Superoptimal analytic approximation and exterior products

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Abstract

This dissertation concerns the classical problem of finding a bounded analytic function on the unit disc $\mathbb{D}$ which approximates a given essentially bounded function G on the unit circle $\mathbb{T}$ as well as possible in the L^∞ norm.

In the case that G is a continuous $m \times n$ matrix-valued function on $\mathbb{T}$, there is a typically large set of optimal bounded analytic approximants in the L^∞ norm, and it is therefore natural to study bounded analytic approximants Q on $\mathbb{D}$ for which $G - Q$ is minimised in a strengthened sense.

One defines, for $j \geq 0$,

$$s_j^\infty(G - Q) = \operatorname*{ess\,sup}_{z \in \mathbb{T}} s_j(G(z) - Q(z)),$$

where $s_0, s_1, \ldots, s_j$ are the singular values of a matrix. One then says that a bounded analytic matrix function Q is a *superoptimal analytic approximant of* G if Q lexicographically minimises the sequence

$$(s_0^\infty(G - Q), s_1^\infty(G - Q), \ldots,)$$

over all bounded analytic matrix functions.

It is known that every continuous matrix-valued function on $\mathbb{T}$ has a unique superoptimal analytic approximant $\mathcal{A}G$; moreover, for rational G, there are numerical procedures for the calculation of $\mathcal{A}G$. Existing algorithms are computationally intensive.

This thesis introduces a new operator-theoretic technique, based on exterior powers of Hilbert spaces and operators, for the calculation of the superoptimal analytic approximants. The result is a new algorithm which avoids some of the lengthier and potentially more ill-conditioned steps in previously described algorithms. In particular, the present algorithm does not require the spectral factorisation of matrix-valued positive functions on $\mathbb{T}$.

Declaration on collaborative work

My thesis contains collaborative work with my supervisors Dr Z. A. Lykova and Professor N. J. Young. We have submitted 2 joint research papers for publication. An extended abstract on the superoptimality is accepted by the MTNS 2021. The main problems and ideas how to approach them were provided to me by Lykova and Young. We have had weekly meetings to discuss mathematics, methods, new ideas and research papers related to my thesis. We have done research together.

The rest of each week I have worked independently on my thesis. I did calculations which were required in each step of the proofs, searched for research material related to our research project and organised all research material in my thesis. I have given several talks on topics of my thesis to Young Functional Analysts' Workshops in the University of Glasgow and in Leeds University, to Pure PhD workshops in Newcastle University and to the functional analysis seminar in King's College London.

Acknowledgements

First and foremost, I would like to thank my supervisor Dr Zinaida Lykova for the time she devoted to meeting with me and proof reading my dissertation, for her incredible patience when I struggled and for advising me on the research topic along with general matters. Our weekly meetings and her constant availability provided an invaluable support for the progress and completion of this dissertation. In addition, I would like to express my appreciation for my supervisor Professor Nicholas Young for his genuine interest in my progress and the numerous occasions on which we were able to discuss crucial points of this project. His comments provided me with an additional motivation for researching the topic in depth. I also wish to thank my supervisor Dr David Kimsey for always being available, for attending my presentations and for urging me to constantly improve. His help has been priceless for me, especially in the beginning of my PhD when he assisted me significantly in comprehending central parts of the background knowledge needed for the topic.

I am thankful to my examiners, Professors David Seifert and Alexander Pushnitski, for reading and assessing my thesis as well as for their exceptional comments. Their diverse viewpoints have been precious and have shed some new light upon underlying concepts and questions that arise naturally from this project.

Most importantly, I am grateful to my family for supporting me since the very beginning of my studies in mathematics. Without their love and encouragement to pursue knowledge, I would not have been able to start my bachelor's degree in 2010, let alone submit my PhD thesis in Newcastle University. They have always been optimistic and highly appreciative of my work and achievements. Saying farewell every time I have to leave Athens to return to the place I study is becoming more difficult than it was in the beginning. Hence, I wish the completion of my PhD fills them with great joy and pride while it heals some of the melancholy those farewells have caused over the years.

I would like to thank all my colleagues and particularly Matina Trachana, George Sofronis, Nicola Chiodetto, Omar Alsalhi, Hadi Alshammari, Robbie Bickerton, Keaghan Krog and Tao Ding for the mathematical discussions, chats, coffee-lunch breaks, dinners, research trips and many memories we shared together. Each one of them contributed to the exciting experience of being a PhD student in Newcastle University.

Finally, I wish to cordially thank Matina Trachana for her priceless support and companionship, and the imminent impact these have had on my performance, determination and successful completion of my PhD journey. Her presence has been invaluable to me and we will definitely reminisce about our years in Newcastle.

Dedication

I wish to dedicate this dissertation to my late grandmother Vasiliki who sadly passed away a few days after the first submission of this thesis. My academic successes always brought such delight to her. Born in Limnos island in 1937, she had always been an exceptional student and received her degree in Law from the University of Athens. She later married my grandfather Evangelos, who died, alas, in 1993, and gave birth to my mother and my uncle. An inspirational figure, extremely dedicated to her children and grandchildren, loved us with all her heart as we did her. Being raised so close to my grandmother taught me a great deal and I will always remember those years we lived together with joy.

Contents

Chapter 1

Introduction

1.1 Superoptimal analytic approximation

The superoptimal analytic approximation problem entails finding, for a given matrix-valued function G in $L^\infty(\mathbb{T}, \mathbb{C}^{m\times n})$, a bounded analytic matrix-valued function Q in $H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ which simultaneously minimises the essential suprema, taken over all $z \in \mathbb{T}$, of all the singular values of the matrix $G(z) - Q(z)$.

Let us first provide some preliminary definitions and then formulate the problem. Throughout the dissertation, $\mathbb{C}^{m\times n}$ denotes the space of $m \times n$ complex matrices with the operator norm and $\mathbb{D}, \mathbb{T}$ denote the unit disc and the unit circle respectively.

Definition 1.1.1. *$H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ denotes the space of bounded analytic matrix-valued functions on the unit disc with the supremum norm*

$$\|Q\|_{H^\infty} \stackrel{\text{def}}{=} \|Q\|_\infty \stackrel{\text{def}}{=} \sup_{z\in\mathbb{D}} \|Q(z)\|_{\mathbb{C}^{m\times n}}.$$

$L^\infty(\mathbb{T}, \mathbb{C}^{m\times n})$ is the space of essentially bounded Lebesgue measurable matrix-valued functions on the unit circle with the essential supremum norm

$$\|f\|_{L^\infty} = \operatorname*{ess\,sup}_{|z|=1} \|f(z)\|_{\mathbb{C}^{m\times n}}.$$

Also, $C(\mathbb{T}, \mathbb{C}^{m\times n})$ is the space of continuous matrix-valued functions from $\mathbb{T}$ to $\mathbb{C}^{m\times n}$.

Definition 1.1.2. *Let $F \in L^\infty(\mathbb{T}, \mathbb{C}^{m\times n})$ and let $s_j(F(z))$ denote the j-th singular value of the matrix $F(z)$, for $z \in \mathbb{T}$. We define*

$$s_j^\infty(F) = \operatorname*{ess\,sup}_{|z|=1} s_j(F(z))$$

and

$$s^\infty(F) = (s_0^\infty(F), s_1^\infty(F), s_2^\infty(F), \dots).$$

Problem 1.1.3 (The superoptimal analytic approximation problem)**.** *Given a function $G \in L^\infty(\mathbb{T}, \mathbb{C}^{m\times n})$, find a function $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ such that the sequence $s^\infty(G - Q)$ is minimised with respect to the lexicographic ordering.*

In general, the superoptimal analytic approximant might not be unique. However, it has been proved that if the given function G belongs to $H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$, then Problem 1.1.3 has a unique solution. The following theorem, which was obtained by V.V. Peller and N.J. Young in [24], asserts what we have just noted.

Theorem 1.1.4 ([24], p. 303)**.** *Let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$. Then the minimum with respect to the lexicographic ordering of $s^\infty(G-Q)$ over all $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ is attained at a unique function Q_0. Moreover, the singular values $s_j(G(z) - Q_0(z))$ are constant almost everywhere on $\mathbb{T}$ for $j = 0, 1, 2, \ldots$.*

The topic of this dissertation is not the existence and uniqueness of the function $\mathcal{A}G$ described in Theorem 1.1.4, but rather the *construction* of $\mathcal{A}G$. In the proof of the validity of our construction, we have no compunction in making any use of results proved in [24], such as the existence of some special matrix functions. For example, to justify our algorithm we shall prove, using results of [24], that certain operators that we introduce are unitarily equivalent to block Hankel operators, which fact enables us to make use of general properties of Schmidt vectors of Hankel operators, without the need to calculate the symbols of those Hankel operators.

The proof of Theorem 1.1.4 by Peller and Young is based on a process of diagonalisation of the error function $G - Q$, for $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$. They prove the existence of certain unitary-matrix-valued functions V, W on $\mathbb{T}$ such that $W(G - Q)V$ takes a block-diagonal form in which the singular values of $G(z) - Q(z)$ are exhibited. For any particular $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$ the matrix functions V and W can in principle be computed with the aid of Wiener-Masani matrix factorisations, or spectral factorisations, of positive-semi-definite-matrix-valued functions on the circle. However, our aim is to give an algorithm for the construction of $\mathcal{A}G$ which avoids the calculation of Wiener-Masani factorisations of matricial positive semi-definite functions. A key point of the present work is that, even though we make extensive use of the *existence* of the matrix functions V, W, the algorithm does not require us to calculate these matrix functions. This feature of the construction contrasts with the conceptual algorithm put forward by Peller and Young in [25], which does not specify a way of computing the Schmidt pairs occurring in the formula for $\mathcal{A}G$, but describes them in terms of the functions V, W, so that any straightforward way of finding the Schmidt pairs will almost certainly require the calculation of V and W. We *shall*, however, need to calculate the inner and outer factors of some elements of $H^2(\mathbb{D}, \mathbb{C}^n)$. For this purpose we need to find the spectral factors of positive *scalar-valued* functions on $\mathbb{T}$, but this is a simpler and better-conditioned computational task than the corresponding problem for matricial functions.

The purpose of this dissertation is to derive an alternative algorithm to the one given in [25], which avoids the Wiener-Masani factorisations of matrix functions and employs exterior powers of Hilbert spaces along with some of the ideas obtained in [24] and [25]. We adopt the known theory of exterior powers of Hilbert spaces with their established operations and inner product. We also present a notion of pointwise exterior product for mappings that are defined on $\mathbb{D}$ or $\mathbb{T}$ with the same operations and inner product mentioned. It is worth noting that the algorithm we introduce in Section 3.2 produces a similar formula to [25] for the superoptimal analytic approximant $\mathcal{A}G$ of a matrix-valued continuous function G on the circle, to wit

$$G - \mathcal{A}G = \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2}, \tag{1.1}$$

where x_i, y_i are certain vector-valued functions on the circle which are the Schmidt pairs of a succession of Hankel-type operators Γ_i, $t_i = \|\Gamma_i\|$ and $h_i \in H^2(\mathbb{D}, \mathbb{C})$ such that $|h_i(z)| = \|x_i(z)\|_{\mathbb{C}^n}$ almost everywhere on $\mathbb{T}$ for $i = 0, 1, 2 \dots, r-1$. The difference between the two approaches lies in the methods of defining and calculating the vectors x_i, y_i and in characterising the function spaces in which they belong. In [24],[25] the spaces are described by a block-diagonalisation procedure which requires the calculation of a "thematic completion" of an inner column-matrix function, which can itself be constructed from the spectral or Wiener-Masani factorisation of a singular positive-valued function on the circle. In the present approach, the objects x_i, y_i and the spaces in which they lie are described with the aid of wedge products of Hilbert spaces and operators. The new approach enables us to derive the functions x_i, y_i, and hence to calculate $\mathcal{A}G$ from the formula (1.1), without the spectral factorisation step. The results of this thesis are presented in [4], [5] and in the extended abstract in [39]. In the algorithm of [25], the pair of vectors (x_j, y_j) in equation (1.1) is a Schmidt pair of the operator Γ_j defined below in Theorem 3.2.54, corresponding to $\|\Gamma_j\|$. In the present algorithm we give a construction of a suitable Schmidt pair (x_j, y_j) for Γ_j corresponding to $\|\Gamma_j\|$ using exterior powers via the equations

$$x_{j+1} = (I_{\mathbb{C}^n} - \xi_0 \xi_0^* - \cdots - \xi_j \xi_j^*) v_{j+1}, \quad y_{j+1} = (I_{\mathbb{C}^m} - \bar{\eta}_0 \eta_0^T - \cdots - \bar{\eta}_j \eta_j^T) w_{j+1}, \tag{1.2}$$

where the quantities concerned are generated by the algorithm below, without any need for Wiener-Masani factorization of positive matricial functions on $\mathbb{T}$. See Lemmas 3.2.55 and 3.2.56 for the proof. Since the singular value t_i of Γ_i can perfectly well have high multiplicity, there is no sort of uniqueness of Schmidt pairs. Therefore we do not assert that the summands in the right hand side of equation (1.1) are the same in [25] and in the present algorithm, though of course the sums themselves must be, because $\mathcal{A}G$ is uniquely determined.

1.2 Main results

The main outcome of this dissertation is the algorithm for the superoptimal analytic approximation given in Chapter 3 and presented below. The reader can find an application to a concrete example in Chapter 4.

In order to present the algorithm, let us first give some preliminary definitions.

Definition 1.2.1 ([14]). *(i) $L^2(\mathbb{T},\mathbb{C}^n)$ is defined to be the space of square-integrable $\mathbb{C}^n$-valued functions on the unit circle with its natural inner product and norm*

$$\|f\|_{L^2} = \left(\frac{1}{2\pi}\int_0^{2\pi} \|f(e^{i\theta})\|^2_{\mathbb{C}^n} d\theta\right)^{1/2}.$$

(ii) $H^2(\mathbb{D},\mathbb{C}^n)$ is defined to be the space of holomorphic $\mathbb{C}^n$-valued functions on the unit disc such that

$$\lim_{r\to 1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \|f(re^{i\theta})\|^2_{\mathbb{C}^n} d\theta\right)^{1/2} < \infty.$$

(iii) $H^2(\mathbb{D},\mathbb{C}^n)^\perp$ is defined to be the space

$$H^2(\mathbb{D},\mathbb{C}^n)^\perp = \{f \in L^2(\mathbb{T},\mathbb{C}^n) \mid \langle f, g\rangle_{L^2} = 0, \text{ for all } g \in H^2(\mathbb{D},\mathbb{C}^n)\}.$$

Remark 1.2.2. *Let $0 < r < 1$ and let $f \in H^2(\mathbb{D},\mathbb{C}^n)$. By the generalised Fatou's Theorem C.2.5, the radial limits*

$$\lim_{r\to 1} f(re^{i\theta}) \underset{\|\cdot\|_{\mathbb{C}^n}}{=} \tilde{f}(e^{i\theta})$$

exist almost everywhere on $\mathbb{T}$ and define a function $\tilde{f} \in L^2(\mathbb{T},\mathbb{C}^n)$ which satisfies

$$\lim_{r\to 1} \|f(re^{i\theta}) - \tilde{f}(e^{i\theta})\|_{\mathbb{C}^n} = 0 \quad \text{almost everywhere on } \mathbb{T}.$$

Moreover, the space $H^2(\mathbb{D},\mathbb{C}^n)$ is identified isometrically with a closed subspace of $L^2(\mathbb{T},\mathbb{C}^n)$ by the injection $f \mapsto \tilde{f}$.

Definition 1.2.3. *Let $f \in L^2(\mathbb{T},\mathbb{C}^m)$ be given by*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \text{ for all } z \in \mathbb{T}.$$

The projection

$$P_- : L^2(\mathbb{T},\mathbb{C}^m) \to H^2(\mathbb{D},\mathbb{C}^m)^\perp$$

is given by

$$P_-\left(\sum_{n=-\infty}^{\infty} a_n z^n\right) = \sum_{n=-\infty}^{-1} a_n z^n \text{ for all } z \in \mathbb{T}.$$

Definition 1.2.4. *For any $G \in L^\infty(\mathbb{T},\mathbb{C}^{m\times n})$, we define the* Hankel operator *with symbol G to be the operator*

$$H_G : H^2(\mathbb{D},\mathbb{C}^n) \to H^2(\mathbb{D},\mathbb{C}^m)^\perp$$

given by

$$H_G x = P_-(Gx).$$

Definition 1.2.5. *Let H, K be Hilbert spaces. We define by $\mathcal{L}(H,K)$ the space of bounded linear operators from H to K.*

Definition 1.2.6 ([38], p. 204)**.** *Let H, K be Hilbert spaces and let $T \in \mathcal{L}(H,K)$. For any non-negative integer k, let*

$$s_k(T) = \inf\{\|T - R\| : R \in \mathcal{L}(H,K), \text{rank} R \leq k\}.$$

The numbers

$$s_0(T) \geq s_1(T) \geq s_2(T) \geq \cdots \geq 0$$

are called the singular values of T.

Remark 1.2.7. *In this dissertation we call an operator $U \colon H \to K$ between Hilbert spaces H, K a* unitary operator *if U is both isometric and surjective. Some authors restrict the name "unitary operator" to the case that $H = K$. Such authors would use a terminology like "isometric isomorphism" for our "unitary operator" in the case that $H \neq K$.*

Remark 1.2.8. *Suppose s is a singular value for a compact operator $T \in \mathcal{L}(H,K)$. Then s^2 is a singular value of T^*T, and so there is a corresponding eigenvector $x \in H$ such that*

$$T^*Tx = s^2 x.$$

If $s \neq 0$, we can let $y = s^{-1}Tx \in K$, and then

$$T^*y = sx.$$

Definition 1.2.9 ([38], p. 206)**.** *Let H, K be Hilbert spaces and let $T \colon H \to K$ be a compact operator. Suppose that s is a singular value of T. A* Schmidt pair *for T corresponding to s is a pair (x, y) of non-zero vectors $x \in H$ and $y \in K$ such that*

$$Tx = sy, \; T^*y = sx.$$

Lemma 1.2.10. *Let $T \in \mathcal{L}(H,K)$ be a compact operator and let $x \in H$, $y \in K$ be such that (x, y) is a Schmidt pair for T corresponding to $s = \|T\|$. Then x is a maximizing vector for T, y is a maximizing vector for T^*, and $\|x\|_H = \|y\|_K$.*

Proof. Since (x, y) is a Schmidt pair for T corresponding to $s = \|T\|$,

$$Tx = sy, \quad T^*y = sx.$$

Then

$$s\|y\|_K = \|Tx\|_K \le \|T\|\|x\|_H = s\|x\|_H = \|T^*y\|_H \le \|T^*\|\|y\|_K = s\|y\|_K.$$

Thus equality holds throughout, that is,

$$\|Tx\|_K = \|T\|\|x\|_H, \quad \|T^*y\|_H = \|T^*\|\|y\|_K, \quad \|x\|_H = \|y\|_K. \qquad \square$$

Definition 1.2.11 ([14], p. 190)**.** *The matrix-valued bounded analytic function $\Theta \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ is called* inner *if $\Theta(e^{it})$ is an isometry from $\mathbb{C}^n$ to $\mathbb{C}^m$ for almost every e^{it} on $\mathbb{T}$.*

An analytic $m \times n$-matrix-valued function Φ on $\mathbb{D}$ is said to be outer *if $\Phi H^2(\mathbb{D}, \mathbb{C}^n) = \{\Phi f : f \in H^2(\mathbb{D}, \mathbb{C}^n)\}$ is a norm-dense subspace of $H^2(\mathbb{D}, \mathbb{C}^m)$, and* co-outer *if $\Phi^T H^2(\mathbb{D}, \mathbb{C}^m) = \{\Phi^T g : g \in H^2(\mathbb{D}, \mathbb{C}^m)\}$ is dense in $H^2(\mathbb{D}, \mathbb{C}^n)$.*

Definition 1.2.12. *Let E be a Hilbert space and let $(\otimes_H^p, \|\cdot\|_{\otimes_H^p E})$ be the p-fold Hilbert tensor product. Let $S_p \stackrel{\text{def}}{=} \text{Sym}\{1, \cdots, p\}$ be the symmetric group on $\{1, \cdots, p\}$.*

For $\sigma \in S_p$, we define $S_\sigma : \otimes^p E \to \otimes^p E$ on elementary tensors by

$$S_\sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_p) = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(p)},$$

and we extend S_σ to $\otimes^p E$ by linearity, that is, for $u = \sum_{i=1}^n x_1^i \otimes \cdots \otimes x_p^i$, we define

$$S_\sigma(u) = \sum_{i=1}^n S_\sigma(x_1^i \otimes \cdots \otimes x_p^i).$$

A tensor $u \in \otimes_H^p E$ is called antisymmetric *if*

$$u = \epsilon_\sigma S_\sigma u$$

for all $\sigma \in S_p$, where ϵ_σ is the signature of the permutation σ. The space of all antisymmetric tensors in $\otimes_H^p E$ will be denoted by $\wedge^p E$.

Definition 1.2.13. *Let E be a Hilbert space and let $f, g \colon \mathbb{D} \to E$ ($f, g \colon \mathbb{T} \to E$) be E-valued maps. We define the* pointwise wedge product of f and g,

$$f \dot\wedge g \colon \mathbb{D} \to \wedge^2 E \quad (f \dot\wedge g \colon \mathbb{T} \to \wedge^2 E)$$

by

$$(f \dot\wedge g)(z) = f(z) \wedge g(z) \quad \text{for all } z \in \mathbb{D} \quad (\text{for almost all } z \in \mathbb{T}).$$

Recall that, by Theorem 1.1.4, if $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$, then Problem 1.1.3 has a unique solution. Given that, the endeavour to construct an algorithm that determines the unique superoptimal analytic approximant is not void.

Hence we shall devise an algorithm that, given a function

$$G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n}),$$

yields a function $\mathcal{A}G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ such that the sequence

$$s^\infty(G - \mathcal{A}G) = (s_0^\infty(G - \mathcal{A}G), s_1^\infty(G - \mathcal{A}G), \dots)$$

is minimised with respect to the lexicographic ordering.

The following is a brief summary of our algorithm. A full account of all the steps, with definitions and justifications will be given in Section 3.2.

Algorithm: For a given $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$, the superoptimal analytic approximant $\mathcal{A}G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ can be constructed as follows.
i) **Step 0.** Let $T_0 = H_G$ be the Hankel operator with symbol G as defined by Definition 1.2.4. Let $t_0 = \|H_G\|$. If $t_0 = 0$, then $H_G = 0$, which implies $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$. In this case, the algorithm terminates, we define r, which is the least index $j \geq 0$ such that $T_j = 0$, to be zero and the superoptimal approximant $\mathcal{A}G$ is given by $\mathcal{A}G = G$.
Suppose that $t_0 \neq 0$. By Hartman's Theorem 3.1.2, H_G is a compact operator and so there exists a Schmidt pair (x_0, y_0) corresponding to the singular value t_0 of H_G. By the definition of a Schmidt pair (x_0, y_0) for the Hankel operator

$$H_G : H^2(\mathbb{D}, \mathbb{C}^n) \to H^2(\mathbb{D}, \mathbb{C}^m)^\perp,$$

$$x_0 \in H^2(\mathbb{D}, \mathbb{C}^n), \quad y_0 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$$

are non-zero vector-valued functions such that

$$H_G x_0 = t_0 y_0, \quad H_G^* y_0 = t_0 x_0.$$

By Lemma 3.1.12, $x_0 \in H^2(\mathbb{D}, \mathbb{C}^n)$ and $\bar{z}\bar{y}_0 \in H^2(\mathbb{D}, \mathbb{C}^m)$ admit the inner-outer factorisations

$$x_0 = \xi_0 h_0, \quad \bar{z}\bar{y}_0 = \eta_0 h_0 \tag{1.3}$$

for some scalar outer factor $h_0 \in H^2(\mathbb{D}, \mathbb{C})$ and column matrix inner functions $\xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$, $\eta_0 \in H^\infty(\mathbb{D}, \mathbb{C}^m)$. Then,

$$\|x_0(z)\|_{\mathbb{C}^n} = |h_0(z)| = \|y_0(z)\|_{\mathbb{C}^m} \text{ almost everywhere on } \mathbb{T}. \tag{1.4}$$

We write equations (1.3) as

$$\xi_0 = \frac{x_0}{h_0}, \quad \eta_0 = \frac{\bar{z}\bar{y}_0}{h_0}. \tag{1.5}$$

By equations (1.4) and (1.5),

$$\|\xi_0(z)\|_{\mathbb{C}^n} = 1 = \|\eta_0(z)\|_{\mathbb{C}^m} \text{ almost everywhere on } \mathbb{T}. \tag{1.6}$$

By Theorem D.2.4, every function $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ which is at minimal distance from G satisfies

$$(G - Q_1)x_0 = t_0 y_0, \quad y_0^*(G - Q_1) = t_0 x_0^*. \tag{1.7}$$

ii) **Step 1.** Let

$$X_1 \stackrel{\text{def}}{=} \xi_0 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n). \tag{1.8}$$

By Proposition 3.2.2, X_1 is a closed subspace of $H^2(\mathbb{D}, \wedge^2\mathbb{C}^n)$. Moreover

$$\eta_0 \dot\wedge z H^2(\mathbb{D}, \mathbb{C}^m) \subset z H^2(\mathbb{D}, \wedge^2\mathbb{C}^m)$$

and therefore

$$\bar\eta_0 \dot\wedge \overline{z H^2(\mathbb{D}, \mathbb{C}^m)} \subset \overline{z H^2(\mathbb{D}, \wedge^2\mathbb{C}^m)},$$

that is, if

$$Y_1 \stackrel{\text{def}}{=} \bar\eta_0 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^m)^\perp, \tag{1.9}$$

then Y_1 is a closed subspace of $H^2(\mathbb{D}, \wedge^2\mathbb{C}^m)^\perp$.

Choose any function $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ which satisfies equations (1.7). Consider the operator $T_1 : X_1 \to Y_1$ defined by

$$T_1(\xi_0 \dot\wedge x) = P_{Y_1}(\bar\eta_0 \dot\wedge (G - Q_1)x) \text{ for all } x \in H^2(\mathbb{D}, \mathbb{C}^n), \tag{1.10}$$

where P_{Y_1} is the projection from $L^2(\mathbb{T}, \wedge^2\mathbb{C}^m)$ on Y_1. By Corollary 3.2.5 and Proposition 3.2.8, T_1 is well-defined. If $T_1 = 0$, then the algorithm terminates, we define r to be 1 and, in accordance with Theorem 3.2.59, the superoptimal approximant $\mathcal{A}G$ is given by the formula

$$G - \mathcal{A}G = \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2} = \frac{t_0 y_0 x_0^*}{|h_0|^2},$$

and the solution is

$$\mathcal{A}G = G - \frac{t_0 y_0 x_0^*}{|h_0|^2}.$$

Suppose $T_1 \neq 0$ and let $t_1 = \|T_1\| > 0$. By Theorem 3.2.10, T_1 is a compact operator and so there exist $v_1 \in H^2(\mathbb{D}, \mathbb{C}^n)$, $w_1 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ such that $(\xi_0 \dot\wedge v_1, \bar\eta_0 \dot\wedge w_1)$ is a Schmidt pair for T_1 corresponding to t_1. Let h_1 be the scalar outer factor of $\xi_0 \dot\wedge v_1$ and let

$$x_1 = (I_{\mathbb{C}^n} - \xi_0\xi_0^*)v_1, \quad y_1 = (I_{\mathbb{C}^m} - \bar\eta_0\eta_0^T)w_1, \tag{1.11}$$

where $\mathrm{I}_{\mathbb{C}^n}$ and $\mathrm{I}_{\mathbb{C}^m}$ are the identity operators in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. Then, by Proposition

3.2.24,

$$\|x_1(z)\|_{\mathbb{C}^n} = |h_1(z)| = \|y_1(z)\|_{\mathbb{C}^m} \text{ almost everywhere on } \mathbb{T}. \tag{1.12}$$

By Theorem 1.1.4, there exists a function $Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ such that both $s_0^\infty(G - Q_2)$ and $s_1^\infty(G - Q_2)$ are minimised, that is,

$$s_0^\infty(G - Q_2) = t_0, \quad s_1^\infty(G - Q_2) = t_1.$$

By Proposition 3.2.27, any such Q_2 satisfies

$$\begin{aligned} (G - Q_2)x_0 = t_0 y_0, &\quad y_0^*(G - Q_2) = t_0 x_0^* \\ (G - Q_2)x_1 = t_1 y_1, &\quad y_1^*(G - Q_2) = t_1 x_1^*. \end{aligned} \tag{1.13}$$

Define

$$\xi_1 = \frac{x_1}{h_1}, \quad \eta_1 = \frac{\bar{z}\bar{y}_1}{h_1}. \tag{1.14}$$

By equations (1.12) and (1.14), $\|\xi_1(z)\|_{\mathbb{C}^n} = 1 = \|\eta_1(z)\|_{\mathbb{C}^n}$ almost everywhere on $\mathbb{T}$.

Definition 1.2.14. *We say that a finite collection $\gamma_0, \dots, \gamma_j$ of elements of $L^2(\mathbb{T}, \mathbb{C}^n)$ is* pointwise orthonormal on $\mathbb{T}$ *if, for almost all $z \in \mathbb{T}$ with respect to Lebesgue measure, the set $\{\gamma_0(z), \dots, \gamma_j(z)\}$ is orthonormal in $\mathbb{C}^n$.*

iii) **Recursive step.** Suppose that, for $j \le \min(m, n) - 2$, we have constructed

$t_0 \ge t_1 \ge \cdots \ge t_j > 0$
$x_0, x_1, \cdots, x_j \in L^2(\mathbb{T}, \mathbb{C}^n)$
$y_0, y_1, \cdots, y_j \in L^2(\mathbb{T}, \mathbb{C}^m)$
$h_0, h_1, \cdots, h_j \in H^2(\mathbb{D}, \mathbb{C})$ outer
$\xi_0, \xi_1, \cdots, \xi_j \in L^2(\mathbb{T}, \mathbb{C}^n)$ pointwise orthonormal on $\mathbb{T}$
$\eta_0, \eta_1, \cdots, \eta_j \in L^2(\mathbb{T}, \mathbb{C}^m)$ pointwise orthonormal on $\mathbb{T}$
$X_0 = H^2(\mathbb{D}, \mathbb{C}^n), X_1, \cdots, X_j$
$Y_0 = H^2(\mathbb{D}, \mathbb{C}^m)^\perp, Y_1, \cdots, Y_j$
$T_0, T_1, \cdots, T_j$ compact operators.

