A}$ we have $F(\alpha' \circ \alpha) = F(\alpha') \circ F(\alpha)$.

A *contravariant functor* $G : \mathcal{A} \to \mathcal{B}$ between two categories is defined by assigning to each $A \in \mathrm{Ob}\,\mathcal{A}$ some $G(A) \in \mathrm{Ob}\,\mathcal{B}$ and to each $\alpha : A \to A'$ in $\mathrm{Hom}\,\mathcal{A}$ some $G(\alpha) : G(A') \to G(A)$ in $\mathrm{Hom}\,\mathcal{B}$ such that the following hold.

(a) For every $A \in \mathrm{Ob}\,\mathcal{A}$, $G(1_A) = 1_{G(A)}$.

(b) For each pair of morphisms $\alpha : A \to A'$ and $\alpha' : A' \to A''$ in $\mathrm{Hom}\,\mathcal{A}$ we have $G(\alpha' \circ \alpha) = G(\alpha) \circ G(\alpha')$.

Note that covariant functors preserve the direction of the arrows while contravariant functors reverse it.

**Definition 2.1.14.** We say that a covariant functor $F : \mathcal{A} \to \mathcal{B}$ is a *full functor* if for any objects $A$ and $A'$ in $\mathcal{A}$, the map

$$F_{AA'} : \mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(F(A), F(A')),$$

sending a morphism $\alpha$ to $F(\alpha)$ is surjective. Similarly, if the map $F_{AA'}$ is injective for any objects $A$ and $A'$ in $\mathcal{A}$, then we say that $F$ is a *faithful functor*.

**Definition 2.1.15.** Let $\mathcal{A}$ be a category. Then $1_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ sending each object and each morphism to itself is a covariant functor called the *identity functor of* $\mathcal{A}$. A functor $F : \mathcal{A} \to \mathcal{A}$ is an *automorphism* if there exists a functor $F^{-1} : \mathcal{A} \to \mathcal{A}$ such that $F \circ F^{-1} = 1_{\mathcal{A}} = F^{-1} \circ F$.

**Definition 2.1.16.** A functor $F : \mathcal{A} \to \mathcal{B}$ between two additive categories is called an *additive functor* if $F(\alpha + \beta) = F(\alpha) + F(\beta)$ when this equation makes sense. Note that additive functors preserve finite coproducts.

We will use the following result in later sections.

**Lemma 2.1.17.** Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories and $\mathcal{A}$ have split idempotents. Suppose $F : \mathcal{A} \to \mathcal{B}$ is a full additive functor, then $F(\mathcal{A})$ is closed under direct summands.

*Proof.* Let $A \in \mathcal{A}$ satisfy $F(A) = X \oplus Y$. Then, we have a biproduct diagram:

$$\begin{tikzcd} X \arrow[r, shift left, "i"] & F(A) \arrow[l, shift left, "p"] \arrow[r, shift right, "q"'] & Y, \arrow[l, shift right, "j"'] \end{tikzcd} \tag{2.2}$$

where $p \circ i = 1_X$ and $q \circ j = 1_Y$. Also, $e = i \circ p$ and $1_{F(A)} - e = j \circ q$ are idempotents in $\mathrm{End}_{\mathcal{B}}(F(A))$. Now, as $F$ is full, there is an idempotent $e'$ in $\mathrm{End}_{\mathcal{A}}(A)$ such that $F(e') = e$, and $F(1_A - e') = 1_{F(A)} - e$. Since $\mathcal{A}$ has split idempotents, we get a biproduct diagram:

$$\begin{tikzcd} X' \arrow[r, shift left, "i'"] & A \arrow[l, shift left, "p'"] \arrow[r, shift right, "q'"'] & Y', \arrow[l, shift right, "j'"'] \end{tikzcd}$$

where $p' \circ i' = 1_{X'}$, $q' \circ j' = 1_{Y'}$, $i' \circ p' = e'$ and $j' \circ q' = 1_A - e'$. Applying $F$ to this, we get:

$$\begin{tikzcd} F(X') \arrow[r, shift left, "F(i')"] & F(A) \arrow[l, shift left, "F(p')"] \arrow[r, shift right, "F(q')"'] & F(Y'). \arrow[l, shift right, "F(j')"'] \end{tikzcd}$$

We show that $F(p') \circ i : X \to F(X')$ and $p \circ F(i') : F(X') \to X$ are mutually inverse isomorphisms, so that $F(X') \cong X$. First note that

$$i \circ p = e = F(i') \circ F(p'), \text{ and } F(p') \circ F(i') = F(p' \circ i') = F(1_{X'}) = 1_{F(X')}.$$

Then, recalling also that $p \circ i = 1_X$, we have that

$$\begin{aligned} &(F(p') \circ i) \circ (p \circ F(i')) = F(p') \circ (i \circ p) \circ F(i') = F(p') \circ F(i') \circ F(p') \circ F(i') = 1_{F(X')}, \\ &(p \circ F(i')) \circ (F(p') \circ i) = p \circ (F(i') \circ F(p')) \circ i = p \circ i \circ p \circ i = 1_X. \end{aligned}$$

Hence $X \cong F(X')$ and similarly, $Y \cong F(Y')$. $\square$

We omit a description of the theory of algebras and modules in this thesis, see [2, Chapter I] for example for details.

**Notation 2.1.18.** Let $k$ be a field and $\Lambda$ a finite dimensional associative $k$-algebra. Unless otherwise specified, we assume that $\Lambda$-modules of any $k$-algebra $\Lambda$ are right $\Lambda$-modules. Right (respectively left) $\Lambda$-modules together with their module morphisms form a category, denoted by $\operatorname{Mod}\Lambda$ (respectively $\operatorname{Mod}\Lambda^{op}$). Moreover, the category of finitely generated right $\Lambda$-modules is denoted by $\operatorname{mod}\Lambda$ and the one of finitely generated left $\Lambda$-modules is denoted by $\operatorname{mod}\Lambda^{op}$.

**Example 2.1.19.** Let $k$ be a field and $\Lambda$ be an associative algebra over $k$. Let $M$ be a right $\Lambda$-module and note that for any right $\Lambda$-module $N$, we have that the set of morphisms from $M$ to $N$, denoted by $\operatorname{Hom}_\Lambda(M,N)$ is a $k$-vector space and $\operatorname{Mod}\Lambda$ is a $k$-linear category.

(a) Consider $\operatorname{Hom}_\Lambda(M,-) : \operatorname{Mod}\Lambda \to \operatorname{Mod}k$, that is the map sending a right $\Lambda$-module $N$ to the $k$-vector space $\operatorname{Hom}_\Lambda(M,N)$ and a module morphism $f : N \to L$ to the $k$-linear map $\operatorname{Hom}_\Lambda(M,f) : \operatorname{Hom}_\Lambda(M,N) \to \operatorname{Hom}_\Lambda(M,L)$ given by $\operatorname{Hom}_\Lambda(M,f)(g) = f \circ g$, for $g \in \operatorname{Hom}_\Lambda(M,N)$. Then $\operatorname{Hom}_\Lambda(M,-)$ is a covariant functor.

(b) Consider $\operatorname{Hom}_\Lambda(-,M) : \operatorname{Mod}\Lambda \to \operatorname{Mod}k$, that is the map sending a right $\Lambda$-module $N$ to the $k$-vector space $\operatorname{Hom}_\Lambda(N,M)$ and a module morphism $f : N \to L$ to the $k$-linear map $\operatorname{Hom}_\Lambda(f,M) : \operatorname{Hom}_\Lambda(L,M) \to \operatorname{Hom}_\Lambda(N,M)$ given by $\operatorname{Hom}_\Lambda(f,M)(g) = g \circ f$. Then, $\operatorname{Hom}_\Lambda(-,M)$ is a contravariant functor.

**Definition 2.1.20.** Let $\mathcal{A}$ and $\mathcal{B}$ be categories and $F$, $G : \mathcal{A} \to \mathcal{B}$ be functors, say both covariant. Let $\eta = \{\eta_A\}_{A\in\operatorname{Ob}\mathcal{A}}$ be a family of morphisms in $\mathcal{B}$ such that for each $A \in \operatorname{Ob}\mathcal{A}$, we have that $\eta_A \in \operatorname{Hom}_\mathcal{B}(F(A), G(A))$. We say that $\eta$ is a *functorial morphism*, also known as a *natural transformation*, if for each $A$, $A' \in \operatorname{Ob}\mathcal{A}$ and each $\alpha \in \operatorname{Hom}_\mathcal{A}(A, A')$ the following diagram commutes

$$\begin{tikzcd}
F(A) \arrow[r, "\eta_A"] \arrow[d, "F(\alpha)"'] & G(A) \arrow[d, "G(\alpha)"] \\
F(A') \arrow[r, "\eta_{A'}"] & G(A').
\end{tikzcd}$$

If in addition $\eta_A$ is an isomorphism for each $A \in \operatorname{Ob}\mathcal{A}$, we say that $\eta$ is a *functorial isomorphism*, also known as a *natural equivalence.*

**Definition 2.1.21.** We say that a covariant functor $F : \mathcal{A} \to \mathcal{B}$ is an *equivalence of categories* if there is a functor $G : \mathcal{B} \to \mathcal{A}$ and functorial isomorphisms $\psi : F \circ G \xrightarrow{\simeq} 1_\mathcal{B}$ and $\phi : G \circ F \xrightarrow{\simeq} 1_\mathcal{A}$. In this case we say that $G$ is a *quasi-inverse* of $F$ and $\mathcal{A}$ and $\mathcal{B}$ are *equivalent categories* (and write $\mathcal{A} \cong \mathcal{B}$).

**Definition 2.1.22.** We say that a contravariant functor $F : \mathcal{A} \to \mathcal{B}$ is a *duality of categories* if there is a functor $G : \mathcal{B} \to \mathcal{A}$ and functorial isomorphisms $\psi : F \circ G \xrightarrow{\simeq} 1_\mathcal{B}$ and $\phi : G \circ F \xrightarrow{\simeq} 1_\mathcal{A}$. In this case, we say that $G$ is a *quasi-inverse* of $F$.

**Example 2.1.23.** Let $k$ be a field and $\Lambda$ be a finite dimensional associative $k$-algebra. Finitely generated right, respectively left, $\Lambda$-modules together with module morphisms form a category, denoted by $\operatorname{mod}\Lambda$, respectively by $\operatorname{mod}\Lambda^{op}$. Let $D(-) = \operatorname{Hom}_k(-,k) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{op}$ be the contravariant functor which assigns to each $M$ in $\operatorname{mod}\Lambda$ the dual $k$-vector space $D(M) = \operatorname{Hom}_k(M,k)$ together with left $\Lambda$-module structure given by $(af)(m) = f(ma)$ for any $f \in \operatorname{Hom}_k(M,k)$, $a \in \Lambda$ and $m \in M$; and to each $h \in \operatorname{Hom}_\Lambda(M,N)$, the dual $k$-homomorphism $D(h) = \operatorname{Hom}_k(h,k) : D(N) \to D(M)$ given by $D(h)(\varphi) = \varphi \circ h$. Then $D(-)$ is a duality of categories, called the *standard $k$-duality* with quasi-inverse $D(-) = \operatorname{Hom}_k(-,k) : \operatorname{mod}\Lambda^{op} \to \operatorname{mod}\Lambda$ defined similarly.

We end this section by introducing the Serre functor. This functor plays an important role in Auslander-Reiten theory and we will widely use it in later sections and chapters of this thesis.

**Definition 2.1.24.** Let $k$ be a field and $\mathcal{A}$ be a $k$-linear additive category with finite dimensional Hom spaces over $k$. An additive functor $S : \mathcal{A} \to \mathcal{A}$ is called a *Serre functor* if it is an auto-equivalence and there are isomorphisms

$$\operatorname{Hom}_{\mathcal{A}}(A,B) \cong D \circ \operatorname{Hom}_{\mathcal{A}}(B,SA),$$

functorial in $A$, $B \in \operatorname{Ob}\mathcal{A}$, where $D(-) = \operatorname{Hom}_k(-,k)$.

### 2.1.3 Some important morphisms

We define isomorphisms, monomorphisms and epimorphisms and what it means for monomorphisms and epimorphisms to split. We also introduce irreducible morphisms and left and right almost split morphisms, which we will widely use in later sections.

**Definition 2.1.25.** Let $\mathcal{A}$ be a category and $A$, $B$ be objects in $\mathcal{A}$. A morphism $\alpha : A \to B$ is called an *isomorphism* if there exists a morphism $\beta : B \to A$ such that $\alpha \circ \beta = 1_B$ and $\beta \circ \alpha = 1_A$. In this case, the morphism $\beta$ is uniquely determined by $\alpha$, it is called the *inverse* of $\alpha$ and it is denoted by $\alpha^{-1}$. Moreover, we say that $A$ and $B$ are *isomorphic* and write $A \cong B$. A morphism of the form $\alpha : A \to A$ is called an *endomorphism* and we define $\operatorname{End}_{\mathcal{A}}(A) := \operatorname{Hom}_{\mathcal{A}}(A,A)$. An endomorphism that is also an isomorphism is called an *automorphism*.

**Definition 2.1.26.** Let $\mathcal{A}$ be a category and $A$, $B$ be objects in $\mathcal{A}$. A morphism $\alpha : A \to B$ is called a *monomorphism* if for each object $C$ in $\mathcal{A}$ and each pair of morphisms $\gamma, \gamma' \in \operatorname{Hom}_{\mathcal{A}}(C,A)$ such that $\alpha \circ \gamma = \alpha \circ \gamma'$, we have that $\gamma = \gamma'$. Dually, a morphism $\alpha : A \to B$ is called an *epimorphism* if for each object $C$ in $\mathcal{A}$ and each pair of morphisms $\beta, \beta' \in \operatorname{Hom}_{\mathcal{A}}(B,C)$ such that $\beta \circ \alpha = \beta' \circ \alpha$, we have that $\beta = \beta'$.

**Definition 2.1.27.** Let $\mathcal{A}$ be a category and $A$, $B$ be objects in $\mathcal{A}$. A morphism $\alpha : A \to B$ is called a *split monomorphism* (or *section*) if there exists a morphism $\beta : B \to A$ such that $\beta \circ \alpha = 1_A$. Dually, a morphism $\beta : B \to A$ is called a *split epimorphism* (or *retraction*) if there exists a morphism $\alpha : A \to B$ such that $\beta \circ \alpha = 1_A$.

**Remark 2.1.28.** Note that a split monomorphism is a monomorphism and a split epimorphism is an epimorphism.

**Definition 2.1.29.** Let $\mathcal{A}$ be a category and $A$, $B$ be objects in $\mathcal{A}$. A morphism $\alpha : A \to B$ is called an *irreducible morphism* if

(a) $\alpha$ is neither a split monomorphism nor a split epimorphism,

(b) whenever $\alpha = \alpha_2 \circ \alpha_1$ for some object $C \in \mathcal{A}$ and morphisms $\alpha_1 : A \to C$ and $\alpha_2 : C \to B$, then either $\alpha_1$ is a split monomorphism or $\alpha_2$ is a split epimorphism.

We define left (and right) minimal morphism, see for example [2, Definition 1.1, Chapter IV].

**Definition 2.1.30.** A morphism $\alpha : A \to B$ in a category $\mathcal{A}$ is *left minimal* if each morphism $\eta : B \to B$ which satisfies $\eta \circ \alpha = \alpha$ is an isomorphism. Dually, $\alpha$ is *right minimal* if each morphism $\varphi : A \to A$ which satifies $\alpha \circ \varphi = \alpha$ is an isomorphism.

**Definition 2.1.31.** Let $\mathcal{A}$ be a category and $A$, $B$ and $C$ be objects in $\mathcal{A}$.

(a) A morphism $\alpha : A \to B$ is *left almost split in* $\mathcal{A}$ if it is not a split monomorphism and for every $A'$ in $\mathcal{A}$, every morphism $\alpha' : A \to A'$ which is not a split monomorphism factors through $\alpha$, **i.e.** there exists a morphism $B \to A'$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "\alpha'"'] & & B. \arrow[dl, dashed, "\exists"] \\ & A' & \end{tikzcd}$$

(b) A morphism $\beta : B \to C$ is *right almost split in* $\mathcal{A}$ if it is not a split epimorphism and for every $C'$ in $\mathcal{C}$, every morphism $\gamma : C' \to C$ which is not a split epimorphism factors through $\beta$, **i.e.** there exists a morphism $C' \to B$ such that the following diagram commutes:

$$\begin{tikzcd} B \arrow[rr, "\beta"] & & C. \\ & C' \arrow[ul, dashed, "\exists"] \arrow[ur, "\gamma"'] & \end{tikzcd}$$

**Definition 2.1.32.** A morphism in a category $\mathcal{A}$ is *minimal left almost split in* $\mathcal{A}$ if it is both left minimal and left almost split in $\mathcal{A}$. Similarly, a morphism is *minimal right almost split in* $\mathcal{A}$ if it is both right minimal and right almost split in $\mathcal{A}$.

**Notation 2.1.33.** If the category $\mathcal{A}$ we are working in is clear, we sometimes omit it and just say that a morphism is (minimal) left or right almost split.

**Lemma 2.1.34.** Let $\mathcal{A}$ be a category and $A$ be an object in $\mathcal{A}$.

(a) Suppose there exists a minimal left almost split morphism $\alpha : A \to B$ in $\mathcal{A}$. Then $\alpha$ is unique up to isomorphism in the sense that if $\alpha' : A \to B'$ is another minimal left almost split morphism in $\mathcal{A}$, then there exists an isomorphism $\varphi : B \to B'$ such that $\varphi \circ \alpha = \alpha'$.

(b) Suppose there exists a minimal right almost split morphism $\gamma : C \to A$ in $\mathcal{A}$. Then $\gamma$ is unique up to isomorphism in the sense that if $\gamma' : C' \to A$ is another minimal right almost split morphism in $\mathcal{A}$, then there exists an isomorphism $\psi : C \to C'$ such that $\gamma' \circ \psi = \gamma$.

*Proof.* We only prove (a) as (b) follows by a dual argument. Since $\alpha$ and $\alpha'$ are left almost split in $\mathcal{A}$, there are morphisms $\varphi : B \to B'$ and $\varphi' : B' \to B$ such that $\varphi \circ \alpha = \alpha'$ and $\varphi' \circ \alpha' = \alpha$. Then $\alpha = \varphi' \circ \varphi \circ \alpha$ and $\alpha' = \varphi \circ \varphi' \circ \alpha'$. By left minimality of $\alpha$ and $\alpha'$, it follows that $\varphi \circ \varphi'$ and $\varphi' \circ \varphi$ are isomorphisms and so $\varphi$ is an isomorphism. □

### 2.1.4 Subcategories

In this thesis, we will often work with subcategories and we will usually assume these subcategories are full.

**Definition 2.1.35.** Let $\mathcal{A}$ be a category. A category $\mathcal{X}$ is called a *subcategory of* $\mathcal{A}$ if the following are satisfied.

(a) The class $\operatorname{Ob}\mathcal{X}$ is a subclass of the class $\operatorname{Ob}\mathcal{A}$.

(b) If $X$, $Y$ are objects in $\mathcal{X}$, then $\operatorname{Hom}_{\mathcal{X}}(X,Y) \subseteq \operatorname{Hom}_{\mathcal{A}}(X,Y)$.

(c) The composition of morphisms in $\mathcal{X}$ is the same as in $\mathcal{A}$.

(d) For each object $X$ in $\mathcal{X}$, the identity morphism in $\operatorname{Hom}_{\mathcal{X}}(X,X)$ coincides with the identity morphism in $\operatorname{Hom}_{\mathcal{A}}(X,X)$.

If, in addition, we have that $\operatorname{Hom}_{\mathcal{X}}(X,Y) = \operatorname{Hom}_{\mathcal{A}}(X,Y)$ for all objects $X$, $Y$ in $\mathcal{X}$, then $\mathcal{X}$ is called a *full subcategory of* $\mathcal{A}$.

**Example 2.1.36.** Let $k$ be a field and $\Lambda$ a finite dimensional associative $k$-algebra. Following Notation 2.1.18, we have that $\operatorname{mod}\Lambda \subseteq \operatorname{Mod}\Lambda$ and $\operatorname{mod}\Lambda^{op} \subseteq \operatorname{Mod}\Lambda^{op}$ are full subcategories.

We will also often assume that subcategories are precovering and/or preenveloping. We hence recall the definitions of precovers, covers, precovering subcategories and their dual notions, see for example [37, Definition 1.4].

**Definition 2.1.37.** Let $\mathcal{A}$ be a category and $\mathcal{X}$ a full subcategory of $\mathcal{A}$. An *$\mathcal{X}$-precover* (or *right $\mathcal{X}$-approximation*) of an object $A$ in $\mathcal{A}$ is a morphism of the form $\xi : X \to A$ with $X \in \mathcal{X}$ such that every morphism $\xi' : X' \to A$ with $X' \in \mathcal{X}$ factorizes as:

$$\begin{tikzcd} X' \arrow[rr, "\xi'"] \arrow[dr, dashed, "\exists"'] & & A. \\ & X \arrow[ur, "\xi"'] & \end{tikzcd}$$

An *$\mathcal{X}$-cover* (or *minimal right $\mathcal{X}$-approximation*) of $A$ is an $\mathcal{X}$-precover of $A$ which is also a right minimal morphism.

An *$\mathcal{X}$-preenvelope* (or *left $\mathcal{X}$-approximation*) of an object $A$ in $\mathcal{A}$ is a morphism of the form $\eta : A \to Y$ with $Y \in \mathcal{X}$ such that every morphism $\eta' : A \to Y'$ with $Y' \in \mathcal{X}$ factorizes as:

$$\begin{tikzcd} A \arrow[rr, "\eta'"] \arrow[dr, "\eta"'] & & Y'. \\ & Y \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

An *$\mathcal{X}$-envelope* (or *minimal left $\mathcal{X}$-approximation*) of $A$ is an $\mathcal{X}$-preenvelope of $A$ which is also a left minimal morphism.

**Definition 2.1.38.** The full subcategory $\mathcal{X}$ of $\mathcal{A}$ is called *precovering* (or *contravariantly finite*) if every object in $\mathcal{A}$ has an $\mathcal{X}$-precover. Dually, it is called *preenveloping* (or *covariantly finite*) if every object in $\mathcal{A}$ has an $\mathcal{X}$-preenvelope. If $\mathcal{X}$ is both precovering and preenveloping, it is called *functorially finite* in $\mathcal{A}$.

Often, subcategories of additive subcategories will be assumed to be additive subcategories in the following sense.

**Definition 2.1.39.** Let $\mathcal{A}$ be an additive category. An *additive subcategory of $\mathcal{A}$* is a full subcategory which is closed under direct sums, direct summands and isomorphisms in $\mathcal{A}$.

An important example of an additive subcategory we will often use is the following.

**Definition 2.1.40.** Let $\mathcal{A}$ be an additive category and $A$ be an object in $\mathcal{A}$. Then add$(A)$ is defined to be the additive subcategory of $\mathcal{A}$ whose objects are direct summands of direct sums of copies of $A$.

## 2.2 Homological algebra

Auslander-Reiten theory plays an important role in the study of abelian and triangulated categories and their higher analogues. In this section, we work in the setup of classic homological algebra, with abelian and triangulated categories.

### 2.2.1 Abelian categories

We first define general abelian categories and we then focus on module categories.

**Definition 2.2.1.** Let $\mathcal{A}$ be an additive category and $\alpha : A \to B$ be a morphism in $\mathcal{A}$. A *kernel of* $\alpha$ is an object $\operatorname{Ker}\alpha$ in $\mathcal{A}$ together with a morphism $\iota : \operatorname{Ker}\alpha \to A$ satisfying the following two conditions:

(a) $\alpha \circ \iota = 0$,

(b) for any object $C$ of $\mathcal{A}$ and any morphism $\gamma : C \to A$ such that $\alpha \circ \gamma = 0$, there exists a unique morphism $\gamma' : C \to \operatorname{Ker}\alpha$ such that the following diagram commutes

$$\begin{tikzcd} C \arrow[rr, "\gamma"] \arrow[dr, dashed, "\gamma'"'] & & A \arrow[r, "\alpha"] & B. \\ & \operatorname{Ker}\alpha \arrow[ur, "\iota"'] & & \end{tikzcd}$$

A *cokernel of* $\alpha$ is an object $\operatorname{Coker}\alpha$ in $\mathcal{A}$ together with a morphism $\pi : B \to \operatorname{Coker}\alpha$ satisfying the following two conditions:

(a) $\pi \circ \alpha = 0$,

(b) for any object $C$ of $\mathcal{A}$ and any morphism $\beta : B \to C$ such that $\beta \circ \alpha = 0$, there exists a unique morphism $\gamma : \operatorname{Coker}\alpha \to C$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\alpha"] & B \arrow[rr, "\beta"] \arrow[dr, "\pi"'] & & C. \\ & & \operatorname{Coker}\alpha \arrow[ur, dashed, "\gamma"'] & \end{tikzcd}$$

**Remark 2.2.2** ([2, pp 408, Appendix A.1])**.** Let $\mathcal{A}$ be an additive category such that each morphism in $\mathcal{A}$ admits a kernel and a cokernel. For any morphism $\alpha : A \to B$ in

$\mathcal{A}$, by Definition 2.2.1, there exists a unique morphism $\overline{\alpha}$ such that the following diagram commutes

$$\begin{tikzcd}
\operatorname{Ker}\alpha \arrow[r, "\iota"] & A \arrow[r, "\alpha"] \arrow[d, "\pi'"'] & B \arrow[r, "\pi"] & \operatorname{Coker}\alpha. \\
& \operatorname{Coker}\iota \arrow[r, "\overline{\alpha}"'] & \operatorname{Ker}\pi \arrow[u, "\iota'"'] &
\end{tikzcd}$$

Moreover, the object $\operatorname{Ker}\pi$ is called the *image of* $\alpha$ and it is denoted by $\operatorname{Im}\alpha$.

**Definition 2.2.3.** A category $\mathcal{A}$ is called an *abelian category* if

(a) $\mathcal{A}$ is additive,

(b) each morphism $\alpha : A \to B$ in $\mathcal{A}$ admits a kernel $\iota : \operatorname{Ker}\alpha \to A$ and a cokernel $\pi : B \to \operatorname{Coker}\alpha$ and the induced morphism $\overline{\alpha} : \operatorname{Coker}\iota \to \operatorname{Ker}\pi$ from Remark 2.2.2 is an isomorphism.

**Definition 2.2.4.** Let $\mathcal{A}$ be an abelian category. We say that a sequence

$$A = \cdots \to A_{i+1} \xrightarrow{\alpha_{i+1}} A_i \xrightarrow{\alpha_i} A_{i-1} \to \cdots$$

of objects and morphisms in $\mathcal{A}$ is a *chain complex* if $\alpha_i \circ \alpha_{i+1} = 0$ for all $i$. A *chain map* between two chain complexes $A$ and $B$ over $\mathcal{A}$ is a collection of morphisms $f = \{f_i : A_i \to B_i\}$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & A_{i+1} \arrow[r, "\alpha_{i+1}"] \arrow[d, "f_{i+1}"] & A_i \arrow[r, "\alpha_i"] \arrow[d, "f_i"] & A_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\
\cdots \arrow[r] & B_{i+1} \arrow[r, "\beta_{i+1}"'] & B_i \arrow[r, "\beta_i"'] & B_{i-1} \arrow[r] & \cdots.
\end{tikzcd}$$

Chain complexes together with chain maps form an abelian category, denoted by $C(\mathcal{A})$, see [56, Chapter 1, Theorem 1.2.3].

**Definition 2.2.5.** Let $\mathcal{A}$ be an abelian category and consider a chain complex

$$A = \cdots \to A_{i+1} \xrightarrow{\alpha_{i+1}} A_i \xrightarrow{\alpha_i} A_{i-1} \to \cdots$$

in $\mathcal{A}$. The $i^{th}$ *homology of* $A$ is defined to be

$$H_i(A) := \operatorname{Coker}(\operatorname{Im}\alpha_{i+1} \xrightarrow{\varphi} \operatorname{Ker}\alpha_i),$$

where $\varphi$ is the morphism induced by $\alpha_{i+1}$ and $\alpha_i$. Note that if $f : A \to B$ is a chain map, then $f$ induces a morphism $H_i(f) : H_i(A) \to H_i(B)$ for every $i$ and $H_i(-) : C(\mathcal{A}) \to \mathcal{A}$

is a well-defined covariant functor. We say that $A$ is an *exact sequence* if each morphism $\varphi : \operatorname{Im}\alpha_{i+1} \to \operatorname{Ker}\alpha_i$ is an isomorphism, or equivalently $H_i(A) \cong 0$, for all $i$. An exact sequence of the form

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is called a *short exact sequence.*

**Remark 2.2.6.** Let $k$ be a field and $\Lambda$ a finite dimensional $k$-algebra. Then the category of right $\Lambda$-modules, denoted by $\operatorname{Mod}\Lambda$, is abelian. Consider a chain complex

$$A = \cdots \to A_{i+1} \xrightarrow{\alpha_{i+1}} A_i \xrightarrow{\alpha_i} A_{i-1} \to \cdots$$

in $\operatorname{Mod}\Lambda$. We have that $\alpha_i \circ \alpha_{i+1} = 0$ implies that $\operatorname{Im}\alpha_{i+1}$ is isomorphic to a submodule of $\operatorname{Ker}\alpha_i$ and the $i^{th}$ homology of $A$ is the quotient module $H_i(A) = \operatorname{Ker}\alpha_i / \operatorname{Im}\alpha_{i+1}$. Then, for any integer $i$, the $i^{th}$ *homology functor* is the covariant functor $H_i(-) : C(\operatorname{Mod}\Lambda) \to \operatorname{Mod}\Lambda$ sending a chain complex

$$A = \cdots \to A_{i+1} \xrightarrow{\alpha_{i+1}} A_i \xrightarrow{\alpha_i} A_{i-1} \to \cdots$$

to the module $H_i(A) = \operatorname{Ker}\alpha_i / \operatorname{Im}\alpha_{i+1}$ and a chain map

$$\begin{tikzcd} A = \arrow[d, "f"] & \cdots \arrow[r] & A_{i+1} \arrow[r, "\alpha_{i+1}"] \arrow[d, "f_{i+1}"] & A_i \arrow[r, "\alpha_i"] \arrow[d, "f_i"] & A_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\ B = & \cdots \arrow[r] & B_{i+1} \arrow[r, "\beta_{i+1}"'] & B_i \arrow[r, "\beta_i"'] & B_{i-1} \arrow[r] & \cdots \end{tikzcd}$$

to the morphism

$$H_i(f) : H_i(A) \to H_i(B) : (a + \operatorname{Im}\alpha_{i+1} \mapsto f_i(a) + \operatorname{Im}\beta_{i+1}),$$

see [25, Definition 7.1].

**Notation 2.2.7.** If instead of subscripts and descending indices, we use superscripts and ascending indices for our sequences, then we talk about *cochain complexes in $\mathcal{A}$* of the form

$$A = \cdots \to A^{i-1} \xrightarrow{\alpha^{i-1}} A^i \xrightarrow{\alpha^i} A^{i+1} \to \cdots$$

and cochain maps. Moreover, in this case we use the $i^{th}$ *cohomology of* $A$, denoted by $H^i(A)$ instead of the homology.

**Definition 2.2.8.** Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $F : \mathcal{A} \to \mathcal{B}$ be a covariant, additive functor. We say that $F$ is *left exact* (respectively *right exact*) if for every short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$, the sequence $0 \to F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C)$ (respectively $F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C) \to 0$) is exact.

Similarly if $F : \mathcal{A} \to \mathcal{B}$ is a contravariant functor, $F$ is *left exact* (respectively *right exact*) if for every short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$, the sequence $0 \to F(C) \xrightarrow{F(\beta)} F(B) \xrightarrow{F(\alpha)} F(A)$ (respectively $F(C) \xrightarrow{F(\beta)} F(B) \xrightarrow{F(\alpha)} F(A) \to 0$) is exact.

A covariant or contravariant functor which is both left and right exact is called an *exact functor.*

**Example 2.2.9.** Let $\mathcal{A}$ be an abelian category. Then for any object $A$ in $\mathcal{A}$, the covariant functor $\mathrm{Hom}_{\mathcal{A}}(A, -)$ and the contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-, A)$ are left exact.

### Derived functors and Ext

In this section we fix an abelian category $\mathcal{A}$ and we define derived functors.

**Definition 2.2.10.** An object $P$ in $\mathcal{A}$ is called a *projective object* if the covariant functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact. Dually, an object $I$ in $\mathcal{A}$ is called an *injective object* if the contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact.

**Definition 2.2.11.** A *projective presentation of an object $A$ in $\mathcal{A}$* is a short exact sequence in $\mathcal{A}$ of the form

$$0 \to K \to P \to A \to 0,$$

where $P$ is a projective object. If every object in $\mathcal{A}$ has a projective presentation, we say that $\mathcal{A}$ *has enough projectives.*

Dually, one can define *injective presentations of objects* and $\mathcal{A}$ *having enough injectives.*

**Definition 2.2.12.** A *projective resolution of an object $A$ in $\mathcal{A}$* is a chain complex in $\mathcal{A}$ consisting of projective objects of the form

$$P = \cdots \to P_2 \to P_1 \to P_0 \to 0,$$

such that $H_0(P) \cong A$ and $H_i(P) = 0$ for $i \geq 1$.

**Remark 2.2.13.** Assume that $\mathcal{A}$ has enough projectives and let $A$ be an object in $\mathcal{A}$.

Then we can construct a commutative diagram of the form

$$\begin{tikzcd}
 & & K_2 \arrow[dr] & & & & \\
\cdots \arrow[r] & P_2 \arrow[ur, "\pi_2"] \arrow[rr, "\overline{\pi_2}"] & & P_1 \arrow[rr, "\overline{\pi_1}"] \arrow[dr, "\pi_1"'] & & P_0 \arrow[r, "\pi_0"] & A, \\
 & & & & K_1 \arrow[ur] & &
\end{tikzcd}$$

where, fixing $K_0 = A$, we have that $0 \to K_{i+1} \to P_i \xrightarrow{\pi_i} K_i \to 0$ is a projective presentation of $K_i$ for $i \geq 0$. Then

$$P = \cdots \to P_2 \xrightarrow{\overline{\pi_2}} P_1 \xrightarrow{\overline{\pi_1}} P_0 \to 0$$

is a projective resolution of $A$.

If instead of having enough projectives, the category $\mathcal{A}$ has enough injectives, then one can construct an injective presentation of a given object using a dual construction.

**Definition 2.2.14** ([56, Section 2.4])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be an abelian categories, $\mathcal{A}$ have enough projectives and $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. For each object $A$ in $\mathcal{A}$, we fix a projective resolution $P(A)$ of $A$. Note that a morphism $\alpha : A \to A'$ in $\mathcal{A}$ can be lifted to a morphism $P(\alpha) : P(A) \to P(A')$ of the projective resolutions, that is a a chain map such that $H_0(P(\alpha)) = \alpha$.

If $F$ is a covariant functor, we define the *left derived functors of* $F$ to be $L_iF(-) := H_i(F(P(-)))$ for $i \geq 0$. Note that if $F$ is right exact, then $L_0F(A) \cong F(A)$. If $F$ is a contravariant functor, we define the *right derived functors of* $F$ to be $R^iF(-) := H^i(F(P(-)))$ for $i \geq 0$. Note that if $F$ is left exact, then $R^0F(A) \cong F(A)$. Note that, as pointed out in [56, Lemma 2.4.1], the choice of the projective resolutions and how morphisms are lifted do not matter.

Dually, if instead of having enough projectives, the category $\mathcal{A}$ has enough injectives, then one can define left derived functors of a contravariant functor and right derived functors of a covariant functor using injective resolutions.

We introduce an important example of right derived functors that we will often use in later sections, that is the Ext-functors.

**Definition 2.2.15.** Let $\mathcal{A}$ be an abelian category with enough projectives and enough injectives and let $A$ and $B$ be objects in $\mathcal{A}$. For $i \geq 1$, we define $\operatorname{Ext}^i_{\mathcal{A}}(-, B) := R^i \operatorname{Hom}_{\mathcal{A}}(-, B)$, that is the right derived functors of the contravariant, left exact, additive functor $\operatorname{Hom}_{\mathcal{A}}(-, B)$ using projective resolutions. Similarly, we define $\operatorname{Ext}^i_{\mathcal{A}}(A, -) := R^i \operatorname{Hom}_{\mathcal{A}}(A, -)$, that is the right derived functors of the covariant, left exact, additive functor $\operatorname{Hom}_{\mathcal{A}}(A, -)$, using injective resolutions. Note that the two definitions give natural equivalent functors, see [56, Theorem 2.7.6].

**Auslander-Reiten theory in module categories**

We now focus on some module categories and we give an overview of Auslander-Reiten theory for these abelian categories. We follow Notation 2.1.18.

**Setup 2.2.16.** Let $k$ be a field and $\Lambda$ a finite dimensional $k$-algebra.

**Definition 2.2.17.** A short exact sequence of the form

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{2.3}$$

in $\operatorname{mod}\Lambda$ is called an *Auslander-Reiten sequence* if $\alpha$ is a left almost split morphism in $\operatorname{mod}\Lambda$ and $\beta$ is a right almost split morphism in $\operatorname{mod}\Lambda$.

The following result presents equivalent definitions to the one of Auslander-Reiten sequence, see [2, Theorem IV.1.13] and [5, Proposition V.1.14].

**Theorem 2.2.18.** *Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence in* $\operatorname{mod}\Lambda$. *The following are equivalent:*

*(a) the sequence is an Auslander-Reiten sequence,*

*(b) $A$ is indecomposable and $\beta$ is right almost split,*

*(c) $C$ is indecomposable and $\alpha$ is left almost split,*

*(d) $\alpha$ is minimal left almost split,*

*(e) $\beta$ is minimal right almost split,*

*(f) $A$ and $C$ are indecomposable and $\alpha$ and $\beta$ are irreducible.*

**Remark 2.2.19.** Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an Auslander-Reiten sequence. Then, by [5, Theorem V.5.3], the components of $\alpha$ are, up to isomorphism and scalar multiple, all the irreducible morphisms starting at $A$ and the components of $\beta$ are, up to isomorphism and scalar multiple, all the irreducible morphisms ending at $C$.

Hence Auslander-Reiten sequences collect important information both about the building blocks of objects in $\operatorname{mod}\Lambda$, that is indecomposable modules, and about the building blocks of morphisms in $\operatorname{mod}\Lambda$, that is irreducible morphisms. It is now natural to ask when it is possible to construct an Auslander-Reiten sequence ending (or starting) at a given indecomposable in $\operatorname{mod}\Lambda$. This question has been answered, and there is also a recipe to find the other end term of such an Auslander-Reiten sequence.

**Definition 2.2.20.** Let $\operatorname{proj}\Lambda$ be the full subcategory of $\operatorname{mod}\Lambda$ whose objects are the finitely generated projective modules. Considering the module $\Lambda$ in $\operatorname{mod}\Lambda$, note that $\operatorname{proj}\Lambda = \operatorname{add}(\Lambda)$, where add is defined in Definition 2.1.40. Moreover, note that $\operatorname{proj}\Lambda$ is a precovering subcategory. Let $C$ be an object in $\operatorname{mod}\Lambda$. A *minimal projective presentation* of $C$ is a sequence $P_1 \xrightarrow{\pi_1} P_0 \xrightarrow{\pi_0} C$ where $\pi_0$ is a $\operatorname{proj}\Lambda$-cover of $C$ and $\pi_1$ factors as

$$\begin{tikzcd} P_1 \arrow[rr, "\pi_1"] \arrow[dr, "\overline{\pi_1}"'] & & P_0, \\ & \operatorname{Ker}\pi_0 \arrow[ur, "\iota"'] & \end{tikzcd}$$

where $\iota$ is the kernel of $\pi_0$ and $\overline{\pi_1}$ is a $\operatorname{proj}\Lambda$-cover of $\operatorname{Ker}\pi_0$.

**Definition 2.2.21.** Let $C$ be a module in $\operatorname{mod}\Lambda$ and $P_1 \xrightarrow{\pi_1} P_0 \xrightarrow{\pi_0} C$ be a minimal projective presentation of $C$. The *transpose of* $C$ is defined to be

$$\operatorname{Tr}(C) := \operatorname{Coker}(\operatorname{Hom}_\Lambda(P_0,\Lambda) \xrightarrow{\operatorname{Hom}_\Lambda(\pi_1,\Lambda)} \operatorname{Hom}_\Lambda(P_1,\Lambda)) \in \operatorname{mod}\Lambda^{op}.$$

The *Auslander-Reiten translation of* $C$ and the inverse Auslander-Reiten translation of $C$ are $\tau(C) := D \circ \operatorname{Tr}(C)$ and $\tau^{-1}(C) = \operatorname{Tr} \circ D(C)$ respectively, where $D(-) := \operatorname{Hom}_k(-,k) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{op}$ is the standard $k$-duality.

**Remark 2.2.22.** By definition, if $P$ is an indecomposable projective module in $\operatorname{mod}\Lambda$, then it cannot appear as the last term of an Auslander-Reiten sequence. In fact all epimorphisms ending at $P$ split. Similarly, if $I$ is an indecomposable injective module in $\operatorname{mod}\Lambda$, then it cannot appear as the first term of an Auslander-Reiten sequence. As pointed out in the following result, these are the only cases for which there is no Auslander-Reiten sequence ending (respectively starting) at a given indecomposable module in $\operatorname{mod}\Lambda$.

**Theorem 2.2.23** ([5, Proposition V.1.14 and Theorem V.1.15])**.** *(a) If $C$ is an indecomposable non-projective module in* $\operatorname{mod}\Lambda$*, then there is an Auslander-Reiten sequence in* $\operatorname{mod}\Lambda$ *of the form* $0 \to \tau(C) \to B \to C \to 0$.

*(b) If $A$ is an indecomposable non-injective module in* $\operatorname{mod}\Lambda$*, then there is an Auslander-Reiten sequence in* $\operatorname{mod}\Lambda$ *of the form* $0 \to A \to B \to \tau^{-1}(A) \to 0$.

We end this section by recalling stable categories and the Auslander-Reiten duality, see for example [2, Theorem IV.2.13], linking Hom spaces in stable categories and $\operatorname{Ext}^1$, see Definition 2.2.15. We will also see a higher version of this in a later section of this chapter, see Theorem 2.3.26.

**Definition 2.2.24.** For $A$ and $B$ in $\operatorname{mod}\Lambda$, we define $\mathcal{P}(A,B)$ to be the subset of $\operatorname{Hom}_\Lambda(A,B)$ consisting of morphisms factoring through a projective $\Lambda$-module. Note that this defines an ideal $\mathcal{P}$ in $\operatorname{mod}\Lambda$, see [2, p. 109]. We define the *projectively stable category* to be the quotient category

$$\underline{\operatorname{mod}}\,\Lambda := \operatorname{mod}\Lambda/\mathcal{P}.$$

The objects of $\underline{\operatorname{mod}}\,\Lambda$ are the same as the objects of $\operatorname{mod}\Lambda$ and, for $A, B \in \underline{\operatorname{mod}}\,\Lambda$, we define

$$\underline{\operatorname{Hom}}_\Lambda(A,B) = \operatorname{Hom}_{\underline{\operatorname{mod}}\,\Lambda}(A,B) = \operatorname{Hom}_\Lambda(A,B)/\mathcal{P}(A,B),$$

with the composition of morphisms induced from the one in $\operatorname{mod}\Lambda$.

The *injectively stable category* $\overline{\operatorname{mod}}\,\Lambda$ is defined dually and the morphism space between $A, B \in \overline{\operatorname{mod}}\,\Lambda$ is denoted by $\overline{\operatorname{Hom}}_\Lambda(A,B)$.

**Theorem 2.2.25** ([2, Theorem IV.2.13])**.** *For all $A$ and $B$ in $\operatorname{mod}\Lambda$, there are functorial isomorphisms*

$$\operatorname{Ext}^1_\Lambda(A,B) \cong D \circ \underline{\operatorname{Hom}}_\Lambda(\tau^{-1}(B), A) \cong D \circ \overline{\operatorname{Hom}}_\Lambda(B, \tau(A)).$$

### 2.2.2 Triangulated categories

We start by recalling the definition of a triangulated category.

**Definition 2.2.26** ([22, Definition in I.1])**.** A *triangulated category* is a triple $(\mathcal{T}, \Sigma, \Delta)$, where

- $\mathcal{T}$ is an additive category,
- $\Sigma : \mathcal{T} \to \mathcal{T}$ is an automorphism, called *suspension*, (with inverse denoted by $\Sigma^{-1}$),
- $\Delta$ is a class of diagrams in $\mathcal{T}$ of the form $A \to B \to C \to \Sigma A$ called *triangles*,

satisfying the following axioms.

(TR1)
- Each morphism $A \xrightarrow{\alpha} B$ in $\mathcal{T}$ is part of a triangle $(A \xrightarrow{\alpha} B \to C \to \Sigma A) \in \Delta$.
- For each object $A$, the following is a triangle: $A \xrightarrow{1_A} A \to 0 \to \Sigma A$.
- For each commutative diagram as below, with vertical arrows isomorphisms,

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"', "\wr"] & B \arrow[r] \arrow[d, "\wr"] & C \arrow[r] \arrow[d, "\wr"] & \Sigma A \arrow[d, "\Sigma f"', "\wr"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & \Sigma A', \end{tikzcd}$$

  if the top row is a triangle, then so is the bottom row.

(TR2) $(A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A) \in \Delta$ if and only if $(B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A \xrightarrow{-\Sigma\alpha} \Sigma B) \in \Delta$.

(TR3) For every diagram as below, where the rows are triangles and $f$, $g$ are such that the left square is commutative, there is a morphism $h$ making the other two squares commute:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, dashed, "\exists h"'] & \Sigma A \arrow[d, "\Sigma f"'] & \in \Delta \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & \Sigma A' & \in \Delta.
\end{tikzcd}$$

(TR4) *Octahedral axiom:* for any two morphisms $\alpha : A \to B$ and $\beta : B \to C$ in $\mathcal{T}$, there is a commutative diagram of the form

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "1_A"'] & B \arrow[r] \arrow[d, "\beta"] & C' \arrow[r] \arrow[d] & \Sigma A \arrow[d, "1_{\Sigma A}"] \\
A \arrow[r, "\beta\circ\alpha"] \arrow[d] & C \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & \Sigma A \arrow[d] \\
0 \arrow[r] \arrow[d] & A' \arrow[r, "1_A"] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma A \arrow[r] & \Sigma B \arrow[r] & \Sigma C' \arrow[r] & \Sigma^2 A,
\end{tikzcd}$$

where each row and each column is a triangle.

We recall two lemmas proven by Krause about left and right minimal morphisms. Note that we prove the higher version of the first one in Lemma 2.3.36 and the second one is a consequence of Lemma 2.3.34 in the case when $d = 1$.

**Lemma 2.2.27** ([44, Lemma 2.4])**.** Let $\alpha : A \to B$ be a non-zero morphism in a triangulated category $\mathcal{T}$. If $B$ has local endomorphism ring, then $\alpha$ is left minimal and if $A$ has local endomorphism ring, then $\alpha$ is right minimal.

**Lemma 2.2.28** ([44, Lemma 2.5])**.** Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$ be a triangle in a triangulated category $\mathcal{T}$. Then $\beta$ is right minimal if and only if $\gamma$ is left minimal.

Important functors between triangulated categories are the triangulated functors.

**Definition 2.2.29** ([48, Definition 2.1.1])**.** Let $(\mathcal{T}, \Sigma, \Delta)$ and $(\mathcal{T}', \Sigma', \Delta')$ be triangulated categories. A *triangulated functor* from $(\mathcal{T}, \Sigma, \Delta)$ to $(\mathcal{T}', \Sigma', \Delta')$ is a pair $(F, \varphi)$, where $F : \mathcal{T} \to \mathcal{T}'$ is an additive functor and $\varphi : F \circ \Sigma \to \Sigma' \circ F$ a natural equivalence, such that if $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$ is in $\Delta$, then

$$F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C) \xrightarrow{F(\gamma)} \Sigma' F(A)$$

is in $\Delta'$.

We will work in the following setup.

**Setup 2.2.30.** Let $k$ be a field, $\mathcal{T}$ be a skeletally small $k$-linear triangulated category with split idempotents in which each Hom space is finite dimensional over $k$. Note that this implies that $\mathcal{T}$ is a Krull-Schmidt category by Remark 2.1.12.

### The bounded derived category of $\operatorname{mod} \Lambda$

Let $k$ be a field and $\Lambda$ be a finite dimensional $k$-algebra. An important example of a triangulated category satisfying Setup 2.2.30 is the bounded derived category of $\operatorname{Mod} \Lambda$. In this section, we give an overview of this category, see for example [25, Section 7] or [56, Chapter 10] for more details.

Recall the category of chain complexes of an abelian category $\mathcal{A}$ and the homology of a chain complex, see Definitions 2.2.4 and 2.2.5. Here we fix $\mathcal{A} = \operatorname{Mod} \Lambda$, see Remark 2.2.6.

**Definition 2.2.31** ([25, Definition 7.4])**.** A chain map $f : A \to B$ in $C(\operatorname{Mod} \Lambda)$ is called a *quasi-isomorphism* if $H_i(f) : H_i(A) \to H_i(B)$ is an isomorphism for all $i$.

**Definition 2.2.32** ([25, Definition 1.6])**.** Let

$$A = \cdots \to A_{i+1} \xrightarrow{\alpha_{i+1}} A_i \xrightarrow{\alpha_i} A_{i-1} \to \cdots,$$
$$B = \cdots \to B_{i+1} \xrightarrow{\beta_{i+1}} B_i \xrightarrow{\beta_i} B_{i-1} \to \cdots$$

be chain complexes in $C(\operatorname{Mod} \Lambda)$ and $f$, $g : A \to B$ be chain maps. We say that $f$ and $g$ are *homotopic* and write $f \sim g$ if there are morphisms $s_i : A_i \to B_{i+1}$ in $\operatorname{Mod} \Lambda$ such that $f_i - g_i = \beta_{i+1} \circ s_i + s_{i-1} \circ \alpha_i$ for all $i$. In the case when $g = 0$, we say that $f$ is *null-homotopic*. Note that $\sim$ is an equivalence relation.

The *homotopy category* $K(\operatorname{Mod} \Lambda)$ is then defined to be the category with the same objects as $C(\operatorname{Mod} \Lambda)$ and whose morphisms are the equivalence classes of the morphisms in $C(\operatorname{Mod} \Lambda)$ modulo homotopy, that is for $A$, $B \in K(\operatorname{Mod} \Lambda)$, we have

$$\operatorname{Hom}_{K(\operatorname{Mod} \Lambda)}(A, B) := \operatorname{Hom}_{C(\operatorname{Mod} \Lambda)}(A, B) / \sim .$$

**Lemma 2.2.33** ([25, Proposition 7.3])**.** Let $f$ and $g$ be chain maps in $C(\operatorname{Mod} \Lambda)$ such that $f \sim g$. Then $H_i(f) = H_i(g)$ for all $i$. As a consequence, the homology functors induce well-defined homology functors on the category $K(\operatorname{Mod}\Lambda)$.

**Definition 2.2.34** ([25, Definition 6.1])**.** We define an automorphism $\Sigma : C(\operatorname{Mod}\Lambda) \to C(\operatorname{Mod}\Lambda)$ sending

- a chain complex

$$A = \cdots \to A_{i+1} \xrightarrow{\alpha_{i+1}} A_i \xrightarrow{\alpha_i} A_{i-1} \to \cdots,$$

  to the chain complex

$$\Sigma(A) = \cdots \to A_i \xrightarrow{-\alpha_i} A_{i-1} \xrightarrow{-\alpha_{i-1}} A_{i-2} \to \cdots,$$

  that is the chain complex with $(\Sigma(A))_i = A_{i-1}$ and $(\Sigma\alpha)_i = -\alpha_{i-1}$;

- a chain map

$$\begin{tikzcd}
A = \arrow[d, "f"] & \cdots \arrow[r] & A_{i+1} \arrow[r, "\alpha_{i+1}"] \arrow[d, "f_{i+1}"] & A_i \arrow[r, "\alpha_i"] \arrow[d, "f_i"] & A_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\
B = & \cdots \arrow[r] & B_{i+1} \arrow[r, "\beta_{i+1}"'] & B_i \arrow[r, "\beta_i"'] & B_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

  to the chain map $\Sigma(f) : \Sigma(A) \to \Sigma(B)$, where $(\Sigma(f))_i = f_{i-1}$.

**Definition 2.2.35** ([25, Definitions 6.3 and 6.6])**.** Let

$$\begin{tikzcd}
A = \arrow[d, "f"] & \cdots \arrow[r] & A_{i+1} \arrow[r, "\alpha_{i+1}"] \arrow[d, "f_{i+1}"] & A_i \arrow[r, "\alpha_i"] \arrow[d, "f_i"] & A_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\
B = & \cdots \arrow[r] & B_{i+1} \arrow[r, "\beta_{i+1}"'] & B_i \arrow[r, "\beta_i"'] & B_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

be a chain map in $C(\operatorname{Mod}\Lambda)$. Then the *mapping cone of $f$* is the chain complex

$$M(f) = \cdots \to A_i \oplus B_{i+1} \xrightarrow{\begin{pmatrix} -\alpha_i & 0 \\ f_i & \beta_{i+1} \end{pmatrix}} A_{i-1} \oplus B_i \xrightarrow{\begin{pmatrix} -\alpha_{i-1} & 0 \\ f_{i-1} & \beta_i \end{pmatrix}} A_{i-2} \oplus B_{i-1} \to \cdots.$$

Then, we have a short exact sequence in $C(\operatorname{Mod}\Lambda)$ of the form

$$0 \to B \xrightarrow{\binom{0}{1}} M(f) \xrightarrow{(1\ 0)} \Sigma(A) \to 0.$$

Viewed as a sequence in $K(\operatorname{Mod}\Lambda)$, then

$$A \xrightarrow{f} B \xrightarrow{\binom{0}{1}} M(f) \xrightarrow{(1\ 0)} \Sigma(A)$$

is called a *standard triangle in* $K(\operatorname{Mod}\Lambda)$.

**Proposition 2.2.36** ([25, Theorem 6.7])**.** The category $K(\operatorname{Mod}\Lambda)$ is triangulated with suspension $\Sigma$ as defined in Definition 2.2.34 and triangles as diagrams that are isomorphic to standard triangles as described in Definition 2.2.35.

We now introduce the derived category $\mathcal{D}(\operatorname{Mod}\Lambda)$, which is obtained by inverting the quasi-isomorphisms in $K(\operatorname{Mod}\Lambda)$. The process to build $\mathcal{D}(\operatorname{Mod}\Lambda)$ is called *localisation* of $K(\operatorname{Mod}\Lambda)$ with respect to the quasi-isomorphisms, for more details on the construction of this category see for example [25, Sections 7.2 and 7.3].

**Theorem 2.2.37** ([25, Theorems 7.10 and 7.18])**.** *There exists a category* $\mathcal{D}(\operatorname{Mod}\Lambda)$*, called the derived category of* $\operatorname{Mod}\Lambda$*, which has the same objects as* $K(\operatorname{Mod}\Lambda)$ *and is equipped with a functor* $L : K(\operatorname{Mod}\Lambda) \to \mathcal{D}(\operatorname{Mod}\Lambda)$ *with the following properties.*

*(a) For every quasi-isomorphism* $q$ *in* $K(\operatorname{Mod}\Lambda)$*, then* $L(q)$ *is an isomorphism in* $\mathcal{D}(\operatorname{Mod}\Lambda)$*.*

*(b) If* $\mathcal{D}$ *is a category and* $F : K(\operatorname{Mod}\Lambda) \to \mathcal{D}$ *is a functor sending all quasi-isomorphisms to isomorphisms, then there is a unique functor* $F' : \mathcal{D}(\operatorname{Mod}\Lambda) \to \mathcal{D}$ *making the following diagram commutative*

$$\begin{tikzcd} K(\operatorname{Mod}\Lambda) \arrow[r, "L"] \arrow[d, "F"'] & \mathcal{D}(\operatorname{Mod}\Lambda). \arrow[dl, dashed, "F'"] \\ \mathcal{D} & \end{tikzcd}$$

*Then* $\mathcal{D}(\operatorname{Mod}\Lambda)$ *is a triangulated category, where triangles are the isomorphism closure of* $L$ *applied to triangles in* $K(\operatorname{Mod}\Lambda)$ *and so* $L$ *is a triangulated functor in the sense of Definition 2.2.29.*

**Remark 2.2.38.** The functor $G : \operatorname{Mod}\Lambda \to \mathcal{D}(\operatorname{Mod}\Lambda)$ obtained by composing the inclusion of $\operatorname{Mod}\Lambda$ into $K(\operatorname{Mod}\Lambda)$ with the functor $L : K(\operatorname{Mod}\Lambda) \to \mathcal{D}(\operatorname{Mod}\Lambda)$ from Theorem 2.2.37, is full and faithful in the sense of Definition 2.1.14. Moreover, for a positive integer $i$, we have an equivalence

$$\operatorname{Ext}^i_{\operatorname{Mod}\Lambda}(-,-) \cong \operatorname{Hom}_{\mathcal{D}(\operatorname{Mod}\Lambda)}(G(-), \Sigma^i G(-)),$$

see [22, Section I.6]. The functor $G$ sends a short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ corresponding to the class $\epsilon \in \operatorname{Ext}^1_{\operatorname{Mod}\Lambda}(C, A)$, to a triangle

$$G(A) \xrightarrow{G(\alpha)} G(B) \xrightarrow{G(\beta)} G(C) \xrightarrow{G(\epsilon)} \Sigma G(A)$$

in $\mathcal{D}(\operatorname{Mod}\Lambda)$.

Instead of working with $\mathcal{D}(\operatorname{Mod}\Lambda)$, we will usually work with the full subcategory $\operatorname{mod}\Lambda \subseteq \operatorname{Mod}\Lambda$ of finitely generated right $\Lambda$-modules and the following full subcategory of the derived category.

**Definition 2.2.39.** The *bounded derived category*, denoted by $\mathcal{D}^b(\operatorname{mod}\Lambda)$ is the full subcategory of $\mathcal{D}(\operatorname{Mod}\Lambda)$ whose objects are bounded chain complexes of finitely generated $\Lambda$-modules, that is chain complexes of the form

$$A = \cdots \to 0 \to \cdots \to 0 \to A_i \xrightarrow{\alpha_i} A_{i-1} \to \cdots \to A_{j+1} \xrightarrow{\alpha_{j+1}} A_j \to 0 \to \cdots \to 0 \to \cdots.$$

### Auslander-Reiten theory in triangulated categories

In this section, we study Auslander-Reiten triangles in triangulated categories satisfying Setup 2.2.30. These were first studied by Happel, see [21, Section I.4], but we present some of the results as stated by Krause in [44].

**Definition 2.2.40** ([21, Definition I.4.1])**.** A triangle in $\mathcal{T}$ of the form

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$$

is an *Auslander-Reiten triangle in* $\mathcal{T}$ if the following are satisfied:

(a) the morphism $\gamma$ is non-zero,

(b) the morphism $\alpha$ is left almost split in $\mathcal{T}$,

(c) the morphism $\beta$ is right almost split in $\mathcal{T}$.

**Remark 2.2.41.** Note that in the above definition, condition (a) is implied by both of the other two conditions.

**Lemma 2.2.42** ([44, Lemma 2.3])**.** (a) Let $\beta : B \to C$ be right almost split in $\mathcal{T}$, then $C$ is indecomposable.

(b) Let $\alpha : A \to B$ be left almost split in $\mathcal{T}$, then $A$ is indecomposable.

The following lemma presents equivalent definitions to the one of Auslander-Reiten triangle. Note that its dual is also true.

**Lemma 2.2.43** ([44, Lemma 2.6])**.** Let $\epsilon : A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$ be a triangle in $\mathcal{T}$ and suppose that $\beta$ is right almost split. Then the following are equivalent:

(a) $A$ in indecomposable,

(b) $\beta$ is right minimal,

(c) $\alpha$ is left almost split,

(d) $\epsilon$ is an Auslander-Reiten triangle.

Like in the abelian setup, the Ext-functor plays an important role in triangulated categories and it will often be used in this thesis. We have seen in Remark 2.2.38 that in the special case when our triangulated category is $\mathcal{D}(\operatorname{mod}\Lambda)$, then for $A$, $B \in \operatorname{mod}\Lambda$ and a positive integer $i$, we have that

$$\operatorname{Hom}_{\mathcal{D}(\operatorname{mod}\Lambda)}(A, \Sigma^i B) \cong \operatorname{Ext}^i_{\operatorname{mod}\Lambda}(A, B),$$

where we dropped the functor $G : \operatorname{mod}\Lambda \to \mathcal{D}(\operatorname{mod}\Lambda)$ because it can be viewed as an inclusion. We now define Ext for a more general triangulated category satisfying our setup.

**Definition 2.2.44.** Let $i$ be a positive integer. We define

$$\operatorname{Ext}^i_{\mathcal{T}}(A, B) := \operatorname{Hom}_{\mathcal{T}}(A, \Sigma^i B),$$

for objects $A$, $B$ in $\mathcal{T}$. If the triangulated category we are working in is clear, we sometimes omit the subscript $\mathcal{T}$ and simply write $\operatorname{Ext}^i(A, B)$.

We will often assume that $\mathcal{T}$ has a Serre functor, see Definition 2.1.24, since this implies the existence of Auslander-Reiten triangles in $\mathcal{T}$.

**Definition 2.2.45.** Suppose that $\mathcal{T}$ has a Serre functor $S : \mathcal{T} \to \mathcal{T}$. Then the functor $\tau := S \circ \Sigma^{-1} : \mathcal{T} \to \mathcal{T}$ is called *Auslander-Reiten translation* and it is invertible with inverse $\tau^{-1} = S^{-1} \circ \Sigma$.

The following was first proved in [52, Theorem I.2.4] with the extra assumption that $k$ is an algebraically closed field. The result for a general field $k$ corresponds to the case $n = 1$ of [32, Theorem 3.10], as the only 1-cluster tilting subcategory of $\mathcal{T}$ is $\mathcal{T}$ itself.

**Theorem 2.2.46.** *Suppose that the category $\mathcal{T}$ has a Serre functor $S : \mathcal{T} \to \mathcal{T}$. Then $\mathcal{T}$ has Auslander-Reiten triangles, in the sense that if $X$ is an indecomposable in $\mathcal{T}$, then there exists an Auslander-Reiten triangle in $\mathcal{T}$ starting at $X$ and one ending at $X$. Moreover, these Auslander-Reiten triangles in $\mathcal{T}$ have the form*

$$X \to Y \to \tau^{-1}X \to \Sigma X \text{ and } \tau X \to Z \to X \to \Sigma(\tau X).$$

Sometimes, we also assume that $\mathcal{T}$ has a Serre functor with some extra property.

**Definition 2.2.47.** Assume that the category $\mathcal{T}$ has a Serre functor $S$. If for some integer $n \geq 2$, we have that $S \cong \Sigma^n$, then $\mathcal{T}$ is called an *$n$-Calabi-Yau category* .

## 2.3 Higher homological algebra

In this section, we introduce the generalisation of homological algebra, that is higher homological algebra. Let $d$ be a positive integer. Jasso generalised abelian categories to $d$-abelian categories in [34], where kernels and cokernels are replaced by complexes of $d+1$ objects, called $d$-kernels and $d$-cokernels respectively, and short exact sequences by complexes of $d+2$ objects, called $d$-exact sequences. In [19], Geiss, Keller and Oppermann likewise generalised triangulated categories to $(d+2)$-angulated categories, where triangles are replaced by complexes consisting of $d+2$ objects.

Note that the base case $d = 1$ recovers classic homological algebra as 1-abelian categories are abelian categories and 3-angulated categories are triangulated categories.

### 2.3.1 $d$-abelian categories

Let $d$ be a fixed positive integer. In this section we present the definitions of $d$-abelian categories and $d$-cluster tilting subcategories of the category of finitely generated right $\Lambda$-modules and we present some of their properties.

**Definition 2.3.1** ([34, Definitions 2.2, 2.4 and 2.9])**.** Let $\mathcal{A}$ be an additive category.

(a) A sequence of objects and morphisms in $\mathcal{A}$ of the form

$$A^0 \longrightarrow A^1 \longrightarrow A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \longrightarrow A^d$$

is a *$d$-kernel* of a morphism $A^d \longrightarrow A^{d+1}$ if

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(B, A^0) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{\mathcal{A}}(B, A^d) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(B, A^{d+1})$$

is an exact sequence for each $B$ in $\mathcal{A}$.

(b) A sequence of objects and morphisms in $\mathcal{A}$ of the form

$$A^1 \longrightarrow A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \longrightarrow A^d \longrightarrow A^{d+1}$$

is a *$d$-cokernel* of a morphism $A^0 \longrightarrow A^1$ if

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A^{d+1}, B) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A^1, B) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A^0, B)$$

is an exact sequence for each $B$ in $\mathcal{A}$.

(c) A *$d$-exact sequence* is a sequence of objects and morphisms in $\mathcal{A}$ of the form

$$0 \longrightarrow A^0 \xrightarrow{\alpha^0} A^1 \longrightarrow A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \longrightarrow A^d \xrightarrow{\alpha^d} A^{d+1} \longrightarrow 0,$$

such that $A^0 \xrightarrow{\alpha^0} A^1 \longrightarrow A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \longrightarrow A^d$ is a $d$-kernel of $\alpha^d$ and $A^1 \longrightarrow A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \longrightarrow A^d \xrightarrow{\alpha^d} A^{d+1}$ is a $d$-cokernel of $\alpha^0$.

(d) A *morphism of $d$-exact sequences* is a commutative diagram of the form:

$$\begin{tikzcd}
0 \arrow[r] & A^0 \arrow[r] \arrow[d] & A^1 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & A^{d-1} \arrow[r] \arrow[d] & A^d \arrow[r] \arrow[d] & A^{d+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B^0 \arrow[r] & B^1 \arrow[r] & B^2 \arrow[r] & \cdots \arrow[r] & B^{d-1} \arrow[r] & B^d \arrow[r] & B^{d+1} \arrow[r] & 0,
\end{tikzcd}$$

in which each row is a $d$-exact sequence.

**Definition 2.3.2** ([34, Definition 3.1])**.** A *$d$-abelian category* is an additive category $\mathcal{F}$ which satisfies the following axioms:

(A0) The category $\mathcal{F}$ has split idempotents.

(A1) Each morphism in $\mathcal{F}$ has a $d$-kernel and a $d$-cokernel.

(A2) If $\alpha^0 : A^0 \longrightarrow A^1$ is a monomorphism and $A^1 \longrightarrow A^2 \longrightarrow \cdots \longrightarrow A^{d+1}$ is a $d$-cokernel of $\alpha^0$, then

$$0 \longrightarrow A^0 \xrightarrow{\alpha^0} A^1 \longrightarrow A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \longrightarrow A^d \longrightarrow A^{d+1} \longrightarrow 0$$

is a $d$-exact sequence.

(A2$^{\mathrm{op}}$) If $\alpha^d : A^d \longrightarrow A^{d+1}$ is an epimorphism and $A^0 \longrightarrow \cdots \longrightarrow A^{d-1} \longrightarrow A^d$ is a $d$-kernel of $\alpha^d$, then

$$0 \longrightarrow A^0 \longrightarrow A^1 \longrightarrow A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \longrightarrow A^d \xrightarrow{\alpha^d} A^{d+1} \longrightarrow 0$$

is a $d$-exact sequence.

**Setup 2.3.3.** Let $d$ be a fixed positive integer and $\mathcal{F}$ be a $d$-abelian category.

In [34], Jasso generalised the idea of pushout to $d$-pushout of a $d$-exact sequence along a morphism from its first term. We recall Jasso's definition and see how these higher-pushouts can be used to construct morphisms of $d$-exact sequences in $\mathcal{F}$.

**Definition 2.3.4** ([34, Definition 2.11])**.** Consider a complex in $\mathcal{F}$ of the form

$$A: \qquad A^0 \xrightarrow{\alpha^0} A^1 \xrightarrow{\alpha^1} A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \xrightarrow{\alpha^{d-1}} A^d$$

and a morphism $f^0 : A^0 \to B^0$ in $\mathcal{F}$. A *$d$-pushout diagram of $A$ along $f^0$* is a cochain map

$$\begin{tikzcd}
A: \arrow[d, "\varphi"] & A^0 \arrow[r, "\alpha^0"] \arrow[d, "f^0"] & A^1 \arrow[r, "\alpha^1"] \arrow[d, "f^1"] & A^2 \arrow[r] \arrow[d, "f^2"] & \cdots \arrow[r] & A^{d-1} \arrow[r, "\alpha^{d-1}"] \arrow[d, "f^{d-1}"] & A^d \arrow[d, "f^d"] \\
B: & B^0 \arrow[r, "\beta^0"'] & B^1 \arrow[r, "\beta^1"'] & B^2 \arrow[r] & \cdots \arrow[r] & B^{d-1} \arrow[r, "\beta^{d-1}"'] & B^d
\end{tikzcd} \tag{2.4}$$

with $B^1, \ldots, B^d$ in $\mathcal{F}$ such that in the *mapping cone*

$$C(\varphi): \qquad A^0 \xrightarrow{\gamma^{-1}} A^1 \oplus B^0 \xrightarrow{\gamma^0} A^2 \oplus B^1 \longrightarrow \cdots \longrightarrow A^d \oplus B^{d-1} \xrightarrow{\gamma^{d-1}} B^d,$$

the sequence $(\gamma^0, \ldots, \gamma^{d-1})$ is a $d$-cokernel of $\gamma^{-1}$, where we define

$$\gamma^i = \begin{pmatrix} -\alpha^{i+1} & 0 \\ f^{i+1} & \beta^i \end{pmatrix} : A^{i+1} \oplus B^i \to A^{i+2} \oplus B^{i+1},$$

for $i = -1, 0, \ldots, d-1$ and where $B^{-1}$ and $A^{d+1}$ are fixed to be zero. The concept of *$d$-pullback diagram* is defined in a dual way.

**Remark 2.3.5.** By [34, Theorem 3.8], for a complex in $\mathcal{F}$ of the form:

$$A: \qquad A^0 \xrightarrow{\alpha^0} A^1 \xrightarrow{\alpha^1} A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \xrightarrow{\alpha^{d-1}} A^d$$

and a morphism $f^0 : A^0 \to B^0$ in $\mathcal{F}$, there is always a $d$-pushout diagram of $A$ along $f^0$ of the form (2.4). Moreover, if $\alpha^0$ is a monomorphism, then $\beta^0$ is a monomorphism.

The next lemma follows from the dual of [36, Proposition 2.12].

**Lemma 2.3.6.** Consider a $d$-exact sequence in $\mathcal{F}$ of the form

$$\delta : \qquad 0 \longrightarrow A^0 \xrightarrow{\alpha^0} A^1 \xrightarrow{\alpha^1} A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \xrightarrow{\alpha^{d-1}} A^d \xrightarrow{\alpha^d} A^{d+1} \longrightarrow 0$$

and a morphism $f^0 : A^0 \to B^0$ in $\mathcal{F}$. Then there is a $d$-pushout diagram of

$$A^0 \xrightarrow{\alpha^0} \cdots \xrightarrow{\alpha^{d-1}} A^d$$

along $f^0$ and it induces a morphism of $d$-exact sequences of the form:

$$\begin{tikzcd}
\delta : \arrow[d, "f"] & 0 \arrow[r] & A^0 \arrow[r, "\alpha^0"] \arrow[d, "f^0"] & A^1 \arrow[r, "\alpha^1"] \arrow[d, "f^1"] & A^2 \arrow[r] \arrow[d, "f^2"] & \cdots \arrow[r] & A^{d-1} \arrow[r, "\alpha^{d-1}"] \arrow[d, "f^{d-1}"] & A^d \arrow[r, "\alpha^d"] \arrow[d, "f^d"] & A^{d+1} \arrow[r] \arrow[d, equal] & 0 \\
\epsilon : & 0 \arrow[r] & B^0 \arrow[r, "\beta^0"'] & B^1 \arrow[r, "\beta^1"'] & B^2 \arrow[r] & \cdots \arrow[r] & B^{d-1} \arrow[r, "\beta^{d-1}"'] & B^d \arrow[r, "\beta^d"'] & A^{d+1} \arrow[r] & 0.
\end{tikzcd} \tag{2.5}$$

**Lemma 2.3.7.** Consider a morphism $h$ of $d$-exact sequences in $\mathcal{F}$ of the form:

$$\begin{tikzcd}
\delta : \arrow[d, "h"] & 0 \arrow[r] & A^0 \arrow[r, "\alpha^0"] \arrow[d, "h^0"] & A^1 \arrow[r, "\alpha^1"] \arrow[d, "h^1"] \arrow[dl, dashed, "s^1"] & A^2 \arrow[r] \arrow[d, "h^2"] \arrow[dl, dashed, "s^2"] & \cdots \arrow[r] & A^{d-1} \arrow[r, "\alpha^{d-1}"] \arrow[d, "h^{d-1}"] & A^d \arrow[r, "\alpha^d"] \arrow[d, "h^d"] \arrow[dl, dashed, "s^d"] & A^{d+1} \arrow[r] \arrow[d, "h^{d+1}"] \arrow[dl, dashed, "s^{d+1}"] & 0 \\
\epsilon : & 0 \arrow[r] & B^0 \arrow[r, "\beta^0"'] & B^1 \arrow[r, "\beta^1"'] & B^2 \arrow[r] & \cdots \arrow[r] & B^{d-1} \arrow[r, "\beta^{d-1}"'] & B^d \arrow[r, "\beta^d"'] & B^{d+1} \arrow[r] & 0.
\end{tikzcd}$$

Then, the following are equivalent.

(a) There is a morphism $s^{d+1} : A^{d+1} \to B^d$ such that $\beta^d \circ s^{d+1} = h^{d+1}$.

(b) There is a morphism $s^1 : A^1 \to B^0$ such that $s^1 \circ \alpha^0 = h^0$.

(c) The morphism $h : \delta \to \epsilon$ is *null-homotopic*, that is there are morphisms $s^i : A^i \to B^{i-1}$ such that $h^i = \beta^{i-1} \circ s^i + s^{i+1} \circ \alpha^i$ for $i = 0, \ldots, d+1$ and where $B^{-1}$ and $A^{d+2}$ are set to be zero.

*Proof.* It is clear that (c) implies both (a) and (b). Assume (a) holds. By the definition of $d$-kernel, applying $\operatorname{Hom}_{\mathcal{F}}(A^d, -)$ to $\epsilon$, we obtain the exact sequence:

$$\operatorname{Hom}_{\mathcal{F}}(A^d, B^{d-1}) \xrightarrow{\beta_*^{d-1}} \operatorname{Hom}_{\mathcal{F}}(A^d, B^d) \xrightarrow{\beta_*^d} \operatorname{Hom}_{\mathcal{F}}(A^d, B^{d+1}).$$

Note that

$$\beta_*^d(h^d - s^{d+1} \circ \alpha^d) = \beta^d \circ h^d - \beta^d \circ s^{d+1} \circ \alpha^d = \beta^d \circ h^d - h^{d+1} \circ \alpha^d = 0,$$

so that $h^d - s^{d+1} \circ \alpha^d$ is in $\operatorname{Ker} \beta_*^d = \operatorname{Im} \beta_*^{d-1}$. So there exists a morphism $s^d : A^d \to B^{d-1}$ such that $\beta^{d-1} \circ s^d = h^d - s^{d+1} \circ \alpha^d$. Inductively, for $i = d-1, d-2, \dots, 1$, we obtain $s^i : A^i \to B^{i-1}$ such that $h^i = \beta^{i-1} \circ s^i + s^{i+1} \circ \alpha^i$. Then,

$$\beta^0 \circ s^1 \circ \alpha^0 = h^1 \circ \alpha^0 - s^2 \circ \alpha^1 \circ \alpha^0 = h^1 \circ \alpha^0 = \beta^0 \circ h^0.$$

Since $\beta^0$ is a monomorphism, it follows that $s^1 \circ \alpha^0 = h^0$. So (b) and (c) hold. Dually, (b) implies both (a) and (c). □

The special case when $\delta = \epsilon$ and $h$ is the identity on $\delta$ in Lemma 2.3.7 gives the following.

**Corollary 2.3.8.** Consider a $d$-exact sequence in $\mathcal{F}$ of the form

$$\delta : \qquad 0 \longrightarrow A^0 \xrightarrow{\alpha^0} A^1 \xrightarrow{\alpha^1} A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \xrightarrow{\alpha^{d-1}} A^d \xrightarrow{\alpha^d} A^{d+1} \longrightarrow 0.$$

The following are equivalent:

(a) $\alpha^0$ is a split monomorphism,

(b) $\alpha^d$ is a split epimorphism,

(c) the identity on $\delta$ is null-homotopic.

**Definition 2.3.9.** Consider a $d$-exact sequence in $\mathcal{F}$ of the form

$$\delta : \qquad 0 \longrightarrow A^0 \xrightarrow{\alpha^0} A^1 \xrightarrow{\alpha^1} A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \xrightarrow{\alpha^{d-1}} A^d \xrightarrow{\alpha^d} A^{d+1} \longrightarrow 0.$$

If any, and so all, of the conditions in Corollary 2.3.8 hold, we say that $\delta$ is a *split $d$-exact sequence.*

Given two $d$-exact sequences in $\mathcal{F}$, we see that it is possible to complete partial morphisms between them to morphisms of $d$-exact sequences. Recall the notion of radical of the category $\mathcal{F}$ from Definition 2.1.8.

**Lemma 2.3.10.** Suppose there are $d$-exact sequences $\delta$ and $\epsilon$ in $\mathcal{F}$ and, for some $0 \le i < j \le d$, there are morphims $f^i, f^{i+1}, \dots, f^j$ such that $\beta^l \circ f^l = f^{l+1} \circ \alpha^l$ for $i \le l \le j-1$, **i.e.**

the following diagram commutes:

$$\begin{tikzcd}
\delta : 0 \arrow[r] & A^0 \arrow[r, "\alpha^0"] \arrow[d, dashed, "f^0"] & \cdots \arrow[r] & A^{i-1} \arrow[r, "\alpha^{i-1}"] \arrow[d, dashed, "f^{i-1}"] & A^i \arrow[r, "\alpha^i"] \arrow[d, "f^i"] & \cdots \arrow[r, "\alpha^{j-1}"] & A^j \arrow[r, "\alpha^j"] \arrow[d, "f^j"] & A^{j+1} \arrow[r, "\alpha^{j+1}"] \arrow[d, dashed, "f^{j+1}"] & \cdots \arrow[r, "\alpha^d"] & A^{d+1} \arrow[r] \arrow[d, dashed, "f^{d+1}"] & 0 \\
\epsilon : 0 \arrow[r] & B^0 \arrow[r, "\beta^0"'] & \cdots \arrow[r] & B^{i-1} \arrow[r, "\beta^{i-1}"'] & B^i \arrow[r, "\beta^i"'] & \cdots \arrow[r, "\beta^{j-1}"'] & B^j \arrow[r, "\beta^j"'] & B^{j+1} \arrow[r, "\beta^{j+1}"'] & \cdots \arrow[r, "\beta^d"'] & B^{d+1} \arrow[r] & 0.
\end{tikzcd}$$

Then, for $0 \leq l \leq i-1$ and $j+1 \leq l \leq d+1$, there exist morphisms $f^l : A^l \to B^l$ completing $f^i, f^{i+1}, \ldots, f^j$ to a morphism of $d$-exact sequences.

*Proof.* The existence of the morphisms $f^l$ for $0 \leq l \leq i-1$ follows from the fact that

$$0 \longrightarrow B^0 \xrightarrow{\beta^0} B^1 \longrightarrow \cdots \xrightarrow{\beta^{d-1}} B^d$$

is a $d$-kernel of $\beta^d : B^d \to B^{d+1}$. The existence of the morphisms $f^l$ for $j+1 \leq l \leq d+1$, follows from the fact that

$$A^1 \xrightarrow{\alpha^1} A^2 \longrightarrow \cdots \xrightarrow{\alpha^d} A^{d+1} \longrightarrow 0$$

is a $d$-cokernel of $\alpha^0 : A^0 \to A^1$. □

**Lemma 2.3.11.** Consider a $d$-exact sequence in $\mathcal{F}$ of the form

$$\delta : \qquad 0 \longrightarrow A^0 \xrightarrow{\alpha^0} A^1 \xrightarrow{\alpha^1} A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \xrightarrow{\alpha^{d-1}} A^d \xrightarrow{\alpha^d} A^{d+1} \longrightarrow 0.$$

For $i = 1, \ldots, d$, we have that $\alpha^i$ is right minimal if and only if $\alpha^{i-1}$ is in $\mathrm{rad}_{\mathcal{F}}$.

*Proof.* Suppose that $\alpha^i$ is right minimal. For any $f : A^i \to A^{i-1}$, we have

$$\alpha^i \circ (1_{A^i} - \alpha^{i-1} \circ f) = \alpha^i - \alpha^i \circ \alpha^{i-1} \circ f = \alpha^i.$$

Since $\alpha^i$ is right minimal, it follows that $1_{A^i} - \alpha^{i-1} \circ f$ is invertible and hence $\alpha^{i-1}$ is in $\mathrm{rad}_{\mathcal{F}}$.

Suppose now that $\alpha^{i-1}$ is in $\mathrm{rad}_{\mathcal{F}}$ and let $h : A^i \to A^i$ be such that $\alpha^i \circ h = \alpha^i$. Then $\alpha^i \circ (h - 1_{A^i}) = 0$ and, since the first part of $\delta$ is a $d$-kernel of $\alpha^d$, there exists a morphism $g : A^i \to A^{i-1}$ such that $h - 1_{A^i} = \alpha^{i-1} \circ g$. Hence $h = 1_{A^i} + \alpha^{i-1} \circ g$ and, since $\alpha^{i-1} \in \mathrm{rad}_{\mathcal{F}}$, it follows that $h$ is invertible. □

**Lemma 2.3.12.** Consider a $d$-exact sequence in $\mathcal{F}$ of the form

$$\delta : \qquad 0 \longrightarrow A^0 \xrightarrow{\alpha^0} A^1 \xrightarrow{\alpha^1} A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \xrightarrow{\alpha^{d-1}} A^d \xrightarrow{\alpha^d} A^{d+1} \longrightarrow 0,$$

with $\alpha^0, \dots, \alpha^{d-1}$ in $\mathrm{rad}_{\mathcal{F}}$ and a morphism of $d$-exact sequences:

$$\begin{tikzcd}
\delta : \arrow[d, "f"] & 0 \arrow[r] & A^0 \arrow[r, "\alpha^0"] \arrow[d, "f^0"] & A^1 \arrow[r, "\alpha^1"] \arrow[d, "f^1"] & A^2 \arrow[r] \arrow[d, "f^2"] & \cdots \arrow[r] & A^{d-1} \arrow[r, "\alpha^{d-1}"] \arrow[d, "f^{d-1}"] & A^d \arrow[r, "\alpha^d"] \arrow[d, "f^d"] & A^{d+1} \arrow[r] \arrow[d, equal] & 0 \\
\delta : & 0 \arrow[r] & A^0 \arrow[r, "\alpha^0"'] & A^1 \arrow[r, "\alpha^1"'] & A^2 \arrow[r] & \cdots \arrow[r] & A^{d-1} \arrow[r, "\alpha^{d-1}"'] & A^d \arrow[r, "\alpha^d"'] & A^{d+1} \arrow[r] & 0,
\end{tikzcd}$$

where $f^d$ is an isomorphism. Then $f^0, \dots, f^{d-1}$ are all isomorphisms.

*Proof.* First note that, by Lemma 2.3.11, since $\alpha^0, \dots, \alpha^{d-1}$ are in $\mathrm{rad}_{\mathcal{F}}$ then $\alpha^1, \dots, \alpha^d$ are right minimal. Since $f^d$ is invertible, $\alpha^d \circ f^d = \alpha^d$ implies that $\alpha^d = \alpha^d \circ (f^d)^{-1}$. Then, using Lemma 2.3.10, we can construct a commutative diagram of the form:

$$\begin{tikzcd}
\delta : \arrow[d, "f"] & 0 \arrow[r] & A^0 \arrow[r, "\alpha^0"] \arrow[d, "f^0"] & A^1 \arrow[r, "\alpha^1"] \arrow[d, "f^1"] & A^2 \arrow[r] \arrow[d, "f^2"] & \cdots \arrow[r] & A^{d-1} \arrow[r, "\alpha^{d-1}"] \arrow[d, "f^{d-1}"] & A^d \arrow[r, "\alpha^d"] \arrow[d, "f^d"] & A^{d+1} \arrow[r] \arrow[d, equal] & 0 \\
\delta : \arrow[d, "g"] & 0 \arrow[r] & A^0 \arrow[r, "\alpha^0"] \arrow[d, "g^0"] & A^1 \arrow[r, "\alpha^1"] \arrow[d, "g^1"] & A^2 \arrow[r] \arrow[d, "g^2"] & \cdots \arrow[r] & A^{d-1} \arrow[r, "\alpha^{d-1}"] \arrow[d, "g^{d-1}"] & A^d \arrow[r, "\alpha^d"] \arrow[d, "(f^d)^{-1}"] & A^{d+1} \arrow[r] \arrow[d, equal] & 0 \\
\delta : & 0 \arrow[r] & A^0 \arrow[r, "\alpha^0"'] & A^1 \arrow[r, "\alpha^1"'] & A^2 \arrow[r] & \cdots \arrow[r] & A^{d-1} \arrow[r, "\alpha^{d-1}"'] & A^d \arrow[r, "\alpha^d"'] & A^{d+1} \arrow[r] & 0.
\end{tikzcd}$$

Hence $\alpha^{d-1} = \alpha^{d-1} \circ g^{d-1} \circ f^{d-1}$ and as $\alpha^{d-1}$ is right minimal, it follows that $g^{d-1} \circ f^{d-1}$ is an isomorphism. Similarly, looking at $f \circ g$ we conclude that $f^{d-1} \circ g^{d-1}$ is an isomorphism and hence $f^{d-1}$ is an isomorphism. Letting $h^{d-1} := (g^{d-1} \circ f^{d-1})^{-1}$, we can construct a commutative diagram of the form:

$$\begin{tikzcd}
\delta : \arrow[d, "g \circ f"] & 0 \arrow[r] & A^0 \arrow[r, "\alpha^0"] \arrow[d, "g^0 \circ f^0"] & A^1 \arrow[r, "\alpha^1"] \arrow[d, "g^1 \circ f^1"] & \cdots \arrow[r] & A^{d-2} \arrow[r, "\alpha^{d-2}"] \arrow[d, "g^{d-2} \circ f^{d-2}"] & A^{d-1} \arrow[r, "\alpha^{d-1}"] \arrow[d, "g^{d-1} \circ f^{d-1}"] & A^d \arrow[r, "\alpha^d"] \arrow[d, equal] & A^{d+1} \arrow[r] \arrow[d, equal] & 0 \\
\delta : \arrow[d, "h"] & 0 \arrow[r] & A^0 \arrow[r, "\alpha^0"] \arrow[d, dashed, "h^0"] & A^1 \arrow[r, "\alpha^1"] \arrow[d, dashed, "h^1"] & \cdots \arrow[r] & A^{d-2} \arrow[r, "\alpha^{d-2}"] \arrow[d, dashed, "h^{d-2}"] & A^{d-1} \arrow[r, "\alpha^{d-1}"] \arrow[d, "h^{d-1}"] & A^d \arrow[r, "\alpha^d"] \arrow[d, equal] & A^{d+1} \arrow[r] \arrow[d, equal] & 0 \\
\delta : & 0 \arrow[r] & A^0 \arrow[r, "\alpha^0"'] & A^1 \arrow[r, "\alpha^1"'] & \cdots \arrow[r] & A^{d-2} \arrow[r, "\alpha^{d-2}"'] & A^{d-1} \arrow[r, "\alpha^{d-1}"'] & A^d \arrow[r, "\alpha^d"'] & A^{d+1} \arrow[r] & 0.
\end{tikzcd}$$

Then

$$\alpha^{d-2} = h^{d-1} \circ g^{d-1} \circ f^{d-1} \circ \alpha^{d-2} = \alpha^{d-2} \circ h^{d-2} \circ g^{d-2} \circ f^{d-2},$$

and, as $\alpha^{d-2}$ is right minimal, we have that $h^{d-2} \circ g^{d-2} \circ f^{d-2}$ is an isomorphism. Similarly, $g^{d-2} \circ f^{d-2} \circ h^{d-2}$ is an isomorphism. Then $g^{d-2} \circ f^{d-2}$ is an isomorphism. Since also $f^{d-1} \circ g^{d-1}$ is an isomorphism, by a similar argument we have that $f^{d-2} \circ g^{d-2}$ is an isomorphism. Hence $f^{d-2}$ is an isomorphism. Proceeding by induction, we conclude that $f^1, \dots, f^{d-2}$

are all isomorphisms. Then also $f^0$ is forced to be an isomorphism, because $\alpha^0$ is a monomorphism. □

We focus on some $d$-abelian categories that arise as subcategories of module categories, namely $d$-cluster tilting subcategories.

**Definition 2.3.13** ([28, Definition 2.2])**.** Let $k$ be a field, $\Lambda$ a finite dimensional $k$-algebra and $\mathcal{F}$ be a full subcategory of $\operatorname{mod}\Lambda$. We say that $\mathcal{F}$ is a *$d$-cluster tilting subcategory of* $\operatorname{mod}\Lambda$ if:

(a) $\mathcal{F} = \{A \in \operatorname{mod}\Lambda \mid \operatorname{Ext}_\Lambda^{1\cdots d-1}(\mathcal{F}, A) = 0\} = \{A \in \mathcal{A} \mid \operatorname{Ext}_\Lambda^{1\cdots d-1}(A, \mathcal{F}) = 0\}$,

(b) $\mathcal{F}$ is functorially finite in $\operatorname{mod}\Lambda$.

**Theorem 2.3.14.** *Let $k$ be a field and $\Lambda$ a finite dimensional $k$-algebra. If $\mathcal{F} \subseteq \operatorname{mod}\Lambda$ is $d$-cluster tilting, then it is a $d$-abelian category. Moreover, a diagram in $\mathcal{F}$ of the form*

$$0 \longrightarrow A^0 \longrightarrow A^1 \longrightarrow A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \longrightarrow A^d \longrightarrow A^{d+1} \longrightarrow 0$$

*is a $d$-exact sequence in $\mathcal{F}$ if and only if it is an exact sequence in* $\operatorname{mod}\Lambda$.

*Proof.* The subcategory $\mathcal{F}$ is $d$-abelian by [34, Theorem 3.16]. The second part of the theorem follows combining [34, Theorem 3.16 and Proposition 3.18]. □

For the rest of this section, we work in the following setup.

**Setup 2.3.15.** Let $d$ be a fixed positive integer, $k$ a field, $\Lambda$ a finite dimensional $k$-algebra and $\mathcal{F} \subseteq \operatorname{mod}\Lambda$ a $d$-cluster tilting subcategory.

We introduce Yoneda equivalence on exact sequences in $\operatorname{mod}\Lambda$ and its connection with $\operatorname{Ext}_\Lambda^d$, see [24, Chapter IV.9].

**Definition 2.3.16.** Consider two exact sequences in $\operatorname{mod}\Lambda$ with the same end terms:

$$\begin{array}{ll} \epsilon: & 0 \longrightarrow B \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \cdots \longrightarrow C^{d-1} \longrightarrow C^d \longrightarrow A \longrightarrow 0, \\ \epsilon': & 0 \longrightarrow B \longrightarrow D^1 \longrightarrow D^2 \longrightarrow \cdots \longrightarrow D^{d-1} \longrightarrow D^d \longrightarrow A \longrightarrow 0. \end{array}$$

We say that $\epsilon$ and $\epsilon'$ satisfy the relation $\epsilon \rightsquigarrow \epsilon'$ if there exists a commutative diagram of the form:

$$\begin{tikzcd}
\epsilon: \arrow[d] & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & C^1 \arrow[r] \arrow[d] & C^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & C^{d-1} \arrow[r] \arrow[d] & C^d \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
\epsilon': & 0 \arrow[r] & B \arrow[r] & D^1 \arrow[r] & D^2 \arrow[r] & \cdots \arrow[r] & D^{d-1} \arrow[r] & D^d \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

We say that $\epsilon$ and $\epsilon'$ are *Yoneda equivalent* if there exists a chain of exact sequences of the above form $\epsilon = \epsilon_0, \epsilon_1, \dots, \epsilon_t = \epsilon'$ with

$$\epsilon_0 \rightsquigarrow \epsilon_1 \leftsquigarrow \epsilon_2 \rightsquigarrow \cdots \leftsquigarrow \epsilon_t.$$

We denote the equivalence class of $\epsilon$ by $[\epsilon]$ and the set of all equivalence classes of exact sequences of the above form by $\operatorname{Yext}^d_\Lambda(A, B)$.

**Remark 2.3.17.** Note that $\operatorname{Yext}^d_\Lambda(A, B)$ has an abelian group structure, see [24, Chapter IV.9]. Moreover, by [24, Theorem IV.9.1], there is a functorial isomorphism of set-valued bifunctors $\operatorname{Yext}^d_\Lambda(-,-) \cong \operatorname{Ext}^d_\Lambda(-,-)$. By [28, Appendix A], if $A$, $B \in \mathcal{F}$, then each equivalence class in $\operatorname{Yext}^d_\Lambda(A, B)$ contains a $d$-exact sequence in $\mathcal{F}$ of the form:

$$0 \longrightarrow B \longrightarrow F^1 \xrightarrow{\varphi^1} F^2 \xrightarrow{\varphi^2} \cdots \xrightarrow{\varphi^{d-2}} F^{d-1} \xrightarrow{\varphi^{d-1}} F^d \longrightarrow A \longrightarrow 0,$$

with $\varphi^1, \dots, \varphi^{d-1}$ in $\operatorname{rad}_{\mathcal{F}}$, called *almost minimal*, which is unique up to isomorphism. So, from now on, we will talk about equivalence classes of $d$-exact sequences in $\operatorname{Ext}^d_\Lambda$-groups.

**Proposition 2.3.18.** Let $A^0$, $A^{d+1} \in \mathcal{F}$, then every element in $\operatorname{Ext}^d_\Lambda(A^{d+1}, A^0)$ is given by a $d$-exact sequence in $\mathcal{F}$. Consider a $d$-exact sequence in $\mathcal{F}$ of the form

$$\delta: \qquad 0 \longrightarrow A^0 \xrightarrow{\alpha^0} A^1 \xrightarrow{\alpha^1} A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \xrightarrow{\alpha^{d-1}} A^d \xrightarrow{\alpha^d} A^{d+1} \longrightarrow 0.$$

(a) We have that $[\delta] = 0$ in $\operatorname{Ext}^d_\Lambda(A^{d+1}, A^0)$ if and only if $\delta$ is a split $d$-exact sequence.

(b) Given a morphism $f^0 : A^0 \to B^0$ in $\mathcal{F}$, we can look at the morphism

$$\operatorname{Ext}^d_\Lambda(A^{d+1}, f^0) : \operatorname{Ext}^d_\Lambda(A^{d+1}, A^0) \to \operatorname{Ext}^d_\Lambda(A^{d+1}, B^0)$$

in terms of $d$-exact sequences in $\mathcal{F}$. For $\delta$ as above, $f^0 \cdot \delta := \operatorname{Ext}^d_\Lambda(A^{d+1}, f^0)(\delta)$ is given by extending a $d$-pushout diagram as in (2.5) from Lemma 2.3.6:

$$\begin{tikzcd}
\delta: \arrow[d, "f"] & 0 \arrow[r] & A^0 \arrow[r, "\alpha^0"] \arrow[d, "f^0"] & A^1 \arrow[r, "\alpha^1"] \arrow[d, "f^1"] & A^2 \arrow[r] \arrow[d, "f^2"] & \cdots \arrow[r] & A^{d-1} \arrow[r, "\alpha^{d-1}"] \arrow[d, "f^{d-1}"] & A^d \arrow[r, "\alpha^d"] \arrow[d, "f^d"] & A^{d+1} \arrow[r] \arrow[d, equal] & 0 \\
f^0 \cdot \delta: & 0 \arrow[r] & B^0 \arrow[r, "\beta^0"'] & B^1 \arrow[r, "\beta^1"'] & B^2 \arrow[r] & \cdots \arrow[r] & B^{d-1} \arrow[r, "\beta^{d-1}"'] & B^d \arrow[r, "\beta^d"'] & A^{d+1} \arrow[r] & 0.
\end{tikzcd}$$

Dually, for $g^{d+1} : B^{d+1} \to A^{d+1}$ in $\mathcal{F}$, we have that $\delta \cdot g^{d+1} := \operatorname{Ext}^d_\Lambda(g^{d+1}, A^0)(\delta) \in \operatorname{Ext}^d_\Lambda(B^{d+1}, A^0)$ is given by a $d$-pullback diagram.

*Proof.* First note that by Remark 2.3.17 every element in $\operatorname{Ext}^d_\Lambda(A^{d+1}, A^0)$ is given by a $d$-exact sequence in $\mathcal{F}$.

(a) If $[\delta] = 0$ in $\operatorname{Ext}^d_\Lambda(A^{d+1}, A^0)$, then $\delta$ is a split $d$-exact sequence by [33, Lemma 1.6]. The other direction follows by a simple argument.

(b) This construction can be seen in the $d = 1$ case in [24, Section III.1 and Theorem III.2.4]. The case for general $d \geq 1$ follows by methods similar to those used in [24, Section IV.9].

$\square$

**Lemma 2.3.19.** Any $d$-exact sequence in $\mathcal{F}$ of the form:

$$\delta: \qquad 0 \longrightarrow A^0 \xrightarrow{\alpha^0} A^1 \xrightarrow{\alpha^1} A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \xrightarrow{\alpha^{d-1}} A^d \xrightarrow{\alpha^d} A^{d+1} \longrightarrow 0,$$

induces the exact sequences

$$0 \longrightarrow (B, A^0) \longrightarrow \cdots \to (B, A^d) \to (B, A^{d+1}) \longrightarrow \operatorname{Ext}^d_\Lambda(B, A^0) \longrightarrow \operatorname{Ext}^d_\Lambda(B, A^1),$$

$$0 \to (A^{d+1}, B) \to \cdots \to (A^1, B) \longrightarrow (A^0, B) \longrightarrow \operatorname{Ext}^d_\Lambda(A^{d+1}, B) \to \operatorname{Ext}^d_\Lambda(A^d, B),$$

for any $B$ in $\mathcal{F}$ and where we used the notation $(-,-) := \operatorname{Hom}_{\mathcal{F}}(-,-)$.

*Proof.* See [36, Proposition 2.2]. $\square$

**Lemma 2.3.20.** Consider a $d$-exact sequence in $\mathcal{F}$ of the form

$$\delta: \qquad 0 \longrightarrow A^0 \xrightarrow{\alpha^0} A^1 \xrightarrow{\alpha^1} A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \xrightarrow{\alpha^{d-1}} A^d \xrightarrow{\alpha^d} A^{d+1} \longrightarrow 0$$

and a morphism $f^0 : A^0 \to B^0$ in $\mathcal{F}$. Let $f : \delta \to f^0 \cdot \delta$ be as described in Proposition 2.3.18(b). Suppose there is a morphism of $d$-exact sequences of the form:

$$\begin{tikzcd}
\delta: \arrow[d, "g"] & 0 \arrow[r] & A^0 \arrow[r, "\alpha^0"] \arrow[d, "g^0=f^0"'] & A^1 \arrow[r, "\alpha^1"] \arrow[d, "g^1"] & A^2 \arrow[r] \arrow[d, "g^2"] & \cdots \arrow[r] & A^{d-1} \arrow[r, "\alpha^{d-1}"] \arrow[d, "g^{d-1}"] & A^d \arrow[r, "\alpha^d"] \arrow[d, "g^d"] & A^{d+1} \arrow[r] \arrow[d, equal] & 0 \\
\epsilon': & 0 \arrow[r] & B^0 \arrow[r, "\gamma^0"'] & C^1 \arrow[r, "\gamma^1"'] & C^2 \arrow[r] & \cdots \arrow[r] & C^{d-1} \arrow[r, "\gamma^{d-1}"'] & C^d \arrow[r, "\gamma^d"'] & A^{d+1} \arrow[r] & 0.
\end{tikzcd}$$

Then $[f^0 \cdot \delta] = [\epsilon']$ in $\operatorname{Ext}^d_\Lambda(A^{d+1}, B^0)$.

*Proof.* Note that $f^0 \cdot \delta$ as described in Proposition 2.3.18(b) is obtained by extending a $d$-pushout diagram. The result then follows using [34, Proposition 4.8]. $\square$

#### Higher Auslander-Reiten theory

In [28, Section 3], Iyama introduced a higher analogue of Auslander-Reiten sequences in module categories, namely $d$-Auslander-Reiten sequences in $d$-cluster tilting subcategories $\mathcal{F}$ of module categories. The main result in [28, Section 3] is the proof of the existence of these sequences. An important property of these sequences is that their end terms are indecomposables determining each other, in the same way as in the classic module category setup.

In this section we recall Iyama's result, focusing on the right end term of $d$-Auslander-Reiten sequences in $\mathcal{F}$. Moreover, in [28, Theorem 1.5], Iyama also proved a higher version of the Auslander-Reiten duality, called $d$-Auslander-Reiten duality. We also recall this result and give an alternative proof for it.

**Definition 2.3.21.** We say that a $d$-exact sequence in $\mathcal{F}$ of the form

$$\epsilon : 0 \longrightarrow A^0 \xrightarrow{\alpha^0} A^1 \xrightarrow{\alpha^1} \cdots \longrightarrow A^{d-1} \xrightarrow{\alpha^{d-1}} A^d \xrightarrow{\alpha^d} A^{d+1} \longrightarrow 0,$$

is a *$d$-Auslander-Reiten sequence in $\mathcal{F}$* if the morphism $\alpha^0$ is left almost split in $\mathcal{F}$, the morphism $\alpha^d$ is right almost split in $\mathcal{F}$ and, when $d \geq 2$, also $\alpha^1, \ldots, \alpha^{d-1} \in \mathrm{rad}_{\mathcal{F}}$.

**Remark 2.3.22.** Note that if $\epsilon$ as above is a $d$-Auslander-Reiten sequence in $\mathcal{F}$, then $\mathrm{End}_\Lambda(A^0)$ and $\mathrm{End}_\Lambda(A^{d+1})$ are local and $\alpha^0$, $\alpha^d$ are in $\mathrm{rad}_{\mathcal{F}}$.

The following lemma presents equivalent definitions to the one of $d$-Auslander-Reiten sequence in $\mathcal{F}$. Instead of proving it here, we will later prove the more general Lemma 5.2.7, where the case $\mathcal{X} = \mathcal{F}$ corresponds to the following.

**Lemma 2.3.23.** Consider a $d$-exact sequence in $\mathcal{F}$ of the form:

$$\epsilon : 0 \longrightarrow A^0 \xrightarrow{\alpha^0} A^1 \xrightarrow{\alpha^1} \cdots \longrightarrow A^{d-1} \xrightarrow{\alpha^{d-1}} A^d \xrightarrow{\alpha^d} A^{d+1} \longrightarrow 0.$$

The following are equivalent:

(a) $\epsilon$ is a $d$-Auslander-Reiten sequence in $\mathcal{F}$,

(b) $\alpha^0, \alpha^1, \ldots, \alpha^{d-1}$ are in $\mathrm{rad}_{\mathcal{F}}$ and $\alpha^d$ is right almost split in $\mathcal{F}$,

(c) $\alpha^1, \ldots, \alpha^{d-1}$, $\alpha^d$ are in $\mathrm{rad}_{\mathcal{F}}$ and $\alpha^0$ is left almost split in $\mathcal{F}$.

**Definition 2.3.24** ([28, 1.4.1])**.** Let $M \in \operatorname{mod}\Lambda$ and consider an augmented minimal projective resolution of $M$ of the form:

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

The *dth transpose of* $M$ is $\operatorname{Tr}_d(M) := \operatorname{Coker}(\operatorname{Hom}_\Lambda(P_{d-1},\Lambda) \to \operatorname{Hom}_\Lambda(P_d,\Lambda))$. The *dth Auslander-Reiten translation of* $M$ and the *inverse dth Auslander-Reiten translation of* $M$ are $\tau_d(M) := D \circ \operatorname{Tr}_d(M)$ and $\tau_d^{-1}(M) := \operatorname{Tr}_d \circ D(M)$ respectively, where $D(-) := \operatorname{Hom}_k(-,k) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{op}$ is the standard $k$-duality from Example 2.1.23.

**Proposition 2.3.25** ([28, Theorem 3.3.1])**.** For each non-projective indecomposable object $A^{d+1}$ in $\mathcal{F}$, there exists a $d$-Auslander-Reiten sequence in $\mathcal{F}$ of the form:

$$\delta : \qquad 0 \longrightarrow A^0 \xrightarrow{\alpha^0} A^1 \xrightarrow{\alpha^1} A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \xrightarrow{\alpha^{d-1}} A^d \xrightarrow{\alpha^d} A^{d+1} \longrightarrow 0.$$

Moreover, if $\delta$ is a $d$-Auslander-Reiten sequence in $\mathcal{F}$, then $A^0 = \tau_d(A^{d+1})$.

We now give an alternative proof for the $d$-Auslander-Reiten duality, first proved in [28, Theorem 1.5]. This alternative proof is the higher analogue of Schiffler's proof of the Auslander-Reiten duality presented in [53, Theorem 7.18].

**Theorem 2.3.26.** *For all $M$ and $N$ in $\mathcal{F}$ there are functorial isomorphisms*

$$\operatorname{Ext}^d_\Lambda(M,N) \cong D\underline{\operatorname{Hom}}_\Lambda(\tau_d^{-1}(N),M) \cong D\overline{\operatorname{Hom}}_\Lambda(N,\tau_d(M)),$$

*where $\underline{\operatorname{Hom}}_\Lambda(-,-)$, respectively $\overline{\operatorname{Hom}}_\Lambda(-,-)$, denote* Hom*-spaces in the projectively, respectively injectively, stable category of* $\operatorname{mod}\Lambda$*, see Definition 2.2.24.*

*Proof.* We only prove that $\operatorname{Ext}^d_\Lambda(M,N) \cong D\underline{\operatorname{Hom}}_\Lambda(\tau_d^{-1}(N),M)$, the second isomorphism follows by a dual argument. Without loss of generality, assume that $N$ has no injective direct summands, so that $N = \tau_d(L)$ for some $A$-module $L$, by [28, Theorem 1.4.1]. Consider an augmented projective resolution of $L$:

$$\cdots \to P_d \xrightarrow{p_d} P_{d-1} \xrightarrow{p_{d-1}} \cdots \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} L \to 0. \tag{2.6}$$

As $P_{d-1} \xrightarrow{p_{d-1}} \cdots \to P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} L \to 0$ is a $d$-cokernel of $p_d$ in $\mathcal{F}$, writing $(-)^* = \operatorname{Hom}_\Lambda(-,\Lambda)$, we get the exact sequence:

$$0 \to L^* \to P_0^* \to P_1^* \to \cdots \to P_{d-1}^* \to P_d^* \to \operatorname{Tr}_d(L) \to 0. \tag{2.7}$$

Applying the exact functor $D(-)$ to (2.7), we then obtain the exact sequence:

$$0 \to \tau_d(L) \to \nu P_d \to \nu P_{d-1} \to \cdots \to \nu P_1 \to \nu P_0 \to \nu L \to 0,$$

where $\nu = D \circ \operatorname{Hom}_\Lambda(-, \Lambda)$ is the Nakayama functor and $\nu P_d, \dots, \nu P_0$ are injective $\Lambda$-modules. In particular, we have an injective resolution of $N = \tau_d(L)$ which starts:

$$\begin{tikzcd} 0 \arrow[r] & \tau_d(L) \arrow[r] & \nu P_d \arrow[r] & \nu P_{d-1} \arrow[r] & \cdots \arrow[r] & \nu P_1 \arrow[r] & \nu P_0 \arrow[rr, "p"] \arrow[dr, two heads] & & I^{d+1}. \\ & & & & & & & \nu L \arrow[ur, hook, "\iota"'] & \end{tikzcd}$$

Using the notation $\operatorname{Hom}_\Lambda(-,-) = (-,-)$, let $\overline{p_i} \coloneqq (M, \nu p_i)$ for $i = 0, \dots, d$. In order to compute $\operatorname{Ext}^d_\Lambda(M, N)$, we look at the complex:

$$\begin{tikzcd} (M, \nu P_1) \arrow[r, "\overline{p_1}"] & (M, \nu P_0) \arrow[rr, "{(M,p)}"] \arrow[dr, "\overline{p_0}"'] & & (M, I^{d+1}). \\ & & (M, \nu L) \arrow[ur, hook, "{(M,\iota)}"'] & \end{tikzcd}$$

Note that, as $\iota$ is injective and $(M, -)$ is left exact, then $(M, \iota)$ is injective and $\operatorname{Ker}(M, p) = \operatorname{Ker} \overline{p_0}$. Hence

$$\operatorname{Ext}^d_\Lambda(M, N) = \operatorname{Ker}(M, p)/\operatorname{Im} \overline{p_1} = \operatorname{Ker} \overline{p_0}/\operatorname{Im} \overline{p_1}. \tag{2.8}$$

Using again the fact that $P_{d-1} \xrightarrow{p_{d-1}} \cdots \to P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} L \to 0$ is a $d$-cokernel of $p_d$ in $\mathcal{F}$, and as $M \in \mathcal{F}$, we have the exact sequence:

$$0 \to (L, M) \to (P_0, M) \to (P_1, M) \to \cdots \to (P_{d-1}, M) \to (P_d, M).$$

Applying the exact functor $D(-)$ to it, we obtain the exact sequence:

$$D \circ (P_d, M) \xrightarrow{\widetilde{p_d}} D \circ (P_{d-1}, M) \xrightarrow{\widetilde{p_{d-1}}} \cdots \xrightarrow{\widetilde{p_1}} D \circ (P_0, M) \xrightarrow{\widetilde{p_0}} D \circ (L, M) \longrightarrow 0,$$

where $\widetilde{p_i} \coloneqq D \circ (p_i, M)$, for $i = 0, \dots, d$.

From now on, the proof proceeds as the proof of [53, Theorem 7.18]. Using [53, Lemma

7.22], we get the commutative diagram:

$$\begin{tikzcd} D\circ(P_1,M) \arrow[r, "\widetilde{p_1}"] \arrow[d, "\omega_M^{P_1}", "\wr"'] & D\circ(P_0,M) \arrow[r, "\widetilde{p_0}"] \arrow[d, "\omega_M^{P_0}", "\wr"'] & D\circ(L,M) \arrow[r] \arrow[d, "\omega_M^{L}"] & 0 \\ (M,\nu P_1) \arrow[r, "\overline{p_1}"'] & (M,\nu P_0) \arrow[r, "\overline{p_0}"'] & (M,\nu L), \end{tikzcd}$$

where the top row is exact, the bottom row is a complex, and the two vertical maps on the left are isomorphisms since $P_1$ and $P_0$ are projective modules. Define a morphism

$$\Psi = \widetilde{p_0}\circ(\omega_M^{P_0})^{-1}\,|_{\operatorname{Ker}\overline{p_0}}\colon \operatorname{Ker}\overline{p_0} \to D\circ(L,M).$$

**Claim 1:** $\operatorname{Ker}\Psi = \operatorname{Im}\overline{p_1}$.

We first show $\operatorname{Ker}\Psi \subseteq \operatorname{Im}\overline{p_1}$. Let $\overline{x}\in\operatorname{Ker}\Psi$, then $(\omega_M^{P_0})^{-1}(\overline{x})$ is in $\operatorname{Ker}\widetilde{p_0} = \operatorname{Im}\widetilde{p_1}$. Hence there is some $\widetilde{y}\in D\circ(P_1,M)$ such that $\widetilde{p_1}(\widetilde{y}) = (\omega_M^{P_0})^{-1}(\overline{x})$. So that

$$\overline{x} = \omega_M^{P_0}\circ\widetilde{p_1}(\widetilde{y}) = \overline{p_1}\circ\omega_M^{P_1}(\widetilde{y})$$

and $\overline{x}$ is in $\operatorname{Im}\overline{p_1}$.

In order to show the other inclusion, let $\overline{x}\in\operatorname{Ker}\overline{p_0}$ be such that $\overline{x} = \overline{p_1}(\overline{y})$ for some $\overline{y}$ in $(M,\nu P_1)$. Then

$$\Psi(\overline{x}) = \Psi\circ\overline{p_1}(\overline{y}) = \widetilde{p_0}\circ(\omega_M^{P_0})^{-1}\circ\overline{p_1}(\overline{y}) = \widetilde{p_0}\circ\widetilde{p_1}\circ(\omega_M^{P_1})^{-1}(\overline{y}) = 0,$$

so that $\overline{x}\in\operatorname{Ker}\Psi$.

**Claim 2:** $\operatorname{Im}\Psi = \operatorname{Ker}\omega_M^L$.

Note that if $\overline{x}\in\operatorname{Ker}\overline{p_0}$, then $\omega_M^L\circ\Psi(\overline{x}) = \omega_M^L\circ\widetilde{p_0}\circ(\omega_M^{P_0})^{-1}(\overline{x}) = \overline{p_0}(\overline{x}) = 0$ and so $\operatorname{Im}\Psi\subseteq\operatorname{Ker}\omega_M^L$.

Now suppose that $\widetilde{u}$ is in $\operatorname{Ker}\omega_M^L$. Since $\widetilde{p_0}$ is surjective, there exists some $\widetilde{x}$ in $D\circ(P_0,M)$ such that $\widetilde{u} = \widetilde{p_0}(\widetilde{x}) = \Psi\circ\omega_M^{P_0}(\widetilde{x})\in\operatorname{Im}\Psi$. Note that the last equality makes sense as $\omega_M^{P_0}(\widetilde{x})$ is in $\operatorname{Ker}\overline{p_0}$ since

$$\overline{p_0}\omega_M^{P_0}(\widetilde{x}) = \omega_M^L\circ\widetilde{p_0}(\widetilde{x}) = \omega_M^L(\widetilde{u}) = 0.$$

Now, using [53, Lemma 7.22], our two claims, the first isomorphism theorem and (2.8), we have that

$$D\circ\underline{\operatorname{Hom}}_\Lambda(L,M) = \operatorname{Ker}\omega_M^L = \operatorname{Im}\Psi \cong \operatorname{Ker}\overline{p_0}/\operatorname{Ker}\Psi = \operatorname{Ker}\overline{p_0}/\operatorname{Im}\overline{p_1} = \operatorname{Ext}_\Lambda^d(M,N).$$

□

### 2.3.2 $(d+2)$-angulated categories

In this section, we introduce $(d+2)$-angulated categories and some of their properties.

**Definition 2.3.27** ([19, Definition 1.1])**.** Let $\mathcal{M}$ be an additive category and $\Sigma^d$ be an automorphism of $\mathcal{M}$ with inverse $\Sigma^{-d}$. A $\Sigma^d$-*sequence* is a sequence of morphisms in $\mathcal{M}$ of the form

$$\epsilon: \qquad X^0 \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^d \xrightarrow{\xi^d} X^{d+1} \xrightarrow{\xi^{d+1}} \Sigma^d X^0. \tag{2.9}$$

A *morphism of* $\Sigma^d$-*sequences* is given by a sequence of morphisms $\varphi = (\varphi^0, \dots, \varphi^{d+1})$ such that the following diagram commutes:

$$\begin{tikzcd}
\epsilon: \arrow[d, "\varphi"] & X^0 \arrow[r, "\xi^0"] \arrow[d, "\varphi^0"] & X^1 \arrow[r, "\xi^1"] \arrow[d, "\varphi^1"] & X^2 \arrow[r] \arrow[d, "\varphi^2"] & \cdots \arrow[r] & X^d \arrow[r, "\xi^d"] \arrow[d, "\varphi^d"] & X^{d+1} \arrow[r, "\xi^{d+1}"] \arrow[d, "\varphi^{d+1}"] & \Sigma^d X^0 \arrow[d, "\Sigma^d(\varphi^0)"] \\
\epsilon': & Y^0 \arrow[r, "\eta^0"'] & Y^1 \arrow[r, "\eta^1"'] & Y^2 \arrow[r] & \cdots \arrow[r] & Y^d \arrow[r, "\eta^d"'] & Y^{d+1} \arrow[r, "\eta^{d+1}"'] & \Sigma^d Y^0.
\end{tikzcd}$$

**Definition 2.3.28** ([19, Definition 1.1])**.** A $(d+2)$-*angulated category* is a triple $(\mathcal{M}, \Sigma^d, \pentagon)$, where $\mathcal{M}$ and $\Sigma^d$ are as above and $\pentagon$ is a collection of $\Sigma^d$-sequences, called $(d+2)$-*angles*, satisfying the following axioms.

(N1) The collection $\pentagon$ is closed under isomorphisms, direct sums and direct summands and, for every $X \in \mathcal{M}$, the *trivial* $\Sigma^d$-*sequence*

$$\epsilon: \qquad X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \Sigma^d X$$

is in $\pentagon$. For each morphism $\xi^0 : X^0 \to X^1$ in $\mathcal{M}$, there is a $(d+2)$-angle in $\pentagon$ of the form (2.9).

(N2) A $\Sigma^d$-sequence (2.9) is in $\pentagon$ if and only if so is its *left rotation*:

$$X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^d \xrightarrow{\xi^d} X^{d+1} \xrightarrow{\xi^{d+1}} \Sigma^d X^0 \xrightarrow{(-1)^d \Sigma^d(\xi^0)} \Sigma^d X^1.$$

(N3) Each commutative diagram of solid arrows, with rows in $\pentagon$

$$\begin{tikzcd}
X^0 \arrow[r, "\xi^0"] \arrow[d, "\varphi^0"] & X^1 \arrow[r, "\xi^1"] \arrow[d, "\varphi^1"] & X^2 \arrow[r] \arrow[d, dashed, "\varphi^2"] & \cdots \arrow[r] & X^d \arrow[r, "\xi^d"] \arrow[d, dashed, "\varphi^d"] & X^{d+1} \arrow[r, "\xi^{d+1}"] \arrow[d, dashed, "\varphi^{d+1}"] & \Sigma^d X^0 \arrow[d, "\Sigma^d(\varphi^0)"] \\
Y^0 \arrow[r, "\eta^0"'] & Y^1 \arrow[r, "\eta^1"'] & Y^2 \arrow[r] & \cdots \arrow[r] & Y^d \arrow[r, "\eta^d"'] & Y^{d+1} \arrow[r, "\eta^{d+1}"'] & \Sigma^d Y^0,
\end{tikzcd}$$

can be completed as indicated to a morphism of $\Sigma^d$-sequences.

(N4) In the situation of (N3), the morphisms $\varphi^2, \dots, \varphi^{d+1}$ can be chosen such that

$$X^1 \oplus Y^0 \xrightarrow{\begin{pmatrix} -\xi^1 & 0 \\ \varphi^1 & \eta^0 \end{pmatrix}} X^2 \oplus Y^1 \longrightarrow \cdots \longrightarrow \Sigma^d X^0 \oplus Y^{d+1} \xrightarrow{\begin{pmatrix} -\Sigma^d(\xi^0) & 0 \\ \Sigma(\varphi^0) & \eta^{d+1} \end{pmatrix}} \Sigma^d X^1 \oplus \Sigma^d Y^0$$

belongs to $\pentagon$.

**Setup 2.3.29.** Let $d$ be a fixed positive integer and $\mathcal{M}$ be a $(d+2)$-angulated category.

**Remark 2.3.30.** Note that by [19, Proposition 2.5(a)], any $(d+2)$-angle in $\mathcal{M}$ of the form

$$X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots \longrightarrow X^d \longrightarrow X^{d+1} \longrightarrow \Sigma^d X^0,$$

is such that the induced sequence

$$\cdots \to \mathcal{M}(-, \Sigma^{-d} X^{d+1}) \to \mathcal{M}(-, X^0) \to \cdots \to \mathcal{M}(-, X^{d+1}) \to \mathcal{M}(-, \Sigma^d X^0) \to \cdots$$

is exact, where we used the notation $\mathrm{Hom}_{\mathcal{M}}(-,-) = \mathcal{M}(-,-)$. Moreover, note that the dual of the above is also true by [19, Remark 2.2(c) and Proposition 2.5(a)].

**Lemma 2.3.31.** Any two consecutive morphisms in a $(d+2)$-angle compose to zero.

*Proof.* By (N2), it is enough to prove $\xi^1 \circ \xi^0 = 0$. By (N3), we have a commutative diagram of the form:

$$\begin{tikzcd}
X^0 \arrow[r, "1_{X^0}"] \arrow[d, equal] & X^0 \arrow[r] \arrow[d, "\xi^0"] & 0 \arrow[r] \arrow[d, dashed] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d, dashed] & 0 \arrow[r, "\xi^{d+1}"] \arrow[d, dashed] & \Sigma^d X^0 \arrow[d, equal] \\
X^0 \arrow[r, "\xi^0"'] & X^1 \arrow[r, "\xi^1"'] & X^2 \arrow[r] & \cdots \arrow[r] & X^d \arrow[r, "\xi^d"'] & X^{d+1} \arrow[r, "\xi^{d+1}"'] & \Sigma^d X^0.
\end{tikzcd}$$

In particular, $\xi^1 \circ \xi^0 = 0$. $\square$

The following two lemmas are reformulations of Remark 2.3.30.

**Lemma 2.3.32.** Consider

$$
\begin{tikzcd}
 & & A \arrow[dll, dashed] \arrow[d, "\alpha"] \arrow[dr, "0"] & & & & \\
\epsilon: & X^0 \arrow[r, "\xi^0"'] & X^1 \arrow[r, "\xi^1"'] & X^2 \arrow[r] & \cdots \arrow[r] & X^{d+1} \arrow[r, "\xi^{d+1}"'] & \Sigma^d X^0,
\end{tikzcd}
$$

where $\epsilon$ is a $(d+2)$-angle in $\mathcal{M}$. Then $\xi^1 \circ \alpha = 0$ if and only if there exists a morphism $\delta : A \to X^0$ such that $\xi^0 \circ \delta = \alpha$.

**Lemma 2.3.33.** Consider

$$
\begin{tikzcd}
\epsilon: & X^0 \arrow[r, "\xi^0"] & X^1 \arrow[r] & \cdots \arrow[r] & X^d \arrow[r, "\xi^d"] \arrow[dr, "0"'] & X^{d+1} \arrow[r, "\xi^{d+1}"] \arrow[d, "\varphi"'] & \Sigma^d X^0, \arrow[dl, dashed] \\
 & & & & & A &
\end{tikzcd}
$$

where $\epsilon$ is a $(d+2)$-angle in $\mathcal{M}$. Then $\varphi \circ \xi^d = 0$ if and only if there exists a morphism $\phi : \Sigma^d X^0 \to A$ such that $\phi \circ \xi^{d+1} = \varphi$.

The following lemmas are well-known in the triangulated case. Here we present and prove their higher-angulated analogues. In the first one, we use the radical of the category $\mathcal{M}$, see Definition 2.1.8.

**Lemma 2.3.34.** Consider a $(d+2)$-angle of the form

$$X^0 \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^d \xrightarrow{\xi^d} X^{d+1} \xrightarrow{\xi^{d+1}} \Sigma^d X^0.$$

Then:

(a) $\xi^1$ is right minimal if and only if $\xi^0 \in \mathrm{rad}_{\mathcal{M}}$,

(b) $\xi^d$ is left minimal if and only if $\xi^{d+1} \in \mathrm{rad}_{\mathcal{M}}$.

*Proof.* We only prove (a), then (b) follows by a similar argument. Suppose that $\xi^1$ is right minimal. Then, for any $\alpha : X^1 \to X^0$, we have

$$\xi^1 \circ (1_{X^1} - \xi^0 \circ \alpha) = \xi^1 - \xi^1 \circ \xi^0 \circ \alpha = \xi^1,$$

where the last step follows from Lemma 2.3.31. Then, as $\xi^1$ is right minimal, we have that $1_{X^1} - \xi^0 \circ \alpha$ is invertible and so $\xi^0$ is in $\mathrm{rad}_{\mathcal{M}}$.

Suppose now that $\xi^0$ is in $\mathrm{rad}_{\mathcal{M}}$. Given $\varphi : X^1 \to X^1$ such that $\xi^1 \circ \varphi = \xi^1$, we have $\xi^1 \circ (\varphi - 1_{X^1}) = 0$. Then, by Lemma 2.3.32, there exists a morphism $\delta : X^1 \to X^0$ such that

$\xi^0 \circ \delta = \varphi - 1_{X^1}$. Hence, since $\xi^0 \in \mathrm{rad}_{\mathcal{M}}$, we have that $\varphi = 1_{X^1} + \xi^0 \circ \delta$ is invertible and so $\xi^1$ is right minimal. □

**Lemma 2.3.35.** Consider a $(d+2)$-angle of the form

$$X^0 \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^d \xrightarrow{\xi^d} X^{d+1} \xrightarrow{\xi^{d+1}} \Sigma^d X^0.$$

The following are equivalent:

(a) $\xi^{d+1} = 0$,

(b) $\xi^d$ is a split epimorphism,

(c) $\xi^0$ is a split monomorphism.

*Proof.* Suppose (a) holds and consider the morphism of $(d+2)$-angles:

$$\begin{tikzcd}
\Sigma^{-d}X^{d+1} \arrow[r, "0"] \arrow[d] & X^0 \arrow[r, "\xi^0"] \arrow[d, equal] & X^1 \arrow[r, "\xi^1"] \arrow[d, dashed, "\delta"] & X^2 \arrow[r] \arrow[d, dashed] & \cdots \arrow[r] & X^d \arrow[r, "\xi^d"] \arrow[d, dashed] & X^{d+1} \arrow[d] \\
0 \arrow[r] & X^0 \arrow[r, "1_{X^0}"] & X^0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

Then, $\delta \circ \xi^0 = 1_{X^0}$ and (c) holds.

Suppose now that (c) holds. Hence there exists a morphism $\delta : X^1 \to X^0$ such that $\delta \circ \xi^0 = 1_{X^0}$ and we have morphism of $(d+2)$-angles:

$$\begin{tikzcd}
X^0 \arrow[r, "\xi^0"] \arrow[d, equal] & X^1 \arrow[r, "\xi^1"] \arrow[d, "\delta"] & X^2 \arrow[r] \arrow[d, dashed] & \cdots \arrow[r] & X^d \arrow[r, "\xi^d"] \arrow[d, dashed] & X^{d+1} \arrow[r, "\xi^{d+1}"] \arrow[d, dashed] & \Sigma^d X^0 \arrow[d, equal] \\
X^0 \arrow[r, "1_{X^0}"] & X^0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \Sigma^d X^0.
\end{tikzcd}$$

Hence $1_{\Sigma^d X^0} \circ \xi^{d+1} = 0$ and so $\xi^{d+1} = 0$, that is (a) holds.

By a dual argument, (a) and (b) are equivalent. □

**Lemma 2.3.36.** Let $\alpha : A \to B$ be a non-zero morphism in $\mathcal{M}$. If $B$ has local endomorphism ring, then $\alpha$ is left minimal and if $A$ has local endomorphism ring, then $\alpha$ is right minimal.

*Proof.* Suppose that $\mathrm{End}(B)$ is a local ring and let $\varphi : B \to B$ be a morphism such that $\varphi \circ \alpha = \alpha$. Consider the finitely generated $\mathrm{End}(B)$-module $M := \mathrm{End}(B) \circ \alpha \subseteq \mathrm{Hom}(A, B)$. If $\varphi \in \mathrm{rad}_{\mathcal{M}}$, then $\varphi \circ \alpha = \alpha \in \mathrm{rad}_{\mathcal{M}}$ and $M \subseteq \mathrm{rad}_{\mathcal{M}} \circ M$. Then, by Nakayama's Lemma, see [1, Corollary 15.13], we have that $M = 0$. This is a contradiction to $\alpha$ being non-zero.

Hence $\varphi$ is not in $\mathrm{rad}_{\mathcal{M}}$ and, since $\mathrm{End}(B)$ is local, we have that $\varphi$ is an isomorphism by [45, Section 4] and $\alpha$ is left minimal. The rest of the lemma is true by a similar argument. $\square$

For the rest of this section, we work in the following setup.

**Setup 2.3.37.** Let $d$ be a fixed positive integer and $k$ a field. Let $\mathcal{M}$ be a skeletally small $k$-linear Hom-finite $(d+2)$-angulated category with split idempotents. Note that this implies that $\mathcal{M}$ is Krull-Schmidt by Remark 2.1.12.

In the case when $d = 1$, so in the case of a triangulated category, a morphism can be extended to a triangle in a unique way up to isomorphism. On the other hand, for $d > 1$, a morphism can be extendend to a $(d+2)$-angle in different non-isomorphic ways. However, we still have a unique "minimal" $(d+2)$-angle extending any given morphism. The following was first proven in [49, Lemma 5.18], but we present here a detailed proof.

**Lemma 2.3.38.** Let $d > 1$ and $\delta : M'' \to \Sigma^d M'$ be a morphism in $\mathcal{M}$. Then, up to isomorphism, there exists a unique $(d+2)$-angle of the form

$$M' \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} M'' \xrightarrow{\delta} \Sigma^d M',$$

with $\xi^1, \ldots, \xi^{d-1}$ in $\mathrm{rad}_{\mathcal{M}}$.

We present some lemmas that will then be used in the proof of Lemma 2.3.38.

**Lemma 2.3.39.** A $(d+2)$-angle of the form

$$\epsilon : A \oplus X^0 \xrightarrow{\begin{pmatrix} 1_A & \beta \\ \alpha & \xi^0 \end{pmatrix}} A \oplus X^1 \xrightarrow{(\psi, \xi^1)} X^2 \xrightarrow{\xi^2} \cdots \longrightarrow X^d \xrightarrow{\xi^d} X^{d+1} \xrightarrow{\begin{pmatrix} \varphi \\ \xi^{d+1} \end{pmatrix}} \Sigma^d (A \oplus X^0)$$

is isomorphic to the direct sum of the two $(d+2)$-angles

$$A \xrightarrow{1_A} A \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \Sigma^d A \qquad \text{and}$$

$$X^0 \xrightarrow{\overline{\xi^0}} X^1 \xrightarrow{\xi^1} X^2 \xrightarrow{\xi^2} \cdots \longrightarrow X^d \xrightarrow{\xi^d} X^{d+1} \xrightarrow{\xi^{d+1}} \Sigma^d X^0,$$

where $\overline{\xi^0} = -\alpha \circ \beta + \xi^0$.

*Proof.* It is easy to check that the following are isomorphisms of $(d+2)$-angles:

$$\begin{tikzcd}
\epsilon: & A\oplus X^0 \arrow[r, "\begin{pmatrix}1_A & \beta\\ \alpha & \xi^0\end{pmatrix}"] \arrow[d, equal] & A\oplus X^1 \arrow[r, "(\psi,\xi^1)"] \arrow[d, "\begin{pmatrix}1_A & 0\\ -\alpha & 1_{X^1}\end{pmatrix}"] & X^2 \arrow[r, "\xi^2"] \arrow[d, equal] & \cdots \arrow[r] & X^{d+1} \arrow[r, "\begin{pmatrix}\varphi\\ \xi^{d+1}\end{pmatrix}"] \arrow[d, equal] & \Sigma^d(A\oplus X^0) \arrow[d, equal] \\
\epsilon': & A\oplus X^0 \arrow[r, "\begin{pmatrix}1_A & \beta\\ 0 & -\alpha\circ\beta+\xi^0\end{pmatrix}"] \arrow[d, "\begin{pmatrix}1_A & \beta\\ 0 & 1_{X^0}\end{pmatrix}"'] & A\oplus X^1 \arrow[r, "(0,\xi^1)"] \arrow[d, equal] & X^2 \arrow[r, "\xi^2"] \arrow[d, equal] & \cdots \arrow[r] & X^{d+1} \arrow[r, "\begin{pmatrix}\varphi\\ \xi^{d+1}\end{pmatrix}"] \arrow[d, equal] & \Sigma^d(A\oplus X^0) \arrow[d, "\begin{pmatrix}1_{\Sigma^d A} & \Sigma^d(\beta)\\ 0 & 1_{\Sigma^d X^0}\end{pmatrix}"'] \\
\epsilon'': & A\oplus X^0 \arrow[r, "\begin{pmatrix}1_A & 0\\ 0 & -\alpha\circ\beta+\xi^0\end{pmatrix}"'] & A\oplus X^1 \arrow[r, "(0,\xi^1)"'] & X^2 \arrow[r, "\xi^2"'] & \cdots \arrow[r] & X^{d+1} \arrow[r, "\begin{pmatrix}0\\ \xi^{d+1}\end{pmatrix}"'] & \Sigma^d(A\oplus X^0).
\end{tikzcd}$$

So $\epsilon \cong \epsilon''$ and $\epsilon''$ is clearly isomorphic to the direct sum of the two given $(d+2)$-angles. □

**Lemma 2.3.40.** Let $\delta : M'' \to \Sigma^d M'$ be any morphism in $\mathcal{M}$ and consider a $(d+2)$-angle extending it:

$$M' \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} M'' \xrightarrow{\delta} \Sigma^d M'.$$

Then $\xi^1, \dots, \xi^{d-1}$ are in $\mathrm{rad}_{\mathcal{M}}$ if and only if for $i = 1, \dots, d-1$ there is no $A \ncong 0$ in $\mathcal{M}$ such that

$$\xi^i \cong \begin{pmatrix}1_A & \beta\\ \alpha & \overline{\xi^i}\end{pmatrix} : A \oplus \overline{X^i} \to A \oplus \overline{X^{i+1}}.$$

*Proof.* Suppose first that $\xi^1, \dots, \xi^{d-1}$ are in $\mathrm{rad}_{\mathcal{M}}$. Then, for $i = 1, \dots, d-1$, we have that each component of $\xi^i$ is in $\mathrm{rad}_{\mathcal{M}}$ by [2, Lemma A.3.4(b)] and so for every $A \in \mathcal{M}$, we have

$$\xi^i \ncong \begin{pmatrix}1_A & \beta\\ \alpha & \overline{\xi^i}\end{pmatrix},$$

as $1_A$ is not in $\mathrm{rad}_{\mathcal{M}}$.

Suppose now that for every $i = 1, \dots, d-1$ there is no $A \ncong 0$ in $\mathcal{M}$ such that

$$\xi^i \cong \begin{pmatrix}1_A & \beta\\ \alpha & \overline{\xi^i}\end{pmatrix} : A \oplus \overline{X^i} \to A \oplus \overline{X^{i+1}}.$$

Suppose that for some $i$, we have $\xi^i$ not in $\mathrm{rad}_{\mathcal{M}}$. Then, there is a component $\eta$ of $\xi^i$, from an indecomposable direct summand $Y^i$ of $X^i$ to an indecomposable direct summand $Y^{i+1}$ of $X^{i+1}$, that is not in $\mathrm{rad}_{\mathcal{M}}$. Then, $\eta$ is an isomorphism, and without loss of generality we may assume it is $1_{Y^i}$. So

$$\xi^i \cong \begin{pmatrix} 1_{Y^i} & \beta \\ \alpha & \overline{\xi^i} \end{pmatrix} : Y^i \oplus \overline{X^i} \to Y^i \oplus \overline{X^{i+1}},$$

contradicting our initial assumption. Hence $\xi^i \in \mathrm{rad}_{\mathcal{M}}$ for all $i = 1, \ldots, d-1$. □

We present a lemma that will be useful to prove uniqueness of the $(d+2)$-angle from Lemma 2.3.38.

**Lemma 2.3.41.** Consider $(d+2)$-angles and morphisms in $\mathcal{M}$ of the form:

$$\begin{tikzcd}
A^0 \arrow[r, "\alpha^0"] \arrow[d, "\varphi^0"] & A^1 \arrow[r, "\alpha^1"] \arrow[d, "\varphi^1"] & A^2 \arrow[r] \arrow[d, "\varphi^2"] & \cdots \arrow[r] & A^d \arrow[r, "\alpha^d"] \arrow[d, "\varphi^d"] & A^{d+1} \arrow[r, "\alpha^{d+1}"] \arrow[d, "\varphi^{d+1}"] & \Sigma^d A^0, \arrow[d, "\Sigma^d(\varphi^0)"] \\
B^0 \arrow[r, "\beta^0"] \arrow[d, "\theta^0"] & B^1 \arrow[r, "\beta^1"] \arrow[d, "\theta^1"] & B^2 \arrow[r] \arrow[d, "\theta^2"] & \cdots \arrow[r] & B^d \arrow[r, "\beta^d"] \arrow[d, "\theta^d"] & B^{d+1} \arrow[r, "\beta^{d+1}"] \arrow[d, "\theta^{d+1}"] & \Sigma^d B^0, \arrow[d, "\Sigma^d(\theta^0)"] \\
A^0 \arrow[r, "\alpha^0"] & A^1 \arrow[r, "\gamma^1"] & C^2 \arrow[r] & \cdots \arrow[r] & C^d \arrow[r, "\gamma^d"] & C^{d+1} \arrow[r, "\gamma^{d+1}"] & \Sigma^d A^0.
\end{tikzcd}$$

Suppose that $\theta^0 \circ \varphi^0 = 1_{A^0}$ and $\alpha^1$ is in $\mathrm{rad}_{\mathcal{M}}$, then $\theta^1 \circ \varphi^1$ is an isomorphism.

*Proof.* Since $\alpha^1 \in \mathrm{rad}_{\mathcal{M}}$, by Lemma 2.3.34 it follows that $\alpha^0$ is left minimal. Then, since

$$\theta^1 \circ \varphi^1 \circ \alpha^0 = \alpha^0 \circ \theta^0 \circ \varphi^0 = \alpha^0 \circ 1_{A^0} = \alpha^0,$$

we have that $\theta^1 \circ \varphi^1$ is an isomorphism. □

*Proof of Lemma 2.3.38.* We first discuss the existence of such a $(d+2)$-angle. By (N1) and (N2), it is possible to extend $\delta$ to a $(d+2)$-angle of the form

$$\epsilon : \qquad M' \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} M'' \xrightarrow{\delta} \Sigma^d M'.$$

If for some $i = 1, \ldots, d-1$, there is a direct summand $A$ of $X^i$ isomorphic to a direct summand of $X^{i+1}$, then without loss of generality we have

$$\xi^i = \begin{pmatrix} 1_A & \beta \\ \alpha & \overline{\xi^i} \end{pmatrix} : A \oplus \overline{X^i} \to A \oplus \overline{X^{i+1}}.$$

By Lemma 2.3.39, $\epsilon$ is isomorphic to the direct sum of a rotation of

$$A \xrightarrow{1_A} A \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \Sigma^d A \qquad \text{and}$$

$$\epsilon' : M' \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} \cdots \longrightarrow X^{i-1} \xrightarrow{\xi^{i-1}} \overline{X^i} \xrightarrow{\overline{\xi^i}} \overline{X^{i+1}} \longrightarrow \cdots \longrightarrow M'' \xrightarrow{\delta} \Sigma^d M'.$$

Now, starting from $\epsilon'$, repeat this process. By Lemma 2.3.40, we will eventually end up with a $(d+2)$-angle with the required properties.

We now prove that such a $(d+2)$-angle is unique up to isomorphism. Suppose that

$$\epsilon : M' \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} M'' \xrightarrow{\delta} \Sigma^d M',$$

$$\epsilon' : M' \xrightarrow{\eta^0} Y^1 \xrightarrow{\eta^1} Y^2 \longrightarrow \cdots \longrightarrow Y^{d-1} \xrightarrow{\eta^{d-1}} Y^d \xrightarrow{\eta^d} M'' \xrightarrow{\delta} \Sigma^d M'$$

are $(d+2)$-angles of the desired form. By (N3), we obtain the following two morphisms

$$\begin{tikzcd}
\epsilon : \arrow[d, "\psi"] & M' \arrow[r, "\xi^0"] \arrow[d, equal] & X^1 \arrow[r, "\xi^1"] \arrow[d, "\psi^1"] & X^2 \arrow[r] \arrow[d, "\psi^2"] & \cdots \arrow[r] & X^{d-1} \arrow[r, "\xi^{d-1}"] \arrow[d, "\psi^{d-1}"] & X^d \arrow[r, "\xi^d"] \arrow[d, "\psi^d"] & M'' \arrow[r, "\delta"] \arrow[d, equal] & \Sigma^d M' \arrow[d, equal] \\
\epsilon' : \arrow[d, "\phi"] & M' \arrow[r, "\eta^0"] \arrow[d, equal] & Y^1 \arrow[r, "\eta^1"] \arrow[d, "\phi^1"] & Y^2 \arrow[r] \arrow[d, "\phi^2"] & \cdots \arrow[r] & Y^{d-1} \arrow[r, "\eta^{d-1}"] \arrow[d, "\phi^{d-1}"] & Y^d \arrow[r, "\eta^d"] \arrow[d, "\phi^d"] & M'' \arrow[r, "\delta"] \arrow[d, equal] & \Sigma^d M' \arrow[d, equal] \\
\epsilon : & M' \arrow[r, "\xi^0"] & X^1 \arrow[r, "\xi^1"] & X^2 \arrow[r] & \cdots \arrow[r] & X^{d-1} \arrow[r, "\xi^{d-1}"] & X^d \arrow[r, "\xi^d"] & M'' \arrow[r, "\delta"] & \Sigma^d M'.
\end{tikzcd}$$

By Lemma 2.3.41, we have that $\phi^1 \circ \psi^1$ is an isomorphism. Using $(\phi^1 \circ \psi^1)^{-1}$ and (N3) we obtain the following morphism $\theta$:

$$\begin{tikzcd}
\epsilon : \arrow[d, "\theta"] & M' \arrow[r, "\xi^0"] \arrow[d, equal] & X^1 \arrow[r, "\xi^1"] \arrow[d, "(\phi^1\circ\psi^1)^{-1}"] & X^2 \arrow[r] \arrow[d, "\theta^2"] & \cdots \arrow[r] & X^{d-1} \arrow[r, "\xi^{d-1}"] \arrow[d, "\theta^{d-1}"] & X^d \arrow[r, "\xi^d"] \arrow[d, "\theta^d"] & M'' \arrow[r, "\delta"] \arrow[d, "\theta^{d+1}"] & \Sigma^d M' \arrow[d, equal] \\
\epsilon : & M' \arrow[r, "\xi^0"] & X^1 \arrow[r, "\xi^1"] & X^2 \arrow[r] & \cdots \arrow[r] & X^{d-1} \arrow[r, "\xi^{d-1}"] & X^d \arrow[r, "\xi^d"] & M'' \arrow[r, "\delta"] & \Sigma^d M'.
\end{tikzcd}$$

Applying Lemma 2.3.41 to $\epsilon \xrightarrow{\phi\circ\psi} \epsilon \xrightarrow{\theta} \epsilon$ and $\epsilon \xrightarrow{\theta} \epsilon \xrightarrow{\phi\circ\psi} \epsilon$, we have that $\theta^2 \circ (\phi^2 \circ \psi^2)$ and $(\phi^2 \circ \psi^2) \circ \theta^2$ are isomorphisms. Hence $\phi^2 \circ \psi^2$ is an isomorphism. The same argument can be repeated to prove that

$$\phi^3 \circ \psi^3, \ldots, \phi^{d-2} \circ \psi^{d-2}, \phi^{d-1} \circ \psi^{d-1}$$

are isomorphisms. Since $\xi^{d-1} \in \operatorname{rad}_{\mathcal{M}}$, by Lemma 2.3.34(a) we have that $\xi^d$ is right minimal and so $\xi^d = \xi^d \circ \phi^d \circ \psi^d$ implies that $\phi^d \circ \psi^d$ is an isomorphism. Hence $\phi \circ \psi$ is an isomorphism in $\mathcal{M}$. By a dual argument, we have that $\psi \circ \phi$ is an isomorphism in $\mathcal{M}$. Hence $\psi$ and $\phi$ are isomorphisms in $\mathcal{M}$ and the $(d+2)$-angles $\epsilon$ and $\epsilon'$ are isomorphic. $\square$

We now present some special $(d+2)$-angulated categories, arising from $d$-cluster tilting subcategories of module categories. In a similar way to $d$-cluster tilting subcategories of $\operatorname{mod}\Lambda$, we can define $d$-cluster tilting subcategories of a triangulated category $\mathcal{T}$.