By Theorem 1.1.4, there exists a function $Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ such that

$$\left(s_0^\infty(G - Q_{j+1}), s_1^\infty(G - Q_{j+1}), \cdots, s_j^\infty(G - Q_{j+1})\right)$$

is lexicographically minimised. By Proposition 3.2.47, any such function satisfies

$$(G - Q_{j+1})x_i = t_i y_i, \quad y_i^*(G - Q_{j+1}) = t_i x_i^*, \quad i = 0, 1, \cdots, j. \tag{1.15}$$

Define

$$X_{j+1} = \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \cdots \dot{\wedge} \xi_j \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n) \tag{1.16}$$

$$Y_{j+1} = \bar{\eta}_0 \dot{\wedge} \bar{\eta}_1 \dot{\wedge} \cdots \dot{\wedge} \bar{\eta}_j \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp. \tag{1.17}$$

Note that, by Proposition 3.2.3, X_{j+1} is a closed subspace of $H^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^n)$, and, by Proposition 3.2.6, Y_{j+1} is a closed subspace of $H^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^m)^\perp$.
Choose any function $Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ which satisfies equations (1.15). Consider the operator

$$T_{j+1} : X_{j+1} \to Y_{j+1}$$

given by

$$T_{j+1}(\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \cdots \dot{\wedge} \xi_j \dot{\wedge} x) = P_{Y_{j+1}} (\bar{\eta}_0 \dot{\wedge} \bar{\eta}_1 \dot{\wedge} \cdots \dot{\wedge} \bar{\eta}_j \dot{\wedge} (G - Q_{j+1})x) \tag{1.18}$$

for all $x \in H^2(\mathbb{D}, \mathbb{C}^n)$. By Corollary 3.2.7 and by Proposition 3.2.8, T_{j+1} is well-defined.
If $T_{j+1} = 0$, then the algorithm terminates, we define r to be $j+1$, and, in accordance with Theorem 3.2.59, the superoptimal approximant $\mathcal{A}G$ is given by the formula

$$G - \mathcal{A}G = \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2} = \sum_{i=0}^{j} \frac{t_i y_i x_i^*}{|h_i|^2}.$$

Otherwise, we define $t_{j+1} = \|T_{j+1}\| > 0$. By Theorem 3.2.54, T_{j+1} is a compact operator and hence there exist $v_{j+1} \in H^2(\mathbb{D}, \mathbb{C}^n)$, $w_{j+1} \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ such that

$$(\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \cdots \dot{\wedge} \xi_j \dot{\wedge} v_{j+1}, \bar{\eta}_0 \dot{\wedge} \bar{\eta}_1 \dot{\wedge} \cdots \dot{\wedge} \bar{\eta}_j \dot{\wedge} w_{j+1}) \tag{1.19}$$

is a Schmidt pair for T_{j+1} corresponding to the singular value t_{j+1}.
Let h_{j+1} be the scalar outer factor of $\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \cdots \dot{\wedge} \xi_j \dot{\wedge} v_{j+1}$, and let

$$x_{j+1} = (I_{\mathbb{C}^n} - \xi_0 \xi_0^* - \cdots - \xi_j \xi_j^*) v_{j+1}, \quad y_{j+1} = (I_{\mathbb{C}^m} - \bar{\eta}_0 \eta_0^T - \cdots - \bar{\eta}_j \eta_j^T) w_{j+1}, \tag{1.20}$$

and define

$$\xi_{j+1} = \frac{x_{j+1}}{h_{j+1}}, \quad \eta_{j+1} = \frac{\bar{z}\bar{y}_{j+1}}{h_{j+1}}. \tag{1.21}$$

One can show that $\|\xi_{j+1}(z)\|_{\mathbb{C}^n} = 1$ and $\|\eta_{j+1}(z)\|_{\mathbb{C}^m} = 1$ almost everywhere on $\mathbb{T}$. This completes the recursive step. The algorithm terminates after at most $\min(m, n)$ steps, so that, $r \leq \min(m, n)$ and, in accordance with Theorem 3.2.59, the superoptimal approximant $\mathcal{A}G$ is given by the formula

$$G - \mathcal{A}G = \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2}. \qquad \square$$

Remark 1.2.15. Observe that, in step j of the algorithm, we define an operator T_j in terms of any function $Q_j \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ that satisfies the equations

$$(G - Q_j)x_i = t_i y_i, \quad y_i^*(G - Q_j) = t_i x_i^*, \quad i = 0, 1, \cdots, j-1. \tag{1.22}$$

This constitutes a system of linear equations for Q_j in terms of the computed quantities

x_i, t_i and y_i for $i = 0, \ldots, j-1$, and we know, from Proposition 3.2.47, that the system has a solution for Q_j in $H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$. By Proposition 3.2.8, T_j is independent of the choice of Q_j that satisfies equations (1.22).

Remark 1.2.16. At each step we need to find $\|T_j\|$ and a Schmidt pair

$$(\xi_0 \dot\wedge \xi_1 \dot\wedge \cdots \dot\wedge \xi_{j-1} \dot\wedge v_j, \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \cdots \dot\wedge \bar\eta_{j-1} \dot\wedge w_j) \tag{1.23}$$

for T_j corresponding to the singular value t_j. Then we compute the scalar outer factor h_j of $\xi_0 \dot\wedge \xi_1 \dot\wedge \cdots \dot\wedge \xi_{j-1} \dot\wedge v_j \in H^2(\mathbb{D}, \wedge^{j+1}\mathbb{C}^n)$. These are the only spectral factorisations needed in the algorithm. Note that if $f \in H^2(\mathbb{D}, \mathbb{C}^n)$ has the inner-outer factorisation $f = hg$, with $h \in H^2(\mathbb{D}, \mathbb{C})$ a scalar outer function and $g \in H^\infty(\mathbb{D}, \mathbb{C}^n)$ inner, then $(f^*f)(z) = |h(z)|^2$ almost everywhere on $\mathbb{T}$, and so the calculation of h requires us to find a spectral factorisation of the positive *scalar-valued* function f^*f on the circle.

Remark 1.2.17. In a numerical implementation of the algorithm one would need to find a way to compute the norms and Schmidt vectors of the compact operators T_j. For this purpose it would be natural to choose convenient orthonormal bases of the cokernel $X_j \ominus \ker T_j$ and the range $\operatorname{ran} T_j$. It is safe to assume that in most applications G will be a rational function, in which case the cokernel and range will be finite-dimensional. At step 0, T_0 is a Hankel operator, and the calculation of the matrix of T_0 with respect to suitable orthonormal bases is a known task [36]; we believe that similar methods will work for step j.

We arrive at the following conclusion about the superoptimal approximant $\mathcal{A}G$.

Theorem 3.2.59. *Let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$. Let T_i, x_i, y_i, h_i for $i \geq 0$ be defined by the algorithm above. Let r be the least index $j \geq 0$ such that $T_j = 0$. Then $r \leq \min(m,n)$ and the superoptimal approximant $\mathcal{A}G$ is given by the formula*

$$G - \mathcal{A}G = \sum_0^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2}.$$

Wedge products, and in particular pointwise wedge products, along with their properties are studied in detail in Chapter 2.

1.3 Motivation for the development of an algorithm

Motivation derives from the problem of designing automatic controllers for linear time-invariant plants with multiple inputs and outputs. Such design problems are often formulated in the frequency domain, that is, in terms of the Laplace or $z-$transform of signals. By this means the problem becomes to construct an analytic matrix-valued function in a disc or half-plane, subject to various constraints. An important constraint is usually to minimise, or at least bound, some cost or penalty function. In practical engineering problems a wide variety of constraints and cost functions arise, and the engineer must take account of many complications, such as the physical limitations of devices and the imprecision of models. Engineers have developed numerous ways to cope with these complications. One of them, developed in the 1980s, is H^∞ control theory [9]. It is a wide-ranging theory. It makes pleasing contact with some problems and results of classical analysis; a seminal role was played by Nehari's theorem on the best approximation of a bounded function on the circle by an analytic function in the disc. Also important in the development of the theory were a series of deep papers by Adamyan, Arov and Krein [1],[2] which greatly extend Nehari's theorem and apply to matrix-valued functions.

In this context the notion of a superoptimal analytic approximation arose very naturally. Simple diagonal examples of a 2×2-matrix-valued function G on $\mathbb{T}$ show that the set of best analytic approximants to G in the L^∞ norm typically comprises an entire ball of functions, and so one is driven to ask for a stronger optimality criterion, and preferably one which will provide a unique optimum. The term "superoptimal" was coined by engineers even before its existence had been proved in generality. The paper [24] proved that the superoptimal approximant does indeed exist, and moreover is unique, as long as the approximand G is the sum of a continuous function and an H^∞ function on the circle. In engineering examples G is usually rational and so continuous on the circle.

Naturally engineers need to be able to compute the superoptimal approximant of G. The existence proof in [24] can in principle be turned into an algorithm, but into a very computationally intensive one. The construction is recursive, and at each step of the recursion one must augment a column-matrix function to a unitary matrix-valued function on the circle with some special properties. Computationally this step requires a *spectral factorisation* of a positive semi-definite matrix-valued function on the circle. There are indeed algorithms for this step, but they involve an iteration which may be slow to converge and badly conditioned, especially if some function values have eigenvalues on or close to the unit circle.

It is certainly desirable to avoid the matricial spectral factorisation step if it is possible to do so. Our aim in this project was to devise an algorithm in which the iterative procedures are as few and as well-conditioned as possible. Iteration cannot be completely avoided; even in the scalar case, the optimal error is the norm of a certain operator, and the best approximant is given by a simple formula involving the corresponding Schmidt vectors. Thus one has to perform a singular value decomposition of matrix-valued functions. In the case that the approximand G is of type $m \times n$ one must expect to solve $\min(m, n)$ successive singular value problems. However, from the point of view of numerical linear algebra, singular value decomposition is a fast, accurate and well-behaved operation. In this paper we describe an algorithm that is, in a sense, parallel to the construction of [25] and that requires only rational arithmetic and singular-value decompositions and the spectral factorisation of *scalar* functions.

Several engineers have developed alternative approaches [13],[27] based on state-space methods. These too are computationally intensive. We believe that our method, which makes use of exterior powers of Hilbert spaces and operators, provides a more conceptual approach to the construction of superoptimal approximants. It will be very interesting to see whether it leads to an efficient numerical method.

1.4 History and recent work

The Nehari problem of approximating an essentially bounded Lebesgue measurable function on the unit circle $\mathbb{T}$ by a bounded analytic function on the unit disc $\mathbb{D}$ in the L^∞ norm, has been attracting the interest of both pure mathematicians and engineers since the second half of the 20th century. The problem was initially formulated and solved from the scalar-valued viewpoint, with Adamjan, Arov, Krein and Sarason contributing greatly. In the years that followed, the operator-valued perspective was also explored, subsequently motivating research into the superoptimal analytic approximation problem, which we consider in this dissertation.

The initial inspiration for the study of the Nehari problem in the scalar case was the paper of Nehari [15]. Given an essentially bounded complex valued function g on $\mathbb{T}$, determine: its distance from H^∞ with respect to the essential supremum norm, for which elements this distance is attained and whether this element is uniquely determined. These problems have been studied in detail by Nehari in [15], Sarason [30] and Adamjan, Arov and Krein in [1] and [2]. Adamjan, Arov and Krein obtained significant results studying these problems; they proved that the distance is equal to the norm of the Hankel operator with symbol g, namely H_g. Moreover, if H_g has a maximizing vector in H^2, then the bounded analytic complex-valued function q that minimises the essential supremum norm $\|g-q\|_{L^\infty}$ is uniquely determined and can be explicitly calculated (see Theorem D.1.24). Furthermore, they proved that if the essential norm $\|H_g\|_e$ is less than $\|H_g\|$, then g also has a unique best approximant.

Pure mathematicians and engineers started seeking operator-valued analogues for these results. These generalisations are not only mathematically interesting. In engineering, and especially in control theory, various approximation problems arise for operator-valued functions, which enhances the motivation for the research of generalised Nehari problems in both scientific fields.

Page in [16] and Treil in [34] gave various operator-valued extensions of the obtained results by Adamjan, Arov and Krein. Page proved that for operator valued mappings $T \in L^\infty(\mathbb{T}, \mathcal{L}(E_1, E_2))$, $\inf\{\|T - \Phi\| : \Phi \in H^\infty(\mathbb{D}, \mathcal{L}(E_1, E_2)\} = \|H_T\|$. Here E_1, E_2 are Hilbert spaces and $\mathcal{L}(E_1, E_2)$ denotes the Banach space of bounded linear operators from E_1 to E_2. Treil extended the Adamjan, Arov and Krein theorem in [2] to its operator-valued analogue.

On the other hand, in the matrix-valued setting there can be infinitely many functions that best approximate a given function with respect to the L^∞ norm. This can be illustrated by considering Example D.2.6. Let $G(z) = \mathrm{diag}\{\bar{z}, 0\}, z \in \mathbb{T}$. The norm of H_G in this case is equal to 1, hence *all* the matrix-valued functions $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{2\times 2})$ of the form $Q(z) = \mathrm{diag}\{0, q(z)\}$, where $\|q\|_{H^\infty} \leq 1$, clearly minimise the norm $\|G - Q\|_{L^\infty}$.

The question that naturally arises here is whether one can determine the "very best" among those best approximants. Let us see what can be gained if one considers minimizing the essential suprema of both singular values of $G(z) - Q(z)$ instead of minimizing only the largest of them. It may easily be deduced that such a minimisation occurs when $q(z)$ is equal to 0 and the "very best approximant" in this case is the zero 2×2 matrix, that is, $Q(z) = \mathbb{O}_{2\times 2}$. Consequently, the latter reasoning strengthens the approximation criterion and one can indeed determine the "very best" amongst the best approximants.

This led to the formulation of a strengthened approximation problem, the *superoptimal analytic approximation problem.* For $G \in L^\infty(\mathbb{T}, \mathbb{C}^{m\times n})$ one defines, for $j = 0, 1, 2, \ldots$,

$$s_j^\infty(G) = \operatorname*{ess\,sup}_{|z|=1} s_j(G(z))$$

and

$$s^\infty(G) = (s_0^\infty(G), s_1^\infty(G), \ldots),$$

where $s_j(G(z))$ denotes the j-th singular value of the matrix $G(z)$. In [37] N.J. Young introduced the notion of superoptimal analytic approximation. Given a G as above, find a function $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ such that the sequence $s^\infty(G - Q)$ is lexicographically minimised. This obviously constitutes a strengthening of optimality, as one needs to determine a function $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ that not only minimises $\|G - Q\|_{L^\infty}$, but minimises the L^∞ norm of all the subsequent singular values of $G(z) - Q(z)$ over $\mathbb{T}$.

The starting point for the superoptimal analytic approximation of matrix functions is [24]. The problem is to determine, given a (matrix-valued) function

$$G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n}),$$

a function $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ such that the sequence $s^\infty(G - Q)$ is lexicographically minimised. Peller and Young obtained significant results on thematic factorisations, on the analyticity of the minors of unitary completions of inner matrix-columns and on the compactness of Hankel operators with matrix symbols. These provided the foundation for their notable result, namely if G belongs to $H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$, there exists a unique $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ such that the sequence $s^\infty(G - Q)$ is lexicographically minimised. Moreover, the singular values $s_j(G(z) - Q(z)$ are constant almost everywhere on $\mathbb{T}$ for all $j = 0, 1, 2, \ldots$.

Later, in [25] Peller and Young presented a conceptual algorithm for the computation of the superoptimal approximant. Their algorithm is based on the theory developed in [24]. Also in [25], the algorithm was applied to a specific matrix-valued G in $H^\infty(\mathbb{D}, \mathbb{C}^{2\times 2}) + C(\mathbb{T}, \mathbb{C}^{2\times 2})$ and the superoptimal approximant $\mathcal{A}G$ was calculated. It is worth noting that the thematic completions described in [24] and [25] invoke spectral (or Wiener-Masani) factorisations of positive matrix functions and the corona theorem.

Furthermore, Peller and Young in [26] studied the superoptimal approximation by meromorphic matrix-valued functions, that is, matrix-valued functions in $H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ that have at most k poles in the open unit disc. They adjusted the results obtained in [24] and established a uniqueness criterion in the case where the given matrix-valued function G is in $H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$ and has at most k poles in the open unit disc. In addition, they provided a different algorithm in order to calculate the superoptimal approximant.

Towards the extension of the results in the operator case, the operator-valued superoptimal approximation problem was studied by Peller in [21]. The author generalised the notions of [24] and proved there exists a unique superoptimal approximant in $H^\infty(\mathcal{B})$ for functions that belong to $H^\infty(\mathcal{B}) + C(\mathcal{K})$, where $\mathcal{B}$ denotes the space of bounded linear operators and $\mathcal{K}$ denotes the space of compact operators.

Very badly approximable functions, that is, functions that have the zero function as a superoptimal approximant, were studied in the years that followed and a considerable amount of work was published. Peller and Young's paper [24] provided the motivation for the study of this problem, where they were able to algebraically characterise the very badly approximable matrix functions of class $H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$. Their results were extended in [22] to the case of matrix functions G for which the essential norm $\|H_G\|_e$ is less than the smallest non-zero superoptimal singular value of G. Very badly approximable matrix functions with entries in $H^\infty + C$ were completely characterised in [23].

Recent work in [3] by Baratchart, Nazarov and Peller explores the analytic approximation of matrix-valued functions in L^p of the unit circle by matrix-valued functions from H^p of the unit disc in the L^p norm for $2 \le p < \infty$. They proved that if a given matrix-valued function $\Psi \in L^p(\mathbb{T}, \mathbb{C}^{m\times n})$ is a 'respectable' matrix function, then its distance from $H^p(\mathbb{D}, \mathbb{C}^{m\times n})$ is equal to $\|H_\Psi\|$, and they obtained a characterisation of that distance also in the case Ψ is a 'weird' matrix-valued function. Furthermore, they established the notion of p-superoptimal approximation and illustrated that every $n \times n$ rational matrix function has a unique p-superoptimal approximant for $2 \le p < \infty$. However, for p-approximable functions with $p = \infty$, they provided an example of a function that has two different p-superoptimal approximants.

In a more recent paper of Condori [6], the author considered the relation between the sum of the superoptimal singular values of admissible functions in $L^\infty(\mathbb{T}, \mathbb{C}^{m\times n})$ and the superoptimal analytic approximation problem in the space $L^\infty(\mathbb{T}, S_p^{m,n})$, where $S_p^{m,n}$ denotes the space of $m \times n$ matrices endowed with the Schatten-von-Neumann norm $\|\cdot\|_{S_p^{m,n}}$. Condori illustrated that if $\Phi \in L^\infty(\mathbb{T}, \mathbb{C}^{n\times n})$ is an admissible matrix function of order k, then $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{n\times n})$ is a best approximant function under the $L^\infty(\mathbb{T}, S_1^{n,n})$-norm and the singular values $s_j((\phi - Q)(z))$ are constant almost everywhere on $\mathbb{T}$ for all $0 \le j \le k-1$ if and only if Q is a superoptimal approximant to Φ, $\operatorname{ess\,sup}_{z\in\mathbb{T}} s_j((\Phi - Q)(z)) = 0$ for $j \ge k$, and the sum of the superoptimal values of Φ is equal to

$$\sup \left| \int_{\mathbb{T}} \operatorname{trace}(\Phi(\zeta)\Psi(\zeta))\, dm(\zeta) \right|,$$

where $m, n > 1$, $1 \leq k \leq \min(m,n)$ and the supremum is taken over all $\Psi \in H^1_0(\mathbb{D}, \mathbb{C}^{n\times m})$ for which $\|\Psi\|_{L^1(\mathbb{T},\mathbb{C}^{n\times m})} \leq 1$ and $\operatorname{rank}\Psi(\zeta) \leq k$ almost everywhere on $\mathbb{T}$.

1.5 Description of results by sections

In Chapter 2, we recall the long-established notion of the wedge product of Hilbert spaces. We define an inner product on the p-fold wedge product of Hilbert spaces and study the notion of pointwise wedge product of operator- or vector-valued functions on $\mathbb{D}$ or $\mathbb{T}$. We study numerous properties of it and we formulate a concise theory specifically for multiplication, block diagonal and creation operators. Towards the end of the chapter, we examine in detail the characteristics of the pointwise orthogonal complement and pointwise linear span. Some of the main results of Chapter 2 are the following.

Proposition 2.2.8. *Let E be a Hilbert space and let $x_i \colon \mathbb{D} \to E$ be analytic E-valued maps on $\mathbb{D}$ for all $i = 0, \dots, k$. Then*

$$x_0 \dot\wedge \cdots \dot\wedge x_k \colon \mathbb{D} \to \wedge^{k+1} E$$

is also analytic on $\mathbb{D}$ and

$$\begin{aligned}(x_0 \dot\wedge \cdots \dot\wedge x_k)'(z) \; &= x_0'(z) \wedge x_1(z) \wedge \cdots \wedge x_k(z) + x_0(z) \wedge x_1'(z) \wedge x_2(z) \wedge \cdots \wedge x_k(z) \\ &+ \cdots + x_0(z) \wedge x_1(z) \wedge \cdots x_k'(z).\end{aligned}$$

Proposition 2.2.13. *Let E be a Hilbert space, let $x \in H^2(\mathbb{D}, E)$ and let $y \in H^\infty(\mathbb{D}, E)$. Then*

$$x \dot\wedge y \in H^2(\mathbb{D}, \wedge^2 E).$$

Proposition 2.2.40. *Let E be a separable Hilbert space and let $\xi \in H^\infty(\mathbb{D}, E)$ be an inner function. Consider the* pointwise creation operator

$$C_\xi : H^2(\mathbb{D}, E) \to H^2(\mathbb{D}, \wedge^2 E),$$

given by

$$C_\xi f = \xi \dot\wedge f, \text{ for } f \in H^2(\mathbb{D}, E),$$

and let $P_+ \colon L^2(\mathbb{T}, E) \to H^2(\mathbb{D}, E)$ be the orthogonal projection operator. Then, for any $h \in H^2(\mathbb{D}, E)$,

$$C_\xi^* C_\xi h = P_+ \alpha,$$

where $\alpha = h - \xi\xi^* h$. Moreover

$$C_\xi^* C_\xi h = h - T_{\xi\xi^*} h,$$

where $T_{\xi\xi^*} : H^2(\mathbb{D}, E) \to H^2(\mathbb{D}, E)$ is the Toeplitz operator with symbol $\xi\xi^*$.

Furthermore, in Chapter 3 we present our main result; the superoptimal analytic approximation algorithm. At first, we describe the algorithm and then we prove its validity. The purpose of the algorithm is the determination of the unique superoptimal approximant. Let us give an overview of the main results obtained in Chapter 3. We first prove the pointwise orthonormality of the sets $\{\xi_i\}_{i=1}^j$, $\{\overline{\eta_i}\}_{i=1}^j$ almost everywhere on $\mathbb{T}$. Next, we show that the spaces X_i, Y_i are closed linear subspaces of $H^2(\mathbb{D}, \wedge^{i+1}\mathbb{C}^n)$ and $H^2(\mathbb{D}, \wedge^{i+1}\mathbb{C}^m)^\perp$ respectively. Hence we prove that the projections P_{Y_i} are well-defined, and consequently, that the operators T_i are well-defined for all $i \geq 0$. For the latter, one also has to show that each operator T_i is independent of the choice of $Q_i \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$. Moreover, we prove that the operator T_i is compact for $i = 0, 1, \ldots, r-1$ and the superoptimal analytic approximant $\mathcal{A}G$ is given by the formula

$$G - \mathcal{A}G = \sum_{0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2},$$

where each term in this sum can in principle be calculated.

In Chapter 4 we apply the algorithm obtained in Chapter 3 to calculate the superoptimal approximant of the matrix-valued functions

$$G(z) = \begin{pmatrix} 2/z & 0 \\ 0 & 1/z \end{pmatrix} \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$$

and

$$G(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{3}\bar{z}^2 + 2\bar{z} & \bar{z} \\ \sqrt{3}\bar{z} + 2 & -1 \end{pmatrix} \in H^\infty(\mathbb{D}, \mathbb{C}^{2\times 2}) + C(\mathbb{T}, \mathbb{C}^{2\times 2}).$$

The former is a relatively simple example which involves trivial operations and enables the reader to familiarise themselves with our algorithm, while the latter is a more elaborate one and its superoptimal approximant is calculated to be

$$\mathcal{A}G = \frac{\sqrt{2}}{1-\gamma z} \begin{pmatrix} -\gamma & \sqrt{3} + 4\gamma \\ 2 + \gamma\sqrt{3} - \gamma z & -(\sqrt{3}+4\gamma)(\sqrt{3}+z) \end{pmatrix},$$

where $\gamma = -\dfrac{5-\sqrt{13}}{2\sqrt{3}}$.

Regarding the appendices, in Appendix A we give a well known construction of the algebraic tensor product using the universal property. We then consider the tensor product of Hilbert spaces and define an inner product. In appendices B and C we review scalar inner and outer functions and Fatou's theorem, and in appendices D and E we recall the established notions of operator valued inner and outer functions, we present the established generalised Fatou's theorem in the matricial setting and we describe the Nehari problem both in the scalar and in the matrix-valued setting.

1.6 Future work

The algorithm introduced in the present dissertation establishes a new approach to the computation of the superoptimal analytic approximant in the problem of best analytic approximation of matrix-valued functions. Let us briefly refer to research topics which could arise from this project.

Immediate tasks this dissertation could inspire is the construction of similar algorithms for the meromorphic and operator-valued cases, as these are studied by Peller and Young [26] and Peller [21] respectively. Wedge products of Hilbert spaces could be implemented in order to obtain an alternative algorithm in the meromorphic case, since Proposition 2.2.8 asserts the analyticity of the pointwise wedge product. Moreover, wedge products of infinite-dimensional Hilbert spaces may provide a plausible alternative to Peller's methods in [21].

In addition, our project could steer one's interests towards the direction of investigating the algorithm's advantages and disadvantages to previous algorithms, especially to the ones facilitated in control engineering [13], [27]. It would be of great significance to numerically implement our algorithm and perform a rather deeper comparison with past algorithms in this respect. Such a project would entail a certain comprehension of numerical methods and, quite possibly, a collaboration between control engineers and functional analysts.

Furthermore, we trust that consideration of the wedge product of Hilbert spaces could potentially lead to research in different topics in analysis, one of them being reproducing kernel Hilbert spaces. Reproducing kernels play a prominent role in the study of Hilbert spaces of functions, such as Hardy spaces and Dirichlet spaces, as well as in Statistics and certain physical problems. In particular they can be used to prove some classical interpolation problems, such as Pick-type theorems, which are theorems giving necessary and sufficient conditions for the existence of multipliers of norm at most one that satisfy some interpolation conditions.

The techniques introduced in the subsequent chapters of this dissertation illustrate the fact that exterior products of Hilbert spaces and operators thereon are naturally well adapted to the analysis of *matrix-valued* functions on the circle, disc or line, and therefore to questions arising from problems in engineering design. Though a long established theory [7], [17], [35], exterior products of Hilbert spaces and operators deserve in our view to be better exploited in functional analysis than they have been hitherto. An initial orientation could be given by a number of concrete questions, as follows.

(1) Give concrete descriptions of the exterior product $\wedge^p H$ as a reproducing kernel Hilbert space for various standard Hilbert function spaces H, such as Hardy, Bergman, Dirichlet and Hardy-Sobolev spaces.

(2) Explore best approximation problems associated with the spaces described in item (1).

(3) Try to prove a "super-Pick Theorem" for bounded analytic matrix-valued functions in the disc. *Given distinct points $\lambda_1, \dots, \lambda_N \in \mathbb{D}, m \times n$ matrices $W_1, \dots, W_N$ and positive numbers $t_0 > \dots > t_k > 0$, find a necessary and sufficient condition for the existence of a bounded analytic matrix-valued function F in $\mathbb{D}$ such that $F(\lambda_j) = W_j$ for $j = 1, \dots, N$*

and $s_j^\infty(F) \le t_j$ *for* $j = 0, 1, \ldots, k,$ *where*

$$s_j^\infty(F) \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{z\in\mathbb{T}} s_j(F(z))$$

and, for any matrix A, $s_j(A)$ denotes the jth singular value of A.

(4) Explore natural variants of item (3).

(5) Can one prove a Parrott theorem for s_1 of completions of a partially specified operator T by using $\wedge^2 T$?

Chapter 2

Exterior powers of Hilbert spaces and operators

2.1 Exterior powers

In this section, we first present some results concerning the action of permutation operators on tensors, then we recall a well-known definition of the antisymmetric tensors and we define an inner product on the space of all antisymmetric tensors. Basic definitions and properties of exterior products can be found in a S. Winitzki's book [35] as well as in [32], [33].

Below E denotes a Hilbert space.

Definition 2.1.1. *$\otimes^p E$ is the algebraic p-fold tensor product of E and is spanned by tensors of the form $x_1 \otimes x_2 \otimes \cdots \otimes x_p$, where $x_j \in E$, for $j = 1, \ldots, p$.*

Definition 2.1.2. *We define an inner product on $\otimes^p E$ on elementary tensors by*

$$\langle x_1 \otimes x_2 \otimes \cdots \otimes x_p, y_1 \otimes y_2 \otimes \cdots \otimes y_p \rangle_{\otimes^p E} = p! \langle x_1, y_1 \rangle_E \cdots \langle x_p, y_p \rangle_E,$$

for any $x_1, \ldots, x_p,\ y_1, \ldots, y_p \in E$, and extend $\langle \cdot, \cdot \rangle$ to $\otimes^p E$ by sesqui-linearity.

Remark 2.1.3. *The space $(\otimes^p E, \| \cdot \|)$, where $\|u\| = \langle u, u \rangle_{\otimes^p E}^{1/2}$, is a normed space.*

Definition 2.1.4. *$\otimes_H^p E$ is the completion of $\otimes^p E$ with respect to the norm*

$$\|u\| = \langle u, u \rangle_{\otimes^p E}^{1/2},$$

for $u \in \otimes^p E$.

Definition 2.1.5. *Let $S_p \stackrel{\text{def}}{=} \mathrm{Sym}\{1, \ldots, p\}$ be the symmetric group on $\{1, \ldots, p\}$, with the operation of composition. For $\sigma \in S_p$, we define $S_\sigma : \otimes^p E \to \otimes^p E$ on elementary tensors by*

$$S_\sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_p) = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(p)},$$

and we extend S_σ to $\otimes^p E$ by linearity, that is, for $u = \sum_{i=1}^n x_1^i \otimes \cdots \otimes x_p^i$, we define

$$S_\sigma(u) = \sum_{i=1}^n S_\sigma(x_1^i \otimes \cdots \otimes x_p^i).$$

Proposition 2.1.6. *Let E be a Hilbert space, and let p be a positive integer. Then, for any $\sigma \in S_p$, S_σ is a linear operator on the normed space $(\otimes^p E, \|\cdot\|)$, which extends to an isometry $\mathbf{S}_\sigma$ on $(\otimes_H^p E, \|\cdot\|)$.*

Proof. To prove linearity, let $u = \sum\limits_{i=1}^n x_1^i \otimes \cdots \otimes x_p^i \in \otimes^p E$ and $v = \sum\limits_{j=1}^n y_1^j \otimes \cdots \otimes y_p^j \in \otimes^p E$, then

$$\begin{aligned} S_\sigma(\lambda u + \mu v) &= S_\sigma\left(\sum_{i=1}^n \lambda x_1^i \otimes \cdots \otimes x_p^i + \sum_{j=1}^n \mu y_1^j \otimes \cdots \otimes y_p^j\right) \\ &= \sum_{i=1}^n \lambda x_{\sigma(1)}^i \otimes \cdots \otimes x_{\sigma(p)}^i + \sum_{j=1}^n \mu y_{\sigma(1)}^j \otimes \cdots \otimes y_{\sigma(p)}^j \\ &= \lambda S_\sigma\left(\sum_{i=1}^n \lambda x_1^i \otimes \cdots \otimes x_p^i\right) + \mu S_\sigma\left(\sum_{j=1}^n \mu y_1^j \otimes \cdots \otimes y_p^j\right) \\ &= \lambda S_\sigma(u) + \mu S_\sigma(v) \end{aligned}$$

for scalars $\lambda, \mu \in \mathbb{C}$.
Furthermore, for an elementary tensor $w = x_1 \otimes x_2 \otimes \cdots \otimes x_p$, we need to prove that $\|S_\sigma w\|^2_{\otimes^p E} = \|w\|^2_{\otimes^p E}$. By the definition of the inner product on $\otimes^p E$, we get

$$\begin{aligned} \|S_\sigma w\|^2_{\otimes^p E} &= \langle S_\sigma w, S_\sigma w\rangle_{\otimes^p E} \\ &= \langle x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(p)}, x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(p)}\rangle_{\otimes^p E} \\ &= p!\langle x_{\sigma(1)}, x_{\sigma(1)}\rangle_E \ldots \langle x_{\sigma(p)}, x_{\sigma(p)}\rangle_E \\ &= p!\|x_{\sigma(1)}\|_E^2 \ldots \|x_{\sigma(p)}\|_E^2 \\ &= \|w\|_E^2. \end{aligned}$$

Also, for a tensor ω of the form $\omega = \sum\limits_{i=1}^n x_1^i \otimes \cdots \otimes x_p^i$, we have

$$\begin{aligned} \|\omega\|^2_{\otimes^p E} &= \left\langle \sum_{i=1}^n x_1^i \otimes \cdots \otimes x_p^i \,,\, \sum_{j=1}^n x_1^j \otimes \cdots \otimes x_p^j \right\rangle_{\otimes^p E} \\ &= p! \sum_{i=1}^n \sum_{j=1}^n \langle x_1^i, x_1^j\rangle_E \cdots \langle x_p^i, x_p^j\rangle_E \end{aligned}$$

and

$$\begin{aligned}\langle S_\sigma\omega, S_\sigma\omega\rangle_{\otimes^p E} &= \left\langle S_\sigma\left(\sum_{i=1}^n x_1^i\otimes\cdots\otimes x_p^i\right)\ ,\ S_\sigma\left(\sum_{j=1}^n x_1^j\otimes\cdots\otimes x_p^j\right)\right\rangle_{\otimes^p E}\\ &= \left\langle \sum_{i=1}^n x_{\sigma(1)}^i\otimes\cdots\otimes x_{\sigma(p)}^i\ ,\ \sum_{j=1}^n x_{\sigma(1)}^j\otimes\cdots\otimes x_{\sigma(p)}^j\right\rangle_{\otimes^p E}\\ &= p!\sum_{i=1}^n\sum_{j=1}^n \langle x_{\sigma(1)}^i, x_{\sigma(1)}^j\rangle_E\cdots\langle x_{\sigma(p)}^i, x_{\sigma(p)}^j\rangle_E.\end{aligned}$$

So, $\|S_\sigma\omega\|^2_{\otimes^p E} = \|\omega\|^2_{\otimes^p E}$. Hence S_σ is also a surjective self-map of $\otimes^p E$.

Thus one can extend S_σ by continuity to an isometric linear self-map $\mathbf{S}_\sigma$ of the completion $\otimes^p_H E$ of $\otimes^p E$. □

Proposition 2.1.7. $\mathbf{S}_\sigma$ *is a bounded linear operator from* $\otimes^p_H E$ *to* $\otimes^p_H E$*. Furthermore,* $\mathbf{S}_\sigma$ *is a unitary operator on* $\otimes^p_H E$*.*

Proof. Since by Proposition 2.1.6 $\mathbf{S}_\sigma$ is isometric, its range is complete, hence closed in $\otimes^p_H E$. Since the range of $\mathbf{S}_\sigma$ contains that of S_σ, $\operatorname{ran}\mathbf{S}_\sigma = \otimes^p_H E$. Being both surjective and isometric, $\mathbf{S}_\sigma$ is a unitary operator on $\otimes^p_H E$. □

Henceforth we shall denote the extended operator $\mathbf{S}_\sigma$ by S_σ.

Definition 2.1.8. *A tensor* $u \in \otimes^p_H E$ *is called* symmetric *if* $S_\sigma(u) = u$ *for all* $\sigma \in S_p$*. A tensor* $u \in \otimes^p_H E$ *is called* antisymmetric *if* $u = \epsilon_\sigma S_\sigma u$ *for all* $\sigma \in S_p$*, where* ϵ_σ *is the signature of* σ*.*

Definition 2.1.9. *The space of all antisymmetric tensors in* $\otimes^p_H E$ *will be denoted by* $\wedge^p E$*.*

Remark 2.1.10. $(S_p, \circ)$ *is a group so, for every permutation* $\sigma \in S_p$*, there exists* $\sigma^{-1} \in S_p$ *such that*

$$\sigma\circ\sigma^{-1} = \mathrm{id} = \sigma^{-1}\circ\sigma,$$

where id $\in S_p$ *is the identity map on* $\{1,\dots,p\}$*. Then,*

$$\epsilon_{\sigma\circ\sigma^{-1}} = \epsilon_\sigma\epsilon_{\sigma^{-1}} = 1,$$

hence $\epsilon_\sigma = \epsilon_{\sigma^{-1}}$*.*

Example 2.1.11. *Let* E *be a Hilbert space and let* $x_1, x_2 \in E$*. In* $\otimes^2_H E$*, the elementary tensor*

$$x_1\otimes x_2 + x_2\otimes x_1$$

is symmetric, whereas the elementary tensor

$$x_1\otimes x_2 - x_2\otimes x_1$$

is antisymmetric.

Theorem 2.1.12. *Let E be a Hilbert space. Then $\bigwedge^p E$ is a closed linear subspace of the Hilbert space $\otimes_H^p E$ for any $p \geq 2$.*

Proof. For $\sigma \in S_p$ we define the operator

$$f_\sigma \stackrel{def}{=} (S_\sigma - \epsilon_\sigma \cdot I) \ : \otimes_H^p E \to \otimes_H^p E,$$

where $I : \otimes_H^p E \to \otimes_H^p E$ is given by $I(u) = u$, for $u \in \otimes_H^p E$.
Since S_σ is a continuous linear operator on $\otimes_H^p E$, f_σ is a continuous linear operator. The kernel of the operator f_σ is

$$\begin{aligned} \ker f_\sigma &= \{u \in \otimes_H^p E : (S_\sigma - \epsilon_\sigma \cdot I)(u) = 0\} \\ &= \{u \in \otimes_H^p E : S_\sigma(u) = \epsilon_\sigma u\} \\ &= \{u \in \otimes_H^p E : \epsilon_\sigma S_\sigma(u) = u\}. \end{aligned}$$

Since f_σ is a continuous linear operator on $\otimes_H^p E$, $\ker f_\sigma$ is a closed linear subspace of $\otimes_H^p E$. Thus $\wedge^p E$ is a closed linear subspace of $\otimes_H^p E$, since

$$\wedge^p E = \bigcap_{\sigma \in S_p} \ker f_\sigma = \{u \in \otimes_H^p E \ : \epsilon_\sigma S_\sigma(u) = u, \ \text{for all } \sigma \in S_p\}. \qquad \square$$

Theorem 2.1.12 implies that the orthogonal projection onto $\wedge^p E$ is well-defined on $\otimes_H^p E$.

Definition 2.1.13. *Let E be a Hilbert space. For $x_1, \ldots, x_p \in E$, define $x_1 \wedge x_2 \wedge \cdots \wedge x_p$ to be the orthogonal projection of the elementary tensor $x_1 \otimes x_2 \otimes \cdots \otimes x_p$ onto $\wedge^p E$, that is,*

$$x_1 \wedge x_2 \wedge \cdots \wedge x_p = P_{\wedge^p E}(x_1 \otimes \cdots \otimes x_p).$$

Remark 2.1.14. *For any tensor of the form*

$$u = \sum_{i=1}^{n} \lambda^i x_1^i \otimes \cdots \otimes x_p^i \in \otimes_H^p E,$$

the orthogonal projection of u onto $\wedge^p E$ is given by

$$\mathrm{P}_{\wedge^p E}\left(\sum_{i=1}^{n} \lambda^i x_1^i \otimes \cdots \otimes x_p^i\right) = \sum_{i=1}^{n} \lambda^i x_1^i \wedge \cdots \wedge x_p^i,$$

where $\lambda^i \in \mathbb{C}$ for all $i = 1, \cdots, n$.

Theorem 2.1.15. *Let E be a Hilbert space. For all $u \in \otimes_H^p E$,*

$$P_{\wedge^p E}(u) = \frac{1}{p!} \sum_{\sigma \in S_p} \epsilon_\sigma S_\sigma(u).$$

Proof. Let $u \in \otimes_H^p E$. Then, for any $\sigma \in S_p$, $u = \epsilon_\sigma S_\sigma(u) + (u - \epsilon_\sigma S_\sigma(u))$, and so

$$p!u = \sum_{\sigma \in S_p} \epsilon_\sigma S_\sigma(u) + \sum_{\sigma \in S_p} (u - \epsilon_\sigma S_\sigma(u)).$$

It suffices to show that $\sum_{\sigma \in S_p} \epsilon_\sigma S_\sigma(u) \in \wedge^p E$ and

$$\sum_{\sigma \in S_p} (u - \epsilon_\sigma S_\sigma(u))$$

is orthogonal to the set of antisymmetric tensors, in other words, if $v \in \wedge^p E$ then

$$\left\langle v, \sum_{\sigma \in S_p} (u - \epsilon_\sigma S_\sigma(u)) \right\rangle_{\otimes_H^p E} = 0.$$

Let $w = \sum_{\sigma \in S_p} \epsilon_\sigma S_\sigma(u) \in \otimes_H^p E$. For every $\tau \in S_p$, we have

$$\begin{aligned} \epsilon_\tau S_\tau(w) &= \epsilon_\tau S_\tau \left(\sum_{\sigma \in S_p} \epsilon_\sigma S_\sigma(u) \right) \\ &= \sum_{\tau \circ \sigma \in S_p} \epsilon_{\tau \circ \sigma} S_{\tau \circ \sigma}(u) \\ &= \sum_{\sigma' \in S_p} \epsilon_{\sigma'} S_{\sigma'}(u) \\ &= w, \end{aligned}$$

where $\tau \circ \sigma = \sigma'$. Hence $\sum_{\sigma \in S_p} \epsilon_\sigma S_\sigma(u) \in \wedge^p E$.

Furthermore, for every $v \in \wedge^p E$, we have $v = \epsilon_\sigma S_\sigma v$ for all $\sigma \in S_p$, and

$$\begin{aligned} \left\langle v, \sum_{\sigma \in S_p} (u - \epsilon_\sigma S_\sigma(u)) \right\rangle_{\otimes_H^p E} &= \sum_{\sigma \in S_p} \langle v, u \rangle_{\otimes_H^p E} - \sum_{\sigma \in S_p} \epsilon_\sigma \langle v, S_\sigma(u) \rangle_{\otimes_H^p E} \\ &= \sum_{\sigma \in S_p} \langle v, u \rangle_{\otimes_H^p E} - \sum_{\sigma \in S_p} \epsilon_\sigma \langle S_\sigma{}^* v, u \rangle_{\otimes_H^p E} \\ &= \sum_{\sigma \in S_p} \langle v - \epsilon_{\sigma^{-1}} S_{\sigma^{-1}} v, u \rangle_{\otimes_H^p E} \\ &= \langle \sum_{\sigma \in S_p} (v - \epsilon_{\sigma^{-1}} S_{\sigma^{-1}} v), u \rangle_{\otimes_H^p E} \\ &= 0. \end{aligned}$$

Thus, for all $u \in \otimes_H^p E$,

$$P_{\wedge^p E}(p!u) = P_{\wedge^p E}\left(\sum_{\sigma\in S_p} \epsilon_\sigma S_\sigma(u) + \sum_{\sigma\in S_p}(u - \epsilon_\sigma S_\sigma(u))\right) = \sum_{\sigma\in S_p} \epsilon_\sigma S_\sigma(u),$$

and so,

$$P_{\wedge^p E}(u) = \frac{1}{p!}\sum_{\sigma\in S_p} \epsilon_\sigma S_\sigma(u)$$

as required. □

Example 2.1.16. *Let E be a Hilbert space. If $x_1 \otimes x_2 \in \otimes_H^2 E$, then*

$$x_1 \wedge x_2 = P_{\wedge^2 E}(x_1 \otimes x_2) = \frac{1}{2!}(x_1 \otimes x_2 - x_2 \otimes x_1).$$

Remark 2.1.17. *If $p > 1$, then S_p contains a transposition, for instance $\sigma = (1\ 2)$, and $\epsilon_\sigma = -1$. If $p = 1$, then $\wedge^1 E = E$.*

Proposition 2.1.18. *Let E be a Hilbert space and let $p \geq 2$. The set of antisymmetric tensors and the set of symmetric tensors are orthogonal in $\otimes_H^p E$.*

Proof. Suppose that u is a symmetric tensor and v is an antisymmetric tensor, that is, $S_\sigma u = u$ and $S_\sigma v = \epsilon_\sigma v$ respectively for all $\sigma \in S_p$. By Proposition 2.1.7, S_σ is a unitary operator on $\otimes_H^p E$, for all $\sigma \in S_p$. Thus

$$\langle u, v\rangle_{\otimes_H^p E} = \langle S_\sigma u, S_\sigma v\rangle_{\otimes_H^p E} = \langle u, \epsilon_\sigma v\rangle_{\otimes_H^p E}, \text{ for all } \sigma \in S_p.$$

The equality holds for all $\sigma \in S_p$, thus it is true for $\epsilon_\sigma = -1$. Then

$$\langle u, v\rangle_{\otimes_H^p E} = -\langle u, v\rangle_{\otimes_H^p E},$$

and so $\langle u, v\rangle_{\otimes_H^p E} = 0$. □

Proposition 2.1.19. *Let E be a Hilbert space. The inner product in $\wedge^p E$ is given by*

$$\langle x_1 \wedge \cdots \wedge x_p, y_1 \wedge \cdots \wedge y_p\rangle_{\wedge^p E} = \det\begin{pmatrix} \langle x_1, y_1\rangle_E & \ldots & \langle x_1, y_p\rangle_E \\ \vdots & \ddots & \vdots \\ \langle x_p, y_1\rangle_E & \ldots & \langle x_p, y_p\rangle_E \end{pmatrix}$$

for all $x_1, \ldots, x_p,\ y_1, \ldots, y_p \in E$.

Proof. By Theorem 2.1.15, we have

$$\begin{aligned}
&\langle x_1 \wedge \cdots \wedge x_p, y_1 \wedge \cdots \wedge y_p\rangle_{\wedge^p E} = \\
&= \left\langle \frac{1}{p!} \sum_{\sigma \in S_p} \epsilon_\sigma S_\sigma (x_1 \otimes x_2 \otimes \cdots \otimes x_p), \frac{1}{p!} \sum_{\tau \in S_p} \epsilon_\tau S_\tau (y_1 \otimes y_2 \otimes \cdots \otimes y_p) \right\rangle_{\otimes_H^p E} \\
&= \frac{1}{p!^2} \sum_{\sigma,\tau \in S_p} \langle \epsilon_\sigma S_\sigma (x_1 \otimes x_2 \otimes \cdots \otimes x_p), \epsilon_\tau S_\tau (y_1 \otimes y_2 \otimes \cdots \otimes y_p)\rangle_{\otimes_H^p E} \\
&= \frac{1}{p!^2} \sum_{\sigma,\tau \in S_p} \epsilon_\sigma \epsilon_\tau \langle x_1 \otimes x_2 \otimes \cdots \otimes x_p, S_\sigma^* S_\tau (y_1 \otimes y_2 \otimes \cdots \otimes y_p)\rangle_{\otimes_H^p E} \\
&= \frac{1}{p!^2} \sum_{\sigma,\tau \in S_p} \epsilon_{\sigma^{-1}} \epsilon_\tau \langle x_1 \otimes x_2 \otimes \cdots \otimes x_p, S_{\sigma^{-1}} S_\tau (y_1 \otimes y_2 \otimes \cdots \otimes y_p)\rangle_{\otimes_H^p E} \\
&= \frac{1}{p!} \sum_{\sigma' \in S_p} \epsilon_{\sigma'} \langle x_1 \otimes x_2 \otimes \cdots \otimes x_p, S_{\sigma'} (y_1 \otimes y_2 \otimes \cdots \otimes y_p)\rangle_{\otimes_H^p E} \\
&= \sum_{\sigma' \in S_p} \epsilon_{\sigma'} \prod_{i=1}^p \langle x_i, y_{\sigma'(i)}\rangle_E \\
&= \det \begin{pmatrix} \langle x_1, y_1\rangle_E & \cdots & \langle x_1, y_p\rangle_E \\ \vdots & \ddots & \vdots \\ \langle x_p, y_1\rangle_E & \cdots & \langle x_p, y_p\rangle_E \end{pmatrix}.
\end{aligned}$$ □

Corollary 2.1.20. *Let E be a Hilbert space and let $x_1, \ldots, x_p \in E$. Then $x_1 \wedge \cdots \wedge x_p = 0$ if and only if $x_1, \ldots, x_p$ are linearly dependent.*

Proof. Note that $x_1 \wedge \cdots \wedge x_p = 0$ if and only if $\|x_1 \wedge \cdots \wedge x_p\|^2_{\wedge^p E} = 0$, which, by Proposition 2.1.19, holds if and only if

$$\det[\langle x_i, x_j\rangle]_{i,j=1}^p = 0.$$

Thus $x_1 \wedge \cdots \wedge x_p = 0$ if and only if there exist complex numbers $\lambda_1, \ldots, \lambda_p$, which are not all zero, such that

$$\begin{pmatrix} \langle x_1, y_1\rangle_E & \ldots & \langle x_1, y_p\rangle_E \\ \vdots & \ddots & \vdots \\ \langle x_p, y_1\rangle_E & \ldots & \langle x_p, y_p\rangle_E \end{pmatrix} \begin{pmatrix} \bar{\lambda}_1 \\ \vdots \\ \bar{\lambda}_p \end{pmatrix} = 0.$$

This holds if and only if there exist complex numbers $\lambda_1, \ldots, \lambda_p$, which are not all zero, such that

$$\langle x_i, \sum_{j=1}^p \lambda_j x_j\rangle_E = 0 \quad \text{for } i = 1, \ldots, p.$$

The latter statement is equivalent to the assertion that there exist complex numbers $\lambda_1, \ldots, \lambda_p$, which are not all zero, such that

$$\langle \sum_{i=1}^p \lambda_i x_i, \sum_{j=1}^p \lambda_j x_j\rangle_E = 0,$$

which in turn is equivalent to the condition that there exist complex numbers $\lambda_1, \ldots, \lambda_p$, not all zero, such that

$$\sum_{j=1}^{p} \lambda_j x_j = 0.$$

The latter statement is equivalent to the linear dependence of $x_1, \ldots, x_p$ as required. □

Corollary 2.1.21. *Let E be a Hilbert space. Suppose $x, y \in E$, and x,y are orthogonal in E, that is, $\langle x, y\rangle_E = 0$. Then*

$$\|x \wedge y\|_{\wedge^2 E} = \|x\|_E \|y\|_E.$$

Since we have already shown that $\wedge^p E$ is a closed linear subspace of the Hilbert space $\otimes_H^p E$, the space $(\wedge^p E, \langle \cdot, \cdot \rangle_{\wedge^p E})$ with inner product given by Proposition 2.1.19 is itself a Hilbert space.

Proof. By Proposition 2.1.19,

$$\|x \wedge y\|^2_{\wedge^2 E} = \langle x \wedge y, x \wedge y\rangle_{\wedge^2 E} = \det \begin{pmatrix} \langle x, x\rangle_E & \langle x, y\rangle_E \\ \langle y, x\rangle_E & \langle y, y\rangle_E \end{pmatrix}.$$

If x is orthogonal to y in E, the off-diagonal entries are zero and thus

$$\|x \wedge y\|^2_{\wedge^2 E} = \|x\|^2_E \|y\|^2_E.$$

□

Lemma 2.1.22. *Suppose $\{u_1, \cdots, u_n\}$ is an orthonormal set in $\mathbb{C}^n$. Then, for $j = 1, \ldots, n-1$ and for every $x \in E$,*

$$\|u_1 \wedge \cdots \wedge u_j \wedge x\|_{\wedge^{j+1}\mathbb{C}^n} = \|x - \sum_{i=1}^{j} \langle x, u_i\rangle u_i\|_{\mathbb{C}^n}.$$

Proof. Let $x \in \mathbb{C}^n$. We may write

$$x = x - \sum_{i=1}^{j} \langle x, u_i\rangle u_i + \sum_{i=1}^{j} \langle x, u_i\rangle u_i.$$

By Proposition 2.1.19, we get

$$\begin{aligned} \|u_1 \wedge \cdots \wedge u_j \wedge x\|^2_{\wedge^{j+1}\mathbb{C}^n} &= \langle u_1 \wedge \cdots \wedge u_j \wedge x, u_1 \wedge \cdots \wedge u_j \wedge x\rangle_{\wedge^{j+1}\mathbb{C}^n} \\ &= \det \begin{pmatrix} \langle u_1, u_1\rangle_{\mathbb{C}^n} & \langle u_1, u_2\rangle_{\mathbb{C}^n} & \cdots & \cdots & \langle u_1, x\rangle_{\mathbb{C}^n} \\ \langle u_2, u_1\rangle_{\mathbb{C}^n} & \langle u_2, u_2\rangle_{\mathbb{C}^n} & \cdots & \cdots & \langle u_2, x\rangle_{\mathbb{C}^n} \\ \vdots & \cdots & \ddots & \cdots & \cdots \\ \langle u_j, u_1\rangle_{\mathbb{C}^n} & \langle u_j, u_2\rangle_{\mathbb{C}^n} & \cdots & \langle u_j, u_j\rangle_{\mathbb{C}^n} & \langle u_j, x\rangle_{\mathbb{C}^n} \\ \langle x, u_1\rangle_{\mathbb{C}^n} & \langle x, u_2\rangle_{\mathbb{C}^n} & \cdots & \cdots & \langle x, x\rangle_{\mathbb{C}^n} \end{pmatrix}. \end{aligned}$$

By assumption,

$$\langle u_i, u_k\rangle = \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases},$$

and hence,

$$\|u_1 \wedge \cdots \wedge u_j \wedge x\|^2_{\wedge^{j+1}\mathbb{C}^n} = \det \begin{pmatrix} 1 & 0 & \cdots & \langle u_1, x\rangle_{\mathbb{C}^n} \\ 0 & 1 & \cdots & \langle u_2, x\rangle_{\mathbb{C}^n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & \langle u_j, x\rangle_{\mathbb{C}^n} \\ \langle x, u_1\rangle_{\mathbb{C}^n} & \langle x, u_2\rangle_{\mathbb{C}^n} & \cdots & \langle x, x\rangle_{\mathbb{C}^n} \end{pmatrix}.$$

If, for every $k = 1, \cdots, j$ we multiply the k-th column of the determinant by $\langle u_k, x\rangle_{\mathbb{C}^n}$ and subtract it from the $(j+1)$-th column, we find that

$$\begin{aligned} \|u_1 \wedge \cdots \wedge u_j \wedge x\|^2_{\wedge^{j+1}\mathbb{C}^n} &= \det \begin{pmatrix} 1 & 0 & & \cdots & 0 \\ 0 & 1 & & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \cdots & 1 & 0 \\ \langle x, u_1\rangle_{\mathbb{C}^n} & \langle x, u_2\rangle_{\mathbb{C}^n} & \cdots & \cdots & \langle x, x\rangle_{\mathbb{C}^n} - \sum_{i=1}^{j} |\langle x, u_i\rangle_{\mathbb{C}^n}|^2 \end{pmatrix} \\ &= \|x\|^2_{\mathbb{C}^n} - \sum_{i=1}^{j} |\langle x, u_i\rangle_{\mathbb{C}^n}|^2 \\ &= \|x - \sum_{i=1}^{j} \langle x, u_i\rangle_{\mathbb{C}^n} u_i\|^2_{\mathbb{C}^n}, \end{aligned}$$

the latter equality following by Pythagoras theorem. □

Suppose E is a separable Hilbert space with an orthonormal basis. In what follows, we derive an orthonormal basis for the space $\wedge^p E$.

Theorem 2.1.23 ([17], p. 47)**.** *Let E be a separable Hilbert space with* $\dim E = m$ *and let* $(e_n)_{n=1}^m$ *be a basis of E. Then the set*

$$B = \{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p} : 1 \leq i_1 < \cdots < i_p \leq m\}$$

is linearly independent in $\wedge^p E$.

Proposition 2.1.24. *Let E be a separable Hilbert space with* $\dim E = m$ *and let $(e_n)_{n=1}^m$ be an orthonormal basis of E. If $x, y \in E$ with $x = \sum_{i=1}^m x_i e_i$ and $y = \sum_{j=1}^m y_j e_j$, then*

$$x \wedge y = \sum_{i<j} (x_i y_j - x_j y_i) e_i \wedge e_j$$

and

$$\|x \wedge y\|^2_{\wedge^2 E} = \sum_{i<j} \left| \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} \right|^2.$$

Proof. Let $x = \sum_{i=1}^{m} x_i e_i$ and $y = \sum_{j=1}^{m} y_j e_j$. We know that $e_i \wedge e_i = 0$ and $e_i \wedge e_j = -e_j \wedge e_i$, for $i \neq j$. Hence

$$\begin{aligned} x \wedge y &= \sum_{i=1}^{m} x_i e_i \wedge \sum_{j=1}^{m} y_j e_j \\ &= \sum_{i<j} (x_i y_j)(e_i \wedge e_j) + \sum_{j<i} (x_j y_i)(e_j \wedge e_i) + \sum_{i=1}^{m} (x_i y_i)(e_i \wedge e_i) \\ &= \sum_{i<j} (x_i y_j - x_j y_i) e_i \wedge e_j \end{aligned}$$

and

$$\begin{aligned} \langle x \wedge y, x \wedge y \rangle_{\wedge^2 E} &= \langle \sum_{i<j} (x_i y_j - x_j y_i)(e_i \wedge e_j), \sum_{k<l} (x_k y_l - x_l y_k)(e_k \wedge e_l) \rangle_{\wedge^2 E} \\ &= \sum_{i<j} \sum_{k<l} (x_i y_j - x_j y_i) \overline{(x_k y_l - x_l y_k)} \langle e_i \wedge e_j, e_k \wedge e_l \rangle_{\wedge^2 E}. \end{aligned}$$

Since $(e_n)_{n=1}^{\infty}$ is an orthonormal basis of E,

$$\langle e_i \wedge e_j, e_k \wedge e_l \rangle_E = \det \begin{pmatrix} \langle e_i, e_k \rangle_E & \langle e_i, e_l \rangle_E \\ \langle e_j, e_k \rangle_E & \langle e_j, e_l \rangle_E \end{pmatrix} = \begin{cases} 1 & \text{if } i = k \text{ and } j = l, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\langle x \wedge y, x \wedge y \rangle_{\wedge^2 E} = \sum_{i<j} |x_i y_j - x_j y_i|^2 = \sum_{i<j} \left| \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} \right|^2 .$$

□

Corollary 2.1.25. *Let E be a separable Hilbert space with $\dim E = m$ and let $(e_n)_{n=1}^{m}$ be an orthonormal basis of E. Then, the set $\mathcal{B} = \{e_i \wedge e_j : 1 \leq i < j \leq m\}$ is an orthonormal basis of $\wedge^2 E$.*

Proof. By Theorem 2.1.23, the set $\mathcal{B}$ is a linearly independent set in $\wedge^2 E$. Also, by Proposition 2.1.24, the set $\mathcal{B}$ spans $\wedge^2 E$ and is an orthonormal set. Hence $\mathcal{B}$ is an orthonormal basis of $\wedge^2 E$. □

Proposition 2.1.26. *If E is an m-dimensional Hilbert space with orthonormal basis $\{e_1, \cdots, e_m\}$, then, for $0 < p \leq m$, $\wedge^p E$ is an $\binom{m}{p}$-dimensional Hilbert space with orthonormal basis*

$$B = \{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p},\ 1 \leq i_1 < \cdots < i_p \leq m\}.$$

Proof. By Theorem 2.1.12, $\wedge^p E$ is a Hilbert space. Let us first show that the set $\{e_{i_1}, \cdots, e_{i_p} : 1 \leq i_1 < \cdots i_p \leq m\}$ spans $\wedge^p E$. Suppose that $x_j = \sum_{i=1}^{m} \alpha_{i_j} e_i$. Then

$$x_1 \wedge x_2 \wedge \cdots \wedge x_p = \sum_{i=1}^{m} \alpha_{i_1} e_i \wedge \sum_{i=1}^{m} \alpha_{i_2} e_i \wedge \cdots \wedge \sum_{i=1}^{m} \alpha_{i_p} e_i.$$

By multilinearity of the wedge product we have

$$x_1 \wedge x_2 \wedge \cdots \wedge x_p = \sum_{1 \le i_1 < \cdots < i_p \le m} \alpha_{i_1,\cdots,i_p} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}.$$

Hence the set B spans $\wedge^p E$. Since $(e_i)_{i=1}^m$ is an orthonormal basis, for $1 \le i_1 < \cdots < i_p \le m$ we have

$$\begin{aligned} \|e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_p}\|^2_{\wedge^p E} &= \det \begin{pmatrix} \langle e_{i_1}, e_{i_1}\rangle_E & \langle e_{i_1}, e_{i_2}\rangle_E & \cdots & \langle e_{i_1}, e_{i_p}\rangle_E \\ \langle e_{i_2}, e_{i_1}\rangle_E & \langle e_{i_2}, e_{i_2}\rangle & \cdots & \langle e_{i_2}, e_{i_p}\rangle_E \\ \vdots & \cdots & \ddots & \vdots \\ \langle e_{i_p}, e_{i_1}\rangle_E & \cdots & \cdots & \langle e_{i_p}, e_{i_p}\rangle_E \end{pmatrix} \\ &= 1. \end{aligned}$$

Furthermore, for $\{j_1, \cdots, j_p : \ 1 \le j_1 < \cdots < j_p \le m\}$ such that

$$\{i_1, \cdots, i_p\} \cap \{j_1, \cdots, j_p\} = \emptyset,$$

we get

$$\langle e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_p}, e_{j_1} \wedge \cdots \wedge e_{j_p}\rangle_{\wedge^p E} = \det \begin{pmatrix} \langle e_{i_1}, e_{j_1}\rangle_E & \langle e_{i_1}, e_{j_2}\rangle_E & \cdots & \langle e_{i_1}, e_{j_p}\rangle_E \\ \langle e_{i_2}, e_{j_1}\rangle_E & \langle e_{i_2}, e_{j_2}\rangle_E & \cdots & \langle e_{i_2}, e_{j_p}\rangle_E \\ \vdots & \cdots & \ddots & \vdots \\ \langle e_{i_p}, e_{j_1}\rangle_E & \cdots & \cdots & \langle e_{i_p}, e_{j_p}\rangle_E \end{pmatrix} = 0.$$

Moreover, by Theorem 2.1.23, the set B is a linearly independent set in $\wedge^p E$, thus the set B is an orthonormal basis for $\wedge^p E$. Finally, the cardinality of the basis is equal to $\binom{m}{p}$ since this is the number of possible choices of p elements out of m elements, and so $\wedge^p E$ is an $\binom{m}{p}$-dimensional Hilbert space. □

Example 2.1.27.

$$\wedge^2 \mathbb{C}^2 \cong \mathbb{C}^{\binom{2}{2}} = \mathbb{C}$$

and

$$\wedge^p \mathbb{C}^n \cong \mathbb{C}^{\binom{n}{p}}.$$

Next, we study properties of wedge products of bounded linear operators. Detailed information is included in Appendix A.