**Definition 2.3.42** ([32, Section 3])**.** A *d-cluster tilting subcategory of a triangulated category* $\mathcal{T}$ is a functorially finite, full subcategory $\mathcal{C}$ of $\mathcal{T}$ satisfying

$$\mathcal{C} = \{X \in \mathcal{T} \mid \operatorname{Ext}_{\mathcal{T}}^{1\ldots d-1}(\mathcal{C}, X) = 0\} = \{X \in \mathcal{T} \mid \operatorname{Ext}_{\mathcal{T}}^{1\ldots d-1}(X, \mathcal{C}) = 0\}.$$

We see how some of these categories can be used to construct $(d+2)$-angulated categories. The following is a result by Geiss, Keller and Opperman, see [19, Theorem 1].

**Theorem 2.3.43.** *Let $\mathcal{D}$ be a triangulated category with suspension functor $\Sigma$. Let $\mathcal{C} \subseteq \mathcal{D}$ be a d-cluster tilting subcategory satisfying $\Sigma^d(\mathcal{C}) \subseteq \mathcal{C}$. Then $(\mathcal{C}, \Sigma^d, \pentagon)$ is a $(d+2)$-angulated category, where $\pentagon$ constists of the $\Sigma^d$-sequences of the form*

$$C^0 \xrightarrow{\gamma^0} C^1 \xrightarrow{\gamma^1} C^2 \to \cdots \to C^d \xrightarrow{\gamma^d} C^{d+1} \xrightarrow{\gamma^{d+1}} \Sigma^d C^0$$

*coming from diagrams in $\mathcal{D}$ of the form*

$$\begin{tikzcd}
& C^1 \arrow[rr, "\gamma^1"] \arrow[dr] & & C^2 \arrow[r] \arrow[dr] & {} & \cdots & {} \arrow[r] & C^{d-1} \arrow[rr, "\gamma^{d-1}"] \arrow[dr] & & C^d \arrow[dr, "\gamma^d"] & \\
C^0 \arrow[ur, "\gamma^0"] & & X^1 \arrow[ll, squiggly] \arrow[ur] & & X^2 \arrow[ll, squiggly] & \cdots & X^{d-2} \arrow[ur] & & X^{d-1} \arrow[ll, squiggly] \arrow[ur] & & C^{d+1}. \arrow[ll, squiggly]
\end{tikzcd} \tag{2.10}$$

*In the above, by $X \rightsquigarrow Y$ , we mean a morphism $X \to \Sigma Y$ and the composition of all the wavy arrows is $\gamma^{d+1}$. Each oriented triangle is a triangle in $\mathcal{D}$ and each non-oriented triangle is commutative.*

**Remark 2.3.44.** Let $\Lambda$ be a finite dimensional $k$-algebra with global dimension at most $d$. By [29, Theorem 1.6], if $\operatorname{mod}\Lambda$ has a $d$-cluster tilting subcategory, then this is unique and it is

$$\mathcal{F} = \operatorname{add}\{\tau_d^j(I) \mid I \text{ is injective in } \operatorname{mod}\Lambda \text{ and } j \geq 0\}.$$

Moreover, Iyama proved in [29, Theorem 1.21] that

$$\overline{\mathcal{F}} \coloneqq \operatorname{add}\{\Sigma^{di}\mathcal{F} \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}^b(\operatorname{mod}\Lambda)$$

is $d$-cluster tilting in $\mathcal{D}^b(\operatorname{mod}\Lambda)$. Then, since $\Sigma^d(\overline{\mathcal{F}}) \subseteq \overline{\mathcal{F}}$, we have that $\overline{\mathcal{F}}$ is a $(d+2)$-angulated category by Theorem 2.3.43. Note that this is the only known example of a $d$-abelian category $\mathcal{F}$ embedded into a $(d+2)$-angulated category $\overline{\mathcal{F}}$ where

$$\operatorname{Ext}^{di}_{\mathcal{F}}(A, B) \cong \operatorname{Hom}_{\overline{\mathcal{F}}}(A, \Sigma^{di} B),$$

for $A$, $B \in \mathcal{F}$ and $i \in \mathbb{Z}$.

In this situation, the $d$-abelian category $\mathcal{F}$ plays the role of a higher $\operatorname{mod}\Lambda$ and the $(d+2)$-angulated category $\overline{\mathcal{F}}$ of a higher derived category of $\mathcal{F}$.

**Remark 2.3.45.** In the situation of Remark 2.3.44, we have that any $d$-exact sequence in $\mathcal{F}$ induces a $(d+2)$-angle in $\overline{\mathcal{F}}$. In fact, any $d$-exact sequence in $\mathcal{F}$:

$$0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^d \longrightarrow C^{d+1} \longrightarrow 0,$$

can be decomposed into short exact sequences which correspond to triangles in $\mathcal{D}^b(\operatorname{mod}\Lambda)$. Hence, we obtain a diagram of the form (2.10), and so a $(d+2)$-angle in $\overline{\mathcal{F}}$ of the form

$$C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^d \longrightarrow C^{d+1} \longrightarrow \Sigma^d C^0.$$

### Higher-angulated Auslander-Reiten theory

The definition of Auslander-Reiten $(d+2)$-angles was first introduced by Iyama and Yoshino in [32, Definition 3.8]. In this thesis, we use a modified definition since we force the end terms of any Auslander-Reiten $(d+2)$-angle to be indecomposable, or equivalently to have local endomorphism rings as pointed out in Lemma 2.3.47. This change has been made to match with the classic homological algebra theory. In fact, the end terms of an Auslander-Reiten triangle are always indecomposable objects, see Definition 2.2.40 and Lemma 2.2.42.

We do not prove the results presented in this section now. Instead, we will later prove the more general Lemmas 4.4.4, 4.4.6 and 4.4.7. In these more general lemmas, $\mathcal{W}$ will be an additive subcategory of $\mathcal{M}$ closed under $d$-extensions and the case $\mathcal{W} = \mathcal{M}$ will respectively give us Lemmas 2.3.47, 2.3.49 and 2.3.50.

**Definition 2.3.46.** A $(d+2)$-angle in $\mathcal{M}$ of the form

$$\epsilon : \qquad X^0 \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^d \xrightarrow{\xi^d} X^{d+1} \xrightarrow{\xi^{d+1}} \Sigma^d X^0$$

is an *Auslander-Reiten* $(d+2)$*-angle* if $\xi^0$ is left almost split, $\xi^d$ is right almost split and, when $d \geq 2$, also $\xi^1, \ldots, \xi^{d-1} \in \operatorname{rad}_{\mathcal{M}}$.

**Lemma 2.3.47.** (a) Let $\xi^0 : X^0 \to X^1$ be left almost split, then $\mathrm{End}(X^0)$ is local and $\xi^0 \in \mathrm{rad}_{\mathcal{M}}$.

(b) Let $\xi^d : W^d \to W^{d+1}$ be right almost split, then $\mathrm{End}(X^{d+1})$ is local and $\xi^d \in \mathrm{rad}_{\mathcal{M}}$.

**Remark 2.3.48.** Suppose $\epsilon$ as in Definition 2.3.46 is an Auslander-Reiten $(d+2)$-angle. When $d = 1$, we have $\xi^0$ and $\xi^d$ in $\mathrm{rad}_{\mathcal{M}}$, so that $\xi^d$ is right minimal and $\xi^0$ is left minimal. When $d \geq 2$, since $\xi^{d-1} \in \mathrm{rad}_{\mathcal{M}}$, by Lemma 2.3.34 we have that $\xi^d$ is right minimal and similarly $\xi^0$ is left minimal.

We now give equivalent definitions for Auslander-Reiten $(d+2)$-angles.

**Lemma 2.3.49.** Let

$$\epsilon : \qquad X^0 \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^d \xrightarrow{\xi^d} X^{d+1} \xrightarrow{\xi^{d+1}} \Sigma^d X^0$$

be a $(d+2)$-angle. Then the following are equivalent:

(a) $\epsilon$ is an Auslander-Reiten $(d+2)$-angle,

(b) $\xi^0, \xi^1, \ldots, \xi^{d-1}$ are in $\mathrm{rad}_{\mathcal{M}}$ and $\xi^d$ is right almost split,

(c) $\xi^1, \ldots, \xi^{d-1}, \xi^d$ are in $\mathrm{rad}_{\mathcal{M}}$ and $\xi^0$ is left almost split.

The following lemma is the generalisation of Lemma 2.2.43 to $(d+2)$-angles.

**Lemma 2.3.50.** Consider a $(d+2)$-angle of the form

$$\epsilon : \qquad X^0 \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^d \xrightarrow{\xi^d} X^{d+1} \xrightarrow{\xi^{d+1}} \Sigma^d X^0,$$

and suppose that $\xi^d$ is right almost split and, if $d \geq 2$, also that $\xi^1, \ldots, \xi^{d-1}$ are in $\mathrm{rad}_{\mathcal{M}}$. Then the following are equivalent:

(a) $\mathrm{End}(X^0)$ is local,

(b) $\xi^{d+1}$ is left minimal,

(c) $\xi^0$ is in $\mathrm{rad}_{\mathcal{M}}$,

(d) $\epsilon$ is an Auslander-Reiten $(d+2)$-angle.

# Chapter 3

# Some examples

In this chapter, we introduce some examples of the categories defined in Chapter 2. These examples will also be used in later chapters to show applications of our results.

The first two examples are triangulated categories. In the first, we introduce the cluster category of Dynkin type $A_n$, denoted by $\mathcal{C}_{A_n}$, and describe its Auslander-Reiten triangles. Our second example is a generalisation of the first one: the triangulated $q$-cluster category of Dynkin type $A_n$, denoted by $\mathcal{C}_q(A_n)$, where in the case $q = 1$ we have $\mathcal{C}_1(A_n) = \mathcal{C}_{A_n}$.

Finally, we introduce a class of examples defined by Vaso. These are $d$-abelian categories and $(d+2)$-angulated categories arising from $d$-cluster tilting subcategories of module categories constructed as described in Remark 2.3.44.

## 3.1 The cluster category of $Q$

Let $k$ be a field, $Q$ a finite quiver with no loops and cycles and $kQ$ its path algebra. We describe the cluster category $\mathcal{C}_Q$, see [11, pp 577] for more details.

Let $\mathcal{D}_Q := \mathcal{D}^b(\operatorname{mod} kQ)$ be the derived category of bounded complexes of right modules over $kQ$, with suspension $\Sigma$, see Theorem 2.2.37 and Definition 2.2.39. Let $M$ and $N$ be objects in $\operatorname{mod} kQ$ and $i \in \mathbb{Z}$, then we have that

$$\operatorname{Hom}_{\mathcal{D}_Q}(M, \Sigma^i N) \cong \operatorname{Ext}^i_{kQ}(M, N)$$

by Remark 2.2.38.

The category $\mathcal{D}_Q$ has a Serre functor, denoted by $S$. Then, as in Definition 2.2.45, we can define the Auslander-Reiten translation $\tau = S \circ \Sigma^{-1} : \mathcal{D}_Q \to \mathcal{D}_Q$, with inverse $\tau^{-1} = S^{-1} \circ \Sigma$.

**Definition 3.1.1.** The *cluster category* of $Q$ is the orbit category

$$\mathcal{C}_Q = \mathcal{D}_Q/(\tau^{-1}\Sigma)^{\mathbb{Z}} = \mathcal{D}_Q/(S^{-1}\Sigma^2)^{\mathbb{Z}}.$$

Hence $\mathcal{C}_Q$ is the category with the same objects as $\mathcal{D}_Q$ and, for $X, Y$ objects in $\mathcal{C}_Q$,

$$\mathrm{Hom}_{\mathcal{C}_Q}(X,Y) = \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}_Q}(X,(\tau^{-1}\Sigma)^p(Y)).$$

Note that objects of $\mathcal{D}_Q$ that are in the same orbit of $\tau^{-1}\Sigma$ become isomorphic in $\mathcal{C}_Q$. It is possible to turn $\mathcal{C}_Q$ into a $k$-linear triangulated category with finite dimensional Hom-spaces, see [41]. Moreover, $\mathcal{C}_Q$ has Serre functor $S$, suspension $\Sigma$ and Auslander-Reiten translation $\tau$ induced by the ones in $\mathcal{D}_Q$ and $\mathcal{C}_Q$ is 2-Calabi-Yau, see Definition 2.2.47. Then it follows that in $\mathcal{C}_Q$ we have

$$\Sigma^2 \cong S \cong \tau\Sigma,$$

and hence $\Sigma \cong \tau$.

### 3.1.1 The cluster category of Dynkin type $A_n$

We focus on the case $Q = A_n$, **i.e.**

$$Q : \overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longleftarrow \cdots \longleftarrow \overset{n-1}{\bullet} \longleftarrow \overset{n}{\bullet},$$

where $n$ is a fixed positive integer.

The Auslander-Reiten quiver of $\mathcal{D}_{A_n}$ is the infinite quiver illustrated in Figure 3.1, see [20]. Note that $X$ stands for the $kA_n$-module $X$ viewed as a complex concentrated in degree zero, and we can apply the suspension to it to obtain the complex $\Sigma X$ concentrated in homological degree one. As usual with Auslander-Reiten quivers, $\tau X$ is drawn in the same row and one step to the left of $X$, so for example $\tau I(2) = P(n-1)$.

In $\mathcal{C}_{A_n}$, objects in the same $(\tau^{-1}\Sigma)$-orbit are isomorphic. For example $P(1)$ and $\tau^{-1}\Sigma P(1) = \Sigma M$ are isomorphic. Then, from the infinite quiver in Figure 3.1, we obtain the finite quiver in Figure 3.2, representing the Auslander-Reiten quiver of $\mathcal{C}_{A_n}$. Note that this can be drawn on a Möbius strip.

We study $\mathcal{C}_{A_n}$ through a geometric realisation of it. Let $P$ be the regular polygon with $n+3$ vertices. The following can be proved using the results in [13].

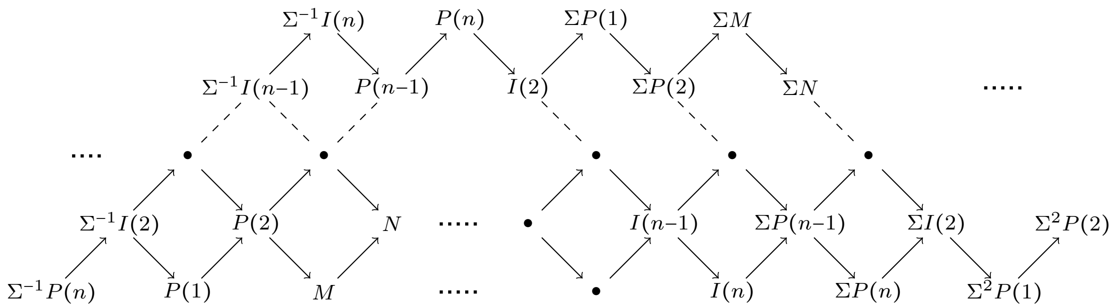



Figure 3.1: Auslander-Reiten quiver of $\mathcal{D}_{A_n}$.

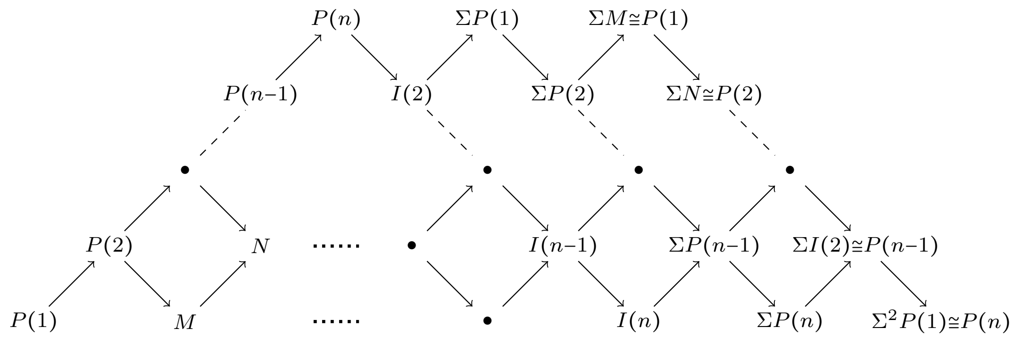



Figure 3.2: Auslander-Reiten quiver of $\mathcal{C}_{A_n}$.

(I) There is a bijection

$$\left\{\begin{matrix}\text{diagonals in } P \text{ between}\\ \text{non-neighbouring vertices}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{isomorphism classes of}\\ \text{indecomposables in } \mathcal{C}_{A_n}\end{matrix}\right\},$$

where $\operatorname{Ind}\mathcal{C}_{A_n}$ is the full subcategory of $\mathcal{D}_{A_n}$ whose objects are the indecomposable objects. We identify $\operatorname{Ind}\mathcal{C}_{A_n}$ and the diagonals of $P$, so given an indecomposable $x \in \mathcal{C}_{A_n}$ it makes sense to write $x = \{x_0, x_1\}$, for $x_0, x_1$ its endpoints as a diagonal in $P$.

(II) Let the diagonals $\mathfrak{a}, \mathfrak{c}$ correspond respectively to the indecomposables $a, c$ under the bijection from (I). Then

$$\dim_k(\operatorname{Ext}^1(a,c)) = \dim_k(\operatorname{Hom}(a, \Sigma c)) = \begin{cases} 1 & \text{if } \mathfrak{a}, \mathfrak{c} \text{ cross,} \\ 0 & \text{otherwise,} \end{cases}$$

where we say that two diagonals *cross* if they intersect in the interior of $P$ (so excluding the endpoints).

(III) If $a \in \operatorname{Ind} \mathcal{C}_{A_n}$ corresponds to the diagonal $\mathfrak{a} = \{a_0, a_1\}$, then $\Sigma a$ corresponds to the

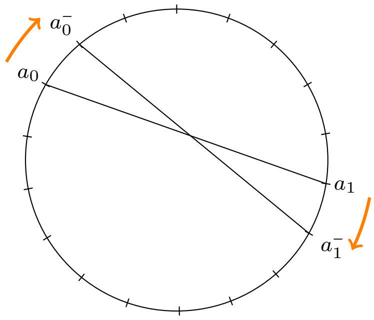



Figure 3.3: $\mathfrak{a} = \{a_0, a_1\}$ and $\Sigma\mathfrak{a} = \{a_0^-, a_1^-\}$ .

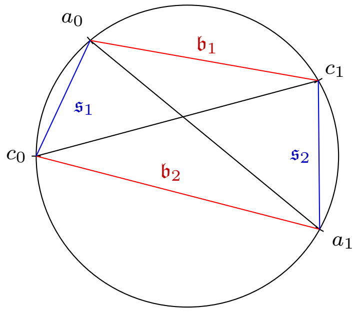



Figure 3.4: There are triangles $a \to b_1 \oplus b_2 \to c \to$ and $c \to s_1 \oplus s_2 \to a \to \Sigma c$ in $\mathcal{C}_{A_n}$.

diagonal $\{a_0^-, a_1^-\}$ obtained by moving the endpoints of $\mathfrak{a}$ by one clockwise step, see Figure 3.3.

(IV) In (II), suppose $\mathfrak{a}, \mathfrak{c}$ cross, so $\dim_k(\mathrm{Hom}(a, \Sigma c)) = \dim_k(\mathrm{Hom}(c, \Sigma a)) = 1$. Then we can complete the non-zero morphisms $c \to \Sigma a$ and $a \to \Sigma c$ to obtain the two triangles

$$a \to b_1 \oplus b_2 \to c \to \Sigma a$$
$$c \to s_1 \oplus s_2 \to a \to \Sigma c,$$

where $b_1$, $b_2$, $s_1$ and $s_2$ are the indecomposables corresponding to the diagonals $\mathfrak{b}_1$, $\mathfrak{b}_2$, $\mathfrak{s}_1$ and $\mathfrak{s}_2$ respectively in Figure 3.4. If $s_i$ or $b_i$ corresponds to an edge of $P$, then it is zero in $\mathcal{T}$.

(V) Of the triangles from (IV), the Auslander-Reiten triangles are exactly those in which either $a = \Sigma c$ or $c = \Sigma a$. For example, if $a = \Sigma c$, then we have an Auslander-Reiten triangle and a trivial triangle, respectively

$$\Sigma c \to b_1 \oplus b_2 \to c \to \Sigma^2 c \text{ and } c \to 0 \to \Sigma c \xrightarrow{\cong} \Sigma c.$$

Note that in this case $\mathfrak{s}_1, \mathfrak{s}_2$ are edges of $P$ and hence zero in $\mathcal{C}_{A_n}$.

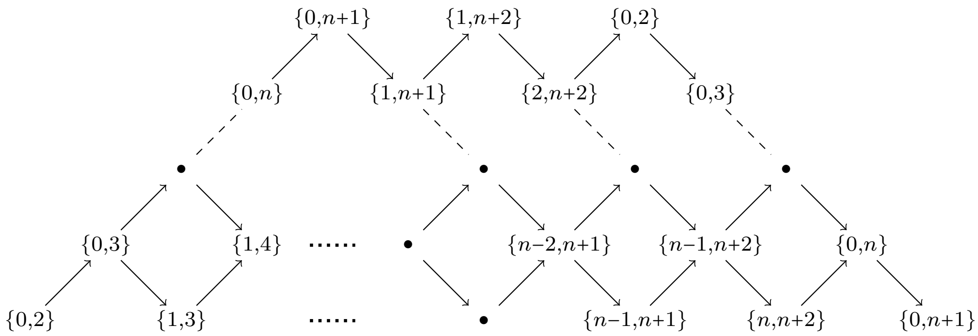



Figure 3.5: Auslander-Reiten quiver of $\mathcal{C}_{A_n}$.

(VI) Labelling the vertices of $P$ from 0 to $n+2$ anticlockwise and using (I)-(V), the Auslander-Reiten quiver of $\mathcal{C}_{A_n}$ is as shown in Figure 3.5. For a diagonal $\mathfrak{a}$, we have that $\Sigma\mathfrak{a}$ is placed in the same row and to the left of $\mathfrak{a}$ and the Auslander-Reiten triangle $\Sigma a \to s_1 \oplus s_2 \to a \to \Sigma^2 a$ corresponds to the mesh:

$$\begin{tikzcd} & \mathfrak{s}_1 \arrow[dr] & \\ \Sigma\mathfrak{a} \arrow[ur] \arrow[dr] & & \mathfrak{a}. \\ & \mathfrak{s}_2 \arrow[ur] & \end{tikzcd}$$

(VII) We can define a cyclic order on the vertices of $P$ as follows. Given three vertices $u, v, w$ of $P$, we write $u < v < w$ if they appear in the order $u, v, w$ when going through the vertices of $P$ in the positive (= anticlockwise) direction. Moreover, if we choose two distinct vertices $u$ and $v$, we can consider the interval of vertices $[u, v]$ and in this "<" is a total order.

(VIII) Let $x = \{x_0, x_1\} \in \operatorname{Ind} \mathcal{C}_{A_n}$. Then, by [27, Lemma 2.4.2], we have that $y = \{y_0, y_1\} \in \operatorname{Ind} \mathcal{C}_{A_n}$ is such that $\operatorname{Hom}(x, y) \neq 0$ if and only if $y$ has one endpoint in each of the intervals $[x_0, x_1^{--}]$ and $[x_1, x_0^{--}]$, **i.e.** the blue arcs in Figure 3.6.

Moreover, for such a $y$, the indecomposables $s = \{s_0, s_1\}$ such that the morphism $x \to y$ factors through $s$ are exactly those having one endpoint in each of the intervals $[x_0, y_0]$ and $[x_1, y_1]$, **i.e.** the two red arcs in Figure 3.6.

(IX) Let $x = \{x_0, x_1\} \in \operatorname{Ind} \mathcal{C}_{A_n}$. Then, by [27, Lemma 2.4.2], we have that $z = \{z_0, z_1\} \in \operatorname{Ind} \mathcal{C}_{A_n}$ is such that $\operatorname{Hom}(z, x) \neq 0$ if and only if $z$ has one endpoint in each of the intervals $[x_0^{++}, x_1]$ and $[x_1^{++}, x_0]$, **i.e.** the two green arcs in Figure 3.7.

Moreover, for such a $z$, the indecomposables $s = \{s_0, s_1\}$ such that the morphism

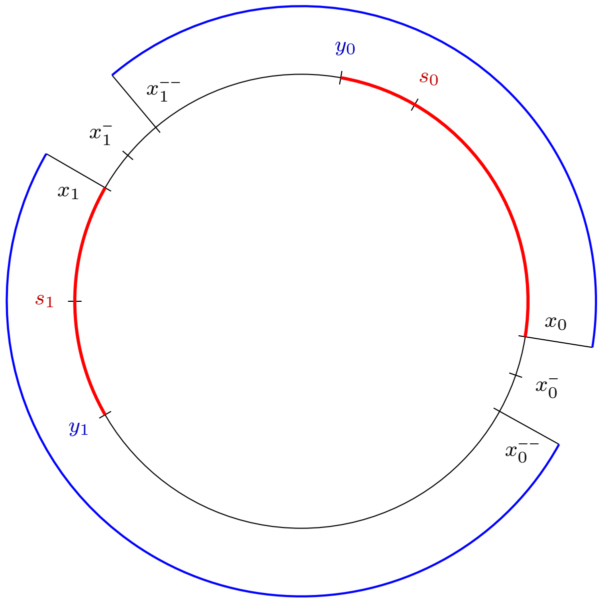



Figure 3.6: There is a non-zero morphism $x = \{x_0, x_1\} \to \{y_0, y_1\} = y$ if and only if $y$ has one endpoint below each blue arc. Moreover, $x \to y$ factors through $s = \{s_0, s_1\}$ if and only if $s$ has an endpoint below each red arc.

$z \to x$ factors through $s$ are exactly those having one endpoint in each of the intervals $[z_0, x_1]$ and $[z_1, x_0]$, **i.e.** the two red arcs in Figure 3.7.

## 3.2 The $q$-cluster category of Dynkin type $A_p$

Let $k$ be a field and $p$ and $q$ be fixed positive integers. We describe the triangulated $q$-cluster category of Dynkin type $A_p$, denoted by $\mathcal{C}_q(A_p)$ and first defined in [54], and its geometric realisation, see [47] and [8] for more details.

**Remark 3.2.1.** The properties we present in this section express the fact that the Auslander-Reiten quiver of $\mathcal{C}_q(A_p)$ has a certain shape and do not rely on $k$ being algebraically closed. Hence, even if [47] and [8] assume that $k$ is an algebraically closed field, we remove this assumption and work with a general field $k$, as done in [54].

Consider the coordinate system on the translation quiver $\mathbb{Z}A_p$ illustrated in Figure 3.8.

**Definition 3.2.2** ([47, Remark 2.3])**.** Define the following automorphisms on $\mathbb{Z}A_p$:

$$\Sigma : \mathbb{Z}A_p \to \mathbb{Z}A_p, \qquad (i,j) \mapsto (j-1, i+(p+1)q+1),$$
$$\tau : \mathbb{Z}A_p \to \mathbb{Z}A_p, \qquad (i,j) \mapsto (i-q, j-q),$$

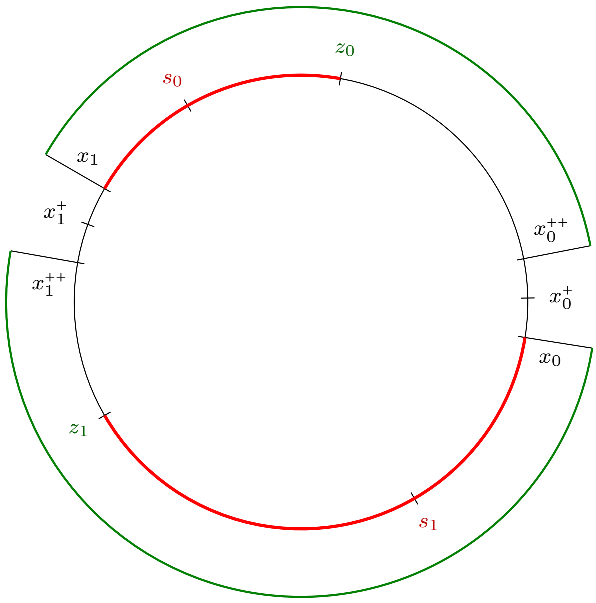



Figure 3.7: There is a non-zero morphism $z = \{z_0, z_1\} \to \{x_0, x_1\} = x$ if and only if $z$ has one endpoint below each green arc. Moreover $z \to x$ factors through $s = \{s_0, s_1\}$ if and only if $s$ has an endpoint below each red arc.

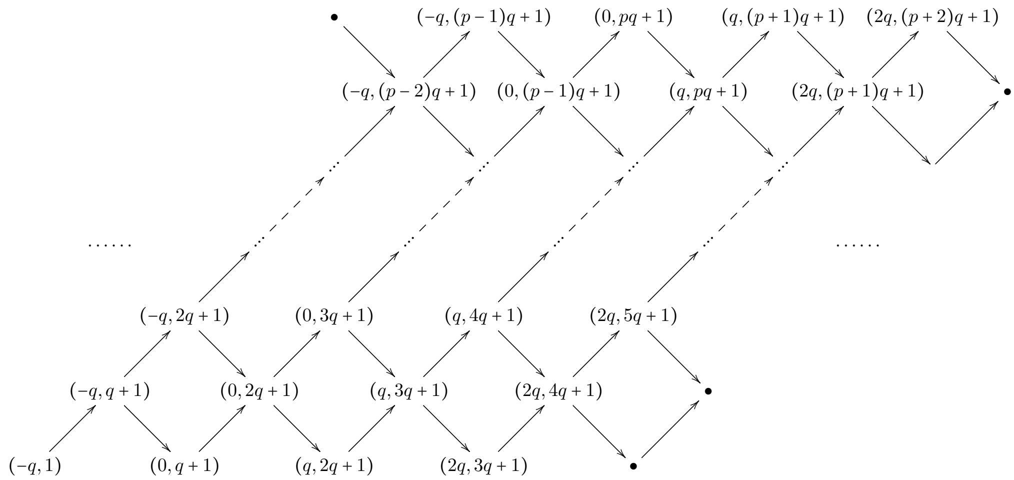



Figure 3.8: Coordinate system on $\mathbb{Z}A_p$.

and let $\tau_{q+1} = \tau \circ \Sigma^{-q}$.

Note that $(\mathbb{Z}A_p, \tau)$ is a translation quiver in the sense of [47, Definition 2.2]. Hence there exists a *mesh category* associated to it. The objects of this category are the vertices of $\mathbb{Z}A_p$ and the morphisms are linear combinations of paths in $\mathbb{Z}A_p$ subject to the *mesh relations*. For each arrow $\alpha : x \to y$, let $\sigma(\alpha)$ be the unique arrow $\sigma(\alpha) : \tau(y) \to x$. The

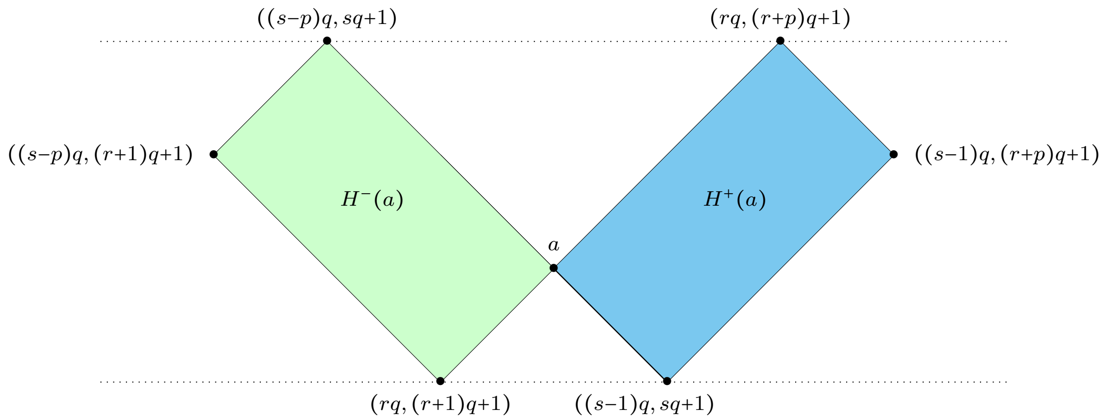



Figure 3.9: The regions $H^+(a)$ and $H^-(a)$ for $a = (rq, sq+1)$.

mesh relations are given by

$$\sum_{\alpha: x \to y} \alpha\sigma(\alpha) = 0,$$

for each vertex $y$ in $\mathbb{Z}A_p$.

**Definition 3.2.3.** Let $a = (rq, sq+1)$ be a vertex in $\mathbb{Z}A_p$, for $r$ and $s$ integers such that $r+1 \leq s \leq r+p$. We denote by

- $H^+(a)$ the set of vertices of $\mathbb{Z}A_p$ of the form $(iq, jq+1)$ for integers $r \leq i \leq s-1$ and $s \leq j \leq r+p$;
- $H^-(a)$ the set of vertices of $\mathbb{Z}A_p$ of the form $(iq, jq+1)$ for integers $s-p \leq i \leq r$ and $r+1 \leq j \leq s$.

Note that the sets of vertices $H^-(a)$ and $H^+(a)$ are those in the "hammocks" spanned from $a$, see Figure 3.9.

**Remark 3.2.4.** By [47, Remark 2.3], the regions $H^-(a)$ and $H^+(a)$ describe the set of vertices from which (respectively, to which) there is a non-zero morphism in the mesh category associated to $\mathbb{Z}A_p$.

**Remark 3.2.5** ([47, Section 2])**.** As in the previous section, let $\mathcal{D}_{A_p} := \mathcal{D}^b(\operatorname{mod} kA_p)$. The Auslander-Reiten quiver of $\mathcal{D}_{A_p}$ is isomorphic, as a stable translation quiver, to $\mathbb{Z}A_p$. The automorphisms $\Sigma$ and $\tau$ from Definition 3.2.2 are the action of the suspension and the Auslander-Reiten translation in $\mathcal{D}_{A_p}$ respectively, expressed in terms of the coordinate system from Figure 3.8. Moreover, the mesh category $k(\mathbb{Z}A_p)$ is equivalent to $\operatorname{Ind}\mathcal{D}_{A_p}$, **i.e.** the full subcategory of $\mathcal{D}_{A_p}$ whose objects are the indecomposable objects.

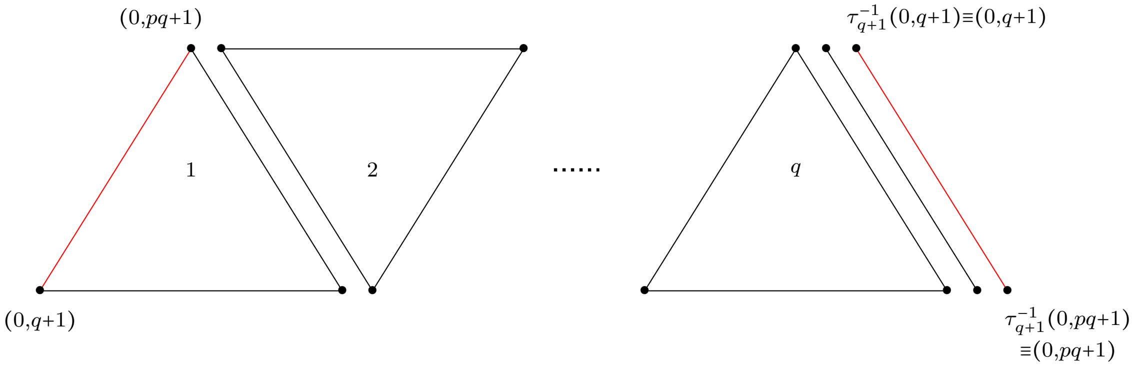



Figure 3.10: The quotient translation quiver $\mathbb{Z}A_p/\langle\tau_{q+1}\rangle$ when $q$ is odd.

The quotient translation quiver $\mathbb{Z}A_p/\langle\tau_{q+1}\rangle$ is obtained by identifying the vertices and arrows of $\mathbb{Z}A_p$ with their $\tau_{q+1}$-shifts. It is the Auslander-Reiten quiver of

$$\mathcal{C}_q(A_p) := \mathcal{D}_{A_p}/\tau \circ \Sigma^{-q},$$

the *triangulated $q$-cluster category of Dynkin type $A_p$*. Figure 3.10 shows the identification on $\mathbb{Z}A_p$ when $q$ is odd. Note that in this case, the quiver can be drawn on a Möbius strip. Moreover, by [47, Remark 2.12] we have that $\mathcal{C}_q(A_p)$ is a category whose Hom-spaces between indecomposables are either zero or one dimensional over $k$ and by [8, Introduction] it is $(q+1)$-Calabi-Yau, that is it has a Serre functor that is isomorphic to $\Sigma^{q+1}$, see Definition 2.2.47.

We present a geometric realisation of $\mathbb{Z}A_p/\langle\tau_{q+1}\rangle$. Let $N = (p+1)q+2$ and $P$ be a regular convex $N$-gon. Label the vertices of $P$ from 0 to $N-1$ in an anticlockwise direction. We denote the diagonal joining vertices $i$ and $j$ by $\{i,j\}$.

**Definition 3.2.6** ([47, Definition 2.5])**.** A *$q$-allowable diagonal in $P$* is a diagonal joining two non-adjacent boundary vertices which divides $P$ into two smaller polygons which can themselves be subdivided into $(q+2)$-gons by non-crossing diagonals. Note that these are the diagonals of $P$ spanning $1+lq$ vertices, for $l$ a positive integer.

**Proposition 3.2.7** ([47, Proposition 2.9])**.** There is a bijection

$$\left\{\begin{matrix}\text{isomorphism classes of indecomposables in } \mathcal{C}_q(A_p)\\ (=\text{vertices of } \mathbb{Z}A_p/\langle\tau_{q+1}\rangle)\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}q\text{-allowable diagonals}\\ \text{in } P\end{matrix}\right\}$$

given by $(i,j) \mapsto \{i\,(\mathrm{mod}\,N), j\,(\mathrm{mod}\,N)\}$.

From now on, $q$-allowable diagonals in $P$ and indecomposable objects in $\mathcal{C}_q(A_p)$ are identified. Hence it makes sense to talk about morphisms between two $q$-allowable diagonals.

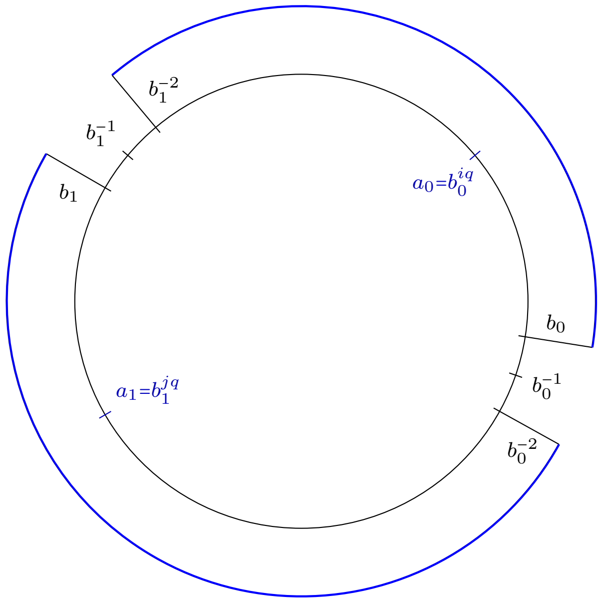



Figure 3.11: There is a non-zero morphism $b = \{b_0, b_1\} \to \{a_0, a_1\} = a$.

**Notation 3.2.8.** Given a vertex $v$ of $P$ and an integer $r$, we denote by $v^r$ its $r^{th}$ successor in the anticlockwise direction if $r$ is positive and its $(-r)^{th}$ successor in the clockwise direction if $r$ is negative. We also use the convention $v^0 = v$.

**Remark 3.2.9.** We can define a cyclic order "<" on the vertices of $P$ as described in (VII) and for any two distinct vertices $u$ and $v$, we have that "<" is a total order in the interval of vertices $[u, v]$.

The next two lemmas follow by describing the $H^+(b)$, respectively $H^-(b)$, region from Definition 3.2.3 in terms of $q$-allowable diagonals in $P$.

**Lemma 3.2.10.** Consider a $q$-allowable diagonal $b = \{b_0, b_1\}$ in $P$. Then a $q$-allowable diagonal $a$ is such that $\mathrm{Hom}(b, a) \neq 0$ if and only if there are some non-negative integers $i$, $j$ such that $a = \{a_0, a_1\}$ for

$$a_0 = b_0^{iq} \in [b_0, b_1^{-2}] \text{ and } a_1 = b_1^{jq} \in [b_1, b_0^{-2}].$$

See Figure 3.11.

**Lemma 3.2.11.** Consider a $q$-allowable diagonal $b = \{b_0, b_1\}$ in $P$. Then a $q$-allowable diagonal $a$ is such that $\mathrm{Hom}(a, b) \neq 0$ if and only if there are some non-negative integers $i$, $j$ such that $a = \{a_0, a_1\}$ for

$$a_0 = b_0^{-iq} \in [b_1^2, b_0] \text{ and } a_1 = b_1^{-jq} \in [b_0^2, b_1].$$

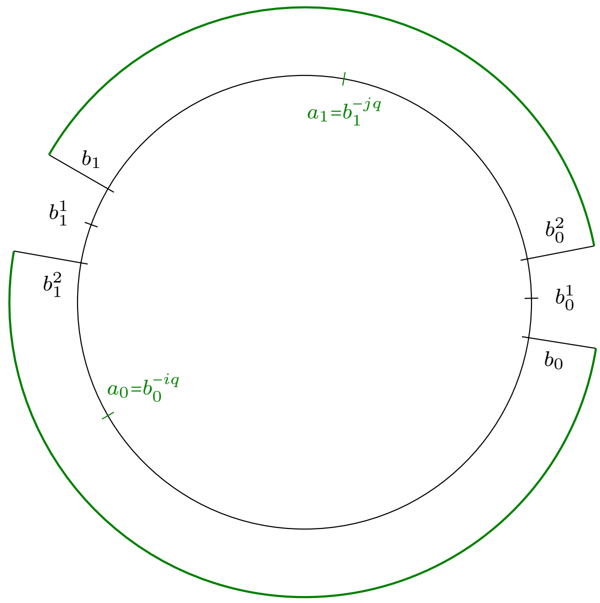



Figure 3.12: There is a non-zero morphism $a = \{a_0, a_1\} \to \{b_0, b_1\} = b$.

See Figure 3.12.

The following is a consequence of [47, Proposition 2.9].

**Lemma 3.2.12.** Two $q$-allowable diagonals $a$ and $b$ in $P$ cross if and only if there exists an integer $1 \leq i \leq q$ such that $\mathrm{Ext}^i(a, b) \neq 0$.

### 3.2.1 Triangles in $\mathcal{C}_q(A_p)$

In this section, we describe all the triangles in $\mathcal{C}_q(A_p)$ with indecomposable end terms in terms of $q$-allowable diagonals in $P$. In order to do this, we use a similar method to the one used by Pescod in [51, Chapter 4]. The following two lemmas are inspired by [51, Lemmas 4.1.1 and 4.1.2]

**Lemma 3.2.13.** Consider a triangle in $\mathcal{C}_q(A_p)$ of the form

$$\Delta = a \to e \to b \to \Sigma a,$$

with $a$ and $b$ indecomposable. If $c$ is an indecomposable in $\mathcal{C}_q(A_p)$ such that there exists an integer $1 \leq i \leq q$ with $\mathrm{Ext}^i(c, e) \neq 0$, then at least one of $\mathrm{Ext}^i(c, a)$ and $\mathrm{Ext}^i(c, b)$ is non-zero.

In terms of $q$-allowable diagonals in $P$, we have that if $c$ crosses a direct summand of $e$, then $c$ crosses at least one of $a$ and $b$.

*Proof.* The triangle $\Delta$ induces the exact sequence:

$$\mathrm{Ext}^i(c,a) \to \mathrm{Ext}^i(c,e) \to \mathrm{Ext}^i(c,b).$$

Since $\mathrm{Ext}^i(c,e) \neq 0$, it follows that at least one of $\mathrm{Ext}^i(c,a)$ and $\mathrm{Ext}^i(c,b)$ is non-zero. $\square$

**Lemma 3.2.14.** Let $a$ and $b \in \mathcal{C}_q(A_p)$ be indecomposable and assume that $\mathrm{Ext}^1(b,a) \neq 0$. Let

$$\Delta = a \xrightarrow{\alpha} e \xrightarrow{\epsilon} b \xrightarrow{\beta} \Sigma a$$

be the ensuing non-split triangle. Then

$$\mathrm{Ext}^{1\cdots q}(e,a) = \mathrm{Ext}^{1\cdots q}(b,e) = 0.$$

In terms of $q$-allowable diagonals in $P$, there is no direct summand of $e$ crossing $a$ or $b$.

*Proof.* Note that the identification on $\mathbb{Z}A_p$ to obtain $\mathbb{Z}A_p/\langle\tau_{q+1}\rangle \cong \mathcal{C}_q(A_p)$ is such that

$$H^+(\Sigma^{-q}b),\, H^+(\Sigma^{-q+1}b),\, \dots,\, H^+(\Sigma^{-2}b),\, H^+(\Sigma^{-1}b)$$

are all disjoint. See Figure 3.10 for $\mathbb{Z}A_p/\langle\tau_{q+1}\rangle$ when $q$ is odd; the case when $q$ is even is similar. By Remark 3.2.4, we have that at most one of

$$\mathrm{Hom}(\Sigma^{-q}b,a),\, \mathrm{Hom}(\Sigma^{-q+1}b,a),\, \dots,\, \mathrm{Hom}(\Sigma^{-2}b,a),\, \mathrm{Hom}(\Sigma^{-1}b,a)$$

is non-zero. Equivalently, at most one of

$$\mathrm{Ext}^q(b,a),\, \mathrm{Ext}^{q-1}(b,a),\, \dots,\, \mathrm{Ext}^2(b,a),\, \mathrm{Ext}^1(b,a)$$

is non-zero. Since $\mathrm{Ext}^1(b,a)$ is non-zero by assumption, we have that $\mathrm{Ext}^{2\cdots q}(b,a) = 0$. Consider the following exact sequence induced by $\Delta$:

$$\mathrm{Hom}(b,b) \xrightarrow{\beta_*} \mathrm{Hom}(b,\Sigma a) \xrightarrow{(-\Sigma\alpha)_*} \mathrm{Hom}(b,\Sigma e) \to \mathrm{Hom}(b,\Sigma b).$$

Since $b$ does not cross itself, by Lemma 3.2.12, we have that $\mathrm{Hom}(b,\Sigma^i b) = 0$ for any $1 \leq i \leq q$. Moreover, as $\mathrm{Hom}(b,b)$ is non-zero and $\beta \neq 0$, we have that $\beta_* \neq 0$. Hence $\mathrm{Hom}(b,\Sigma a)$ is one-dimensional over $k$ and $\beta_*$ is surjective, so that $(-\Sigma\alpha)_* = 0$. Then $\mathrm{Ext}^1(b,e) = \mathrm{Hom}(b,\Sigma e) = 0$. For $2 \leq i \leq q$, consider the following exact sequence induced by $\Delta$ :

$$\mathrm{Hom}(b, \Sigma^i a) \to \mathrm{Hom}(b, \Sigma^i e) \to \mathrm{Hom}(b, \Sigma^i b) = 0.$$

For $2 \leq i \leq q$, we have

$$0 = \mathrm{Ext}^i(b, a) = \mathrm{Hom}(b, \Sigma^i a),$$

and so $\mathrm{Ext}^i(b, e) = \mathrm{Hom}(b, \Sigma^i e) = 0$ for $2 \leq i \leq q$.
A similar argument shows that $\mathrm{Ext}^{1\cdots q}(e, a) = 0$. □

Note that the results from [51, Chapter 4.2] are valid in $\mathcal{C}_q(A_p)$ since this is a $k$-linear, Hom-finite, Krull-Schmidt triangulated category. We state them again for convenience of the reader.

**Lemma 3.2.15** ([51, Lemmas 4.2.1 and 4.2.2])**.** Let $b \in \mathcal{C}_q(A_p)$ be indecomposable. Assume there exists a non-split triangle

$$a \xrightarrow{\alpha} e \xrightarrow{\epsilon} b \to \Sigma a,$$

then each row of the matrix $\alpha$ has a non-zero entry and each column of the matrix $\epsilon$ has a non-zero entry.

**Remark 3.2.16.** Note that since in this setup Hom-spaces between indecomposables in $\mathcal{C}_q(A_p)$ are either zero or one dimensional over $k$, we can state [51, Lemma 4.2.3] as follows.

**Lemma 3.2.17.** For $a$ and $b$ indecomposables in $\mathcal{C}_q(A_p)$, let

$$a \to e \to b \to \Sigma a$$

be a triangle in $\mathcal{C}_q(A_p)$. Then, $e$ has no repeated indecomposable summands.

**Lemma 3.2.18** ([51, Lemma 4.2.4])**.** For $a$ and $b$ indecomposables in $\mathcal{C}_q(A_p)$, let

$$a \to e \to b \to \Sigma a$$

be a triangle in $\mathcal{C}_q(A_p)$. Then

$$\mathrm{Ext}^1(\Sigma a, e_i) \neq 0 \text{ and } \mathrm{Ext}^1(e_i, \Sigma^{-1} b) \neq 0,$$

for each indecomposable summand $e_i$ of $e$.

**Lemma 3.2.19.** Consider two crossing $q$-allowable diagonals $a = \{a_0, a_1\}$ and $b = \{b_0, b_1\}$ in $P$, where $b_0 < a_0 < b_1 < a_1$. Then $\{a_0, b_0\}$ is $q$-allowable or an edge if and only if $\{a_1, b_1\}$ is $q$-allowable or an edge.

*Proof.* Since $a$ and $b$ are $q$-allowable diagonals, there are positive integers $r$, $t$ such that

$$a_1 = a_0^{1+rq} \text{ and } b_1 = b_0^{1+tq}.$$

Assume that $\{a_0, b_0\}$ is $q$-allowable or an edge in $P$. Then $a_0 = b_0^{1+sq}$ for some integer $s \geq 0$, see Figure 3.13. We have

$$a_1 = a_0^{1+rq} = b_0^{2+(r+s)q} = b_1^{2+(r+s)q-1-tq} = b_1^{1+(r+s-t)q},$$

where $r + s \geq t$. Hence $\{a_1, b_1\}$ is $q$-allowable or an edge in $P$. The other direction is proved in a similar way. □

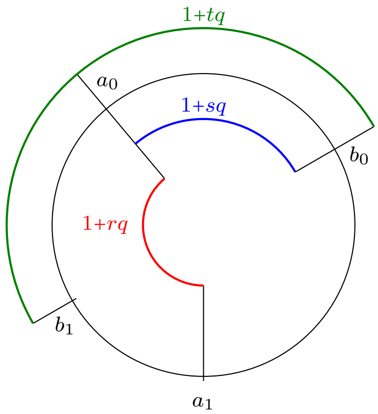



Figure 3.13: The diagonals $\{a_0, a_1\}$ and $\{b_0, b_1\}$ are $q$-allowable and $\{a_0, b_0\}$ is either $q$-allowable or an edge.

**Proposition 3.2.20.** Consider two crossing $q$-allowable diagonals $a = \{a_0, a_1\}$ and $b = \{b_0, b_1\}$ in $P$, where $b_0 < a_0 < b_1 < a_1$.

- There exists exactly one integer $0 \leq l \leq q-1$ such that $\mathrm{Hom}(b, \Sigma^{l+1} a) \neq 0$. Then the non-split triangle extending $\beta : b \to \Sigma^{l+1} a$ is

$$\Delta = \Sigma^l a \to e \to b \xrightarrow{\beta} \Sigma^{l+1} a,$$

where $e = e_1 \oplus e_2$ for $e_1 = \{a_0^{-l}, b_0\}$ and $e_2 = \{b_1, a_1^{-l}\}$.

- If $0 \leq i \leq q-1$ is an integer such that $\{a_0^{-i}, b_0\}$ is a $q$-allowable diagonal or an edge in $P$, then $i = l$.

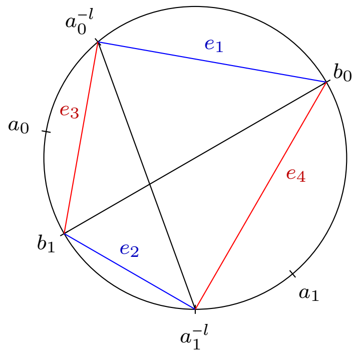



Figure 3.14: The triangle $\Sigma^l a \to e_1 \oplus e_2 \to b \xrightarrow{\beta} \Sigma^{l+1} a$.

*Proof.* Assume that $\Delta$ is a non-split triangle and let $\overline{e}$ be a direct summand of the middle term $e$. By Lemma 3.2.14, we have that $\overline{e}$ does not cross $\Sigma^l a$ or $b$. Moreover, if $c$ is a $q$-allowable diagonal crossing $\overline{e}$, by Lemma 3.2.13, we have that $c$ crosses at least one of $\Sigma^l a$ and $b$. Hence the only possibilities for $\overline{e}$ are the diagonals

$$e_1 = \{a_0^{-l}, b_0\},\ e_2 = \{b_1, a_1^{-l}\},\ e_3 = \{a_0^{-l}, b_1\},\ e_4 = \{b_0, a_1^{-l}\},$$

see Figure 3.14. By Lemma 3.2.18, we have that

$$\operatorname{Ext}^1(\Sigma^{l+1} a, \overline{e}) \neq 0 \text{ and } \operatorname{Ext}^1(\overline{e}, \Sigma^{-1} b) \neq 0.$$

Hence $\overline{e}$ crosses both $\Sigma^{l+1} a$ and $\Sigma^{-1} b$, which implies that $e_3$ and $e_4$ must be excluded from the possible summands of $e$. Moreover, $e$ has no repeated summands by Lemma 3.2.17, and so

$$e \in \{0, e_1, e_2, e_1 \oplus e_2\}.$$

We claim that $e = e_1 \oplus e_2$. We prove this claim by dealing with the cases $e_1 = e_2 = 0$, one of $e_1$, $e_2$ zero and $e_1$, $e_2$ both non-zero separately. First, note that if $b = \Sigma^{l+1} a$, then

$$\Delta = \Sigma^l a \to 0 \to \Sigma^{l+1} a \xrightarrow{\cong} \Sigma^{l+1} a.$$

Note that in this case $b_0^1 = a_0^{-l}$ and $b_1^1 = a_1^{-l}$ so that $e_1 = e_2 = 0$ and $e = e_1 \oplus e_2 = 0$. Assume now that $b$ is not $\Sigma^{l+1} a$, so that $e \neq 0$. Note that if $e_1$ (respectively $e_2$) is zero, then $e = e_1 \oplus e_2 = e_2$ (respectively $e = e_1$) is the only option and we are done. Moreover, since $e \neq 0$, we have that at least one of $e_1$, $e_2$ is $q$-allowable or an edge in $P$. But then, by Lemma 3.2.19, we have that $e_1$ and $e_2$ are both $q$-allowable diagonals or edges in $P$. The last case to deal with is when $e_1$ and $e_2$ are both non-zero, **i.e.** they both are $q$-allowable diagonals in $P$. Suppose for a contradiction that $e = e_1$. Consider the triangle

$$e \to b \to \Sigma^{l+1}a \to \Sigma e,$$

and set $c = \{a_0^{-l}, b_1^1\}$. Note that $b_1 < b_1^1 < a_1^{-l}$ since $e_2 \neq 0$. Since $\Sigma^l a$ and $e_2$ are $q$-allowable diagonals, there are integers $s > r > 0$ such that

$$a_1^{-l} = a_0^{-l+1+sq} = b_1^{1+rq}.$$

Hence $b_1^1 = a_0^{-l+1+(s-r)q}$ and so $c$ is $q$-allowable. So $c$ is a $q$-allowable diagonal crossing $b$ but not crossing neither $e = e_1$ nor $\Sigma^{l+1}a$, see Figure 3.15, contracting Lemma 3.2.13. Then $e \neq e_1$.

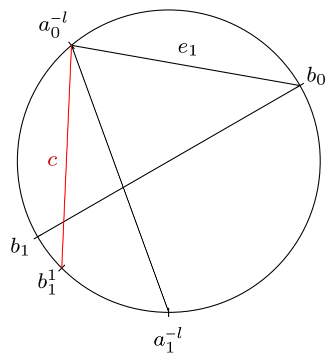



Figure 3.15: The $q$-allowable diagonal $c$ crosses $b$ but does not cross neither $e_1$ nor $\Sigma^{l+1}a$.

By a similar argument, assuming that $e = e_2$ leads to a contradiction. Hence we have that $e = e_1 \oplus e_2$.

We now prove the second part of the proposition. Assume $0 \leq i \leq q-1$ is an integer such that $\{a_0^{-i}, b_0\}$ is a $q$-allowable diagonal or an edge in $P$. Then there are integers $r$, $s \geq 0$ such that

$$a_0^{-i} = b_0^{1+sq} \text{ and } a_0^{-l} = b_0^{1+rq}.$$

Then $b_0^{1+rq} = a_0^{-l} = b_0^{1+sq+i-l}$. So $i - l = (r-s)q$.

If $0 \leq l \leq i \leq q-1$, then $r \geq s \geq 0$. Note that $0 \leq i - l \leq q - l - 1 < q$, so $i = l$ is the only option. If $0 \leq i \leq l \leq q-1$, then $s \geq r \geq 0$. Note that $0 \leq l - i \leq q - i - 1 < q$, so $i = l$ is again the only option. $\square$

## 3.3 A class of examples by Vaso

In this section, we present a class of examples due to Vaso. We summarise results from [55, Section 4] and [23, Section 7].

**Definition 3.3.1.** Let $d \geq 2$, $l \geq 2$, $m \geq 3$ be integers such that $d$ is even and

$$\frac{m-1}{l} = \frac{d}{2}.$$

Let $Q = A_m$ be the quiver

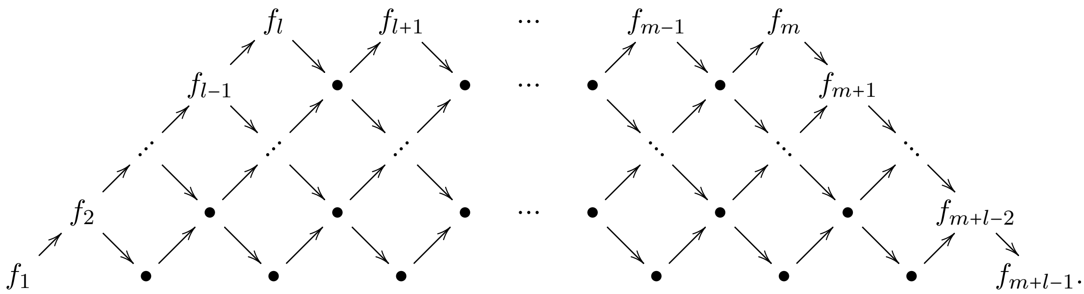
$$m \to m-1 \to \cdots \to 2 \to 1,$$

$k$ be a field and set $\Lambda := kQ/(\mathrm{rad}_{kQ})^l$. In other words, $\Lambda$ is the path algebra of the quiver $Q$ with the relation that $l$ consecutive arrows compose to zero.

As representations of the quiver $Q$, the indecomposable projectives and injectives in $\operatorname{mod}\Lambda$ are the following:

$$\begin{aligned}
f_1 &= 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to k,\\
f_2 &= 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to k \to k,\\
&\vdots\\
f_{l-1} &= 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to \underbrace{k \to \cdots \to k}_{l-1},\\
f_l &= 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to \underbrace{k \to k \to \cdots \to k}_{l},\\
f_{l+1} &= 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to \underbrace{k \to k \to \cdots \to k}_{l} \to 0,\\
&\vdots\\
f_{m-1} &= 0 \to \underbrace{k \to k \to \cdots \to k}_{l} \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to 0,\\
f_m &= \underbrace{k \to k \to \cdots \to k}_{l} \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to 0,\\
f_{m+1} &= \underbrace{k \to \cdots \to k}_{l-1} \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to 0,\\
&\vdots\\
f_{m+l-2} &= k \to k \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to 0,\\
f_{m+l-1} &= k \to 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to 0,
\end{aligned}$$

where each morphism $k \to k$ is the identity on $k$. The indecomposable projectives are $f_i$ for $1 \leq i \leq m$ and the indecomposable injectives are $f_i$ for $l \leq i \leq m+l-1$. Moreover, we set $f_i = 0$ for $i \leq 0$ and $i \geq m+l$. These modules appear in the Auslander-Reiten quiver of $\operatorname{mod}\Lambda$ as follows.




It can be checked that $\operatorname{gldim}\Lambda \leq d$ and $\operatorname{mod}\Lambda$ has a $d$-cluster tilting subcategory

$$\mathcal{F} = \operatorname{add}(\Lambda \oplus D\Lambda) = \operatorname{add}\{f_i \mid 1 \leq i \leq m+l-1\},$$

which is unique by Remark 2.3.44. Recall that, by Theorem 2.3.14, we have that $\mathcal{F}$ is a $d$-abelian category. We have that

$$\dim_k \mathcal{F}(f_i, f_j) = \begin{cases} 1 & \text{if } 0 \leq j-i \leq l-1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if $i \leq r \leq j$, then each morphism of the form $f_i \to f_j$ factors through $f_r$. The quiver of $\mathcal{F}$ is then

$$f_1 \to f_2 \to f_3 \to \cdots \to f_{m+l-3} \to f_{m+l-2} \to f_{m+l-1},$$

where the composition of $l$ consecutive arrows is zero.

Consider now

$$\overline{\mathcal{F}} = \operatorname{add}\{\Sigma^{id}\mathcal{F} \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}^b(\operatorname{mod}\Lambda).$$

By Remark 2.3.44 this is a $(d+2)$-angulated category. Moreover, $\overline{\mathcal{F}}$ has quiver

$$\cdots\Sigma^{-d}f_1 \to \cdots \to \Sigma^{-d}f_{m+l-1} \to f_1 \to \cdots \to f_{m+l-1} \to \Sigma^d f_1 \to \cdots, \tag{3.1}$$

where the composition of $l$ consecutive arrows is zero, see [39, Proposition A.11].

We conclude this section by describing some $d$-exact sequences in $\mathcal{F}$ and the corresponding $(d+2)$-angles in $\overline{\mathcal{F}}$.

**Remark 3.3.2.** For any non-zero morphism $\mu : f_i \to f_j$, where $i \neq j$, there is an exact sequence of the form:

$$\cdots \to f_{j-2l} \to f_{i-l} \to f_{j-l} \to f_i \xrightarrow{\mu} f_j \to f_{i+l} \to f_{j+l} \to f_{i+2l} \to \cdots. \tag{3.2}$$

This sequence terminates both on the right and on the left giving a $d$-exact sequence containing $\mu$:

$$0 \to f_x \to \cdots \to f_{j-l} \to f_i \xrightarrow{\mu} f_j \to f_{i+l} \to \cdots \to f_y \to 0.$$

Note that the cases when $f_{i+l} = 0$ (or $f_{j-l} = 0$) are allowed and correspond to $f_i$ being injective non-projective (or $f_j$ being projective non-injective, respectively). In these cases, $\mu$ is surjective (or injective, respectively).

By Remark 2.3.45, the $d$-exact sequence (3.2) gives a $(d+2)$-angle in $\overline{\mathcal{F}}$ of the form:

$$f_x \to \cdots \to f_{j-l} \to f_i \xrightarrow{\mu} f_j \to f_{i+l} \to \cdots \to f_y \to \Sigma^d f_x.$$

Note that we can rotate this $(d+2)$-angle to make $f_j$ the end-term on the right.

# Chapter 4

# Auslander-Reiten $(d+2)$-angles in subcategories and a $(d+2)$-angulated generalisation of a theorem by Brüning

## 4.1 Introduction

Let $d$ be a fixed positive integer, $k$ an algebraically closed field and $\Lambda$ a finite dimensional $k$-algebra with global dimension at most $d$. As in previous chapters, the category of finitely generated right $\Lambda$-modules is denoted by $\operatorname{mod}\Lambda$ and its bounded derived category by $\mathcal{D}^b(\operatorname{mod}\Lambda)$, with suspension functor $\Sigma$. Moreover, for an additive subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$, we define an additive subcategory

$$\overline{\mathcal{C}} := \operatorname{add}\{\Sigma^{id}\mathcal{C} \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}^b(\operatorname{mod}\Lambda).$$

For $d \geq 2$, suppose there is a $d$-cluster tilting subcategory $\mathcal{F} \subseteq \operatorname{mod}\Lambda$. Then $\mathcal{F}$ plays the role of a higher $\operatorname{mod}\Lambda$ and $\overline{\mathcal{F}}$ of a higher derived category of $\mathcal{F}$, see Remark 2.3.44.

We generalise Brüning's result on wide subcategories of $\mathcal{D}^b(\operatorname{mod}\Lambda)$ and Jørgensen's result on Auslander-Reiten triangles in extension closed subcategories of triangulated categories to higher homological algebra.

### 4.1.1 Classic background ($d = 1$ case).

In the case $d = 1$, the global dimension of $\Lambda$ is at most 1, that is $\operatorname{mod}\Lambda$ is *hereditary.* So [10, Theorem 1.1] can be stated as follows in this case.

**Theorem (Brüning).** *There is a bijection*

$$\left\{\begin{matrix}\textit{wide subcategories}\\ \textit{of } \operatorname{mod}\Lambda\end{matrix}\right\} \to \left\{\begin{matrix}\textit{wide subcategories}\\ \textit{of } \mathcal{D}^b(\operatorname{mod}\Lambda)\end{matrix}\right\}$$

*sending a wide subcategory $\mathcal{W}$ of* $\operatorname{mod}\Lambda$ *to* $\overline{\mathcal{W}}$.

Happel introduced Auslander-Reiten triangles in triangulated categories in [21, Chapter I.4] and Jørgensen studied Auslander-Reiten triangles in their extension closed subcategories in [37]. Whenever $\mathcal{M}$ is a skeletally small Hom-finite $k$-linear triangulated category with split idempotents and $\mathcal{W} \subseteq \mathcal{M}$ is an additive subcategory closed under extensions, [37, Theorem 3.1] states the following.

**Theorem (Jørgensen).** *Let $W$ be in $\mathcal{W}$ and suppose that there exists $U'$ in $\mathcal{W}$ and a non-zero morphism $W \to \Sigma U'$. Let*

$$X \longrightarrow Y \longrightarrow W \longrightarrow \Sigma X$$

*be an Auslander-Reiten triangle in $\mathcal{M}$. Then the following are equivalent.*

*(a) $X$ has a $\mathcal{W}$-cover of the form $U \to X$,*

*(b) there is an Auslander-Reiten triangle in $\mathcal{W}$ of the form*

$$U \longrightarrow V \longrightarrow W \longrightarrow \Sigma U.$$

Note that the above theorem can be applied to any wide subcategory of the triangulated category $\mathcal{D}^b(\operatorname{mod}\Lambda)$. So, given a wide subcategory of $\operatorname{mod}\Lambda$, one can find a wide subcategory $\mathcal{W}$ of $\mathcal{D}^b(\operatorname{mod}\Lambda)$ using the theorem by Brüning and then use the theorem by Jørgensen to find Auslander-Reiten triangles in $\mathcal{W}$.

### 4.1.2 This chapter ($d \geq 1$ case).

We now allow $d$ to be bigger than 1 and work in higher homological algebra, see Section 2.3. Suppose $\operatorname{mod}\Lambda$ has a $d$-cluster tilting subcategory $\mathcal{F}$, see Definition 2.3.13. By Remark 2.3.44, we have that $\overline{\mathcal{F}}$ is $d$-cluster tilting in $\mathcal{D}^b(\operatorname{mod}\Lambda)$ and $\overline{\mathcal{F}}$ is a $(d+2)$-angulated category. Note that the $d$-abelian category $\mathcal{F}$ plays the role of a higher $\operatorname{mod}\Lambda$ and $\overline{\mathcal{F}}$ of a higher derived category of $\mathcal{F}$.

Keeping in mind the above, we generalise the theorem by Brüning to higher homological algebra as follows.

**Theorem 4.3.2.** *There is a bijection*

$$\left\{\begin{matrix} \textit{functorially finite} \\ \textit{wide subcategories} \\ \textit{of } \mathcal{F} \end{matrix}\right\} \to \left\{\begin{matrix} \textit{functorially finite} \\ \textit{wide subcategories} \\ \textit{of } \overline{\mathcal{F}} \end{matrix}\right\}$$

*sending a wide subcategory $\mathcal{W}$ of $\mathcal{F}$ to $\overline{\mathcal{W}}$.*

In the above, by a wide subcategory of a $d$-abelian category, we mean an additive subcategory closed under $d$-kernels and $d$-cokernels, and such that every $d$-exact sequence with end terms in the subcategory is Yoneda equivalent, in the sense of Defintion 2.3.16, to a $d$-exact sequence having all terms in the subcategory. By a wide subcategory of a $(d+2)$-angulated category with automorphism $\Sigma^d$, we mean an additive subcategory closed under $d$-extensions and $\Sigma^{\pm d}$.

Iyama and Yoshino defined Auslander-Reiten $(d+2)$-angles in $(d+2)$-angulated categories in [32, Definition 3.8] and we proposed a modified definition, see Definition 2.3.46. Here, we define Auslander-Reiten $(d+2)$-angles in additive subcategories of $(d+2)$-angulated categories closed under $d$-extensions, an example of which are wide subcategories. We generalise the theorem by Jørgensen as follows.

**Theorem 4.5.5.** *Let $\mathcal{M}$ be a skeletally small* Hom*-finite $k$-linear $(d+2)$-angulated category with split idempotents. Let $\mathcal{W}$ be an additive subcategory of $\mathcal{M}$ closed under $d$-extensions. Let $W$ be in $\mathcal{W}$ and suppose that there exists $U^0$ in $\mathcal{W}$ and a non-zero morphism $\gamma^{d+1} : W \to \Sigma^d U^0$. Let*

$$\epsilon : \qquad X^0 \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^d \xrightarrow{\xi^d} W \xrightarrow{\xi^{d+1}} \Sigma^d X^0$$

*be an Auslander-Reiten $(d+2)$-angle in $\mathcal{M}$. Then the following are equivalent:*

*(a) $X^0$ has a $\mathcal{W}$-cover of the form $\varphi : W^0 \to X^0$,*

*(b) there is an Auslander-Reiten $(d+2)$-angle in $\mathcal{W}$ of the form*

$$\epsilon' : \qquad W^0 \xrightarrow{\omega^0} W^1 \xrightarrow{\omega^1} W^2 \longrightarrow \cdots \longrightarrow W^d \xrightarrow{\omega^d} W \xrightarrow{\omega^{d+1}} \Sigma^d W^0.$$

Note that for $d=1$, the above becomes exactly the theorem by Jørgensen.

**Remark.** We will apply Theorems 4.3.2 and 4.5.5 to the class of examples introduced in

Section 3.3. For positive integers $m$, $l$ and $d$ such that $(m-1)/l = d/2$, consider

$$\Lambda = kA_m/(\mathrm{rad}_{kA_m})^l.$$

There is a unique $d$-cluster tilting subcategory $\mathcal{F}$ of $\operatorname{mod}\Lambda$ with Auslander-Reiten quiver

$$f_1 \longrightarrow f_2 \longrightarrow \cdots \longrightarrow f_l \longrightarrow \cdots \longrightarrow f_m \longrightarrow \cdots \longrightarrow f_{m+l-2} \longrightarrow f_{m+l-1},$$

where $f_1, \dots, f_m$ are the indecomposable projectives and $f_l, \dots, f_{m+l-1}$ the indecomposable injectives in $\operatorname{mod}\Lambda$. The wide subcategories of $\mathcal{F}$ are fully described in [23]. We consider the $(d+2)$-angulated category $\overline{\mathcal{F}}$. We give a full description of the wide subcategories of $\overline{\mathcal{F}}$, using Theorem 4.3.2, and a recipe to construct Auslander-Reiten $(d+2)$-angles in these subcategories, using Theorem 4.5.5.