Definition 2.1.28. *Suppose $H_1, \ldots, H_p, K_1, \ldots, K_p$ are Hilbert spaces and $T_i : H_i \to K_i$, $i = 1, \ldots, p$, are bounded linear operators. Then, on algebraic tensor products, we define the operator*

$$T_1 \otimes \cdots \otimes T_p : H_1 \otimes \cdots \otimes H_p \to K_1 \otimes \cdots \otimes K_p$$

on elementary tensors by

$$(T_1 \otimes \cdots \otimes T_p)(u_1 \otimes \cdots \otimes u_p) = T_1(u_1) \otimes \cdots \otimes T_p(u_p), \tag{2.1}$$

and we extend $T_1 \otimes \cdots \otimes T_p$ to $H_1 \otimes \cdots \otimes H_p$ by linearity.

Proposition 2.1.29 ([8], Chapitre 1, Section 2)**.** *Let $(H_i, \langle \cdot, \cdot \rangle_{H_i})$ and $(G_i, \langle \cdot, \cdot \rangle_{G_i})$ be Hilbert spaces, and let $T_i : H_i \to G_i$ be bounded linear operators for $i = 1, \dots, p$. Then, the operator $T_1 \otimes \cdots \otimes T_p$ of equation* (2.1) *has a continuous extension*

$$T_1 \otimes \cdots \otimes T_p : H_1 \otimes_H \cdots \otimes_H H_p \to G_1 \otimes_H \cdots \otimes_H G_p$$

to a bounded linear operator on the completion $H_1 \otimes_H \cdots \otimes_H H_p$ of $H_1 \otimes \cdots \otimes H_p$.

Proposition 2.1.30. *Let E, K be Hilbert spaces and let $T : E \to K$ be a bounded linear operator. Let $\wedge^p T$ be the restriction of*

$$\underbrace{T \otimes \cdots \otimes T}_{p-times} : \otimes_H^p E \to \otimes_H^p K$$

to $\wedge^p E$. Then the image of $\wedge^p T$ is in $\wedge^p K$.

Proof. Let $\lambda^i \in \mathbb{C}$ for all $i = 1, \dots, n$ and let

$$u = \sum_{i=1}^n \lambda^i x_1^i \otimes \cdots \otimes x_p^i$$

be in $\wedge^p E$. Then $u = \epsilon_\sigma S_\sigma u$ for all $\sigma \in S_p$. Therefore, for $\sigma \in S_\sigma$,

$$\begin{aligned}
(T \otimes \cdots \otimes T)(u) &= (T \otimes \cdots \otimes T) \left(\epsilon_\sigma S_\sigma \sum_{i=1}^n \lambda^i x_1^i \otimes \cdots \otimes x_p^i \right) \\
&= (T \otimes \cdots \otimes T) \left(\epsilon_\sigma \sum_{i=1}^n \lambda^i x_{\sigma(1)}^i \otimes \cdots \otimes x_{\sigma(p)}^i \right) \\
&= \epsilon_\sigma \sum_{i=1}^n \lambda^i T(x_{\sigma(1)}^i) \otimes \cdots \otimes T(x_{\sigma(p)}^i) \\
&= \epsilon_\sigma S_\sigma \sum_{i=1}^n \lambda^i T(x_1^i) \otimes \cdots \otimes T(x_p^i) \\
&= \epsilon_\sigma S_\sigma (T \otimes \cdots \otimes T) \left(\sum_{i=1}^n \lambda^i x_1^i \otimes \otimes x_p^i \right) \\
&= \epsilon_\sigma S_\sigma ((T \otimes \cdots \otimes T)(u)).
\end{aligned}$$

Thus, for $u \in \wedge^p E$, $(T \otimes \cdots \otimes T)(u)$ is an antisymmetric tensor in $\otimes_H^p K$, that is, a member of $\wedge^p K$. □

Definition 2.1.31. *Let H, K be Hilbert spaces and $T \colon H \to K$ be a bounded linear operator. We define the operator*

$$\wedge^p T \colon \wedge^p H \to \wedge^p K$$

to be the restriction of $T \otimes \cdots \otimes T$ *to* $\wedge^p H$.

Definition 2.1.32. *Let* $(E, \|\cdot\|_E)$ *be a Hilbert space. The* p-fold Cartesian product of E *is defined to be the set*

$$\underbrace{E \times \cdots \times E}_{p-times} = \{(x_1, \ldots, x_p) : x_i \in E\}.$$

Moreover, we define a norm on $\underbrace{E \times \cdots \times E}_{p-times}$ *by*

$$\|(x_1, \ldots, x_p)\| = \{\sum_{i=1}^{p} \|x_i\|_E^2\}^{\frac{1}{2}}.$$

Definition 2.1.33. *Let* E *be a Hilbert space. We define the multilinear operator*

$$\Lambda : \underbrace{E \times \cdots \times E}_{p-times} \to \wedge^p E$$

by

$$\Lambda(x_1, \ldots, x_p) = x_1 \wedge \ldots \wedge x_p \quad \textit{for all} \quad x_1, \ldots, x_p \in E.$$

Proposition 2.1.34. *[Hadamard's inequality, [12], p. 477] For any matrix*

$$A = (a_{ij}) \in \mathbb{C}^{n \times n},$$

$$|\det(A)| \leq \prod_{j=1}^{n} \left(\sum_{i=1}^{n} |a_{ij}|^2 \right)^{1/2} \quad \textit{and} \quad |\det(A)| \leq \prod_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

Proposition 2.1.35. *Let* E *be a Hilbert space. Then the multilinear mapping*

$$\Lambda \colon \underbrace{E \times \cdots \times E}_{p-times} \to \wedge^p E$$

is bounded.

Proof. Let $x_i \in E$ for all $i = 1, \ldots, p$. Then $\Lambda(x_1, \ldots, x_p) = x_1 \wedge \ldots \wedge x_p$ and

$$\begin{aligned}
\|\Lambda(x_1, \ldots, x_p)\|_{\wedge^p E}^2 &= \|x_1 \wedge \ldots \wedge x_p\|_{\wedge^p E}^2 \\
&= \langle x_1 \wedge \ldots \wedge x_p, x_1 \wedge \ldots \wedge x_p \rangle_{\wedge^p E} \\
&= \det \begin{pmatrix} \langle x_1, x_1 \rangle_E & \langle x_1, x_2 \rangle_E & \ldots & \langle x_1, x_p \rangle_E \\ \langle x_2, x_1 \rangle_E & \langle x_2, x_2 \rangle_E & \ldots & \langle x_2, x_p \rangle_E \\ \vdots & \vdots & \ddots & \ldots \\ \langle x_p, x_1 \rangle_E & \ldots & \ldots & \langle x_p, x_p \rangle_E \end{pmatrix} \geq 0.
\end{aligned}$$

Observe that the matrix

$$X = \begin{pmatrix} \langle x_1, x_1\rangle_E & \langle x_1, x_2\rangle_E & \dots & \langle x_1, x_p\rangle_E \\ \langle x_2, x_1\rangle_E & \langle x_2, x_2\rangle_E & \dots & \langle x_2, x_p\rangle_E \\ \vdots & \vdots & \ddots & \dots \\ \langle x_p, x_1\rangle_E & \dots & \dots & \langle x_p, x_p\rangle_E \end{pmatrix}$$

is Hermitian, and so its determinant is real. By Hadamard's inequality,

$$|\det(X)| \le \prod_{j=1}^{p} \left(\sum_{i=1}^{p} |\langle x_i, x_j\rangle_E|^2 \right)^{1/2} \quad \text{and} \quad |\det(X)| \le \prod_{i=1}^{p} \left(\sum_{j=1}^{p} |\langle x_i, x_j\rangle_E|^2 \right)^{1/2}.$$

Moreover, by the Cauchy-Schwartz inequality,

$$|\det(X)| \le \prod_{j=1}^{p} \|x_j\|_E \left(\sum_{i=1}^{p} \|x_i\|_E^2 \right)^{1/2}.$$

Therefore

$$\|\Lambda(x_1, \dots, x_p)\|^2_{\wedge^p E} \le \prod_{j=1}^{p} \|x_j\|_E \left(\sum_{i=1}^{p} \|x_i\|_E^2 \right)^{1/2}. \tag{2.2}$$

Let $\|(x_1, \dots, x_p)\|_{E^p} \le 1$. Since $\|x_j\|_E \le \|(x_1, \dots, x_p)\|_{E^p} \le 1$ for each j, we have

$$\|\Lambda(x_1, \dots, x_p)\|^2_{\wedge^p E} \le 1.$$

Hence the p-linear operator Λ is bounded. □

Lemma 2.1.36. *Let H, K be Hilbert spaces and let $S, T \colon H \to K$ be bounded linear operators. Then,*

(i) $\wedge^p(ST) = (\wedge^p S)(\wedge^p T)$.

(ii) $(\wedge^p T)^* = \wedge^p(T^*)$.

Proof. (i). By Definition 2.1.31, for all $x_i \in H$, where $i = 1, \dots, p$, we have

$$\begin{aligned} \wedge^p(S) \wedge^p (T)(x_1 \wedge \dots \wedge x_p) &= \wedge^p(S)(Tx_1 \wedge \dots \wedge Tx_p) \\ &= STx_1 \wedge \dots \wedge STx_p \\ &= \wedge^p(ST)(x_1 \wedge \dots \wedge x_p). \end{aligned}$$

(ii). By Definition 2.1.31 and by Proposition 2.1.19, for all $x_i \in H$ and all $y_i \in K$, where $i = 1, \dots, p$,

$$
\begin{aligned}
\langle (\wedge^p T^*)(y_1 \wedge \cdots \wedge y_p), (x_1 \wedge \cdots \wedge x_p) \rangle_{\wedge^p H} &= \langle T^*(y_1) \wedge \cdots \wedge T^*(y_p), x_1 \wedge \cdots \wedge x_p \rangle_{\wedge^p H} \\
&= \det \begin{pmatrix} \langle T^*(y_1), x_1 \rangle_H & \dots & \langle T^*(y_1), x_p \rangle_H \\ \langle T^*(y_2), x_1 \rangle_H & \dots & \langle T^*(y_2), x_p \rangle_H \\ \vdots & \ddots & \vdots \\ \langle T^*(y_p), x_1 \rangle_H & \dots & \langle T^*(y_p), x_p \rangle_H \end{pmatrix} \\
&= \det \begin{pmatrix} \langle y_1, T(x_1) \rangle_K & \dots & \langle y_1, (Tx_p) \rangle_K \\ \langle y_2, T(x_1) \rangle_K & \dots & \langle y_2, T(x_p) \rangle_K \\ \vdots & \dots & \vdots \\ \langle y_p, T(x_1) \rangle_K & \dots & \langle y_p, T(x_p) \rangle_K \end{pmatrix} \\
&= \langle y_1 \wedge \cdots \wedge y_p, (\wedge^p T)(x_1 \wedge \cdots \wedge x_p) \rangle_{\wedge^p K} \\
&= \langle (\wedge^p T)^*(y_1 \wedge \cdots \wedge y_p), (x_1 \wedge \cdots \wedge x_p) \rangle_{\wedge^p H} \\
&= \langle (\wedge^p T)^*(y_1 \wedge \cdots \wedge y_p), (x_1 \wedge \cdots \wedge x_p) \rangle_{\wedge^p H}.
\end{aligned}
$$

Hence $\wedge^p(T^*) = (\wedge^p T)^*$. □

2.2 Pointwise wedge products

For the purposes of this dissertation, we wish to consider the wedge product of mappings defined on the unit circle or in the unit disc that take values in Hilbert spaces. To this end, we introduce the notion of pointwise wedge product and we study various properties of it.

Definition 2.2.1. *Let E be a Hilbert space and let $f, g \colon \mathbb{D} \to E$ $(f, g \colon \mathbb{T} \to E)$ be E-valued maps. We define the* pointwise wedge product of f and g,

$$f \dot\wedge g \colon \mathbb{D} \to \wedge^2 E \quad (f \dot\wedge g \colon \mathbb{T} \to \wedge^2 E)$$

by

$$(f \dot\wedge g)(z) = f(z) \wedge g(z) \quad \text{for all } z \in \mathbb{D} \quad (\text{for almost all } z \in \mathbb{T}).$$

Definition 2.2.2. *Let E be a Hilbert space and let $\chi, \psi \colon \mathbb{D} \to E$ $(\chi, \psi \colon \mathbb{T} \to E)$ be E-valued maps. We call χ and ψ* pointwise linearly dependent *on $\mathbb{D}$ (respectively on $\mathbb{T}$) if there exist non-zero mappings $\kappa, \nu \colon \mathbb{D} \to \mathbb{C}$ $(\kappa, \nu \colon \mathbb{T} \to \mathbb{C})$, which do not simultaneously vanish at any point of $\mathbb{D}$ (of $\mathbb{T}$), such that*

$$\kappa(z)\chi(z) = \nu(z)\psi(z)$$

for all $z \in \mathbb{D}$ (for almost all $z \in \mathbb{T}$).

Remark 2.2.3. *Corollary 2.1.20 asserts that $x_1, \dots, x_n \colon \mathbb{T} \to E$ are pointwise linearly dependent on $\mathbb{T}$ if and only if*

$$(x_1 \dot\wedge \dots \dot\wedge x_n)(z) = 0 \quad \text{for almost all} \;\; z \in \mathbb{T}.$$

Henceforth we consider vector-valued L^p spaces as they are presented in [14].

Definition 2.2.4. *Let E be a separable Hilbert space and let $1 \le p < \infty$. Define*

(i) $L^p(\mathbb{T}, E)$ to be the normed space of measurable (weakly or strongly, which amounts to the same thing, in view of the separability of E) E-valued maps $f \colon \mathbb{T} \to E$ such that

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(e^{i\theta})\|_E^p d\theta \right)^{1/p} < \infty.$$

(ii) $H^p(\mathbb{D}, E)$ to be the normed space of analytic E-valued maps $f \colon \mathbb{D} \to E$ such that

$$\|f\|_p = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_E^p d\theta \right)^{1/p} < \infty.$$

(iii) $L^\infty(\mathbb{T}, E)$ to be the space of essentially bounded measurable E-valued functions on the unit circle with the essential supremum norm

$$\|f\|_{L^\infty} = \operatorname{ess\,sup}_{|z|=1} \|f(z)\|_E,$$

and with functions equal almost everywhere identified.

(iv) $H^\infty(\mathbb{D}, E)$ to be the space of bounded analytic E-valued functions on the unit disc with the supremum norm

$$\|f\|_{H^\infty} = \sup_{z \in \mathbb{D}} \|f(z)\|_E.$$

Lemma 2.2.5 ([18], p. 242)**.** [Hölder's inequality] *Let $f \in L^p(\mathbb{T})$ and let $g \in L^q(\mathbb{T})$, where $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\|fg\|_{L^1} \le \|f\|_{L^p} \|g\|_{L^q}.$$

Proposition 2.2.6. *Let E be a separable Hilbert space and let $\dfrac{1}{p} + \dfrac{1}{q} = 1$, where $1 \le p, q \le \infty$. Suppose that $x \in L^p(\mathbb{T}, E)$, $y \in L^q(\mathbb{T}, E)$. Then*

$$x \dot\wedge y \in L^1(\mathbb{T}, \wedge^2 E)$$

and

$$\|x \dot\wedge y\|_{L^1(\mathbb{T}, \wedge^2 E)} \le \|x\|_{L^p(\mathbb{T}, E)} \|y\|_{L^q(\mathbb{T}, E)}. \tag{2.3}$$

Proof. By Proposition 2.1.19, for all $z \in \mathbb{T}$,

$$\begin{aligned}\|(x\dot{\wedge}y)(z)\|^2_{\wedge^2 E} &= \langle x(z)\wedge y(z), x(z)\wedge y(z)\rangle_{\wedge^2 E}\\ &= \langle x(z), x(z)\rangle_E \cdot \langle y(z), y(z)\rangle_E - |\langle x(z), y(z)\rangle_E|^2\\ &\le \|x(z)\|_E^2 \|y(z)\|_E^2.\end{aligned}$$

Thus, for all $z \in \mathbb{T}$,

$$\|(x\dot{\wedge}y)(z)\|_{\wedge^2 E} \le \|x(z)\|_E \|y(z)\|_E.$$

By Definition 2.2.4,

$$\|x\dot{\wedge}y\|_{L^1(\mathbb{T},\wedge^2 E)} = \frac{1}{2\pi}\int_0^{2\pi} \|(x\dot{\wedge}y)(e^{i\theta})\|_{\wedge^2 E}\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \|x(e^{i\theta})\|_E \|y(e^{i\theta})\|_E\, d\theta. \tag{2.4}$$

Now by Hölder's inequality,

$$\frac{1}{2\pi}\int_0^{2\pi} \|x(e^{i\theta})\|_E \|y(e^{i\theta})\|_E\, d\theta \le \left(\frac{1}{2\pi}\int_0^{2\pi} \|x(e^{i\theta})\|_E^p\, d\theta\right)^{1/p} \left(\frac{1}{2\pi}\int_0^{2\pi} \|y(e^{i\theta})\|_E^q\, d\theta\right)^{1/q}. \tag{2.5}$$

Hence, by inequalities (2.4) and (2.5), $x\dot{\wedge}y \in L^1(\mathbb{T},\wedge^2 E)$ and the inequality (2.3) holds. □

Proposition 2.2.7. *Let E be a Hilbert space and $x, y\colon \mathbb{D} \to E$ be two analytic E-valued maps on $\mathbb{D}$. Then,*

$$x\dot{\wedge}y\colon \mathbb{D} \to \wedge^2 E$$

is also analytic on $\mathbb{D}$ and

$$(x\dot{\wedge}y)'(z) = x'(z)\wedge y(z) + x(z)\wedge y'(z) \quad \text{for all} \quad z \in \mathbb{D}.$$

Proof. For E-valued maps x, y, being analytic on $\mathbb{D}$ means that for every $z_0 \in \mathbb{D}$ there exist $x'(z_0) \in E$ and $y'(z_0) \in E$ such that

$$\lim_{h\to 0}\left\|\frac{x(z_0+h) - x(z_0)}{h} - x'(z_0)\right\|_E = 0$$

and

$$\lim_{h\to 0}\left\|\frac{y(z_0+h) - y(z_0)}{h} - y'(z_0)\right\|_E = 0.$$

Note that

$$\frac{(x\dot{\wedge}y)(z_0+h) - (x\dot{\wedge}y)(z_0)}{h} \in \wedge^2 E.$$

One can see that, for $h \in \mathbb{D}$ such that $z_0 + h \in \mathbb{D}$,

$$\begin{aligned}
&\frac{(x\dot{\wedge}y)(z_0+h)-(x\dot{\wedge}y)(z_0)}{h}\\
&=\frac{x(z_0+h)\wedge y(z_0+h)-x(z_0)\wedge y(z_0)+x(z_0)\wedge y(z_0+h)-x(z_0)\wedge y(z_0+h)}{h}\\
&=\frac{(x(z_0+h)-x(z_0))\wedge y(z_0+h)}{h}+\frac{x(z_0)\wedge (y(z_0+h)-y(z_0))}{h}\\
&=\frac{x(z_0+h)-x(z_0)}{h}\wedge y(z_0+h)+x(z_0)\wedge\frac{y(z_0+h)-y(z_0)}{h}\\
&\overset{h\to 0}{\to} x'(z_0)\wedge y(z_0)+x(z_0)\wedge y'(z_0).
\end{aligned}$$

Now,

$$\begin{aligned}
&\left\|\frac{(x\dot{\wedge}y)(z_0+h)-(x\dot{\wedge}y)(z_0)}{h}-(x'(z_0)\wedge y(z_0)+x(z_0)\wedge y'(z_0))\right\|_{\wedge^2 E}\\
&=\left\|\frac{(x(z_0+h)-x(z_0))}{h}\wedge y(z_0+h)+x(z_0)\wedge\frac{(y(z_0+h)-y(z_0))}{h}\right.\\
&\qquad\qquad\left.-(x'(z_0)\wedge y(z_0)+x(z_0)\wedge y'(z_0))\right\|_{\wedge^2 E}\\
&\le\left\|\frac{(x(z_0+h)-x(z_0))}{h}\wedge y(z_0+h)-x'(z_0)\wedge y(z_0)\right\|_{\wedge^2 E}\\
&+\left\|x(z_0)\wedge\frac{(y(z_0+h)-y(z_0))}{h}-x(z_0)\wedge y'(z_0)\right\|_{\wedge^2 E}.
\end{aligned}$$

Let us consider each term separately.

$$\begin{aligned}
&\left\|\frac{(x(z_0+h)-x(z_0))}{h}\wedge y(z_0+h)-x'(z_0)\wedge y(z_0)\right\|_{\wedge^2 E}\\
&=\left\|\frac{(x(z_0+h)-x(z_0))}{h}\wedge y(z_0+h)-x'(z_0)\wedge y(z_0+h)\right.\\
&\qquad\qquad\left.+x'(z_0)\wedge y(z_0+h)-x'(z_0)\wedge y(z_0)\right\|_{\wedge^2 E}\\
&\le\left\|\left(\frac{x(z_0+h)-x(z_0)}{h}-x'(z_0)\right)\wedge y(z_0+h)\right\|_{\wedge^2 E}+\|x'(z_0)\wedge(y(z_0+h)-y(z_0))\|_{\wedge^2 E}.
\end{aligned}$$

By Proposition 2.1.19,

$$\begin{aligned}
&\left\|\left(\frac{x(z_0+h)-x(z_0)}{h}-x'(z_0)\right)\wedge y(z_0+h)\right\|_{\wedge^2 E}+\|x'(z_0)\wedge(y(z_0+h)-y(z_0))\|_{\wedge^2 E}\\
&\le\left\|\frac{(x(z_0+h)-x(z_0))}{h}-x'(z_0)\right\|_E\cdot\|y(z_0+h)\|_E+\|x'(z_0)\|_E\cdot\|y(z_0+h)-y(z_0)\|_E
\end{aligned}$$

which tends to 0 as $h\to 0$. For the other term we have

$$\left\|x(z_0)\wedge\frac{(y(z_0+h)-y(z_0))}{h}-x(z_0)\wedge y'(z_0)\right\|_{\wedge^2 E}=\left\|x(z_0)\wedge\left(\frac{y(z_0+h)-y(z_0)}{h}-y'(z_0)\right)\right\|_{\wedge^2 E}.$$

By Proposition 2.1.19,

$$\left\| x(z_0) \wedge \left(\frac{y(z_0+h) - y(z_0)}{h} - y'(z_0) \right) \right\|_{\wedge^2 E} \leq \|x(z_0)\|_E \cdot \left\| \frac{y(z_0+h) - y(z_0)}{h} - y'(z_0) \right\|_E$$

which tends to 0 as $h \to 0$.
Thus we get

$$\left\| \frac{(x\dot{\wedge}y)(z_0+h) - (x\dot{\wedge}y)(z_0)}{h} - (x'(z_0) \wedge y(z_0) + x(z_0) \wedge y'(z_0)) \right\|_{\wedge^2 E} \to 0$$

as $h \to 0$.
Therefore $x\dot{\wedge}y$ is analytic on $\mathbb{D}$ and, at every point $z \in \mathbb{D}$, the derivative is given by

$$(x\dot{\wedge}y)'(z) = (x'\dot{\wedge}y)(z) + (x\dot{\wedge}y')(z).$$ □

Proposition 2.2.8. *Let E be a Hilbert space and let $x_i \colon \mathbb{D} \to E$ be analytic E-valued maps on $\mathbb{D}$ for all $i = 0, \dots, k$. Then*

$$x_0\dot{\wedge} \cdots \dot{\wedge}x_k \colon \mathbb{D} \to \wedge^{k+1} E$$

is also analytic on $\mathbb{D}$ and

$$\begin{aligned}(x_0\dot{\wedge} \cdots \dot{\wedge}x_k)'(z) &= x_0'(z) \wedge x_1(z) \wedge \cdots \wedge x_k(z) + x_0(z) \wedge x_1'(z) \wedge x_2(z) \wedge \cdots \wedge x_k(z) \\ &+ \cdots + x_0(z) \wedge x_1(z) \wedge \cdots \wedge x_k'(z).\end{aligned}$$

Proof. The E-valued maps x_i being analytic on $\mathbb{D}$ means that, for every $z_0 \in \mathbb{D}$, there exist $x_i'(z) \in E$ such that

$$\lim_{h\to 0} \left\| \frac{x_i(z+h) - x_i(x)}{h} - x_i'(z) \right\|_E = 0.$$

Notice

$$X = \frac{(x_0\dot{\wedge} \cdots \dot{\wedge}x_k)(z_0+h) - (x_0\dot{\wedge} \cdots \dot{\wedge}x_k)(z_0)}{h} \tag{2.6}$$

is an element of $\wedge^{k+1} E$. One can see that, for $h \in \mathbb{D}$ such that $z_0 + h \in \mathbb{D}$, expression (2.6) yields

$$X = \frac{1}{h}\Big(x_0(z_0+h) \wedge \cdots \wedge x_k(z_0+h) - x_0(z_0) \wedge \cdots \wedge x_k(z_0) \Big). \tag{2.7}$$

If we add and subtract

$$\frac{1}{h} x_0(z_0) \wedge x_1(z_0+h) \wedge x_2(z_0+h) \wedge \cdots \wedge x_k(z_0+h)$$

to expression (2.7), we obtain

$$\begin{aligned}X &= \tfrac{1}{h}\Big([x_0(z_0+h) - x_0(z_0)] \wedge x_1(z_0+h) \wedge \cdots \wedge x_k(z_0+h) - x_0(z_0) \wedge \cdots \wedge x_k(z_0) \\ &\quad + x_0(z_0) \wedge x_1(z_0+h) \wedge \cdots \wedge x_k(z_0+h)\Big).\end{aligned}$$

Adding and subtracting

$$\frac{1}{h}x_0(z_0) \wedge x_1(z_0) \wedge x_2(z_0+h) \wedge \cdots \wedge x_k(z_0+h)$$

to the latter expression, we get

$$\begin{aligned} X \quad &= \tfrac{1}{h}\Big([x_0(z_0+h) - x_0(z_0)] \wedge x_1(z_0+h) \wedge \cdots \wedge x_k(z_0+h) - x_0(z_0) \wedge \cdots \wedge x_k(z_0) \\ &+ x_0(z_0) \wedge [x_1(z_0+h) - x_1(z_0)] \wedge x_2(z_0+h) \wedge \cdots \wedge x_k(z_0+h) \\ &+ x_0(z_0) \wedge x_1(z_0) \wedge \cdots \wedge x_k(z_0+h)\Big). \end{aligned}$$