In this chapter, we will use many of the results presented in 2.3.2. The chapter is organised as follows. Section 4.2 introduces our setup and defines wide subcategories. Section 4.3 proves Theorem 4.3.2. Section 4.4 studies Auslander-Reiten $(d+2)$-angles in $\mathcal{W}$. Section 4.5 proves Theorem 4.5.5. Finally, Section 4.6 is an application of Theorems 4.3.2 and 4.5.5 to the class of examples from [55], presented in Section 3.3.

## 4.2 Setup and definition of wide subcategories

In this section, we present the setup we will be working in and the definition of wide subcategories in this setup.

**Setup 4.2.1.** Let $k$ be a field and $\mathcal{M}$ be a skeletally small $k$-linear Hom-finite $(d+2)$-angulated category with split idempotents. Note that this implies that $\mathcal{M}$ is Krull-Schmidt by Remark 2.1.12.

**Definition 4.2.2.** Let $\mathcal{W}$ be an additive subcategory of $\mathcal{M}$. We say that $\mathcal{W}$ *is closed under $d$-extensions* if given any morphism in $\mathcal{M}$ of the form $\delta : W'' \to \Sigma^d W'$ with $W', W'' \in \mathcal{W}$, there is a $(d+2)$-angle in $\mathcal{M}$ of the form

$$W' \longrightarrow W^1 \longrightarrow \cdots \longrightarrow W^d \longrightarrow W'' \xrightarrow{\delta} \Sigma^d W'$$

with $W^i \in \mathcal{W}$ for any $i \in \{1, \dots, d\}$.

**Remark 4.2.3.** Let $\mathcal{W} \subseteq \mathcal{M}$ be closed under $d$-extensions. Note that when $d > 1$, for a $(d+2)$-angle in $\mathcal{M}$ of the form

$$W' \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} W'' \xrightarrow{\delta} \Sigma^d W',$$

with $W', W''$ in $\mathcal{W}$, it is not necessarily true that $X^1, \dots, X^d$ are in $\mathcal{W}$. However, if $\xi^1, \dots, \xi^{d-1}$ are in $\mathrm{rad}_{\mathcal{M}}$, then $X^1, \dots, X^d$ are in $\mathcal{W}$.

**Definition 4.2.4.** An additive subcategory $\mathcal{W}$ of a $(d+2)$-angulated category $\mathcal{M}$ is called *wide* if it is closed under $d$-extensions and satisfies $\Sigma^d(\mathcal{W}) \subseteq \mathcal{W}$ and $\Sigma^{-d}(\mathcal{W}) \subseteq \mathcal{W}$.

## 4.3 Functorially finite wide subcategories of $\mathcal{F}$ and $\overline{\mathcal{F}}$

The aim of this section is to prove Theorem 4.3.2. Note that we assume that the field $k$ is algebraically closed because our arguments rely on results from [23], where this assumption is made. We start by presenting the setup we will be working in this section and stating the theorem.

**Setup 4.3.1.** Let $d$ be a fixed positive integer, $k$ an algebraically closed field and $\Lambda$ a finite dimensional $k$-algebra with global dimension at most $d$. Assume that there is a $d$-cluster tilting subcategory $\mathcal{F} \subseteq \mathrm{mod}\,\Lambda$, see Definition 2.3.13.

Note that $\overline{\mathcal{F}}$ is $d$-cluster tilting in $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$, see Definition 2.3.42, and so it is $(d+2)$-angulated, see Remark 2.3.44. So, using the notation in Setup 4.2.1, in this section we have $\mathcal{M} = \overline{\mathcal{F}}$.

**Theorem 4.3.2.** *There is a bijection*

$$\left\{\begin{matrix}\textit{functorially finite}\\ \textit{wide subcategories}\\ \textit{of } \mathcal{F}\end{matrix}\right\} \to \left\{\begin{matrix}\textit{functorially finite}\\ \textit{wide subcategories}\\ \textit{of } \overline{\mathcal{F}}\end{matrix}\right\}$$

*sending a wide subcategory $\mathcal{W}$ of $\mathcal{F}$ to $\overline{\mathcal{W}}$.*

We build the proof of Theorem 4.3.2 by first proving a more general bijection, then proving this bijection respects "functorially finite". Proving Theorem 4.3.2 will then amount to proving the bijection respects "wide".

**Lemma 4.3.3.** There is a bijection

$$\left\{\begin{matrix}\text{additive subcategories}\\ \text{of } \mathcal{F}\end{matrix}\right\} \to \left\{\begin{matrix}\text{additive subcategories}\\ \text{of } \overline{\mathcal{F}}\\ \text{closed under } \Sigma^{\pm d}\end{matrix}\right\}$$

sending an additive subcategory $\mathcal{W}$ of $\mathcal{F}$ to $\overline{\mathcal{W}}$.

*Proof.* Let $\mathcal{W} \subseteq \mathcal{F}$ be an additive subcategory of $\mathcal{F}$, then $\overline{\mathcal{W}} \subseteq \overline{\mathcal{F}}$ is clearly additive and closed under $\Sigma^{\pm d}$.

Suppose now that $\mathcal{X} \subseteq \overline{\mathcal{F}}$ is an additive subcategory closed under $\Sigma^{\pm d}$. Let $x$ be an indecomposable in $\mathcal{X}$, then $x = \Sigma^{id} f$ for some $f \in \mathcal{F}$ and integer $i$. Since $\mathcal{X}$ is closed under $\Sigma^{\pm d}$ and under direct summands, then

$$\operatorname{add}\{\Sigma^{id} f \mid i \in \mathbb{Z}\} \subseteq \mathcal{X}.$$

Take $\mathcal{W} := \mathcal{F} \cap \mathcal{X}$ and note that, by the above, we have $\operatorname{add}\{\Sigma^{id}\mathcal{W} \mid i \in \mathbb{Z}\} \subseteq \mathcal{X}$. Moreover, if $x$ is an indecomposable in $\mathcal{X}$, say $x = \Sigma^{id} f$, then $\Sigma^{(-i)d} x = f \in \mathcal{X}$ and so $f \in \mathcal{W}$. Hence $x \in \operatorname{add}\{\Sigma^{id}\mathcal{W} \mid i \in \mathbb{Z}\}$ and $\overline{\mathcal{W}} = \operatorname{add}\{\Sigma^{id}\mathcal{W} \mid i \in \mathbb{Z}\} = \mathcal{X}$. □

**Lemma 4.3.4.** The bijection from Lemma 4.3.3 respects “functorially finite”.

*Proof.* Suppose first that $\overline{\mathcal{W}}$ is functorially finite in $\overline{\mathcal{F}}$ and take any $f \in \mathcal{F}$. Then, there is a $\overline{\mathcal{W}}$-precover of $f$ of the form $\overline{\omega} : \overline{w} \to f$. Since $f$ is a complex concentrated in degree zero, then we may assume $\overline{w} = w_0 \oplus \Sigma^{-d} w_{-d}$ for some $w_0, w_{-d} \in \mathcal{W}$, as any other summand of $\overline{w}$ would have zero Hom space to $f$. Then,

$$\overline{\omega} = (\omega_0, \omega_{-d}) : w_0 \oplus \Sigma^{-d} w_{-d} \to f.$$

Let $w \in \mathcal{W}$ and $\alpha : w \to f$. Since $\mathcal{W} \subseteq \overline{\mathcal{W}}$, there is a morphism $\gamma : w \to w_0 \oplus \Sigma^{-d} w_{-d}$ such that $\overline{\omega} \circ \gamma = \alpha$. Since there are no non-zero maps of the form $w \to \Sigma^{-d} w_{-d}$, then

$$\alpha = \overline{\omega} \circ \gamma = (\omega_0, \omega_{-d}) \circ \begin{pmatrix} \gamma_0 \\ 0 \end{pmatrix} = \omega_0 \circ \gamma_0.$$

Hence $\omega_0$ is a $\mathcal{W}$-precover of $f$ and $\mathcal{W}$ is precovering in $\mathcal{F}$. Dually, $\mathcal{W}$ is preenveloping in $\mathcal{F}$.

Suppose now that $\mathcal{W}$ is functorially finite in $\mathcal{F}$. Note that, in order to prove that $\overline{\mathcal{W}}$ is precovering in $\overline{\mathcal{F}}$, it is enough to find a $\overline{\mathcal{W}}$-precover of any $f \in \mathcal{F}$. We have that $\mathcal{W} \subseteq \mathcal{F}$ is functorially finite, $\mathcal{F} \subseteq \operatorname{mod}\Lambda$ is functorially finite since $\mathcal{F}$ is $d$-cluster tilting in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda \subseteq \mathcal{D}^b(\operatorname{mod}\Lambda)$ is functorially finite by [29, Theorem 5.1]. Hence $\mathcal{W} \subseteq \mathcal{D}^b(\operatorname{mod}\Lambda)$ is functorially finite. Moreover, for any integer $i$, applying the automorphism $\Sigma^i$ to

$$\mathcal{W} \subseteq \mathcal{F} \subseteq \operatorname{mod}\Lambda \subseteq \mathcal{D}^b(\operatorname{mod}\Lambda),$$

we conclude that $\Sigma^i\mathcal{W} \subseteq \mathcal{D}^b(\operatorname{mod}\Lambda)$ is functorially finite. For $f \in \mathcal{F}$, note that the only non-zero morphisms from $\overline{\mathcal{W}}$ to $f$ are from objects in $\mathcal{W} \oplus \Sigma^{-d}\mathcal{W}$. Take a $\mathcal{W}$-precover of

$f$, say $\omega_0 : w_0 \to f$, and a $\Sigma^{-d}\mathcal{W}$-precover of $f$, say $\omega_{-d} : \Sigma^{-d} w_{-d} \to f$. Consider

$$\overline{\omega} := (\omega_0, \omega_{-d}) : w_0 \oplus \Sigma^{-d} w_{-d} \to f.$$

Given any $\overline{v}$ in $\overline{\mathcal{W}}$ and $\overline{\nu} : \overline{v} \to f$, without loss of generality, let $\overline{v} = v_0 \oplus \Sigma^{-d} v_{-d}$ for some $v_0, v_{-d} \in \mathcal{W}$. So

$$\overline{\nu} = (\nu_0, \nu_{-d}) : v_0 \oplus \Sigma^{-d} v_{-d} \to f.$$

Then, there are $\gamma_0 : v_0 \to w_0$ and $\gamma_{-d} : \Sigma^{-d} v_{-d} \to \Sigma^{-d} w_{-d}$ such that $\nu_0 = \omega_0 \circ \gamma_0$ and $\nu_{-d} = \omega_{-d} \circ \gamma_{-d}$. Hence

$$\overline{\omega} \circ \begin{pmatrix} \gamma_0 & 0 \\ 0 & \gamma_{-d} \end{pmatrix} = (\omega_0 \circ \gamma_0, \omega_{-d} \circ \gamma_{-d}) = \nu$$

and $\overline{\omega}$ is a $\overline{\mathcal{W}}$-precover of $f$. Dually, $\overline{\mathcal{W}}$ is preenveloping in $\overline{\mathcal{F}}$. $\square$

**Lemma 4.3.5.** Suppose we have an exact sequence with terms in $\mathcal{F}$ of the form:

$$0 \longrightarrow f^0 \xrightarrow{\varphi^0} f^1 \xrightarrow{\varphi^1} \cdots \xrightarrow{\varphi^{d-2}} f^{d-1} \xrightarrow{\varphi^{d-1}} f^d \xrightarrow{\varphi^d} f^{d+1}. \tag{4.1}$$

Then

$$0 \longrightarrow f^0 \xrightarrow{\varphi^0} f^1 \xrightarrow{\varphi^1} \cdots \xrightarrow{\varphi^{d-2}} f^{d-1} \xrightarrow{\varphi^{d-1}} f^d$$

is a $d$-kernel in $\mathcal{F}$ of $\varphi^d$.

*Proof.* Using the notation $\mathcal{F}(-,-) = \mathrm{Hom}_{\mathcal{F}}(-,-)$ and applying $\mathcal{F}(f,-)$ to (4.1), for any $f$ in $\mathcal{F}$, we obtain:

$$0 \longrightarrow \mathcal{F}(f,f^0) \xrightarrow{\varphi^{0*}} \mathcal{F}(f,f^1) \xrightarrow{\varphi^{1*}} \cdots \xrightarrow{\varphi^{d-1*}} \mathcal{F}(f,f^d) \xrightarrow{\varphi^{d*}} \mathcal{F}(f,f^{d+1}). \tag{4.2}$$

First, note that this is a complex, since $\varphi^i \circ \varphi^{i-1} = 0$ for all $i = 1, \ldots, d$. Moreover, since $\mathcal{F}(f,-)$ is left exact, $\varphi^{0*}$ is injective. It remains to show that $\mathrm{Ker}(\varphi^{i*}) \subseteq \mathrm{Im}(\varphi^{i-1*})$ for all $i = 1, \ldots, d$.

We have a splitting of (4.1) into short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & f^0 \arrow[r, hook, "\varphi^0"] & f^1 \arrow[rr, "\varphi^1"] \arrow[dr, "\pi^1"'] & & f^2 \arrow[rr] \arrow[dr, "\pi^2"'] & & \cdots \arrow[r] & f^{d-1} \arrow[rr, "\varphi^{d-1}"] \arrow[dr, "\pi^{d-1}"'] & & f^d \arrow[rr, "\varphi^d"] \arrow[dr, "\pi^d"'] & & f^{d+1}. \\
& & & k^2 \arrow[ur, hook, "\iota^2"'] & & k^3 & & & k^d \arrow[ur, hook, "\iota^d"'] & & k^{d+1} \arrow[ur, hook, "\iota^{d+1}"'] &
\end{tikzcd}$$

For $\alpha : f \to f^1$ in $\operatorname{Ker}(\varphi^{1*})$, since $\operatorname{Ker}\varphi^1 = f^0$, there is a morphism $\beta : f \to f^0$ such that $\alpha = \varphi^0 \circ \beta = \varphi^{0*}(\beta)$. So we may assume $i = 2, \dots, d$. We have exact sequence

$$\cdots \to \mathcal{F}(f, f^{i-1}) \xrightarrow{\pi^{i-1*}} \mathcal{F}(f, k^i) \to \operatorname{Ext}^1(f, k^{i-1}),$$

induced by the short exact sequence $0 \to k^{i-1} \to f^{i-1} \to k^i \to 0$. We show that $\pi^{i-1*}$ is surjective by showing that $\operatorname{Ext}^1(f, k^{i-1}) = 0$. The case $i = 2$ is trivial, since $k^1 = f^0 \in \mathcal{F}$ and $\mathcal{F}$ is $d$-cluster tilting. So assume $i > 2$.

The short exact sequences in the splitting of (4.1) induce the following exact sequences:

$$\begin{array}{ccccc}
\operatorname{Ext}^1(f, f^{i-2}) & \longrightarrow & \operatorname{Ext}^1(f, k^{i-1}) & \longrightarrow & \operatorname{Ext}^2(f, k^{i-2}) \\
\operatorname{Ext}^2(f, f^{i-3}) & \longrightarrow & \operatorname{Ext}^2(f, k^{i-2}) & \longrightarrow & \operatorname{Ext}^3(f, k^{i-3}) \\
\vdots & & \vdots & & \vdots \\
\operatorname{Ext}^{i-3}(f, f^2) & \longrightarrow & \operatorname{Ext}^{i-3}(f, k^3) & \longrightarrow & \operatorname{Ext}^{i-2}(f, k^2) \\
\operatorname{Ext}^{i-2}(f, f^1) & \longrightarrow & \operatorname{Ext}^{i-2}(f, k^2) & \longrightarrow & \operatorname{Ext}^{i-1}(f, f^0).
\end{array}$$

Since $i - 1 \le d - 1$ and $\mathcal{F}$ is $d$-cluster tilting, all the objects in the left column and $\operatorname{Ext}^{i-1}(f, f^0)$ are zero. Note that this forces all the objects in the middle column to be zero and in particular $\operatorname{Ext}^1(f, k^{i-1}) = 0$. Hence $\pi^{i-1*}$ is surjective.

Take $\alpha : f \to f^i \in \operatorname{Ker}(\varphi^{i*})$. Then, by definition of kernel, there is a morphism $\gamma : f \to k^i$ such that $\alpha = \iota^i \circ \gamma$. Since $\pi^{i-1*}$ is surjective, there is a morphism $\beta : f \to f^{i-1}$ such that $\pi^{i-1} \circ \beta = \gamma$. Then,

$$\alpha = \iota^i \circ \gamma = \iota^i \circ \pi^{i-1} \circ \beta = \varphi^{i-1} \circ \beta = \varphi^{i-1*}(\beta).$$

Hence $\operatorname{Ker}(\varphi^{i*}) \subseteq \operatorname{Im}(\varphi^{i-1*})$ for all $i = 1, \dots, d$ as we wished to prove. □

**Lemma 4.3.6.** Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated categories with suspension functors $\Sigma$ and $\Sigma'$ respectively. Suppose there are $d$-cluster tilting subcategories $\mathcal{C} \subseteq \mathcal{D}$ and $\mathcal{C}' \subseteq \mathcal{D}'$ such that $\Sigma^d(\mathcal{C}) \subseteq \mathcal{C}$ and $(\Sigma')^d(\mathcal{C}') \subseteq \mathcal{C}'$. Suppose $F : \mathcal{D} \to \mathcal{D}'$ is a triangulated functor, see Definition 2.2.29, such that $F(\mathcal{C}) \subseteq \mathcal{C}'$. Then $F$ sends $(d+2)$-angles in $\mathcal{C}$ to $(d+2)$-angles in $\mathcal{C}'$.

*Proof.* First note that $(\mathcal{C}, \Sigma^d)$ and $(\mathcal{C}', (\Sigma')^d)$ are $(d+2)$-angulated categories by Theorem 2.3.43. Take any $(d+2)$-angle in $\mathcal{C}$, say

$$c^0 \longrightarrow c^1 \longrightarrow \cdots \longrightarrow c^d \longrightarrow c^{d+1} \xrightarrow{\gamma} \Sigma^d c^0.$$

This comes from a diagram in $\mathcal{D}$ of the form

$$\begin{tikzcd}
 & c^1 \arrow[rr] \arrow[dr] & & c^2 \arrow[r] \arrow[dr] & \cdots \arrow[r] & c^{d-1} \arrow[rr] \arrow[dr] & & c^d \arrow[dr] & \\
c^0 \arrow[ur] & & x^1 \arrow[ll, squiggly] \arrow[ur] & & x^2 \arrow[ll, squiggly] \quad \cdots \quad x^{d-2} \arrow[ur] & & x^{d-1} \arrow[ll, squiggly] \arrow[ur] & & c^{d+1}, \arrow[ll, squiggly]
\end{tikzcd} \tag{4.3}$$

where by $x \rightsquigarrow y$ , we mean a morphism $x \to \Sigma y$ and the composition of all the wavy arrows is $\gamma$. Each oriented triangle is a triangle in $\mathcal{D}$ and each non-oriented triangle is commutative. Applying the functor $F$ to (4.3), we get the diagram:

$$\begin{tikzcd}
 & F(c^1) \arrow[rr] \arrow[dr] & & F(c^2) \arrow[r] \arrow[dr] & \cdots \arrow[r] & F(c^{d-1}) \arrow[rr] \arrow[dr] & & F(c^d) \arrow[dr] & \\
F(c^0) \arrow[ur] & & F(x^1) \arrow[ll, squiggly] \arrow[ur] & & F(x^2) \arrow[ll, squiggly] \quad \cdots \quad F(x^{d-2}) \arrow[ur] & & F(x^{d-1}) \arrow[ll, squiggly] \arrow[ur] & & F(c^{d+1}), \arrow[ll, squiggly]
\end{tikzcd}$$

where each non-oriented triangle is commutative, and since $F$ is triangulated, each oriented triangle is a triangle in $\mathcal{D}'$ and by $F(x) \rightsquigarrow F(y)$ we mean a morphism $F(x) \to F(\Sigma y) = \Sigma' F(y)$. Then, by Theorem 2.3.43, we obtain a $(d+2)$-angle in $\mathcal{C}'$:

$$F(c^0) \longrightarrow F(c^1) \longrightarrow \cdots \longrightarrow F(c^d) \longrightarrow F(c^{d+1}) \xrightarrow{F(\gamma)} (\Sigma')^d F(c^0).$$

$\square$

Using the above lemmas and [23, Theorem A], we prove there is a bijection between functorially finite wide subcategories of $\mathcal{F}$ as defined in [23, Definition 2.11], and functorially finite wide subcategories of $\overline{\mathcal{F}}$ as defined below.

**Definition 4.3.7** ([23, Section 1])**.** Let $\Gamma$ be a finite dimensional $k$-algebra and $\mathcal{G} \subseteq \operatorname{mod} \Gamma$ be a $d$-cluster tilting subcategory. We say that $(\Gamma, \mathcal{G})$ is a *$d$-homological pair.*

If $\lambda : \Lambda \to \Gamma$ is a homomorphism of algebras, then we denote by $\lambda_* : \operatorname{mod} \Gamma \to \operatorname{mod} \Lambda$ the functor given by restriction of scalars from $\Gamma$ to $\Lambda$. Moreover, if $\lambda$ is an epimorphism of algebras such that $\lambda_*(\mathcal{G}) \subseteq \mathcal{F}$ and $\operatorname{Tor}_d^{\Lambda}(\Gamma, \Gamma) = 0$, then we say that $\lambda : (\Lambda, \mathcal{F}) \to (\Gamma, \mathcal{G})$ is a *$d$-pseudoflat epimorphism of $d$-homological pairs.*

**Remark 4.3.8.** In the situation of Definition 4.3.7, we also denote by $\lambda_*$ the induced functor on the level of bounded derived categories:

$$\lambda_* : \mathcal{D}^b(\operatorname{mod} \Gamma) \to \mathcal{D}^b(\operatorname{mod} \Lambda).$$

This is full, faithful and triangulated, since $\lambda$ is a homological epimorphism by [23, Proposition 5.8]. Note that, since $\lambda_*$ is triangulated, it commutes with $\Sigma$. Moreover, by Lemma

2.1.17, $\lambda_*(\overline{\mathcal{G}})$ is closed under direct summands. Hence

$$\lambda_*(\overline{\mathcal{G}}) = \lambda_*(\operatorname{add}(\Sigma^{\mathbb{Z}d}\mathcal{G})) = \operatorname{add}(\Sigma^{\mathbb{Z}d}(\lambda_*(\mathcal{G}))) = \overline{\lambda_*(\mathcal{G})}.$$

*Proof of Theorem 4.3.2.* We start by showing that if $\mathcal{W} \subseteq \mathcal{F}$ is a functorially finite wide subcategory of $\mathcal{F}$, then $\overline{\mathcal{W}}$ is a functorially finite wide subcategory of $\overline{\mathcal{F}}$. By [23, Theorem A], there is a $d$-pseudoflat epimorphism of $d$-homological pairs $\lambda : (\Lambda, \mathcal{F}) \to (\Gamma, \mathcal{G})$ such that $\lambda_*(\mathcal{G}) = \mathcal{W}$. Then, by Remark 4.3.8, we have

$$\lambda_*(\overline{\mathcal{G}}) = \overline{\lambda_*(\mathcal{G})} = \overline{\mathcal{W}} \subseteq \overline{\mathcal{F}}.$$

Note that $\lambda_*(\overline{\mathcal{G}})$ is functorially finite in $\overline{\mathcal{F}}$ by Lemma 4.3.4. Then, to complete the first part of the proof, it remains to show that $\lambda_*(\overline{\mathcal{G}})$ is closed under $d$-extensions. Take any morphism $\delta : \lambda_*(\overline{g}) \to \lambda_*(\overline{g'})$ in $\lambda_*(\overline{\mathcal{G}})$. Since $\lambda_*$ is full and faithful, then $\delta = \lambda_*(\overline{g} \xrightarrow{\gamma} \overline{g'})$, for some morphism $\gamma$ in $\overline{\mathcal{G}}$. As $\overline{\mathcal{G}}$ is $(d+2)$-angulated, we can extend $\gamma$ to a $(d+2)$-angle in $\overline{\mathcal{G}}$ of the form:

$$\Sigma^{-d}(\overline{g'}) \longrightarrow \overline{g^1} \longrightarrow \cdots \longrightarrow \overline{g^d} \longrightarrow \overline{g} \xrightarrow{\gamma} \overline{g'}.$$

Then, by Lemma 4.3.6, we obtain a $(d+2)$-angle in $\overline{\mathcal{F}}$ with objects from $\overline{\mathcal{W}}$:

$$\Sigma^{-d}\lambda_*(\overline{g'}) \longrightarrow \lambda_*(\overline{g^1}) \longrightarrow \cdots \longrightarrow \lambda_*(\overline{g^d}) \longrightarrow \lambda_*(\overline{g}) \xrightarrow{\delta} \lambda_*(\overline{g'}).$$

Hence $\lambda_*(\overline{\mathcal{G}})$ is closed under $d$-extensions.

Now let $\mathcal{X} \subseteq \overline{\mathcal{F}}$ be a functorially finite wide subcategory. Then, by Lemmas 4.3.3 and 4.3.4, we have that $\mathcal{X} = \overline{\mathcal{V}}$ for some functorially finite subcategory $\mathcal{V} \subseteq \mathcal{F}$. It remains to show that $\mathcal{V} \subseteq \mathcal{F}$ is wide, in the sense of [23, Definition 2.11]. Let $\nu : v \to v'$ be a morphism in $\mathcal{V}$. Since $\mathcal{X} \subseteq \overline{\mathcal{F}}$ is wide, there is a $(d+2)$-angle in $\overline{\mathcal{F}}$ with objects from $\mathcal{X}$ of the form:

$$\Sigma^{-d}v' \longrightarrow x^1 \xrightarrow{\xi^1} x^2 \xrightarrow{\xi^2} \cdots \xrightarrow{\xi^{d-1}} x^d \xrightarrow{\xi^d} v \xrightarrow{\nu} v'. \tag{4.4}$$

Note that $v$, $v'$ are chain complexes concentrated in degree zero since they are in $\mathcal{V}$. Also, as $\mathcal{X} = \overline{\mathcal{V}}$, any $x \in \mathcal{X}$ is isomorphic to a complex with zero differentials and so $H(x) \cong x$. For $i = 1, \ldots, d$, let $v^i$ and $\nu^i$ be the components at degree zero of $x^i$ and $\xi^i$ respectively, and note that $v^i \in \mathcal{V}$.

Note that $H^0(-) = \operatorname{Hom}_{\mathcal{D}^b}(\Lambda, -)$. Since $\Lambda$ is a projective module in $\operatorname{mod}\Lambda$, then $\Lambda \in \overline{\mathcal{F}}$. Applying $H^0(-) = \operatorname{Hom}_{\overline{\mathcal{F}}}(\Lambda, -)$ to (4.4), by [19, Proposition 2.5] we obtain the exact sequence:

$$0 \longrightarrow v^1 \xrightarrow{\nu^1} v^2 \xrightarrow{\nu^2} \cdots \xrightarrow{\nu^{d-1}} v^d \xrightarrow{\nu^d} v \xrightarrow{\nu} v',$$

where we have used the fact that $H^0(\Sigma^{-d}v') = 0$, since $v'$ is concentrated in degree zero. Then, by Lemma 4.3.5, we conclude that

$$0 \longrightarrow v^1 \xrightarrow{\nu^1} v^2 \xrightarrow{\nu^2} \cdots \xrightarrow{\nu^{d-1}} v^d \xrightarrow{\nu^d} v$$

is a $d$-kernel of $\nu$ in $\mathcal{F}$ with objects from $\mathcal{V}$. The existence of a $d$-cokernel of $\nu$ in $\mathcal{F}$ with objects from $\mathcal{V}$ follows by a dual argument.

Consider a $d$-exact sequence in $\mathcal{F}$ of the form:

$$0 \longrightarrow v' \xrightarrow{\varphi^0} f^1 \xrightarrow{\varphi^1} \cdots \xrightarrow{\varphi^{d-2}} f^{d-1} \xrightarrow{\varphi^{d-1}} f^d \xrightarrow{\varphi^d} v \longrightarrow 0,$$

with $v$ and $v'$ in $\mathcal{V}$. Then, by Remark 2.3.45, there is a $(d+2)$-angle in $\overline{\mathcal{F}}$ of the form:

$$v' \xrightarrow{\varphi^0} f^1 \xrightarrow{\varphi^1} f^2 \xrightarrow{\varphi^2} \cdots \xrightarrow{\varphi^{d-2}} f^{d-1} \xrightarrow{\varphi^{d-1}} f^d \xrightarrow{\varphi^d} v \xrightarrow{\alpha} \Sigma^d v'.$$

Since $\mathcal{X}$ is closed under $d$-extensions and $v, v' \in \mathcal{X}$, there is a $(d+2)$-angle in $\overline{\mathcal{F}}$ with objects from $\mathcal{X}$:

$$v' \xrightarrow{\xi^0} x^1 \xrightarrow{\xi^1} x^2 \xrightarrow{\xi^2} \cdots \xrightarrow{\xi^{d-2}} x^{d-1} \xrightarrow{\xi^{d-1}} x^d \xrightarrow{\xi^d} v \xrightarrow{\alpha} \Sigma^d v'. \tag{4.5}$$

For $i = 0, \ldots, d$, let $v^i$ and $\nu^i$ be the components at degree zero of $x^i$ and $\xi^i$ respectively, and note that $v^i \in \mathcal{V}$. Applying $H^0(-) = \mathrm{Hom}_{\overline{\mathcal{F}}}(\Lambda, -)$ to (4.5), we obtain the exact sequence:

$$0 \longrightarrow v' \xrightarrow{\nu^0} v^1 \xrightarrow{\nu^1} v^2 \xrightarrow{\nu^2} \cdots \xrightarrow{\nu^{d-1}} v^d \xrightarrow{\nu^d} v \longrightarrow 0.$$

By Lemma 4.3.5 and its dual, this is a $d$-exact sequence. Moreover, by axiom (N3) from Definition 2.3.28, we have the morphism of $(d+2)$-angles in $\overline{\mathcal{F}}$:

$$\begin{tikzcd}
v' \arrow[r, "\varphi^0"] \arrow[d, equal] & f^1 \arrow[r, "\varphi^1"] \arrow[d, dashed, "\zeta^1"] & f^2 \arrow[r, "\varphi^2"] \arrow[d, dashed, "\zeta^2"] & \cdots \arrow[r, "\varphi^{d-1}"] & f^d \arrow[r, "\varphi^d"] \arrow[d, dashed, "\zeta^d"] & v \arrow[r, "\alpha"] \arrow[d, equal] & \Sigma^d v' \arrow[d, equal] \\
v' \arrow[r, "\xi^0"'] & x^1 \arrow[r, "\xi^1"'] & x^2 \arrow[r, "\xi^2"'] & \cdots \arrow[r, "\xi^{d-1}"'] & x^d \arrow[r, "\xi^d"'] & v \arrow[r, "\alpha"'] & \Sigma^d v'.
\end{tikzcd}$$

Applying $H^0(-)$ to the above, we obtain the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & v' \arrow[r, "\varphi^0"] \arrow[d, equal] & f^1 \arrow[r, "\varphi^1"] \arrow[d, "\psi^1"] & f^2 \arrow[r, "\varphi^2"] \arrow[d, "\psi^2"] & \cdots \arrow[r, "\varphi^{d-1}"] & f^d \arrow[r, "\varphi^d"] \arrow[d, "\psi^d"] & v \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & v' \arrow[r, "\nu^0"'] & v^1 \arrow[r, "\nu^1"'] & v^2 \arrow[r, "\nu^2"'] & \cdots \arrow[r, "\nu^{d-1}"'] & v^d \arrow[r, "\nu^d"'] & v \arrow[r] & 0.
\end{tikzcd}$$

Hence, the first and second row in the above diagram are two Yoneda equivalent $d$-exact sequences, and the second row has objects in $\mathcal{V}$. □

## 4.4 Auslander-Reiten $(d+2)$-angles in $\mathcal{W}$

**Setup 4.4.1.** Let us go back to Setup 4.2.1 and let $\mathcal{W}$ be an additive subcategory of $\mathcal{M}$ closed under $d$-extensions.

We have seen the Definition of Auslander-Reiten $(d+2)$-angulated categories in Definition 2.3.46. In this section, we introduce and study Auslander-Reiten $(d+2)$-angles in the subcategory $\mathcal{W}$.

**Definition 4.4.2.** A $(d+2)$-angle in $\mathcal{M}$ of the form

$$\epsilon: \qquad W^0 \xrightarrow{\omega^0} W^1 \xrightarrow{\omega^1} W^2 \longrightarrow \cdots \longrightarrow W^d \xrightarrow{\omega^d} W^{d+1} \xrightarrow{\omega^{d+1}} \Sigma^d W^0,$$

with $W^0, W^1, \dots, W^{d+1}$ in $\mathcal{W}$ is an *Auslander-Reiten* $(d+2)$*-angle in* $\mathcal{W}$ if the morphism $\omega^0$ is left almost split in $\mathcal{W}$, the morphism $\omega^d$ is right almost split in $\mathcal{W}$ and, when $d \geq 2$, also $\omega^1, \dots, \omega^{d-1}$ are in $\mathrm{rad}_{\mathcal{W}}$.

**Remark 4.4.3.** Note that since $\mathcal{W}$ is a full subcategory of $\mathcal{M}$, then $\mathrm{rad}_{\mathcal{W}}$ is equal to the restriction of $\mathrm{rad}_{\mathcal{M}}$ to $\mathcal{W}$.

**Lemma 4.4.4.** (a) Let $\omega^0 : W^0 \to W^1$ be left almost split in $\mathcal{W}$, then $\mathrm{End}(W^0)$ is local and $\omega^0 \in \mathrm{rad}_{\mathcal{W}}$.

(b) Let $\omega^d : W^d \to W^{d+1}$ be right almost split in $\mathcal{W}$, then $\mathrm{End}(W^{d+1})$ is local and $\omega^d \in \mathrm{rad}_{\mathcal{W}}$.

*Proof.* We only prove (a), the proof for (b) is then dual. Suppose $\omega^0 : W^0 \to W^1$ is left almost split in $\mathcal{W}$. Let $\mu, \nu : W^0 \to W^0$ be morphisms that are not split monomorphisms. Then there are morphisms $\mu', \nu' : W^1 \to W^0$ such that $\mu = \mu' \circ \omega^0$ and $\nu = \nu' \circ \omega^0$. By [1, Proposition 15.15], in order to prove that $\mathrm{End}(W^0)$ is local, it is enough to prove that $\mu + \nu$ is not a split monomorphism.

Suppose for a contradiction that $\mu+\nu$ is a split monomorphism. Hence there is a morphism $\gamma : W^0 \to W^0$ such that $\gamma \circ (\mu + \nu) = 1_{W^0}$. Then

$$\gamma \circ (\mu' + \nu') \circ \omega^0 = \gamma \circ (\mu + \nu) = 1_{W^0}.$$

Hence $\omega^0$ is a split monomorphism, contradicting our initial assumption. So $\mathrm{End}(W^0)$ is local.

Since $\mathrm{End}(W^0)$ is local, it follows that $W^0$ is indecomposable. Since $\mathcal{M}$ is Krull-Schmidt, there are indecomposable objects $W_1, \dots, W_t$ such that

$$W^1 = W_1 \oplus \cdots \oplus W_t.$$

Moreover, $W_1, \dots, W_t$ are in $\mathcal{W}$ since $\mathcal{W}$ is closed under summands. Then we have

$$\omega^0 = \begin{pmatrix} \omega_1^0 \\ \vdots \\ \omega_t^0 \end{pmatrix} : W^0 \to W_1 \oplus \cdots \oplus W_t.$$

Suppose there is some $i \in \{1, \dots, t\}$ such that $\omega_i^0 : W^0 \to W_i$ is not in $\mathrm{rad}_{\mathcal{W}}$. Then, since $W^0$ and $W_i$ are both indecomposable, it follows that $\omega_i^0$ is invertible. Hence

$$\begin{pmatrix} 0 & \cdots & 0 & (\omega_i^0)^{-1} & 0 & \cdots & 0 \end{pmatrix} \circ \omega^0 = (\omega_i^0)^{-1} \circ \omega_i^0 = 1_{W^0},$$

contradicting the fact that $\omega^0$ is not a split monomorphism. Hence such an $i$ does not exist and $\omega^0 \in \mathrm{rad}_{\mathcal{W}}$. □

**Lemma 4.4.5.** Let

$$\epsilon : \qquad W^0 \xrightarrow{\omega^0} W^1 \xrightarrow{\omega^1} W^2 \longrightarrow \cdots \longrightarrow W^d \xrightarrow{\omega^d} W^{d+1} \xrightarrow{\omega^{d+1}} \Sigma^d W^0$$

be a $(d+2)$-angle with $W^0, W^1, \dots, W^{d+1}$ in $\mathcal{W}$. If $\omega^d$ is right almost split in $\mathcal{W}$ and $\omega^{d+1}$ is left minimal, then $\omega^0$ is left almost split in $\mathcal{W}$.

*Proof.* Since $\omega^d$ is not a split epimorphism, Lemma 2.3.35 implies that $\omega^0$ is not a split monomorphism. Let $\phi^0 : W^0 \to V^0$ be a morphism in $\mathcal{W}$ that is not a split monomorphism. Extend $\Sigma^d(\phi^0) \circ \omega^{d+1}$ to a $(d+2)$-angle and consider the following commutative diagram, built using axiom (N3) from Definition 2.3.28:

$$\begin{tikzcd}
W^0 \arrow[r, "\omega^0"] \arrow[d, "\phi^0"] & W^1 \arrow[r, "\omega^1"] \arrow[d, dashed, "\phi^1"] & W^2 \arrow[r] \arrow[d, dashed, "\phi^2"] & \cdots \arrow[r] & W^d \arrow[r, "\omega^d"] \arrow[d, dashed, "\phi^d"] & W^{d+1} \arrow[r, "\omega^{d+1}"] \arrow[d, equal] & \Sigma^d W^0 \arrow[d, "\Sigma^d(\phi^0)"] \\
V^0 \arrow[r, "\eta^0"'] & V^1 \arrow[r, "\eta^1"'] & V^2 \arrow[r] & \cdots \arrow[r] & V^d \arrow[r, "\eta^d"'] & W^{d+1} \arrow[r, "\Sigma^d(\phi^0)\circ\omega^{d+1}"'] & \Sigma^d V^0,
\end{tikzcd}$$

where, as $V^0$ and $W^{d+1}$ are in $\mathcal{W}$, which is closed under $d$-extensions, we can choose $V^1, \ldots, V^d$ in $\mathcal{W}$.

Suppose for a contradiction that $\eta^0$ is not a split monomorphism. Then $\eta^d$ is not a split epimorphism by Lemma 2.3.35. As $\omega^d$ is right almost split in $\mathcal{W}$ and $\eta^d : V^d \to W^{d+1}$ is a morphism in $\mathcal{W}$, then there is a morphism $\psi^d : V^d \to W^d$ such that $\eta^d = \omega^d \circ \psi^d$. So we can construct a commutative diagram of the form:

$$\begin{tikzcd}
V^0 \arrow[r, "\eta^0"] \arrow[d, dashed, "\psi^0"] & V^1 \arrow[r] \arrow[d, dashed, "\psi^1"] & \cdots \arrow[r] & V^{d-1} \arrow[r, "\eta^{d-1}"] \arrow[d, dashed, "\psi^{d-1}"] & V^d \arrow[r, "\eta^d"] \arrow[d, "\psi^d"] & W^{d+1} \arrow[r, "\Sigma^d(\phi^0)\circ\omega^{d+1}"] \arrow[d, equal] & \Sigma^d V^0 \arrow[d, dashed, "\Sigma^d(\psi^0)"] \\
W^0 \arrow[r, "\omega^0"'] & W^1 \arrow[r] & \cdots \arrow[r] & W^{d-1} \arrow[r, "\omega^{d-1}"'] & W^d \arrow[r, "\omega^d"'] & W^{d+1} \arrow[r, "\omega^{d+1}"'] & \Sigma^d W^0.
\end{tikzcd}$$

Hence we have

$$\Sigma^d(\psi^0 \circ \phi^0) \circ \omega^{d+1} = \Sigma^d(\psi^0) \circ \Sigma^d(\phi^0) \circ \omega^{d+1} = \omega^{d+1}.$$

Since $\omega^{d+1}$ is left minimal, then $\Sigma^d(\psi^0 \circ \phi^0)$ is an isomorphism, and so also $\psi^0 \circ \phi^0$ is an isomorphism, contradicting our assumption that $\phi^0$ is not a split monomorphism. Hence $\eta^0$ is a split monomorphism and there is a morphism $\gamma : V^1 \to V^0$ such that $\gamma \circ \eta^0 = 1_{V^0}$. Then

$$\gamma \circ \phi^1 \circ \omega^0 = \gamma \circ \eta^0 \circ \phi^0 = 1_{V^0} \circ \phi^0 = \phi^0,$$

and so $\omega^0$ is left almost split in $\mathcal{W}$. $\square$

**Lemma 4.4.6.** Let

$$\epsilon : \qquad W^0 \xrightarrow{\omega^0} W^1 \xrightarrow{\omega^1} W^2 \longrightarrow \cdots \longrightarrow W^d \xrightarrow{\omega^d} W^{d+1} \xrightarrow{\omega^{d+1}} \Sigma^d W^0$$

be a $(d+2)$-angle in $\mathcal{M}$ with $W^0, W^1, \ldots, W^{d+1}$ in $\mathcal{W}$. Then the following are equivalent:

(a) $\epsilon$ is an Auslander-Reiten $(d+2)$-angle in $\mathcal{W}$,

(b) $\omega^0, \omega^1, \ldots, \omega^{d-1}$ are in $\operatorname{rad}_{\mathcal{W}}$ and $\omega^d$ is right almost split in $\mathcal{W}$.

(c) $\omega^1, \ldots, \omega^{d-1}, \omega^d$ are in $\operatorname{rad}_{\mathcal{W}}$ and $\omega^0$ is left almost split in $\mathcal{W}$.

*Proof.* Note that (a) implies both (b) and (c) by Lemma 4.4.4 and Definition 4.4.2. Suppose now that (b) holds. Since $\omega^0$ is in $\operatorname{rad}_{\mathcal{W}}$ and so in $\operatorname{rad}_{\mathcal{M}}$, then so is $(-1)^d\Sigma^d(\omega^0)$ and $\omega^{d+1}$ is left minimal by Lemma 2.3.34. Then, by Lemma 4.4.5, it follows that $\omega^0$ is left almost split in $\mathcal{W}$, so (c) holds as $\omega^d \in \operatorname{rad}_{\mathcal{M}}$ by Lemma 4.4.4.

The fact that (c) implies (b) follows by a dual argument and so it is now clear that they both imply (a). $\square$

**Lemma 4.4.7.** Consider a $(d+2)$-angle of the form

$$\epsilon: \qquad W^0 \xrightarrow{\omega^0} W^1 \xrightarrow{\omega^1} W^2 \longrightarrow \cdots \longrightarrow W^d \xrightarrow{\omega^d} W^{d+1} \xrightarrow{\omega^{d+1}} \Sigma^d W^0,$$

with $W^0$, $W^1, \dots, W^{d+1}$ in $\mathcal{W}$ and suppose that $\omega^d$ is right almost split in $\mathcal{W}$ and, if $d \geq 2$, also that $\omega^1, \dots, \omega^{d-1}$ are in $\mathrm{rad}_{\mathcal{W}}$. Then the following are equivalent:

(a) $\mathrm{End}(W^0)$ is local,

(b) $\omega^{d+1}$ is left minimal,

(c) $\omega^0$ is in $\mathrm{rad}_{\mathcal{W}}$,

(d) $\epsilon$ is an Auslander-Reiten $(d+2)$-angle in $\mathcal{W}$.

*Proof.* (a)⇒(b). Suppose $\mathrm{End}(W^0)$ is local. By Lemma 2.3.35, since $\omega^d$ is not a split epimorphism, it follows that $\omega^{d+1}$ is non-zero. Then, as $\mathrm{End}(W^0) \cong \mathrm{End}(\Sigma^d W^0)$ is local, it follows that $\omega^{d+1}$ is left minimal by Lemma 2.3.36.

(d)⇒(a). Suppose $\epsilon$ is an Auslander-Reiten $(d+2)$-angle in $\mathcal{W}$. Then $\omega^0$ is left almost split in $\mathcal{W}$ and by Lemma 4.4.4, we have that $\mathrm{End}(W^0)$ is local.

(c)⇒(d). Suppose $\omega^0$ is in $\mathrm{rad}_{\mathcal{W}}$. Then, by Lemma 4.4.6, it follows that $\epsilon$ is an Auslander-Reiten $(d+2)$-angle in $\mathcal{W}$.

(b)⇒(c). Suppose $\omega^{d+1}$ is left minimal. Lemma 2.3.34 implies that $(-1)^d \Sigma^d(\omega^0) \in \mathrm{rad}_{\mathcal{M}}$, so $\omega^0 \in \mathrm{rad}_{\mathcal{M}}$ and $\omega^0 \in \mathrm{rad}_{\mathcal{W}}$ by Remark 4.4.3. $\square$

## 4.5 $\mathcal{W}$-covers and Auslander-Reiten $(d+2)$-angles in $\mathcal{W}$

In this section, we generalise [37, Theorem 3.1] to any $d \geq 1$, see Theorem 4.5.5. To do so, we start by proving the higher version of [37, Lemmas 2.2 and 2.3] and another lemma. We work in Setup 4.4.1.

**Lemma 4.5.1.** Consider an Auslander-Reiten $(d+2)$-angle in $\mathcal{M}$ of the form

$$\epsilon: \qquad X^0 \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^d \xrightarrow{\xi^d} X^{d+1} \xrightarrow{\xi^{d+1}} \Sigma^d X^0.$$

View the abelian group $\mathrm{Hom}(X^{d+1}, \Sigma^d X^0)$ as an $\mathrm{End}(X^{d+1})$-right-module via composition of morphisms. The socle of this module is simple and equal to the submodule generated by $\xi^{d+1}$.

*Proof.* Let $M$ be a non-zero submodule of $\operatorname{Hom}(X^{d+1}, \Sigma^d X^0)$ and pick a non-zero element $\mu : X^{d+1} \to \Sigma^d X^0$ in $M$. Extend $\mu$ to a $(d+2)$-angle:

$$X^0 \xrightarrow{\eta^0} Y^1 \xrightarrow{\eta^1} Y^2 \longrightarrow \cdots \longrightarrow Y^d \xrightarrow{\eta^d} X^{d+1} \xrightarrow{\mu} \Sigma^d X^0.$$

Since $\mu$ is non-zero, then $\eta^0$ is not a split monomorphism by Lemma 2.3.35. Then, as $\xi^0$ is left almost split, there is a morphism $\psi^1 : X^1 \to Y^1$ such that $\psi^1 \circ \xi^0 = \eta^0$. So, by axiom (N3) from Definition 2.3.28, there exist morphisms $\psi^2, \dots, \psi^{d+1}$ making the following diagram commutative:

$$\begin{tikzcd}
X^0 \arrow[r, "\xi^0"] \arrow[d, equal] & X^1 \arrow[r, "\xi^1"] \arrow[d, "\psi^1"] & X^2 \arrow[r] \arrow[d, dashed, "\psi^2"] & \cdots \arrow[r] & X^d \arrow[r, "\xi^d"] \arrow[d, dashed, "\psi^d"] & X^{d+1} \arrow[r, "\xi^{d+1}"] \arrow[d, dashed, "\psi^{d+1}"] & \Sigma^d X^0 \arrow[d, equal] \\
X^0 \arrow[r, "\eta^0"'] & Y^1 \arrow[r, "\eta^1"'] & Y^2 \arrow[r] & \cdots \arrow[r] & Y^d \arrow[r, "\eta^d"'] & X^{d+1} \arrow[r, "\mu"'] & \Sigma^d X^0.
\end{tikzcd}$$

In particular, we have $\mu \circ \psi^{d+1} = \xi^{d+1}$. So, in the $\operatorname{End}(X^{d+1})$-module $\operatorname{Hom}(X^{d+1}, \Sigma^d X^0)$, the element $\xi^{d+1}$ is a multiple of $\mu$. So $\xi^{d+1}$ is in $M$ and the non-zero submodule of $\operatorname{Hom}(X^{d+1}, \Sigma^d X^0)$ generated by $\xi^{d+1}$ is contained in $M$. Then, the socle of the $\operatorname{End}(X^{d+1})$-module $\operatorname{Hom}(X^{d+1}, \Sigma^d X^0)$ is the submodule generated by $\xi^{d+1}$.

Note that $\operatorname{End}(X^{d+1})$ is local, as $\epsilon$ is an Auslander-Reiten $(d+2)$-angle. Since the socle of $\operatorname{Hom}(X^{d+1}, \Sigma^d X^0)$ is generated by the single element $\xi^{d+1}$, it follows that it is simple if it is annihilated by the Jacobson radical of $\operatorname{End}(X^{d+1})$. Let $\rho : X^{d+1} \to X^{d+1}$ be in the radical of $\operatorname{End}(X^{d+1})$, then by the dual of [1, Proposition 15.15(e)], we have that $\rho$ has no right inverse. Hence $\rho$ is not a split epimorphism and, since $\xi^d$ is right almost split, there is a morphism $\rho' : X^{d+1} \to X^d$ such that $\rho = \xi^d \circ \rho'$. Then, by Lemma 2.3.31 we have

$$\xi^{d+1} \circ \rho = \xi^{d+1} \circ \xi^d \circ \rho' = 0 \circ \rho' = 0,$$

as we wished to prove. $\square$

**Definition 4.5.2** ([38, Section 0])**.** For an additive subcategory $\mathcal{U} \subseteq \mathcal{M}$, we define

$$\mathcal{U}\text{-exact} = \left\{ \begin{array}{c} M^1 \to \cdots \to M^d \\ \text{is a complex in } \mathcal{M} \end{array} \middle| \begin{array}{c} 0 \to \operatorname{Hom}_{\mathcal{M}}(U, M^1) \to \cdots \to \operatorname{Hom}_{\mathcal{M}}(U, M^d) \to 0 \\ \text{is exact for each } U \in \mathcal{U} \end{array} \right\}.$$

**Lemma 4.5.3.** Let $W$ be in $\mathcal{W}$ and let

$$\epsilon : \qquad X^0 \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^d \xrightarrow{\xi^d} W \xrightarrow{\xi^{d+1}} \Sigma^d X^0$$

be an Auslander-Reiten $(d+2)$-angle in $\mathcal{M}$. Suppose $\nu : V \to X^0$ is a $\mathcal{W}$-cover. Then $V$ is either zero or indecomposable.

*Proof.* Suppose $V$ is non-zero and recall that $\mathcal{M}$ is Krull-Schmidt. Let $V_i$ be an indecomposable direct summand of $V$, let $\iota_i : V_i \to V$ be the inclusion of $V_i$ into $V$ and $\nu_i := \nu \circ \iota_i$. Extend $\Sigma^d(\nu_i)$ to a $(d+2)$-angle:

$$X^0 \xrightarrow{\eta^0} Y^1 \xrightarrow{\eta^1} Y^2 \longrightarrow \cdots \longrightarrow Y^d \xrightarrow{\eta^d} \Sigma^d V_i \xrightarrow{\Sigma^d(\nu_i)} \Sigma^d X^0.$$

Since $\nu$ is a $\mathcal{W}$-cover, then $\nu_i$ is non-zero and so $\Sigma^d(\nu_i)$ is non-zero. Hence $\eta^0$ is not a split monomorphism by Lemma 2.3.35 and, as $\xi^0$ is left almost split, there exists a morphism $\psi^1 : X^1 \to Y^1$ such that $\psi^1 \circ \xi^0 = \eta^0$. Then, by axiom (N3) from Definition 2.3.28, there are morphisms $\psi^2, \ldots, \psi^{d+1}$ making the following diagram commutative:

$$\begin{tikzcd}
X^0 \arrow[r, "\xi^0"] \arrow[d, equal] & X^1 \arrow[r, "\xi^1"] \arrow[d, "\psi^1"] & X^2 \arrow[r] \arrow[d, dashed, "\psi^2"] & \cdots \arrow[r] & X^d \arrow[r, "\xi^d"] \arrow[d, dashed, "\psi^d"] & W \arrow[r, "\xi^{d+1}"] \arrow[d, dashed, "\psi^{d+1}"] & \Sigma^d X^0 \arrow[d, equal] \\
X^0 \arrow[r, "\eta^0"'] & Y^1 \arrow[r, "\eta^1"'] & Y^2 \arrow[r] & \cdots \arrow[r] & Y^d \arrow[r, "\eta^d"'] & \Sigma^d V_i \arrow[r, "\Sigma^d(\nu_i)"'] & \Sigma^d X^0.
\end{tikzcd}$$

In particular, we have $\Sigma^d(\nu_i) \circ \psi^{d+1} = \xi^{d+1}$. Then, letting $\varphi := \Sigma^{-d}(\psi^{d+1}) : \Sigma^{-d}W \to V_i$, we have $\nu_i \circ \varphi = \Sigma^{-d}(\xi^{d+1})$. As $\xi^{d+1}$ is non-zero, it follows that $\varphi$ is non-zero.

Hence every indecomposable direct summand $V_i$ of $V$ permits a non-zero morphism $\Sigma^{-d}W \to V_i$. We complete the proof by showing that at most one indecomposable direct summand of $V$ can permit such a morphism.

Extend $\nu$ to a $(d+2)$-angle of the form

$$V \xrightarrow{\nu} X^0 \xrightarrow{\omega^0} Z^1 \xrightarrow{\omega^1} Z^2 \longrightarrow \cdots \longrightarrow Z^{d-1} \xrightarrow{\omega^{d-1}} Z^d \xrightarrow{\omega^d} \Sigma^d V.$$

Consider the exact sequence

$$\mathrm{Hom}(W, Z^{d-1}) \xrightarrow{\widetilde{\omega^{d-1}}} \mathrm{Hom}(W, Z^d) \xrightarrow{\widetilde{\omega^d}} \mathrm{Hom}(W, \Sigma^d V) \xrightarrow{\phi} \mathrm{Hom}(W, \Sigma^d X^0),$$

where, for a morphism $\eta$ we use the notation $\widetilde{\eta} := \mathrm{Hom}(W, \eta)$ and $\phi := \widetilde{(-1)^d \Sigma^d(\nu)}$ for readability. Note that $Z^1 \to \cdots \to Z^d$ is in $\mathcal{W}$-exact by [38, Lemma 2.1]. Hence $\widetilde{\omega^{d-1}}$ is surjective, so that $\widetilde{\omega^d}$ is the zero map and $\phi$ is injective. Viewing $\phi$ as a homomorphism of finite dimensional right modules over the finite dimensional $k$-algebra $\mathrm{End}(W)$, the target $\mathrm{Hom}(W, \Sigma^d X^0)$ has simple socle by Lemma 4.5.1. Hence the image is either zero or indecomposable. Since $\phi$ is injective, then the same is true for the source $\mathrm{Hom}(W, \Sigma^d V)$. So, if $V = V_1 \oplus \cdots \oplus V_t$, there can be at most one $i \in \{1, \ldots, t\}$ such that $\mathrm{Hom}(W, \Sigma^d V_i) \cong \mathrm{Hom}(\Sigma^{-d} W, V_i)$ is non-zero.

Hence, as we claimed, there is at most one indecomposable summand $V_i$ of $V$ permitting a non-zero morphism $\Sigma^{-d}W \to V_i$. □

**Lemma 4.5.4.** Consider an Auslander-Reiten $(d+2)$-angle in $\mathcal{M}$ of the form

$$Y^0 \xrightarrow{\eta^0} Y^1 \xrightarrow{\eta^1} Y^2 \longrightarrow \cdots \longrightarrow Y^d \xrightarrow{\eta^d} W \xrightarrow{\eta^{d+1}} \Sigma^d Y^0,$$

and any $U^0 \in \mathcal{M}$.

(a) For every non-zero morphism $\delta : \Sigma^d U^0 \to \Sigma^d Y^0$, there is a morphism $\phi : W \to \Sigma^d U^0$ such that $\delta \circ \phi = \eta^{d+1}$.

(b) For every non-zero morphism $\phi : W \to \Sigma^d U^0$, there is a morphism $\delta : \Sigma^d U^0 \to \Sigma^d Y^0$ such that $\delta \circ \phi = \eta^{d+1}$.

*Proof.* (a) Extend $\delta$ to a $(d+2)$-angle of the form

$$Y^0 \xrightarrow{\delta^0} M^1 \xrightarrow{\delta^1} M^2 \longrightarrow \cdots \longrightarrow M^d \xrightarrow{\delta^d} \Sigma^d U^0 \xrightarrow{\delta} \Sigma^d Y^0.$$

Since $\delta$ is non-zero, then $\delta^0$ is not a split monomorphism by Lemma 2.3.35, so there is $\phi^1 : Y^1 \to M^1$ such that $\delta^0 = \phi^1 \circ \eta^0$. So, by axiom (N3) from Definition 2.3.28, there exist morphisms $\phi^2, \ldots, \phi^{d+1}$ making the following diagram commutative:

$$\begin{tikzcd}
Y^0 \arrow[r, "\eta^0"] \arrow[d, equal] & Y^1 \arrow[r, "\eta^1"] \arrow[d, "\phi^1"] & Y^2 \arrow[r] \arrow[d, dashed, "\phi^2"] & \cdots \arrow[r] & Y^d \arrow[r, "\eta^d"] \arrow[d, dashed, "\phi^d"] & W \arrow[r, "\eta^{d+1}"] \arrow[d, dashed, "\phi^{d+1}"] & \Sigma^d Y^0 \arrow[d, equal] \\
Y^0 \arrow[r, "\delta^0"'] & M^1 \arrow[r, "\delta^1"'] & M^2 \arrow[r] & \cdots \arrow[r] & M^d \arrow[r, "\delta^d"'] & \Sigma^d U^0 \arrow[r, "\delta"'] & \Sigma^d Y^0.
\end{tikzcd}$$

Then $\phi := \phi^{d+1}$ is such that $\delta \circ \phi = \eta^{d+1}$.

(b) Follows by a dual argument.

□

**Theorem 4.5.5.** *Let $\mathcal{M}$ be a skeletally small* Hom*-finite $k$-linear $(d+2)$-angulated category with split idempotents. Let $\mathcal{W}$ be an additive subcategory of $\mathcal{M}$ closed under $d$-extensions. Let $W$ be in $\mathcal{W}$ and suppose that there exists $U^0$ in $\mathcal{W}$ and a non-zero morphism $\gamma^{d+1} : W \to \Sigma^d U^0$. Let*

$$\epsilon : \qquad X^0 \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^d \xrightarrow{\xi^d} W \xrightarrow{\xi^{d+1}} \Sigma^d X^0$$

*be an Auslander-Reiten $(d+2)$-angle in $\mathcal{M}$. Then the following are equivalent:*

*(a)* $X^0$ *has a* $\mathcal{W}$*-cover of the form* $\varphi : W^0 \to X^0$*,*

*(b) there is an Auslander-Reiten* $(d+2)$*-angle in* $\mathcal{W}$ *of the form*

$$\epsilon' : \qquad W^0 \xrightarrow{\omega^0} W^1 \xrightarrow{\omega^1} W^2 \longrightarrow \cdots \longrightarrow W^d \xrightarrow{\omega^d} W \xrightarrow{\omega^{d+1}} \Sigma^d W^0.$$

*Proof.* We first prove that (a) implies (b). Suppose $\varphi : W^0 \to X^0$ is a $\mathcal{W}$-cover. Extend the non-zero morphism $\gamma^{d+1}$ to a $(d+2)$-angle:

$$U^0 \xrightarrow{\gamma^0} U^1 \xrightarrow{\gamma^1} U^2 \longrightarrow \cdots \longrightarrow U^d \xrightarrow{\gamma^d} W \xrightarrow{\gamma^{d+1}} \Sigma^d U^0,$$

where we can choose $U^1, \ldots, U^d$ in $\mathcal{W}$. Note that $\gamma^d$ is not a split epimorphism by Lemma 2.3.35. Since $\xi^d$ is right almost split, there is a morphism $\psi^d : U^d \to X^d$ such that $\gamma^d = \xi^d \circ \psi^d$. Then, by axioms (N2) and (N3) from Definition 2.3.28, there exist morphisms $\psi^0, \ldots, \psi^{d-1}$ making the following diagram commutative:

$$\begin{tikzcd}
U^0 \arrow[r, "\gamma^0"] \arrow[d, dashed, "\psi^0"] & U^1 \arrow[r, "\gamma^1"] \arrow[d, dashed, "\psi^1"] & \cdots \arrow[r] & U^{d-1} \arrow[r, "\gamma^{d-1}"] \arrow[d, dashed, "\psi^{d-1}"] & U^d \arrow[r, "\gamma^d"] \arrow[d, "\psi^d"] & W \arrow[r, "\gamma^{d+1}"] \arrow[d, equal] & \Sigma^d U^0 \arrow[d, dashed, "\Sigma^d(\psi^0)"] \\
X^0 \arrow[r, "\xi^0"'] & X^1 \arrow[r, "\xi^1"'] & \cdots \arrow[r] & X^{d-1} \arrow[r, "\xi^{d-1}"'] & X^d \arrow[r, "\xi^d"'] & W \arrow[r, "\xi^{d+1}"'] & \Sigma^d X^0.
\end{tikzcd}$$

In particular, we have $\Sigma^d(\psi^0) \circ \gamma^{d+1} = \xi^{d+1}$. Since $\varphi : W^0 \to X^0$ is a $\mathcal{W}$-cover, there is a morphism $\nu : U^0 \to W^0$ such that $\varphi \circ \nu = \psi^0$. Consider a $(d+2)$-angle extending $\Sigma^d(\nu) \circ \gamma^{d+1}$:

$$\epsilon' : \qquad W^0 \xrightarrow{\omega^0} W^1 \xrightarrow{\omega^1} W^2 \longrightarrow \cdots \xrightarrow{\omega^{d-1}} W^d \xrightarrow{\omega^d} W \xrightarrow{\Sigma^d(\nu)\circ\gamma^{d+1}} \Sigma^d W^0,$$

where, as $W, W^0 \in \mathcal{W}$, we can choose $W^1, \ldots, W^d$ in $\mathcal{W}$ and by Lemma 2.3.38, when $d \geq 2$, we can also choose $\omega^1, \ldots, \omega^{d-1}$ in $\operatorname{rad}_{\mathcal{M}}$ and so in $\operatorname{rad}_{\mathcal{W}}$. We will show that $\epsilon'$ is an Auslander-Reiten $(d+2)$-angle in $\mathcal{W}$.

By Lemma 4.5.3, we have that $W^0$ is either zero or indecomposable. Since

$$0 \neq \xi^{d+1} = \Sigma^d(\psi^0) \circ \gamma^{d+1} = \Sigma^d(\varphi \circ \nu) \circ \gamma^{d+1} = \Sigma^d(\varphi) \circ \Sigma^d(\nu) \circ \gamma^{d+1}, \tag{4.6}$$

it follows that $\Sigma^d(\nu) \circ \gamma^{d+1}$ is non-zero. Then $\Sigma^d W^0$ is non-zero and so $W^0$ is non-zero, hence it is indecomposable, so $\operatorname{End}(W^0)$ is local.

Now, by Lemma 4.4.7, in order to prove that $\epsilon'$ is an Auslander-Reiten $(d+2)$-angle in $\mathcal{W}$, it is enough to prove that $\omega^d$ is right almost split in $\mathcal{W}$.

Extend $\varphi : W^0 \to X^0$ to a $(d+2)$-angle:

$$W^0 \xrightarrow{\varphi} X^0 \xrightarrow{\delta^0} Y^1 \longrightarrow \cdots \xrightarrow{\delta^{d-1}} Y^d \xrightarrow{\delta^d} \Sigma^d W^0.$$

Since $\Sigma^d(\nu) \circ \gamma^{d+1}$ is non-zero, Lemma 2.3.35 implies that $\omega^d$ is not a split epimorphism. By (4.6), we have that $\varphi \circ \nu \circ \Sigma^{-d}(\gamma^{d+1}) = \Sigma^{-d}(\xi^{d+1})$ and so there are morphisms $\alpha^1, \ldots, \alpha^d$ making the following diagram commutative:

$$\begin{tikzcd}
\Sigma^{-d}W \arrow[r, "(-1)^d\Sigma^{-d}(\xi^{d+1})"] \arrow[d, "(-1)^d\nu\circ\Sigma^{-d}(\gamma^{d+1})"] & X^0 \arrow[r, "\xi^0"] \arrow[d, equal] & X^1 \arrow[r] \arrow[d, dashed, "\alpha^1"] & \cdots \arrow[r, "\xi^{d-1}"] & X^d \arrow[r, "\xi^d"] \arrow[d, dashed, "\alpha^d"] & W \arrow[d, "(-1)^d\Sigma^d(\nu)\circ\gamma^{d+1}"] \\
W^0 \arrow[r, "\varphi"'] & X^0 \arrow[r, "\delta^0"'] & Y^1 \arrow[r] & \cdots \arrow[r, "\delta^{d-1}"'] & Y^d \arrow[r, "\delta^d"'] & \Sigma^d W^0.
\end{tikzcd} \tag{4.7}$$

For any $W'$ in $\mathcal{W}$, consider the exact sequence

$$\mathrm{Hom}(W', Y^{d-1}) \xrightarrow{\widetilde{\delta^{d-1}}} \mathrm{Hom}(W', Y^d) \xrightarrow{\widetilde{\delta^d}} \mathrm{Hom}(W', \Sigma^d W^0) \xrightarrow{\widetilde{(-1)^d\Sigma^d}(\varphi)} \mathrm{Hom}(W', \Sigma^d X^0),$$

where, for a morphism $\eta$ we use the notation $\widetilde{\eta} \coloneqq \mathrm{Hom}(W', \eta)$ for readability. Note that $Y^1 \to \cdots \to Y^d$ is in $\mathcal{W}$-exact by [38, Lemma 2.1]. Hence $\widetilde{\delta^{d-1}}$ is surjective, so $\widetilde{\delta^d}$ is the zero map and $\widetilde{(-1)^d\Sigma^d}(\varphi)$ is injective.

Let $\phi : W' \to W$ be a morphism in $\mathcal{W}$ which is not a split epimorphism. As $\xi^d$ is right almost split, there exists a morphism $\eta : W' \to X^d$ such that $\phi = \xi^d \circ \eta$. Consider $\delta^d \circ \alpha^d \circ \eta \in \mathrm{Hom}(W', \Sigma^d W^0)$ and note that

$$\widetilde{(-1)^d\Sigma^d}(\varphi)(\delta^d \circ \alpha^d \circ \eta) = (-1)^d \Sigma^d(\varphi) \circ \delta^d \circ \alpha^d \circ \eta = 0 \circ \alpha^d \circ \eta = 0,$$

where $(-1)^d\Sigma^d(\varphi) \circ \delta^d = 0$ by Lemma 2.3.31. Then, by injectivity of $\widetilde{(-1)^d\Sigma^d}(\varphi)$, we conclude that $\delta^d \circ \alpha^d \circ \eta = 0$.

By commutativity of (4.7), we have

$$0 = \delta^d \circ \alpha^d \circ \eta = (-1)^d \Sigma^d(\nu) \circ \gamma^{d+1} \circ \xi^d \circ \eta = (-1)^d \Sigma^d(\nu) \circ \gamma^{d+1} \circ \phi.$$

Then we obtain a commutative diagram:

$$\begin{tikzcd}
 & & & & W' \arrow[dll, dashed, "\exists\beta'"'] \arrow[d, "(-1)^d\phi"] \arrow[dr, "0"] & \\
W^0 \arrow[r, "\omega^0"'] & W^1 \arrow[r, "\omega^1"'] & W^2 \arrow[r] & \cdots \arrow[r] W^d & W \arrow[r, "\Sigma^d(\nu)\circ\gamma^{d+1}"'] & \Sigma^d W^0,
\end{tikzcd}$$

where $\beta' : W' \to W^d$ exists by Lemma 2.3.32. Letting $\beta \coloneqq (-1)^d\beta'$, we have $\phi = \omega^d \circ \beta$.

Hence $\omega^d$ is right almost split in $\mathcal{W}$ as we wished and (a) implies (b).

We now prove that (b) implies (a). Suppose that we have an Auslander-Reiten $(d+2)$-angle in $\mathcal{W}$ of the form

$$\epsilon' : \qquad W^0 \xrightarrow{\omega^0} W^1 \xrightarrow{\omega^1} W^2 \longrightarrow \cdots \longrightarrow W^d \xrightarrow{\omega^d} W \xrightarrow{\omega^{d+1}} \Sigma^d W^0.$$

Since $\omega^d$ is not a split epimorphism and $\xi^d$ is right almost split, there is a morphism $\varphi^d : W^d \to X^d$ such that $\xi^d \circ \varphi^d = \omega^d$. Then there are morphisms $\varphi^0, \ldots, \varphi^{d-1}$ making the following diagram commutative:

$$\begin{tikzcd}
\Sigma^{-d}W \arrow[r, "(-1)^d\Sigma^{-d}(\omega^{d+1})"] \arrow[d, equal] & W^0 \arrow[r, "\omega^0"] \arrow[d, dashed, "\varphi^0"] & W^1 \arrow[r] \arrow[d, dashed, "\varphi^1"] & \cdots \arrow[r] & W^{d-1} \arrow[r, "\omega^{d-1}"] \arrow[d, dashed, "\varphi^{d-1}"] & W^d \arrow[r, "\omega^d"] \arrow[d, "\varphi^d"] & W \arrow[d, equal] \\
\Sigma^{-d}W \arrow[r, "(-1)^d\Sigma^{-d}(\xi^{d+1})"'] & X^0 \arrow[r, "\xi^0"'] & X^1 \arrow[r] & \cdots \arrow[r] & X^{d-1} \arrow[r, "\xi^{d-1}"'] & X^d \arrow[r, "\xi^d"'] & W.
\end{tikzcd} \tag{4.8}$$

We show that $\varphi^0 : W^0 \to X^0$ is a $\mathcal{W}$-cover. First note that commutativity of (4.8) and the fact that $\xi^{d+1}$ is non-zero implies that $\varphi^0$ is non-zero. Moreover, by Lemma 4.4.7, we know that $\mathrm{End}(W^0)$ is local. Hence, by Lemma 2.2.27, it follows that $\varphi^0$ is right minimal. So it remains to show that $\varphi^0$ is a $\mathcal{W}$-precover.

Suppose that $U^0$ in $\mathcal{W}$ and a morphism $\gamma^0 : U^0 \to X^0$ are given. We want to prove that $\gamma^0$ factors through $\varphi^0$. The case $U^0 = 0$ is trivial, so suppose that $U^0$ is non-zero.

Take a linear map $\psi : \mathrm{Hom}(\Sigma^{-d}W, X^0) \to k$ with $\psi(\Sigma^{-d}(\xi^{d+1})) \neq 0$. Define a bilinear map

$$q : \mathrm{Hom}(\Sigma^{-d}W, U^0) \times \mathrm{Hom}(U^0, W^0) \to k,$$
$$q(\phi, \alpha) = \psi(\varphi^0 \circ \alpha \circ \phi).$$

We show that if $\phi \neq 0$, then there exists an $\alpha$ such that $q(\phi, \alpha) \neq 0$. Let $\phi \in \mathrm{Hom}(\Sigma^{-d}W, U^0)$ be non-zero and extend $\Sigma^d(\phi)$ to a $(d+2)$-angle of the form

$$U^0 \xrightarrow{\nu^0} U^1 \xrightarrow{\nu^1} U^2 \longrightarrow \cdots \longrightarrow U^d \xrightarrow{\nu^d} W \xrightarrow{\Sigma^d(\phi)} \Sigma^d U^0,$$

where, since $W$, $U^0$ are in $\mathcal{W}$, we can choose $U^1, \ldots, U^d$ in $\mathcal{W}$. Note that, as $\Sigma^d(\phi)$ is non-zero and by Lemma 2.3.35, then $\nu^d$ is not a split epimorphism. So there is $\eta^d : U^d \to W^d$ such that $\nu^d = \omega^d \circ \eta^d$. Hence $\omega^{d+1} \circ \nu^d = \omega^{d+1} \circ \omega^d \circ \eta^d = 0$ by Lemma 2.3.31. Then we have a commutative diagram:

$$\begin{tikzcd} U^0 \arrow[r, "\nu^0"] & U^1 \arrow[r, "\nu^1"] & U^2 \arrow[r] & \cdots \arrow[r] & U^d \arrow[r, "\nu^d"] \arrow[dr, "0"'] & W \arrow[r, "\Sigma^d(\phi)"] \arrow[d, "\omega^{d+1}"] & \Sigma^d U^0, \arrow[dl, dashed, "\Sigma^d(\eta^0)"] \\ & & & & & \Sigma^d W^0 & \end{tikzcd}$$

where $\Sigma^d(\eta^0)$ exists by Lemma 2.3.33. Note that $\eta^0 \circ \phi = \Sigma^{-d}(\omega^{d+1})$. Then the element $\eta^0$ in $\operatorname{Hom}(U^0, W^0)$ is such that

$$q(\phi, \eta^0) = \psi(\varphi^0 \circ \eta^0 \circ \phi) = \psi(\varphi^0 \circ \Sigma^{-d}(\omega^{d+1})) = \psi(\Sigma^{-d}(\xi^{d+1})) \neq 0,$$

so we have established the desired property of $q$.

Consider the linear map

$$\varphi : \operatorname{Hom}(\Sigma^{-d}W, U^0) \to k, \qquad \varphi(\phi) = \psi(\gamma^0 \circ \phi).$$

By [37, Lemma 2.5], there is an element $\alpha \in \operatorname{Hom}(U^0, W^0)$ such that $\varphi(-) = q(-, \alpha)$. Then, by the definitions of $\varphi$ and $q$, for any $\phi \in \operatorname{Hom}(\Sigma^{-1}W, U^0)$, we have

$$\psi(\gamma^0 \circ \phi) = \psi(\varphi^0 \circ \alpha \circ \phi). \tag{4.9}$$

Since $\epsilon$ is an Auslander-Reiten $(d+2)$-angle in $\mathcal{M}$, then so is $\overline{\epsilon} := (-1)^d \Sigma^{-d}(\epsilon)$. By Lemma 4.5.4, we conclude that the bilinear map

$$p : \operatorname{Hom}(\Sigma^{-d}W, U^0) \times \operatorname{Hom}(U^0, X^0) \to k, \qquad p(\phi, \delta) = \psi(\delta \circ \phi)$$

is non-degenerate. Hence (4.9) implies that $\gamma^0 = \varphi^0 \circ \alpha$, that is $\gamma^0$ factors through $\varphi^0$ as we wished. □

## 4.6 A class of examples

In this section, we further study the class of examples by Vaso that we introduced in Section 3.3, adding the extra assumption that $k$ is an algebraically closed field in order to be able to apply Theorem 4.3.2. So let $d \geq 2$, $l \geq 2$ and $m \geq 3$ be integers such that

$$\frac{m-1}{l} = \frac{d}{2},$$

$Q = A_m$ and $\mathcal{F}$ be the unique $d$-cluster tilting subcategory of $\operatorname{mod}\Lambda$ for $\Lambda = kQ/(\operatorname{rad}_{kQ})^l$.

We first give a full description of the wide subcategories $\overline{\mathcal{W}}$ of the $(d+2)$-angulated category $\overline{\mathcal{F}} = \operatorname{add}\{\Sigma^{id}\mathcal{F} \mid i \in \mathbb{Z}\}$, using Theorem 4.3.2. We then apply Theorem 4.5.5 to find the Auslander-Reiten $(d+2)$-angles in these subcategories $\overline{\mathcal{W}}$.

**Lemma 4.6.1.** Let $1 \le j \le m+l-1$. Then there is an Auslander-Reiten $(d+2)$-angle in $\overline{\mathcal{F}}$ ending at $f_j$ of the form:

$$\Sigma^{-d} f_l \to \Sigma^{-d} f_{l+1} \to \cdots \to \Sigma^{-d} f_m \to \Sigma^{-d} f_{m+l-1} \xrightarrow{\mu} f_1 \to f_l \qquad \text{if } j = 1; \qquad \text{(a)}$$

$$\Sigma^{-d} f_{j+l-1} \to \Sigma^{-d} f_{j+l} \to \cdots \to f_{j-1} \xrightarrow{\mu} f_j \to f_{j+l-1} \qquad \text{if } 2 \le j \le m; \qquad \text{(b)}$$

$$f_{j-m} \to f_{j+1-m} \to \cdots \to f_{j-l} \to f_{j-1} \xrightarrow{\mu} f_j \to \Sigma^d f_{j-m} \qquad \text{if } j \ge m+1. \qquad \text{(c)}$$

*Proof.* First note that in any case, the complex is a $(d+2)$-angle in $\overline{\mathcal{F}}$ by Remark 3.3.2. In fact, we can extend and rotate $\mu$ in cases (b) and (c) and $f_1 \to f_l$ in case (a).

Moreover, since any morphism between two non-isomorphic indecomposable objects is in $\operatorname{rad}_{\overline{\mathcal{F}}}$, all the morphisms of each $(d+2)$-angle are in $\operatorname{rad}_{\overline{\mathcal{F}}}$.

In cases (b) and (c), we have $\mu : f_{j-1} \to f_j$. This is not a split epimorphism as the only morphism of the form $f_j \to f_{j-1}$ is the zero morphism. Let $\overline{f} \in \overline{\mathcal{F}}$ and $\alpha : \overline{f} \to f_j$ be a non-zero morphism which is not a split epimorphism. Without loss of generality, assume that $\overline{f}$ is indecomposable. Note that, since $\alpha$ is not an isomorphism, then $\overline{f} \neq f_j$. Hence $\overline{f}$ is an object to the left of $f_j$ in the quiver (3.1) and $\alpha$ factors through $\mu$. Similarly, in case (a), we have that $\mu : \Sigma^{-d} f_{m+l-1} \to f_1$ is not a split epimorphism and any morphism in $\overline{\mathcal{F}}$ ending at $f_1$ that is not a split epimorphism factors through $\mu$.

Hence in any case, $\mu$ is right almost split and the $(d+2)$-angle is an Auslander-Reiten $(d+2)$-angle by Lemma 4.4.6. $\square$

**Example 4.6.2.** Let us fix $d = 4$, $l = 4$ and $m = 9$. Using Lemma 4.6.1, the following are Auslander-Reiten 6-angles in $\overline{\mathcal{F}}$:

$$\Sigma^{-4} f_4 \to \Sigma^{-4} f_5 \to \Sigma^{-4} f_8 \to \Sigma^{-4} f_9 \to \Sigma^{-4} f_{12} \xrightarrow{\mu} f_1 \to f_4, \qquad \text{where } j = 1; \qquad \text{(a)}$$

$$\Sigma^{-4} f_8 \to \Sigma^{-4} f_9 \to \Sigma^{-4} f_{12} \to f_1 \to f_4 \xrightarrow{\mu} f_5 \to f_8, \qquad \text{where } j = 5; \qquad \text{(b)}$$

$$f_1 \to f_2 \to f_5 \to f_6 \to f_9 \xrightarrow{\mu} f_{10} \to \Sigma^4 f_1, \qquad \text{where } j = 10. \qquad \text{(c)}$$

**Lemma 4.6.3.** Let $\mathcal{V} \subseteq \overline{\mathcal{F}}$ be a wide subcategory. Then, $\mathcal{V} = \overline{\mathcal{W}} = \operatorname{add}\{\Sigma^{id}\mathcal{W} \mid i \in \mathbb{Z}\}$ for some wide subcategory $\mathcal{W}$ of $\mathcal{F}$. Moreover,

(a) $\mathcal{W}$ is semisimple if and only if for all distinct $f_i$, $f_j$ in $\mathcal{W}$, we have $l \le | i - j | \le m-1$;

(b) $\mathcal{W}$ is non-semisimple if and only if it is $l$*-periodic*, **i.e.** $0 \neq f_q \in \mathcal{W}$ implies $f_{q+rl} \in \mathcal{W}$ for all $r \in \mathbb{Z}$.

*Proof.* The fact that $\mathcal{V} = \overline{\mathcal{W}}$ follows from Theorem 4.3.2. The rest of the lemma follows from [23, Section 7]. □

**Lemma 4.6.4.** Let $\overline{\mathcal{W}} \subseteq \overline{\mathcal{F}}$ be a wide subcategory, where $\mathcal{W} \subseteq \mathcal{F}$ is semisimple. Let $f_j \in \mathcal{W}$ and suppose $\overline{f}$ is the initial object of the Auslander-Reiten $(d+2)$-angle in $\overline{\mathcal{F}}$ ending at $f_j$. Then

$$\Sigma^{-d} f_j \to 0 \to \cdots \to 0 \to f_j \xrightarrow{1_{f_j}} f_j$$

is an Auslander-Reiten $(d+2)$-angle in $\overline{\mathcal{W}}$ and $\Sigma^{-d} f_j \to \overline{f}$ is a $\overline{\mathcal{W}}$-cover.

*Proof.* We claim that the only non-zero morphisms in $\overline{\mathcal{W}}$ are scalar multiples of the identity morphisms. If $f_i$, $f_k$ are two distinct objects in $\mathcal{W}$, then

$$\overline{\mathcal{W}}(f_i, f_k) = 0 \text{ since } l \leq \mid i-k \mid .$$

Suppose for a contradiction that $\overline{\mathcal{W}}(f_i, \Sigma^d f_k)$ is non-zero. Then, since the composition of $l$ consecutive arrows in diagram (3.1) is zero, we must have $k < i$ and there is a sequence of at most $l$ objects of the form:

$$f_i \to f_{i+1} \to \cdots \to f_{m+l-2} \to f_{m+l-1} \to \Sigma^d f_1 \to \Sigma^d f_2 \to \cdots \to \Sigma^d f_{k-1} \to \Sigma^d f_k.$$

But then, as there are at most $l-1$ arrows in the above sequence, we have that $m+l-i+k \leq l$ and so $m-1 < i-k$, contradicting the fact that $i-k \leq m-1$. Hence we proved our claim and so $0 \to f_j$ is right almost split in $\overline{\mathcal{W}}$. Then

$$\Sigma^{-d} f_j \to 0 \to \cdots \to 0 \to f_j \xrightarrow{1_{f_j}} f_j$$

is an Auslander-Reiten $(d+2)$-angle in $\overline{\mathcal{W}}$ and $\Sigma^{-d} f_j \to \overline{f}$ is a $\overline{\mathcal{W}}$-cover, by Theorem 4.5.5. □

**Lemma 4.6.5.** Let $\overline{\mathcal{W}} \subseteq \overline{\mathcal{F}}$ be a wide subcategory, where $\mathcal{W} \subseteq \mathcal{F}$ is non-semisimple. Let $f_j \in \mathcal{W}$ and suppose $\overline{f}$ is the initial object of the Auslander-Reiten $(d+2)$-angle in $\overline{\mathcal{F}}$ ending at $f_j$.

Starting from $\overline{f}$ and moving left in the quiver (3.1), let $w$ be the first object found which is in $\overline{\mathcal{W}}$. Then there is a $\overline{\mathcal{W}}$-cover $w \to \overline{f}$, and we have an Auslander-Reiten $(d+2)$-angle in $\overline{\mathcal{W}}$ of the form:

$$w \to \cdots \to f_j \to \Sigma^d w.$$

**Remark 4.6.6.** Note that $\overline{f}$ is described in cases (a), (b), (c) of Lemma 4.6.1 for all possible values of $j$. In cases (b) and (c), so $\overline{f} = \Sigma^{-d} f_{j+l-1}$, we have $w = \Sigma^{-d} f_p$, for

$$p := \max\{n \in \mathbb{Z}^{>0} \mid n \leq j + l - 1,\ f_p \in \mathcal{W}\}.$$

In case (a), so $\overline{f} = f_{j-m}$, then $w$ can be either of the form $f_p$ or $\Sigma^{-d} f_q$.

Moreover, once $w$ is found, Remark 3.3.2 can be used to find the Auslander-Reiten $(d+2)$-angle in $\overline{\mathcal{W}}$. Note that the latter has half of its objects equal to the ones in the Auslander-Reiten $(d+2)$-angle in $\overline{\mathcal{F}}$ ending at $f_j$, **i.e.** $f_j$ and every second of the terms to its left. The remaining objects are obtained by replacing $\overline{f}$ with $w$ and, at every step, shifting by $l$ objects in diagram (3.1).

*Proof of Lemma 4.6.5.* Given any object $g$ in (3.1), the indecomposable objects in $\overline{\mathcal{F}}$ having non-zero morphism into $g$ are exactly $g$ and the $l-1$ objects to its left in the quiver.

Consider $g = \overline{f}$ and note that, since $\mathcal{W} \neq 0$ is $l$-periodic, then at least one of these $l$ objects is in $\overline{\mathcal{W}}$. Hence $w$ can be chosen as described in Lemma 4.6.5 with $w \to \overline{f}$ non-zero. Moreover, $\delta : w \to \overline{f}$ is a $\overline{\mathcal{W}}$-cover since all other morphisms from $\overline{\mathcal{W}}$ to $\overline{f}$ factor through $\delta$. The last part of the lemma follows from Theorem 4.5.5. □

**Example 4.6.7** (Continuing Example 4.6.2)**.** Let $\mathcal{W} = \operatorname{add}\{f_1, f_2, f_5, f_6, f_9, f_{10}\}$. Note this is 4-periodic and hence $\overline{\mathcal{W}} \subseteq \overline{\mathcal{F}}$ is wide. Consider the 6-angle (a) from Example 4.6.2, where $f_1 \in \overline{\mathcal{W}}$. Here, $\overline{f} = \Sigma^{-4} f_4$ has $\overline{\mathcal{W}}$-cover $w = \Sigma^{-4} f_2 \to \Sigma^{-4} f_4$. Then, we obtain the Auslander-Reiten 6-angle in $\overline{\mathcal{W}}$:

$$\Sigma^{-4} f_2 \to \Sigma^{-4} f_5 \to \Sigma^{-4} f_6 \to \Sigma^{-4} f_9 \to \Sigma^{-4} f_{10} \to f_1 \to f_2.$$

Similarly, starting from the 6-angle (b) from Example 4.6.2, we obtain the Auslander-Reiten 6-angle in $\overline{\mathcal{W}}$:

$$\Sigma^{-4} f_6 \to \Sigma^{-4} f_9 \to \Sigma^{-4} f_{10} \to f_1 \to f_2 \to f_5 \to f_6.$$

Finally, note that since all the objects in the 6-angle (c) in Example 4.6.2 are in $\overline{\mathcal{W}}$, then

$$f_1 \to f_2 \to f_5 \to f_6 \to f_9 \xrightarrow{\mu} f_{10} \to \Sigma^4 f_1$$

is also an Auslander-Reiten 6-angle in $\overline{\mathcal{W}}$.

# Chapter 5

# $d$-Auslander-Reiten sequences in subcategories

## 5.1 Introduction

Let $d$ be a fixed positive integer, $k$ a field and $\Lambda$ a finite dimensional $k$-algebra. As in previous chapters, let $\operatorname{mod}\Lambda$ denote the category of finitely generated right $\Lambda$-modules.

### 5.1.1 Classic background ($d = 1$ case).

We have introduced Auslander-Reiten sequences in $\operatorname{mod}\Lambda$ in Section 2.2.1. As stated in Theorem 2.2.23, if $M \in \operatorname{mod}\Lambda$ is an indecomposable non-projective module, then there is an Auslander-Reiten sequence in $\operatorname{mod}\Lambda$ of the form:

$$0 \longrightarrow \tau M \longrightarrow N \longrightarrow M \longrightarrow 0,$$

where $\tau M = D \circ \operatorname{Tr} M$ is the Auslander-Reiten translation of $M$.

Let $\mathcal{X} \subseteq \operatorname{mod}\Lambda$ be a full subcategory closed under summands and extensions, in the sense that if $0 \to X \to Y \to Z \to 0$ is a short exact sequence in $\operatorname{mod}\Lambda$ with $X, Z \in \mathcal{X}$, then $Y \in \mathcal{X}$. Auslander and Smalø introduced the notion of almost split sequences in subcategories and, in [7, Theorem 2.4], showed a weaker version of the following theorem introduced later by Kleiner in [43, Corollary 2.8].

**Theorem A (Kleiner).** *Assume $\mathcal{X}$ is precovering in $\operatorname{mod}\Lambda$ and let $X$ be an indecomposable in $\mathcal{X}$.*

*(a) There exists a right almost split morphism $W \to X$ in $\mathcal{X}$.*

*(b) If* $\mathrm{Ext}^1_\Lambda(X,\mathcal{X})$ *is non-zero, there is an Auslander-Reiten sequence in* $\mathcal{X}$*, see Definition 5.2.4 with* $d=1$ *and so* $\mathcal{F}=\mathrm{mod}\,\Lambda$*, of the form:*

$$0 \longrightarrow \zeta X \longrightarrow X^1 \longrightarrow X \longrightarrow 0,$$

*where* $\zeta X$ *is the unique indecomposable direct summand of the* $\mathcal{X}$*-cover of* $\tau X$ *such that* $\mathrm{Ext}^1_\Lambda(X,\zeta X)\neq 0$.

For $M\in\mathrm{mod}\,\Lambda$, let $\underline{\mathrm{End}}_\Lambda(M)$ denote the factor ring of $\mathrm{End}_\Lambda(M)$ modulo the ideal of morphisms $M\to M$ that factor through a projective module. Then, Auslander, Reiten and Smalø's argument in [5, proof of Corollary V.2.4] can be easily modified to prove the following.

**Theorem B.** *Assume* $\mathcal{X}$ *is precovering in* $\mathrm{mod}\,\Lambda$*. Let* $X\in\mathcal{X}$ *be an indecomposable such that* $\underline{\mathrm{End}}_\Lambda(X)$ *is a division ring. For a short exact sequence of the form*

$$\delta: \qquad 0 \longrightarrow \zeta X \longrightarrow X^1 \longrightarrow X \longrightarrow 0,$$

*with* $\zeta X$*,* $X^1$ *and* $X$ *in* $\mathcal{X}$*, the following are equivalent:*

*(a)* $\delta$ *is an Auslander-Reiten sequence in* $\mathcal{X}$*,*

*(b)* $\delta$ *does not split.*

As a corollary of the above, one can prove the following result by Kleiner, see [43, Proposition 2.10].

**Corollary C (Kleiner).** *Assume* $\mathcal{X}$ *is precovering in* $\mathrm{mod}\,\Lambda$*. Let* $g:Y\to\tau X$ *be an* $\mathcal{X}$*-cover, where* $X$ *is an indecomposable in* $\mathcal{X}$ *with* $\underline{\mathrm{End}}_\Lambda(X)$ *a division ring. Consider a non-split short exact sequence with terms in* $\mathcal{X}$ *of the form*

$$0 \longrightarrow Y \longrightarrow Y^1 \xrightarrow{\eta} X \longrightarrow 0.$$

*Then the bottom row of the pushout diagram*

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d] & Y^1 \arrow[r, "\eta"] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \tau X \arrow[r] & N \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

*is an Auslander-Reiten sequence in* $\mathrm{mod}\,\Lambda$ *and* $\eta$ *is right almost split in* $\mathcal{X}$*.*

### 5.1.2 This chapter ($d \geq 1$ case).

Assume now that there is a $d$-cluster tilting subcategory $\mathcal{F} \subseteq \operatorname{mod}\Lambda$, see Definition 2.3.13. As seen in Section 2.3.1, Jasso generalised abelian categories to $d$-abelian categories in [34]: kernels and cokernels are replaced by complexes of $d$ objects, called $d$-kernels and $d$-cokernels respectively, and short exact sequences by complexes of $d+2$ objects, called $d$-exact sequences, see Definition 2.3.1. By Theorem 2.3.14, we have that $\mathcal{F}$ is a $d$-abelian category and it plays the role of a higher version of the abelian category $\operatorname{mod}\Lambda$. Note that for $d = 1$, the only possible choice is $\mathcal{F} = \operatorname{mod}\Lambda$.

In [28], Iyama generalised Auslander-Reiten sequences in $\operatorname{mod}\Lambda$ to $d$-Auslander Reiten sequences in $\mathcal{F}$. Moreover, as seen in Proposition 2.3.25, he proved that if $A^{d+1}$ is an indecomposable non-projective in $\mathcal{F}$, then there exists a $d$-Auslander-Reiten sequence in $\mathcal{F}$, see Definition 2.3.21, of the form:

$$0 \longrightarrow \tau_d(A^{d+1}) \longrightarrow A^1 \longrightarrow A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \longrightarrow A^d \longrightarrow A^{d+1} \longrightarrow 0,$$

where $\tau_d$ is the $d$-Auslander-Reiten translation. Let $\mathcal{X} \subseteq \mathcal{F}$ be an additive subcategory in the sense of Definition 2.1.39 that is closed under $d$-extensions, see Definition 5.2.1. We define $d$-Auslander-Reiten sequences in $\mathcal{X}$, see Definition 5.2.4, and prove a higher version of Theorem A.

**Theorem 5.3.13.** *Assume $\mathcal{X}$ is precovering in $\mathcal{F}$ and let $X$ be an indecomposable in $\mathcal{X}$.*

*(a) There exists a right almost split morphism $W \to X$ in $\mathcal{X}$.*

*(b) If $\operatorname{Ext}^d_\Lambda(X, \mathcal{X})$ is non-zero, there is a d-Auslander-Reiten sequence in $\mathcal{X}$ of the form:*

$$0 \longrightarrow \sigma X \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} X \longrightarrow 0,$$

*where $\sigma X$ is the unique indecomposable direct summand of the $\mathcal{X}$-cover of $\tau_d(X)$ such that $\operatorname{Ext}^d_\Lambda(X, \sigma X) \neq 0$.*

We prove a higher version of Theorem B.

**Theorem 5.4.4.** *Assume $\mathcal{X}$ is precovering in $\mathcal{F}$. Let $X$ be an indecomposable in $\mathcal{X}$ such that $\underline{\operatorname{End}}_\Lambda(X)$ is a division ring. Let*

$$\delta : 0 \longrightarrow \sigma X \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} X \longrightarrow 0$$

*be a d-exact sequence with terms in $\mathcal{X}$ and such that $\xi^1, \ldots, \xi^{d-1}$ are in $\operatorname{rad}_{\mathcal{X}}$ when $d \geq 2$. Then the following are equivalent:*

*(a) $\delta$ is a d-Auslander-Reiten sequence in $\mathcal{X}$,*

*(b) $\delta$ is not a split d-exact sequence, see Definition 2.3.9.*

As we have seen in Definition 2.3.4 and Lemma 2.3.6, in [34] Jasso generalised the idea of pushout to $d$-pushout of a $d$-exact sequence along a morphism from its first term. Using these, we obtain a higher version of Corollary C as a corollary of Theorem 5.4.4.

**Corollary 5.4.5.** *Assume $\mathcal{X}$ is precovering in $\mathcal{F}$. Let $g : Y \to \tau_d(X)$ be an $\mathcal{X}$-cover, where $X$ is an indecomposable in $\mathcal{X}$ with $\underline{\mathrm{End}}_\Lambda(X)$ a division ring. Consider a non-split d-exact sequence with terms in $\mathcal{X}$ of the form:*

$$\begin{tikzcd} \epsilon : & 0 \arrow[r] & Y \arrow[r, "\eta^0"] & Y^1 \arrow[r, "\eta^1"] & \cdots \arrow[r] & Y^d \arrow[r, "\eta^d"] & X \arrow[r] & 0, \end{tikzcd}$$

*where, if $d \geq 2$, we also have $\eta^1, \dots, \eta^{d-1} \in \mathrm{rad}_{\mathcal{X}}$. Consider a morphism induced by a d-pushout diagram:*

$$\begin{tikzcd} \epsilon : \arrow[d] & 0 \arrow[r] & Y \arrow[r, "\eta^0"] \arrow[d, "g"] & Y^1 \arrow[r, "\eta^1"] \arrow[d, "g^1"] & \cdots \arrow[r] & Y^d \arrow[r, "\eta^d"] \arrow[d, "g^d"] & X \arrow[r] \arrow[d, equal] & 0 \\ \delta : & 0 \arrow[r] & \tau_d(X) \arrow[r, "\alpha^0"'] & A^1 \arrow[r, "\alpha^1"'] & \cdots \arrow[r] & A^d \arrow[r, "\alpha^d"'] & X \arrow[r] & 0, \end{tikzcd}$$

*where, if $d \geq 2$, we have that $\alpha^1, \dots, \alpha^{d-1} \in \mathrm{rad}_{\mathcal{F}}$. Then $\delta$ is a d-Auslander-Reiten sequence in $\mathcal{F}$ and $\eta^d$ is right almost split in $\mathcal{X}$.*

We illustrate Theorem 5.3.13 in the following example with $d = 2$. Let $\Lambda$ be the algebra defined by the following quiver with relations:

$$\begin{tikzcd}[column sep=small] & & & 10 \arrow[dr] & & & \\ & & 9 \arrow[ur] \arrow[dr] \arrow[rr, dash, dotted] & & 8 \arrow[dr] & & \\ & 7 \arrow[ur] \arrow[dr] \arrow[rr, dash, dotted] & & 6 \arrow[ur] \arrow[dr] \arrow[rr, dash, dotted] & & 5 \arrow[dr] & \\ 4 \arrow[ur] \arrow[rr, dash, dotted] & & 3 \arrow[ur] \arrow[rr, dash, dotted] & & 2 \arrow[ur] \arrow[rr, dash, dotted] & & 1. \end{tikzcd}$$

The Auslander-Reiten quiver of the unique 2-cluster tilting subcategory $\mathcal{F}$ of $\mathrm{mod}\,\Lambda$ is shown in Figure 5.1 on page 118. Choosing a subcategory $\mathcal{X} \subseteq \mathcal{F}$ satisfying our setup, namely add of the vertices coloured red in Figure 5.1, we use Theorem 5.3.13 to describe the 2-Auslander-Reiten sequences in $\mathcal{X}$.

This chapter is organised as follows. Section 5.2 studies $d$-Auslander-Reiten sequences in $\mathcal{X}$. Section 5.3 proves higher analogues to some of Kleiner's results from [43, Section 2], including Theorem 5.3.13. Section 5.4 proves Theorem 5.4.4 and Corollary 5.4.5. Finally, Section 5.5 illustrates an example of Theorem 5.3.13.

## 5.2 $d$-Auslander-Reiten sequences in $\mathcal{X}$

We aim to study additive subcategories of $\mathcal{F}$ closed under $d$-extensions. We define these subcategories using the notion of Yoneda equivalence, see Definition 2.3.16.

**Definition 5.2.1.** Let $\mathcal{F} \subseteq \operatorname{mod}\Lambda$ be $d$-cluster tilting. We say that an additive subcategory $\mathcal{X} \subseteq \mathcal{F}$ is *closed under $d$-extensions* if each $d$-exact sequence in $\mathcal{F}$ of the form:

$$0 \longrightarrow X^0 \longrightarrow A^1 \longrightarrow A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \longrightarrow A^d \longrightarrow X^{d+1} \longrightarrow 0,$$

with $X^0$, $X^{d+1}$ in $\mathcal{X}$ is Yoneda equivalent to a $d$-exact sequence in $\mathcal{F}$,

$$0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots \longrightarrow X^{d-1} \longrightarrow X^d \longrightarrow X^{d+1} \longrightarrow 0,$$

with all terms in $\mathcal{X}$.

**Setup 5.2.2.** Let $d$ be a fixed positive integer, $k$ a field, $\Lambda$ a finite dimensional $k$-algebra and $\mathcal{F} \subseteq \operatorname{mod}\Lambda$ a $d$-cluster tilting subcategory. Then $\mathcal{F}$ is $d$-abelian by Theorem 2.3.14. Moreover, let $\mathcal{X} \subseteq \mathcal{F}$ be an additive subcategory closed under $d$-extensions.

**Remark 5.2.3.** By Remark 2.3.17, in every Yoneda equivalence class, there is a unique almost minimal sequence up to isomorphism. Consider a $d$-exact sequence in $\mathcal{F}$ of the form:

$$\delta: \qquad 0 \longrightarrow X^0 \longrightarrow A^1 \longrightarrow A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \longrightarrow A^d \longrightarrow X^{d+1} \longrightarrow 0,$$

with $X^0$, $X^{d+1}$ in $\mathcal{X}$. The almost minimal sequence in the equivalence class $[\delta]$ has all the terms in $\mathcal{X}$. In fact, since $\mathcal{X}$ is closed under $d$-extensions, we know there is a $d$-exact sequence with all terms in $\mathcal{X}$ in $[\delta]$, and dropping extra direct summands of the form $X \xrightarrow{\sim} X$ in the middle terms of this, we obtain the unique almost minimal sequence in $[\delta]$, say

$$\delta': \qquad 0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots \longrightarrow X^{d-1} \longrightarrow X^d \longrightarrow X^{d+1} \longrightarrow 0,$$

with all terms in $\mathcal{X}$. Note that dropping extra direct summands of the form $A \xrightarrow{\sim} A$ in the middle terms of $\delta$, we also obtain an almost minimal sequence

$$\epsilon: \qquad 0 \longrightarrow X^0 \longrightarrow \overline{A^1} \longrightarrow \overline{A^2} \longrightarrow \cdots \longrightarrow \overline{A^{d-1}} \longrightarrow \overline{A^d} \longrightarrow X^{d+1} \longrightarrow 0.$$

By uniqueness, $\delta' \cong \epsilon$ and so $\epsilon$ has all terms in $\mathcal{X}$. Note that $[\delta] = [\epsilon]$ and, since $\overline{A^i}$ is a direct summand of $A^i$ for any $i = 1, \ldots, d$, there are morphisms of $d$-exact sequences $\epsilon \to \delta$ and $\delta \to \epsilon$.

We introduce $d$-Auslander-Reiten sequences in the subcategory $\mathcal{X}$ and give equivalent definitions. Note that the case $\mathcal{X} = \mathcal{F}$ will give the corresponding results in the ambient category $\mathcal{F}$.

**Definition 5.2.4.** We say that a $d$-exact sequence in $\mathcal{F}$ with all terms from $\mathcal{X}$ of the form

$$\epsilon : 0 \longrightarrow X^0 \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} X^{d+1} \longrightarrow 0,$$

is a *$d$-Auslander-Reiten sequence in $\mathcal{X}$* if the morphism $\xi^0$ is left almost split in $\mathcal{X}$, the morphism $\xi^d$ is right almost split in $\mathcal{X}$ and, when $d \geq 2$, also $\xi^1, \ldots, \xi^{d-1} \in \operatorname{rad}_{\mathcal{X}}$.

The following is a well known result, see [5, Lemma V.1.7]. Recall that for a module in $\operatorname{mod}\Lambda$, having local endomorphism ring is equivalent to being indecomposable.

**Lemma 5.2.5.** Let $\xi^0 : X^0 \to X^1$ be left almost split in $\mathcal{X}$. Then $\operatorname{End}_\Lambda(X^0)$ is local and $\xi^0$ is in $\operatorname{rad}_{\mathcal{X}}$.

**Remark 5.2.6.** Note that if $\epsilon$ is a $d$-Auslander-Reiten sequence in $\mathcal{X}$, Lemma 5.2.5 and its dual imply that $\operatorname{End}_\Lambda(X^0)$ and $\operatorname{End}_\Lambda(X^{d+1})$ are local and $\xi^0$, $\xi^d$ are in $\operatorname{rad}_{\mathcal{X}}$.

**Lemma 5.2.7.** Consider a $d$-exact sequence in $\mathcal{F}$ with all terms in $\mathcal{X}$ of the form:

$$\epsilon : 0 \longrightarrow X^0 \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} X^{d+1} \longrightarrow 0.$$

The following are equivalent:

(a) $\epsilon$ is a $d$-Auslander-Reiten sequence in $\mathcal{X}$,

(b) $\xi^0, \xi^1, \ldots, \xi^{d-1}$ are in $\operatorname{rad}_{\mathcal{X}}$ and $\xi^d$ is right almost split in $\mathcal{X}$,

(c) $\xi^1, \ldots, \xi^{d-1}, \xi^d$ are in $\operatorname{rad}_{\mathcal{X}}$ and $\xi^0$ is left almost split in $\mathcal{X}$.

*Proof.* By Lemma 5.2.5 and its dual, it is clear that (a) implies both (b) and (c). Suppose now that (b) holds. By the dual of Lemma 5.2.5, it follows that $\xi^d \in \operatorname{rad}_{\mathcal{X}}$. Let $f^0 : X^0 \to Y^0$ be a morphism in $\mathcal{X}$ that is not a split monomorphism. By Lemma 2.3.6, there is a morphism of $d$-exact sequences of the form:

$$\begin{tikzcd}
\epsilon: \arrow[d, "f"] & 0 \arrow[r] & X^0 \arrow[r, "\xi^0"] \arrow[d, "f^0"] & X^1 \arrow[r, "\xi^1"] \arrow[d, "f^1"] & \cdots \arrow[r] & X^{d-1} \arrow[r, "\xi^{d-1}"] \arrow[d, "f^{d-1}"] & X^d \arrow[r, "\xi^d"] \arrow[d, "f^d"] & X^{d+1} \arrow[r] \arrow[d, equal] & 0 \\
\delta: & 0 \arrow[r] & Y^0 \arrow[r, "\eta^0"'] & Y^1 \arrow[r, "\eta^1"'] & \cdots \arrow[r] & Y^{d-1} \arrow[r, "\eta^{d-1}"'] & Y^d \arrow[r, "\eta^d"'] & X^{d+1} \arrow[r] & 0,
\end{tikzcd}$$

where we may assume $Y^1, \ldots, Y^d$ are in $\mathcal{X}$ by Remark 5.2.3. Suppose for a contradiction that $\eta^0$ is not a split monomorphism. Then $\eta^d$ is not a split epimorphism by Corollary 2.3.8 and, since $\xi^d$ is right almost split in $\mathcal{X}$, then there exists $g^d : Y^d \to X^d$ such that $\xi^d \circ g^d = \eta^d$. By Lemma 2.3.10, there is a morphism of $d$-exact sequences of the form:

$$\begin{tikzcd}
\delta: \arrow[d, "g"] & 0 \arrow[r] & Y^0 \arrow[r, "\eta^0"] \arrow[d, "g^0"] & Y^1 \arrow[r, "\eta^1"] \arrow[d, "g^1"] & \cdots \arrow[r] & Y^{d-1} \arrow[r, "\eta^{d-1}"] \arrow[d, "g^{d-1}"] & Y^d \arrow[r, "\eta^d"] \arrow[d, "g^d"] & X^{d+1} \arrow[r] \arrow[d, equal] & 0 \\
\epsilon: & 0 \arrow[r] & X^0 \arrow[r, "\xi^0"'] & X^1 \arrow[r, "\xi^1"'] & \cdots \arrow[r] & X^{d-1} \arrow[r, "\xi^{d-1}"'] & X^d \arrow[r, "\xi^d"'] & X^{d+1} \arrow[r] & 0.
\end{tikzcd}$$

Note that $\xi^d \circ g^d \circ f^d = \xi^d$ and, since Lemma 2.3.11 implies that $\xi^d$ is right minimal, it follows that $g^d \circ f^d$ is an isomorphism. Hence, Lemma 2.3.12 implies that $g^0 \circ f^0$ is an isomorphism. so that $f^0$ is a split monomorphism, contradicting our assumption. So $\eta^0$ is a split monomorphism and there is a morphism $\mu : Y^1 \to Y^0$ such that $\mu \circ \eta^0 = 1_{Y^0}$. Then

$$\mu \circ f^1 \circ \xi^0 = \mu \circ \eta^0 \circ f^0 = f^0,$$

so $\xi^0$ is left almost split in $\mathcal{X}$ and we have proved (c). Dually, (c) implies (b) and it is hence clear that both (b) and (c) imply (a). □

## 5.3 $\mathcal{X}$-covers and the left end term of a $d$-Auslander-Reiten sequence in $\mathcal{X}$

In this section, we generalise the results in [43, Section 2] on $\operatorname{mod} \Lambda$ to its higher analogue $\mathcal{F}$. Iyama proved in [28, Theorem 3.3.1] that if $A^{d+1} \in \mathcal{F}$ is an indecomposable non-projective, then there exists a $d$-Auslander-Reiten sequence in $\mathcal{F}$ ending at $A^{d+1}$ and starting at $\tau_d(A^{d+1})$, see Proposition 2.3.25. The idea is to give an analogue of this result for $d$-Auslander-Reiten sequences in $\mathcal{X}$. Consider an indecomposable $X$ in $\mathcal{X}$ that admits non-split $d$-exact sequences ending at it with terms in $\mathcal{X}$. We "approximate" $\tau_d(X)$ with an indecomposable $\sigma X$ in $\mathcal{X}$. We show there is a $d$-Auslander-Reiten sequence in $\mathcal{X}$ ending in $X$ and that this sequence is forced to start at $\sigma X$.

Recall the definition of $\mathcal{X}$-cover from Definition 2.1.37. Note that the duals of all the results presented in this section are also true.

**Lemma 5.3.1.** Let $A \in \mathcal{F}$ and $g : X \to A$ be an $\mathcal{X}$-cover. Then,

$$\operatorname{Ext}^d_\Lambda(-, g) \mid_{\mathcal{X}} : \operatorname{Ext}^d_\Lambda(-, X) \mid_{\mathcal{X}} \longrightarrow \operatorname{Ext}^d_\Lambda(-, A) \mid_{\mathcal{X}}$$

is a monomorphism of contravariant functors.

*Proof.* Consider a $d$-exact sequence in $\mathcal{F}$ of the form

$$\delta : \qquad 0 \longrightarrow X \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} X^{d+1} \longrightarrow 0,$$

where $X^{d+1}$ is in $\mathcal{X}$. Since $\mathcal{X}$ is closed under $d$-extensions, we may assume that $X^1, \ldots, X^d$ are also in $\mathcal{X}$. Consider the morphism of $d$-exact sequences in $\mathcal{F}$ obtained as in Proposition 2.3.18(b):

$$\begin{tikzcd}
\delta \arrow[d] & 0 \arrow[r] & X \arrow[r, "\xi^0"] \arrow[d, "g"] & X^1 \arrow[r, "\xi^1"] \arrow[d, "g^1"] & \cdots \arrow[r] & X^d \arrow[r, "\xi^d"] \arrow[d, "g^d"] & X^{d+1} \arrow[r] \arrow[d, equal] & 0 \\
g \cdot \delta : & 0 \arrow[r] & A \arrow[r, "\alpha^0"'] & A^1 \arrow[r, "\alpha^1"'] & \cdots \arrow[r] & A^d \arrow[r, "\alpha^d"'] & X^{d+1} \arrow[r] & 0.
\end{tikzcd}$$

Suppose that $g \cdot \delta$ splits, **i.e.** $[g \cdot \delta] = 0$. By Proposition 2.3.18(a), we want to prove that also $\delta$ splits so that $\operatorname{Ext}^d_\Lambda(-, g) \mid_{\mathcal{X}}$ is a monomorphism. By Proposition 2.3.18(a), there exists a morphism $\gamma : A^1 \to A$ such that $\gamma \circ \alpha^0 = 1_A$. Then

$$g = \gamma \circ \alpha^0 \circ g = \gamma \circ g^1 \circ \xi^0.$$

Moreover, since $X^1$ is in $\mathcal{X}$ and $g$ is an $\mathcal{X}$-cover, there is a morphism $\eta : X^1 \to X$ such that $g \circ \eta = \gamma \circ g^1$. Then, we have

$$g = \gamma \circ g^1 \circ \xi^0 = g \circ \eta \circ \xi^0.$$

As $g$ is right minimal, it follows that $\eta \circ \xi^0$ is an isomorphism. This implies that $\xi^0$ is a split monomorphism and so $\delta$ splits, **i.e.** $[\delta] = 0$ in $\operatorname{Ext}^d_\Lambda(X^{d+1}, X)$. □

**Lemma 5.3.2.** Let $X$ in $\mathcal{X}$ be an indecomposable such that $\operatorname{Ext}^d_\Lambda(X, \mathcal{X})$ is non-zero. Suppose $\tau_d(X)$ has an $\mathcal{X}$-cover of the form $g : Y \to \tau_d(X)$. Then, for any non-split $d$-exact sequence in $\mathcal{F}$ of the form

$$\delta : 0 \longrightarrow X^0 \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} X \longrightarrow 0,$$

with all terms in $\mathcal{X}$, there is a morphism $h : X^0 \to Y$ such that $h \cdot \delta$ is a non-split $d$-exact sequence in $\mathcal{F}$. In particular, $\operatorname{Ext}^d_\Lambda(X, Y) \neq 0$.

*Proof.* First note that such a $\delta$ exists since $\operatorname{Ext}^d_\Lambda(X, \mathcal{X}) \neq 0$. Moreover, by Proposition 2.3.25, there is a $d$-Auslander-Reiten sequence in $\mathcal{F}$ of the form:

$$\epsilon : 0 \longrightarrow \tau_d(X) \xrightarrow{\alpha^0} A^1 \xrightarrow{\alpha^1} \cdots \longrightarrow A^{d-1} \xrightarrow{\alpha^{d-1}} A^d \xrightarrow{\alpha^d} X \longrightarrow 0.$$

Since $\xi^d$ is not a split epimorphism and $\alpha^d$ is right almost split in $\mathcal{F}$, there is a morphism $f^d : X^d \to A^d$ such that $\alpha^d \circ f^d = \xi^d$. Then, by Lemma 2.3.10, we can construct a morphism of $d$-exact sequences of the form:

$$\begin{tikzcd}
\delta : \arrow[d, "f"] & 0 \arrow[r] & X^0 \arrow[r, "\xi^0"] \arrow[d, "f^0"] & X^1 \arrow[r, "\xi^1"] \arrow[d, "f^1"] & \cdots \arrow[r] & X^{d-1} \arrow[r, "\xi^{d-1}"] \arrow[d, "f^{d-1}"] & X^d \arrow[r, "\xi^d"] \arrow[d, "f^d"] & X \arrow[r] \arrow[d, equal] & 0 \\
\epsilon : & 0 \arrow[r] & \tau_d X \arrow[r, "\alpha^0"'] & A^1 \arrow[r, "\alpha^1"'] & \cdots \arrow[r] & A^{d-1} \arrow[r, "\alpha^{d-1}"'] & A^d \arrow[r, "\alpha^d"'] & X \arrow[r] & 0.
\end{tikzcd}$$

Since $g$ is an $\mathcal{X}$-cover, there is a morphism $h : X^0 \to Y$ such that $f^0 = g \circ h$. Then, applying $\operatorname{Ext}^d_\Lambda(X, -)$, we obtain the commutative diagram:

$$\begin{tikzcd}
\operatorname{Ext}^d_\Lambda(X, X^0) \arrow[rr, "\operatorname{Ext}^d_\Lambda(X,f^0)"] \arrow[dr, "\operatorname{Ext}^d_\Lambda(X,h)"'] & & \operatorname{Ext}^d_\Lambda(X, \tau_d(X)). \\
& \operatorname{Ext}^d_\Lambda(X, Y) \arrow[ur, "\operatorname{Ext}^d_\Lambda(X,g)"'] &
\end{tikzcd} \tag{5.1}$$

Considering the morphism $\delta \to f^0 \cdot \delta$ obtained as in Proposition 2.3.18(b) and $f : \delta \to \epsilon$, Lemma 2.3.20 implies that $0 \neq [\epsilon] = [f^0 \cdot \delta]$ in $\operatorname{Ext}^d_\Lambda(X, \tau_d(X))$, so that $f^0 \cdot \delta$ is non-split by Proposition 2.3.18(a). Then, in diagram (5.1), we have $\operatorname{Ext}^d_\Lambda(X, g \circ h)(\delta) = g \circ h \cdot \delta = f^0 \cdot \delta$ is non-split and so $[h \cdot \delta] \neq 0$, **i.e.** $h \cdot \delta$ is non-split. In particular $\operatorname{Ext}^d_\Lambda(X, Y) \neq 0$. □

The following is a higher version of [5, proof of Proposition V.2.1]. Recall that modules are assumed to be right-modules. Instead of proving it here, we later prove the more general Lemma 5.4.1. The case $\mathcal{X} = \mathcal{F}$ in the latter, corresponds to the following.

**Lemma 5.3.3.** Let $A$ be an indecomposable non-projective in $\mathcal{F}$. Then $\operatorname{Ext}^d_\Lambda(A, \tau_d(A))$, as an $\operatorname{End}_\Lambda(A)$-module, has a simple socle generated by a $d$-Auslander-Reiten sequence in $\mathcal{F}$ of the form

$$\delta: \qquad 0 \longrightarrow \tau_d(A) \xrightarrow{\alpha^0} A^1 \xrightarrow{\alpha^1} A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \xrightarrow{\alpha^{d-1}} A^d \xrightarrow{\alpha^d} A \longrightarrow 0.$$

**Proposition 5.3.4.** (a) Let $X$ in $\mathcal{X}$ be an indecomposable such that $\mathrm{Ext}^d_\Lambda(X,\mathcal{X})$ is non-zero. If $\tau_d(X)$ has an $\mathcal{X}$-cover of the form $g: Y \to \tau_d(X)$, then $Y = Z \oplus Z'$, where $Z$ is an indecomposable such that $\mathrm{Ext}^d_\Lambda(X,Z) \neq 0$ and $\mathrm{Ext}^d_\Lambda(X,Z') = 0$. The module $Z$ is unique up to isomorphism.

(b) In the setting of (a), a non-split $d$-exact sequence of the form

$$\epsilon: \qquad 0 \longrightarrow Y \xrightarrow{\eta^0} Y^1 \xrightarrow{\eta^1} Y^2 \xrightarrow{\eta^2} \cdots \longrightarrow Y^{d-1} \xrightarrow{\eta^{d-1}} Y^d \xrightarrow{\eta^d} X \longrightarrow 0$$

is isomorphic to the direct sum of the split $d$-exact sequence:

$$0 \longrightarrow Z' \xrightarrow{1_{Z'}} Z' \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

and a non-split $d$-exact sequence of the form

$$0 \longrightarrow Z \xrightarrow{\zeta^0} V \xrightarrow{\zeta^1} Y^2 \xrightarrow{\eta^2} \cdots \longrightarrow Y^{d-1} \xrightarrow{\eta^{d-1}} Y^d \xrightarrow{\eta^d} X \longrightarrow 0.$$

*Proof.* (a) Let $Y = Z_1 \oplus \cdots \oplus Z_m$ be the indecomposable decomposition of $Y$. By Lemma 5.3.1, we have a monomorphism:

$$\mathrm{Ext}^d_\Lambda(X,g) : \mathrm{Ext}^d_\Lambda(X,Y) \longrightarrow \mathrm{Ext}^d_\Lambda(X,\tau_d(X)),$$

which is also a monomorphism of $\mathrm{End}_\Lambda(X)$-modules. Hence $\mathrm{Im}\,\mathrm{Ext}^d_\Lambda(X,g)$ is an $\mathrm{End}_\Lambda(X)$-submodule of $\mathrm{Ext}^d_\Lambda(X,\tau_d(X))$ isomorphic to

$$\mathrm{Ext}^d_\Lambda(X,Y) \cong \bigoplus_{j=1}^m \mathrm{Ext}^d_\Lambda(X,Z_j).$$

Since $\mathrm{Ext}^d_\Lambda(X,\mathcal{X}) \neq 0$, it follows that $X$ is not projective in $\mathrm{mod}\,\Lambda$. Then, viewed as an $\mathrm{End}_\Lambda(X)$-module, $\mathrm{Ext}^d_\Lambda(X,\tau_d(X))$ has simple socle by Lemma 5.3.3. Hence $\mathrm{Im}\,\mathrm{Ext}^d_\Lambda(X,g)$ is either zero or an indecomposable $\mathrm{End}_\Lambda(X)$-module. So there is at most one $j \in \{1,\ldots,m\}$ such that $\mathrm{Ext}^d_\Lambda(X,Z_j)$ is non-zero. Note that $\mathrm{Ext}^d_\Lambda(X,Y)$ is non-zero by Lemma 5.3.2. Hence there is exactly one $j \in \{1,\ldots,m\}$ such that $\mathrm{Ext}^d_\Lambda(X,Z_j)$ is non-zero.

(b) By Lemma 2.3.6, there is a morphism of $d$-exact sequences of the form:

$$\begin{tikzcd}
\epsilon: \arrow[d] & 0 \arrow[r] & Z' \oplus Z \arrow[r, "\eta^0=(\xi',\xi)"] \arrow[d, "(1{,}0)"] & Y^1 \arrow[r, "\eta^1"] \arrow[d] & Y^2 \arrow[r, "\eta^2"] \arrow[d] & \cdots \arrow[r, "\eta^{d-1}"] & Y^d \arrow[r, "\eta^d"] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
\overline{\epsilon}: & 0 \arrow[r] & Z' \arrow[r, "\omega^0"'] & W^1 \arrow[r, "\omega^1"'] & W^2 \arrow[r, "\omega^2"'] & \cdots \arrow[r, "\omega^{d-1}"'] & W^d \arrow[r, "\omega^d"'] & X \arrow[r] & 0.
\end{tikzcd} \tag{5.2}$$

Since $\mathrm{Ext}^d_\Lambda(X, Z') = 0$, the bottom row is a split $d$-exact sequence by Proposition 2.3.18(a). Hence, we have that $\overline{\epsilon}$ is isomorphic to:

$$0 \longrightarrow Z' \xrightarrow{\binom{1_{Z'}}{0}} Z' \oplus \overline{W^1} \xrightarrow{(0,\gamma)} W^2 \xrightarrow{\omega^2} \cdots \xrightarrow{\omega^{d-1}} W^d \xrightarrow{\omega^d} X \longrightarrow 0.$$

Then the morphism (5.2) is isomorphic to the morphism:

$$\begin{tikzcd}
\epsilon: \arrow[d] & 0 \arrow[r] & Z' \oplus Z \arrow[r, "(\xi'{,}\xi)"] \arrow[d, "(1{,}0)"] & Y^1 \arrow[r, "\eta^1"] \arrow[d, "\binom{\mu'}{\mu}"] & Y^2 \arrow[r, "\eta^2"] \arrow[d] & \cdots \arrow[r, "\eta^{d-1}"] & Y^d \arrow[r, "\eta^d"] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
\overline{\epsilon}: & 0 \arrow[r] & Z' \arrow[r, "\binom{1_{Z'}}{0}"'] & Z' \oplus \overline{W^1} \arrow[r, "(0{,}\gamma)"'] & W^2 \arrow[r, "\omega^2"'] & \cdots \arrow[r, "\omega^{d-1}"'] & W^d \arrow[r, "\omega^d"'] & X \arrow[r] & 0.
\end{tikzcd}$$

In particular, $\mu' \circ \xi' = 1_{Z'}$ and so $Y^1 = Z' \oplus V$ and $\epsilon$ isomorphic to a $d$-exact sequence of the form:

$$\epsilon : 0 \longrightarrow Z' \oplus Z \xrightarrow{\begin{pmatrix} 1_{Z'} & 0 \\ 0 & \zeta^0 \end{pmatrix}} Z' \oplus V \xrightarrow{(0,\zeta^1)} Y^2 \xrightarrow{\eta^2} \cdots \xrightarrow{\eta^{d-1}} Y^d \xrightarrow{\eta^d} X \longrightarrow 0.$$

Clearly, this is isomorphic to the direct sum of the two $d$-exact sequences we wanted, where the one starting at $Z$ does not split since $\epsilon$ does not split. $\square$

**Definition 5.3.5.** Suppose that $\mathcal{X}$ is precovering in $\mathcal{F}$ and let $X$ be an indecomposable in $\mathcal{X}$. If $\mathrm{Ext}^d_\Lambda(X, \mathcal{X}) = 0$ we put $\sigma X = 0$. Otherwise, letting $g : Y \to \tau_d(X)$ be an $\mathcal{X}$-cover, we denote by $\sigma X$ the unique indecomposable direct summand $Z$ of $Y$ such that $\mathrm{Ext}^d_\Lambda(X, Z) \neq 0$.

**Corollary 5.3.6.** Let $\mathcal{X}$ be precovering in $\mathcal{F}$ and let

$$\delta : 0 \longrightarrow X^0 \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} X \longrightarrow 0$$

be a $d$-Auslander-Reiten sequence in $\mathcal{X}$. Then $X^0 \cong \sigma X$.

*Proof.* Note that the existence of $\delta$ implies that $\mathrm{Ext}^d_\Lambda(X, \mathcal{X})$ is non-zero. As $\mathcal{X}$ is precovering in $\mathcal{F}$, there is an $\mathcal{X}$-cover $g : Y \to \tau_d(X)$. Then, by Lemma 5.3.2, there is a morphism of non-split $d$-exact sequences in $\mathcal{F}$ of the form:

$$\begin{tikzcd}
\delta : \arrow[d] & 0 \arrow[r] & X^0 \arrow[r, "\xi^0"] \arrow[d, "h"] & X^1 \arrow[r, "\xi^1"] \arrow[d, "h^1"] & \cdots \arrow[r] & X^{d-1} \arrow[r, "\xi^{d-1}"] \arrow[d, "h^{d-1}"] & X^d \arrow[r, "\xi^d"] \arrow[d, "h^d"] & X \arrow[r] \arrow[d, equal] & 0 \\
h \cdot \delta : & 0 \arrow[r] & Y \arrow[r, "\eta^0"'] & Y^1 \arrow[r, "\eta^1"'] & \cdots \arrow[r] & Y^{d-1} \arrow[r, "\eta^{d-1}"'] & Y^d \arrow[r, "\eta^d"'] & X \arrow[r] & 0.
\end{tikzcd}$$

Since $\eta^d$ is not a split epimorphism, Lemma 2.3.7 implies that $h$ does not factor through $\xi^0$. As $\xi^0$ is a left almost split morphism in $\mathcal{X}$, it follows that $h$ is a split monomorphism. Hence $X^0$ is an indecomposable direct summand of $Y$ such that $\mathrm{Ext}^d_\Lambda(X, X^0) \neq 0$ and Proposition 5.3.4(a) implies that $X^0 \cong \sigma X$. $\square$

**Notation 5.3.7.** For $A$ and $B$ in $\mathcal{F}$, we use the notation $(A, B) := \mathrm{Hom}_{\mathcal{F}}(A, B)$.

**Definition 5.3.8** ([36, Definition 3.1])**.** Consider a $d$-exact sequence in $\mathcal{F}$ of the form:

$$\delta : \qquad 0 \longrightarrow A^0 \xrightarrow{\alpha^0} A^1 \xrightarrow{\alpha^1} A^2 \longrightarrow \cdots \longrightarrow A^{d-1} \xrightarrow{\alpha^{d-1}} A^d \xrightarrow{\alpha^d} A^{d+1} \longrightarrow 0.$$

We define $\delta^*$, the *contravariant defect of* $\delta$ *on* $\mathcal{F}$, by the exact sequence of functors

$$(-, A^d) \to (-, A^{d+1}) \to \delta^*(-) \to 0.$$

Dually, we define $\delta_*$, the *covariant defect of* $\delta$ *on* $\mathcal{F}$, by the exact sequence of functors

$$(A^1, -) \to (A^0, -) \to \delta_*(-) \to 0.$$

**Remark 5.3.9.** By Lemma 2.3.19, we have that $\delta^*(-)$ is a subfunctor of $\mathrm{Ext}^d_\Lambda(-, A^0)\mid_{\mathcal{F}}$ and $\delta_*(-)$ is a subfunctor of $\mathrm{Ext}^d_\Lambda(A^{d+1}, -)\mid_{\mathcal{F}}$.

**Lemma 5.3.10.** Consider a $d$-exact sequence in $\mathcal{F}$ with all terms in $\mathcal{X}$ of the form:

$$\delta : 0 \longrightarrow X^0 \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} X^{d+1} \longrightarrow 0$$

and an $\mathcal{X}$-cover $g : X \to A$ for some $A \in \mathcal{F}$. The $k$-linear map $(X^0, g) : (X^0, X) \to (X^0, A)$ induces an isomorphism of $k$-vector spaces:

$$\delta_*(g) : \delta_*(X) \xrightarrow{\sim} \delta_*(A).$$

In particular, $\dim_k(\delta_*(X)) = \dim_k(\delta_*(A))$.

*Proof.* By Definition 5.3.8, we have

$$\delta_*(g) : (X^0, X)/\operatorname{Im}(\xi^0, X) \to (X^0, A)/\operatorname{Im}(\xi^0, A).$$

Note that since $g : X \to A$ is an $\mathcal{X}$-cover, the map $(X^0, g)$ is surjective. Hence it is enough to show that $\operatorname{Im}(\xi^0, X)$ is the full preimage of $\operatorname{Im}(\xi^0, A)$ under $(X^0, g)$. It is clear that

$$(X^0, g)(\operatorname{Im}(\xi^0, X)) \subseteq \operatorname{Im}(\xi^0, A).$$

It remains to show that if $h : X^0 \to X$ is such that $g \circ h : X^0 \to A$ factors through $\xi^0$, then $h$ factors through $\xi^0$. Consider the following morphisms of $d$-exact sequences:

$$\begin{tikzcd}
\delta \arrow[d] & 0 \arrow[r] & X^0 \arrow[r, "\xi^0"] \arrow[d, "h"] & X^1 \arrow[r, "\xi^1"] \arrow[d] \arrow[ddl] & \cdots \arrow[r] & X^{d-1} \arrow[r, "\xi^{d-1}"] \arrow[d] & X^d \arrow[r, "\xi^d"] \arrow[d] & X^{d+1} \arrow[r] \arrow[d, equal] \arrow[ddl, dashed] & 0 \\
h \cdot \delta : \arrow[d] & 0 \arrow[r] & X \arrow[r] \arrow[d, "g"] & Y^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & Y^{d-1} \arrow[r] \arrow[d] & Y^d \arrow[r] \arrow[d] & X^{d+1} \arrow[r] \arrow[d, equal] & 0 \\
gh \cdot \delta : & 0 \arrow[r] & A \arrow[r] & A^1 \arrow[r] & \cdots \arrow[r] & A^{d-1} \arrow[r] & A^d \arrow[r] & X^{d+1} \arrow[r] & 0.
\end{tikzcd}$$

Since $g \circ h$ factors through $\xi^0$, Lemma 2.3.7 implies that the bottom row splits. Hence, we have that $[\operatorname{Ext}^d_\Lambda(X^{d+1}, g)(h \cdot \delta)] = 0$. Since $\operatorname{Ext}^d_\Lambda(X^{d+1}, g)$ is a monomorphism by Lemma 5.3.1, it follows that the middle row splits. Hence $h$ factors through $\xi^0$ by Lemma 2.3.7. □

**Remark 5.3.11.** Let $X \in \mathcal{X}$ be indecomposable and assume that $\tau_d(X)$ has an $\mathcal{X}$-cover, say $g : Y \to \tau_d(X)$. Given any $d$-exact sequence with terms in $\mathcal{X}$ of the form

$$\delta : 0 \longrightarrow X^0 \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} X^{d+1} \longrightarrow 0,$$

we have that

$$\dim_k(\delta_*(Y)) = \dim_k(\delta_*(\tau_d(X))) = \dim_k(\delta^*(X)),$$

where the first equality holds by Lemma 5.3.10 and the second by [36, Theorem 3.8].

**Proposition 5.3.12.** Assume $\mathcal{X}$ is precovering in $\mathcal{F}$. Let $X \in \mathcal{X}$ be an indecomposable such that $\operatorname{Ext}^d_\Lambda(X, \mathcal{X}) \neq 0$ and $g : Y \to \tau_d(X)$ be an $\mathcal{X}$-cover. Then there is a $d$-exact sequence with terms in $\mathcal{X}$ of the form:

$$\epsilon : \qquad 0 \longrightarrow Y \xrightarrow{\eta^0} Y^1 \xrightarrow{\eta^1} Y^2 \xrightarrow{\eta^2} \cdots \longrightarrow Y^{d-1} \xrightarrow{\eta^{d-1}} Y^d \xrightarrow{\eta^d} X \longrightarrow 0,$$

with $\eta^d$ right almost split in $\mathcal{X}$.

*Proof.* Since $\operatorname{Ext}^d_\Lambda(X, \mathcal{X}) \neq 0$, there exists a non-split $d$-exact sequence with terms from $\mathcal{X}$ of the form:

$$\delta : 0 \longrightarrow X^0 \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} X \longrightarrow 0.$$

As not every endomorphism of $X$ factors through $\xi^d$, we have that $\dim_k(\delta^*(X)) \neq 0$. By Remark 5.3.11, we have that

$$\dim_k(\delta_*(Y)) = \dim_k(\delta_*(\tau_d(X))) = \dim_k(\delta^*(X)) \neq 0.$$

So $\operatorname{Ext}^d_\Lambda(\mathcal{X}, Y)$ is non-zero by Remark 5.3.9 and there is a non-split $d$-exact sequence with terms in $\mathcal{X}$ of the form:

$$\zeta : 0 \longrightarrow Y \xrightarrow{\zeta^0} Z^1 \xrightarrow{\zeta^1} \cdots \longrightarrow Z^{d-1} \xrightarrow{\zeta^{d-1}} Z^d \xrightarrow{\zeta^d} Z^{d+1} \longrightarrow 0.$$

Since not every endomorphism of $Y$ factors through $\zeta^0$, then $\dim_k(\zeta_*(Y))$ is non-zero and so, by Remark 5.3.11, we have

$$0 \neq \dim_k(\zeta_*(Y)) = \dim_k(\zeta_*(\tau_d(X))) = \dim_k(\zeta^*(X)).$$

Hence not every morphism of the form $X \to Z^{d+1}$ factors through $\zeta^d$. So there is a morphism $t : X \to Z^{d+1}$ such that its image in $\zeta^*(X) = (X, Z^{d+1})/\operatorname{Im}(X, \zeta^d)$ generates a simple $\operatorname{End}_\Lambda(X)$-module. Thus, by the dual of Proposition 2.3.18(b), we have a morphism of $d$-exact sequences in $\mathcal{F}$ of the form:

$$\begin{tikzcd}
\zeta \cdot t: \arrow[d] & 0 \arrow[r] & Y \arrow[r, "\eta^0"] \arrow[d, equal] & Y^1 \arrow[r, "\eta^1"] \arrow[d, "t^1"] & \cdots \arrow[r] & Y^{d-1} \arrow[r, "\eta^{d-1}"] \arrow[d, "t^{d-1}"] & Y^d \arrow[r, "\eta^d"] \arrow[d, "t^d"] & X \arrow[r] \arrow[d, "t"] & 0 \\
\zeta: & 0 \arrow[r] & Y \arrow[r, "\zeta^0"'] & Z^1 \arrow[r, "\zeta^1"'] & \cdots \arrow[r] & Z^{d-1} \arrow[r, "\zeta^{d-1}"'] & Z^d \arrow[r, "\zeta^d"'] & Z^{d+1} \arrow[r] & 0,
\end{tikzcd}$$

where we can assume $Y^1, \ldots, Y^d$ are in $\mathcal{X}$ by Remark 5.2.3. We claim that $\epsilon := \zeta \cdot t$ is such that $\eta^d$ is right almost split in $\mathcal{X}$. First note that since $t$ does not factor through $\zeta^d$, then $\epsilon$ is not a split $d$-exact sequence by Lemma 2.3.7. In particular, $\eta^d$ is not a split epimorphism. Suppose that $s : W \to X$ in $\mathcal{X}$ is not a split epimorphism. We need to show that $s$ factors through $\eta^d$. Consider the morphism obtained by the dual of Proposition

2.3.18(b):

$$\begin{tikzcd}
\epsilon \cdot s : \arrow[d] & 0 \arrow[r] & Y \arrow[r, "\omega^0"] \arrow[d, equal] & W^1 \arrow[r, "\omega^1"] \arrow[d, "s^1"] & \cdots \arrow[r] & W^{d-1} \arrow[r, "\omega^{d-1}"] \arrow[d, "s^{d-1}"] & W^d \arrow[r, "\omega^d"] \arrow[d, "s^d"] & W \arrow[r] \arrow[d, "s"] & 0 \\
\epsilon : & 0 \arrow[r] & Y \arrow[r, "\eta^0"'] & Y^1 \arrow[r, "\eta^1"'] & \cdots \arrow[r] & Y^{d-1} \arrow[r, "\eta^{d-1}"'] & Y^d \arrow[r, "\eta^d"'] & X \arrow[r] & 0.
\end{tikzcd}$$

By Lemma 2.3.7, we have that $s$ factoring through $\eta^d$ is equivalent to $\epsilon \cdot s$ splitting. Note that if every morphism $r : X \to W$ factors through $\omega^d$, then $(\epsilon \cdot s)^*(X) = 0$ and, by Remark 5.3.11, we have that $(\epsilon \cdot s)_*(Y) = 0$, that is $(\omega_0, Y)$ is surjective and $\epsilon \cdot s$ splits. Hence it is enough to show that every morphism $r : X \to W$ factors through $\omega^d$.

Note that since $s$ is not a split epimorphism, $s \circ r : X \to X$ is not an isomorphism. Hence, $t \circ s \circ r : X \to Z^{d+1}$ is in $t \circ \mathrm{End}_\Lambda(X) \circ \mathrm{rad}_{\mathrm{End}_\Lambda(X)}$. Since the image of $t \circ \mathrm{End}_\Lambda(X)$ in $\zeta^*(X)$ is a simple module, it follows that $t \circ s \circ r$ projects to zero in $\zeta^*(X)$. In other words, $t \circ s \circ r$ factors through $\zeta^d$, so there is a morphism $\alpha : X \to Z^d$ such that $\zeta^d \circ \alpha = t \circ s \circ r$. Consider:

$$\begin{tikzcd}
\epsilon \cdot sr : \arrow[d] & 0 \arrow[r] & Y \arrow[r, "\mu^0"] \arrow[d, equal] & U^1 \arrow[r, "\mu^1"] \arrow[d, "r^1"] \arrow[ddl, dashed] & \cdots \arrow[r] & U^{d-1} \arrow[r, "\mu^{d-1}"] \arrow[d, "r^{d-1}"] & U^d \arrow[r, "\mu^d"] \arrow[d, "r^d"] & X \arrow[r] \arrow[d, "r"] \arrow[ddl] & 0 \\
\epsilon \cdot s : \arrow[d] & 0 \arrow[r] & Y \arrow[r, "\omega^0"] \arrow[d, equal] & W^1 \arrow[r, "\omega^1"] \arrow[d, "t^1 \circ s^1"] & \cdots \arrow[r] & W^{d-1} \arrow[r, "\omega^{d-1}"] \arrow[d, "t^{d-1} \circ s^{d-1}"'] & W^d \arrow[r, "\omega^d"] \arrow[d, "t^d \circ s^d"'] & W \arrow[r] \arrow[d, "t \circ s"] & 0 \\
\zeta : & 0 \arrow[r] & Y \arrow[r, "\zeta^0"'] & Z^1 \arrow[r, "\zeta^1"'] & \cdots \arrow[r] & Z^{d-1} \arrow[r, "\zeta^{d-1}"'] & Z^d \arrow[r, "\zeta^d"'] & Z^{d+1} \arrow[r] & 0.
\end{tikzcd}$$

Then, by Lemma 2.3.7, there is a morphism $\alpha^1 : U^1 \to Y$ such that $\alpha^1 \circ \mu^0 = 1_Y$. Hence the top row of the above diagram splits. So there is a morphism $\phi : X \to U^d$ such that $\mu^d \circ \phi = 1_X$. Note that

$$\omega^d \circ r^d \circ \phi = r \circ \mu^d \circ \phi = r.$$

Hence $r$ factors through $\omega^d$ as we wished to prove. □

**Theorem 5.3.13.** *Assume $\mathcal{X}$ is precovering in $\mathcal{F}$ and let $X$ be an indecomposable in $\mathcal{X}$.*

*(a) There exists a right almost split morphism $W \to X$ in $\mathcal{X}$.*

*(b) If $\mathrm{Ext}^d_\Lambda(X, \mathcal{X})$ is non-zero, there is a d-Auslander-Reiten sequence in $\mathcal{X}$ of the form:*

$$0 \longrightarrow \sigma X \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} X \longrightarrow 0. \tag{5.3}$$

*Proof.* (a) This follows from [6, Proposition 3.10].

(b) Let $g : Y \to \tau_d(X)$ be an $\mathcal{X}$-cover. Then, by Proposition 5.3.12, there exists a $d$-exact sequence with terms in $\mathcal{X}$ of the form

$$\epsilon : 0 \longrightarrow Y \xrightarrow{\eta^0} Y^1 \xrightarrow{\eta^1} Y^2 \xrightarrow{\eta^2} \cdots \longrightarrow Y^{d-1} \xrightarrow{\eta^{d-1}} Y^d \xrightarrow{\eta^d} X \longrightarrow 0,$$

with $\eta^d$ right almost split in $\mathcal{X}$. By Proposition 5.3.4, $\epsilon$ has a non-split $d$-exact sequence with terms in $\mathcal{X}$ of the form

$$\delta : 0 \longrightarrow \sigma X \xrightarrow{\zeta^0} V \xrightarrow{\zeta^1} Y^2 \xrightarrow{\eta^2} \cdots \longrightarrow Y^{d-1} \xrightarrow{\eta^{d-1}} Y^d \xrightarrow{\eta^d} X \longrightarrow 0$$

as a direct summand. If $d \geq 2$, we may also assume that $\zeta^1, \eta^2, \ldots, \eta^{d-1}$ are in $\mathrm{rad}_{\mathcal{X}}$. Moreover, since $\sigma X$ is indecomposable and $\zeta^0$ is not a split monomorphism, it follows that $\zeta^0$ is in $\mathrm{rad}_{\mathcal{X}}$. Hence, by Lemma 5.2.7, we conclude that $\delta$ is a $d$-Auslander-Reiten sequence in $\mathcal{X}$. □

## 5.4 More on $d$-Auslander-Reiten sequences in $\mathcal{X}$

In this section, we study the case when, for an indecomposable $X \in \mathcal{X}$, the factor ring of $\mathrm{End}_\Lambda(X)$ modulo the morphisms factoring through a projective is a division ring. Generalising [5, Corollary V.2.4], we prove that an almost minimal $d$-exact sequence with terms in $\mathcal{X}$ ending at $X$ is a $d$-Auslander-Reiten sequence if and only if it does not split. As a consequence of this result, we prove a higher version of [43, Proposition 2.10].

The argument from [5, proof of Proposition V.2.1] can be modified to prove the following result, we present here the argument for convenience of the reader. Note that this differs from the original result in two ways: it is a higher version and we work in the subcategory $\mathcal{X}$. The condition on an indecomposable $C$ in $\mathrm{mod}\,\Lambda$ to be non-projective is hence substituted with the condition on an indecomposable $X \in \mathcal{X}$ to be such that $\mathrm{Ext}^d_\Lambda(X, \mathcal{X}) \neq 0$ and $\tau C$ with $\sigma X$.