It becomes evident that, if we continue accordingly, we obtain

$$\begin{aligned} X = \frac{1}{h}\Big(&[x_0(z_0+h) - x_0(z_0)] \wedge x_1(z_0+h) \wedge \cdots \wedge x_k(z_0+h) \\ &+ x_0(z_0) \wedge [x_1(z_0+h) - x_1(z_0)] \wedge x_2(z_0+h) \wedge \cdots \wedge x_k(z_0+h) \\ &+ x_0(z_0) \wedge x_1(z_0) \wedge [x_2(z_0+h) - x_2(z_0)] \wedge \cdots \wedge x_k(z_0+h) \\ &+ \cdots + x_0(z_0) \wedge x_1(z_0) \wedge \cdots \wedge [x_k(z_0+h) - x_k(z_0)]\Big) \\ &= \frac{1}{h}[x_0(z_0+h) - x_0(z_0)] \wedge x_1(z_0+h) \cdots \wedge x_k(z_0+h) \\ &+ x_0(z_0) \wedge \frac{1}{h}[x_1(z_0+h) - x_1(z_0)] \wedge x_2(z_0+h) \wedge \cdots \wedge x_k(z_0+h) \\ &+ x_0(z_0) \wedge x_1(z_0) \wedge \frac{1}{h}[x_2(z_0+h) - x_2(z_0)] \wedge \cdots \wedge x_k(z_0+h) \\ &+ \cdots + x_0(z_0) \wedge x_1(z_0) \wedge \cdots \wedge \frac{1}{h}[x_k(z_0+h) - x_k(z_0)]. \end{aligned} \tag{2.8}$$

Let us show that

$$\begin{aligned} \tfrac{1}{h}[(x_0\dot{\wedge}\ldots\dot{\wedge}x_k)(z_0+h) - (x_0\dot{\wedge}\ldots\dot{\wedge}x_k)(z_0)] \quad &\overset{h\to 0}{\to} x_0'(z_0) \wedge x_1(z_0) \cdots \wedge x_k(z_0) \\ &+ x_0(z_0) \wedge x_1'(z_0) \wedge x_2(z_0) \wedge \cdots \wedge x_k(z_0) \\ &+ \cdots + x_0(z_0) \wedge x_1(z_0) \wedge \cdots \wedge x_k'(z_0). \end{aligned}$$

Thus, by equation (2.8),

$$\begin{aligned} &\Bigg\| \frac{(x_0\dot{\wedge}\cdots\dot{\wedge}x_k)(z_0+h) - (x_0\dot{\wedge}\cdots\dot{\wedge}x_k)(z_0)}{h} - \Big(x_0'(z_0) \wedge x_1(z_0) \cdots \wedge x_k(z_0) \\ &+ x_0(z_0) \wedge x_1'(z) \wedge x_2(z_0) \wedge \cdots \wedge x_k(z_0) + x_0(z_0) \wedge x_1(z_0) \wedge x_2'(z_0) \wedge \cdots \wedge x_k(z_0) \\ &+ \cdots + x_0(z_0) \wedge x_1(z_0) \wedge \cdots \wedge x_k'(z_0)\Big)\Bigg\|_{\wedge^{k+1}E} \end{aligned}$$

is equal to

$$\Bigg\|\frac{(x_0\dot\wedge\cdots\dot\wedge x_k)(z_0+h)-(x_0\dot\wedge\cdots\dot\wedge x_k)(z_0)}{h}-\Big(x_0'(z_0)\wedge x_1(z_0)\cdots\wedge x_k(z_0)$$
$$+x_0(z_0)\wedge x_1'(z)\wedge x_2(z_0)\wedge\cdots\wedge x_k(z_0)+x_0(z_0)\wedge x_1(z_0)\wedge x_2'(z_0)\wedge\cdots\wedge x_k(z_0)$$
$$+\cdots+x_0(z_0)\wedge x_1(z_0)\wedge\cdots\wedge x_k'(z_0)\Big)\Bigg\|_{\wedge^{k+1}E}$$
$$=\|\frac{1}{h}[x_0(z_0+h)-x_0(z_0)]\wedge x_1(z_0+h)\wedge\cdots\wedge x_k(z_0+h) \tag{2.9}$$
$$+\ x_0(z_0)\wedge\frac{1}{h}[x_1(z_0+h)-x_1(z_0)]\wedge x_2(z_0+h)\wedge\cdots\wedge x_k(z_0+h)$$
$$+\ \cdots+x_0(z_0)\wedge x_1(z_0)\wedge\cdots\frac{1}{h}[x_k(z_0+h)-x_k(z_0)]$$
$$-\ x_0'(z_0)\wedge x_1(z_0)\cdots\wedge x_k(z_0)+x_0(z_0)\wedge x_1'(z)\wedge x_2(z_0)\wedge\cdots\wedge x_k(z_0)$$
$$-\cdots-x_0(z_0)\wedge x_1(z_0)\wedge\cdots\wedge x_k'(z_0)\|_{\wedge^{k+1}E}$$
$$\le\|\frac{1}{h}[x_0(z_0+h)-x_0(z_0)]\wedge x_1(z_0+h)\wedge\cdots\wedge x_k(z_0+h)-x_0'(z_0)\wedge x_1(z_0)\cdots\wedge x_k(z_0)\|_{\wedge^{k+1}E}$$
$$+\cdots+\|x_0(z_0)\wedge x_1(z_0)\wedge\cdots\wedge\frac{1}{h}[x_k(z_0+h)-x_k(z_0)]-x_0(z_0)\wedge x_1(z_0)\wedge\cdots\wedge x_k'(z_0)\|_{\wedge^{k+1}E}. \tag{2.10}$$

Considering, for instance, the first term of the sum (2.10), we have

$$\|\frac{1}{h}[x_0(z_0+h)-x_0(z_0)]\wedge x_1(z_0+h)\cdots\wedge x_k(z_0+h)-x_0'(z_0)\wedge x_1(z_0)\wedge\cdots\wedge x_k(z_0)\|_{\wedge^{k+1}E}$$
$$=\|\frac{1}{h}[x_0(z_0+h)-x_0(z_0)]\wedge x_1(z_0+h)\wedge\cdots\wedge x_k(z_0+h)-x_0'(z)\wedge x_1(z_0+h)\wedge\cdots\wedge x_k(z_0+h)$$
$$+\ x_0'(z)\wedge x_1(z_0+h)\wedge\cdots\wedge x_k(z_0+h)-x_0'(z)\wedge x_1(z_0)\cdots\wedge x_k(z_0)\|_{\wedge^{k+1}E}$$
$$\le\|[\frac{1}{h}[x_0(z_0+h)-x_0(z_0)]-x_0'(z)]\wedge x_1(z_0+h)\cdots\wedge x_k(z_0+h)\|_{\wedge^{k+1}E}$$
$$+\ \|x_0'(z)\wedge(x_1(z_0+h)-x_1(z_0))\wedge\cdots\wedge x_k(z_0+h)\|_{\wedge^{k+1}E}$$
$$+\cdots+\|x_0'(z)\wedge x_1(z_0)\wedge\ldots\wedge(x_k(z_0+h)-x_k(z_0))\|_{\wedge^{k+1}E}. \tag{2.11}$$

Recall,

$$\lim_{h\to 0}\left\|\frac{1}{h}[x_0(z_0+h)-x_0(z_0)]-x_0'(z)\right\|_E=0.$$

Hence, by Hadamard's inequality (2.2),

$$\|[\frac{1}{h}[x_0(z_0+h)-x_0(z_0)]-x_0'(z)]\wedge x_1(z_0+h)\cdots\wedge x_k(z_0+h)\|^2_{\wedge^{k+1}E}$$
$$\le\|[\frac{1}{h}[x_0(z_0+h)-x_0(z_0)]-x_0'(z)]\|_E\|x_1(z_0+h)\|_E\cdots\|x_k(z_0+h)\|_E$$
$$\left(\|[\frac{1}{h}[x_0(z_0+h)-x_0(z_0)]-x_0'(z)]\|^2_E+\|x_1(z_0+h)\|^2_E+\cdots+\|x_k(z_0+h)\|^2_E\right)^{1/2}$$

tends to 0 as $h\to 0$, and

$$\|x_0'(z) \wedge x_1(z_0+h) - x_1(z_0) \wedge \cdots \wedge x_k(z_0+h)\|^2_{\wedge^{k+1}E}$$

$$\leq \|x_0'(z)\|_E \|x_1(z_0+h) - x_1(z_0)\|_E \cdots \|x_k(z_0+h)\|_E$$
$$(\|x_0'(z)\|_E^2 + \|x_1(z_0+h) - x_1(z_0)\|_E^2 + \cdots + \|x_k(z_0+h)\|_E^2)^{1/2}$$

tends to 0 as $h \to 0$. Similarly, we infer that the sum (2.11), and consequently, the sum (2.10) tend to 0 as $h \to 0$. Thus

$$\Bigg\| \frac{(x_0\dot\wedge \cdots \dot\wedge x_k)(z_0+h) - (x_0\dot\wedge \cdots \dot\wedge x_k)(z_0)}{h} - \Bigg(x_0'(z_0) \wedge x_1(z_0) \cdots \wedge x_k(z_0)$$
$$+ x_0(z_0) \wedge x_1'(z) \wedge x_2(z_0) \wedge \cdots \wedge x_k(z_0) + x_0(z_0) \wedge x_1(z_0) \wedge x_2'(z_0) \wedge \cdots \wedge x_k(z_0)$$
$$+ \cdots + x_0(z_0) \wedge x_1(z_0) \wedge \cdots \wedge x_k'(z_0) \Bigg) \Bigg\|_{\wedge^{k+1}E} \to 0$$

as $h \to 0$. Hence

$$x_0\dot\wedge \cdots \dot\wedge x_k \colon \mathbb{D} \to \wedge^{k+1}E$$

is analytic on $\mathbb{D}$ and

$$(x_0\dot\wedge \cdots \dot\wedge x_k)'(z) = x_0'(z) \wedge x_1(z) \cdots \wedge x_k(z) + x_0(z) \wedge x_1'(z) \wedge x_2(z) \wedge \cdots \wedge x_k(z)$$
$$+ \cdots + x_0(z) \wedge x_1(z) \wedge \cdots x_k'(z).$$ □

Proposition 2.2.9. *Let E be a separable Hilbert space. Suppose $x, y \in H^2(\mathbb{D}, E)$. Then*

$$x\dot\wedge y \in H^1(\mathbb{D}, \wedge^2 E).$$

Proof. By Proposition 2.2.7, $x\dot\wedge y$ is analytic on $\mathbb{D}$. By Proposition 2.1.19, for $0 < r < 1$ and $0 \leq \theta \leq 2\pi$,

$$\|(x\dot\wedge y)(re^{i\theta})\|_{\wedge^2 E} \leq \|x(re^{i\theta})\|_E \|y(re^{i\theta})\|_E.$$

By Proposition 2.1.19 and by Definition 2.2.4,

$$\begin{aligned} \|x\dot\wedge y\|_{H^1(\mathbb{D},\wedge^2 E)} &= \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|(x\dot\wedge y)(re^{i\theta})\|_{\wedge^2 E}\, d\theta \right) \\ &\leq \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|x(re^{i\theta})\|_E \|y(re^{i\theta})\|_E\, d\theta \right), \end{aligned}$$

for $0 < r < 1$ and $0 \leq \theta \leq 2\pi$. Also, by Hölder's inequality, for $0 < r < 1$ and $0 \leq \theta \leq 2\pi$,

$$\frac{1}{2\pi} \int_0^{2\pi} \|x(re^{i\theta})\|_E \|y(re^{i\theta})\|_E\, d\theta \leq \left(\frac{1}{2\pi} \int_0^{2\pi} \|x(re^{i\theta})\|_E^2\, d\theta \right)^{1/2} \left(\frac{1}{2\pi} \int_0^{2\pi} \|y(re^{i\theta})\|_E^2\, d\theta \right)^{1/2},$$

hence $\|x\dot\wedge y\|_{H^1(\mathbb{D},\wedge^2 E)} \leq \|x\|_{H^2(\mathbb{D},E)} \|y\|_{H^2(\mathbb{D},E)}$. Consequently, $x\dot\wedge y \in H^1(\mathbb{D}, \wedge^2 E)$. □

Remark 2.2.10. *Let E be a finite dimensional Hilbert space. For $1 \leq p \leq \infty$, we will regard $x \in H^p(\mathbb{D}, E)$ as a column-vector valued function on $\mathbb{D}$ or $\mathbb{T}$ and x^* as the row-vector valued function, $x^*(z) = x(z)^*$, for all $z \in \mathbb{D}$ or $\mathbb{T}$.*

Example 2.2.11. *If $E = \mathbb{C}^n$, and if*

$$x(z) = \begin{pmatrix} x_1(z) & x_2(z) & \dots & x_n(z) \end{pmatrix}^T$$

for all $z \in \mathbb{T}$, then

$$x^*(z) = \begin{pmatrix} \overline{x_1(z)} & \cdots & \overline{x_n(z)} \end{pmatrix}.$$

Example 2.2.12. *Suppose $E = \mathbb{C}^n$ and let $x \in H^2(\mathbb{D}, \mathbb{C}^n)$, $y \in H^\infty(\mathbb{D}, \mathbb{C}^n)$. Then, for all $z \in \mathbb{D}$,*

$$\langle x(z), y(z) \rangle_{\mathbb{C}^n} = \sum_{i=1}^n x_i(z)\overline{y_i(z)} = \begin{pmatrix} \overline{y_1(z)} & \cdots & \overline{y_n(z)} \end{pmatrix} \begin{pmatrix} x_1(z) \\ x_2(z) \\ \vdots \\ x_n(z) \end{pmatrix}$$

Proposition 2.2.13. *Let E be a separable Hilbert space, let $x \in H^2(\mathbb{D}, E)$ and let $y \in H^\infty(\mathbb{D}, E)$. Then*

$$x \dot{\wedge} y \in H^2(\mathbb{D}, \wedge^2 E).$$

Proof. By Proposition 2.2.7, $x \dot{\wedge} y$ is analytic on $\mathbb{D}$. By Proposition 2.1.19, for

$$0 < r < 1, \quad 0 \leq \theta \leq 2\pi,$$

we have

$$\|(x \dot{\wedge} y)(re^{i\theta})\|_{\wedge^2 E} \leq \|x(re^{i\theta})\|_E \|y(re^{i\theta})\|_E.$$

Thus,

$$\begin{aligned} \|x \dot{\wedge} y\|_{H^2(\mathbb{D}, \wedge^2 E)} &= \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|(x \dot{\wedge} y)(re^{i\theta})\|^2_{\wedge^2 E} \, d\theta \right)^{1/2} \\ &\leq \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|x(re^{i\theta})\|^2_E \|y(re^{i\theta})\|^2_E d\theta \right)^{1/2} \\ &\leq \|y\|_\infty \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|x(re^{i\theta})\|^2_E d\theta \right)^{1/2} < \infty. \end{aligned}$$

□

Proposition 2.2.14. *Suppose $\{\xi_0, \dots, \xi_j\} \subset L^\infty(\mathbb{T}, \mathbb{C}^n)$ is a pointwise orthonormal set. Then*

$$\|\xi_0 \dot{\wedge} \cdots \dot{\wedge} \xi_j \dot{\wedge} x\|_{L^2(\mathbb{T}, \wedge^{j+2}\mathbb{C}^n)} < \infty$$

for all $x \in L^2(\mathbb{T}, \mathbb{C}^n)$.

Proof. By Proposition 2.1.19, $\|\xi_0\dot\wedge\cdots\dot\wedge\xi_j\dot\wedge x\|^2_{L^2(\mathbb{T},\wedge^{j+2}\mathbb{C}^n)}$ is equal to

$$\frac{1}{2\pi}\int_0^{2\pi}\det\begin{pmatrix}\langle\xi_0(e^{i\theta}),\xi_0(e^{i\theta})\rangle_{\mathbb{C}^n} & \langle\xi_0(e^{i\theta}),\xi_1(e^{i\theta})\rangle_{\mathbb{C}^n} & \dots & \langle\xi_0(e^{i\theta}),x(e^{i\theta})\rangle_{\mathbb{C}^n}\\ \langle\xi_0(e^{i\theta}),\xi_1(e^{i\theta})\rangle_{\mathbb{C}^n} & \langle\xi_1(e^{i\theta}),\xi_1(e^{i\theta})\rangle_{\mathbb{C}^n} & \dots & \langle\xi_1(e^{i\theta}),x(e^{i\theta})\rangle_{\mathbb{C}^n}\\ \vdots & & \ddots & \vdots\\ \langle x(e^{i\theta}),\xi_0(e^{i\theta})\rangle_{\mathbb{C}^n} & \langle x(e^{i\theta}),\xi_1(e^{i\theta})\rangle_{\mathbb{C}^n} & \dots & \langle x(e^{i\theta}),x(e^{i\theta})\rangle_{\mathbb{C}^n}\end{pmatrix} d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\det\begin{pmatrix}1 & 0 & \dots & \langle\xi_0(e^{i\theta}),x(e^{i\theta})\rangle\\ 0 & 1 & \dots & \langle\xi_1(e^{i\theta}),x(e^{i\theta})\rangle\\ \vdots & & \ddots & \vdots\\ \langle x(e^{i\theta}),\xi_0(e^{i\theta})\rangle_{\mathbb{C}^n} & \langle x(e^{i\theta}),\xi_1(e^{i\theta})\rangle_{\mathbb{C}^n} & \dots & \langle x(e^{i\theta}),x(e^{i\theta})\rangle_{\mathbb{C}^n}\end{pmatrix} d\theta,$$

the last equality following by the pointwise orthonormality of the set $\{\xi_k(z)\}_{k=0}^j$ on $\mathbb{T}$. Multiplying the l-th column by $-\langle\xi_l(e^{i\theta}),x(e^{i\theta})\rangle_{\mathbb{C}^n}$ and adding it to the last column, we get

$$\frac{1}{2\pi}\int_0^{2\pi}\det\begin{pmatrix}1 & 0 & \dots & 0\\ 0 & 1 & \dots & 0\\ \vdots & & \ddots & \vdots\\ \langle x(e^{i\theta}),\xi_0(e^{i\theta})\rangle_{\mathbb{C}^n} & \langle x(e^{i\theta}),\xi_1(e^{i\theta})\rangle_{\mathbb{C}^n} & \dots & \|x(e^{i\theta})\|^2_{\mathbb{C}^n} \\ & & & -\sum_{k=0}^{j}|\langle x(e^{i\theta}),\xi_k(e^{i\theta})\rangle_{\mathbb{C}^n}|^2\end{pmatrix} d\theta$$

$$\begin{aligned}&=\frac{1}{2\pi}\int_0^{2\pi}\|x(e^{i\theta})\|^2_{\mathbb{C}^n}-\sum_{k=0}^{j}|\langle x(e^{i\theta}),\xi_k(e^{i\theta})\rangle_{\mathbb{C}^n}|^2\,d\theta\\ &\le\frac{1}{2\pi}\int_0^{2\pi}\|x(e^{i\theta})\|^2_{\mathbb{C}^n}\,d\theta\\ &<\infty.\end{aligned}$$

□

2.2.1 Multiplication operators

Definition 2.2.15. *Let E,F be Hilbert spaces and let $G\in L^\infty(\mathbb{T},\mathcal{L}(E,F))$. For every $z\in\mathbb{T}$, we define*

$$\wedge^2G(z)\colon\ \wedge^2E\to\wedge^2F$$

on elements $x\wedge y$ by

$$[\wedge^2G(z)](x\wedge y)=G(z)x\wedge G(z)y.$$

Lemma 2.2.16. *Let E,F be Hilbert spaces and let $G\in L^\infty(\mathbb{T},\mathcal{L}(E,F))$. Then, for almost all $z\in\mathbb{T}$, $\wedge^2G(z)\colon\ \wedge^2E\to\wedge^2F$ is a bounded linear operator.*

Proof. By Proposition 2.1.30, for every $z\in\mathbb{T}$, $\wedge^2G(z)\colon\ \wedge^2E\to\wedge^2F$ is well-defined.
Now, let $x\wedge y,\ w\wedge v\in\wedge^2E$. Linearity of $\wedge^2G(z)$ follows from Proposition A.1.6 and the fact that $\wedge^2G(z)$ is a restriction of $G(z)\otimes G(z)$ to $\wedge^2E$. Let us show that, for almost all $z\in\mathbb{T}$, $\wedge^2G(z)$ is a bounded linear operator. By Proposition 2.1.19, for all $x,y\in E$,

$$\begin{aligned}\| \wedge^2 G(z)(x\wedge y)\|^2_{\wedge^2 F} &= \langle G(z)x\wedge G(z)y, G(z)x\wedge G(z)y\rangle_{\wedge^2 F}\\ &= \|G(z)x\|_F^2\|G(z)y\|_F^2 - |\langle G(z)x, G(z)y\rangle_F|^2\ ,\ \text{for all } x,y\in E.\end{aligned}$$

Thus

$$\| \wedge^2 G(z)(x\wedge y)\|_{\wedge^2 F} \le \|G(z)x\|_F\|G(z)y\|_F\ ,\ \text{for all } x,y\in E.$$

By the assumption, $G(z)$ is a bounded linear operator from E to F, hence there exists some $M>0$ such that

$$\|G(z)x\|_F \le M\|x\|_E\ ,\ \|G(z)y\|_F\le M\|y\|_E$$

for all $x,y\in E$.
Consequently, for each $z\in\mathbb{T}$,

$$\begin{aligned}\| \wedge^2 G(z)\| &= \sup_{\|x\wedge y\|_{\wedge^2 E}\le 1} \| \wedge^2 G(z)(x\wedge y)\|_{\wedge^2 F}\\ &\le M^2\|x\|_E\|y\|_E\ ,\ \text{for all } x,y\in E.\end{aligned}$$

Hence $\wedge^2 G(z)$ is a bounded linear operator. □

Corollary 2.2.17. *Let E,F be Hilbert spaces and let $G\in L^\infty(\mathbb{T},\mathcal{L}(E,F))$. Then, for almost all $z\in\mathbb{T}$,*

$$\wedge^2 G(z)\colon\ \wedge^2 E\to\wedge^2 F$$

is a continuous linear operator.

Proposition 2.2.18. *Let E,F be Hilbert spaces and let $G\in L^\infty(\mathbb{T},\mathcal{L}(E,F))$. Then*

$$(M_G x)(z) = G(z)\cdot x(z)\in F$$

for almost all $z\in\mathbb{T}$, and $M_G x\in L^2(\mathbb{T},F)$ for all $x\in L^2(\mathbb{T},E)$.

Proof. Since G is a bounded linear operator, by Lemma 2.2.16, there exists an $N>0$ such that

$$\|(M_G x)(z)\|_F = \|G(z)x(z)\|_F \le N\|x(z)\|_E \quad \text{for almost all } z\in\mathbb{T}.$$

Furthermore

$$\begin{aligned}\|M_G x\|_{L^2} &= \left(\frac{1}{2\pi}\int_0^{2\pi} \|(M_G x)(e^{i\theta})\|_F^2\, d\theta\right)^{1/2}\\ &\le \left(\frac{1}{2\pi}\int_0^{2\pi} (N\|x(e^{i\theta})\|_E)^2\, d\theta\right)^{1/2} = N\|x\|_{L^2} < \infty.\end{aligned}$$ □

Definition 2.2.19. *Let E,F be Hilbert spaces. For the operator $G\in L^\infty(\mathbb{T},\mathcal{L}(E,F))$, we define an operator $M_G\colon L^2(\mathbb{T},E)\to L^2(\mathbb{T},F)$ by*

$$(M_G x)(z) = G(z)\cdot x(z)$$

for almost all $z \in \mathbb{T}$ *and for all* $x \in L^2(\mathbb{T}, E)$.

Definition 2.2.20. *Let* E, F *be Hilbert spaces and let* $G \in L^\infty(\mathbb{T}, \mathcal{L}(E, F))$. *We define an operator*

$$M_G|_{H^2(\mathbb{D},E)} \colon H^2(\mathbb{D}, E) \to L^2(\mathbb{T}, F)$$

by

$$(M_G|_{H^2(\mathbb{D},E)} x)(z) = G(z)x(z) \quad \text{for all} \quad z \in \mathbb{T}, x \in H^2(\mathbb{D}, E).$$

Remark 2.2.21. *Let* $\wedge^2 G \in L^\infty(\mathbb{T}, \mathcal{L}(\wedge^2 E, \wedge^2 F))$. *The restriction of* $M_{\wedge^2 G}$ *to* $H^2(\mathbb{D}, E)$ *is the operator*

$$M_{\wedge^2 G}|_{H^2(\mathbb{D},E)} \colon H^2(\mathbb{D}, \wedge^2 E) \to L^2(\mathbb{T}, \wedge^2 F),$$

given by

$$\begin{aligned} (M_{\wedge^2 G}(x \dot{\wedge} y))(z) &= (\wedge^2 G)(z) \cdot (x(z) \wedge y(z)) \\ &= (G(z) \wedge G(z)) \cdot (x(z) \wedge y(z)) \\ &= (G(z) \cdot x(z)) \wedge (G(z) \cdot y(z)) \end{aligned}$$

for all $z \in \mathbb{T}$.

2.2.2 Pointwise creation operators, orthogonal complements and linear spans

Below, let E denote a separable Hilbert space.

Definition 2.2.22. *Let* $\xi \in H^\infty(\mathbb{D}, E)$. *We define the* pointwise creation operator

$$C_\xi \colon H^2(\mathbb{D}, E) \to H^2(\mathbb{D}, \wedge^2 E)$$

by

$$C_\xi f = \xi \dot{\wedge} f, \text{ for } f \in H^2(\mathbb{D}, E).$$

Remark 2.2.23. *Let* E *be a separable Hilbert space. Let* $\xi \in H^\infty(\mathbb{D}, E)$ *and let* $f \in H^2(\mathbb{D}, E)$. *By the generalised Fatou's Theorem C.2.5, the radial limits*

$$\lim_{r \to 1} \xi(re^{i\theta}) \underset{\|\cdot\|_E}{=} \tilde{\xi}(e^{i\theta}), \quad \lim_{r \to 1} f(re^{i\theta}) \underset{\|\cdot\|_E}{=} \tilde{f}(e^{i\theta}) \quad (0 < r < 1)$$

exist almost everywhere on $\mathbb{T}$ *and define functions* $\tilde{\xi} \in L^\infty(\mathbb{T}, E)$ *and* $\tilde{f} \in L^2(\mathbb{T}, E)$ *respectively, which satisfy the relations*

$$\lim_{r \to 1} \|\xi(re^{i\theta}) - \tilde{\xi}(e^{i\theta})\|_E = 0, \quad \lim_{r \to 1} \|f(re^{i\theta}) - \tilde{f}(e^{i\theta})\|_E = 0$$

for almost all $e^{i\theta} \in \mathbb{T}$.

Lemma 2.2.24. *Let E be a separable Hilbert space. Let $\xi \in H^\infty(\mathbb{D}, E)$ and let $f \in H^2(\mathbb{D}, E)$. Then the radial limits $\lim_{r\to 1}(\xi(re^{i\theta}) \wedge f(re^{i\theta}))$ exist for almost all $e^{i\theta} \in \mathbb{T}$ and define a function in $L^2(\mathbb{T}, \wedge^2 E)$.*

Proof. By Proposition 2.1.35, the bilinear operator $\Lambda\colon E \times E \to \wedge^2 E$ is a continuous operator for the norms of E and $\wedge^2 E$. By Remark 2.2.23, the functions $\xi \in H^\infty(\mathbb{D}, E)$ and $f \in H^2(\mathbb{D}, E)$ have radial limit functions $\tilde{\xi} \in L^\infty(\mathbb{T}, E)$ and $\tilde{f} \in L^2(\mathbb{T}, E)$ respectively. Also, by Proposition 2.2.13, $\xi \dot{\wedge} f \in H^2(\mathbb{D}, \wedge^2 E)$. Hence

$$\lim_{r\to 1} \|\xi(re^{i\theta}) \wedge f(re^{i\theta}) - \tilde{\xi}(e^{i\theta}) \wedge \tilde{f}(e^{i\theta})\|_{\wedge^2 E} = 0 \quad \text{almost everywhere on} \quad \mathbb{T}$$

and we conclude that

$$\lim_{r\to 1}(\xi(re^{i\theta}) \wedge f(re^{i\theta})) \underset{\|\cdot\|_{\wedge^2 E}}{=} \tilde{\xi}(e^{i\theta}) \wedge \tilde{f}(e^{i\theta}) \text{ almost everywhere on } \mathbb{T}.$$

This shows that the radial limits

$$\lim_{r\to 1}(\xi(re^{i\theta}) \wedge f(re^{i\theta}))$$

exist almost everywhere on $\mathbb{T}$ and, by Lemma 2.2.6, define a function in $L^2(\mathbb{T}, \wedge^2 E)$. Hence one can consider $(C_\xi f)(z) = (\xi \dot{\wedge} f)(z)$ to be defined for either all $z \in \mathbb{D}$ or for almost all $z \in \mathbb{T}$. □

Proposition 2.2.25. *Let $\mathcal{H}$ be a separable Hilbert space. The space $H^2(\mathbb{D}, \mathcal{H})$ can be identified with a closed linear subspace of $L^2(\mathbb{T}, \mathcal{H})$.*

Proof. By Remark C.2.3, for any separable Hilbert space $\mathcal{H}$ and $f \in H^2(\mathbb{D}, \mathcal{H})$, the map $f \mapsto \tilde{f}$ is an isometric embedding of $H^2(\mathbb{D}, \mathcal{H})$ in $L^2(\mathbb{T}, \mathcal{H})$ as a subspace, where $\tilde{f}$ is the radial limit function

$$\tilde{f}(e^{i\theta}) \underset{\|\cdot\|_{\mathcal{H}}}{=} \lim_{r\to 1} f(re^{i\theta}).$$

Since $H^2(\mathbb{D}, \mathcal{H})$ is complete and the embedding is isometric, the image of the embedding is complete, and therefore closed in $L^2(\mathbb{T}, \mathcal{H})$. □

Remark 2.2.26. *In future statements we shall to use the same notation for f and $\tilde{f}$.*

Definition 2.2.27. *Let E be a separable Hilbert space. Let F be a subspace of $L^2(\mathbb{T}, E)$ and let X be a subset of $L^2(\mathbb{T}, E)$. We define the* pointwise linear span *of X in F to be the set*

$$\mathrm{PLS}(X, F) = \{f \in F : f(z) \in \operatorname{span}\{x(z) : x \in X\} \textit{ for almost all } z \in \mathbb{T}\}.$$

We define the pointwise orthogonal complement *of X in F to be the set*

$$\mathrm{POC}(X, F) = \{f \in F : f(z) \perp \{x(z) : x \in X\} \textit{ for almost all } z \in \mathbb{T}\}.$$

Our next aim is to show that POC(X, F) is a closed subspace of F. We are going to need the following Lemma.

Lemma 2.2.28. *Let E be a Hilbert space and let $x \in L^2(\mathbb{T}, E)$. The function $\phi\colon L^2(\mathbb{T}, E) \to \mathbb{C}$ given by*

$$\phi(g) = \frac{1}{2\pi}\int_0^{2\pi} |\langle g(e^{i\theta}), x(e^{i\theta})\rangle_E| \, d\theta$$

is continuous.

Proof. Consider $g_0 \in L^2(\mathbb{T}, E)$. For any $\epsilon > 0$, we are looking for a $\delta > 0$ such that

$$\|g - g_0\|_{L^2(\mathbb{T},E)} = \left(\frac{1}{2\pi}\int_0^{2\pi} \|g(e^{i\theta}) - g_0(e^{i\theta})\|_E^2 \, d\theta\right)^{1/2} < \delta$$

implies

$$|\phi(g) - \phi(g_0)| < \epsilon.$$

Note that

$$\begin{aligned} |\phi(g) - \phi(g_0)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} |\langle g(e^{i\theta}), x(e^{i\theta})\rangle_E| d\theta - \frac{1}{2\pi}\int_0^{2\pi} |\langle g_0(e^{i\theta}), x(e^{i\theta})\rangle_E| d\theta\right| \\ &= \left|\frac{1}{2\pi}\int_0^{2\pi} \left(|\langle g(e^{i\theta}), x(e^{i\theta})\rangle_E| - |\langle g_0(e^{i\theta}), x(e^{i\theta})\rangle_E|\right) d\theta\right|. \end{aligned}$$

For each $e^{i\theta} \in \mathbb{T}$, by the reverse triangle inequality, the integrand satisfies

$$\begin{aligned} \Big||\langle g(e^{i\theta}), x(e^{i\theta})\rangle_E| - |\langle g_0(e^{i\theta}), x(e^{i\theta})\rangle_E|\Big| &\leq |\langle g(e^{i\theta}), x(e^{i\theta})\rangle_E - \langle g_0(e^{i\theta}), x(e^{i\theta})\rangle_E| \\ &= |\langle (ge^{i\theta}) - g_0(e^{i\theta}), x(e^{i\theta})\rangle_E|, \end{aligned}$$

hence

$$|\phi(g) - \phi(g_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |\langle g(e^{i\theta}) - g_0(e^{i\theta}), x(e^{i\theta})\rangle_E| d\theta. \tag{2.12}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} |\langle g(e^{i\theta}) - g_0(e^{i\theta}), x(e^{i\theta})\rangle_E| \, d\theta \\ &\leq \left(\frac{1}{2\pi}\int_0^{2\pi} \|g(e^{i\theta}) - g_0(e^{i\theta})\|_E^2 \, d\theta\right)^{1/2} \left(\frac{1}{2\pi}\int_0^{2\pi} \|x(e^{i\theta})\|_E^2 \, d\theta\right)^{1/2}. \end{aligned} \tag{2.13}$$

For the given $\epsilon > 0$, let δ be equal to $\dfrac{\epsilon}{\|x\|_{L^2(\mathbb{T},E)} + 1}$, and let

$$\left(\frac{1}{2\pi} \int_0^{2\pi} \|g(e^{i\theta}) - g_0(e^{i\theta})\|_E^2 \, d\theta \right)^{1/2} < \delta.$$

By equations (2.12) and (2.13),

$$\begin{aligned} |\phi(g) - \phi(g_0)| &\leq \left(\frac{1}{2\pi} \int_0^{2\pi} \|g(e^{i\theta}) - g_0(e^{i\theta})\|_E^2 \, d\theta \right)^{1/2} \|x\|_{L^2(\mathbb{T},E)} \\ &< \frac{\epsilon}{\|x\|_{L^2(\mathbb{T},E)} + 1} \|x\|_{L^2(\mathbb{T},E)} < \epsilon. \end{aligned}$$

Hence ϕ is a continuous function. □

Proposition 2.2.29. *Let E be a separable Hilbert space and let $\varphi \in L^2(\mathbb{D}, E)$. Then*

(i) The space $V = \{f \in H^2(\mathbb{D}, E) \ : \ \langle f(z), \varphi(z)\rangle_E = 0$ for almost all $z \in \mathbb{T}\}$ is a closed subspace of $H^2(\mathbb{D}, E)$.

(ii) The space $V = \{f \in L^2(\mathbb{T}, E) : \langle f(z), \phi(z)\rangle_E = 0$ for almost all $z \in \mathbb{T}\}$ is a closed subspace of $L^2(\mathbb{T}, E)$.

Proof. (i). V is a linear subspace of $H^2(\mathbb{D}, E)$ since for $\lambda, \mu \in \mathbb{C}$, $\psi, k \in V$ and for almost all $z \in \mathbb{T}$,

$$\langle \lambda\psi(z) + \mu k(z), \varphi(z)\rangle_E = \lambda\langle \psi(z), \varphi(z)\rangle_E + \mu\langle k(z), \varphi(z)\rangle_E = 0,$$

hence $\lambda\psi + \mu k \in V$.
Now, suppose that the sequence of functions $(g_n)_{n=1}^{\infty}$ in V converges to a function g. We need to show that $g \in V$. Since $g_n \in V$ for all $n \in \mathbb{N}$, we have

$$\langle g_n(z), \varphi(z)\rangle_E = 0 \text{ for almost all } z \in \mathbb{T}. \tag{2.14}$$

Consider the function $\phi\colon H^2(\mathbb{D}, E) \to \mathbb{C}$ given by

$$\phi(f) = \frac{1}{2\pi} \int_0^{2\pi} |\langle f(e^{i\theta}), \varphi(e^{i\theta})\rangle_E| \, d\theta.$$

Then, by equation (2.14), we have

$$\phi(g_n) = \frac{1}{2\pi} \int_0^{2\pi} |\langle g_n(e^{i\theta}), \varphi(e^{i\theta})\rangle_E| \, d\theta = 0.$$

Note that by Fatou's theorem, for each function $f \in H^2(\mathbb{D}, E)$, the radial limit

$$\lim_{r\to 1} f(re^{i\theta})$$

exists almost everywhere and defines a function in $L^2(\mathbb{T}, E)$. This way, $H^2(\mathbb{D}, E)$ can be identified with a closed subspace of $L^2(\mathbb{T}, E)$. Hence by Lemma 2.2.28, ϕ is a continuous function on $H^2(\mathbb{D}, E)$, thus $\phi(g) = \lim_{n\to\infty} \phi(g_n)$, and so

$$\frac{1}{2\pi}\int_0^{2\pi} |\langle g(e^{i\theta}), \varphi(e^{i\theta})\rangle_E|\, d\theta = \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} |\langle g_n(e^{i\theta}), \varphi(e^{i\theta})\rangle_E|\, d\theta = 0$$

for almost all $e^{i\theta} \in \mathbb{T}$. Thus $|\langle g(e^{i\theta}), \varphi(e^{i\theta})\rangle_E| = 0$ for almost all $e^{i\theta} \in \mathbb{T}$, and, hence, $g \in V$. Thus we have proved that V is a closed subspace of $H^2(\mathbb{D}, E)$.
(ii). The proof is similar to (i). □

Lemma 2.2.30. *Let E be a separable Hilbert space, let F be a subspace of $L^2(\mathbb{T}, E)$ and let X be a subset of $L^2(\mathbb{T}, E)$. The space*

$$\mathrm{POC}(X, F) = \{f \in F \ : \ f(z) \perp \{x(z) : x \in X\} \textit{ for almost all } z \in \mathbb{T}\}$$

is a closed subspace of F.

Proof. The assertion follows from Proposition 2.2.29, since $\mathrm{POC}(X, F)$ is an intersection of closed subspaces

$$V_x = \{f \in F : \langle f(z), x(z)\rangle_E = 0 \text{ for almost all } z \in \mathbb{T}\}$$

over $x \in F$. □

Definition 2.2.31. *Let E be a separable Hilbert space. Let $f \in H^p(\mathbb{D}, E)$, for $1 \le p \le \infty$. By the generalised Fatou's Theorem C.2.5, the radial limit*

$$\lim_{r\to 1} f(re^{i\theta}) \underset{\|\cdot\|_E}{=} \tilde{f}(e^{i\theta}) \quad (0 < r < 1)$$

exists almost everywhere on $\mathbb{T}$ and defines a function $\tilde{f} \in L^p(\mathbb{T}, E)$. The set of points on $\mathbb{T}$ at which the above limit does not exist, will be called the singular set *of the function f and will be denoted by N_f.*

Note that the singular sets of functions in $H^p(\mathbb{D}, E)$ for $1 \le p \le \infty$ are null sets with respect to Lebesgue measure.

Lemma 2.2.32. *Let E be a separable Hilbert space. Let $\xi \in H^\infty(\mathbb{D}, E)$. For every $f \in H^\infty(\mathbb{D}, E)$ and $g \in H^2(\mathbb{D}, E)$, the function*

$$f \dot\wedge g \colon \mathbb{D} \to \wedge^2 E$$

defined by

$$(f\dot{\wedge}g)(z) = f(z) \wedge g(z) \quad \textit{for all} \quad z \in \mathbb{D}$$

belongs to $H^2(\mathbb{D}, \wedge^2 E)$, *and moreover the operator*

$$C_\xi^* \colon H^2(\mathbb{D}, \wedge^2 E) \to H^2(\mathbb{D}, E)$$

is given by the formula

$$C_\xi^*(f\dot{\wedge}g) = P_+\alpha,$$

where $\alpha \in L^2(\mathbb{T}, E)$ *is defined by*

$$\alpha(e^{i\theta}) = \langle f(e^{i\theta}), \xi(e^{i\theta})\rangle_E g(e^{i\theta}) - \langle g(e^{i\theta}), \xi(e^{i\theta})\rangle_E f(e^{i\theta})$$

for all $e^{i\theta} \in \mathbb{T} \setminus (N_f \cup N_\xi \cup N_g)$, *and* P_+ *is the orthogonal projection*

$$P_+ \colon L^2(\mathbb{T}, E) \to H^2(\mathbb{D}, E).$$

Here N_f, N_g, N_ξ *are the singular sets of the functions* f, g *and* ξ *respectively.*

Proof. By Proposition 2.2.13, $f\dot{\wedge}g \in H^2(\mathbb{D}, \wedge^2 E)$. Now, for all $f \in H^\infty(\mathbb{D}, E)$, all $g, h \in H^2(\mathbb{D}, E)$ and all $e^{i\theta} \in \mathbb{T} \setminus (N_\xi \cup N_g \cup N_f)$, we have

$$\begin{aligned}
\langle C_\xi^*(f\dot{\wedge}g), h\rangle_{H^2(\mathbb{D},E)} &= \langle f\dot{\wedge}g, C_\xi h\rangle_{H^2(\mathbb{D},\wedge^2 E)} \\
&= \langle f\dot{\wedge}g, \xi\dot{\wedge}h\rangle_{L^2(\mathbb{T},\wedge^2 E)} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \langle f(e^{i\theta}) \wedge g(e^{i\theta}), \xi(e^{i\theta}) \wedge h(e^{i\theta})\rangle_{\wedge^2 E}\, d\theta,
\end{aligned}$$

which, by Proposition 2.1.19, is equal to

$$\frac{1}{2\pi}\int_0^{2\pi} \det \begin{pmatrix} \langle f(e^{i\theta}), \xi(e^{i\theta})\rangle_E & \langle f(e^{i\theta}), h(e^{i\theta})\rangle_E \\ \langle g(e^{i\theta}), \xi(e^{i\theta})\rangle_E & \langle g(e^{i\theta}), h(e^{i\theta})\rangle_E \end{pmatrix} d\theta.$$

The latter in turn is equal to

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\langle f(e^{i\theta}), \xi(e^{i\theta})\rangle_E \langle g(e^{i\theta}), h(e^{i\theta})\rangle_E - \langle f(e^{i\theta}), h(e^{i\theta})\rangle_E \langle g(e^{i\theta}), \xi(e^{i\theta})\rangle_E\right) d\theta,$$

which equals

$$\frac{1}{2\pi}\int_0^{2\pi} \left\langle \langle f(e^{i\theta}), \xi(e^{i\theta})\rangle_E g(e^{i\theta}) - \langle g(e^{i\theta}), \xi(e^{i\theta})\rangle_E f(e^{i\theta}), h(e^{i\theta})\right\rangle_E d\theta,$$

and so

$$\langle C_\xi^*(f\dot{\wedge}g), h\rangle_{H^2(\mathbb{D},E)} = \frac{1}{2\pi}\int_0^{2\pi} \langle \alpha(e^{i\theta}), h(e^{i\theta})\rangle_E\, d\theta = \langle \alpha, h\rangle_{L^2(\mathbb{T},E)} = \langle P_+(\alpha), h\rangle_{H^2(\mathbb{D},E)},$$

where

$$\alpha(e^{i\theta}) = \langle f(e^{i\theta}), \xi(e^{i\theta})\rangle_E g(e^{i\theta}) - \langle g(e^{i\theta}), \xi(e^{i\theta})\rangle_E f(e^{i\theta})$$

for all $e^{i\theta} \in \mathbb{T} \setminus (N_\xi \cup N_f \cup N_g)$. Hence $C_\xi^*(f \dot{\wedge} g) = P_+\alpha$ as required. □

Proposition 2.2.33. *Let E be a separable Hilbert space. For $\xi \in H^\infty(\mathbb{D}, E)$,*

$$\ker C_\xi \subset \mathrm{PLS}(\{\xi\}, H^2(\mathbb{D}, E)).$$

Proof. We have

$$\begin{aligned}\ker C_\xi &= \{f \in H^2(\mathbb{D}, E) : (\xi \dot{\wedge} f)(z) = 0 \text{ for all } z \in \mathbb{D}\}\\ &= \{f \in H^2(\mathbb{D}, E) : \xi(z) \wedge f(z) = 0 \text{ for all } z \in \mathbb{D}\}\\ &= \{f \in H^2(\mathbb{D}, E) : \xi(z), f(z) \text{ are pointwise linearly dependent for all } z \in \mathbb{D}\}.\end{aligned}$$

By Remark 2.2.23, the functions $\xi \in H^\infty(\mathbb{D}, E)$ and $f \in H^2(\mathbb{D}, E)$ have radial limit functions $\xi \in L^\infty(\mathbb{T}, E)$ and $f \in L^2(\mathbb{T}, E)$ respectively, hence the radial limit functions will be linearly dependent almost everywhere on $\mathbb{T}$. Thus

$$\begin{aligned}\ker C_\xi &\subset \{f \in H^2(\mathbb{D}, E) : \xi(z), f(z) \text{ are pointwise linearly dependent for almost all } z \in \mathbb{T}\}\\ &= \mathrm{PLS}(\{\xi\}, H^2(\mathbb{D}, E)).\end{aligned}$$

□

Example 2.2.34. Let $E = \mathbb{C}^2$. We can find functions $f, g \in H^2(\mathbb{D}, E)$ such that $f \in \mathrm{POC}(\{g\}, H^2(\mathbb{D}, E))$ but it is false that $\langle f(z), g(z)\rangle_E = 0$ for all $z \in \mathbb{D}$. Choose

$$g(z) = \begin{pmatrix} z \\ z^2 \end{pmatrix}, \quad f(z) = \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} \quad \text{for} \quad z \in \mathbb{D}.$$

Then

$$f \in \mathrm{POC}(\{g\}, H^2(\mathbb{D}, E))$$

is equivalent to

$$\langle \tilde{f}(z), \tilde{g}(z)\rangle_E = 0 \quad \text{for almost all} \quad z \in \mathbb{T}.$$

The later is equivalent to

$$\left\langle \begin{pmatrix} z \\ z^2 \end{pmatrix}, \begin{pmatrix} \tilde{f}_1(z) \\ \tilde{f}_2(z) \end{pmatrix} \right\rangle_E = 0 \quad \text{for almost all} \quad z \in \mathbb{T},$$

which holds if and only if

$$\bar{z}\tilde{f}_1(z) + \bar{z}^2\tilde{f}_2(z) = 0 \quad \text{for almost all} \quad z \in \mathbb{T}.$$

Equivalently

$$\tilde{f}_1(z) = -\bar{z}\tilde{f}_2(z) \quad \text{for almost all} \quad z \in \mathbb{T},$$

which in turn is equivalent to

$$f(z) = \begin{pmatrix} \tilde{f}_1(z) \\ -z\tilde{f}_1(z). \end{pmatrix} \quad \text{for almost all} \quad z \in \mathbb{T}.$$

Now for $z \in \mathbb{D}$,

$$\begin{aligned} \langle f(z), g(z) \rangle_E &= \left\langle \begin{pmatrix} f_1(z) \\ -zf_1(z) \end{pmatrix}, \begin{pmatrix} z \\ z^2 \end{pmatrix} \right\rangle_E \\ &= \bar{z}f_1(z) - \bar{z}|z|^2 f_1(z) \\ &= \bar{z}(1-|z|^2)f_1(z). \end{aligned}$$

So if we take $f_1(z) = 1, f(z) = \begin{pmatrix} 1 \\ -z \end{pmatrix}$ for all $z \in \mathbb{D}$, then $f \in \mathrm{POC}(\{g\}, H^2(\mathbb{D}, E))$ but $\langle f(z), g(z) \rangle_E \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$.

Thus it is not true in general that $\mathrm{POC}(\{g\}, H^2(\mathbb{D}, E)) \subset \{g\}^\perp$.

Lemma 2.2.35. *Let E be a separable Hilbert space. For $\xi \in H^\infty(\mathbb{D}, E)$,*

$$\mathrm{POC}(\{\xi\}, H^2(\mathbb{D}, E)) \subset H^2(\mathbb{D}, E) \ominus \mathrm{PLS}(\{\xi\}, H^2(\mathbb{D}, E)).$$

Proof. Let $f \in \mathrm{POC}(\{\xi\}, H^2(\mathbb{D}, E))$. This is equivalent to $f \in H^2(\mathbb{D}, E)$ and

$$\tilde{f}(z) \perp \tilde{\xi}(z) \text{ for all } z \in \mathbb{T} \setminus (N_f \cup N_\xi),$$

where N_f, N_ξ are the singular sets for the functions f, ξ respectively. This in turn is equivalent to $f \in H^2(\mathbb{D}, E)$ and

$$\langle \tilde{f}(z), \tilde{\xi}(z) \rangle_E = 0 \text{ for all } z \in \mathbb{T} \setminus (N_f \cup N_\xi).$$

The latter implies the condition

$$f \in H^2(\mathbb{D}, E) \text{ and } \langle \tilde{f}(z), \tilde{g}(z) \rangle_E = 0 \text{ for almost all } z \in \mathbb{T} \text{ and all } g \in \mathrm{PLS}(\{\xi\}, H^2(\mathbb{D}, E)).$$

Thus

$$f \in H^2(\mathbb{D}, E) \ominus \mathrm{PLS}(\{\xi\}, H^2(\mathbb{D}, E)).$$ □

Lemma 2.2.36. *Let E and F be separable Hilbert spaces, and let $G \in L^\infty(\mathbb{T}, \mathcal{B}(F, E))$. For every $x \in L^2(\mathbb{T}, E)$, the function Gx, defined almost everywhere on $\mathbb{T}$ by*

$$(Gx)(z) = G(z)(x(z)),$$

belongs to $L^2(\mathbb{T}, E)$.

Proof. For almost all $z \in \mathbb{T}$,

$$\|(Gx)(z)\|_E = \|G(z)x(z)\|_E \le \|G\|_{L^\infty(\mathbb{T},\mathcal{B}(F,E))}\|x(z)\|_F.$$

Thus

$$\begin{aligned}\|Gx\|^2_{L^2(\mathbb{T},E)} &= \frac{1}{2\pi}\int_0^{2\pi} \|Gx(e^{i\theta})\|^2_E \, d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \|G\|^2_{L^\infty(\mathbb{T},\mathcal{B}(F,E))}\|x(e^{i\theta})\|^2_F \, d\theta \\ &\le \|G\|^2_{L^\infty(\mathbb{T},\mathcal{B}(F,E))}\|x\|^2_{L^2(\mathbb{T},F)} < \infty.\end{aligned}$$ □

Definition 2.2.37. *Let E and F be separable Hilbert spaces. Let $P_+ \colon L^2(\mathbb{T},E) \to H^2(\mathbb{D},E)$ be the orthogonal projection operator. Corresponding to any $G \in L^\infty(\mathbb{T},\mathcal{B}(F,E))$ we define the* Toeplitz operator *with symbol G to be the operator*

$$T_G \colon H^2(\mathbb{D},F) \to H^2(\mathbb{D},E)$$

given by

$$T_G x = P_+(Gx) \quad \text{for any} \quad x \in H^2(\mathbb{D},F).$$

Definition 2.2.38 ([24]). *For a separable Hilbert space E, a function $\xi \in H^\infty(\mathbb{D},E)$ will be called* inner *if for almost every $z \in \mathbb{T}$,*

$$\|\xi(z)\|_E = 1.$$

Definition 2.2.39. *Let E be a separable Hilbert space and let $\xi, \eta \in L^\infty(\mathbb{T},E)$. We define $\xi\eta^* \in L^\infty(\mathbb{T},\mathcal{B}(E,E))$ by*

$$(\xi\eta^*)(z)x = \langle x, \eta(z)\rangle \xi(z) \text{ for all } x \in E \text{ and for almost every } z \text{ on } \mathbb{T}.$$

Proposition 2.2.40. *Let E be a separable Hilbert space. Let $\xi \in H^\infty(\mathbb{D},E)$ be an inner function. Then, for any $h \in H^2(\mathbb{D},E)$,*

$$C_\xi^* C_\xi h = P_+\alpha,$$

where $\alpha = h - \xi\xi^* h$ and $P_+ \colon L^2(\mathbb{T},E) \to H^2(\mathbb{D},E)$ is the orthogonal projection. Moreover

$$C_\xi^* C_\xi h = h - T_{\xi\xi^*} h,$$

where $T_{\xi\xi^*} \colon H^2(\mathbb{D},E) \to H^2(\mathbb{D},E)$ is the Toeplitz operator with symbol $\xi\xi^*$.

Proof. For all $g, h \in H^2(\mathbb{D},E)$ and for N_ξ, N_h, N_g singular sets of ξ, h, g respectively,

$$\langle (C_\xi^* C_\xi)h, g\rangle_{H^2(\mathbb{D},E)} = \langle C_\xi h, C_\xi g\rangle_{H^2(\mathbb{D},\wedge^2 E)} = \langle \xi \dot\wedge h, \xi \dot\wedge g\rangle_{L^2(\mathbb{T},\wedge^2 E)},$$

and, by Proposition 2.1.19, we get

$$\langle \xi \dot\wedge h, \xi \dot\wedge g \rangle_{L^2(\mathbb{T}, \wedge^2 E)} = \frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix} \langle \xi(e^{i\theta}), \xi(e^{i\theta}) \rangle_E & \langle \xi(e^{i\theta}), g(e^{i\theta}) \rangle_E \\ \langle h(e^{i\theta}), \xi(e^{i\theta}) \rangle_E & \langle h(e^{i\theta}), g(e^{i\theta}) \rangle_E \end{pmatrix} d\theta.$$

Since ξ is inner, $\|\xi(e^{i\theta})\|_E = 1$ almost everywhere on $\mathbb{T}$, and so,

$$\begin{aligned} \langle (C_\xi^* C_\xi) h, g \rangle_{H^2(\mathbb{D}, E)} &= \frac{1}{2\pi} \int_0^{2\pi} \langle h(e^{i\theta}), g(e^{i\theta}) \rangle_E - \langle \xi(e^{i\theta}), g(e^{i\theta}) \rangle_E \langle h(e^{i\theta}), \xi(e^{i\theta}) \rangle_E \, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \langle h(e^{i\theta}) - \langle h(e^{i\theta}), \xi(e^{i\theta}) \rangle_E \xi(e^{i\theta}), g(e^{i\theta}) \rangle_E \, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \langle \alpha(e^{i\theta}), g(e^{i\theta}) \rangle_E \, d\theta \\ &= \langle \alpha, g \rangle_{L^2(\mathbb{T}, E)} \\ &= \langle P_+ \alpha, g \rangle_{H^2(\mathbb{D}, E)}, \end{aligned}$$

where $\alpha(e^{i\theta}) = h(e^{i\theta}) - \langle h(e^{i\theta}), \xi(e^{i\theta}) \rangle_E \xi(e^{i\theta})$ for all $e^{i\theta} \in \mathbb{T} \setminus (N_\xi \cup N_h \cup N_g)$. Thus $C_\xi^* C_\xi h = P_+ \alpha$, where $\alpha = h - \xi \xi^* h$. Hence

$$C_\xi^* C_\xi h = P_+(h - \xi \xi^* h) = h - T_{\xi \xi^*} h,$$

where $T_{\xi \xi^*} h = P_+(\xi \xi^* h)$ is a Toeplitz operator. □

Example 2.2.41. *In this example we show that there exists an inner function $\xi \in H^\infty(\mathbb{D}, \mathbb{C}^2)$ such that, for some $h \in H^2(\mathbb{D}, \mathbb{C}^2)$, $C_\xi^* C_\xi h$ is not in the pointwise orthogonal complement of ξ in E.*
Let $\xi \in H^\infty(\mathbb{D}, \mathbb{C}^2)$ be an inner function and let $h \in H^2(\mathbb{D}, \mathbb{C}^2)$. Let ξ, h be given by

$$\xi(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ z \end{pmatrix}, \quad h(z) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ for all } z \in \mathbb{D}.$$

By Proposition 2.2.40,

$$C_\xi^* C_\xi h = P_+ \alpha,$$

where, for all $z \in \mathbb{T}$,

$$\alpha(z) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ z \end{pmatrix} \right\rangle_{\mathbb{C}^2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ z \end{pmatrix}.$$

Calculations yield, for all $z \in \mathbb{T}$,

$$\alpha(z) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} (1 + \bar{z}) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 + \bar{z} \\ z(1 + \bar{z}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{1}{2}\bar{z} \\ 1 - \frac{1}{2} z - \frac{1}{2}|z|^2 \end{pmatrix},$$

and so

$$\alpha(z) = \frac{1}{2}\begin{pmatrix} 1-\bar{z} \\ 1-z \end{pmatrix}.$$

Thus

$$(P_+\alpha)(z) = \frac{1}{2}\begin{pmatrix} 1 \\ 1-z \end{pmatrix} \quad \textit{for all} \quad z \in \mathbb{T} \setminus (N_\xi \cup N_g).$$

The latter expression is not in the pointwise orthogonal complement of ξ in $\mathbb{C}^2$, since for all $z \in \mathbb{T}$,

$$\left\langle \frac{1}{2}\begin{pmatrix} 1 \\ 1-z \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ z \end{pmatrix} \right\rangle_{\mathbb{C}^2} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & \bar{z} \end{pmatrix}\begin{pmatrix} 1 \\ 1-z \end{pmatrix} = \frac{1}{2\sqrt{2}}(1+\bar{z}-|z|^2) = \frac{\bar{z}}{2\sqrt{2}} \neq 0.$$

The next lemma shows that C_ξ is an isometry on $\mathrm{POC}\{\xi_0, H^2(\mathbb{D}, E)\}$.