**Lemma 5.4.1.** Assume that $\mathcal{X}$ is precovering in $\mathcal{F}$ and let $X$ be an indecomposable in $\mathcal{X}$ such that $\mathrm{Ext}^d_\Lambda(X, \mathcal{X}) \neq 0$. Then $\mathrm{Ext}^d(X, \sigma X)$, as an $\mathrm{End}_\Lambda(X)$-module, has a simple socle generated by a $d$-Auslander-Reiten sequence in $\mathcal{X}$ of the form

$$\delta : \qquad 0 \longrightarrow \sigma X \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} X \longrightarrow 0.$$

*Proof.* By Theorem 5.3.13, there is a $d$-Auslander-Reiten sequence in $\mathcal{X}$ of the form

$$\delta : \qquad 0 \longrightarrow \sigma X \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} X^2 \longrightarrow \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} X \longrightarrow 0.$$

Consider a non-zero element in the $\mathrm{End}_\Lambda(X)$-socle of $\mathrm{Ext}^d_\Lambda(X, \sigma X)$ of the form

$$\epsilon : \qquad 0 \longrightarrow \sigma X \xrightarrow{\eta^0} Y^1 \xrightarrow{\eta^1} Y^2 \longrightarrow \cdots \longrightarrow Y^{d-1} \xrightarrow{\eta^{d-1}} Y^d \xrightarrow{\eta^d} X \longrightarrow 0.$$

We prove that $\mathrm{rad}_{\mathrm{End}_\Lambda(X)}$ annihilates $\delta$. Let $h \in \mathrm{rad}_{\mathrm{End}_\Lambda(X)}$, and note that since $X$ is indecomposable, $\mathrm{End}_\Lambda(X)$ is local, so this is the same as saying that $h$ is not an isomorphism by [45, Section 4]. Applying $\mathrm{Ext}^d_\Lambda(h, \sigma X)$ to $\delta$, we get a morphism of $d$-exact sequences dual to the one from Proposition 2.3.18(b):

$$\begin{tikzcd}
\delta \cdot h: \arrow[d] & 0 \arrow[r] & \sigma X \arrow[r, "\zeta^0"] \arrow[d, equal] & Z^1 \arrow[r, "\zeta^1"] \arrow[d, "h^1"] & Z^2 \arrow[r] \arrow[d, "h^2"] & \cdots \arrow[r] & Z^d \arrow[r, "\zeta^d"] \arrow[d, "h^d"] & X \arrow[r] \arrow[d, "h"] & 0 \\
\delta: & 0 \arrow[r] & \sigma X \arrow[r, "\xi^0"'] & X^1 \arrow[r, "\xi^1"'] & X^2 \arrow[r] & \cdots \arrow[r] & X^d \arrow[r, "\xi^d"'] & X \arrow[r] & 0.
\end{tikzcd}$$

Since $h$ is not a split epimorphism and $\xi^d$ is right almost split in $\mathcal{X}$, there exists a morphism $s^{d+1} : X \to X^d$ such that $\xi^d \circ s^{d+1} = h$. Then, by Lemma 2.3.7, there is a morphism $s^1 : Z^1 \to \sigma X$ such that $s^1 \circ \zeta^0 = 1_{\sigma X}$. This means that $\delta \cdot h$ splits, so $\mathrm{rad}_{\mathrm{End}_\Lambda(X)}$ annihilates $\delta$. Hence $\delta$ is in the socle. Moreover, since $\delta$ is a $d$-Auslander-Reiten sequence in $\mathcal{X}$ and $\epsilon$ is non-split, there exists a morphism $t^1 : X^1 \to Y^1$ such that $t^1 \circ \xi^0 = \eta^0$. Then, by Lemma 2.3.10, we have a morphism of $d$-exact sequences:

$$\begin{tikzcd}
\delta: \arrow[d] & 0 \arrow[r] & \sigma X \arrow[r, "\xi^0"] \arrow[d, equal] & X^1 \arrow[r, "\xi^1"] \arrow[d, "t^1"] & X^2 \arrow[r] \arrow[d, "t^2"] & \cdots \arrow[r] & X^{d-1} \arrow[r, "\xi^{d-1}"] \arrow[d, "t^{d-1}"] & X^d \arrow[r, "\xi^d"] \arrow[d, "t^d"] & X \arrow[r] \arrow[d, "t:=t^{d+1}"] & 0 \\
\epsilon: & 0 \arrow[r] & \sigma X \arrow[r, "\eta^0"'] & Y^1 \arrow[r, "\eta^1"'] & Y^2 \arrow[r] & \cdots \arrow[r] & Y^{d-1} \arrow[r, "\eta^{d-1}"'] & Y^d \arrow[r, "\eta^d"'] & X \arrow[r] & 0.
\end{tikzcd}$$

By the dual of Lemma 2.3.20, we have that $[\delta] = [\epsilon \cdot t]$ in $\mathrm{Ext}^d_\Lambda(X, \sigma X)$. Then, as $\epsilon$ is in the $\mathrm{End}_\Lambda(X)$-socle of $\mathrm{Ext}^d_\Lambda(X, \sigma X)$, then $t$ is not in $\mathrm{rad}_{\mathrm{End}_\Lambda(X)}$ and so it is an isomorphism by [45, Section 4]. Hence $[\epsilon] = [\delta \cdot t^{-1}]$ in $\mathrm{Ext}^d_\Lambda(X, \sigma X)$. Then, the $\mathrm{End}_\Lambda(X)$-socle is cyclic and hence simple and generated by $\delta$. $\square$

**Lemma 5.4.2.** Assume $\mathcal{X}$ is precovering in $\mathcal{F}$. Let $X$ be an indecomposable in $\mathcal{X}$ such that $\mathrm{Ext}^d_\Lambda(X, \mathcal{X}) \neq 0$. Consider a non-split $d$-exact sequence of the form:

$$\delta : 0 \longrightarrow \sigma X \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} X \longrightarrow 0,$$

with $X^1, \ldots, X^d$ in $\mathcal{X}$ and, when $d \geq 2$, also $\xi^1, \ldots, \xi^{d-1}$ in $\mathrm{rad}_{\mathcal{X}}$. Then, the following are equivalent:

(a) $\delta$ is a $d$-Auslander-Reiten sequence in $\mathcal{X}$,

(b) $\xi^d$ is right almost split in $\mathcal{X}$,

(c) $\mathrm{Im}(X,\xi^d) = \mathrm{rad}_{\mathrm{End}_\Lambda(X)}$,

(d) $\delta^*(X)$ is a simple $\mathrm{End}_\Lambda(X)$-module.

*Proof.* By Definition 5.2.4, we have that (a) implies (b). Assume now that (b) holds and note that since $X$ is indecomposable, then $\mathrm{End}_\Lambda(X)$ is local. Consider

$$(X,\xi^d) : (X,X^d) \to (X,X) : \alpha \mapsto \xi^d \circ \alpha.$$

Assume $\beta : X \to X$ is in $\mathrm{rad}_{\mathrm{End}_\Lambda(X)}$.Then, since $X$ is indecomposable, it follows that $\beta$ is not an isomorphism and so $\beta$ is not a split epimorphism. As $\xi^d$ is right almost split in $\mathcal{X}$, there exists a morphism $\alpha : X \to X^d$ such that

$$\beta = \xi^d \circ \alpha = (X,\xi^d)(\alpha),$$

and so $\beta \in \mathrm{Im}(X,\xi^d)$. Assume now that $\beta : X \to X$ is in $\mathrm{Im}(X,\xi^d)$, **i.e.** $\beta = \xi^d \circ \alpha$ for some $\alpha \in (X,X)$. Then, since $\xi^d$ is not a split epimorphism, it follows that $\beta$ is not an isomorphism and so $\beta$ is in $\mathrm{rad}_{\mathrm{End}_\Lambda(X)}$. Hence (b) implies (c).

Recall that $\delta^*(X) = (X,X)/\mathrm{Im}(X,\xi^d)$. Assume (c) holds. Then we have that $\delta^*(X) = \mathrm{End}_\Lambda(X)/\mathrm{rad}_{\mathrm{End}_\Lambda(X)}$ and this is simple as $\mathrm{rad}_{\mathrm{End}_\Lambda(X)}$ is the maximal ideal of the algebra $\mathrm{End}_\Lambda(X)$. So (c) implies (d).

Assume now that (d) holds. Then, $\delta^*(X)$ is generated by the image of $1_X$, and this image is sent to $\delta$ as an element of $\mathrm{Ext}^d_\Lambda(X,\sigma X)$. Moreover, by Lemma 5.4.1, we have that $\delta^*(X)$ is the socle of $\mathrm{Ext}^d_\Lambda(X,\sigma X)$ as an $\mathrm{End}_\Lambda(X)$-module and so, as $\delta$ generates $\delta^*(X)$, Lemma 5.4.1 implies that $\delta$ is a $d$-Auslander-Reiten sequence in $\mathcal{X}$. So (d) implies (a). □

**Notation 5.4.3.** For a module $A$ in $\mathcal{F}$, we denote by $\mathcal{P}(A)$ the ideal of all morphisms of the form $A \to A$ that factor through a projective module. The factor ring of $\mathrm{End}_\Lambda(A)$ modulo $\mathcal{P}(A)$ is then denoted by $\underline{\mathrm{End}}_\Lambda(A)$.

**Theorem 5.4.4.** *Assume $\mathcal{X}$ is precovering in $\mathcal{F}$. Let $X$ be an indecomposable in $\mathcal{X}$ such that $\underline{\mathrm{End}}_\Lambda(X)$ is a division ring. For a $d$-exact sequence of the form:*

$$\delta : 0 \longrightarrow \sigma X \xrightarrow{\xi^0} X^1 \xrightarrow{\xi^1} \cdots \longrightarrow X^{d-1} \xrightarrow{\xi^{d-1}} X^d \xrightarrow{\xi^d} X \longrightarrow 0,$$

*with terms in $\mathcal{X}$ and, when $d \geq 2$, also $\xi^1, \dots, \xi^{d-1}$ in $\mathrm{rad}_{\mathcal{X}}$, the following are equivalent:*

*(a) $\delta$ is a $d$-Auslander-Reiten sequence in $\mathcal{X}$,*

*(b) $\delta$ does not split.*

*Proof.* Note that as $\xi^d$ is an epimorphism, $\operatorname{Im}(X,\xi^d)$ contains $\mathcal{P}(X)$. Since $\underline{\operatorname{End}}_\Lambda(X) = \operatorname{End}_\Lambda(X)/\mathcal{P}(X)$ is a division ring, it is simple as an $\operatorname{End}_\Lambda(X)$-module. Then $\mathcal{P}(X)$ is a maximal submodule of $\operatorname{End}_\Lambda(X)$ and, as $\operatorname{End}_\Lambda(X)$ is local, we have that $\mathcal{P}(X) = \operatorname{rad}_{\operatorname{End}_\Lambda(X)}$. Hence, maximality and

$$\operatorname{rad}_{\operatorname{End}_\Lambda(X)} = \mathcal{P}(X) \subseteq \operatorname{Im}(X,\xi^d) \subseteq \operatorname{End}_\Lambda(X),$$

imply that we have the following two cases:

1. $\operatorname{Im}(X,\xi^d) = \operatorname{rad}_{\operatorname{End}_\Lambda(X)}$, **i.e.** $\delta^*(X)$ is a simple $\operatorname{End}_\Lambda(X)$-module, in which case $\delta$ is non-split as $1_X \notin \operatorname{Im}(X,\xi^d)$;
2. $\operatorname{Im}(X,\xi^d) = \operatorname{End}_\Lambda(X)$, **i.e.** $\delta^*(X) = 0$ is not a simple $\operatorname{End}_\Lambda(X)$-module, in which case $\delta$ splits as $1_X \in \operatorname{Im}(X,\xi^d)$.

Hence $\delta^*(X)$ is a simple $\operatorname{End}_\Lambda(X)$-module if and only if $\delta$ does not split. Then, by Lemma 5.4.2, we conclude that $\delta$ does not split if and only if $\delta$ is a $d$-Auslander-Reiten sequence in $\mathcal{X}$. □

**Corollary 5.4.5.** Assume $\mathcal{X}$ is precovering in $\mathcal{F}$. Let $g : Y \to \tau_d(X)$ be an $\mathcal{X}$-cover, where $X$ is an indecomposable in $\mathcal{X}$ with $\underline{\operatorname{End}}_\Lambda(X)$ a division ring. Consider a non-split $d$-exact sequence with terms in $\mathcal{X}$ of the form:

$$\epsilon : \qquad 0 \longrightarrow Y \xrightarrow{\eta^0} Y^1 \xrightarrow{\eta^1} \cdots \longrightarrow Y^d \xrightarrow{\eta^d} X \longrightarrow 0,$$

where, if $d \geq 2$, we also have $\eta^1, \dots, \eta^{d-1} \in \operatorname{rad}_{\mathcal{X}}$. Consider a morphism induced by a $d$-pushout diagram:

$$\begin{tikzcd}
\epsilon : \arrow[d] & 0 \arrow[r] & Y \arrow[r, "\eta^0"] \arrow[d, "g"] & Y^1 \arrow[r, "\eta^1"] \arrow[d, "g^1"] & \cdots \arrow[r] & Y^d \arrow[r, "\eta^d"] \arrow[d, "g^d"] & X \arrow[r] \arrow[d, equal] & 0 \\
\delta : & 0 \arrow[r] & \tau_d(X) \arrow[r, "\alpha^0"'] & A^1 \arrow[r, "\alpha^1"'] & \cdots \arrow[r] & A^d \arrow[r, "\alpha^d"'] & X \arrow[r] & 0,
\end{tikzcd}$$

where, if $d \geq 2$, we have that $\alpha^1, \dots, \alpha^{d-1} \in \operatorname{rad}_{\mathcal{F}}$. Then $\delta$ is a $d$-Auslander-Reiten sequence in $\mathcal{F}$ and $\eta^d$ is right almost split in $\mathcal{X}$.

*Proof.* First note that in a $d$-pushout diagram of $\epsilon$ along $g$, the middle morphisms $\alpha^1, \dots, \alpha^{d-1}$ are not necessarily in $\operatorname{rad}_{\mathcal{F}}$. However, dropping extra direct summands of the form $A \xrightarrow{\cong} A$, we obtain a $d$-pushout diagram with middle morphisms in $\operatorname{rad}_{\mathcal{F}}$.

Considering Theorem 5.4.4 in the case when $\mathcal{X} = \mathcal{F}$, so that $\sigma X = \tau_d(X)$, we have that $\delta$ is a $d$-Auslander-Reiten sequence in $\mathcal{F}$ if it does not split. Suppose for a contradiction that $\delta$ is a split $d$-exact sequence. Then Lemma 2.3.7 implies that there is a morphism $h : Y^1 \to \tau_d(X)$ such that $h \circ \eta^0 = g$. Moreover, since $Y^1 \in \mathcal{X}$ and $g$ is an $\mathcal{X}$-cover, there is a morphism $\phi : Y^1 \to Y$ such that $h = g \circ \phi$. Hence

$$g = h \circ \eta^0 = g \circ \phi \circ \eta^0,$$

and $\phi \circ \eta^0$ is an isomorphism as $g$ is right minimal. But this implies that $\eta^0$ is a split monomorphism, contradicting our initial assumption. So $\delta$ does not split.

By Proposition 5.3.4(b), we have that $\epsilon$ is isomorphic to the direct sum of a split $d$-exact sequence:

$$0 \longrightarrow Y' \xrightarrow{1_{Y'}} Y' \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

and a non-split $d$-exact sequence:

$$\zeta : \qquad 0 \longrightarrow \sigma X \xrightarrow{\zeta^0} W \xrightarrow{\zeta^1} Y^2 \xrightarrow{\eta^2} \cdots \xrightarrow{\eta^{d-1}} Y^d \xrightarrow{\eta^d} X \longrightarrow 0,$$

where, for $d \geq 2$, we have that $\zeta^1, \eta^2, \ldots, \eta^{d-1}$ are in $\mathrm{rad}_{\mathcal{X}}$. Note that, by Theorem 5.4.4, we have that $\zeta$ is a $d$-Auslander-Reiten sequence in $\mathcal{X}$ and in particular $\eta^d$ is a right almost split morphism in $\mathcal{X}$. □

## 5.5 Example

In this section, we illustrate the results from Section 5.3 for a 2-representation finite algebra $\Lambda$. Here we assume that $\Lambda$ is an algebra over an algebraically closed field $k$ in order to be able to apply [23, Theorem B].

**Definition 5.5.1** ([30, Definition 2.2])**.** The algebra $\Lambda$ is called *d-representation finite* if $\mathrm{gldim}\Lambda \leq d$ and $\Lambda$ has a $d$-cluster tilting object.

Let $\Lambda$ be the algebra defined by the following quiver with relations:

$$\begin{tikzcd}
 & & & 10 \arrow[dr] & & & \\
 & & 9 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & 8 \arrow[dr] & & \\
 & 7 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & 6 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & 5 \arrow[dr] & \\
4 \arrow[ur] \arrow[rr, dotted, no head] & & 3 \arrow[ur] \arrow[rr, dotted, no head] & & 2 \arrow[ur] \arrow[rr, dotted, no head] & & 1.
\end{tikzcd}$$

**Remark 5.5.2.** Note that the algebra $\Lambda$ is 2-representation finite by [29, Theorem 1.18]. Moreover, by Remark 2.3.44, we have that $\operatorname{mod}\Lambda$ has the unique 2-cluster tilting subcategory

$$\mathcal{F} = \operatorname{add}\{\tau_2^j(i) \mid i \text{ injective in } \operatorname{mod}\Lambda \text{ and } j \geq 0\}.$$

Denoting the indecomposable modules in $\operatorname{mod}\Lambda$ by their radical series, we find the Auslander-Reiten quiver of $\mathcal{F}$ is as illustrated in Figure 5.1, see [49, Theorems 3.3 and 3.4], where the dashed arrows show the action of $\tau_2$. We will use [23, theorem B] to find an additive subcategory $\mathcal{X} \subseteq \mathcal{F}$ closed under 2-extensions. Using Theorem 5.3.13, we will then describe the 2-Auslander-Reiten sequences in $\mathcal{X}$.

**Theorem 5.5.3** ([23, Theorem B])**.** *Let $\mathcal{X} \subseteq \mathcal{F}$ be a full subcategory closed under isomorphisms in $\mathcal{F}$. Let $s \in \mathcal{X}$ be a module. Set $\Gamma = \operatorname{End}_\Lambda(s)$ so $s$ acquires the structure ${}_\Gamma s_\Lambda$. Assume the following:*

*(i) the projective dimension satisfies projdim$(s_\Lambda) < \infty$,*

*(ii)* $\operatorname{Ext}_\Lambda^{\geq 1}(s, s) = 0$*,*

*(iii) each $x \in \mathcal{X}$ permits an exact sequence $0 \to p_m \to \cdots \to p_1 \to p_0 \to x \to 0$ in $\operatorname{mod}\Lambda$ with $p_i \in \operatorname{add}(s)$,*

*(iv) $\mathcal{G} := \operatorname{Hom}_\Lambda(s, \mathcal{X}) \subseteq \operatorname{mod}\Gamma$ is d-cluster tilting.*

*Then $\mathcal{X}$ is a wide subcategory of $\mathcal{F}$,* ***i.e.*** *an additive subcategory closed under d-extensions such that every morphism in $\mathcal{X}$ has a d-kernel and a d-cokernel in $\mathcal{F}$ consisting of objects from $\mathcal{X}$.*

Consider the full subcategory of $\mathcal{F}$ closed under isomorphisms in $\mathcal{F}$:

$$\mathcal{X} := \operatorname{add}\left\{ 1,\ \begin{smallmatrix} 8 \\ 5 \\ 1 \end{smallmatrix},\ \begin{smallmatrix} 10 \\ 8 \\ 5 \\ 1 \end{smallmatrix},\ \begin{smallmatrix} & 9 & & \\ 10 & & 6 & \\ & 8 & & 2 \\ & & 5 & \end{smallmatrix},\ \begin{smallmatrix} 4 \\ 7 \\ 9 \\ 10 \end{smallmatrix},\ \begin{smallmatrix} & 6 & \\ 2 & & 8 \\ & 5 & \end{smallmatrix},\ \begin{smallmatrix} 6 \\ 2 \end{smallmatrix},\ \begin{smallmatrix} 9 \\ 6 \\ 2 \end{smallmatrix},\ \begin{smallmatrix} 4 \\ 7 \\ 9 \end{smallmatrix},\ \begin{smallmatrix} 4 \\ 7 \end{smallmatrix} \right\},$$

**i.e.** add of the vertices coloured red in Figure 5.1. Using the following module in $\mathcal{X}$:

$$s := 1 \oplus \begin{smallmatrix} 8 \\ 5 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 10 \\ 8 \\ 5 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 9 & & \\ 10 & & 6 & \\ & 8 & & 2 \\ & & 5 & \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 7 \\ 9 \\ 10 \end{smallmatrix} \oplus \begin{smallmatrix} & 6 & \\ 2 & & 8 \\ & 5 & \end{smallmatrix},$$

let $\Gamma := \operatorname{End}_\Lambda(s)$. We check that the conditions (i)-(iv) from Theorem 5.5.3 hold.

(i) Since $\Lambda$ has finite global dimension, then $s$ has finite projective dimension.

(ii) As $s$ is projective in $\operatorname{mod}\Lambda$, it follows that $\operatorname{Ext}_\Lambda^{\geq 1}(s, s) = 0$.

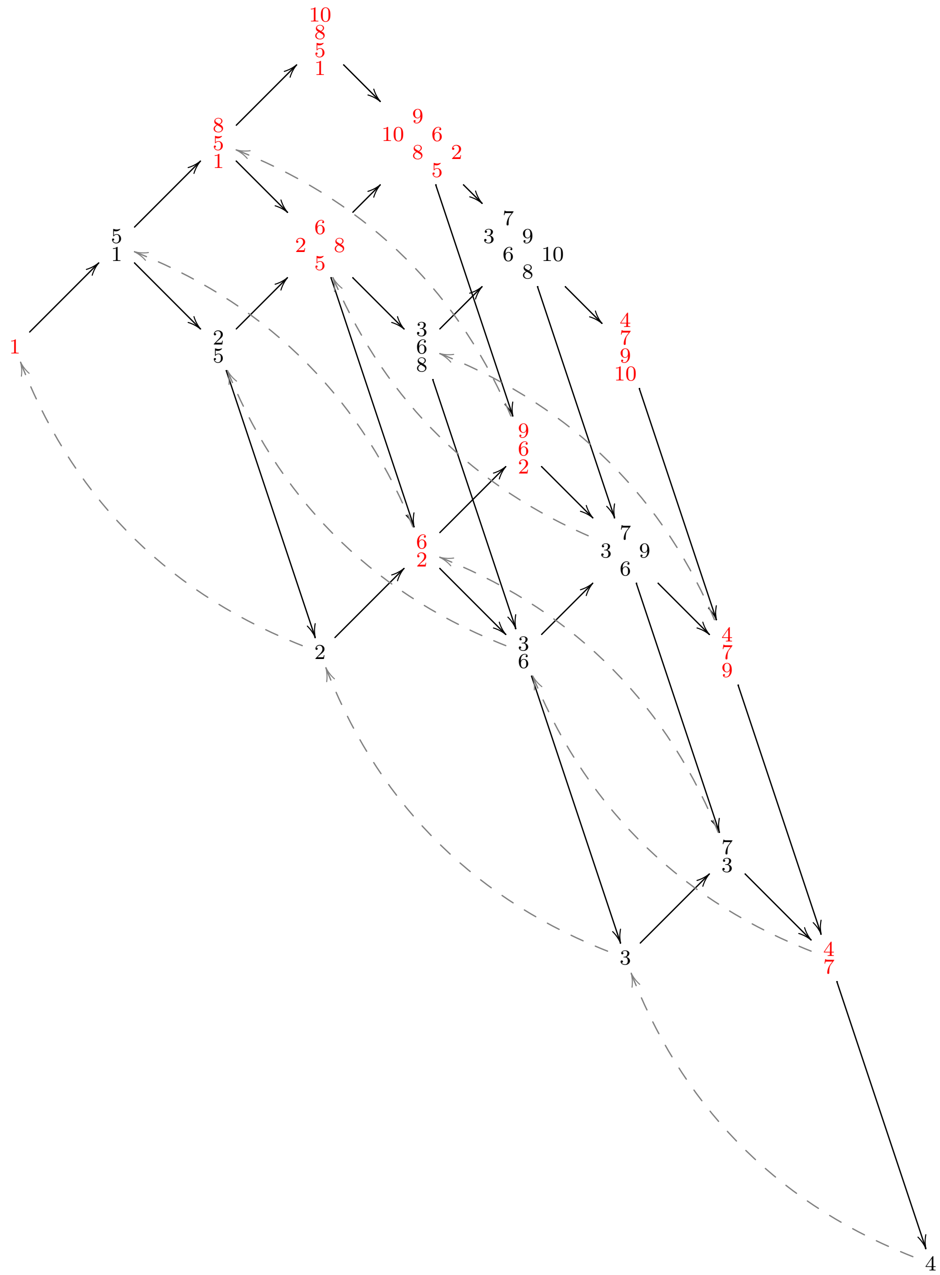


Figure 5.1: The Auslander-Reiten quiver of $\mathcal{F}$.

(iii) When $x \in \mathcal{X}$ is a direct summand of $s$, we have a trivial exact sequence. Moreover, we have the following exact sequences:

$$0 \longrightarrow 1 \longrightarrow \begin{smallmatrix}8\\5\\1\end{smallmatrix} \longrightarrow \begin{smallmatrix} & 6 & \\ 2 & & 8 \\ & 5 & \end{smallmatrix} \longrightarrow \begin{smallmatrix}6\\2\end{smallmatrix} \longrightarrow 0,$$

$$0 \longrightarrow 1 \longrightarrow \begin{smallmatrix}10\\8\\5\\1\end{smallmatrix} \longrightarrow \begin{smallmatrix} & 9 & & \\ 10 & & 6 & \\ & 8 & & 2 \\ & & 5 & \end{smallmatrix} \longrightarrow \begin{smallmatrix}9\\6\\2\end{smallmatrix} \longrightarrow 0,$$

$$0 \longrightarrow \begin{smallmatrix}8\\5\\1\end{smallmatrix} \longrightarrow \begin{smallmatrix}10\\8\\5\\1\end{smallmatrix} \longrightarrow \begin{smallmatrix}4\\7\\9\\10\end{smallmatrix} \longrightarrow \begin{smallmatrix}4\\7\\9\end{smallmatrix} \longrightarrow 0,$$

$$0 \longrightarrow \begin{smallmatrix} & 6 & \\ 2 & & 8 \\ & 5 & \end{smallmatrix} \longrightarrow \begin{smallmatrix} & 9 & & \\ 10 & & 6 & \\ & 8 & & 2 \\ & & 5 & \end{smallmatrix} \longrightarrow \begin{smallmatrix}4\\7\\9\\10\end{smallmatrix} \longrightarrow \begin{smallmatrix}4\\7\end{smallmatrix} \longrightarrow 0,$$

so (iii) is satisfied.

(iv) Consider $\mathcal{G} := \operatorname{Hom}_\Lambda(s, \mathcal{X}) \subseteq \operatorname{mod}\Gamma$. In addition to the idempotents in $\Gamma$ corresponding to the identity morphisms, we have the following irreducible morphisms of add $(s)$:

$$\alpha : \begin{smallmatrix} & 9 & & \\ 10 & & 6 & \\ & 8 & & 2 \\ & & 5 & \end{smallmatrix} \to \begin{smallmatrix}4\\7\\9\\10\end{smallmatrix}, \ \beta : \begin{smallmatrix} & 6 & \\ 2 & & 8 \\ & 5 & \end{smallmatrix} \to \begin{smallmatrix} & 9 & & \\ 10 & & 6 & \\ & 8 & & 2 \\ & & 5 & \end{smallmatrix}, \ \gamma : \begin{smallmatrix}8\\5\\1\end{smallmatrix} \to \begin{smallmatrix} & 6 & \\ 2 & & 8 \\ & 5 & \end{smallmatrix}, \ \delta : 1 \to \begin{smallmatrix}8\\5\\1\end{smallmatrix},$$
$$\epsilon : \begin{smallmatrix}10\\8\\5\\1\end{smallmatrix} \to \begin{smallmatrix} & 9 & & \\ 10 & & 6 & \\ & 8 & & 2 \\ & & 5 & \end{smallmatrix}, \ \zeta : \begin{smallmatrix}8\\5\\1\end{smallmatrix} \to \begin{smallmatrix}10\\8\\5\\1\end{smallmatrix}, \ \beta \circ \gamma = \epsilon \circ \zeta : \begin{smallmatrix}8\\5\\1\end{smallmatrix} \to \begin{smallmatrix} & 9 & & \\ 10 & & 6 & \\ & 8 & & 2 \\ & & 5 & \end{smallmatrix},$$

with $\alpha \circ \beta = 0$ and $\gamma \circ \delta = 0$. Then, using [2, Theorem 3.7, Chapter II], we have that $\Gamma$ is isomorphic to the algebra $\Psi$ defined by the following quiver with relations:

$$\begin{tikzcd}
 & & f \arrow[dr, "\zeta"] & & \\
 & e \arrow[ur, "\epsilon"] \arrow[dr, "\beta"'] \arrow[rr, dotted, no head] & & d \arrow[dr, "\delta"] & \\
c \arrow[ur, "\alpha"] \arrow[rr, dotted, no head] & & b \arrow[ur, "\gamma"'] \arrow[rr, dotted, no head] & & a.
\end{tikzcd}$$

We look at $\mathcal{G}$ in terms of quiver representations, using [2, Theorem 1.6, Chapter III]. So for example, using again the radical series notation, we have

$$\operatorname{Hom}_\Lambda\left(s, \begin{smallmatrix} & 9 & & \\ 10 & & 6 & \\ & 8 & & 2 \\ & & 5 & \end{smallmatrix}\right) = \begin{smallmatrix} & e & \\ b & & f \\ & d & \end{smallmatrix}.$$

Similarly, we find the radical series of $\operatorname{Hom}_\Lambda(s, x)$ for each indecomposable $x \in \mathcal{X}$. Then, using these, it is easy to see that the Auslander-Reiten quiver of $\mathcal{G}$ is as shown in Figure 5.2. By [39, Remark B.5], we conclude that, viewed in terms of quiver representations, $\mathcal{G}$ is the unique 2-cluster tilting subcategory of $\operatorname{mod}\Psi$. Hence $\mathcal{G} \subseteq \operatorname{mod}\Gamma$ is 2-cluster tilting.

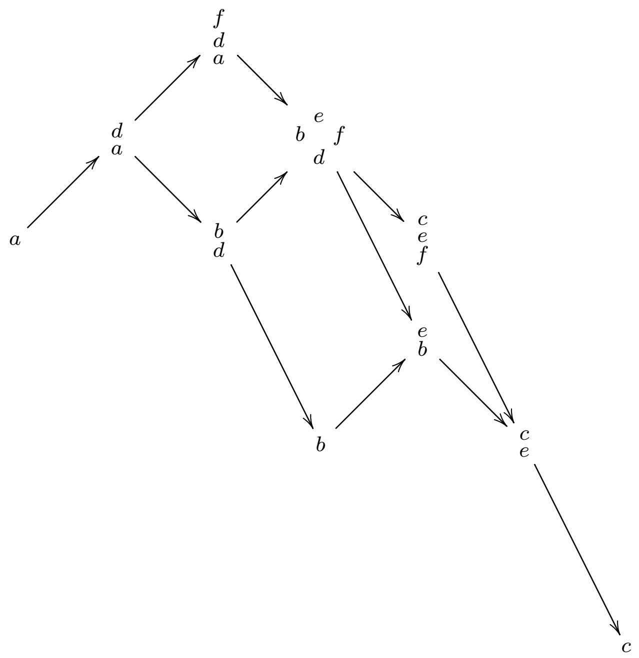



Figure 5.2: The Auslander-Reiten quiver of $\mathcal{G}$.

So (i)-(iv) from Theorem 5.5.3 hold and we have that $\mathcal{X}$ is a wide subcategory of $\mathcal{F}$ in the sense of [23, Definition 2.11]. In particular, $\mathcal{X} \subseteq \mathcal{F}$ is an additive subcategory closed under 2-extensions.

Looking at the Auslander-Reiten quiver of $\mathcal{F}$, we see that the following are the 2-Auslander-Reiten sequences in $\mathcal{F}$ with right end term in $\mathcal{X}$:

$$0 \longrightarrow \begin{smallmatrix}8\\5\\1\end{smallmatrix} \longrightarrow \begin{smallmatrix}&6&\\2&&8\\&5&\end{smallmatrix} \oplus \begin{smallmatrix}10\\8\\5\\1\end{smallmatrix} \longrightarrow \begin{smallmatrix}6\\2\end{smallmatrix} \oplus \begin{smallmatrix}&9&&\\10&&6&\\&8&&2\\&&5&\end{smallmatrix} \longrightarrow \begin{smallmatrix}9\\6\\2\end{smallmatrix} \longrightarrow 0 \tag{a}$$

$$0 \longrightarrow \begin{smallmatrix}5\\1\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\5\end{smallmatrix} \oplus \begin{smallmatrix}8\\5\\1\end{smallmatrix} \longrightarrow 2 \oplus \begin{smallmatrix}&6&\\2&&8\\&5&\end{smallmatrix} \longrightarrow \begin{smallmatrix}6\\2\end{smallmatrix} \longrightarrow 0, \tag{b}$$

$$0 \longrightarrow \begin{smallmatrix}3\\6\\8\end{smallmatrix} \longrightarrow \begin{smallmatrix}3\\6\end{smallmatrix} \oplus \begin{smallmatrix}&7&&\\3&&9&\\&6&&10\\&&8&\end{smallmatrix} \longrightarrow \begin{smallmatrix}&7&\\3&&9\\&6&\end{smallmatrix} \oplus \begin{smallmatrix}4\\7\\9\\10\end{smallmatrix} \longrightarrow \begin{smallmatrix}4\\7\\9\end{smallmatrix} \longrightarrow 0, \tag{c}$$

$$0 \longrightarrow \begin{smallmatrix}3\\6\end{smallmatrix} \longrightarrow 3 \oplus \begin{smallmatrix}&7&\\3&&9\\&6&\end{smallmatrix} \longrightarrow \begin{smallmatrix}7\\3\end{smallmatrix} \oplus \begin{smallmatrix}4\\7\\9\end{smallmatrix} \longrightarrow \begin{smallmatrix}4\\7\end{smallmatrix} \longrightarrow 0. \tag{d}$$

Note that all the terms in (a) are in $\mathcal{X}$, so (a) is also a 2-Auslander-Reiten sequence in $\mathcal{X}$.

Moreover, the following are $\mathcal{X}$-covers:

$$1 \to \begin{smallmatrix}5\\1\end{smallmatrix},\ \ \begin{smallmatrix}&6&\\2&&8\\&5&\end{smallmatrix} \to \begin{smallmatrix}3\\6\\8\end{smallmatrix},\ \ \begin{smallmatrix}6\\2\end{smallmatrix} \to \begin{smallmatrix}3\\6\end{smallmatrix}.$$

Then, using these covers, (b), (c), (d), Theorem 5.3.13 and the fact that the relevant $\mathrm{Ext}^2$-spaces are one dimensional by [49, Theorem 3.6], we find the 2-Auslander-Reiten sequences in $\mathcal{X}$:

$$0 \longrightarrow 1 \longrightarrow \begin{smallmatrix}8\\5\\1\end{smallmatrix} \longrightarrow \begin{smallmatrix}&6&\\2&&8\\&5&\end{smallmatrix} \longrightarrow \begin{smallmatrix}6\\2\end{smallmatrix} \longrightarrow 0, \tag{b'}$$

$$0 \longrightarrow \begin{smallmatrix}&6&\\2&&8\\&5&\end{smallmatrix} \longrightarrow \begin{smallmatrix}&9&&\\10&&6&\\&8&&2\\&&5&\end{smallmatrix} \oplus \begin{smallmatrix}6\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}9\\6\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\7\\9\\10\end{smallmatrix} \longrightarrow \begin{smallmatrix}4\\7\\9\end{smallmatrix} \longrightarrow 0, \tag{c'}$$

$$0 \longrightarrow \begin{smallmatrix}6\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}9\\6\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}4\\7\\9\end{smallmatrix} \longrightarrow \begin{smallmatrix}4\\7\end{smallmatrix} \longrightarrow 0. \tag{d'}$$

# Chapter 6

# Grothendieck groups of triangulated categories via cluster tilting subcategories

## 6.1 Introduction

Let $k$ be a field and $\mathcal{C}$ be a $k$-linear, Hom-finite triangulated category with split idempotents and suspension functor $\Sigma$. We denote the split Grothendieck group of an additive category $\mathcal{A}$ by $K_0^{sp}(\mathcal{A})$ and the Grothendieck group of an abelian or triangulated category $\mathcal{B}$ by $K_0(\mathcal{B})$.

We first present two previous results, one by Xiao and Zhu and the other one by Palu that in some sense are the base case of the results we present in this chapter. Note that both Xiao and Zhu and Palu assume that $k$ is an algebraically closed field. However, this assumption is not needed for our higher versions and $k$ is a general field in our setup.

**Previous results.** In [57, Theorem 2.1], Xiao and Zhu presented triangulated analogues of results of Auslander [4, Theorems 2.2 and 2.3] and Butler [12, Theorem in introduction].

**Theorem (Xiao and Zhu).** *Let $\mathcal{C}$ be a triangulated category of finite type, then*

$$K_0(\mathcal{C}) \cong K_0^{sp}(\mathcal{C})/\langle -[A] + [B] - [C] \mid C \in \operatorname{Ind}\mathcal{C} \textit{ with Auslander-Reiten triangle } A \to B \to C \rangle.$$

We can think of the $\mathcal{C}$ appearing on the right side as the only possible 1-cluster tilting subcategory of $\mathcal{C}$. In this chapter, we are interested in higher-cluster tilting subcategories.

The first higher case occurs when $\mathcal{C}$ has a (2-)cluster tilting subcategory. Palu studied this case, in a more specific setup, in [50]. In fact, Palu assumes that $\mathcal{C}$ is the stable category of a Frobenius $k$-linear category with split idempotents, and that $\mathcal{C}$ is 2-Calabi-Yau with a (2-)cluster tilting subcategory $\mathcal{T}$.

Given an indecomposable $M$ in $\mathcal{T}$, let $\widetilde{\mathcal{T}}$ be the additive subcategory of $\mathcal{T}$ whose indecomposables are the same as $\mathcal{T}$, excluding those isomorphic to $M$. Then, up to isomorphism, there is a unique indecomposable $M^* \notin \mathcal{T}$ such that $\operatorname{add}(\widetilde{\mathcal{T}} \cup M^*) \subseteq \mathcal{C}$ is (2-)cluster tilting. Moreover, $M$ and $M^*$ appear in two triangles with certain properties, called *exchange triangles*, of the form:

$$M^* \to B_M \to M \to \Sigma M^* \text{ and } M \to B_{M^*} \to M^* \to \Sigma M,$$

where $B_M$ and $B_{M^*}$ are in $\widetilde{\mathcal{T}}$. Palu proved the following in [50, Theorem 10].

**Theorem (Palu).** *We have that* $K_0(\mathcal{C}) \cong K_0^{sp}(\mathcal{T})/\langle [B_{M^*}] - [B_M]\rangle_M$.

Note that if the Auslander-Reiten quiver of $\mathcal{T}$ has no loops, then the indecomposable $M \in \mathcal{T}$ has Auslander-Reiten 4-angle $M \to B_{M^*} \to B_M \to M$ in the sense of [32, Theorem 3.8]. So Palu's theorem is a higher version of Xiao and Zhu's theorem.

We present "higher cluster tilting" versions of Xiao and Zhu and Palu's results. Moreover, we present a "higher angulated" version of Palu's result.

**Higher-cluster tilting versions.** Let $n \geq 2$ be an integer and assume that $\mathcal{C}$ has a Serre functor $\mathbb{S}$ and an $n$-cluster tilting subcategory $\mathcal{T}$, see Definition 2.3.42. Let $\operatorname{Ind}\mathcal{T}$ be a full subcategory of $\mathcal{T}$ containing precisely one object from each isomorphism class of indecomposables in $\mathcal{T}$ and assume that $\operatorname{Ind}\mathcal{T}$ is locally bounded, see Definition 6.4.2. Recall that the functor $\mathbb{S}_n := \mathbb{S}\Sigma^{-n}$ and Auslander-Reiten $(n+2)$-angles in $\mathcal{T}$ were introduced in [32, Section 3].

**Theorem 6.4.9.** *We have that* $K_0(\mathcal{C})$ *is isomorphic to*

$$K_0^{sp}(\mathcal{T}) \Big/ \left\langle \begin{matrix} -[M] + (-1)^n[\mathbb{S}_n(M)] + \\ \sum_{i=0}^{n-1}(-1)^i [T_i] \end{matrix} \;\middle|\; \begin{matrix} M \in \operatorname{Ind}\mathcal{T} \textit{ with Auslander-Reiten } (n+2)\textit{-angle} \\ \mathbb{S}_n(M) \to T_{n-1} \to \cdots \to T_0 \to M \to \mathbb{S}(M) \end{matrix} \right\rangle.$$

The arguments we use to prove Theorem 6.4.9 rely on $n \geq 2$. However, note that when $k$ is an algebraically closed field, the case $n = 1$ is still true and it is an instance of Xiao and Zhu's theorem.

If we add the extra assumptions that $n$ is even and $\mathcal{C}$ is $n$-Calabi-Yau, we obtain the following because $\mathbb{S}_n \cong 1_{\mathcal{C}}$.

**Corollary 6.4.11.** *We have that*

$$K_0(\mathcal{C}) \cong K_0^{sp}(\mathcal{T}) \Big/ \left\langle \sum_{i=0}^{n-1} (-1)^i [T_i] \;\middle|\; \begin{array}{c} M \in \operatorname{Ind}\mathcal{T} \textit{ with Auslander-Reiten } (n+2)\textit{-angle} \\ M \to T_{n-1} \to \cdots \to T_0 \to M \to \Sigma^n M \end{array} \right\rangle.$$

When $n = 2$ and the Auslander-Reiten quiver of $\mathcal{T}$ has no loops, then Corollary 6.4.11 and Palu's theorem coincide.

**Higher-angulated version.** Let $d \geq 1$ be an integer and assume that $\mathcal{C}$ has a $d$-cluster tilting subcategory $\mathcal{S}$ such that $\Sigma^d \mathcal{S} = \mathcal{S}$. Note that $\mathcal{S}$ is a $(d+2)$-angulated category with $d$-suspension $\Sigma^d$, by Theorem 2.3.43. Similarly to the way $K_0(\mathcal{C})$ is defined, one can define the Grothendieck group of the $(d+2)$-angulated category $\mathcal{S}$ as

$$K_0(\mathcal{S}) \coloneqq K_0^{sp}(\mathcal{S}) \Big/ \left\langle \sum_{i=0}^{d+1} (-1)^i [S_i] \mid S_{d+1} \to \cdots \to S_0 \to \Sigma^d S_{d+1} \text{ is a } (d+2)\text{-angle in } \mathcal{S} \right\rangle,$$

see [9, Definition 2.1]. We prove that this is isomorphic to the Grothendieck group of $\mathcal{C}$.

**Theorem 6.5.7.** *We have that $K_0(\mathcal{C}) \cong K_0(\mathcal{S})$.*

Let $n = 2d$. We now add the assumptions that $\mathcal{C}$ has a Serre functor $\mathbb{S}$ and that there is an $n$-cluster tilting subcategory $\mathcal{T} \subseteq \mathcal{C}$ such that $\mathcal{T} \subseteq \mathcal{S}$ and $\operatorname{Ind}\mathcal{T}$ is locally bounded. By [49, Theorem 5.26], we have that $\mathcal{T} \subseteq \mathcal{S}$ is an Oppermann-Thomas cluster tilting subcategory, **i.e.** the corresponding concept in a $(d+2)$-angulated category of a cluster tilting subcategory in a triangulated category. Theorems 6.4.9 and 6.5.7 have the following immediate consequence.

**Theorem 6.6.4.** *We have that $K_0(\mathcal{C}) \cong K_0(\mathcal{S})$ and*

$$K_0(\mathcal{S}) \cong K_0^{sp}(\mathcal{T}) \Big/ \left\langle \begin{array}{c} -[M] + [\mathbb{S}_n(M)] + \\ \sum_{i=0}^{n-1} (-1)^i [T_i] \end{array} \;\middle|\; \begin{array}{c} M \in \operatorname{Ind}\mathcal{T} \textit{ with Auslander-Reiten } (n+2)\textit{-angle} \\ \mathbb{S}_n(M) \to T_{n-1} \to \cdots \to T_0 \to M \to \mathbb{S}(M) \end{array} \right\rangle.$$

If we add the extra assumption that $\mathcal{C}$ is $n$-Calabi-Yau, we obtain the following.

**Corollary 6.6.5.** We have that

$$K_0(\mathcal{S}) \cong K_0^{sp}(\mathcal{T}) \Big/ \left\langle \sum_{i=0}^{n-1} (-1)^i [T_i] \;\middle|\; \begin{array}{c} M \in \operatorname{Ind}\mathcal{T} \text{ with Auslander-Reiten } (n+2)\text{-angle} \\ M \to T_{n-1} \to \cdots \to T_0 \to M \to \Sigma^n M \end{array} \right\rangle.$$

When $d = 1$, we have that $\mathcal{S} = \mathcal{C}$ is a triangulated category with (2-)cluster tilting subcategory $\mathcal{T}$ and, adding the extra assumption that the Auslander-Reiten quiver of $\mathcal{T}$ has no loops, Corollary 6.6.5 becomes Palu's theorem. For higher values of $d$, Corollary 6.6.5 proves a higher angulated version of Palu's theorem.

We conclude this chapter by illustrating our results in two examples: one for each of the higher versions of Palu's theorem. Let $q$ and $p$ be integers and $q$ be odd. Consider the triangulated $q$-cluster category of Dynkin type $A_p$, denoted by $\mathcal{C}_q(A_p)$ and introduced in Section 3.2, and note that this is a $(q+1)$-Calabi-Yau category. Since $q+1$ is even by assumption, we can apply Corollary 6.4.11 to show that

$$K_0(\mathcal{C}_q(A_p)) \cong \begin{cases} 0, & \text{if } p \text{ is even,} \\ \mathbb{Z}, & \text{if } p \text{ is odd.} \end{cases}$$

We then consider a higher Auslander algebra $A_3^2$ of Dynkin type $A$ and its Amiot cluster category $\mathcal{C}^4(A_3^2)$, to find an example of categories

$$\mathcal{T} \subseteq \mathcal{O}(A_3^2) \subseteq \mathcal{C}^4(A_3^2),$$

such that $\mathcal{C}^4(A_3^2)$ is triangulated and 4-Calabi-Yau, $\mathcal{O}(A_3^2)$ is closed under $\Sigma^2$ and 2-cluster tilting in $\mathcal{C}^4(A_3^2)$ and $\mathcal{T}$ is 4-cluster tilting in $\mathcal{C}^4(A_3^2)$. Applying Theorem 6.5.7 and Corollary 6.6.5 to this example, we find that

$$K_0(\mathcal{C}^4(A_3^2)) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

This chapter is organised as follows. Section 6.2 recalls some definitions and results and presents our setup. Section 6.3 introduces some morphisms between Grothendieck groups that will be useful in the rest of the chapter. Section 6.4 proves Theorem 6.4.9. Section 6.5 proves Theorem 6.5.7. Section 6.6 presents Corollary 6.6.5. Finally, Sections 6.7 and 6.8 illustrate our two examples.

## 6.2 Setup and definitions

**Definition 6.2.1.** Let $\mathcal{A}$ be an essentially small additive category and $G(\mathcal{A})$ be the free abelian group on isomorphism classes $[A]$ of objects $A \in \mathcal{A}$. We define the *split Grothendieck group of* $\mathcal{A}$ to be

$$K_0^{sp}(\mathcal{A}) \coloneqq G(\mathcal{A})/\langle [A \oplus B] - [A] - [B] \rangle.$$

When $\mathcal{A}$ is abelian or triangulated, we can also define the *Grothendieck group of* $\mathcal{A}$ as

$$K_0(\mathcal{A}) := K_0^{sp}(\mathcal{A})/\langle [A]-[B]+[C] \mid 0 \to A \to B \to C \to 0 \text{ is a short exact sequence in } \mathcal{A}\rangle \text{ or}$$
$$K_0(\mathcal{A}) := K_0^{sp}(\mathcal{A})/\langle [A]-[B]+[C] \mid A \to B \to C \to \Sigma A \text{ is a triangle in } \mathcal{A}\rangle,$$

respectively.

In a similar way, one can define the Grothendieck group of a $(d+2)$-angulated category $\mathcal{S}$ as follows.

**Definition 6.2.2.** Let $d$ be a positive integer. The *Grothendieck group of a* $(d+2)$-*angulated category* $\mathcal{S}$ with $d$-suspension functor $\Sigma^d$ is defined to be

$$K_0(\mathcal{S}) := K_0^{sp}(\mathcal{S}) \Big/ \Big\langle \sum_{i=0}^{d+1} (-1)^i [S_i] \mid S_{d+1} \to \cdots \to S_0 \to \Sigma^d S_{d+1} \text{ is a } (d+2)\text{-angle in } \mathcal{S} \Big\rangle.$$

**Remark 6.2.3.** The definition of the Grothendieck group of a $(d+2)$-angulated category $\mathcal{S}$ we just introduced is different from the original one, see [9, Definition 2.1], however the two definitions are equivalent. In fact, since we define $K_0(\mathcal{S})$ as a quotient of $K_0^{sp}(\mathcal{S})$, the relation $[0]=0$ holds. On the other hand, in [9], $K_0(\mathcal{S})$ is defined as a quotient of the free abelian group on the set of isomorphism classes of objects in $\mathcal{S}$ and the relation $[0]=0$ has to be manually added when $d$ is even.

Recall the concept of $m$-cluster tilting subcategory $\mathcal{U}$ of a triangulated category $\mathcal{C}$, see Definition 2.3.42.

**Remark 6.2.4.** Consider the case of a 1-cluster tilting subcategory $\mathcal{U}$ of $\mathcal{C}$, then the conditions $\operatorname{Ext}_{\mathcal{C}}^{1 \dots 1-1}(\mathcal{U}, C)=0$ and $\operatorname{Ext}_{\mathcal{C}}^{1 \dots 1-1}(C,\mathcal{U})=0$ are empty and $\mathcal{C}=\mathcal{U}$ is the only possible 1-cluster tilting subcategory.

**Setup 6.2.5.** Let $m \geq 2$ be an integer and $\mathcal{U}$ an $m$-cluster tilting subcategory of $\mathcal{C}$.

**Definition 6.2.6** ([32, Definition 2.9])**.** A $\mathcal{U}$-*module* is a contravariant $k$-linear functor of the form $G:\mathcal{U} \to \operatorname{Mod} k$. Then $\mathcal{U}$-modules form an abelian category denoted $\operatorname{Mod}\mathcal{U}$. We say that $G \in \operatorname{Mod}\mathcal{U}$ is *coherent* if there exists an exact sequence of the form

$$\mathcal{U}(-,U_1) \to \mathcal{U}(-,U_0) \to G(-) \to 0,$$

for some $U_1, U_0 \in \mathcal{U}$. We denote by $\operatorname{mod}\mathcal{U}$ the full subcategory of $\operatorname{Mod}\mathcal{U}$ consisting of coherent $\mathcal{U}$-modules.

**Definition 6.2.7.** There is a homological functor

$$F_{\mathcal{U}} : \mathcal{C} \to \operatorname{mod}\mathcal{U}, \;\; C \mapsto \mathcal{C}(-,C)|_{\mathcal{U}}.$$

**Remark 6.2.8.** Note that a priori the target of $F_{\mathcal{U}}$ should be $\operatorname{Mod}\mathcal{U}$. However, since $\mathcal{C} = \mathcal{U} * \Sigma\mathcal{U} * \cdots * \Sigma^{m-1}\mathcal{U}$ by [32, Theorem 3.1], any object $C \in \mathcal{C}$ appears in a triangle of the form

$$\Sigma^{-1}B \to A \to C \to B,$$

where $A \in \mathcal{U} * \Sigma\mathcal{U}$ and $B \in \Sigma^2\mathcal{U} * \cdots * \Sigma^{m-1}\mathcal{U}$. Applying $F_{\mathcal{U}}$ to this triangle, we obtain the exact sequence

$$0 \to F_{\mathcal{U}}(A) \xrightarrow{\cong} F_{\mathcal{U}}(C) \to 0,$$

where $F_{\mathcal{U}}(A) \in \operatorname{mod}\mathcal{U}$ by [32, Proposition 6.2(3)] and so $F_{\mathcal{U}}(C) \in \operatorname{mod}\mathcal{U}$.

**Definition 6.2.9.** If $\mathcal{A}$ and $\mathcal{B}$ are full subcategories of $\mathcal{C}$, then

$$\mathcal{A} * \mathcal{B} = \{C \in \mathcal{C} \mid \text{there is a triangle } A \to C \to B \to \Sigma A \text{ with } A \in \mathcal{A},\ B \in \mathcal{B}\}.$$

We will often use towers of triangles, as defined below. These were first introduced in [32, Notation 3.2], see also [40, Definition 3.1].

**Definition 6.2.10.** A *tower of triangles in* $\mathcal{C}$ is a diagram of the form

$$\begin{tikzcd}
 & C_{l-1} \arrow[rr] \arrow[dr] & & C_{l-2} \arrow[r] \arrow[dr] & {} & \cdots & {} \arrow[r] & C_2 \arrow[rr] \arrow[dr] & & C_1 \arrow[dr] & \\
C_l \arrow[ur] & & X_{l-2} \arrow[ll, squiggly] \arrow[ur] & & X_{l-3} \arrow[ll, squiggly] & \cdots & X_2 \arrow[ur] & & X_1 \arrow[ll, squiggly] \arrow[ur] & & C_0, \arrow[ll, squiggly]
\end{tikzcd}$$

where $l \geq 1$ is an integer, a wavy arrow $X \rightsquigarrow Y$ signifies a morphism $X \to \Sigma Y$, each oriented triangle is a triangle in $\mathcal{C}$ and each non-oriented triangle is commutative.

**Definition 6.2.11** ([40, Definition 3.3])**.** By [32, Corollary 3.3], for $C \in \mathcal{C}$ there is a tower of triangles in $\mathcal{C}$ of the form

$$\begin{tikzcd}
 & U_{m-2} \arrow[rr] \arrow[dr, "\mu_{m-2}"] & & U_{m-3} \arrow[r] \arrow[dr, "\mu_{m-3}"] & {} & \cdots & {} \arrow[r] & U_1 \arrow[rr] \arrow[dr, "\mu_1"] & & U_0 \arrow[dr, "\mu_0"] & \\
U_{m-1} \arrow[ur] & & X_{m-2} \arrow[ll, squiggly] \arrow[ur] & & X_{m-3} \arrow[ll, squiggly] & \cdots & X_2 \arrow[ur] & & X_1 \arrow[ll, squiggly] \arrow[ur] & & C, \arrow[ll, squiggly]
\end{tikzcd}$$

where $U_i \in \mathcal{U}$ and $\mu_i$ is a $\mathcal{U}$-cover for each $i$. In particular, the $U_i$ are determined up to isomorphism. The *index of* $C$ *with respect to* $\mathcal{U}$ is the following element of the Grothendieck group $K_0^{sp}(\mathcal{U})$:

$$\operatorname{index}_{\mathcal{U}}(C) = \sum_{i=0}^{m-1} (-1)^i [U_i].$$

**Remark 6.2.12.** Note that $\mathrm{index}_{\mathcal{U}} : \mathrm{Obj}(\mathcal{C}) \to K_0^{sp}(\mathcal{U})$ induces a homomorphism $K_0^{sp}(\mathcal{C}) \to K_0^{sp}(\mathcal{U})$ which we also denote by $\mathrm{index}_{\mathcal{U}}$.

**Remark 6.2.13.** Note that we work in a more general setup than [40], as we dropped the assumption that $\mathcal{U} = \mathrm{add}\, U$ for some $U \in \mathcal{C}$, or in other words that $\mathcal{U}$ has finitely many indecomposables up to isomorphism. The arguments from [40] can be easily adjusted in this more general setup and the main results still hold. In particular, we state the corresponding definition of the homomorphism $\theta$ from [40, Definition 4.1] and [40, Theorem 4.5] in our setup.

**Definition 6.2.14.** There is a homomorphism

$$\theta_{\mathcal{U}} : K_0(\mathrm{mod}\mathcal{U}) \to K_0^{sp}(\mathcal{U}),$$

defined by

$$\theta_{\mathcal{U}}([F_{\mathcal{U}}N]) = \mathrm{index}_{\mathcal{U}}(\Sigma^{-1}N) + \mathrm{index}_{\mathcal{U}}(N)$$

for $N \in \mathcal{U} * \Sigma\mathcal{U}$.

**Remark 6.2.15.** Note that the fact that $\theta_{\mathcal{U}}$ from Definition 6.2.14 is well-defined can be proved using the equivalence of categories $(\mathcal{U} * \Sigma\mathcal{U})/[\Sigma\mathcal{U}] \cong \mathrm{mod}\mathcal{U}$ from [32, Proposition 6.2(3)] and the general versions of [40, Lemmas 4.3, 4.4], see [40, Remark 4.2].

As mentioned in Remark 6.2.13, the argument proving [40, Theorem 4.5] can be adjusted to prove the same result in our setup. We state it here for the convenience of the reader.

**Proposition 6.2.16** ([40, Theorem 4.5])**.** If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$ is a triangle in $\mathcal{C}$, then

$$\mathrm{index}_{\mathcal{U}}(B) = \mathrm{index}_{\mathcal{U}}(A) + \mathrm{index}_{\mathcal{U}}(C) - \theta_{\mathcal{U}}(\mathrm{Im}\, F_{\mathcal{U}}(\gamma)).$$

## 6.3 Morphisms between Grothendieck groups

**Definition 6.3.1.** There are surjective homomorphisms given by the quotient maps

$$\pi_{\mathcal{C}} : K_0^{sp}(\mathcal{C}) \longrightarrow K_0(\mathcal{C}),\ \pi_{\mathcal{U}} : K_0^{sp}(\mathcal{U}) \longrightarrow K_0^{sp}(\mathcal{U})/\mathrm{Im}\,\theta_{\mathcal{U}},$$

and injective homomorphisms given by the inclusions

$$\iota_{\mathcal{C}} : \mathrm{Ker}\,\pi_{\mathcal{C}} \to K_0^{sp}(\mathcal{C}),\ \iota_{\mathcal{U}} : \mathrm{Ker}\,\pi_{\mathcal{U}} \to K_0^{sp}(\mathcal{U}) \text{ and } j_{\mathcal{U}} : K_0^{sp}(\mathcal{U}) \to K_0^{sp}(\mathcal{C}).$$

**Remark 6.3.2.** Consider the diagram:

$$\begin{tikzcd}
\operatorname{Ker}\pi_{\mathcal{C}} \arrow[d, hook, "\iota_{\mathcal{C}}"'] & & \operatorname{Ker}\pi_{\mathcal{U}} \arrow[d, hook, "\iota_{\mathcal{U}}"] \\
K_0^{sp}(\mathcal{C}) \arrow[rr, bend left, "\mathrm{index}_{\mathcal{U}}"] \arrow[d, two heads, "\pi_{\mathcal{C}}"'] & & K_0^{sp}(\mathcal{U}) \arrow[ll, bend left, "j_{\mathcal{U}}"] \arrow[d, two heads, "\pi_{\mathcal{U}}"] \\
K_0(\mathcal{C}) \arrow[rr, bend left, "f_{\mathcal{U}}"] & & K_0^{sp}(\mathcal{U})/\operatorname{Im}\theta_{\mathcal{U}}. \arrow[ll, bend left, dashed, "\exists\, g_{\mathcal{U}}\,?"]
\end{tikzcd} \tag{6.1}$$

It is clear that $\mathrm{index}_{\mathcal{U}} \circ j_{\mathcal{U}} = 1_{K_0^{sp}(\mathcal{U})}$ and so $\pi_{\mathcal{U}} \circ \mathrm{index}_{\mathcal{U}} \circ j_{\mathcal{U}} = \pi_{\mathcal{U}}$. We will show that $\pi_{\mathcal{C}} \circ j_{\mathcal{U}} \circ \mathrm{index}_{\mathcal{U}} = \pi_{\mathcal{C}}$ and that there exists a morphism $f_{\mathcal{U}} : K_0(\mathcal{C}) \to K_0^{sp}(\mathcal{U})/\operatorname{Im}\theta_{\mathcal{U}}$ such that

$$f_{\mathcal{U}} \circ \pi_{\mathcal{C}} = \pi_{\mathcal{U}} \circ \mathrm{index}_{\mathcal{U}}.$$

Moreover, adding some assumptions on $\mathcal{C}$ and/or $\mathcal{U}$, we will prove that there exists a morphism

$$g_{\mathcal{U}} : K_0^{sp}(\mathcal{U})/\operatorname{Im}\theta_{\mathcal{U}} \to K_0(\mathcal{C})$$

such that $g_{\mathcal{U}} \circ \pi_{\mathcal{U}} = \pi_{\mathcal{C}} \circ j_{\mathcal{U}}$. In this case, $f_{\mathcal{U}}$ and $g_{\mathcal{U}}$ become inverse isomorphisms. In the next sections, we consider different sets of extra assumptions under which such a $g_{\mathcal{U}}$ exists.

**Lemma 6.3.3.** We have that $\pi_{\mathcal{C}} \circ j_{\mathcal{U}} \circ \mathrm{index}_{\mathcal{U}} = \pi_{\mathcal{C}}$.

*Proof.* Given any object $C \in \mathcal{C}$, consider the tower of triangles from Definition 6.2.11. We have that

$$\mathrm{index}_{\mathcal{U}}(C) = \sum_{i=0}^{m-1} (-1)^i [U_i].$$

Using the relations in $K_0(\mathcal{C})$ corresponding to the triangles in the tower, we have that

$$\pi_{\mathcal{C}} \circ j_{\mathcal{U}} \circ \mathrm{index}_{\mathcal{U}}([C]) = \pi_{\mathcal{C}}\left( \sum_{i=0}^{m-1} (-1)^i [U_i] \right) = [C] = \pi_{\mathcal{C}}([C]).$$

Since this is true for arbitrary $C \in \mathcal{C}$, we conclude that $\pi_{\mathcal{C}} \circ j_{\mathcal{U}} \circ \mathrm{index}_{\mathcal{U}} = \pi_{\mathcal{C}}$. $\square$

**Lemma 6.3.4.** There is a homomorphism $f_{\mathcal{U}} : K_0(\mathcal{C}) \to K_0^{sp}(\mathcal{U})/\operatorname{Im}\theta_{\mathcal{U}}$ such that

$$f_{\mathcal{U}} \circ \pi_{\mathcal{C}} = \pi_{\mathcal{U}} \circ \operatorname{index}_{\mathcal{U}}.$$

*Proof.* There exists a homomorphism $f_{\mathcal{U}}$ with the desired property if and only if $\pi_{\mathcal{U}} \circ \operatorname{index}_{\mathcal{U}} \circ \iota_{\mathcal{C}} = 0$. Note that

$$\operatorname{Ker}\pi_{\mathcal{C}} = \langle [A] - [B] + [C] \mid A \to B \to C \xrightarrow{\gamma} \Sigma A \text{ is a triangle in } \mathcal{C} \rangle.$$

For any generator $[A]-[B]+[C]$ of $\operatorname{Ker}\pi_{\mathcal{C}}$ corresponding to a triangle $A \to B \to C \xrightarrow{\gamma} \Sigma A$ in $\mathcal{C}$, we have that

$$\begin{aligned}\pi_{\mathcal{U}} \circ \operatorname{index}_{\mathcal{U}} \circ \iota_{\mathcal{C}}([A] - [B] + [C]) &= \pi_{\mathcal{U}}(\operatorname{index}_{\mathcal{U}}(A) - \operatorname{index}_{\mathcal{U}}(B) + \operatorname{index}_{\mathcal{U}}(C)) \\ &= \pi_{\mathcal{U}}(\theta_{\mathcal{U}}([\operatorname{Im} F_{\mathcal{U}}(\gamma)])) = 0,\end{aligned}$$

where the second equality is obtained by Proposition 6.2.16. Hence $\pi_{\mathcal{U}} \circ \operatorname{index}_{\mathcal{U}} \circ \iota_{\mathcal{C}} = 0$ as desired. □

**Proposition 6.3.5.** Suppose there exists a homomorphism $g_{\mathcal{U}} : K_0^{sp}(\mathcal{U})/\operatorname{Im}\theta_{\mathcal{U}} \to K_0(\mathcal{C})$ such that $g_{\mathcal{U}} \circ \pi_{\mathcal{U}} = \pi_{\mathcal{C}} \circ j_{\mathcal{U}}$. Then $f_{\mathcal{U}}$ and $g_{\mathcal{U}}$ are mutually inverse and

$$K_0^{sp}(\mathcal{U})/\operatorname{Im}\theta_{\mathcal{U}} \cong K_0(\mathcal{C}).$$

*Proof.* Using Lemmas 6.3.3 and 6.3.4 and $g_{\mathcal{U}}$ with the stated property, we have

$$\begin{aligned}f_{\mathcal{U}} \circ g_{\mathcal{U}} \circ \pi_{\mathcal{U}} &= f_{\mathcal{U}} \circ \pi_{\mathcal{C}} \circ j_{\mathcal{U}} = \pi_{\mathcal{U}} \circ \operatorname{index}_{\mathcal{U}} \circ j_{\mathcal{U}} = \pi_{\mathcal{U}} = 1_{K_0^{sp}(\mathcal{U})/\operatorname{Im}\theta_{\mathcal{U}}} \circ \pi_{\mathcal{U}}, \\ g_{\mathcal{U}} \circ f_{\mathcal{U}} \circ \pi_{\mathcal{C}} &= g_{\mathcal{U}} \circ \pi_{\mathcal{U}} \circ \operatorname{index}_{\mathcal{U}} = \pi_{\mathcal{C}} \circ j_{\mathcal{U}} \circ \operatorname{index}_{\mathcal{U}} = \pi_{\mathcal{C}} = 1_{K_0(\mathcal{C})} \circ \pi_{\mathcal{C}}.\end{aligned}$$

Since $\pi_{\mathcal{U}}$ and $\pi_{\mathcal{C}}$ are surjective, and hence right cancellative, we have

$$f_{\mathcal{U}} \circ g_{\mathcal{U}} = 1_{K_0^{sp}(\mathcal{U})/\operatorname{Im}\theta_{\mathcal{U}}} \quad \text{and} \quad g_{\mathcal{U}} \circ f_{\mathcal{U}} = 1_{K_0(\mathcal{C})}.$$

□

## 6.4 $\mathcal{C}$ with Serre functor and $n$-cluster tilting subcategory $\mathcal{T}$

**Notation 6.4.1.** We use the notation $\mathcal{C}(-,-) := \operatorname{Hom}_{\mathcal{C}}(-,-)$.

**Definition 6.4.2** ([35, Section 1.1])**.** Let $\mathcal{T} \subseteq \mathcal{C}$ be a full subcategory and $\operatorname{Ind}\mathcal{T}$ be a full subcategory of $\mathcal{T}$ containing precisely one object from each isomorphism class of indecomposable objects in $\mathcal{T}$. We say that $\operatorname{Ind}\mathcal{T}$ is *locally bounded* if for every object $T \in \operatorname{Ind}\mathcal{T}$, there are only finitely many objects $V \in \operatorname{Ind}\mathcal{T}$ such that $\mathcal{C}(U,V) \neq 0$ and only finitely many objects $W \in \operatorname{Ind}\mathcal{T}$ such that $\mathcal{C}(W,U) \neq 0$.

**Setup 6.4.3.** Assume that $\mathcal{C}$ has a Serre functor $\mathbb{S}$. Let $n \geq 2$ be an integer and let $\mathcal{T} \subseteq \mathcal{C}$ be an $n$-cluster tilting subcategory such that Ind $\mathcal{T}$ is locally bounded.

**Remark 6.4.4.** Using the same notation as in [32, Section 3], we define the functor $\mathbb{S}_n := \mathbb{S} \circ \Sigma^{-n}$ on $\mathcal{C}$. Let $M$ be an indecomposable in $\mathcal{T}$. By [32, Theorem 3.10], there is an Auslander-Reiten $(n+2)$-angle in $\mathcal{T}$, as defined in [32, Definition 3.8], given by a tower of triangles in $\mathcal{C}$ of the form:

$$\begin{tikzcd}
 & T_{n-1} \arrow[rr] \arrow[dr, "\tau_{n-1}"] & & T_{n-2} \arrow[r] \arrow[dr, "\tau_{n-2}"] & {} & \cdots & {} \arrow[r] & T_1 \arrow[rr] \arrow[dr, "\tau_1"] & & T_0 \arrow[dr, "\tau_0"] & \\
\mathbb{S}_n(M) \arrow[ur] & & X_{n-1} \arrow[ll, squiggly, "\xi_{n-1}"] \arrow[ur] & & X_{n-2} \arrow[ll, squiggly, "\xi_{n-2}"] & \cdots & X_2 \arrow[ur] & & X_1 \arrow[ll, squiggly, "\xi_1"] \arrow[ur] & & M. \arrow[ll, squiggly, "\xi_0"]
\end{tikzcd} \tag{6.2}$$

Note that $M$, $\mathbb{S}_n(M)$, $T_0, \ldots, T_{n-1} \in \mathcal{T}$.

**Lemma 6.4.5.** Let $M \in \mathcal{T}$ be an indecomposable with Auslander-Reiten $(n+2)$-angle as in (6.2). Then $F_{\mathcal{T}}(\xi_i) = 0$ for any $i = 1, \ldots, n-1$.

*Proof.* By [32, Definition 3.8], $\tau_i : T_i \to X_i$ is a $\mathcal{T}$-cover of $X_i$. Hence, for every object $\overline{T} \in \mathcal{T}$ and every morphism $\tau \in \mathcal{C}(\overline{T}, X_i)$, there is a morphism $\tau' : \overline{T} \to T_i$ such that $\tau = \tau_i \circ \tau'$. Then,

$$((F_{\mathcal{T}}(\xi_i))(\overline{T}))(\tau) = \xi_i \circ \tau = \xi_i \circ \tau_i \circ \tau' = 0,$$

where $\xi_i \circ \tau_i = 0$ because two consecutive morphisms in a triangle compose to zero. Since this is true for arbitrary $\overline{T} \in \mathcal{T}$ and $\tau \in \mathcal{C}(\overline{T}, X_i)$, we conclude that $F_{\mathcal{T}}(\xi_i) = 0$ for any $i = 1, \ldots, n-1$. $\square$

**Lemma 6.4.6.** Let $M \in \mathcal{T}$ be an indecomposable and consider diagram (6.2). Then, as an element in $K_0^{sp}(\mathcal{T})$, we have

$$-[M] + (-1)^n[\mathbb{S}_n(M)] + [T_0] - [T_1] + \cdots + (-1)^{n-1}[T_{n-1}] = -\theta_{\mathcal{T}}([S_M]),$$

where $S_M$ is the simple in $\operatorname{mod}\mathcal{T}$ that is the top of $\mathcal{C}(-,M)|_{\mathcal{T}}$, the projective in $\operatorname{mod}\mathcal{T}$ corresponding to $M$.