Lemma 2.2.42. *Let E be a separable Hilbert space. For every inner function $\xi \in H^\infty(\mathbb{D}, E)$,*

$$\{x \in H^2(\mathbb{D}, E) : \|C_\xi x\|_{H^2(\mathbb{D}, \wedge^2 E)} = \|x\|_{H^2(\mathbb{D},E)}\} = \mathrm{POC}(\{\xi\}, H^2(\mathbb{D}, E)).$$

Proof. By Proposition 2.2.25, for every $x \in H^2(\mathbb{D}, E)$, $\|x\|_{H^2(\mathbb{D},E)} = \|x\|_{L^2(\mathbb{T},E)}$. Hence $\{x \in H^2(\mathbb{D}, E) : \|C_\xi x\|^2_{H^2(\mathbb{D},\wedge^2 E)} = \|x\|^2_{H^2(\mathbb{D},E)}\} = \{x \in H^2(\mathbb{D}, E) : \|C_\xi x\|^2_{L^2(\mathbb{T},\wedge^2 E)} = \|x\|^2_{L^2(\mathbb{T},E)}\}$. By Proposition 2.1.19, the latter set is equal to

$$\left\{x \in H^2(\mathbb{D}, E) : \frac{1}{2\pi}\int_0^{2\pi} \det\begin{pmatrix} \langle \xi(e^{i\theta}), \xi(e^{i\theta})\rangle_E & \langle \xi(e^{i\theta}), x(e^{i\theta})\rangle_E \\ \langle x(e^{i\theta}), \xi(e^{i\theta})\rangle_E & \langle x(e^{i\theta}), x(e^{i\theta})\rangle_E \end{pmatrix} d\theta \frac{1}{2\pi}\int_0^{2\pi} \|x(e^{i\theta})\|_E^2 \, d\theta\right\}.$$

Since ξ is inner, $\|\xi(e^{i\theta})\|_E = 1$ almost everywhere on $\mathbb{T}$, hence the latter set is equal to

$$\begin{aligned}
&\left\{x \in H^2(\mathbb{D}, E) : \frac{1}{2\pi}\int_0^{2\pi} (\|x(e^{i\theta})\|_E^2 - |\langle \xi(e^{i\theta}), x(e^{i\theta})\rangle_E|^2) \, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \|x(e^{i\theta})\|_E^2 \, d\theta\right\} \\
&= \left\{x \in H^2(\mathbb{D}, E) : \frac{1}{2\pi}\int_0^{2\pi} |\langle \xi(e^{i\theta}), x(e^{i\theta})\rangle_E|^2 \, d\theta = 0\right\} \\
&= \{x \in H^2(\mathbb{D}, E) : \tilde{\xi}(e^{i\theta}) \perp \tilde{x}(e^{i\theta}) \text{ almost everywhere on } \mathbb{T}\} \\
&= \{x \in H^2(\mathbb{D}, E) : x \in \mathrm{POC}(\{\xi\}, H^2(\mathbb{D}, E))\} \\
&= \mathrm{POC}(\{\xi\}, H^2(\mathbb{D}, E)).
\end{aligned}$$

□

Example 2.2.43. *This example shows that $C_\xi^* C_\xi$ fails to be a projection for some inner function $\xi \in H^\infty(\mathbb{D}, \mathbb{C}^2)$. Let us calculate $C_\xi^* C_\xi$, for $\xi(z) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ z \end{pmatrix}$, $z \in \mathbb{D}$. By Proposition 2.2.40, for $h \in H^2(\mathbb{D}, \mathbb{C}^2)$*

$$C_\xi^* C_\xi h = P_+\alpha,$$

where, for all $z \in \mathbb{T}$, $\alpha(z)$ is given by

$$\alpha(z) = h(z) - \langle h(z), \xi(z)\rangle_E \, \xi(z) = \begin{pmatrix} h_1(z) \\ h_2(z) \end{pmatrix} - \frac{1}{\sqrt{2}}(h_1(z) + \bar{z}h_2(z))\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ z \end{pmatrix}.$$

We have

$$\begin{pmatrix} h_1(z) \\ h_2(z) \end{pmatrix} - \tfrac{1}{\sqrt{2}}(h_1(z) + \bar{z}h_2(z))\tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ z \end{pmatrix} = \begin{pmatrix} h_1(z) - \frac{1}{2}(h_1(z) + \bar{z}h_2(z)) \\ h_2(z) - \frac{1}{2}(zh_1(z) + h_2(z) \end{pmatrix}$$

$$= \tfrac{1}{2} \begin{pmatrix} h_1(z) - \bar{z}h_2(z) \\ h_2(z) - zh_1(z) \end{pmatrix}.$$

Thus

$$P_+\alpha = \frac{1}{2} \begin{pmatrix} h_1 - S^*h_2 \\ -Sh_1 + h_2 \end{pmatrix},$$

where S, S^* *denote the shift and the backward shift operators on* $H^2(\mathbb{D}, \mathbb{C})$ *respectively. Hence*

$$C_\xi^* C_\xi = \frac{1}{2} \begin{pmatrix} 1 & -S^* \\ -S & 1 \end{pmatrix}. \tag{2.15}$$

Now, for $h(z) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $z \in \mathbb{D}$,

$$\left(C_\xi^* C_\xi \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)(z) = \frac{1}{2} \begin{pmatrix} 1 \\ 1 - z \end{pmatrix}$$

which is not the projection of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *onto the pointwise orthogonal complement of* $\xi(z)$ *in* E, *since*

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \xi(z) \right\rangle_E \xi(z) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}}(1 + \bar{z})\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{1}{2}(1 + \bar{z}) \\ 1 - \frac{1}{2}(1 + z) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 - \bar{z} \\ 1 - z \end{pmatrix} \neq 0.$$

Alternatively, from equation (2.15),

$$C_\xi^* C_\xi = \frac{1}{2} \begin{pmatrix} 1 & -S^* \\ -S & 1 \end{pmatrix},$$

so

$$(C_\xi^* C_\xi)^2 = \frac{1}{4} \begin{pmatrix} 1 + S^*S & -2S^* \\ -2S & SS^* + 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & -2S^* \\ -2S & 2 - P_0 \end{pmatrix},$$

and thus

$$(C_\xi^* C_\xi)^2 == \frac{1}{2} \begin{pmatrix} 1 & -S^* \\ -S & 1 - \frac{1}{2}P_0 \end{pmatrix} \neq C_\xi^* C_\xi,$$

since $SS^* = 1 - P_0$, *where* $P_0\left(\sum_{n=0}^{\infty} a_n z^n\right) = a_0$.
Consequently $C_\xi^* C_\xi$ *is not a projection and hence* C_ξ *is not a partial isometry on* $H^2(\mathbb{D}, \mathbb{C}^2)$.

Chapter 3

Superoptimal analytic approximation

In this chapter we present the main result of the dissertation, which is the superoptimal analytic approximation algorithm. In Section 3.1, we recall certain known results and Peller and Young's algorithm (Theorem 3.1.19). In Section 3.2, we construct the alternative algorithm for the superoptimal approximant based on exterior powers of Hilbert spaces. The proof of the validity of the new algorithm relies on the cited work given in Section 3.1.

Basic definitions, which we use in this chapter, are given in Chapter 1 and in Appendix D.

3.1 Known results

In this section we present certain established results that we later use to define the steps of the new algorithm and prove their validity.

Definition 3.1.1. *Let E, F be Hilbert spaces. We define by $\mathcal{K}(E,F)$ the Banach space of compact operators from E to F with the operator norm.*

Theorem 3.1.2 (Hartman's Theorem, [19], p. 74)**.** *Let E, F be separable Hilbert spaces and let $\Phi \in L^\infty(\mathbb{T}, \mathcal{L}(E,F))$. The following statements are equivalent*

(i) The Hankel operator H_Φ is compact on $H^2(\mathbb{D}, E)$.

(ii) $\Phi \in H^\infty(\mathbb{D}, \mathcal{L}(E,F)) + C(\mathbb{T}, \mathcal{K}(E,F))$.

(iii) there exists a function $\Psi \in C(\mathbb{T}, \mathcal{K}(E,F))$ such that $\hat{\Phi}(n) = \hat{\Psi}(n)$ for $n < 0$.

Definition 3.1.3 ([24], p. 306)**.** *The class of* quasi-continuous *functions is defined by*

$$QC = (H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})) \cap \overline{(H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n}))}.$$

In other words this class consists of functions on the circle which belong to $H^\infty + C$ and are such that their complex conjugates belong to $H^\infty + C$ as well.

Definition 3.1.4. *Consider a function $f \in L^1(\mathbb{T}, \mathbb{C})$ and an arc I on $\mathbb{T}$. Put*

$$f_I \stackrel{def}{=} \frac{1}{m(I)} \int_I f dm$$

where m is the Lebesgue measure on $\mathbb{T}$. Thus, f_I is the mean of f over I. The function f is said to have vanishing mean oscillation *if*

$$\lim_{m(I)\to 0} \frac{1}{m(I)} \int_I |f - f_I| dm = 0.$$

The space of functions of vanishing mean oscillation on $\mathbb{T}$ is denoted by VMO.

VMO is also related to the compactness of Hankel operators, as the following theorem asserts.

Theorem 3.1.5 ([19], Theorem 5.8)**.** *Let $\phi \in L^2(\mathbb{T}, \mathbb{C})$. Then H_ϕ is compact if and only if* $P_-\phi \in$ VMO.

It is therefore not surprising that the spaces QC and VMO are closely related. The following theorem illustrates such a connection.

Theorem 3.1.6 ([19], p. 729)**.**

$$QC = VMO \cap L^\infty.$$

Theorem 3.1.6 follows from another characterisation of VMO, which was obtained by Sarason in [31], to wit

$$VMO = \{f + \tilde{g} : f, g \in C(\mathbb{T}, \mathbb{C})\},$$

where $\tilde{g}$ denotes the harmonic conjugate of g.

Remark 3.1.7. *Let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$. We will say that every function $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ which minimises the norm $\|G - Q\|_{L^\infty}$, is a function at minimal distance from G. By Nehari's Theorem, all such functions Q satisfy $\|G - Q\|_{L^\infty} = \|H_G\|$.*

Next we describe some properties that a space X of equivalence classes of functions on the unit circle might posses, which were explored in [24]. It should be mentioned that the uniqueness result in Theorem 1.1.4 allows one to define a non-linear operator $\mathcal{A}$ of superoptimal analytic approximation on $H^\infty + C$ as follows.

Definition 3.1.8 ([24], p. 329)**.** *Define $\mathcal{A} = \mathcal{A}^{(m,n)}$ on the space of $m \times n$ functions $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$ by saying that $\mathcal{A}^{(m,n)}G$ is the unique superoptimal approximation in $H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ to G.*

Definition 3.1.9. *We say that a space $X \subseteq L^\infty(\mathbb{T}, \mathbb{C})$ is* hereditary *for $\mathcal{A}$ if, for every scalar function $g \in X$, the best analytic approximation $\mathcal{A}g$ of g belongs to X. For a matrix function $G \in H^\infty(\mathbb{T}, \mathbb{C}^{m\times n})$ we write $G \in X$ if each entry of G is in X.*

We consider spaces X of scalar functions on the circle which satisfy the following axioms.

(α1) X contains trigonometric polynomial functions and $X \subset VMO$;

(α2) X is hereditary for $\mathcal{A}$;

(α3) if $f \in X$ then $\bar{z}\bar{f} \in X$ and $P_+ f \in X$;

(α4) if $f, g \in X \cap L^\infty$ then $fg \in X \cap L^\infty$;

(α5) if $f \in X \cap H^2$ and $h \in H^\infty$ then $T_{\bar{h}} f \in X \cap H^2$.

The relevance of these properties is contained in the following statements. Recall that, according to [20], a function $f \in L^\infty$ is said to be *badly approximable* if the best analytic approximant to f is the zero function. In view of Nehari's Theorem, f is badly approximable if and only if $\|f\|_{L^\infty} = \|H_f\|$.

Theorem 3.1.10 ([24], p. 308). *Let φ be an $n \times 1$ inner matrix function. There exists an inner, co-outer function $\varphi_c \in H^\infty(\mathbb{D}, \mathbb{C}^{n\times(n-1)})$ such that*

$$\Phi = \begin{pmatrix} \varphi & \bar{\varphi}_c \end{pmatrix}$$

is unitary-valued on $\mathbb{T}$ and all minors of Φ on the first column are in H^∞.

Lemma 3.1.11 ([24], p. 332). *Let X satisfy (α1)-(α5) and let ϕ be an $n \times 1$ inner function. Let ϕ_c be an $n\times(n-1)$ function in H^∞ such that $\begin{pmatrix} \phi & \bar{\phi}_c \end{pmatrix}$ is unitary-valued almost everywhere and has all its minors on the first column belonging to H^∞. Then each entry of ϕ_c belongs to X.*

Lemma 3.1.12 ([24], p. 315-316). *Let $m, n > 1$, let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$ and $t_0 = \|H_G\| \neq 0$. Suppose that v is a maximizing vector of H_G and let*

$$H_G v = t_0 w. \tag{3.1}$$

Then $v, \bar{z}\bar{w} \in H^2(\mathbb{D}, \mathbb{C}^n)$ have the factorisations

$$v = v_0 h, \quad \bar{z}\bar{w} = \phi w_0 h \tag{3.2}$$

for some scalar outer function h, some scalar inner ϕ, and column-matrix inner functions v_0, w_0. Moreover there exist unitary-valued functions V, W of types $n\times n$, $m\times m$ respectively, of the form

$$V = \begin{pmatrix} v_0 & \bar{\alpha} \end{pmatrix}, \; W^T = \begin{pmatrix} w_0 & \bar{\beta} \end{pmatrix}, \tag{3.3}$$

where α, β are inner, co-outer, quasi-continuous functions of types $n \times (n-1)$, $m \times (m-1)$ respectively, and all minors on the first columns of V, W^T are in H^∞.

Furthermore every $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ which is at minimal distance from G satisfies

$$W(G-Q)V = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & F \end{pmatrix} \tag{3.4}$$

for some $F \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1)\times(n-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-1)\times(n-1)})$ and some quasi-continuous function u_0 given by

$$u_0 = \frac{\bar{z}\bar{\phi}\bar{h}}{h} \tag{3.5}$$

with $|u_0(z)| = 1$ almost everywhere on $\mathbb{T}$.

Proof. First we construct V and W with the properties (3.1) to (3.4). By Theorem D.2.4 and by equation (3.1), $\|v(z)\| = \|w(z)\|$ almost everywhere, and so the column-vector functions $v, \bar{z}\bar{w} \in H^2$ have the same (scalar) outer factor h. This property yields the inner-outer factorisations (3.2) for some column inner functions v_0, w_0. By Theorem 3.1.10, there exists an inner co-outer function α of type $n \times (n-1)$ such that $V \stackrel{\text{def}}{=} \begin{pmatrix} v_0 & \bar{\alpha} \end{pmatrix}$ is unitary-valued almost everywhere on $\mathbb{T}$ and all minors on the first column of V are in H^∞. Likewise there exists an inner co-outer function β of type $m \times (m-1)$ such that $W \stackrel{\text{def}}{=} \begin{pmatrix} w_0 & \bar{\beta} \end{pmatrix}^T$ is unitary-valued almost everywhere on $\mathbb{T}$ and all minors on the first column of W^T are in H^∞.

Next we show that u_0 given by equation (3.5) is quasi-continuous. Let $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ be at minimal distance from G. Then

$$\|G - Q\|_\infty = \|H_G\| = t_0.$$

By Theorem D.2.4,

$$(G-Q)v = t_0 w$$

and by the factorisations (3.2) we have

$$(G-Q)v_0 h = t_0 \bar{z}\bar{\phi}\bar{h}\bar{w}_0$$

and by equations (3.3) and (3.5)

$$(G-Q)V \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T = W^* \begin{pmatrix} t_0 u_0 & 0 & \cdots & 0 \end{pmatrix}^T.$$

Thus

$$W(G-Q)V = \begin{pmatrix} t_0 u_0 & f \\ 0 & F \end{pmatrix}$$

for some $f \in L^\infty(\mathbb{T}, \mathbb{C}^{1\times(n-1)}),\ F \in L^\infty(\mathbb{T}, \mathbb{C}^{(m-1)\times(n-1)})$.
Because $t_0 = \|H_G\|$, it follows that $|u_0| = 1$ almost everywhere, and from Nehari's Theorem

$$\|W(G-Q)V\|_\infty = \|G-Q\| = \|H_G\| = t_0,$$

and we get that $f = 0$. So, $W(G-Q)V$ is in the form (3.4). Now, $\|H_{u_0}\| \le \|u_0\|_\infty = 1$ and

$\|H_{u_0}h\| = \|\bar{z}\bar{\phi}\bar{h}\| = \|h\|$, which implies that u_0 is badly approximable. Hence

$$\|H_{u_0}\| = 1 = \|u_0\|_\infty.$$

The $(1,1)$ entries of equation (3.4) are

$$w_0^T(G-Q)v_0 = t_0 u_0.$$

Since $v_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$, $w_0 \in H^\infty(\mathbb{D}, \mathbb{C}^m)$ and $H^\infty(\mathbb{D}, \mathbb{C}) + C(\mathbb{T}, \mathbb{C})$ is an algebra, $u_0 \in H^\infty + C$. By a result in [20, Section 3.1], if $u_0 \in H^\infty + C$ and u_0 is badly approximable then $\bar{u_0} \in H^\infty + C$. Thus u_0 is quasi-continuous.

Next we show that $v_0, w_0 \in \mathrm{QC}$. It follows from Nehari's Theorem (see Theorem D.2.4) that

$$(G-Q)^* w = t_0 v.$$

Indeed, since $H_G^* w = t_0 v$ and $H_G^* = P_+ M_{(G-Q)^*}|H^{2^\perp}$, we have (assuming, as we may, that v and w are unit vectors),

$$\begin{aligned} t_0 = \|H_G^* w\| &= \|P_+(G-Q)^* w\| \\ &\leq \|(G-Q)^* w\| \leq \|G-Q\|_{L^\infty}\|w\| = t_0. \end{aligned}$$

It follows that the inequalities hold with equality, and so

$$\|P_+(G-Q)^* w\| = \|(G-Q)^* w\|,$$

whence

$$P_+(G-Q)^* w = (G-Q)^* w,$$

and so

$$(G-Q)^* w = H_G^* w = t_0 v,$$

as claimed.

Taking complex conjugates in the last equation we have

$$(G-Q)^T \bar{w} = t_0 \bar{v}.$$

Thus, by equation (3.2),

$$(G-Q)^T z\phi w_0 h = t_0 \bar{h} \bar{v}_0$$

for some outer function h and scalar inner ϕ. Therefore

$$\bar{v}_0 = \frac{(G-Q)^T z\phi w_0 h}{t_0 \bar{h}}.$$

Recall that $u_0 = \bar{z}\bar{\phi}\bar{h}/h$, and so

$$\bar{v}_0 = \frac{1}{t_0}(G-Q)^T \bar{u}_0 w_0.$$

Since $u_0 \in$ QC, $G - Q \in H^\infty + C$, $w_0 \in H^\infty$ and $H^\infty + C$ is an algebra, it follows that $\bar{v}_0 \in H^\infty + C$. Since also $v_0 \in H^\infty$, we have $v_0 \in$ QC. In an analogous way, one can show that $w_0 \in$ QC.

To complete the proof, all that remains is to show that α, β are quasi-continuous and $F \in H^\infty + C$. This will follow from Lemma 3.1.11 above.

The space VMO satisfies conditions $(\alpha 1)$ to $(\alpha 5)$, and we have $v_0 \in \text{QC} \subset \text{VMO}$. Hence we may apply Lemma 3.1.11 with $\phi = v_0$ to deduce that $\alpha \in$ VMO. Since also $\alpha \in L^\infty$, it follows from Theorem 3.1.6 that $\alpha \in$ QC. Likewise, $\beta \in$ QC.

To show that $F \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1)\times(n-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-1)\times(n-1)})$, for $1 < i \leq m,\ 1 < j \leq n$ consider the 2×2 minor of equation (3.4) with indices $1i, 1j$:

$$\sum_{r<s,\ k<l} W_{1i,rs}(G-Q)_{rs,kl} V_{kl,1j} = t_0 u_0 F_{i-1,j-1}. \tag{3.6}$$

By the analytic minors property of W, V,

$$V_{kl,1j}, W_{1i,rs} \in H^\infty.$$

Since $(G - Q) \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$, all the terms on the left-hand side of equation (3.6) are in $H^\infty + C$ and hence $u_0 F \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1)\times(n-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-1)\times(n-1)})$. Thus

$$F = \bar{u}_0(u_0 F) \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1)\times(n-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-1)\times(n-1)}). \qquad \square$$

Definition 3.1.13. *We say that a unitary-matrix-valued function V is a* thematic completion *of a column-matrix inner function $v_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$, if $V = \begin{pmatrix} v_0 & \bar{\alpha} \end{pmatrix}$ for some co-outer function $\alpha \in H^\infty(\mathbb{D}, \mathbb{C}^{n\times(n-1)})$ such that $V(z)$ is unitary-valued almost everywhere on $\mathbb{T}$ and all minors on the first column of V are analytic.*

Remark 3.1.14. *By Theorem 3.1.10, every column-matrix inner function has a thematic completion. Thematic completions are not unique, for if $V = \begin{pmatrix} v_0 & \bar{\alpha} \end{pmatrix}$ is a thematic completion of v_0, then so is $\begin{pmatrix} v_0 & \bar{\alpha}U \end{pmatrix}$ for any constant $(n-1)-$square unitary matrix U. However, by Corollary* 1.6 *of [24], the thematic completion of v_0* is *unique up to multiplication on the right by a constant unitary matrix of the form* $\text{diag}\{1, U\}$ *for some constant $(n-1)-$ square matrix U, and so it is permissible to speak of "the thematic completion of v_0".*

Furthermore, by Theorem 1.2 *of [24], thematic completions have constant determinants almost everywhere on $\mathbb{T}$, and hence α, β are inner matrix functions. Observe that, as we showed above, if the column v_0 belongs to* VMO, *then the thematic completion of v_0 is quasi-continuous. Similarly, if the column w_0 belongs to* VMO, *then the thematic completion of*

w_0 is quasi-continuous. Thus α, β are inner, co-outer, quasi-continuous functions of types $n \times (n-1)$ and $m \times (m-1)$ respectively.

Lemma 3.1.15 ([24], p. 316)**.** *Let $m, n > 1$, let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$, let $\|H_G\| = t_0$ and let $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ be at minimal distance from G, so that in the notation of Lemma 3.1.12,*

$$W(G - Q_1)V = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & F \end{pmatrix}$$

for some $F \in H^\infty(\mathbb{D}, \mathbb{C}^{m-1\times n-1}) + C(\mathbb{T}, \mathbb{C}^{m-1\times n-1})$. Let

$$\mathcal{E}_0 = \{G - Q : Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}), \|G - Q\|_\infty = t_0\}.$$

Then

$$W\mathcal{E}_0 V = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & F + H^\infty(\mathbb{D}, \mathbb{C}^{m-1\times n-1}) \end{pmatrix} \cap B(t_0),$$

where $B(t_0)$ is the closed ball of radius t_0 in $L^\infty(\mathbb{T}, \mathbb{C}^{m\times n})$.

Lemma 3.1.16 ([25], p. 16)**.** *Let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$ and let (x_0, y_0) be a Schmidt pair for the Hankel operator H_G corresponding to the singular value $t_0 = \|H_G\|$. Let $x_0 = \xi_0 h_0$ be the inner-outer factorisation of x_0, where $\xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$ is the inner and $h_0 \in H^2(\mathbb{D}, \mathbb{C})$ is the scalar outer factor of $x_0 \in H^2(\mathbb{D}, \mathbb{C}^n)$, and let*

$$V_0 = \begin{pmatrix} \xi_0 & \bar{\alpha}_0 \end{pmatrix}$$

be a unitary-valued function on $\mathbb{T}$, where $\alpha_0 \in H^\infty(\mathbb{D}, \mathbb{C}^{n\times(n-1)})$ is inner, co-outer and quasi-continuous. Then

$$V_0 \begin{pmatrix} 0 & H^2(\mathbb{D}, \mathbb{C}^{n-1}) \end{pmatrix}^T$$

is the orthogonal projection of $H^2(\mathbb{D}, \mathbb{C}^n)$ onto the pointwise orthogonal complement of x_0 in $L^2(\mathbb{T}, \mathbb{C}^n)$. Similarly

$$V_0^* \begin{pmatrix} 0 & H^2(\mathbb{D}, \mathbb{C}^{n-1})^\perp \end{pmatrix}$$

is the orthogonal projection of $H^2(\mathbb{D}, \mathbb{C}^n)^\perp$ onto the pointwise orthogonal complement of x_0 in $L^2(\mathbb{T}, \mathbb{C}^n)$.

Lemma 3.1.17 ([25], p. 16)**.** *Let G, x_0, y_0 be defined as in Lemma 3.1.16 and let $\mathcal{K}, \mathcal{L}$ be the projections of $H^2(\mathbb{D}, \mathbb{C}^n), H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ onto the pointwise orthogonal complements of x_0, y_0 in $L^2(\mathbb{T}, \mathbb{C}^n), L^2(\mathbb{T}, \mathbb{C}^m)$ respectively. Let $Q_0 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ be at minimal distance from G, let F be the $(2,2)$ block of $W_0(G - Q_0)V_0$, as in Lemma 3.1.12, that is,*

$$V_0 = \begin{pmatrix} \xi_0 & \bar{\alpha}_0 \end{pmatrix}, \quad W_0 = \begin{pmatrix} \eta_0 & \bar{\beta}_0 \end{pmatrix}^T \tag{3.7}$$

are unitary-valued functions on $\mathbb{T}$, α_0, β_0 are inner, co-outer, quasi-continuous functions of size $n \times (n-1), m \times (m-1)$ respectively and all minors on the first columns of V_0, W_0^T are *in* H^∞. *Let* $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ *satisfy*

$$(G-Q)x_0 = \|H_G\|y_0, \quad y_0^*(G-Q) = \|H_G\|x_0^*.$$

Then H_F *is a unitary multiple of the operator*

$$\Gamma \stackrel{\text{def}}{=} P_{\mathcal{L}} M_{G-Q}|\mathcal{K}, \tag{3.8}$$

where $M_{G-Q}\colon L^2(\mathbb{T}, \mathbb{C}^n) \to L^2(\mathbb{T}, \mathbb{C}^m)$ *is the operator of multiplication by* $G-Q$. *More explicitly, if* $U_1\colon H^2(\mathbb{D}, \mathbb{C}^{n-1}) \to \mathcal{K}$, $U_2\colon H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp \to \mathcal{L}$ *are defined by*

$$U_1\chi = V_0\begin{pmatrix}0\\ \chi\end{pmatrix}, \quad U_2\psi = W_0^*\begin{pmatrix}0\\ \psi\end{pmatrix} \quad \text{for all} \quad \chi \in H^2(\mathbb{D}, \mathbb{C}^{n-1}), \psi \in H^2(\mathbb{D}, \mathbb{C}^{m-1}),$$

then U_1, U_2 *are unitaries and*

$$H_F = U_2^*\Gamma U_1.$$

Lemma 3.1.18 ([24], p. 337)**.** *Let* $\alpha \in$ QC *of type* $m \times n$, *where* $m \geq n$, *be inner and co-outer. There exists* $A \in H^\infty(\mathbb{D}, \mathbb{C}^{n\times m})$ *such that* $A\alpha = I_n$. *Here* I_n *denotes the* $n \times n$ *identity matrix.*

Theorem 3.1.19 gives the algorithm for the superoptimal analytic approximant constructed in [25].

Theorem 3.1.19 ([25], p. 17)**.** *Let* $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$. *The superoptimal approximant* $\mathcal{A}G$ *to* G *is given by the following formula.*
If $H_G = 0$, *then* $\mathcal{A}G = G$. *Otherwise define spaces* $K_j \subset L^2(\mathbb{T}, \mathbb{C}^n), N_j \subset L^2(\mathbb{T}, \mathbb{C}^m)$, *vectors* $\chi_j \in K_j$, $\psi_j \in N_j$, H^∞ *functions* Q_j, *operators* Γ_j *and positive* λ_j *as follows.*
Let

$$K_0 = H^2(\mathbb{D}, \mathbb{C}^n), \quad N_0 = H^2(\mathbb{D}, \mathbb{C}^m)^\perp, \quad Q_0 = 0.$$

Let

$$\Gamma_j = P_{N_j} M_{G-Q_j}|_{K_j}\colon K_j \to N_j, \quad \lambda_j = \|\Gamma_j\|,$$

where P_{N_j} *is the orthogonal projection onto* N_j. *If* $\lambda_j = 0$ *set* $r = j$ *and terminate the construction. Otherwise let* χ_j, ψ_j *be a Schmidt pair for* Γ_j *corresponding to the singular value* λ_j. *Let* K_{j+1} *be the range of the orthogonal projection of* K_j *onto the pointwise orthogonal complement of* $\chi_0, \cdots, \chi_j$ *in* $L^2(\mathbb{T}, \mathbb{C}^n)$. *Let* N_{j+1} *be the projection of* N_j *onto the pointwise orthogonal complement of* $\psi_0, \cdots, \psi_j$ *in* $L^2(\mathbb{T}, \mathbb{C}^m)$. *Let* $Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ *be chosen to satisfy, for* $0 \leq k \leq j$,

$$Q_{j+1}\chi_k = G\chi_k - t_k\psi_k, \quad \psi_k^* Q_{j+1} = \psi_k^* G - t_k\chi_k^*.$$

Then each Γ_j *is a compact operator,* Q_j *with the above properties does exist, the construction terminates with* $r \leq \min(m,n)$ *and*

$$G - \mathcal{A}G = \sum_{j=0}^{r-1} \frac{\lambda_j \psi_j \chi_j^*}{\chi_j^* \chi_j}.$$

We shall derive a similar formula for the superoptimal analytic approximant $\mathcal{A}G$, by making use of exterior products of Hilbert spaces.

3.2 Algorithm for superoptimal analytic approximation

In this section we consider the superoptimal analytic approximation problem 1.1.3 for a matrix-valued function which lies in $H^\infty(\mathbb{D},\mathbb{C}^{m\times n})+C(\mathbb{T},\mathbb{C}^{m\times n})$. In particular, in Subsection 3.2.1 we state the algorithm for calculating the superoptimal approximant in that instance, and moreover, in the subsections that follow, we prove the validity of all the claims which are being made. Throughout we make use of the main result of Peller and Young from [24], which asserts that Problem 1.1.3 is solvable (see Theorem 1.1.4).

3.2.1 The algorithm

Let $G \in H^\infty(\mathbb{D},\mathbb{C}^{m\times n})+C(\mathbb{T},\mathbb{C}^{m\times n})$. In this subsection we shall give a fuller and more precise statement of the algorithm for $\mathcal{A}G$ outlined in Section 1.2, in preparation for a subsequent formal proof of Theorem 3.2.59, which asserts that if entities r, t_i, x_i, y_i, h_i for $i=0,\dots,r-1$, are generated by the algorithm, then the superoptimal approximant is given by equation

$$\mathcal{A}G = G - \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2}.$$

The proof will be by induction on r, which is the least index $j \geq 0$ such that $T_j = 0$, where $T_0 = H_G, T_1, T_2, \dots$ is a sequence of operators recursively generated by the algorithm.