*Proof.* Consider the exact sequence induced by the rightmost triangle in (6.2):

$$\mathcal{C}(-,T_0)|_{\mathcal{T}} \xrightarrow{F_{\mathcal{T}}(\tau_0)} \mathcal{C}(-,M)|_{\mathcal{T}} \xrightarrow{F_{\mathcal{T}}(\xi_0)} \mathcal{C}(-,\Sigma X_1)|_{\mathcal{T}} \to \mathcal{C}(-,\Sigma T_0)|_{\mathcal{T}}.$$

Note that $\mathcal{C}(-,\Sigma T_0)|_{\mathcal{T}} = 0$ and so $\operatorname{Im} F_{\mathcal{T}}(\xi_0) = \operatorname{Coker} F_{\mathcal{T}}(\tau_0) = \mathcal{C}(-,\Sigma X_1)|_{\mathcal{T}}$. By [32, Definition 3.8], we have that $\tau_0 : T_0 \to M$ is minimal right almost split in $\mathcal{T}$ and so using [3, Corollary 2.5], we have that

$$S_M = \mathcal{C}(-,\Sigma X_1)|_{\mathcal{T}}$$

is the simple in $\operatorname{mod}\mathcal{T}$ that is the top of $\mathcal{C}(-,M)|_{\mathcal{T}}$. Then, by Proposition 6.2.16, we have

$$[T_0] = \operatorname{index}_{\mathcal{T}}(T_0) = [M] + \operatorname{index}_{\mathcal{T}}(X_1) - \theta_{\mathcal{T}}([S_M]). \tag{6.3}$$

Moreover, since $\tau_1, \dots, \tau_{n-1}$ are $\mathcal{T}$-covers, by Definition 6.2.11 we have

$$\operatorname{index}_{\mathcal{T}}(X_1) = \sum_{i=1}^{n-1} (-1)^{i-1}[T_i] + (-1)^{n-1}[\mathbb{S}_n(M)].$$

Substituting this into (6.3), we conclude that

$$-[M] + (-1)^n[\mathbb{S}_n(M)] + [T_0] - [T_1] + \dots + (-1)^{n-1}[T_{n-1}] = -\theta_{\mathcal{T}}([S_M]).$$

□

**Remark 6.4.7.** Note that Lemma 6.4.6 can be applied to any indecomposable in $\mathcal{T}$. Moreover, since $\operatorname{Ind}\mathcal{T}$ is locally bounded, then each object in $\operatorname{mod}\mathcal{T}$ has finite length and $K_0(\operatorname{mod}\mathcal{T})$ is generated by the equivalence classes of the simples in $\operatorname{mod}\mathcal{T}$. Since any simple object in $\operatorname{mod}\mathcal{T}$ has the form $S_M$ for some indecomposable $M \in \mathcal{T}$, see [18, Sections 3.1 and 3.2], we have

$$\operatorname{Im}\theta_{\mathcal{T}} = \left\langle -[M] + (-1)^n[\mathbb{S}_n(M)] + \sum_{i=0}^{n-1}(-1)^i[T_i] \;\middle|\; \begin{array}{c} M \in \operatorname{Ind}\mathcal{T} \text{ with Auslander-Reiten} \\ (n+2)\text{-angle (6.2)} \end{array} \right\rangle.$$

**Lemma 6.4.8.** There is a morphism $g_{\mathcal{T}} : K_0^{sp}(\mathcal{T})/\operatorname{Im}\theta_{\mathcal{T}} \to K_0(\mathcal{C})$ such that

$$g_{\mathcal{T}} \circ \pi_{\mathcal{T}} = \pi_{\mathcal{C}} \circ j_{\mathcal{T}}.$$

*Proof.* Consider diagram (6.1) with $\mathcal{U} = \mathcal{T}$. A morphism $g_{\mathcal{U}}$ with the desired property exists if and only if $\pi_{\mathcal{C}} \circ j_{\mathcal{T}} \circ \iota_{\mathcal{T}} = 0$. By Remark 6.4.7, we have

$$\operatorname{Ker}\pi_{\mathcal{T}} = \left\langle -[M] + (-1)^n[\mathbb{S}_n(M)] + \sum_{i=0}^{n-1}(-1)^i[T_i] \;\middle|\; \begin{array}{c} M \in \operatorname{Ind}\mathcal{T} \text{ with Auslander-Reiten} \\ (n+2)\text{-angle (6.2)} \end{array} \right\rangle.$$

Then, for any generator $-[M] + (-1)^n[\mathbb{S}_n(M)] + \sum_{i=0}^{n-1}(-1)^i[T_i]$ of $\operatorname{Ker}\pi_{\mathcal{T}}$ corresponding to the Auslander-Reiten $(n+2)$-angle (6.2), we have

$$\begin{aligned}\pi_{\mathcal{C}} \circ j_{\mathcal{T}} \circ \iota_{\mathcal{T}}\Big( -[M] + (-1)^n[\mathbb{S}_n(M)] + \sum_{i=0}^{n-1}(-1)^i[T_i]\Big) &= -[M] + (-1)^n[\mathbb{S}_n(M)] + \sum_{i=0}^{n-1}(-1)^i[T_i] \\ &= 0,\end{aligned}$$

where all the terms cancel because of the relations in $K_0(\mathcal{C})$ corresponding to the triangles in the tower (6.2). Hence $\pi_{\mathcal{C}} \circ j_{\mathcal{T}} \circ \iota_{\mathcal{T}} = 0$ as desired. □

**Theorem 6.4.9.** *We have that $K_0(\mathcal{C})$ is isomorphic to*

$$K_0^{sp}(\mathcal{T}) \Big/ \left\langle \begin{array}{c} -[M] + (-1)^n[\mathbb{S}_n(M)] + \\ \sum_{i=0}^{n-1}(-1)^i[T_i] \end{array} \;\middle|\; \begin{array}{c} M \in \operatorname{Ind}\mathcal{T} \textit{ with Auslander-Reiten } (n+2)\textit{-angle} \\ \mathbb{S}_n(M) \to T_{n-1} \to \cdots \to T_0 \to M \to \mathbb{S}(M) \end{array} \right\rangle.$$

*Proof.* By Lemma 6.4.8, there exists a homomorphism $g_{\mathcal{T}} : K_0^{sp}(\mathcal{T})/\operatorname{Im}\theta_{\mathcal{T}} \to K_0(\mathcal{C})$ such that $g_{\mathcal{T}} \circ \pi_{\mathcal{T}} = \pi_{\mathcal{C}} \circ j_{\mathcal{T}}$. Then, by Proposition 6.3.5, we have that $K_0(\mathcal{C}) \cong K_0^{sp}(\mathcal{T})/\operatorname{Im}\theta_{\mathcal{T}}$ and, by Remark 6.4.7, this completes the proof. □

**Remark 6.4.10.** Note that our argument does not apply to the case when $n = 1$ as it uses some results, such as Proposition 6.2.16, that rely on $n \geq 2$. However, in the case when $k$ is an algebraically closed field and $n = 1$, Theorem 6.4.9 is an instance of the triangulated analogue by Xiao and Zhu of results of Auslander, see [4, Theorems 2.2 and 2.3], and Butler, see [12, Theorem in introduction], on certain module categories. In this case, the only choice of 1-cluster tilting subcategory is $\mathcal{C} = \mathcal{T}$ and the tower of triangles (6.2) is an Auslander-Reiten triangle in $\mathcal{C}$ of the form

$$\delta : \mathbb{S}\Sigma^{-1}(M) \to T_0 \to M \to \Sigma\mathbb{S}\Sigma^{-1}(M).$$

Note that, since $\operatorname{Ind}\mathcal{T}$ is locally bounded, we have that $\mathcal{C} = \mathcal{T}$ is of finite type. Then, by [57, Theorem 2.1], we have that $K_0(\mathcal{C})$ is isomorphic to the quotient of $K_0^{sp}(\mathcal{C})$ by the elements $[\delta] := -[M] + [T_0] - [\mathbb{S}\Sigma^{-1}(M)]$, where $\delta$ runs through all the Auslander-Reiten triangles in $\mathcal{C}$.

**Corollary 6.4.11.** Assume that $n \geq 2$ is an even integer, $\mathcal{C}$ is $n$-Calabi-Yau and that there is a $\mathcal{T} \subseteq \mathcal{C}$ as in Setup 6.4.3. We then have that

$$K_0(\mathcal{C}) \cong K_0^{sp}(\mathcal{T}) \Big/ \left\langle \sum_{i=0}^{n-1} (-1)^i [T_i] \;\middle|\; \begin{array}{c} M \in \operatorname{Ind}\mathcal{T} \text{ with Auslander-Reiten } (n+2)\text{-angle} \\ M \to T_{n-1} \to \cdots \to T_0 \to M \to \Sigma^n M \end{array} \right\rangle .$$

*Proof.* Since $\mathcal{C}$ is $n$-Calabi-Yau, it has Serre functor $\mathbb{S} = \Sigma^n$ so this situation is a special case of Setup 6.4.3. Note also that in this case $\mathbb{S}_n = \mathbb{S} \circ \Sigma^{-n}$ is the identity functor on $\mathcal{C}$. Hence, since $n$ is even, we have that $-[M] + (-1)^n[\mathbb{S}_n(M)] = -[M] + [M] = 0$ and the result follows from Theorem 6.4.9. □

**Remark 6.4.12.** Note that when $n = 2$ and the Auslander-Reiten quiver of $\mathcal{T}$ has no loops, Corollary 6.4.11 coincides with [50, Theorem 10]. In this case, if $M$ is an indecomposable direct summand of $T$, then its Auslander-Reiten 4-angle is $M \to B_{M^*} \to B_M \to M \to \Sigma^2 M$, where $B_M$, $B_{M^*}$ are defined as in [50, p. 1444]. However, if we do not assume that the Auslander-Reiten quiver of $\mathcal{T}$ has no loops, some of the Auslander-Reiten 4-angles in $\mathcal{T}$ do not come from Palu's exchange triangles and Corollary 6.4.11 and [50, Theorem 10] are different.

## 6.5 A $\Sigma^d$-stable, $d$-cluster tilting subcategory $\mathcal{S} \subseteq \mathcal{C}$

**Setup 6.5.1.** Let $d \geq 1$ be an integer and $\mathcal{S} \subseteq \mathcal{C}$ be a $d$-cluster tilting subcategory. Assume also that $\Sigma^d \mathcal{S} = \mathcal{S}$. Then, by Theorem 2.3.43, we have that $\mathcal{S}$ is a $(d+2)$-angulated category with $d$-suspension functor $\Sigma^d$.

**Lemma 6.5.2.** Consider a $(d+2)$-angle in $\mathcal{S}$ of the form

$$S_{d+1} \longrightarrow S_d \longrightarrow \cdots \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_0 \longrightarrow \Sigma^d S_{d+1}.$$

By Theorem 2.3.43, it corresponds to a tower of triangles in $\mathcal{C}$:

$$\begin{tikzcd}[column sep=small]
& S_d \arrow[rr] \arrow[dr, "\sigma_d"] & & S_{d-1} \arrow[r] \arrow[dr, "\sigma_{d-1}"] & {} & \cdots & {} \arrow[r] & S_3 \arrow[rr] \arrow[dr, "\sigma_3"] & & S_2 \arrow[rr] \arrow[dr, "\sigma_2"] & & S_1 \arrow[dr] & \\
S_{d+1} \arrow[ur] & & Y_{d-1} \arrow[ll, squiggly, "\eta_{d-1}"] \arrow[ur] & & Y_{d-2} \arrow[ll, squiggly, "\eta_{d-2}"] & \cdots & Y_3 \arrow[ur] & & Y_2 \arrow[ll, squiggly, "\eta_2"] \arrow[ur] & & Y_1 \arrow[ll, squiggly, "\eta_1"] \arrow[ur] & & S_0. \arrow[ll, squiggly, "\eta_0"]
\end{tikzcd} \tag{6.4}$$

This tower satisfies $F_{\mathcal{S}}(\eta_l) = 0$ for any integer $1 \leq l \leq d-1$.

*Proof.* Let $Y_d := S_{d+1}$. In order to prove that $F_{\mathcal{S}}(\eta_l) = 0$, we prove that its target $F_{\mathcal{S}}(\Sigma Y_{l+1})$ is zero. More generally, we prove that $F_{\mathcal{S}}(\Sigma^i Y_{d-j}) = 0$ for any integers $0 \leq j \leq d-2$ and $1 \leq i \leq d-j-1$.

First note that for $j=0$, we have $F_{\mathcal{S}}(\Sigma^i Y_d)=0$ for any $1\le i\le d-1$, since $Y_d=S_{d+1}\in\mathcal{S}$. Suppose that for some $0\le j\le d-2$, we proved that

$$F_{\mathcal{S}}(\Sigma^i Y_{d-r})=0 \text{ for any } 0\le r\le j \text{ and } 1\le i\le d-r-1. \tag{$\dagger$}$$

If $j=d-2$, then we are done. So assume $j\le d-3$. We need to prove that

$$F_{\mathcal{S}}(\Sigma^{i'} Y_{d-(j+1)})=0 \text{ for any } 1\le i'\le d-j-2.$$

Given any $1\le i'\le d-j-2$, the triangle $S_{d-j}\to Y_{d-(j+1)}\to\Sigma Y_{d-j}\to\Sigma S_{d-j}$ induces the exact sequence:

$$F_{\mathcal{S}}(\Sigma^{i'}S_{d-j})\to F_{\mathcal{S}}(\Sigma^{i'}Y_{d-(j+1)})\to F_{\mathcal{S}}(\Sigma^{i'+1}Y_{d-j}).$$

Note that $F_{\mathcal{S}}(\Sigma^{i'}S_{d-j})=0$ since $S_{d-j}\in\mathcal{S}$ and $1\le i'<d-1$. Moreover, $F_{\mathcal{S}}(\Sigma^{i'+1}Y_{d-j})=0$ by ($\dagger$) with $r=j$ and as $1<i'+1\le d-j-1$. Hence $F_{\mathcal{S}}(\Sigma^{i'}Y_{d-(j+1)})=0$ for any $1\le i'\le d-j-2$ as we wished to show. □

**Remark 6.5.3.** Consider a $(d+2)$-angle in $\mathcal{S}$ of the form

$$S_{d+1}\longrightarrow S_d\longrightarrow\cdots\longrightarrow S_2\longrightarrow S_1\longrightarrow S_0\longrightarrow\Sigma^d S_{d+1}.$$

By Theorem 2.3.43, it corresponds to a tower of triangles in $\mathcal{C}$ of the form (6.4). By Lemma 6.5.2, we have that $F_{\mathcal{S}}(\eta_l)=0$ for any integer $1\le l\le d-1$. Hence $\sigma_{l+1}:S_{l+1}\to Y_l$ is an $\mathcal{S}$-precover. If it is not right minimal, then $S_{l+1}\cong\overline{S}_{l+1}\oplus S'_{l+1}$ and $\sigma_{l+1}$ is isomorphic to a morphism of the form $(0,\sigma'_{l+1}):\overline{S}_{l+1}\oplus S'_{l+1}\to Y_l$, where $\sigma'_{l+1}$ is an $\mathcal{S}$-cover of $Y_l$. It is then easy to check that $\overline{S}_{l+1}$ appears as a summand of $S_{l+2}$ and so it gets cancelled when computing $[S_{l+1}]-[S_{l+2}]$. Hence, the morphisms $\sigma_l$ do not need to be $\mathcal{S}$-covers for using the tower (6.4) to compute $\operatorname{index}_{\mathcal{S}}(Y_1)$. In other words,

$$\operatorname{index}_{\mathcal{S}}(Y_1)=\sum_{i=2}^{d+1}(-1)^i[S_i].$$

**Proposition 6.5.4.** When $d\ge 2$, we have that

$$\operatorname{Im}\theta_{\mathcal{S}}=\left\langle\sum_{i=0}^{d+1}(-1)^i[S_i]\mid S_{d+1}\to\cdots\to S_0\to\Sigma^d S_{d+1}\text{ is a }(d+2)\text{-angle in }\mathcal{S}\right\rangle.$$

*Proof.* We prove this by proving that the two inclusions hold.

($\subseteq$). Given any $Y \in \mathcal{S} * \Sigma\mathcal{S}$, there is a triangle in $\mathcal{C}$ of the form

$$S_0 \xrightarrow{\eta_0} Y \longrightarrow \Sigma S_1 \longrightarrow \Sigma S_0,$$

where $S_0$, $S_1 \in \mathcal{S}$. Letting $Y_1 := \Sigma^{-1} Y \in \mathcal{C}$, we obtain a triangle in $\mathcal{C}$ of the form

$$\Delta : \qquad Y_1 \longrightarrow S_1 \longrightarrow S_0 \xrightarrow{\eta_0} \Sigma Y_1.$$

Since $\mathcal{S}$ is $d$-cluster tilting in $\mathcal{C}$, by [32, Corollary 3.3], we can construct a tower of triangles in $\mathcal{C}$ of the form

$$\begin{tikzcd}
& S_d \arrow[rr] \arrow[dr] & & S_{d-1} \arrow[r] \arrow[dr] & {} & \cdots & {} \arrow[r] & S_3 \arrow[rr] \arrow[dr] & & S_2 \arrow[dr] & \\
S_{d+1} \arrow[ur] & & Y_{d-1} \arrow[ll, squiggly, "\eta_{d-1}"] \arrow[ur] & & Y_{d-2} \arrow[ll, squiggly, "\eta_{d-2}"] & \cdots & Y_3 \arrow[ur] & & Y_2 \arrow[ll, squiggly, "\eta_2"] \arrow[ur] & & Y_1, \arrow[ll, squiggly, "\eta_1"]
\end{tikzcd}$$

where $S_2, \ldots, S_{d+1}$ are in $\mathcal{S}$. Putting this together with triangle $\Delta$, we obtain the tower of triangles (6.4) in $\mathcal{C}$, which corresponds to the $(d+2)$-angle in $\mathcal{S}$:

$$S_{d+1} \longrightarrow S_d \longrightarrow \cdots \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_0 \longrightarrow \Sigma^d S_{d+1}.$$

By Proposition 6.2.16, we have that in $K_0^{sp}(\mathcal{S})$:

$$[S_1] = \operatorname{index}_{\mathcal{S}}(Y_1) + [S_0] - \theta_{\mathcal{S}}([\operatorname{Im} F_{\mathcal{S}}(\eta_0)]).$$

Moreover, since $F_{\mathcal{S}}(\Sigma S_1) = 0$, we have that $F_{\mathcal{S}}(\eta_0)$ is surjective and so

$$[S_1] = \operatorname{index}_{\mathcal{S}}(Y_1) + [S_0] - \theta_{\mathcal{S}}([F_{\mathcal{S}}(\Sigma Y_1)]).$$

We have that $\operatorname{index}_{\mathcal{S}}(Y_1) = \sum_{i=2}^{d+1}(-1)^i[S_i]$ and so

$$\sum_{i=0}^{d+1}(-1)^i[S_i] = \theta_{\mathcal{S}}([F_{\mathcal{S}}(\Sigma Y_1)]) = \theta_{\mathcal{S}}([F_{\mathcal{S}}(Y)]).$$

($\supseteq$). Given a $(d+2)$-angle in $\mathcal{S}$ of the form

$$S_{d+1} \longrightarrow S_d \longrightarrow \cdots \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_0 \longrightarrow \Sigma^d S_{d+1},$$

consider the corresponding tower (6.4) of triangles in $\mathcal{C}$. By Remark 6.5.3, we have that $\operatorname{index}_{\mathcal{S}}(Y_1) = \sum_{i=2}^{d+1}(-1)^i[S_i]$. Using Proposition 6.2.16 on the rightmost triangle in tower

(6.4), namely $Y_1 \to S_1 \to S_0 \xrightarrow{\eta_0} \Sigma Y_1$, we conclude that

$$\sum_{i=0}^{d+1} (-1)^i [S_i] = \theta_{\mathcal{S}}([\operatorname{Im} F_{\mathcal{S}}(\eta_0)]) \in \operatorname{Im} \theta_{\mathcal{S}}.$$

□

**Remark 6.5.5.** By Proposition 6.5.4 and Definition 6.2.2, for $d \geq 2$, we have that

$$K_0(\mathcal{S}) = K_0^{sp}(\mathcal{S}) / \operatorname{Im} \theta_{\mathcal{S}}.$$

Suppose now that $\operatorname{Ind} \mathcal{S}$ is locally bounded and $\mathcal{C}$ has a Serre functor $\mathbb{S}$. By [32, Theorem 3.10], for every indecomposable $M$ in $\mathcal{S}$, there is an Auslander-Reiten $(d+2)$-angle in $\mathcal{S}$ of the form

$$\mathbb{S}_d(M) \longrightarrow S_{d-1} \longrightarrow \cdots \longrightarrow S_1 \longrightarrow S_0 \longrightarrow M \longrightarrow \Sigma^d \mathbb{S}_d(M), \tag{6.5}$$

where $\mathbb{S}_d = \mathbb{S} \circ \Sigma^{-d}$. Then, by Remark 6.4.7, we have that $K_0(\mathcal{S})$ is equal to

$$K_0^{sp}(\mathcal{S}) \Big/ \left\langle -[M] + (-1)^d [\mathbb{S}_d(M)] + \sum_{i=0}^{d-1} (-1)^i [S_i] \;\middle|\; \begin{array}{c} M \in \operatorname{Ind} \mathcal{S} \text{ with Auslander-Reiten} \\ (d+2)\text{-angle (6.5)} \end{array} \right\rangle.$$

This result agrees with [58, Theorem 3.7]. Note that there are two differences between ours and Zhou's result. The first one is that we do not assume that $d$ is odd, and the second one is that Zhou's $(d+2)$-angulated category is not assumed to arise as a $d$-cluster tilting subcategory of a triangulated category.

**Lemma 6.5.6.** When $d \geq 2$, there is a morphism $g_{\mathcal{S}} : K_0(\mathcal{S}) = K_0^{sp}(\mathcal{S}) / \operatorname{Im} \theta_{\mathcal{S}} \to K_0(\mathcal{C})$ such that

$$g_{\mathcal{S}} \circ \pi_{\mathcal{S}} = \pi_{\mathcal{C}} \circ j_{\mathcal{S}}.$$

*Proof.* Consider diagram (6.1) with $\mathcal{U} = \mathcal{S}$. Note that a morphism $g_{\mathcal{S}}$ with the desired property exists if and only if $\pi_{\mathcal{C}} \circ j_{\mathcal{S}} \circ \iota_{\mathcal{S}} = 0$. Note that, by Proposition 6.5.4, the group $\operatorname{Ker} \pi_{\mathcal{S}}$ is generated by elements of the form

$$\sum_{i=0}^{d+1} (-1)^i [S_i],$$

for some $(d+2)$-angle in $\mathcal{S}$ of the form $S_{d+1} \to \cdots \to S_0 \to \Sigma^d S_{d+1}$. Such a $(d+2)$-angle corresponds to a tower of triangles in $\mathcal{C}$ of the form (6.4). Then, we have

$$\begin{aligned}\pi_{\mathcal{C}} \circ j_{\mathcal{S}} \circ \iota_{\mathcal{S}}\Big(\sum_{i=0}^{d+1}(-1)^i[S_i]\Big) &= [S_0] - ([S_0] + [Y_1]) + \cdots + (-1)^d([Y_{d-1}] + [S_{d+1}]) + (-1)^{d+1}[S_{d+1}] \\ &= 0,\end{aligned}$$

where we have used the relations in $K_0(\mathcal{C})$ corresponding to the triangles in the tower (6.4), for instance $[S_1] = [S_0] + [Y_1]$. Hence $\pi_{\mathcal{C}} \circ j_{\mathcal{S}} \circ \iota_{\mathcal{C}} = 0$ as desired. □

**Theorem 6.5.7.** *We have that*

$$K_0(\mathcal{C}) \cong K_0(\mathcal{S}).$$

*Proof.* If $d = 1$, then $\mathcal{S} = \mathcal{C}$ and the result is clear. So assume $d \geq 2$. By Lemma 6.5.6, there exists a homomorphism $g_{\mathcal{S}} : K_0^{sp}(\mathcal{S})/\operatorname{Im}\theta_{\mathcal{S}} \to K_0(\mathcal{C})$ such that $g_{\mathcal{S}} \circ \pi_{\mathcal{S}} = \pi_{\mathcal{C}} \circ j_{\mathcal{S}}$. Then, by Proposition 6.3.5, we have that $K_0(\mathcal{C}) \cong K_0^{sp}(\mathcal{S})/\operatorname{Im}\theta_{\mathcal{S}}$ and, by Remark 6.5.5, this completes the proof. □

## 6.6 The case when $n = 2d$ and $\mathcal{T} \subseteq \mathcal{S} \subseteq \mathcal{C}$

**Setup 6.6.1.** Let $d \geq 1$ be an integer and $n = 2d$. Assume that $\mathcal{C}$ has a Serre functor $\mathbb{S}$ and $\mathcal{T} \subseteq \mathcal{S} \subseteq \mathcal{C}$ are such that $\mathcal{T}$ is an $n$-cluster tilting subcategory in $\mathcal{C}$ such that $\operatorname{Ind}\mathcal{T}$ is locally bounded and $\mathcal{S}$ is a $d$-cluster tilting subcategory in $\mathcal{C}$ such that $\Sigma^d\mathcal{S} = \mathcal{S}$. Then $\mathcal{S}$ is a $(d+2)$-angulated category with $d$-suspension $\Sigma^d$.

**Definition 6.6.2** ([49, Definition 5.3])**.** A functorially finite, full subcategory $\mathcal{T} \subseteq \mathcal{S}$ is an *Oppermann-Thomas cluster tilting subcategory* if:

(a) $\mathcal{S}(\mathcal{T}, \Sigma^d\mathcal{T}) = 0$,

(b) for each $S' \in \mathcal{S}$, there is a $(d+2)$-angle $T_d \to \cdots \to T_0 \to S' \to \Sigma^d T_d$ in $\mathcal{S}$ with $T_i \in \mathcal{T}$.

**Remark 6.6.3.** The subcategory $\mathcal{T} \subseteq \mathcal{S}$ from Setup 6.6.1 is an Oppermann-Thomas cluster tilting subcategory by [49, Theorem 5.25]. Note that in [49, Definition 5.3 and Theorem 5.25] $\mathcal{T}$ is assumed to have finitely many indecomposables up to isomorphism. We are not restricting to this case and it can be easily checked that the proof of [49, Theorem 5.25] still goes through without this restriction.

**Theorem 6.6.4.** *We have $K_0(\mathcal{C}) \cong K_0(\mathcal{S})$ and*

$$K_0(\mathcal{S}) \cong K_0^{sp}(\mathcal{T}) \Big/ \left\langle \begin{array}{c} -[M] + [\mathbb{S}_n(M)] + \\ \sum_{i=0}^{n-1}(-1)^i[T_i] \end{array} \;\middle|\; \begin{array}{c} M \in \operatorname{Ind}\mathcal{T} \textit{ with Auslander-Reiten } (n+2)\textit{-angle} \\ \mathbb{S}_n(M) \to T_{n-1} \to \cdots \to T_0 \to M \to \mathbb{S}(M) \end{array} \right\rangle.$$

*Proof.* This follows by combining Theorem 6.4.9 and Theorem 6.5.7, and noting that $[\mathbb{S}_n(M)] = (-1)^n[\mathbb{S}_n(M)]$ since $n = 2d$ is even. $\square$

**Corollary 6.6.5.** If in Setup 6.6.1 we also have that $\mathcal{C}$ is $n$-Calabi-Yau, then

$$K_0(\mathcal{S}) \cong K_0^{sp}(\mathcal{T}) \Big/ \left\langle \sum_{i=0}^{n-1}(-1)^i[T_i] \;\middle|\; \begin{array}{c} M \in \operatorname{Ind}\mathcal{T} \text{ with Auslander-Reiten } (n+2)\text{-angle} \\ M \to T_{n-1} \to \cdots \to T_0 \to M \to \Sigma^n M \end{array} \right\rangle.$$

*Proof.* Since $\mathcal{C}$ is $n$-Calabi-Yau, it has Serre functor $\mathbb{S} = \Sigma^n$. Then $\mathbb{S}_n$ is the identity functor on $\mathcal{C}$ and the result follows from Theorem 6.6.4. $\square$

**Remark 6.6.6.** When $d = 1$, we have that $\mathcal{S} = \mathcal{C}$ is a triangulated category with cluster tilting subcategory $\mathcal{T}$ and, adding the extra assumption that the Auslander-Reiten quiver of $\mathcal{T}$ has no loops, Corollary 6.6.5 becomes [50, Theorem 10] by Palu. For higher values of $d$, Corollary 6.6.5 is a higher angulated version of Palu's theorem.

## 6.7 The Grothendieck group associated to $\mathcal{C}_q(A_p)$ for $q$ odd

In this section, we compute the Grothendieck group of the triangulated $q$-cluster category of Dynkin type $A_p$ for $q$ odd. Recall that we introduced this category and its geometric realisation in Section 3.2. Using the described geometric realisation, we can fully describe the $(q+1)$-cluster tilting objects in $\mathcal{C}_q(A_p)$.

**Definition 6.7.1.** A $(q+2)$*-angulation of the $N$-gon $P$*, where $N = (p+1)q+2$, is a maximal collection of non-crossing $q$-allowable diagonals.

**Proposition 6.7.2** ([47, proposition 2.14])**.** There is a bijection

$$\left\{ \begin{array}{c} (q+2)\text{-angulations} \\ \text{of } P \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} (q+1)\text{-cluster tilting objects} \\ \text{in } \mathcal{C}_q(A_p) \end{array} \right\}$$

sending a $(q+2)$-angulation $T = \{T_0, \dots, T_{p-1}\}$ to the $(q+1)$-cluster tilting object $T_0 \oplus \cdots \oplus T_{p-1}$.

**Proposition 6.7.3.** Assume that $q$ is odd. We have that

$$K_0(\mathcal{C}_q(A_p)) \cong \begin{cases} 0, & \text{if } p \text{ is even,} \\ \mathbb{Z}, & \text{if } p \text{ is odd.} \end{cases}$$

*Proof.* Consider the $(q+2)$-angulation $T = T_0, \dots, T_{p-1}$, where

$$T_0 = (0, q+1) \text{ and } T_i = (N-i, (1+i)q+1-i), \text{ for } 1 \le i \le p-1,$$

see Figure 6.1. Note that by Proposition 6.7.2, this corresponds to the $(q+1)$-cluster tilting object $T_0 \oplus \cdots \oplus T_{p-1}$. Let $\mathcal{T} := \operatorname{add}(T_0 \oplus \cdots \oplus T_{p-1}) \subseteq \mathcal{C}_q(A_p)$ be the corresponding $(q+1)$-cluster tilting subcategory. We want to find the Auslander-Reiten $(q+3)$-angle

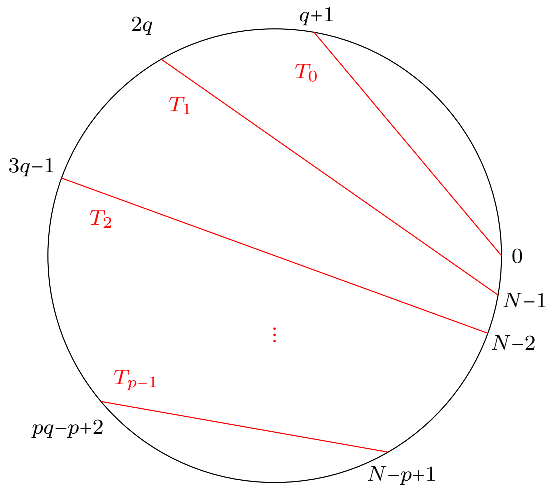



Figure 6.1: The $(q+2)$-angulation $T$.

starting and ending at $T_i$ for $0 \le i \le p-1$.

Consider $i = 0$. By Lemma 3.2.11 there are no non-zero morphisms $T_i \to T_0$ for $i \neq 0$, so that $\tau_0 : 0 \to T_0$ is a right almost split morphism in $\mathcal{T}$. Consider $-q+2 \le j \le -1$. Note that, since $\Sigma^j T_0 = \{-j, q+1-j\}$, Lemma 3.2.10 can be used to check that for all indecomposables $T_i$ in $\mathcal{T}$, we have that $\operatorname{Hom}(T_i, \Sigma^j T_0) = 0$. Hence $\tau_{-j} : 0 \to \Sigma^j T_0$ is a $\mathcal{T}$-cover for any $-q+2 \le j \le -1$. Moreover, by Lemma 3.2.11, we have that $\tau_{q-1} : T_1 \to \Sigma^{-q+1} T_0$ is a $\mathcal{T}$-cover. Using Proposition 3.2.20, the morphism $\tau_{q-1}$ extends to the non-split triangle

$$\Sigma T_0 \to T_1 \xrightarrow{\tau_{q-1}} \Sigma^{-q+1} T_0 \to \Sigma^2 T_0.$$

Applying Lemma 3.2.11 to $\Sigma T_0 = \{N-1, q\}$, we see that $\operatorname{Hom}(\mathcal{T}, \Sigma T_0) = 0$. Hence $\tau_q : 0 \to \Sigma T_0$ is a $\mathcal{T}$-cover. The Auslander-Reiten $(q+3)$-angle starting and ending at $T_0$ is then the one corresponding to the following tower of triangles:

$$\begin{tikzcd}[column sep=small]
& 0 \arrow[rr] \arrow[dr, "\tau_q"] & & T_1 \arrow[rr] \arrow[dr, "\tau_{q-1}"] & & 0 \arrow[r] \arrow[dr, "\tau_{q-2}"] & & & \cdots & \arrow[r] & 0 \arrow[rr] \arrow[dr, "\tau_1"] & & 0 \arrow[dr, "\tau_0"] & \\
T_0 \arrow[ur] & & \Sigma T_0 \arrow[ll, squiggly] \arrow[ur] & & \Sigma^{-q+1} T_0 \arrow[ll, squiggly] \arrow[ur] & & \Sigma^{-q+2} T_0 \arrow[ll, squiggly] & & \cdots & \Sigma^{-2} T_0 \arrow[ur] & & \Sigma^{-1} T_0 \arrow[ll, squiggly] \arrow[ur] & & T_0. \arrow[ll, squiggly]
\end{tikzcd}$$

In a similar way, we can find the remaining Auslander-Reiten $(q+3)$-angles. These are the ones corresponding to the following towers of triangles:

$$
\begin{tikzcd}[column sep=tiny]
 & 0 \arrow[rr] \arrow[dr, "\tau_q"] & & T_{i+1} \arrow[rr] \arrow[dr, "\tau_{q-1}"] & & 0 \arrow[rr] \arrow[dr, "\tau_{q-2}"] & & {} & \cdots & {} \arrow[rr] & & 0 \arrow[rr] \arrow[dr, "\tau_2"] & & T_{i-1} \arrow[rr] \arrow[dr, "\tau_1"] & & 0 \arrow[dr, "\tau_0"] & \\
T_i \arrow[ur] & & \Sigma T_i \arrow[ll, squiggly] \arrow[ur] & & A_i \arrow[ll, squiggly] \arrow[ur] & & \Sigma A_i \arrow[ll, squiggly] & & \cdots & & \Sigma^{q-4} A_i \arrow[ur] & & \Sigma^{q-3} A_i \arrow[ll, squiggly] \arrow[ur] & & \Sigma^{-1} T_i \arrow[ll, squiggly] \arrow[ur] & & T_i, \arrow[ll, squiggly]
\end{tikzcd}
$$

where $A_i := \{(i+2)q - i, (i+1)q - i - 1\}$, for $1 \leq i \leq p-2$, and

$$
\begin{tikzcd}[column sep=tiny]
 & 0 \arrow[rr] \arrow[dr, "\tau_q"] & & 0 \arrow[rr] \arrow[dr, "\tau_{q-1}"] & & {} & \cdots & {} \arrow[rr] & & 0 \arrow[rr] \arrow[dr, "\tau_2"] & & T_{p-2} \arrow[rr] \arrow[dr, "\tau_1"] & & 0 \arrow[dr, "\tau_0"] & \\
T_{p-1} \arrow[ur] & & \Sigma T_{p-1} \arrow[ll, squiggly] \arrow[ur] & & \Sigma^2 T_{p-1} \arrow[ll, squiggly] & & \cdots & & \Sigma^{q-2} T_{p-1} \arrow[ur] & & \Sigma^{q-1} T_{p-1} \arrow[ll, squiggly] \arrow[ur] & & \Sigma^{-1} T_{p-1} \arrow[ll, squiggly] \arrow[ur] & & T_{p-1}. \arrow[ll, squiggly]
\end{tikzcd}
$$

Recall that by Corollary 6.4.11, we have that

$$
K_0(\mathcal{C}_q(A_p)) \cong K_0^{sp}(\mathcal{T}) \Big/ \left\langle \sum_{i=0}^{q} (-1)^i [\overline{T}_i] \;\middle|\; \begin{array}{c} M \in \operatorname{Ind}\mathcal{T} \text{ with Auslander-Reiten } (q+3)\text{-angle} \\ M \to \overline{T}_q \to \cdots \to \overline{T}_0 \to M \to \Sigma^{q+1} M \end{array} \right\rangle.
$$

Using the Auslander-Reiten $(q+3)$-angles found and the fact that $q$ is odd, we obtain that in the quotient group on the right hand side, we have

$$
[T_1] = [T_{p-2}] = 0 \quad \text{and} \quad [T_{i-1}] = [T_{i+1}] \quad \text{for} \quad 1 \leq i \leq p-2.
$$

This implies that

- if $p$ is even, then $[T_i] = 0$ for all $0 \leq i \leq p-1$,
- if $p$ is odd, then $0 = [T_1] = \ldots [T_{p-2}]$ and $0 \neq [T_0] = \cdots = [T_{p-1}]$.

Hence

$$
K_0(\mathcal{C}_q(A_p)) \cong \begin{cases} 0 & \text{if } p \text{ is even,} \\ \mathbb{Z} & \text{if } p \text{ is odd.} \end{cases}
$$

□

## 6.8 A higher angulated cluster category of type $A$

Let $p$ and $d$ be positive integers. We denote by $A_p^d$ the $(d-1)$-st higher Auslander $k$-algebra of linearly oriented $A_p$, see [49, Section 3]. This is a $d$-representation finite algebra, in the sense that it has a $d$-cluster tilting module and $\operatorname{gldim}(A_p^d) \leq d$, see [31, Definition 2.19]. Note that, using this notation, $A_p^1 = kA_p$ is the usual path algebra of $A_p$ with linear orientation. Let $\operatorname{mod} A_p^d$ be the category of finitely generated $A_p^d$-modules and $\mathcal{D}^b(\operatorname{mod} A_p^d)$ be its bounded derived category. We denote by $\mathbb{S}$ its Serre functor and by $\Sigma$ its suspension functor.

**Definition 6.8.1** ([49, Construction 5.13]). For $\delta \geq d$, the *$\delta$-Amiot cluster category of $A_p^d$* is defined to be

$$\mathcal{C}^\delta(A_p^d) = \text{triangulated hull of } \mathcal{D}^b(\operatorname{mod} A_p^d)/(\mathbb{S}_\delta),$$

where $\mathbb{S}_\delta := \mathbb{S} \circ \Sigma^{-\delta}$.

**Remark 6.8.2.** The category $\mathcal{C}^\delta(A_p^d)$ is a triangulated category containing the orbit category $\mathcal{D}^b(\operatorname{mod} A_p^d)/(\mathbb{S}_\delta)$. We do not give a formal definition of triangulated hull here. Note that, by [49, Theorem 5.14], we have that if $\delta > d$, then $\mathcal{C}^\delta(A_p^d)$ is Hom-finite and $\delta$-Calabi-Yau.

**Remark 6.8.3.** Let $M$ be the unique $d$-cluster tilting object in $\operatorname{mod} A_p^d$. Then

$$\mathcal{U} := \operatorname{add}\{\Sigma^{id} M \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}^b(\operatorname{mod} A_p^d)$$

is a $d$-cluster tilting subcategory by Remark 2.3.44.

**Definition 6.8.4** ([49, Definition 5.22]). The $(d+2)$*-angulated cluster category of $A_p^d$* is defined to be the orbit category

$$\mathcal{O}(A_p^d) = \mathcal{U}/(\mathbb{S}_{2d})$$

**Remark 6.8.5.** Note that $\mathcal{O}(A_p^d)$ comes with an inclusion into

$$\mathcal{D}^b(\operatorname{mod} A_p^d)/(\mathbb{S}_{2d}) \subseteq \mathcal{C}^{2d}(A_p^d).$$

Moreover, by [49, Theorem 5.24], we have that $\mathcal{O}(A_p^d) \subseteq \mathcal{C}^{2d}(A_p^d)$ is $d$-cluster tilting and $\mathcal{O}(A_p^d)$ is $(d+2)$-angulated.

**Notation 6.8.6.** Let $Z$ be a *cyclically ordered set* with $p+2d+1$ elements. We can think of $Z$ as marked points on a circle labeled 1 to $p+2d+1$ in the anticlockwise direction. Given three points $u, v, w$, we write $u < v < w$ if they appear in the order $u, v, w$ when going through the points in the anticlockwise direction. Moreover, given two distinct points $u$ and $v$, we can consider the interval of points $[u, v]$ and in this "$<$" is a total order.

For a point $v$, we denote by $v^+$ its successor and by $v^-$ its predecessor in the anticlockwise direction. We say that two points are neighbours if one is the successor of the other.

**Lemma 6.8.7** ([49, Proposition 6.10])**.** There is a bijection

$$\mathrm{Ind}\mathcal{O}(A_p^d) \longleftrightarrow \{X = \{x_0, \dots, x_d\} \subset Z \mid X \text{ contains no neighbouring points}\}.$$

We will use it to identify the indecomposable objects of $\mathcal{O}(A_p^d)$ with the sets $X$. For $X = \{x_0, \dots, x_d\} \in \mathrm{Ind}\mathcal{O}(A_p^d)$, we have that

$$\Sigma^d X = \mathbb{S}_d X = \{x_0^-, \dots, x_d^-\}.$$

**Definition 6.8.8.** For $X, Y \in \mathrm{Ind}\mathcal{O}(A_p^d)$, we say that $X$ *intertwines* $Y$ if we can write $X = \{x_0, \dots, x_d\}$ and $Y = \{y_0, \dots, y_d\}$ such that

$$x_0 < y_0 < x_1 < \dots < y_{d-1} < x_d < y_d < x_0.$$

Note that in this case also $Y$ intertwines $X$.

**Lemma 6.8.9** ([49, proposition 6.1])**.** Given $X$ and $Y$ in $\mathrm{Ind}\mathcal{O}(A_p^d)$, we have that

$$\mathrm{Ext}^d_{\mathcal{O}(A_p^d)}(X, Y) \neq 0$$

if and only if $X$ intertwines $Y$. In this case, $\mathrm{Ext}^d_{\mathcal{O}(A_p^d)}(X, Y)$ is one-dimensional over $k$.

**Lemma 6.8.10** ([49, Proposition 6.11])**.** Let $X = \{x_0, \dots, x_d\}$ and $Y = \{y_0, \dots, y_d\} \in \mathrm{Ind}\mathcal{O}(A_p^d)$ be such that

$$x_0 < y_0 < x_1 < \dots < y_{d-1} < x_d < y_d < x_0,$$

so $X$ intertwines $Y$. Then there is a $(d+2)$-angle in $\mathcal{O}(A_p^d)$ of the form

$$X \longrightarrow E_d \longrightarrow \dots \longrightarrow E_1 \longrightarrow Y \longrightarrow \Sigma^d X \quad \text{with} \quad E_r = \bigoplus_{\substack{I \subseteq \{0, \dots, d\} \\ |I| = r}} \{x_i \mid i \in I\} \cup \{y_j \mid j \notin I\},$$

where $\{x_i \mid i \in I\} \cup \{y_j \mid j \notin I\}$ is interpreted as zero if it contains neighbouring points.

**Lemma 6.8.11.** We have that $\mathcal{T} \subseteq \mathcal{O}(A_p^d)$ is Oppermann-Thomas cluster tilting if and only if $\mathrm{Ind}\mathcal{T}$ is a maximal set of non-intertwining elements in $\mathcal{O}(A_p^d)$ of cardinality

$$\binom{p+d-1}{d}.$$

*Proof.* By [49, Theorem 6.4], we have that $\mathcal{T} \subseteq \mathcal{O}(A_p^d)$ is Oppermann-Thomas cluster tilting if and only if it corresponds to a triangulation of $C(p+2d+1, 2d)$, in the notation of Oppermann and Thomas, see [49, Page 2]. Moreover, by [49, Theorems 2.3 and 2.4] such triangulations are precisely maximal sets of non-intertwining elements in $\mathcal{O}(A_p^d)$ of cardinality

$$\binom{p+d-1}{d}.$$

$\square$

**Remark 6.8.12.** Note that, by [49, Theorem 5.25], an object $T \in \mathcal{O}(A_p^d)$ is Oppermann-Thomas cluster tilting if and only if it is $2d$-cluster tilting when seen as an object in $\mathcal{C}^{2d}(A_p^d)$.

Hence, if we can find $\mathcal{T} = \operatorname{add}(T) \subseteq \mathcal{O}(A_p^d)$ Oppermann-Thomas cluster tilting, we have

$$\mathcal{T} \subseteq \mathcal{O}(A_p^d) \subseteq \mathcal{C}^{2d}(A_p^d),$$

where $\mathcal{C}^{2d}(A_p^d)$ is triangulated and $2d$-Calabi-Yau, $\mathcal{O}(A_p^d)$ is closed under $\Sigma^d$ and $d$-cluster tilting in $\mathcal{C}^{2d}(A_p^d)$ and $\mathcal{T}$ is $2d$-cluster tilting in $\mathcal{C}^{2d}(A_p^d)$. That is, we are in the situation of Setup 6.6.1 with $\mathcal{S} = \mathcal{O}(A_p^d)$ and $\mathcal{C} = \mathcal{C}^{2d}(A_p^d)$. We now choose specific values for $d$ and $p$ and, using our results, we find $K_0(\mathcal{C}^{2d}(A_p^d))$ for these values. The following result will be widely used for the computations in our example.

**Proposition 6.8.13** ([40, Theorem 5.9])**.** If $s_{d+1} \to \cdots \to s_0 \xrightarrow{\gamma} \Sigma^d s_{d+1}$ is a $(d+2)$-angle in $\mathcal{S}$, then

$$\sum_{i=0}^{d+1} (-1)^i \operatorname{index}_{\mathcal{T}}(s_i) = \theta_{\mathcal{T}}([\operatorname{Im} F_{\mathcal{T}}(\gamma)]).$$

**Example 6.8.14.** Let $p = 3$ and $d = 2$, so that $|Z| = p + 2d + 1 = 8$. For simplicity, we write the indecomposable $\{x_0, x_1, x_2\}$ as $x_0x_1x_2$. We have

$$\operatorname{Ind}\mathcal{O}(\mathrm{A}_3^2) = \{135, 136, 137, 146, 147, 157, 246, 247, 248, 257, 258, 268, 357, 358, 368, 468\}.$$

Moreover, the object $T = 135 \oplus 136 \oplus 137 \oplus 146 \oplus 147 \oplus 157 \in \mathcal{O}(A_3^2)$ is such that its indecomposable direct summands are a maximal set of non-intertwining elements in $\mathcal{O}(A_3^2)$ of the overall maximal size $\binom{3+2-1}{2} = \binom{4}{2} = 6$. So $\mathcal{T} = \operatorname{add}(T) \subset \mathcal{O}(A_3^2)$ is Oppermann-Thomas cluster tilting.

Using some 4-angles in $\mathcal{O}(A_3^2)$ obtained as described in Lemma 6.8.10 and [40, Lemma 5.6], we find the index of the indecomposables in $\mathcal{O}(A_3^2)$ with respect to $\mathcal{T}$. For example, considering $X = 135$ and $Y = 246$, by Lemma 6.8.10 we have a 4-angle in $\mathcal{O}(A_3^2)$ of the form

$$135 \to 136 \to 146 \to 246 \xrightarrow{\gamma} \Sigma^2 135.$$

Note that, since $135 \in \mathcal{T}$, by [40, Lemma 5.6], we have that

$$\text{index}_{\mathcal{T}}(246) = \text{index}_{\mathcal{T}}(146) - \text{index}_{\mathcal{T}}(136) + \text{index}_{\mathcal{T}}(135) = [146] - [136] + [135].$$

The other indices can be computed in a similar way, see Table 6.1. Brackets $[-]$ for classes in $K_0^{sp}(\mathcal{T})$ are omitted both in the table and in the rest of this example.

| $s \in \mathcal{O}(A_3^2)$ | $\text{index}_{\mathcal{T}}(s)$ |
|---|---|
| 135 | 135 |
| 136 | 136 |
| 137 | 137 |
| 146 | 146 |
| 147 | 147 |
| 157 | 157 |
| 246 | $146 - 136 + 135$ |
| 247 | $147 + 135 - 137$ |
| 248 | 135 |
| 257 | $157 - 137 + 136$ |
| 258 | 136 |
| 268 | 137 |
| 357 | $157 - 147 + 146$ |
| 358 | 146 |
| 368 | 147 |
| 468 | 157 |

Table 6.1: The index of objects of $\mathcal{O}(A_3^2)$ with respect to $\mathcal{T}$.

Consider the endomorphism algebra $\Gamma := \text{End}_{\mathcal{O}(A_3^2)}(T)$. The indecomposable projective $\Gamma$-modules are $P_x := \text{Hom}_{\mathcal{O}(A_3^2)}(T, x)$, for $x \in \mathcal{T}$ indecomposable. The simple top of $P_x$ is then denoted by $S_x$. We compute $\theta_{\mathcal{T}}([S])$ for every simple $\Gamma$-module $S$. In order to do this, we choose some morphisms $\gamma$ in $\mathcal{T}$, extend them to 4-angles in $\mathcal{O}(A_3^2)$ using Lemma 6.8.10, and compute $\theta_{\mathcal{T}}([\text{Im}\, F_{\mathcal{T}}(\gamma)])$ using Proposition 6.8.13 and Table 6.1, see Table 6.2. For example, consider $\gamma : 135 \to 136$, then we can find $\text{Im}\, F_{\mathcal{T}}(\gamma)$ as a representation

| $s_3 \to s_2 \to s_1 \to s_0 \xrightarrow{\gamma} \Sigma^2 s_3$ | $[\operatorname{Im} F_{\mathcal{T}}(\gamma)] \in K_0(\operatorname{mod}\Gamma)$ | $\theta_{\mathcal{T}}([\operatorname{Im} F_{\mathcal{T}}(\gamma)])$ |
|---|---|---|
| $247 \to 257 \to 357 \to 135 \xrightarrow{\gamma} 136$ | $[S_{135}]$ | $136 - 146$ |
| $257 \to 357 \to 135 \to 136 \xrightarrow{\gamma} 146$ | $[S_{136}]$ | $-135 + 137 + 146 - 147$ |
| $258 \to 358 \to 135 \to 137 \xrightarrow{\gamma} 147$ | $[S_{136}] + [S_{137}]$ | $-135 - 136 + 137 + 146$ |
| $258 \to 268 \to 468 \to 146 \xrightarrow{\gamma} 147$ | $[S_{136}] + [S_{146}]$ | $-136 + 137 + 146 - 157$ |
| $268 \to 468 \to 146 \to 147 \xrightarrow{\gamma} 157$ | $[S_{137}] + [S_{147}]$ | $-137 - 146 + 147 + 157$ |
| $268 \to 0 \to 0 \to 157 \xrightarrow{\gamma} 157$ | $[S_{137}] + [S_{147}] + [S_{157}]$ | $-137 + 157$ |

Table 6.2: Evaluation of $\theta_{\mathcal{T}}$ at some useful values.

of the following quiver with relations:

$$\begin{tikzcd}
 & & 137 \arrow[dr] & & \\
 & 136 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & 147 \arrow[dr] & \\
135 \arrow[ur] \arrow[rr, dotted, no head] & & 146 \arrow[ur] \arrow[rr, dotted, no head] & & 157.
\end{tikzcd}$$

Note that $\operatorname{Hom}_{\mathcal{O}(A_3^2)}(T, 135)$ is 1-dimensional over $k$ and generated by the morphism whose only non-zero component from $T$ is the identity on 135. So we use the notation $\operatorname{Hom}_{\mathcal{O}(A_3^2)}(T, 135) = \langle id_{135} \rangle$ and similarly $\operatorname{Hom}_{\mathcal{O}(A_3^2)}(T, 136) = \langle \gamma, id_{136} \rangle$ is 2-dimensional over $k$. Then, we have that

$$F_{\mathcal{T}}(\gamma) : \langle id_{135} \rangle \to \langle \gamma, id_{136} \rangle$$

and $\operatorname{Im} F_{\mathcal{T}}(\gamma) = \langle \gamma \rangle = S_{135}$, the simple $\Gamma$-module at 135. Moreover, note that $\Sigma^2 247 = 136$ and, using Lemma 6.8.10 we find the 4-angle

$$247 \to 257 \to 357 \to 135 \xrightarrow{\gamma} \Sigma^2 247.$$

By [40, Theorem 5.9] and Table 6.1, we have that

$$\begin{aligned}
\theta_{\mathcal{T}}([S_{135}]) = \theta_{\mathcal{T}}([\operatorname{Im} F_{\mathcal{T}}(\gamma)]) &= 135 - \operatorname{index}_{\mathcal{T}}(357) + \operatorname{index}_{\mathcal{T}}(257) - \operatorname{index}_{\mathcal{T}}(247) \\
&= 136 - 146.
\end{aligned}$$

Since $\theta_{\mathcal{T}}$ is additive, we can compute $\theta_{\mathcal{T}}$ at the simple $\Gamma$-modules using Table 6.2, see Table 6.3. Note that $\langle \theta_{\mathcal{T}}([S]) \mid S \text{ is a simple } \Gamma-\text{module} \rangle$ generates $\operatorname{Im} \theta_{\mathcal{T}}$ since $K_0(\operatorname{mod}\Gamma)$ is generated by the classes of the simple $\Gamma$-modules. Hence, using Table 6.3, in $K_0^{sp}(\mathcal{T}) / \operatorname{Im} \theta_{\mathcal{T}}$

| $[S_x] \in K_0(\operatorname{mod}\Gamma)$ | $\theta_{\mathcal{T}}([S_x])$ |
|---|---|
| $[S_{135}]$ | $136 - 146$ |
| $[S_{136}]$ | $-135 + 137 + 146 - 147$ |
| $[S_{137}]$ | $-135 + 137 + 146 - 147$ |
| $[S_{146}]$ | $135 - 136 + 147 - 157$ |
| $[S_{147}]$ | $136 - 137 - 146 + 157$ |
| $[S_{157}]$ | $146 - 147$ |

Table 6.3: Evaluation of $\theta_{\mathcal{T}}$ at the simple $\Gamma$-modules $S_x$.

we have that

$$136 = 146 = 147 \quad \text{and} \quad 137 = 135 = 157.$$

By Remark 6.4.7, Theorem 6.5.7 and Corollary 6.6.5, we conclude that

$$K_0(\mathcal{C}^4(A_3^2)) \cong K_0(\mathcal{O}(A_3^2)) \cong K_0^{sp}(\mathcal{T})/\operatorname{Im}\theta_{\mathcal{T}} \cong \mathbb{Z} \oplus \mathbb{Z}.$$

# Chapter 7

# Almost split morphisms in subcategories of triangulated categories

## 7.1 Introduction

In Chapter 2, we introduced Auslander-Reiten sequences in abelian categories of the form $\operatorname{mod}\Lambda$ and Auslander-Reiten triangles in triangulated categories $\mathcal{T}$. The theory of Auslander-Reiten sequences has been extended to the study of Auslander-Reiten sequences in subcategories of $\operatorname{mod}\Lambda$ by Auslander and Smalø in [7]. In the same fashion, Jørgensen extended Happel's theory on Auslander-Reiten triangles in triangulated categories to the study of Auslander-Reiten triangles in their non-triangulated subcategories in [37].

Let $k$ be a field and $\mathcal{T}$ a skeletally small $k$-linear Hom-finite triangulated category with split idempotents having a Serre functor $S$. Let $\mathcal{C} \subseteq \mathcal{T}$ be a full subcategory closed under summands and extensions. As seen in Theorem 2.2.46, any indecomposable object $X$ in $\mathcal{T}$ has an Auslander-Reiten triangle in $\mathcal{T}$ of the form

$$\tau X \to Z \to X \to \Sigma\tau X,$$

where $\tau X = S \circ \Sigma^{-1} X$. The main theorem in [37] shows that, if $C$ is an indecomposable in $\mathcal{C}$ such that $\operatorname{Hom}(C, \Sigma\mathcal{C})$ is non-zero, then there is an Auslander-Reiten triangle in $\mathcal{C}$ of the form

$$A \to B \to C \to \Sigma A$$

if and only if there is a $\mathcal{C}$-cover $A \to \tau C$.

Here we focus on the objects for which this theorem cannot be applied, **i.e.** the objects $C$ in $\mathcal{C}$ with $\mathrm{Hom}(C, \Sigma\mathcal{C}) = 0$, which we call Ext-*projectives*. Similarly, the objects for which the dual of the above theorem cannot be applied are called Ext-*injectives*. Some of the results we prove about these objects and the triangles they appear in are inspired by the ones on Ext-projective (and Ext-injective) modules and the properties of the short exact sequences they appear in, proven by Kleiner in [43].

Note that an Ext-projective object $C$ cannot appear in an Auslander-Reiten triangle in $\mathcal{C}$ of the form

$$A \to B \to C \xrightarrow{\gamma} \Sigma A,$$

since $\gamma = 0$ would be forced, contradicting Definition 7.2.4 below. However, as shown in the following theorem, for a suitable subcategory $\mathcal{C}$, we can find something quite similar to an Auslander-Reiten triangle in $\mathcal{C}$.

**Theorem 7.2.6.** *Let $\beta : B \to C$ be a minimal right almost split morphism in $\mathcal{C}$ with $C$* Ext-*projective.*

*(a) The triangle*

$$\Delta : X \xrightarrow{\xi} B \xrightarrow{\beta} C \to \Sigma X$$

*is such that $X$ is an indecomposable object not in $\mathcal{C}$ and $\xi$ is a $\mathcal{C}$-envelope of $X$.*

*(b) In part (a), the end terms $X$ and $C$ determine each other. That is, suppose $\beta' : B' \to C'$ is another minimal right almost split morphism in $\mathcal{C}$ with $C'$* Ext-*projective and extend it to a triangle: $X' \to B' \xrightarrow{\beta'} C' \to \Sigma X'$. Then $C' \cong C$ if and only if $X' \cong X$.*

For their similarity with Auslander-Reiten triangles, we call the triangles of the form $\Delta$ *left-weak Auslander-Reiten triangles in $\mathcal{C}$*. Note that $\beta : B \to C$, and hence $\Delta$, exist in fairly general circumstances, for example if $C$ is indecomposable and $\mathcal{C}$ is functorially finite in $\mathcal{T}$, see [32, Propositions 2.10 and 2.11].

For an algebra $\Lambda$ and a finitely generated $\Lambda$-module $M$, we can construct an injective resolution of $M$ using monomorphisms into injectives and their cokernels:

$$\begin{tikzcd}
 & \mu M \arrow[dr, hook] & & & \\
M \arrow[r, hook] & I^0 \arrow[u] \arrow[r] & I^1 \arrow[r] \arrow[d] & I^2 \arrow[r] & \cdots. \\
 & & \mu^2 M \arrow[ur, hook] & &
\end{tikzcd}$$

In a similar way, whenever $\mathcal{C}$ has $\mathcal{C}$-envelopes, one can construct a minimal right $\mathcal{C}$-resolution of any object $Z$ in $\mathcal{T}$, see Remark 7.3.1. Our second theorem gives us a way to find the Ext-projectives when $\mathcal{C}$ has both $\mathcal{C}$-precovers and $\mathcal{C}$-preenvelopes.

**Theorem 7.3.2.** *Assume $\mathcal{C}$ is functorially finite, let $C \in \mathcal{C}$ be indecomposable. Then $C$ is* Ext*-projective if and only if $C$ is a direct summand of $C^1$, for some $Z$ in $\mathcal{T}$ with minimal right $\mathcal{C}$-resolution*

$$Z \to C^0 \to C^1 \to \cdots,$$

*constructed as described in Remark 7.3.1.*

In [32], Iyama and Yoshino defined the mutation of a subcategory of $\mathcal{T}$ with respect to a rigid subcategory $\mathcal{D}$ of $\mathcal{T}$. Under some assumptions, mutating an extension closed subcategory $\mathcal{C}$ of $\mathcal{T}$ with respect to a rigid $\mathcal{D} \subseteq \mathcal{C}$, gives a new extension closed subcategory of $\mathcal{T}$, see [59, Theorem 3.3]. We introduce a similar process to this and show how, in some cases, removing the third term of a left-weak Auslander-Reiten triangle $\Delta$ in $\mathcal{C}$ and replacing it with the first term of $\Delta$, gives a new extension closed subcategory of $\mathcal{T}$. Let $\mathcal{X}$ be an additive subcategory of $\mathcal{T}$, let $\operatorname{Ind}\mathcal{X}$ denote a maximal set of pairwise non-isomorphic indecomposable objects in $\mathcal{X}$.

**Theorem 7.4.4.** *Assume $\mathcal{C}$ is functorially finite in $\mathcal{T}$ and $C \in \mathcal{C}$ is an indecomposable* Ext*-projective. Then there is a left-weak Auslander-Reiten triangle in $\mathcal{C}$ of the form*

$$X \xrightarrow{\xi} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma X.$$

*Let $\widetilde{\mathcal{C}}$ be the additive subcategory with $Ind\widetilde{\mathcal{C}} = Ind(\mathcal{C}) \smallsetminus C$ and define $\mathcal{C}' := add\ (\widetilde{\mathcal{C}} \cup X)$.*

*(a) If $X$ is* Ext*-injective and* Ext*-projective in $\mathcal{C}'$, then $\mathcal{C}'$ is closed under extensions.*

*(b) If $\operatorname{End}(X) \cong \operatorname{End}(C) \cong k$ and $\mathcal{C}'$ is closed under extensions, then $X$ is* Ext*-injective in $\mathcal{C}'$.*

Moreover, we show that in some cases this process and the classic mutation from [32] coincide.

**Theorem 7.5.4.** *In the setup of Theorem 7.4.4, suppose that $\mathcal{T}$ is 2-Calabi-Yau, $\mathcal{C}$ has finitely many indecomposables and $X$ is* Ext*-projective in $\mathcal{C}'$. Let $\mathcal{D}$ be the additive subcategory generated by the* Ext*-projectives in $\widetilde{\mathcal{C}}$ and $\mu(\mathcal{C};\mathcal{D})$ be the classic (backward) $\mathcal{D}$-mutation of $\mathcal{C}$. Then, we have*

$$\mu(\mathcal{C};\mathcal{D}) = \mathcal{C}',$$

*and this is a subcategory of $\mathcal{T}$ closed under extensions.*

**Remark.** We apply our results to $\mathcal{C}_{A_n}$, the cluster category of Dynkin type $A_n$, introduced in Section 3.1.1. By [26], a subcategory $\mathcal{C} \subseteq \mathcal{C}_{A_n}$ is closed under extensions and direct summands if and only if it corresponds to a so-called Ptolemy diagram of the regular $(n+3)$-gon $P$. Moreover, we show that indecomposable Ext-projectives in such a $\mathcal{C}$ are *dissecting diagonals* in the corresponding Ptolemy diagram, **i.e.** those diagonals dividing $P$ into cells.

We apply [37, Theorem 3.1] to this example to give a full description of the Auslander-Reiten triangles in $\mathcal{C}$ and Theorem 7.2.6 to give a complete description of the left-weak Auslander-Reiten triangles in $\mathcal{C}$. We show that Theorem 7.4.4 can be applied to an indecomposable Ext-projective $C$ in $\mathcal{C}$ if and only if $C$ borders two empty cells in the Ptolemy diagram corresponding to $\mathcal{C}$. Moreover, note that $\mathcal{C}_{A_n}$ is 2-Calabi-Yau and it has finitely many indecomposables. Hence, whenever $C \in \mathcal{C}$ corresponds to a dissecting diagonal bordering two empty cells, Theorem 7.5.4 implies that $\mathcal{C}'$ is the subcategory obtained by mutating $\mathcal{C}$ with respect to the additive subcategory of $\mathcal{C}$ generated by all the indecomposable Ext-projectives in $\widetilde{\mathcal{C}}$.

The chapter is organized as follows. In Section 7.2 we present Ext-projectives and prove Theorem 7.2.6. In Section 7.3 we prove Theorem 7.3.2 and in Section 7.4 we prove Theorem 7.4.4. In Section 7.5 we recall the classic mutation and prove Theorem 7.5.4. Finally, Section 7.6 is an application of our results to $\mathcal{C}_{A_n}$.

In this chapter, we work in the following setup, where, following Definition 2.1.39, additive subcategory means full subcategory closed under isomorphisms, sums and summands.

**Setup 7.1.1.** Let $k$ be a field, $\mathcal{T}$ be a skeletally small $k$-linear triangulated category with split idempotents in which each Hom space is finite dimensional over $k$. Note that this implies that $\mathcal{T}$ is a Krull-Schmidt category by Remark 2.1.12. Assume that $\mathcal{T}$ has a Serre functor $S$, see Definition 2.1.24, and note that by Theorem 2.2.46, this implies the existence of Auslander-Reiten triangles in $\mathcal{T}$. Also, let $\mathcal{C}$ be an additive subcategory of $\mathcal{T}$ that is *closed under extensions*, in the sense that if $A \to B \to C \to \Sigma A$ is a triangle in $\mathcal{T}$ with $A$ and $C$ in $\mathcal{C}$, then $B$ is also in $\mathcal{C}$.

## 7.2 Ext-projectives and weak Auslander-Reiten triangles in $\mathcal{C}$

In this section we introduce Ext-projective (respectively Ext-injective) objects in $\mathcal{C}$. We study the properties of the triangles they appear in, that we will call left-weak (respectively right-weak) Auslander-Reiten triangles in $\mathcal{C}$.

Recall that for $X$ and $Y$ in $\mathcal{T}$ and $i$ a positive integer, we have that $\operatorname{Ext}^i(X,Y) =$

$\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^i Y)$ by Definition 2.2.44.

**Definition 7.2.1.** An object $C \in \mathcal{C}$ is called Ext-*injective* if $\mathrm{Ext}^1(A, C) = 0$ for all $A \in \mathcal{C}$. An object $D \in \mathcal{C}$ is called Ext-*projective* if $\mathrm{Ext}^1(D, A) = 0$ for all $A \in \mathcal{C}$.

**Lemma 7.2.2.** (a) Let $C \in \mathcal{C}$ be an indecomposable Ext-projective object. For any non-split triangle $X \xrightarrow{\xi} B \xrightarrow{\beta} C \xrightarrow{\gamma \neq 0} \Sigma X$ with $B \in \mathcal{C}$, the morphism $\xi$ is a $\mathcal{C}$-envelope of $X$. If $\beta$ is a right minimal morphism, then $X$ is indecomposable.

(b) Let $A \in \mathcal{C}$ be an indecomposable Ext-injective object. For any non-split triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} Z \xrightarrow{\zeta \neq 0} \Sigma A$ with $B \in \mathcal{C}$, the morphism $\beta$ is a $\mathcal{C}$-cover of $Z$. If $\alpha$ is a left minimal morphism, then $Z$ is indecomposable.

We present a lemma that we will use to prove Lemma 7.2.2.

**Lemma 7.2.3.** Suppose $\xi = (\xi', \xi'') : X' \oplus X'' \to Y$ is a right minimal morphism in $\mathcal{T}$. Then $\xi' : X' \to Y$ is right minimal.

*Proof.* Consider $\varphi' : X' \to X'$ such that $\xi' \circ \varphi' = \xi'$. Taking

$$\varphi := \begin{pmatrix} \varphi' & 0 \\ 0 & 1_{X''} \end{pmatrix} : X' \oplus X'' \to X' \oplus X'',$$

we have

$$\xi \circ \varphi = (\xi' \circ \varphi', \xi'' \circ 1_{X''}) = (\xi', \xi'') = \xi.$$

As $\xi$ is right minimal, then $\varphi$ is an isomorphism and hence $\varphi'$ is also an isomorphism, meaning that $\xi'$ is right minimal. $\square$

*Proof of Lemma 7.2.2.* (a) Let $D \in \mathcal{C}$ and apply $\mathrm{Hom}_{\mathcal{T}}(-, D)$ to the triangle $\Sigma^{-1} C \xrightarrow{-\Sigma^{-1}\gamma} X \xrightarrow{\xi} B \xrightarrow{\beta} C$ to obtain the exact sequence:

$$\mathrm{Hom}_{\mathcal{T}}(B, D) \xrightarrow{\mathrm{Hom}_{\mathcal{T}}(\xi, D)} \mathrm{Hom}_{\mathcal{T}}(X, D) \to \mathrm{Hom}_{\mathcal{T}}(\Sigma^{-1} C, D).$$

Since $C$ is Ext-projective in $\mathcal{C}$, then $\mathrm{Hom}_{\mathcal{T}}(C, \Sigma D) = 0$ and hence $\mathrm{Hom}_{\mathcal{T}}(\Sigma^{-1} C, D) = 0$. Then, $\mathrm{Hom}_{\mathcal{T}}(\xi, D)$ is surjective and so every $\eta \in \mathrm{Hom}_{\mathcal{T}}(X, D)$ factors as $\eta = \epsilon \circ \xi$ for some $\epsilon \in \mathrm{Hom}_{\mathcal{T}}(B, D)$. Since this is true for every $D \in \mathcal{C}$, it follows that $\xi$ is a $\mathcal{C}$-preenvelope of $X$.

We can write $\beta = (\beta_1, \dots, \beta_t) : B = B_1 \oplus \dots \oplus B_t \to C$, where $B_1, \dots, B_t$ are indecomposable. As $C$ is also indecomposable, by [45, Section 4], each $\beta_i$ is either an isomorphism or it is in $\mathrm{rad}_{\mathcal{T}}$. Since the triangle extending $\xi$ does not split, then $\beta$ is not a split epimorphism and each $\beta_i$ is in the radical. Hence $\beta$ is in the radical. This implies that $\xi$ is left minimal and hence it is a $\mathcal{C}$-envelope of $X$.

Suppose now that $\beta$ is a right minimal morphism and let $X = X_1 \oplus \cdots \oplus X_r$ be the indecomposable decomposition of $X$. Note that $\xi$ is the direct sum of $\mathcal{C}$-envelopes $\xi_i : X_i \to B_i$, for $i = 1, \ldots, r$. In fact, by [43, Section 1], $\mathcal{C}$-envelopes are unique up to isomorphism, so the direct sum of $\mathcal{C}$-envelopes of $X_i$'s has to be isomorphic to $\xi$. Then, for each $i = 1, \ldots, r$ we have commutative diagram

$$\begin{tikzcd}
X_i \arrow[r, "\xi_i"] \arrow[d, "\iota_i"] & B_i \arrow[d, "\overline{\iota_i}"] \\
X \arrow[r, "\xi"] & B,
\end{tikzcd}$$

where $\iota_i$, $\overline{\iota_i}$ are the inclusions. Completing $\xi_i$ to a triangle, we get a commutative diagram

$$\begin{tikzcd}
X_i \arrow[r, "\xi_i"] \arrow[d, "\iota_i"] & B_i \arrow[r, "\delta_i"] \arrow[d, "\overline{\iota_i}"] & Z_i \arrow[r, "\zeta_i"] \arrow[d, dashed, "\eta_i"] & \Sigma X_i \arrow[d, "\Sigma\iota_i"] \\
X \arrow[r, "\xi"] & B \arrow[r, "\beta"] & C \arrow[r, "\gamma"] & \Sigma X,
\end{tikzcd} \tag{7.1}$$

where $\eta_i$ exists by the axioms of triangulated categories. Note that the direct sum for $i = 1, \ldots, r$ of triangles of the first row of (7.1) is isomorphic to the triangle in the second row. In particular, $C \cong Z_1 \oplus \cdots \oplus Z_r$, and since $C$ is indecomposable, without loss of generality we have $C \cong Z_1$ and $Z_i = 0$ for $i \neq 1$.

Note that since $\beta$ is right minimal, by Lemma 7.2.3 so is its restriction to $B_i$, say $\beta_i : B_i \to C$. For $i \neq 1$, we then have $\beta_i = \beta \circ \overline{\iota_i} = \eta_i \circ \delta_i = 0$ right minimal and so $B_i = 0$. But then, as $\xi_i$ is an isomorphism, it follows that $X_i = 0$ and hence $X \cong X_1$ is indecomposable.

(b) This is proven in a similar way. □

In Definition 2.2.40, we have seen the definition of Auslander-Reiten triangles in $\mathcal{T}$. We now look at Auslander-Reiten triangles in the subcategory $\mathcal{C}$ of $\mathcal{T}$.

**Definition 7.2.4** ([37, Definition 1.3]). A triangle in $\mathcal{T}$ of the form

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$$

with $A, B, C \in \mathcal{C}$ is an *Auslander-Reiten triangle in* $\mathcal{C}$ if the following are satisfied:

(a) the morphism $\gamma$ is non-zero,

(b) the morphism $\alpha$ is left almost split in $\mathcal{C}$,

(c) the morphism $\beta$ is right almost split in $\mathcal{C}$.

We prove the result corresponding to Lemma 2.2.42 in $\mathcal{C}$.

**Lemma 7.2.5.** (a) Let $\beta : B \to C$ be right almost split in $\mathcal{C}$, then $C$ is indecomposable.

(b) Let $\alpha : A \to B$ be left almost split in $\mathcal{C}$, then $A$ is indecomposable.

*Proof.* (a) In order to prove that $C$ is indecomposable, it is enough to prove that $\operatorname{End} C$, the endomorphism ring of $C$, is local. Let $\gamma_0, \gamma_1 : C \to C$ be elements in $\operatorname{End} C$ without right inverses, **i.e.** $\gamma_0, \gamma_1$ are not split epimorphisms. Then there are $\gamma_0', \gamma_1' : C \to B$ such that $\gamma_i = \beta \circ \gamma_i'$ for $i = 0, 1$ and we get $\gamma_0 + \gamma_1 = \beta \circ (\gamma_0' + \gamma_1')$. If $\gamma_0 + \gamma_1$ had a right inverse $\delta$, then

$$1_C = (\gamma_0 + \gamma_1) \circ \delta = \beta \circ (\gamma_0' + \gamma_1') \circ \delta,$$

so that $\beta$ would also have a right inverse, this is a contradiction. Hence the set of elements of $\operatorname{End} C$ without right inverses is closed under addition and so $\operatorname{End} C$ is local by [1, Proposition 15.15].

(b) This follows by a similar argument. $\square$

**Theorem 7.2.6.** *Let $\beta : B \to C$ be a minimal right almost split morphism in $\mathcal{C}$ with $C$* Ext*-projective.*

*(a) The triangle*

$$X \xrightarrow{\xi} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma X \tag{7.2}$$

*is such that $X$ is an indecomposable not in $\mathcal{C}$ and $\xi$ is a $\mathcal{C}$-envelope of $X$.*

*(b) If $\beta' : B' \to C'$ is a minimal right almost split morphism in $\mathcal{C}$ with $C'$* Ext*-projective, then $C' \cong C$ if and only if $X' \cong X$, where $X' \xrightarrow{\xi'} B' \xrightarrow{\beta'} C' \xrightarrow{\gamma'} \Sigma X'$ is the triangle obtained by extending $\beta'$.*

*Proof.* (a) Note that since $\beta$ is right almost split in $\mathcal{C}$, then $C$ is indecomposable by Lemma 7.2.5. If $X$ was in $\mathcal{C}$, as $C$ is Ext-projective, we would have $\operatorname{Ext}^1(C, X) = \operatorname{Hom}(C, \Sigma X) = 0$ and hence $\gamma = 0$ and the triangle (7.2) splitting, contradicting $\beta$ being right almost split. Hence $X \notin \mathcal{C}$ and $\gamma \neq 0$. Then, by Lemma 7.2.2, it follows that $\xi$ is a $\mathcal{C}$-envelope of $X$ and, since $\beta$ is right minimal, $X$ is indecomposable.

(b) Assume now that $\beta' : B' \to C'$ is a minimal right almost split morphism in $\mathcal{C}$ with $C'$ Ext-projective, and extend it to a triangle:

$$X' \xrightarrow{\xi'} B' \xrightarrow{\beta'} C' \xrightarrow{\gamma'} \Sigma X'.$$

By the argument above, $X' \notin \mathcal{C}$ is indecomposable and $\xi'$ is a $\mathcal{C}$-envelope of $X'$.

Suppose first that $C' \cong C$, say that $\varphi : C \to C'$ is an isomorphism. Since $\beta'$ and $\varphi \circ \beta$ are minimal right almost split morphisms with codomain $C'$, it follows from Lemma 2.1.34(b) that there is an isomorphism $\psi : B \to B'$ with $\beta' \circ \psi = \varphi \circ \beta$. By the axioms of triangulated categories, there is a morphism $\rho : X \to X'$ making the following diagram commutative:

$$\begin{tikzcd}
X \arrow[r, "\xi"] \arrow[d, dashed, "\rho"] & B \arrow[r, "\beta"] \arrow[d, "\psi"] & C \arrow[r, "\gamma"] \arrow[d, "\varphi"] & \Sigma X \arrow[d, dashed, "\Sigma\rho"] \\
X' \arrow[r, "\xi'"] & B' \arrow[r, "\beta'"] & C' \arrow[r, "\gamma'"] & \Sigma X'.
\end{tikzcd}$$

By the 5-Lemma [56, Exercise 10.2.2], it follows that $\rho$ is an isomorphism, so that $X \cong X'$.

Suppose now that $X \cong X'$, say that $\rho : X \to X'$ is an isomorphism. Since a $\mathcal{C}$-envelope of $X$ is unique up to isomorphism and $\xi$, $\xi' \circ \rho$ both are $\mathcal{C}$-envelopes of $X$, there exists an isomorphism $\psi : B \to B'$ such that $\psi \circ \xi = \xi' \circ \rho$. Then, by the axioms of triangulated categories and the 5-Lemma, there is an isomorphism $\varphi$ between $C$ and $C'$. $\square$

We state, without proof, the dual of Theorem 7.2.6.

**Theorem 7.2.7.** *Let $\alpha : A \to B$ be a minimal left almost split morphism in $\mathcal{C}$ with $A$* Ext*-injective.*

*(a) The triangle*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} Z \xrightarrow{\zeta} \Sigma A$$

*is such that $Z$ is an indecomposable not in $\mathcal{C}$ and $\beta$ is a $\mathcal{C}$-cover of $Z$.*

*(b) If $\alpha' : A' \to B'$ is a minimal left almost split morphism in $\mathcal{C}$ with $A'$* Ext*-injective, then $A' \cong A$ if and only if $Z' \cong Z$, where $A' \xrightarrow{\alpha'} B' \xrightarrow{\beta'} Z' \xrightarrow{\zeta'} \Sigma A'$ is the triangle obtained by extending $\alpha'$.*

Note that, even though the second morphism in the triangles from Theorem 7.2.6 is minimal right almost split in $\mathcal{C}$, these are not Auslander-Reiten triangles in $\mathcal{C}$ since the first object in them is not in $\mathcal{C}$. Because of this "weakness" they have, we use the following terminology.

**Definition 7.2.8.** Let $C \in \mathcal{C}$ be an indecomposable Ext-projective. If there exists a minimal right almost split morphism in $\mathcal{C}$ of the form $\beta : B \to C$, then the triangle (7.2) from Theorem 7.2.6 is called a *left-weak Auslander-Reiten triangle in* $\mathcal{C}$.

Dually, for $A \in \mathcal{C}$ indecomposable Ext-injective, if there is a minimal left almost split morphism $\alpha : A \to B$ in $\mathcal{C}$, the triangle from Theorem 7.2.7 is called a *right-weak Auslander-Reiten triangle in* $\mathcal{C}$.

**Remark 7.2.9.** Suppose $\mathcal{C}$ is functorially finite in $\mathcal{T}$. Then, by [32, Propositions 2.10 and 2.11] for any indecomposable object $C$ in $\mathcal{C}$ there is a minimal right almost split morphism in $\mathcal{C}$ ending at it and a minimal left almost split morphism in $\mathcal{C}$ starting at it. Hence, by Theorems 7.2.6 and 7.2.7, there is a left-weak Auslander-Reiten triangle in $\mathcal{C}$ ending at $C$ and a right-weak Auslander-Reiten triangle in $\mathcal{C}$ starting at $\mathcal{C}$.

We end this section by giving equivalent definitions to Ext-projectivity in the case when $\mathcal{C}$ is precovering in $\mathcal{T}$.

**Proposition 7.2.10.** Assume $\mathcal{C}$ is precovering in $\mathcal{T}$. Let $C \in \mathcal{C}$ be indecomposable and $\alpha : A \to \tau C$ be a $\mathcal{C}$-cover. Then, the following are equivalent:

(a) $C$ is Ext-projective in $\mathcal{C}$,

(b) $A = 0$,

(c) $\operatorname{Ext}^1(C, A) = 0$.

*Proof.* Note that (a) implies (c) by definition of Ext-projectivity, since $A \in \mathcal{C}$. The fact that (b) implies (c) is also clear.

To prove that (c) implies (a), assume that $C$ is not Ext-projective. By [37, Theorem 3.1], since $\alpha : A \to \tau C$ is a $\mathcal{C}$-cover, there is an Auslander-Reiten triangle in $\mathcal{C}$ of the form

$$A \to B \to C \xrightarrow{\neq 0} \Sigma A.$$

Hence $\operatorname{Ext}^1(C, A) = \operatorname{Hom}(C, \Sigma A) \neq 0$.

To prove that (c) implies (b), note that $C = \tau^{-1}(\tau C)$. Letting $D(-) = \operatorname{Hom}_k(-, k)$, we have

$$\begin{aligned} 0 = \operatorname{Ext}^1(C, A) = \operatorname{Hom}(C, \Sigma A) = \operatorname{Hom}(\tau^{-1}(\tau C), \Sigma A) &\cong \operatorname{Hom}(\tau C, \tau \circ \Sigma A) \\ \cong \operatorname{Hom}(\tau C, SA) &\cong D \operatorname{Hom}(A, \tau C). \end{aligned}$$

Then, $\operatorname{Hom}(A, \tau C) = 0$ and in particular $\alpha = 0$. Since $\alpha$ is right minimal, it follows that $A = 0$. $\square$

The dual of the above follows in a similar way. Here we state it without proof.

**Proposition 7.2.11.** Assume $\mathcal{C}$ is preenveloping in $\mathcal{T}$. Let $A \in \mathcal{C}$ be indecomposable and $\zeta : \tau^{-1}A \to C$ be a $\mathcal{C}$-envelope. Then, the following are equivalent:

(a) $A$ is Ext-injective in $\mathcal{C}$,

(b) $C = 0$,

(c) $\mathrm{Ext}^1(C, A) = 0$.

## 7.3 $\mathcal{C}$-resolutions and Ext-projectives

In this section, we use a similar idea to injective resolutions of finitely generated modules in module categories to describe Ext-projectives in $\mathcal{C}$. We also state the dual result for Ext-injectives in $\mathcal{C}$.

**Remark 7.3.1.** Suppose $\mathcal{C}$ is preenveloping. Then, in a similar way to the one used to construct injective resolutions of finitely generated modules over algebras, we can construct a *minimal right $\mathcal{C}$-resolution* of any object $Z$ in $\mathcal{T}$. We start by taking a $\mathcal{C}$-envelope of $Z$, say $\zeta^0 : Z \to C^0$ and complete it to a triangle: $Z \to C^0 \to \sigma Z \to \Sigma Z$. Then we take a $\mathcal{C}$-envelope of $\sigma Z$, say $\zeta^1 : \sigma Z \to C^1$ and complete it to a triangle. We repeat this process to obtain the minimal right $\mathcal{C}$-resolution:

$$\begin{tikzcd}
 & & \sigma Z \arrow[dr, "\zeta^1"] & & & \\
Z \arrow[r, "\zeta^0"] & C^0 \arrow[ur] \arrow[rr] & & C^1 \arrow[rr] \arrow[dr] & & C^2 \arrow[r] & \cdots. \\
 & & & & \sigma^2 Z \arrow[ur, "\zeta^2"'] & &
\end{tikzcd}$$

**Theorem 7.3.2.** *Assume $\mathcal{C}$ is functorially finite, let $C \in \mathcal{C}$ be indecomposable. Then $C$ is* Ext*-projective if and only if $C$ is a direct summand of $C^1$, for some $Z$ in $\mathcal{T}$ with minimal right $\mathcal{C}$-resolution*

$$Z \to C^0 \to C^1 \to \cdots,$$

*constructed as described in Remark 7.3.1.*

*Proof.* First note that, as $\mathcal{T}$ has a Serre functor $S : \mathcal{T} \to \mathcal{T}$, by Theorem 2.2.46 there exists an Auslander-Reiten triangle in $\mathcal{T}$ of the form $\tau C \xrightarrow{\zeta} Y \xrightarrow{\epsilon} C \xrightarrow{\gamma} \Sigma(\tau C)$, where $\tau = S \circ \Sigma^{-1}$. Suppose that $C$ is Ext-projective and consider a $\mathcal{C}$-precover $\varphi : A \to Y$ of $Y$. Completing

$\epsilon \circ \varphi$ to a triangle, we obtain a commutative diagram:

$$\begin{tikzcd}
Z \arrow[r, "\zeta'"] \arrow[d, dashed, "\phi"] & A \arrow[r, "\epsilon\circ\varphi"] \arrow[d, "\varphi"] & C \arrow[r, "\gamma'"] \arrow[d, equal] & \Sigma Z \arrow[d, dashed, "\Sigma\phi"] \\
\tau C \arrow[r, "\zeta"'] & Y \arrow[r, "\epsilon"'] & C \arrow[r, "\gamma"'] & \Sigma(\tau C),
\end{tikzcd}$$

where $\phi$ exists by the axioms of triangulated categories. Note that since $\Sigma\phi \circ \gamma' = \gamma \neq 0$, then $\gamma' \neq 0$ and so the triangle in the top row does not split. Hence the triangle in the top row is a non-splitting triangle with $C$ indecomposable Ext-projective and $A \in \mathcal{C}$, so by Lemma 7.2.2, it follows that $\zeta'$ is a $\mathcal{C}$-envelope of $Z$. Then, as $C \in \mathcal{C}$, we have a minimal right $\mathcal{C}$-resolution of $Z$:

$$Z \xrightarrow{\zeta'} A \xrightarrow{\epsilon\circ\varphi} C \to 0 \to 0 \to \dots,$$

where, following the notation of Remark 7.3.1, we have $C^1 = C$.

Suppose now that $C$ is a direct summand of $C^1$ for some $Z$ in $\mathcal{T}$ with minimal right $\mathcal{C}$-resolution:

$$\begin{tikzcd}
 & & \sigma Z \arrow[dr, "\zeta^1"] & & \\
Z \arrow[r, "\zeta^0"] & C^0 \arrow[ur, "\gamma"] \arrow[rr] & & C^1 \arrow[r] & \cdots
\end{tikzcd}$$

constructed as in Remark 7.3.1. By the dual of the Triangulated Wakamatsu's Lemma, see [37, Lemma 2.1], we have $\mathrm{Hom}(\Sigma^{-1}(\sigma Z), B) = 0$ for every $B \in \mathcal{C}$. Consider

$$\mathrm{Hom}(\Sigma^{-1}\zeta^1, B) : \mathrm{Hom}(\Sigma^{-1}C^1, B) \to \mathrm{Hom}(\Sigma^{-1}(\sigma Z), B).$$

This is injective by the dual of the proof of [37, Lemma 2.1], and since $\mathrm{Hom}(\Sigma^{-1}(\sigma Z), B) = 0$, it follows that $\mathrm{Hom}(\Sigma^{-1}C^1, B) = 0$ and so $\mathrm{Ext}^1(C^1, B) = \mathrm{Hom}(C^1, \Sigma B) = 0$. Since this is true for every $B \in \mathcal{C}$, it follows that $C^1$ is Ext-projective. Since $C$ is a direct summand of $C^1$, then $C$ is also Ext-projective. □

We present without proof the dual of Theorem 7.3.2.

**Remark 7.3.3.** Suppose $\mathcal{C}$ is precovering. Dualizing Remark 7.3.1, we can construct a *minimal left $\mathcal{C}$-resolution* of any object $X$ in $\mathcal{T}$:

$$\begin{tikzcd}
 & & \omega^2 X \arrow[dr] & & & \\
\cdots \arrow[r] & C_2 \arrow[ur, "\gamma_2"] \arrow[rr] & & C_1 \arrow[r] \arrow[dr, "\gamma_1"'] & C_0 \arrow[r, "\gamma_0"] & X, \\
 & & & & \omega X \arrow[u] &
\end{tikzcd}$$

where, for $i \geq 0$, we have that $\gamma_i$ is a $\mathcal{C}$-cover and $\omega^{i+1}(X)$ is the first object of the triangle with second morphism $\gamma_i$.

**Theorem 7.3.4.** *Assume $\mathcal{C}$ is functorially finite, let $A \in \mathcal{C}$ be indecomposable. Then $A$ is* Ext*-injective if and only if it is a direct summand of $C_1$, for some $X$ in $\mathcal{T}$ with minimal left $\mathcal{C}$-resolution*

$$\cdots \to C_1 \to C_0 \to X,$$

*constructed as described in Remark 7.3.3.*

**Example 7.3.5.** We illustrate Theorem 7.3.4 in the case that $\mathcal{C}$ is cluster-tilting. We say that the full subcategory $\mathcal{C}$ of $\mathcal{T}$ is *cluster-tilting* if:

(a) $\mathcal{C} = \{X \in \mathcal{T} \mid \operatorname{Hom}(\mathcal{C}, \Sigma X) = 0\} = \{X \in \mathcal{T} \mid \operatorname{Hom}(X, \Sigma\mathcal{C}) = 0\}$,

(b) $\mathcal{C}$ is functorially finite.

Note that when $\mathcal{T} = \mathcal{D}^b(\operatorname{mod}\Lambda)$ for some finite dimensional $k$-algebra $\Lambda$, this coincides with the case $d = 2$ in Definition 2.3.42.

If $\mathcal{C}$ is cluster-tilting, then it is closed under extensions and direct summands, so it satisfies our setup. It is immediate to see from the definition that every object in such a $\mathcal{C}$ is Ext-injective, however we present a proof of this using Theorem 7.3.4.

Let $C$ be an object in $\mathcal{C}$. Note that, since $\operatorname{Hom}(\mathcal{C}, \Sigma C) = 0$, then $0 \to \Sigma C$ is a $\mathcal{C}$-cover of $Z := \Sigma C$. Expanding this, we get the triangle in $\mathcal{T}$:

$$C \to 0 \to \Sigma C \xrightarrow{1_{\Sigma C}} \Sigma C,$$

and, as $C \xrightarrow{1_C} C$ is a $\mathcal{C}$-cover, it follows from Theorem 7.3.4 that $C = C_1$ is Ext-injective.

## 7.4 Extension closed subcategories from weak Auslander-Reiten triangles in $\mathcal{C}$

In this section, we show how, in some cases, it is possible to construct a new extension closed subcategory $\mathcal{C}' \subseteq \mathcal{T}$ modifying $\mathcal{C}$ using the objects that appear in a left-weak (or a right-weak) Auslander-Reiten triangle in $\mathcal{C}$. The idea of how this is done is similar to the mutation from [32].

**Definition 7.4.1.** For an additive subcategory $\mathcal{X}$ of $\mathcal{T}$, we denote by $\operatorname{Ind}\mathcal{X}$ a maximal set of pairwise non-isomorphic indecomposable objects in $\mathcal{X}$.

**Lemma 7.4.2.** Assume $\mathcal{C}$ is functorially finite in $\mathcal{T}$. Let $C \in \mathcal{C}$ be an indecomposable object such that $\operatorname{End}(C) \cong k$. Let $X$ in $\mathcal{T}$ be an indecomposable object. Let $\widetilde{\mathcal{C}}$ be the additive subcategory of $\mathcal{T}$ with $\operatorname{Ind}\widetilde{\mathcal{C}} = \operatorname{Ind}(\mathcal{C}) \smallsetminus C$ and set $\mathcal{C}' := \operatorname{add}(\widetilde{\mathcal{C}} \cup X)$. Then $\mathcal{C}'$ is functorially finite in $\mathcal{T}$.

*Proof.* We show that $\mathcal{C}'$ is preenveloping in $\mathcal{T}$, the proof for $\mathcal{C}' \subseteq \mathcal{T}$ precovering is then dual. By [32, Propositions 2.10, 2.11], there exists a left almost split morphism in $\mathcal{C}$ of the form $\eta : C \to D$. Since $\operatorname{End}(C) \cong k$, then every endomorphism of $C$ is either an isomorphism or it is the zero morphism. Hence $C$ is not a direct summand of $D$ and $D \in \widetilde{\mathcal{C}}$. For any $Z$ in $\mathcal{T}$, consider a $\mathcal{C}$-preenvelope $\zeta : Z \to C^n \oplus \widetilde{C}$, for some non-negative integer $n$ and $\widetilde{C} \in \widetilde{\mathcal{C}}$. Consider

$$Z \xrightarrow{\zeta} C^n \oplus \widetilde{C} \xrightarrow{\begin{pmatrix} G & 0 \\ 0 & 1_{\widetilde{C}} \end{pmatrix}} D^n \oplus \widetilde{C},$$

where $G$ is the $n \times n$ matrix having $\eta$ in the diagonal entries and zero elsewhere. Any morphism $\varphi : Z \to \widetilde{C}'$, where $\widetilde{C}' \in \widetilde{\mathcal{C}}$, factors through $\zeta$. So there exists a morphism $\gamma = (\gamma_1, \dots, \gamma_n, \widetilde{\gamma}) : C^n \oplus \widetilde{C} \to \widetilde{C}'$ such that $\gamma \circ \zeta = \varphi$. Note that, since $C$ is not a direct summand of $\widetilde{C}'$, then $\gamma_i$ is not a split monomorphism for $i = 1, \dots, n$. Hence, since $\eta$ is left almost split in $\mathcal{C}$, there exists $\delta_i : D \to \widetilde{C}'$ such that $\delta_i \circ \eta = \gamma_i$. Let $\delta = (\delta_1, \dots, \delta_n, \widetilde{\gamma}) : D^n \oplus \widetilde{C} \to \widetilde{C}'$. Then

$$\varphi = \gamma \circ \zeta = \delta \circ \zeta', \text{ for } \zeta' := \begin{pmatrix} G & 0 \\ 0 & 1_{\widetilde{C}} \end{pmatrix} \zeta.$$

Hence $\zeta' : Z \to D^n \oplus \widetilde{C}$ is a $\widetilde{\mathcal{C}}$-preenvelope. Adding some copies of $X$ to $D^n \oplus \widetilde{C}$ if necessary, we then obtain a $\mathcal{C}'$-preenvelope of $Z$. □

**Definition 7.4.3.** Let $\mathcal{X} \subset \mathcal{T}$ be an additive subcategory. The additive subcategory of $\mathcal{X}$ consisting of all the Ext-injective (respectively Ext-projective) objects in $\mathcal{X}$ is denoted $I(\mathcal{X})$ (respectively $P(\mathcal{X})$).

**Theorem 7.4.4.** *Assume $\mathcal{C}$ is functorially finite in $\mathcal{T}$ and $C \in P(\mathcal{C})$ is an indecomposable. Then there is a left-weak Auslander-Reiten triangle in $\mathcal{C}$ of the form*

$$X \xrightarrow{\xi} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma X. \tag{7.3}$$

*Let $\widetilde{\mathcal{C}} \subseteq \mathcal{T}$ be the additive subcategory with $\operatorname{Ind}\widetilde{\mathcal{C}} = \operatorname{Ind}(\mathcal{C}) \smallsetminus C$ and define $\mathcal{C}' := \operatorname{add}(\widetilde{\mathcal{C}} \cup X)$.*

*(a) If $X \in P(\mathcal{C}') \cap I(\mathcal{C}')$, then $\mathcal{C}'$ is closed under extensions.*

*(b) If* $\operatorname{End}(X) \cong \operatorname{End}(C) \cong k$ *and* $\mathcal{C}'$ *is closed under extensions, then* $X \in I(\mathcal{C}')$.

*Proof.* First note that (7.3) exists by Remark 7.2.9.

(a) Suppose $X$ is Ext-injective and Ext-projective in $\mathcal{C}'$. Consider first a triangle with end terms $\widetilde{C''}$, $\widetilde{C'}$ in $\widetilde{\mathcal{C}}$:

$$\widetilde{C''} \to A \to \widetilde{C'} \to \Sigma\widetilde{C''}.$$

Since $\mathcal{C}$ is closed under extensions and $\widetilde{\mathcal{C}} \subset \mathcal{C}$, then $A \in \mathcal{C}$. We prove that $C$ is not a direct summand of $A$, so that $A \in \widetilde{\mathcal{C}} \subset \mathcal{C}'$. Suppose for a contradiction that $A \cong \overline{A} \oplus C$ for some $\overline{A} \in \mathcal{C}$. Note that any morphism $\widetilde{C} \to C$ with $\widetilde{C} \in \widetilde{\mathcal{C}} \subset \mathcal{C}$ is not a split epimorphism, so it factors through $\beta$ since $\beta$ is right almost split in $\mathcal{C}$. Hence, by the axioms of triangulated categories, we obtain a morphism of triangles of the form

$$\begin{tikzcd}
\widetilde{C''} \arrow[r] \arrow[d] & \overline{A} \oplus C \arrow[r, "\alpha"] \arrow[d, "{(0,1)}"] & \widetilde{C'} \arrow[r] \arrow[d, dashed, "\delta"] & \Sigma\widetilde{C''} \arrow[d] \\
B \arrow[r, "\beta"'] & C \arrow[r, "\gamma"'] & \Sigma X \arrow[r, "-\Sigma\xi"'] & \Sigma B.
\end{tikzcd}$$

Since $X$ is Ext-injective in $\mathcal{C}'$ and $\widetilde{C'} \in \mathcal{C}'$, then $\delta = 0$. Hence

$$0 = \delta \circ \alpha = \gamma \circ (0,1) = (0,\gamma),$$

contradicting the fact that $\gamma$ is non-zero. So $A \in \widetilde{\mathcal{C}} \subset \mathcal{C}'$.

Consider now a triangle with end terms in $\mathcal{C}'$, say $\epsilon : C'' \to A \to C' \to \Sigma C''$. Then, denoting the direct sum of $i$ copies of $X$ by $X^i$ for a positive $i$, we have

$$\epsilon : X^t \oplus \widetilde{C''} \to A \to X^s \oplus \widetilde{C'} \xrightarrow{\gamma'} \Sigma X^t \oplus \Sigma\widetilde{C''},$$

for some non-negative integers $s$, $t$ and some $\widetilde{C''}, \widetilde{C'} \in \widetilde{\mathcal{C}}$. Note that since $X$ is Ext-injective and Ext-projective in $\mathcal{C}'$, we have

$$\gamma' = \begin{pmatrix} 0 & 0 \\ 0 & \overline{\gamma'} \end{pmatrix} : X^s \oplus \widetilde{C'} \to \Sigma X^t \oplus \Sigma\widetilde{C''}.$$

Hence $\epsilon$ is the direct sum of triangles of the form

$$X^t \xrightarrow{1} X^t \to 0 \to \Sigma X^t, \;\; 0 \to X^s \xrightarrow{1} X^s \to 0 \;\text{ and }\; \widetilde{C''} \to \overline{A} \to \widetilde{C'} \xrightarrow{\overline{\gamma'}} \Sigma\widetilde{C''}.$$

Note that, as $\widetilde{C''}, \widetilde{C'} \in \widetilde{\mathcal{C}}$, then $\overline{A} \in \widetilde{\mathcal{C}}$ and so $A \in \mathcal{C}'$. Hence $\mathcal{C}'$ is closed under extensions.

(b) Suppose now that $\operatorname{End}(X) \cong \operatorname{End}(C) \cong k$ and $\mathcal{C}'$ is closed under extensions. Suppose for a contradiction that $X$ is not Ext-injective in $\mathcal{C}'$. By Theorem 2.2.46, there is an Auslander-Reiten triangle in $\mathcal{T}$ of the form:

$$X \to Y \to \tau^{-1}X \to \Sigma X.$$

Also, since $\mathcal{C} \subseteq \mathcal{T}$ is functorially finite, then $\mathcal{C}' \subseteq \mathcal{T}$ is preenveloping by Lemma 7.4.2. Let $\tau^{-1}X \to D$ be a $\mathcal{C}'$-envelope. Then, by [37, Theorem 3.2], there is an Auslander-Reiten triangle in $\mathcal{C}'$ of the form:

$$X \xrightarrow{\xi'} E \xrightarrow{\epsilon} D \xrightarrow{\delta} \Sigma X.$$

Since $\operatorname{End}(X) \cong k$, then every endomorphism of $X$ is either an isomorphism or it is the zero morphism. Then, since $\xi'$ is not a split monomorphism, we have that $X$ is not a direct summand of $E$ and so $E \in \widetilde{\mathcal{C}} \subset \mathcal{C}$. As $\xi : X \to B$ is a $\mathcal{C}$-envelope, there exists a morphism $\varphi : B \to E$ such that $\varphi \circ \xi = \xi'$. Then, by the axioms of triangulated categories, we obtain a morphism of triangles of the form:

$$\begin{tikzcd}
X \arrow[r, "\xi"] \arrow[d, equal] & B \arrow[r, "\beta"] \arrow[d, "\varphi"] & C \arrow[r, "\gamma"] \arrow[d, dashed, "\phi"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "\xi'"'] & E \arrow[r, "\epsilon"'] & D \arrow[r, "\delta"'] & \Sigma X.
\end{tikzcd}$$

Since $\operatorname{End}(X) \cong k$ and $\xi$ is not a split monomorphism, we have that $B \in \widetilde{\mathcal{C}} \subset \mathcal{C}'$. Then, since $\xi' : X \to E$ is left almost split in $\mathcal{C}'$, there is a morphism $\eta : E \to B$ such that $\eta \circ \xi' = \xi$. By the axioms of triangulated categories, we obtain a morphism of triangles of the form:

$$\begin{tikzcd}
X \arrow[r, "\xi'"] \arrow[d, equal] & E \arrow[r, "\epsilon"] \arrow[d, "\eta"] & D \arrow[r, "\delta"] \arrow[d, dashed, "\nu"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "\xi"'] & B \arrow[r, "\beta"'] & C \arrow[r, "\gamma"'] & \Sigma X.
\end{tikzcd}$$

Consider the composition of these two triangle morphisms. As $\xi'$ is left minimal and $\xi' = \varphi \circ \eta \circ \xi'$, it follows that $\varphi \circ \eta : E \to E$ is an isomorphism. Then, by the 5-Lemma, see [56, Exercise 10.2.2], we have that $\phi \circ \nu : D \to D$ is an isomorphism. In particular, $\nu : D \to C$ is a split monomorphism and $D$ is a direct summand of $C$. As $C$ is indecomposable, this means that $D \cong C$, contradicting the fact that $D$ is in $\mathcal{C}'$ while $C$ is not. Hence $X$ is Ext-injective. □

**Remark 7.4.5.** If $\mathcal{T}$ is 2-Calabi-Yau, and recalling that $D(-) = \operatorname{Hom}_k(-, k)$, for any $X, Y \in \mathcal{T}$, we have that $\operatorname{Ext}^1(X, Y) = \operatorname{Hom}(X, \Sigma Y) \cong D \circ \operatorname{Hom}(Y, \Sigma X) = D \circ \operatorname{Ext}^1(Y, X)$. Hence Ext-projective and Ext-injective objects coincide in additive subcategories of $\mathcal{T}$.

**Corollary 7.4.6.** In the setup of Theorem 7.4.4, suppose that $\mathcal{T}$ is 2-Calabi-Yau and

$$\operatorname{End}(X) \cong \operatorname{End}(C) \cong k.$$

Then $\mathcal{C}'$ is closed under extensions if and only if $X \in I(\mathcal{C}')$.

*Proof.* Since $\mathcal{T}$ is 2-Calabi-Yau, we have that Ext-injective and Ext-projective objects in $\mathcal{C}'$ coincide by Remark 7.4.5. The result then follows directly from Theorem 7.4.4. ☐

We state, without proof, the duals of Theorem 7.4.4 and Corollary 7.4.6.

**Theorem 7.4.7.** *Assume $\mathcal{C}$ is functorially finite in $\mathcal{T}$ and $A \in I(\mathcal{C})$ is indecomposable. Then there is a right-weak Auslander-Reiten triangle in $\mathcal{C}$ of the form*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} Z \xrightarrow{\zeta} \Sigma A.$$

*Let $\underline{\mathcal{C}}$ be the additive category with $Ind\underline{\mathcal{C}} = Ind(\mathcal{C}) \smallsetminus A$ and $\mathcal{C}'' := add\ (\underline{\mathcal{C}} \cup Z)$.*

*(a) If $Z \in P(\mathcal{C}'') \cap I(\mathcal{C}'')$, then $\mathcal{C}''$ is closed under extensions.*

*(b) If $\operatorname{End}(Z) \cong \operatorname{End}(A) \cong k$ and $\mathcal{C}''$ is closed under extensions, then $Z \in P(\mathcal{C}'')$.*

**Corollary 7.4.8.** In the setup of Theorem 7.4.7, suppose that $\mathcal{T}$ is 2-Calabi-Yau and

$$\operatorname{End}(Z) \cong \operatorname{End}(A) \cong k.$$

Then $\mathcal{C}''$ is closed under extensions if and only if $Z \in P(\mathcal{C}'')$.

## 7.5 Subcategories of the form $\mathcal{C}'$ and mutations of $\mathcal{C}$

As mentioned before, the idea of how to construct $\mathcal{C}'$ from $\mathcal{C}$, by removing the third term of a left-weak Auslander-Reiten triangle $\Delta$ in $\mathcal{C}$ and replacing it with the first term of $\Delta$, is similar to the classic mutation from [32]. In general, these two constructions are different. However, they coincide under some extra assumptions, as we show in this section.

**Definition 7.5.1** ([59, Definition 3.1])**.** Let $\mathcal{D} \subseteq \mathcal{C}$ be an additive functorially finite rigid subcategory. For any object $C \in \mathcal{C}$, let $\delta : D \to C$ be a $\mathcal{D}$-cover and complete it to a triangle of the form $\mu_{\mathcal{D}}(C) \to D \xrightarrow{\delta} C \to \Sigma D$. Then $\mu_{\mathcal{D}}(C)$ is the *backward $\mathcal{D}$-mutation of $C$* and the *backward $\mathcal{D}$-mutation of $\mathcal{C}$* is

$$\mu(\mathcal{C};\mathcal{D}) := \operatorname{add}\ (\{\mu_{\mathcal{D}}(C) \mid C \in \mathcal{C}\} \cup \mathcal{D}).$$

**Lemma 7.5.2.** Assume that $\mathcal{T}$ is 2-Calabi-Yau and let $\widetilde{\mathcal{C}} \subseteq \mathcal{T}$ be an additive subcategory closed under extensions which has finitely many indecomposable objects. Letting $\mathcal{D} = P(\widetilde{\mathcal{C}})$, we have that

$$\mu(\widetilde{\mathcal{C}};\mathcal{D}) = \widetilde{\mathcal{C}}.$$

*Proof.* First note that since $\mathcal{C}$ has finitely many indecomposable objects, both $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ are functorially finite in $\mathcal{T}$. Then, since $\widetilde{\mathcal{C}}$ is an extension closed subcategory functorially finite in $\mathcal{T}$, by the dual of [32, Proposition 2.3(1)], there exists a cotorsion pair of the form $(\mathcal{X},\widetilde{\mathcal{C}})$. Since $\mathcal{D} = P(\widetilde{\mathcal{C}})$, by the dual of [59, Proposition 3.7(3)], we have that $\mu(\widetilde{\mathcal{C}};\mathcal{D}) \subseteq \widetilde{\mathcal{C}}$. Moreover, by the dual of [59, Proposition 3.7(1)], there is a bijection between $\operatorname{Ind}\widetilde{\mathcal{C}}$ and $\operatorname{Ind}\mu(\widetilde{\mathcal{C}};\mathcal{D})$ and, since these are finite sets, we conclude that $\mu(\widetilde{\mathcal{C}};\mathcal{D}) = \widetilde{\mathcal{C}}$. ☐

**Lemma 7.5.3.** In the setup of Theorem 7.4.4, suppose that $\mathcal{T}$ is 2-Calabi-Yau, $X \in P(\mathcal{C}')$ and $\operatorname{End}(C) \cong k$. Let $\mathcal{D} \subseteq \mathcal{T}$ be the additive subcategory generated by all the indecomposable Ext-projectives in $\mathcal{C}$ apart from $C$ and note that $\mathcal{D}$ is rigid. Assume $\mathcal{D}$ is functorially finite in $\mathcal{C}$. Then $X \cong \mu_{\mathcal{D}}(C)$.

*Proof.* Consider the triangle (7.3) from Theorem 7.4.4. Let $\delta : D \to C$ be a $\mathcal{D}$-cover and note that $\delta$ is not a split epimorphism since $C$ is not in $\mathcal{D}$. Then, since $\beta$ is right almost split in $\mathcal{C}$ and $D \in \mathcal{D} \subseteq \mathcal{C}$, it follows that $\delta$ factors through $\beta$ and we obtain a morphism of triangles of the form:

$$\begin{tikzcd} \mu_{\mathcal{D}}(C) \arrow[r] \arrow[d, dashed, "\eta"] & D \arrow[r, "\delta"] \arrow[d, "\varphi"] & C \arrow[r] \arrow[d, equal] & \Sigma\mu_{\mathcal{D}}(C) \arrow[d, dashed] \\ X \arrow[r, "\xi"'] & B \arrow[r, "\beta"'] & C \arrow[r, "\gamma"'] & \Sigma X, \end{tikzcd}$$

where $\eta$ exists by the axioms of triangulated categories. For $A \in \mathcal{C}$, consider the exact sequence:

$$\operatorname{Hom}(C,\Sigma A) \to \operatorname{Hom}(B,\Sigma A) \to \operatorname{Hom}(X,\Sigma A), \qquad (\ddagger)$$

and note that $\operatorname{Hom}(C,\Sigma A) = 0$ since $C$ is Ext-projective in $\mathcal{C}$. Without loss of generality, assume that $A$ is indecomposable. If $A \in \widetilde{\mathcal{C}} \subset \mathcal{C}'$, then $\operatorname{Hom}(X,\Sigma A) = 0$ since $X$ is Ext-projective in $\mathcal{C}'$. Then, exactness of (‡) forces $\operatorname{Hom}(B,\Sigma A) = 0$. If $A \notin \widetilde{\mathcal{C}}$, then $A = C$ and since $C$ is Ext-injective in $\mathcal{C}$, then $\operatorname{Hom}(B,\Sigma A) = 0$. Hence $\operatorname{Ext}^1(B,A) = 0$ for any $A$ in $\mathcal{C}$ and so $B$ is Ext-projective in $\mathcal{C}$. Since $\operatorname{End}(C) \cong k$, we have that $C$ is not a direct summand of $B$ and so $B \in \mathcal{D}$. Note that as $\delta$ is a $\mathcal{D}$-cover, then $\beta$ factors through $\delta$ and, by the axioms of triangulated categories, we obtain a morphism of triangles of the form:

$$\begin{tikzcd} X \arrow[r, "\xi"] \arrow[d, dashed, "\nu"] & B \arrow[r, "\beta"] \arrow[d, "\phi"] & C \arrow[r] \arrow[d, equal] & \Sigma X \arrow[d, dashed] \\ \mu_{\mathcal{D}}(C) \arrow[r] & D \arrow[r, "\delta"'] & C \arrow[r] & \Sigma\mu_{\mathcal{D}}(C). \end{tikzcd}$$

Then $\delta = \delta \circ \phi \circ \varphi$ and, as $\delta$ is right minimal, then $\phi \circ \varphi : D \to D$ is an isomorphism. By the 5-Lemma, see [56, Exercise 10.2.2], it follows that $\nu \circ \eta : \mu_{\mathcal{D}}(C) \to \mu_{\mathcal{D}}(C)$ is an isomorphism. Hence $\eta$ is a split monomorphism and $\mu_{\mathcal{D}}(C)$ is a direct summand of $X$. Since $X$ is indecomposable, it follows that $X \cong \mu_{\mathcal{D}}(C)$. □

**Theorem 7.5.4.** *Assume $\mathcal{C}$ has finitely many indecomposables and $C \in P(\mathcal{C})$ is an indecomposable. Then there is a left-weak Auslander-Reiten triangle in $\mathcal{C}$ of the form*

$$X \xrightarrow{\xi} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma X.$$

*Let $\widetilde{\mathcal{C}} \subseteq \mathcal{T}$ be the additive subcategory with $\mathrm{Ind}\widetilde{\mathcal{C}} = \mathrm{Ind}(\mathcal{C}) \smallsetminus C$ and define $\mathcal{C}' := \mathrm{add}\ (\widetilde{\mathcal{C}} \cup X)$. Suppose, moreover, that $\mathcal{T}$ is 2-Calabi-Yau and $X \in P(\mathcal{C}')$. Then, letting $\mathcal{D} = P(\widetilde{\mathcal{C}})$, we have*

$$\mu(\mathcal{C}; \mathcal{D}) = \mathcal{C}',$$

*and this is a subcategory of $\mathcal{T}$ closed under extensions.*

*Proof.* First note that, since $\mathcal{T}$ is 2-Calabi-Yau and $X \in P(\mathcal{C}')$, we have that $X \in P(\mathcal{C}') \cap I(\mathcal{C}')$ by Remark 7.4.5. Then, the proof of Theorem 7.4.4 shows that $\widetilde{\mathcal{C}}$ is closed under extensions. By Lemma 7.5.2, we have that $\mu(\widetilde{\mathcal{C}}; \mathcal{D}) = \widetilde{\mathcal{C}}$. Moreover, we have that $\mathrm{Ind}\, P(\mathcal{C}) = \mathrm{Ind}\, P(\widetilde{\mathcal{C}}) \cup \{C\}$, that is $P(\mathcal{C})$ is the additive category generated by all the indecomposable Ext-projectives in $\widetilde{\mathcal{C}}$ plus the indecomposable $C$. In fact, if $D$ is an indecomposable Ext-projective in $\mathcal{C}$, then either $D = C$ or $D \in \widetilde{\mathcal{C}}$. In the latter case, we have that $\mathrm{Ext}^1(D, \widetilde{\mathcal{C}}) = 0$ since $\mathrm{Ext}^1(D, \mathcal{C}) = 0$ and $\widetilde{\mathcal{C}} \subset \mathcal{C}$ and so $D$ is an indecomposable Ext-projective in $\widetilde{\mathcal{C}}$. On the other hand, if $\widetilde{D}$ is an indecomposable Ext-projective in $\widetilde{\mathcal{C}}$, we have that $\mathrm{Ext}^1(\widetilde{D}, \widetilde{\mathcal{C}}) = 0$ and, since $C$ is Ext-injective in $\mathcal{C}$, also $\mathrm{Ext}^1(\widetilde{D}, C) = 0$ so that $\mathrm{Ext}^1(\widetilde{D}, \mathcal{C}) = 0$.

Hence $\mathcal{D}$ is rigid in $\mathcal{C}$ and we can mutate $\mathcal{C}$ with respect to $\mathcal{D}$. Since $\mathcal{C}$ has finitely many indecomposables, then $\mathcal{D} \subseteq \mathcal{C}$ is functorially finite. Then, by Lemma 7.5.3, we have that $\mu_{\mathcal{D}}(C) \cong X$, where $X$ is the first term of the triangle (7.3) from Theorem 7.4.4. Hence, we conclude that

$$\mu(\mathcal{C}; \mathcal{D}) = \mathrm{add}\ (\mu(\widetilde{\mathcal{C}}; \mathcal{D}) \cup \mu_{\mathcal{D}}(C)) = \mathrm{add}\ (\widetilde{\mathcal{C}} \cup X) = \mathcal{C}',$$

and this subcategory of $\mathcal{T}$ is closed under extensions by Theorem 7.4.4. $\square$

We present the definition of forward $\mathcal{D}$-mutation and state, without proof, the dual of Theorem 7.5.4.

**Definition 7.5.5** ([59, Definition 3.1])**.** Let $\mathcal{D} \subseteq \mathcal{C}$ be an additive functorially finite rigid subcategory. For any object $A \in \mathcal{C}$, let $\alpha : A \to D$ be a $\mathcal{D}$-envelope and complete it to a triangle of the form $A \xrightarrow{\alpha} D \to \mu_{\mathcal{D}}^{-1}(A) \to \Sigma A$. Then $\mu_{\mathcal{D}}^{-1}(A)$ is the *forward $\mathcal{D}$-mutation of* $A$ and we define the *forward $\mathcal{D}$-mutation of $\mathcal{C}$* to be

$$\mu^{-1}(\mathcal{C};\mathcal{D}) \coloneqq \text{add}\left(\{\mu_{\mathcal{D}}^{-1}(A) \mid A \in \mathcal{C}\} \cup \mathcal{D}\right).$$

**Theorem 7.5.6.** *In the setup of Theorem 7.4.7, suppose that $\mathcal{T}$ is 2-Calabi-Yau, $\mathcal{C}$ has finitely many indecomposables and $Z \in I(\mathcal{C}'')$. Then, letting $\mathcal{D} = I(\underline{\mathcal{C}})$, we have*

$$\mu^{-1}(\mathcal{C};\mathcal{D}) = \mathcal{C}'',$$

*and this is a subcategory of $\mathcal{T}$ closed under extensions.*

## 7.6 Example: cluster category of Dynkin type $A_n$

In this section, we fix a positive integer $n$ and study $\mathcal{C}_{A_n}$, that is the cluster category of Dynkin type $A_n$ introduced in Section 3.1.1. Note that $\mathcal{T} \coloneqq \mathcal{C}_{A_n}$ satisfies Setup 7.1.1 and it is 2-Calabi-Yau. We give a full description of the additive subcategories $\mathcal{C}$ of $\mathcal{T}$ that are closed under extensions. For such a subcategory $\mathcal{C}$, we describe the Auslander-Reiten triangles in $\mathcal{C}$ and apply our results to this example.

### 7.6.1 Subcategories arising from Ptolemy diagrams

The additive subcategories of $\mathcal{T}$ closed under extensions are precisely those arising from *Ptolemy diagrams* in our regular $(n+3)$-gon $P$.

**Definition 7.6.1** ([26, Definition 2.1])**.** A *Ptolemy diagram* is a set $\mathcal{S}$ of diagonals of a finite polygon such that if the set contains crossing diagonals $a$ and $b$, then it also contains all the diagonals connecting the endpoints of $a$ and $b$.

Note that if we take $\mathcal{S}$ to be the empty set, then this is a Ptolemy diagram, called an *empty cell.* The set of all diagonals in a given polygon is also a Ptolemy diagram, called a *clique.*

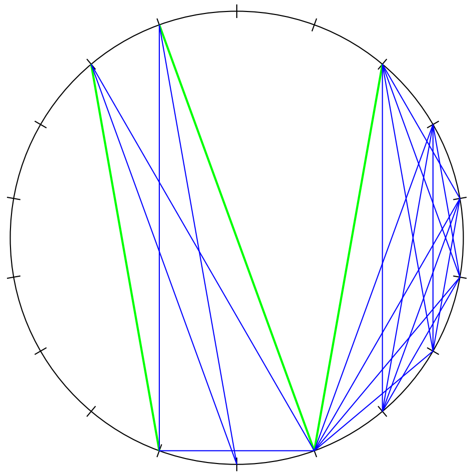


Figure 7.1: Example of a Ptolemy diagram.

**Remark 7.6.2** ([26, Theorem A(ii)]). Each Ptolemy diagram can be obtained by gluing empty cells and cliques. So, for our polygon $P$, a Ptolemy diagram is constructed by first choosing a set of pairwise non-crossing diagonals, called *dissecting diagonals*, that divide $P$ in *cells*, and then deciding whether each cell is empty or a clique.

**Example 7.6.3.** For example, in Figure 7.1 we have chosen the three green diagonals to be the dissecting diagonals and, going left to right, the first and third cells are empty and the second and fourth are cliques.

Recall that by (I) from Section 3.1.1, indecomposables in $\mathcal{T}$ correspond to diagonals in $P$. From now on let $\mathcal{C}$ be a subcategory of $\mathcal{T}$ corresponding to a Ptolemy diagram $\mathcal{S}$ of $P$, that is $\mathcal{C}$ is the additive subcategory of $\mathcal{T}$ generated by the indecomposables in $\mathcal{T}$ corresponding to the diagonals in $\mathcal{S}$. The following result is the reason why this choice satisfies Setup 7.1.1.

**Proposition 7.6.4.** Ptolemy diagrams of $P$ correspond to the additive subcategories of $\mathcal{T}$ closed under extensions.

*Proof.* This follows from [26, Theorem A(i) and Proposition 2.3]. $\square$

### 7.6.2 Auslander-Reiten triangles in $\mathcal{C}$

In this section, we apply a result by Jørgensen, see [37, Theorem 3.1], to this example in order to describe the Auslander-Reiten triangles in $\mathcal{C}$. We first recall the theorem.

**Theorem 7.6.5** ([37, Theorem 3.1]). *Let $c$ be in $\mathcal{C}$ and suppose that it is not* Ext*-projective.*

*Let*

$$x \to y \to c \to \Sigma x$$

*be an Auslander-Reiten triangle in $\mathcal{T}$. Then the following are equivalent:*

*(a) $x$ has a $\mathcal{C}$-cover of the form $\alpha : a \to x$,*

*(b) there is an Auslander-Reiten triangle in $\mathcal{C}$ of the form*

$$a \to b \to c \to \Sigma a.$$

Note that, as proved in [37], the above theorem is not only valid for $\mathcal{T} = \mathcal{C}_{A_n}$ and $\mathcal{C}$ a subcategory of $\mathcal{C}_{A_n}$ corresponding to a Ptolemy diagram of $P$, but also for any $\mathcal{T}$ and $\mathcal{C}$ satisfying Setup 7.1.1.

We aim to find for which choices of $c \in \operatorname{Ind}\mathcal{C}$ we can apply the above theorem, and what the Auslander-Reiten triangles in (b) look like.

**Proposition 7.6.6.** An indecomposable $c$ in $\mathcal{C}$ is Ext-projective if and only if it is a dissecting diagonal.

*Proof.* Suppose $c$ is a dissecting diagonal, so there is no diagonal in $\mathcal{C}$ crossing it. Hence for every $a \in \mathcal{C}$, we have $\dim_k(\operatorname{Ext}^1(c,a)) = 0$ by (II) from Section 3.1.1 and $c$ is Ext-projective.

Suppose now that $c$ is not a dissecting diagonal and let $P'$ be the clique it belongs to. Then there are vertices of $P'$ lying on both sides of $c$. Joining any two vertices of $P'$ lying on different sides of $c$, we obtain a diagonal $a'$ in $\mathcal{C}$ crossing $c$ and hence $\operatorname{Ext}^1(c,a') \neq 0$ by (II) from Section 3.1.1. So $c$ is not Ext-projective. $\square$

Note that Proposition 7.6.6 affirms that we can apply Theorem 7.6.5 to $c \in \operatorname{Ind}\mathcal{C}$ if and only if $c$ is not a dissecting diagonal of the Ptolemy diagram corresponding to $\mathcal{C}$.

**Lemma 7.6.7.** Let $c \in \mathcal{C}$ be an indecomposable which is not a dissecting diagonal. Let $P'$ be the clique to which $c$ belongs and the vertices of $P'$ be $v_1 < v_2 < \cdots < v_{m-1} < v_m$ (for some $4 \leq m \leq n+3$). Then $c = \{v_i, v_j\}$ for certain $i$ and $j$, and $a := \{v_{i-1}, v_{j-1}\}$ is a $\mathcal{C}$-cover of $\Sigma c = \{v_i^-, v_j^-\}$, where $v_0 := v_m$.

*Proof.* Since $c$ is a clique diagonal, the choice of $a$ is such that $a$ and $c$ cross, so that $\dim_k \operatorname{Hom}(a, \Sigma c) = 1$. Also, as $a$ is indecomposable, the non-zero morphism $\alpha : a \to \Sigma c$ is right minimal by Lemma 2.2.27. Hence we only need to prove that $\alpha$ is a $\mathcal{C}$-precover of

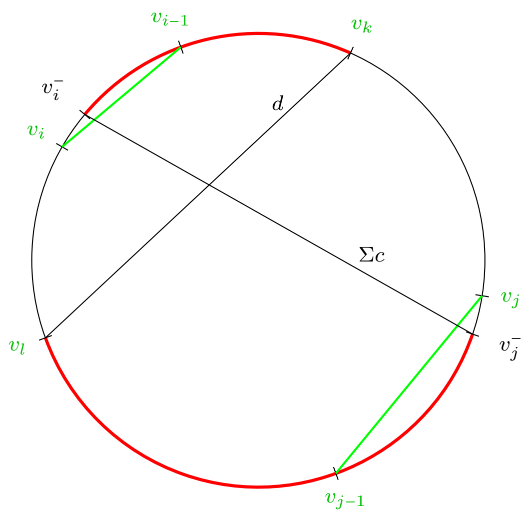



Figure 7.2: Green vertices are in $P'$ and green diagonals are edges of $P'$; diagonals with one endpoint in each of the two red arcs are those through which $d \to \Sigma c$ factors.

$\Sigma c$, **i.e.** for every $d = \{v_k, v_l\}$ in $\mathcal{C}$ and every morphism $\delta : d \to \Sigma c$, we have a factorization of the form

$$\begin{tikzcd} d \arrow[rr, "\delta"] \arrow[dr, dashed, "\exists\epsilon"'] & & \Sigma c. \\ & a \arrow[ur, "\alpha"'] & \end{tikzcd} \tag{7.4}$$

First note that if $d$ and $c$ do not cross, then $\mathrm{Hom}(d, \Sigma c) = 0$ and we have factorization (7.4) with $\delta = \epsilon = 0$. So we may assume that $d$ and $c$ cross and $v_k < v_i < v_l < v_j$. Note that since $d \in \mathcal{C}$ crosses $c$, then $d$ is a diagonal in $P'$ and so it cannot cross $\{v_i, v_{i-1}\}$ and $\{v_j, v_{j-1}\}$, as these are edges of $P'$. We then have

$$v_k \leq v_{i-1} < v_i < v_l \leq v_{j-1} < v_j.$$

Moreover, we have $v_{i-1} \leq v_i^- < v_i$ and $v_{j-1} \leq v_j^- < v_j$ and so

$$v_k \leq v_{i-1} \leq v_i^- \leq v_l^{--} < v_l \leq v_{j-1} \leq v_j^- \leq v_k^{--},$$

see Figure 7.2. Hence, by (VIII) from Section 3.1.1, there exists a factorization of the form (7.4). □

**Theorem 7.6.8.** *Let $c \in \mathcal{C}$ be an indecomposable which is not a dissecting diagonal. Let $P'$ be the clique to which $c$ belongs and the vertices of $P'$ be $v_1 < v_2 < \cdots < v_{m-1} < v_m$ (for some $4 \leq m \leq n+3$). Then $c = \{v_i, v_j\}$ for certain $i$ and $j$, and, taking $a = \{v_{i-1}, v_{j-1}\}$, we*

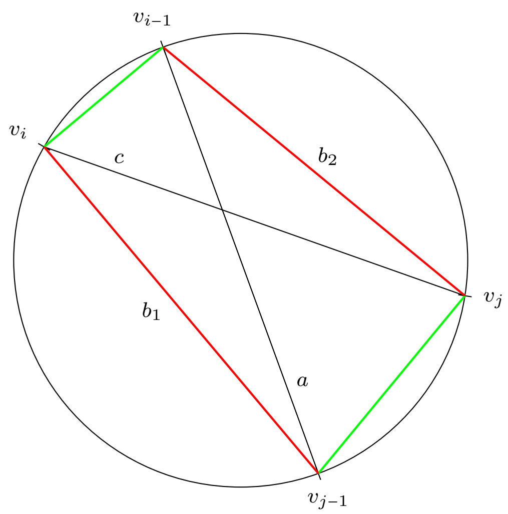



Figure 7.3: The Auslander-Reiten triangle $a \to b_1 \oplus b_2 \to c \to \Sigma a$ in $\mathcal{C}$.

*obtain the Auslander-Reiten triangle in* $\mathcal{C}$*:*

$$a \to b_1 \oplus b_2 \to c \to \Sigma a,$$

*where* $b_1 = \{v_i, v_{j-1}\}$ *and* $b_2 = \{v_j, v_{i-1}\}$*, see Figure 7.3. In the above,* $v_0 := v_m$*.*

*Proof.* By Proposition 7.6.6, $c$ is not Ext-projective. Recall that, by (V) from Section 3.1.1, we have an Auslander-Reiten triangle in $\mathcal{T}$ of the form

$$\Sigma c \to s_1 \oplus s_2 \to c \to \Sigma^2 c,$$

where $s_1$, $s_2$ are obtained as in Figure 7.4. By Lemma 7.6.7, our choice of $a$ is such that $a \to \Sigma c$ is a $\mathcal{C}$-cover. Hence by Theorem 7.6.5 there is an Auslander-Reiten triangle in $\mathcal{C}$ of the form $a \to b \to c \to \Sigma a$. The only possible such triangle is the one with $b = b_1 \oplus b_2$, where $b_1 = \{v_i, v_{j-1}\}$ and $b_2 = \{v_j, v_{i-1}\}$, see Figure 7.3. □

**Remark 7.6.9.** As seen in (V) from Section 3.1.1, for a given indecomposable $c \in \mathcal{T}$, we have an Auslander-Reiten triangle in $\mathcal{T}$ of the form

$$\Sigma c \to s_1 \oplus s_2 \to c \to \Sigma^2 c,$$

see Figure 7.4. Geometrically, the first term of this triangle, namely $\Sigma c$, is obtained by rotating the endpoints of the diagonal $c$ by one clockwise step in $P$. Moreover, $s_1$, $s_2$ are the non-edges of $P$ connecting the endpoints of $\Sigma c$ to the endpoints of $c$.

Now assume $c \in \mathcal{C}$ is not a dissecting diagonal and we follow the same geometrical process

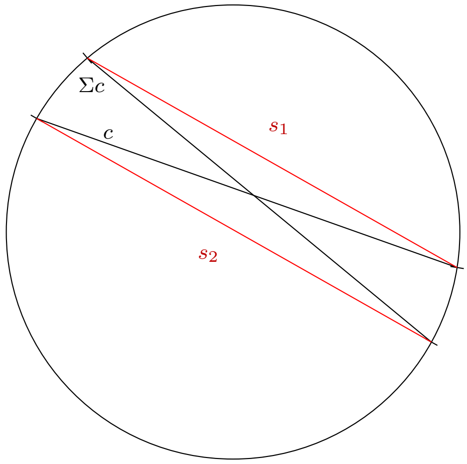



Figure 7.4: Auslander-Reiten triangle $\Sigma c \to s_1 \oplus s_2 \to c \to \Sigma^2 c$ in $\mathcal{T}$.

but looking at the clique $P'$ as an $m$-gon. The operation of rotating the endpoints of the diagonal $c$ by one clockwise step in $P'$ gives the diagonal $a$ defined in Theorem 7.6.8. Moreover, if $b_1$ and $b_2$ are the non-edges of $P'$ connecting the endpoints of $a$ to the endpoints of $c$, see Figure 7.3, Theorem 7.6.8 says that we have an Auslander-Reiten triangle in $\mathcal{C}$ of the form

$$a \to b_1 \oplus b_2 \to c \to \Sigma a.$$

### 7.6.3 Right almost split morphisms in $\mathcal{C}$ ending at Ext-projectives

Note that since $\mathcal{C} \subseteq \mathcal{T}$ has finitely many indecomposables, it is functorially finite and we can apply Theorem 7.2.6. We described the indecomposable Ext-projectives in $\mathcal{C}$ in Proposition 7.6.6. Given any indecomposable Ext-projective in $\mathcal{C}$, we give a way to find a minimal right almost split morphism in $\mathcal{C}$ ending at it and the left-weak Auslander-Reiten triangle in $\mathcal{C}$ completing this morphism.

**Setup 7.6.10.** Let $\mathcal{C}$ correspond to a Ptolemy diagram and $c \in \mathcal{C}$ be Ext-projective and indecomposable. Then $c$ is a dissecting diagonal by Proposition 7.6.6. Let the vertices of the two cells bordered by $c$ be $v_1 < v_2 < \cdots < v_m$ and $c = \{v_i, v_j\}$. Set $v_0 := v_m$.

Choose $v_p$ maximal in $[v_i^+, v_j^-]$ such that $b_0 := \{v_i, v_p\} \in \mathcal{C}$ and $v_q$ maximal in $[v_j^+, v_i^-]$ such that $b_1 := \{v_j, v_q\} \in \mathcal{C}$. An example is shown in Figure 7.5.

**Remark 7.6.11.** Note that the choice of $v_p$ depends on whether $c$ borders a clique or an empty cell in $[v_i, v_j]$. In the first case we have $v_p = v_{j-1}$ while in the second $v_p = v_{i+1}$. Note that in the case when $c$ borders an empty cell with $v_{i+1} = v_i^+$, then $b_0 = \{v_i, v_i^+\} = 0$. The vertex $v_q$ is determined in a similar way, looking at the interval $[v_j, v_i]$.

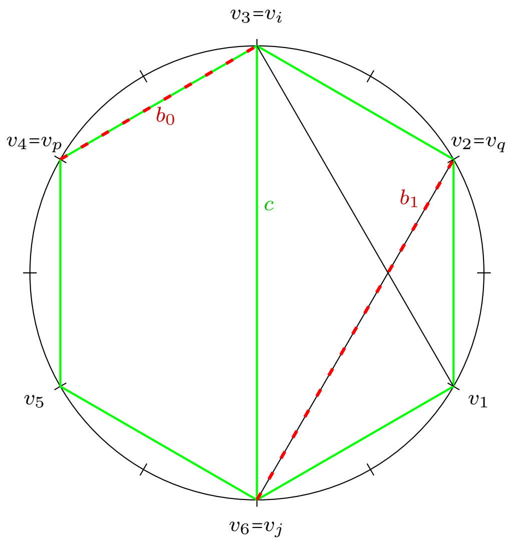



Figure 7.5: Example of Setup 7.6.10 in a Ptolemy diagram of a 12-gon. Green diagonals are dissecting diagonals, with $c = \{v_3, v_6\}$. On the left, $c$ borders an empty cell, so $v_p = v_{3+1}$ and on the right it borders a clique, so $v_q = v_{3-1}$. Then $b_0$ and $b_1$ are the red dashed diagonals.

**Proposition 7.6.12.** In the situation of Setup 7.6.10, let

$$\beta = (\beta_0, \beta_1) : b_0 \oplus b_1 \to c,$$

where $\beta_0$, $\beta_1$ are non-zero morphisms (unless $b_0$ or $b_1$ are zero). Then $\beta$ is a minimal right almost split morphism in $\mathcal{C}$ and

$$x \xrightarrow{\xi = \binom{\xi_0}{\xi_1}} b_0 \oplus b_1 \xrightarrow{\beta = (\beta_0, \beta_1)} c \to \Sigma x \tag{†}$$

is a triangle in $\mathcal{T}$ with $x = \{v_p, v_q\}$ indecomposable not in $\mathcal{C}$ and $\xi$ a $\mathcal{C}$-envelope of $x$. In other words, (†) is a left-weak Auslander-Reiten triangle in $\mathcal{C}$.

**Remark 7.6.13.** In the example illustrated in Figure 7.5, we have $x = \{v_4, v_2\}$. Note that $x$ crosses the dissecting diagonal $c$ and so it is not in $\mathcal{C}$.

*Proof of Proposition 7.6.12.* Consider an indecomposable $d \in \mathcal{C}$. By (IX) from Section 3.1.1, $\mathrm{Hom}_{\mathcal{T}}(d, c) \neq 0$ if and only if $d$ has one endpoint in each of the intervals $[v_i^{++}, v_j]$ and $[v_j^{++}, v_i]$. Since $d \in \mathcal{C}$ and $c$ is a dissecting diagonal, $d$ is not allowed to cross $c$. Hence $d = \{v_i, t\}$ for $t \in [v_i^{++}, v_j]$ or $d = \{v_j, s\}$ for $s \in [v_j^{++}, v_i]$. Note that, whenever they are non-zero, our choices of $b_0$, $b_1$ satisfy this condition and so $\dim_k \mathrm{Hom}_{\mathcal{T}}(b_i, c) = 1$.

We prove $\beta$ is right almost split in $\mathcal{C}$. Take a morphism $\gamma' : c' \to c$ in $\mathcal{C}$ that is not a split epimorphism. If $\gamma' = 0$, then $\gamma'$ clearly factorizes through $\beta$, so assume $\gamma'$ is non-zero and without loss of generality assume that $c'$ is indecomposable. Note that $\gamma'$ being not a split epimorphism forces $c' \neq c$. By the above, we have either $c' = \{v_i, t\}$ for $t \in [v_i^{++}, v_{j-1}]$ or $c' = \{v_j, s\}$ for $s \in [v_j^{++}, v_{i-1}]$. In the first case, note that such a non-zero $c'$ would not exist if $b_0 = 0$. Hence $v_p \neq v_i^+$ and by maximality of $v_p$ in $[v_i^+, v_j^-]$ such that $b_0 = \{v_i, v_p\} \in \mathcal{C}$, we have $t \leq v_p < v_j$. Then, by (IX) from Section 3.1.1, we have that $\gamma'$ factors through $\beta_0$. Similarly, in the second case we have $s \leq v_q < v_i$ and $\gamma'$ factors through $\beta_1$. Hence $\beta$ is right almost split.

We now show that $\beta$ is right minimal. Note that $b_1 = \{v_j, v_q\}$ and $\Sigma^{-1} b_0 = \{v_i^+, v_p^+\}$ do not cross. In fact we have

$$v_i < v_i^+ < v_i^{++} < v_p^+ < v_j < v_j^{++} \leq v_q.$$

Similarly, $b_0$ and $\Sigma^{-1} b_1$ do not cross. Then any morphism $\varphi : b_0 \oplus b_1 \to b_0 \oplus b_1$ has the form

$$\varphi = \begin{pmatrix} \alpha_0 1_{b_0} & 0 \\ 0 & \alpha_1 1_{b_1} \end{pmatrix} : b_0 \oplus b_1 \to b_0 \oplus b_1,$$

where $\alpha_0, \alpha_1 \in k$. If $\varphi$ is such that $\beta \circ \varphi = \beta$, then we must have $\alpha_0 = \alpha_1 = 1$, so that $\varphi$ is an isomorphism. Hence $\beta$ is a minimal right almost split morphism in $\mathcal{C}$. The rest of the proposition follows from (IV) from Section 3.1.1 and Theorem 7.2.6. □

**Remark 7.6.14.** In the situation of Proposition 7.6.12, by Theorem 7.2.6(b) we also have that if $\beta' : b' \to c'$ is a minimal right almost split morphism in $\mathcal{C}$ with $c'$ Ext-projective, then $c' \cong c$ if and only if $x' \cong x$, where $x' \xrightarrow{\xi'} b' \xrightarrow{\beta'} c' \to \Sigma x'$ is the triangle obtained by extending $\beta'$.

We now apply Corollary 7.4.6 to this example.

**Remark 7.6.15.** Note that, by the dimension of Hom spaces over $k$, we have that $\mathrm{End}(z) \cong k$ for every indecomposable $z$ in $\mathcal{T}$. Moreover, since $\mathcal{T}$ is 2-Calabi-Yau, by Remark 7.4.5, we have that Ext-projective and Ext-injective objects coincide in additive subcategories of $\mathcal{T}$.

**Proposition 7.6.16.** In the situation of Setup 7.6.10, consider the triangle $(\dagger) : x \to b_0 \oplus b_1 \to c \to \Sigma x$ from Proposition 7.6.12. Let $\widetilde{\mathcal{C}} \subseteq \mathcal{T}$ be the additive subcategory with $\mathrm{Ind}\,\widetilde{\mathcal{C}} = \mathrm{Ind}\,(\mathcal{C}) \smallsetminus c$ and $\mathcal{C}' := \mathrm{add}\,(\widetilde{\mathcal{C}} \cup x)$. Then $\mathcal{C}' \subseteq \mathcal{T}$ is closed under extensions if and only if $c$ borders two empty cells in the Ptolemy diagram corresponding to $\mathcal{C}$.

*Proof.* Suppose first that $c$ borders two empty cells in $\mathcal{C}$. Then, using the notation in Setup 7.6.10, the only diagonals in $\mathcal{C}'$ that $x = \{v_p, v_q\}$ crosses are from the set of diagonals of the form $\{v_s, v_t\}$ with $s, t \in \{1, \ldots, m\}$ that are in $\widetilde{\mathcal{C}}$. However, since $c$ borders two empty cells, such diagonals $\{v_s, v_t\}$ do not belong to $\widetilde{\mathcal{C}} \subset \mathcal{C}'$. Hence $c$ crosses no diagonals in $\mathcal{C}'$, so that $\mathrm{Ext}^1(x, \mathcal{C}') = 0$ and $x$ is Ext-projective in $\mathcal{C}'$, and so also Ext-injective in $\mathcal{C}'$ by Remark 7.6.15. By Corollary 7.4.6, we then have that $\mathcal{C}'$ is closed under extensions.

Suppose now that one of the cells bordered by $c$ is a clique with at least four vertices. Using the notation in Setup 7.6.10, without loss of generality say that the cell $v_j < v_{j+1} < \cdots < v_{i-1} < v_i$ is a clique with at least four vertices. Then, $x = \{v_p, v_{i-1}\}$ and, since the clique has at least four vertices, we have $v_j < v_{j+1} < v_{i-1} < v_i$. Then the diagonal $\widetilde{c} := \{v_{j+1}, v_i\} \in \widetilde{\mathcal{C}}$ crosses $x$ since

$$v_i < v_{i+1} \leq v_p \leq v_{j-1} < v_j < v_{j+1} < v_{i-1}.$$

Then $\mathrm{Ext}^1(x, \widetilde{c}) \neq 0$ so that $x$ is not Ext-projective, and so also not Ext-injective, in $\mathcal{C}'$. By Corollary 7.4.6, we have that $\mathcal{C}'$ is not closed under extensions. □

**Example 7.6.17.** In the example illustrated in Figure 7.5, we have that the dissecting diagonal $c$ borders an empty cell and a clique with four vertices. Then, by Proposition 7.6.16, the subcategory $\mathcal{C}'$ obtained by removing $c$ and substituting it with $x = \{v_4, v_2\}$ is not closed under extensions. In fact, it is easy to see that this does not correspond to a Ptolemy diagram.

However, if the cell to the right of $c$ was empty, then the subcategory obtained by removing $c$ and substituting it with $x = \{v_1, v_4\}$ would correspond to a Ptolemy diagram and so it would be closed under extensions.

**Remark 7.6.18.** Recall that $\mathcal{C}_{A_n}$ is 2-Calabi-Yau and it has finitely many indecomposables. Hence, whenever $C \in \mathcal{C}$ corresponds to a dissecting diagonal bordering two empty cells, Proposition 7.6.16 and Theorem 7.5.4 imply that $\mathcal{C}'$ is equal to $\mu(\mathcal{C}; \mathcal{D})$, **i.e.** the subcategory obtained by mutating $\mathcal{C}$ with respect to the additive subcategory of $\mathcal{C}$ generated by all the indecomposable Ext-projectives in $\mathcal{C}$ apart from $C$.

### 7.6.4 Left almost split morphisms in $\mathcal{C}$ starting at Ext-injectives

For completeness we state the corresponding results on Ext-injectives. These can be proven using similar arguments to the ones in Section 7.6.3.

**Proposition 7.6.19.** An indecomposable $a$ in $\mathcal{C}$ is Ext-injective if and only if it is a dissecting diagonal.

*Proof.* This follows from Proposition 7.6.6 and Remark 7.4.5. □

We present the setup and the dual of Proposition 7.6.12.

**Setup 7.6.20.** Let $\mathcal{C}$ correspond to a Ptolemy diagram and $a \in \mathcal{C}$ be Ext-injective and indecomposable. Then $a$ is a dissecting diagonal by Proposition 7.6.19. Let the vertices of the two cells bordered by $a$ be $v_1 < v_2 < \cdots < v_m$ and $a = \{v_r, v_s\}$. Set $v_0 := v_m$.

Choose $v_p$ minimal in $[v_r^+, v_s^-]$ such that $b_0 := \{v_s, v_p\} \in \mathcal{C}$ and $v_q$ minimal in $[v_s^+, v_r^-]$ such that $b_1 := \{v_r, v_q\} \in \mathcal{C}$.

**Proposition 7.6.21.** In the situation of Setup 7.6.20, let

$$\alpha = \left(\begin{smallmatrix}\alpha_0\\ \alpha_1\end{smallmatrix}\right) : a \to b_0 \oplus b_1,$$

where $\alpha_0, \alpha_1$ are non-zero morphisms (unless $b_0$ or $b_1$ are zero). Then $\alpha$ is a minimal left almost split morphism in $\mathcal{C}$ and

$$a \xrightarrow{\alpha = \left(\begin{smallmatrix}\alpha_0\\ \alpha_1\end{smallmatrix}\right)} b_0 \oplus b_1 \xrightarrow{\nu = (\nu_0, \nu_1)} z \to \Sigma a \tag{‡}$$

is a triangle in $\mathcal{T}$ with $z = \{v_p, v_q\}$ indecomposable not in $\mathcal{C}$ and $\nu$ a $\mathcal{C}$-cover of $z$. In other words, (‡) is a right-weak Auslander-Reiten triangle in $\mathcal{C}$.

**Proposition 7.6.22.** In the situation of 7.6.20, consider the triangle $a \to b_0 \oplus b_1 \to z \to \Sigma a$ from Proposition 7.6.21. Let Let $\underline{\mathcal{C}}$ be the additive category with $\operatorname{Ind}\underline{\mathcal{C}} = \operatorname{Ind}(\mathcal{C}) \smallsetminus a$ and $\mathcal{C}'' := \operatorname{add}(\underline{\mathcal{C}} \cup z)$. Then $\mathcal{C}'' \subseteq \mathcal{T}$ is closed under extensions if and only if $a$ borders two empty cells in the Ptolemy diagram corresponding to $\mathcal{C}$.

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