Algorithm: For the given $G \in H^\infty(\mathbb{D},\mathbb{C}^{m\times n}) + C(\mathbb{T},\mathbb{C}^{m\times n})$, the superoptimal analytic approximant $\mathcal{A}G \in H^\infty(\mathbb{D},\mathbb{C}^{m\times n})$ can be constructed as follows.
i) **Step 0.** Let $T_0 = H_G$ be the Hankel operator with symbol G as defined by Definition 1.2.4. Let $t_0 = \|H_G\|$. If $t_0 = 0$, then $H_G = 0$, which implies $G \in H^\infty(\mathbb{D},\mathbb{C}^{m\times n})$. In this case, the algorithm terminates, we define r to be zero and the superoptimal approximant $\mathcal{A}G$ is given by $\mathcal{A}G = G$.
Suppose $t_0 \neq 0$. By Hartman's Theorem 3.1.2, H_G is a compact operator and so there exists a Schmidt pair (x_0, y_0) corresponding to the singular value t_0 of H_G. By the definition of a Schmidt pair (x_0, y_0) for the Hankel operator

$$H_G \colon H^2(\mathbb{D},\mathbb{C}^n) \to H^2(\mathbb{D},\mathbb{C}^m)^\perp,$$

$$x_0 \in H^2(\mathbb{D},\mathbb{C}^n), \quad y_0 \in H^2(\mathbb{D},\mathbb{C}^m)^\perp$$

are non-zero vector-valued functions such that

$$H_G x_0 = t_0 y_0, \quad H_G^* y_0 = t_0 x_0.$$

By Lemma 3.1.12, $x_0 \in H^2(\mathbb{D}, \mathbb{C}^n)$ and $\bar{z}\bar{y}_0 \in H^2(\mathbb{D}, \mathbb{C}^m)$ admit the inner-outer factorisations

$$x_0 = \xi_0 h_0, \quad \bar{z}\bar{y}_0 = \eta_0 h_0 \tag{3.9}$$

for some scalar outer factor $h_0 \in H^2(\mathbb{D}, \mathbb{C})$ and column matrix inner functions $\xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$, $\eta_0 \in H^\infty(\mathbb{D}, \mathbb{C}^m)$. Then

$$\|x_0(z)\|_{\mathbb{C}^n} = |h_0(z)| = \|y_0(z)\|_{\mathbb{C}^m} \text{ almost everywhere on } \mathbb{T}. \tag{3.10}$$

We write equations (3.9) as

$$\xi_0 = \frac{x_0}{h_0}, \; \eta_0 = \frac{\bar{z}\bar{y}_0}{h_0}. \tag{3.11}$$

By equations (3.10) and (3.11),

$$\|\xi_0(z)\|_{\mathbb{C}^n} = 1 = \|\eta_0(z)\|_{\mathbb{C}^m} \text{ almost everywhere on } \mathbb{T}.$$

By Theorem D.2.4, every function $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ which is at minimal distance from G satisfies

$$(G - Q_1)x_0 = t_0 y_0, \quad y_0^*(G - Q_1) = t_0 x_0^*. \tag{3.12}$$

ii) **Step 1.** Let

$$X_1 \stackrel{\text{def}}{=} \xi_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n). \tag{3.13}$$

By Proposition 3.2.3, X_1 is a closed subspace of $H^2(\mathbb{D}, \wedge^2\mathbb{C}^n)$.
Moreover

$$\eta_0 \dot{\wedge} z H^2(\mathbb{D}, \mathbb{C}^m) \subset z H^2(\mathbb{D}, \wedge^2\mathbb{C}^m)$$

and therefore

$$\bar{\eta}_0 \dot{\wedge} \overline{z H^2(\mathbb{D}, \mathbb{C}^m)} \subset \bar{z}\overline{H^2(\mathbb{D}, \wedge^2\mathbb{C}^m)},$$

that is, if

$$Y_1 \stackrel{\text{def}}{=} \bar{\eta}_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp, \tag{3.14}$$

then Y_1 is a closed subspace of $H^2(\mathbb{D}, \wedge^2\mathbb{C}^m)^\perp$. Choose any function $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ which satisfies equation (3.12). Consider the operator $T_1 \colon X_1 \to Y_1$ defined by

$$T_1(\xi_0 \dot{\wedge} x) = P_{Y_1}(\bar{\eta}_0 \dot{\wedge} (G - Q_1)x) \text{ for all } x \in H^2(\mathbb{D}, \mathbb{C}^n), \tag{3.15}$$

where P_{Y_1} is the projection from $L^2(\mathbb{T}, \wedge^2\mathbb{C}^m)$ on Y_1. By Corollary 3.2.5 and Proposition 3.2.8, T_1 is well-defined. If $T_1 = 0$, then the algorithm terminates, we define r to be 1 and, in accordance with Theorem 3.2.59, the superoptimal approximant $\mathcal{A}G$ is given by the formula

$$G - \mathcal{A}G = \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2} = \frac{t_0 y_0 x_0^*}{|h_0|^2},$$

and the solution is

$$\mathcal{A}G = G - \frac{t_0 y_0 x_0^*}{|h_0|^2}.$$

Suppose $T_1 \neq 0$ and let $t_1 = \|T_1\| > 0$. By Theorem 3.2.10, T_1 is a compact operator and so there exist $v_1 \in H^2(\mathbb{D}, \mathbb{C}^n)$, $w_1 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ such that $(\xi_0 \dot{\wedge} v_1, \bar{\eta}_0 \dot{\wedge} w_1)$ is a Schmidt pair for T_1 corresponding to t_1. By Proposition 3.2.2, $\xi_0 \dot{\wedge} v_1 \in H^2(\mathbb{D}, \wedge^2 \mathbb{C}^n)$. Let h_1 be the scalar outer factor of $\xi_0 \dot{\wedge} v_1$ and let

$$x_1 = (I_{\mathbb{C}^n} - \xi_0 \xi_0^*) v_1, \quad y_1 = (I_{\mathbb{C}^m} - \bar{\eta}_0 \eta_0^T) w_1, \tag{3.16}$$

where $\mathrm{I}_{\mathbb{C}^n}$ and $\mathrm{I}_{\mathbb{C}^m}$ are the identity operators in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. Then, by Proposition 3.2.24,

$$\|x_1(z)\|_{\mathbb{C}^n} = |h_1(z)| = \|y_1(z)\|_{\mathbb{C}^m} \text{ almost everywhere on } \mathbb{T}. \tag{3.17}$$

By Theorem 1.1.4, there exists a function $Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ such that both $s_0^\infty(G - Q_2)$ and $s_1^\infty(G - Q_2)$ are minimised and

$$s_1^\infty(G - Q_2) = t_1.$$

By Proposition 3.2.27, any such Q_2 satisfies

$$\begin{aligned} (G - Q_2)x_0 &= t_0 y_0, \quad y_0^*(G - Q_2) = t_0 x_0^* \\ (G - Q_2)x_1 &= t_1 y_1, \quad y_1^*(G - Q_2) = t_1 x_1^*. \end{aligned} \tag{3.18}$$

Define

$$\xi_1 = \frac{x_1}{h_1}, \quad \eta_1 = \frac{\bar{z}\bar{y}_1}{h_1}. \tag{3.19}$$

By equations (3.17) and (3.19), $\|\xi_1(z)\|_{\mathbb{C}^n} = 1 = \|\eta_1(z)\|_{\mathbb{C}^n}$ almost everywhere on $\mathbb{T}$.

iii) **Step 2.** Define

$$\begin{aligned} X_2 &\stackrel{\text{def}}{=} \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n) \\ Y_2 &\stackrel{\text{def}}{=} \bar{\eta}_0 \dot{\wedge} \bar{\eta}_1 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp. \end{aligned}$$

Note that, by Proposition 3.2.3, X_2 is a closed linear subspace of $H^2(\mathbb{D}, \wedge^3 \mathbb{C}^n)$, and, by Proposition 3.2.6, Y_2 is a closed linear subspace of $H^2(\mathbb{D}, \wedge^3 \mathbb{C}^m)^\perp$. Choose any function $Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ which satisfies equations (3.18). Consider the operator $T_2 \colon X_2 \to Y_2$ given by

$$T_2(\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} x) = P_{Y_2}(\bar{\eta}_0 \dot{\wedge} \bar{\eta}_1 \dot{\wedge} (G - Q_2)x), \tag{3.20}$$

where P_{Y_2} is the projection from $L^2(\mathbb{T}, \mathbb{C}^m)$ on Y_2.

By Corollary 3.2.7 and by Proposition 3.2.8, T_2 is well-defined, that is, it does not depend on the choice of $Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ satisfying equations (3.18). If $T_2 = 0$, then the algorithm terminates, we define r to be 2 and, in accordance with Theorem 3.2.59, the superoptimal approximant $\mathcal{A}G$ is given by the formula

$$G - \mathcal{A}G = \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2} = \frac{t_0 y_0 x_0^*}{|h_0|^2} + \frac{t_1 y_1 x_1^*}{|h_1|^2}.$$

If $T_2 \neq 0$, then let $t_2 = \|T_2\|$. By Theorem 3.2.37, T_2 is a compact operator and hence there exist $v_2 \in H^2(\mathbb{D}, \mathbb{C}^n),\ w_2 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ such that

$$(\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} v_2,\ \bar{\eta}_0 \dot{\wedge} \bar{\eta}_1 \dot{\wedge} w_2)$$

is a Schmidt pair for T_2 corresponding to $\|T_2\| = t_2$. Define x_2, y_2 by

$$x_2 = (I_{\mathbb{C}^n} - \xi_0 \xi_0^* - \xi_1 \xi_1^*) v_2, \quad y_2 = (I_{\mathbb{C}^m} - \bar{\eta}_0 \eta_0^T - \bar{\eta}_1 \eta_1^T) w_2.$$

By Proposition 3.2.40, $\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} v_2 \in H^2(\mathbb{D}, \wedge^3 \mathbb{C}^n)$. Let $h_2 \in H^2(\mathbb{D}, \mathbb{C})$ be the outer factor of $\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} v_2$. By Proposition 3.2.40,

$$\|x_2(z)\|_{\mathbb{C}^n} = |h_2(z)| = \|y_2(z)\|_{\mathbb{C}^m}$$

almost everywhere on $\mathbb{T}$. Define

$$\xi_2 = \frac{x_2}{h_2}, \quad \eta_2 = \frac{\bar{z}\bar{y}_2}{h_2}.$$

It is easy to see that $\|\xi_2(z)\|_{\mathbb{C}^n} = 1$ and $\|\eta_2(z)\|_{\mathbb{C}^m} = 1$ almost everywhere on $\mathbb{T}$.
iv) **Recursive step.** Suppose that, for $j \leq \min(m, n) - 2$, we have constructed

$$\begin{aligned}
& t_0 \geq t_1 \geq \cdots \geq t_j > 0 \\
& x_0, x_1, \cdots, x_j \in L^2(\mathbb{T}, \mathbb{C}^n) \\
& y_0, y_1, \cdots, y_j \in L^2(\mathbb{T}, \mathbb{C}^m) \\
& h_0, h_1, \cdots, h_j \in H^2(\mathbb{D}, \mathbb{C}) \text{ outer} \\
& \xi_0, \xi_1, \cdots, \xi_j \in L^2(\mathbb{T}, \mathbb{C}^n) \text{ pointwise orthonormal on } \mathbb{T} \\
& \eta_0, \eta_1, \cdots, \eta_j \in L^2(\mathbb{T}, \mathbb{C}^m) \text{ pointwise orthonormal on } \mathbb{T} \\
& X_0 = H^2(\mathbb{D}, \mathbb{C}^n), X_1, \cdots, X_j \\
& Y_0 = H^2(\mathbb{D}, \mathbb{C}^m)^\perp, Y_1, \cdots, Y_j \\
& T_0, T_1, \cdots, T_j \text{ compact operators.}
\end{aligned} \tag{3.21}$$

By Theorem 1.1.4, there exists a function $Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ such that

$$\left(s_0^\infty(G - Q_{j+1}), s_1^\infty(G - Q_{j+1}), \cdots, s_{j+1}^\infty(G - Q_{j+1})\right)$$

is lexicographically minimised. By Proposition 3.2.47, any such function Q_{j+1} satisfies

$$(G - Q_{j+1}) x_i = t_i y_i, \quad y_i^*(G - Q_{j+1}) = t_i x_i^*, \quad i = 0, 1, \cdots, j. \tag{3.22}$$

Define

$$X_{j+1} \stackrel{\text{def}}{=} \xi_0 \dot\wedge \xi_1 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n) \tag{3.23}$$

$$Y_{j+1} \stackrel{\text{def}}{=} \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge H^2(\mathbb{D}, \mathbb{C}^m)^\perp. \tag{3.24}$$

Note that, by Proposition 3.2.3, X_{j+1} is a closed subspace of $H^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^n)$, and, by Proposition 3.2.6, Y_{j+1} is a closed subspace of $H^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^m)^\perp$. Choose any function $Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ which satisfies equations (3.22). Consider the operator

$$T_{j+1} \colon X_{j+1} \to Y_{j+1}$$

given by

$$T_{j+1}(\xi_0 \dot\wedge \xi_1 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge x) = P_{Y_{j+1}}\left(\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge (G - Q_{j+1})x\right) \tag{3.25}$$

for all $x \in H^2(\mathbb{D}, \mathbb{C}^n)$. By Corollary 3.2.7 and by Proposition 3.2.8, T_{j+1} is well-defined. If $T_{j+1} = 0$, then the algorithm terminates, we define r to be $j+1$, and, in accordance with Theorem 3.2.59, the superoptimal approximant $\mathcal{A}G$ is given by the formula

$$G - \mathcal{A}G = \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2} = \sum_{i=0}^{j} \frac{t_i y_i x_i^*}{|h_i|^2}.$$

Otherwise, we define $t_{j+1} = \|T_{j+1}\| > 0$. By Theorem 3.2.54, T_{j+1} is a compact operator and hence there exist $v_{j+1} \in H^2(\mathbb{D}, \mathbb{C}^n)$, $w_{j+1} \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ such that

$$(\xi_0 \dot\wedge \xi_1 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge v_{j+1}, \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge w_{j+1}) \tag{3.26}$$

is a Schmidt pair for T_{j+1} corresponding to the singular value t_{j+1}.
By Proposition 3.2.2,

$$\xi_0 \dot\wedge \xi_1 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge v_{j+1} \in H^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^n).$$

Let h_{j+1} be the scalar outer factor of $\xi_0 \dot\wedge \xi_1 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge v_{j+1}$, and let

$$x_{j+1} = (I_{\mathbb{C}^n} - \xi_0\xi_0^* - \cdots - \xi_j\xi_j^*)v_{j+1}, \quad y_{j+1} = (I_{\mathbb{C}^m} - \bar\eta_0\eta_0^T - \cdots - \bar\eta_j\eta_j^T)w_{j+1}. \tag{3.27}$$

By Proposition 3.2.57,

$$\|x_{j+1}(z)\|_{\mathbb{C}^n} = |h_{j+1}(z)| = \|y_{j+1}(z)\|_{\mathbb{C}^m}$$

almost everywhere on $\mathbb{T}$. Define

$$\xi_{j+1} = \frac{x_{j+1}}{h_{j+1}}, \quad \eta_{j+1} = \frac{\bar z \bar y_{j+1}}{h_{j+1}}. \tag{3.28}$$

It is easy to see that $\|\xi_{j+1}(z)\|_{\mathbb{C}^n} = 1$ and $\|\eta_{j+1}(z)\|_{\mathbb{C}^m} = 1$ almost everywhere on $\mathbb{T}$. This completes the recursive step. The algorithm terminates after at most $\min(m, n)$ steps, so that, $r \le \min(m,n)$ and, in accordance with Theorem 3.2.59, the superoptimal approximant $\mathcal{A}G$ is given by the formula

$$G - \mathcal{A}G = \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2}. \qquad \square$$

3.2.2 Pointwise orthonormality of $\{\xi_i\}_{i=1}^j$ and $\{\bar\eta_i\}_{i=1}^j$ almost everywhere on $\mathbb{T}$

Proposition 3.2.1. *Let m,n be positive integers with $\min(m,n) \ge 2$, let $G \in H^\infty(\mathbb{D},\mathbb{C}^{m\times n}) + C(\mathbb{T},\mathbb{C}^{m\times n})$ and let $0 \le j \le \min(m,n)-2$. Suppose we have applied steps $0,\dots,j$ of the superoptimal analytic approximation algorithm from Section 3.2.1 to G and we have obtained x_i, y_i as in equations (3.27), and ξ_i, η_i as in equations (3.28) for all $i = 0,\cdots,j$. Then*

(i) $\xi_0 \dot\wedge v_1 = \xi_0 \dot\wedge x_1$, $\xi_0 \dot\wedge \cdots \dot\wedge \xi_{j-1} \dot\wedge v_j = \xi_0 \dot\wedge \cdots \dot\wedge \xi_{j-1} \dot\wedge x_j$, $\bar\eta_0 \dot\wedge w_1 = \bar\eta_0 \dot\wedge y_1$, *and* $\bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_{j-1} \dot\wedge w_j = \bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_{j-1} \dot\wedge y_j$;

(ii) for all $i = 0,\cdots,j$, $\|x_i(z)\|_{\mathbb{C}^n} = \|y_i(z)\|_{\mathbb{C}^m} = |h_i(z)|$ *almost everywhere on* $\mathbb{T}$;

(iii) the sets $\{\xi_i(z)\}_{i=0}^j$ *and* $\{\bar\eta_i(z)\}_{i=0}^j$ *are orthonormal in* $\mathbb{C}^n$ *and* $\mathbb{C}^m$ *respectively for almost every* $z \in \mathbb{T}$.

Proof. We will prove statements (ii) in Propositions 3.2.24 and 3.2.40. Statement (i) is proven below in equations (3.32), (3.35), (3.38). Let us prove assertion (iii).

Since the function G belongs to $H^\infty(\mathbb{D},\mathbb{C}^{m\times n}) + C(\mathbb{T},\mathbb{C}^{m\times n})$, by Hartman's theorem, the Hankel operator with symbol G, denoted by H_G, is a compact operator, and so there exist functions

$$x_0 \in H^2(\mathbb{D},\mathbb{C}^n), \quad y_0 \in H^2(\mathbb{D},\mathbb{C}^m)^\perp$$

such that (x_0, y_0) is a Schmidt pair corresponding to the singular value $t_0 = \|H_G\| \ne 0$. By Lemma 3.1.12, $x_0, \bar z \bar y_0$ admit the inner-outer factorisations

$$x_0 = \xi_0 h_0, \quad \bar z \bar y_0 = \eta_0 h_0$$

for column matrix inner functions $\xi_0 \in H^\infty(\mathbb{D},\mathbb{C}^n)$, $\eta_0 \in H^\infty(\mathbb{D},\mathbb{C}^m)$ and some scalar outer factor $h_0 \in H^2(\mathbb{D},\mathbb{C})$. By Theorem D.2.4,

$$\|x_0(z)\|_{\mathbb{C}^n} = |h_0(z)| = \|y_0(z)\|_{\mathbb{C}^m} \text{ almost everywhere on } \mathbb{T}. \tag{3.29}$$

Thus

$$\|\xi_0(z)\|_{\mathbb{C}^n} = 1 \text{ almost everywhere on } \mathbb{T}. \tag{3.30}$$

Hence (iii) of Proposition 3.2.1 holds for $\{\xi_i(z)\}_{i=0}^j$ in the case that $j = 0$.
Let T_1 be given by equation (3.15). By the hypothesis (3.21), T_1 is a compact operator, and if $T_1 \ne 0$, then there exist $v_1 \in H^2(\mathbb{D},\mathbb{C}^n)$ and $w_1 \in H^2(\mathbb{D},\mathbb{C}^m)^\perp$ such that $(\xi_0 \dot\wedge v_1, \bar\eta_0 \dot\wedge w_1)$

is a Schmidt pair corresponding to $\|T_1\| = t_1$. By Proposition 2.2.13, $\xi_0 \dot{\wedge} v_1 \in H^2(\mathbb{D}, \wedge^2\mathbb{C}^n)$. Let h_1 be the scalar outer factor of $\xi_0 \dot{\wedge} v_1$. We define

$$x_1 = (I_{\mathbb{C}^n} - \xi_0 \xi_0^*) v_1 \tag{3.31}$$

and

$$\xi_1 = \frac{x_1}{h_1}.$$

Then, for $z \in \mathbb{D}$,

$$\xi_1(z) = \frac{1}{h_1(z)} v_1(z) - \frac{1}{h_1(z)} \xi_0(z)\xi_0(z)^* v_1(z).$$

Note that by equation (3.30),

$$\xi_0^*(z)\xi_0(z) = \langle \xi_0(z), \xi_0(z) \rangle_{\mathbb{C}^n} = 1 \quad \text{almost everywhere on } \mathbb{T},$$

hence

$$\langle \xi_1(z), \xi_0(z) \rangle_{\mathbb{C}^n} = \xi_0^*(z)\xi_1(z) = \frac{1}{h_1(z)} \xi_0(z)^* v_1(z) - \frac{1}{h_1(z)} \xi_0(z)^* \xi_0(z) \xi_0(z)^* v_1(z) = 0$$

almost everywhere on $\mathbb{T}$. Note that, by equation (3.31), for almost every $z \in \mathbb{T}$,

$$\begin{aligned} \xi_0(z) \wedge v_1(z) &= \xi_0(z) \wedge (x_1(z) + \xi_0(z)\xi_0(z)^* v_1(z)) \\ &= \xi_0(z) \wedge x_1(z) + \xi_0(z) \wedge \xi_0(z)\xi_0(z)^* v_1(z)) \\ &= \xi_0(z) \wedge x_1(z), \end{aligned} \tag{3.32}$$

the last equality following from the pointwise linear dependence of the vectors ξ_0 and $z \mapsto \xi_0(z)\langle v_1(z), \xi_0(z) \rangle_{\mathbb{C}^n}$ almost everywhere on $\mathbb{T}$.

By Proposition 2.2.13, $\xi_0 \dot{\wedge} v_1 \in H^2(\mathbb{D}, \wedge^2\mathbb{C}^n)$. Let h_1 be the scalar outer factor of $\xi_0 \dot{\wedge} v_1$. Then, for almost every $z \in \mathbb{T}$, we have

$$|h_1(z)| \;\; = \|\xi_0(z) \wedge v_1(z)\|_{\wedge^2\mathbb{C}^n} = \|\xi_0(z) \wedge x_1(z)\|_{\wedge^2\mathbb{C}^n},$$

By Lemma 2.1.22,

$$\|\xi_0(z) \wedge x_1(z)\|_{\wedge^2\mathbb{C}^n} = \|x_1(z) - \langle x_1(z), \xi_0(z) \rangle_{\mathbb{C}^n} \xi_0(z)\|_{\mathbb{C}^n} = \|x_1(z)\|_{\mathbb{C}^n}$$

almost everywhere on $\mathbb{T}$. Hence, for almost every $z \in \mathbb{T}$,

$$|h_1(z)| = \|x_1(z)\|_{\mathbb{C}^n} \tag{3.33}$$

and thus

$$\|\xi_1(z)\|_{\mathbb{C}^n} = \frac{\|x_1(z)\|_{\mathbb{C}^n}}{|h_1(z)|} = 1 \quad \text{almost everywhere on } \mathbb{T}.$$

Consequently, $\{\xi_0(z), \xi_1(z)\}$ is an orthonormal set in $\mathbb{C}^n$ for almost every $z \in \mathbb{T}$. Hence (iii) of Proposition 3.2.1 holds for $\{\xi_i(z)\}_{i=0}^{j}$ in the case that $j=1$.

Recursive step: Suppose the entities in equations (3.21) have been constructed and have the stated properties, for $i=0,\dots,j-1$, and that $\{\xi_i(z)\}_{i=0}^{j-1}$ is an orthonormal set almost everywhere on $\mathbb{T}$. Since by the inductive hypothesis T_j is a compact operator, there exist

$$v_j \in H^2(\mathbb{D},\mathbb{C}^n), \quad w_j \in H^2(\mathbb{D},\mathbb{C}^m)^\perp$$

such that

$$(\xi_0\dot\wedge\xi_1\dot\wedge\cdots\dot\wedge\xi_{j-1}\dot\wedge v_j, \overline{\eta}_0\dot\wedge\overline{\eta}_1\dot\wedge\cdots\dot\wedge\overline{\eta}_{j-1}\dot\wedge w_j)$$

is a Schmidt pair for T_j corresponding to $\|T_j\|=t_j$. Let us first prove that $\xi_0\dot\wedge\xi_1\dot\wedge\cdots\dot\wedge\xi_{j-1}\dot\wedge v_j$ is an element of $H^2(\mathbb{D},\wedge^{j+1}\mathbb{C}^n)$. By hypothesis,

$$x_i = (I_n - \xi_0\xi_0^* - \cdots - \xi_{i-1}\xi_{i-1}^*)v_i \quad \text{and} \quad \xi_i = \frac{x_i}{h_i}$$

for $i=0,\dots,j-1$. Then, for all $z\in\mathbb{D}$,

$$\begin{aligned}(\xi_0\dot\wedge\xi_1\dot\wedge\cdots\dot\wedge\xi_{j-1}\dot\wedge v_j)(z) &= \left(\xi_0\dot\wedge\frac{x_1}{h_1}\dot\wedge\cdots\dot\wedge\frac{x_{j-1}}{h_{j-1}}\dot\wedge v_j\right)(z)\\ &= \left(\frac{1}{h_1}\cdots\frac{1}{h_{j-1}}\xi_0\dot\wedge x_1\dot\wedge\cdots\dot\wedge x_{j-1}\dot\wedge v_j\right)(z).\end{aligned}$$

We obtain

$$(\xi_0\dot\wedge\xi_1\dot\wedge\cdots\dot\wedge\xi_{j-1}\dot\wedge v_j)(z) = \left(\frac{1}{h_1}\cdots\frac{1}{h_{j-1}}\xi_0\dot\wedge v_1\dot\wedge\cdots\dot\wedge v_{j-1}\dot\wedge v_j\right)(z), \text{ for all } z\in\mathbb{D},$$

due to pointwise linear dependence of ξ_k and $\xi_k\xi_i^*$ on $\mathbb{D}$, for all $k=0,\dots,i$. By Proposition 2.2.8,

$$\frac{1}{h_1}\cdots\frac{1}{h_{j-1}}\xi_0\dot\wedge v_1\dot\wedge\cdots\dot\wedge v_{j-1}\dot\wedge v_j$$

is analytic on $\mathbb{D}$. Moreover, by Proposition 2.2.14, since $\xi_0,\xi_1,\dots,\xi_{j-1}$ are pointwise orthogonal on $\mathbb{T}$,

$$\|\xi_0\dot\wedge\xi_1\dot\wedge\cdots\dot\wedge\xi_{j-1}\dot\wedge v_j\|_{L^2(\mathbb{T},\wedge^{j+1}\mathbb{C}^n)} < \infty.$$

Therefore

$$\xi_0\dot\wedge\xi_1\dot\wedge\cdots\dot\wedge\xi_{j-1}\dot\wedge v_j \in H^2(\mathbb{D},\wedge^{j+1}\mathbb{C}^n).$$

Let h_j be the scalar outer factor of $\xi_1\dot\wedge\xi_2\dot\wedge\cdots\dot\wedge\xi_{j-1}\dot\wedge v_j$. We define

$$x_j = (I_{\mathbb{C}^n} - \xi_0\xi_0^* - \cdots - \xi_{j-1}\xi_{j-1}^*)v_j \tag{3.34}$$

and

$$\xi_j = \frac{x_j}{h_j}.$$

We will show that $\{\xi_0(z),\cdots,\xi_{j-1}(z),\xi_j(z)\}$ is an orthonormal set in $\mathbb{C}^n$ almost everywhere on $\mathbb{T}$. We have

$$\xi_j = \frac{1}{h_j}v_j - \frac{1}{h_j}\xi_0\xi_0^* v_j - \cdots - \frac{1}{h_j}\xi_{j-1}\xi_{j-1}^* v_j,$$

and so, for all $i = 0, \ldots, j-1$,

$$\langle \xi_j(z), \xi_i(z)\rangle_{\mathbb{C}^n} = \frac{1}{h_j(z)}\xi_i^*(z)v_j(z) - \frac{1}{h_j(z)}\xi_i^*(z)\xi_0(z)\xi_0^*(z)v_1(z) - \cdots$$
$$-\frac{1}{h_j(z)}\xi_i^*(z)\xi_{j-1}(z)\xi_{j-1}^*(z)v_1(z)$$

almost everywhere on $\mathbb{T}$. Note that by the inductive hypothesis, for all $i, k = 0, 1, \cdots, j-1$ and for almost all $z \in \mathbb{T}$,

$$\xi_i^*(z)\xi_k(z) = \begin{cases} 0, & \text{for } i \neq k \\ 1, & \text{for } i = k \end{cases}.$$

Thus, for all $i = 0, \ldots, j-1$,

$$\langle \xi_j(z), \xi_i(z)\rangle_{\mathbb{C}^n} = \frac{1}{h_j(z)}\xi_i^*(z)v_j(z) - \frac{1}{h_j(z)}\xi_i^*(z)\xi_i(z)\xi_i^* v_j(z) = 0$$

almost everywhere on $\mathbb{T}$, and hence, by induction on j and for all integers $j = 0, \ldots, r-1$, $\{\xi_0(z), \cdots, \xi_{j-1}(z), \xi_j(z)\}$ is an orthogonal set in $\mathbb{C}^n$ for almost all $z \in \mathbb{T}$.

Let us show that

$$\xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge v_j(z) = \xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge x_j(z)$$

almost everywhere on $\mathbb{T}$.
Equation (3.34) yields

$$\begin{aligned}\xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge v_j(z) &= \xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge (x_j(z) + \xi_0(z)\xi_0^*(z)v_j(z) \\ &\qquad + \cdots + \xi_{j-1}(z)\xi_{j-1}^*(z)v_j(z)) \\ &= \xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge (x_j(z) + \xi_0(z)\langle v_j(z), \xi_0(z)\rangle_{\mathbb{C}^n} + \cdots \\ &\qquad + \cdots + \xi_{j-1}(z)\langle v_j(z), \xi_{j-1}(z)\rangle_{\mathbb{C}^n})\end{aligned}$$

almost everywhere on $\mathbb{T}$. Notice that, for all $i = 0, \cdots, j-1$, the vectors ξ_i and $z \mapsto \xi_i(z)\langle v_j(z), \xi_i(z)\rangle_{\mathbb{C}^n}$ are pointwise linearly dependent almost everywhere on $\mathbb{T}$. Thus for all $i = 0, \cdots, j-1$,

$$\xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge \xi_i(z)\langle v_{i+1}(z), \xi_i(z)\rangle_{\mathbb{C}^n} = 0$$

almost everywhere on $\mathbb{T}$.
Hence

$$\xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge v_j(z) = \xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge x_j(z) \quad \text{almost everywhere on } \mathbb{T}. \tag{3.35}$$

Next, we shall show that $\|\xi_j(z)\|_{\mathbb{C}^n} = 1$ for almost all $z \in \mathbb{T}$. Recall that h_j is the scalar outer factor of $\xi_1 \dot\wedge \xi_2 \dot\wedge \cdots \dot\wedge \xi_{j-1} \dot\wedge v_j$, and therefore

$$|h_j(z)| = \|\xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge v_j(z)\|_{\wedge^{j+1}\mathbb{C}^n} = \|\xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge x_j(z)\|_{\wedge^{j+1}\mathbb{C}^n}$$

almost everywhere on $\mathbb{T}$.

By the inductive hypothesis, $\{\xi_0(z), \cdots, \xi_{j-1}(z)\}$ is an orthonormal set in $\mathbb{C}^n$ for almost all $z \in \mathbb{T}$, hence, by Lemma 2.1.22,

$$\begin{aligned} |h_j(z)| &= \|\xi_0(z) \wedge \cdots \wedge \xi_{j-1}(z) \wedge x_j(z)\|_{\wedge^{j+1}\mathbb{C}^n} \\ &= \|x_j(z) - \sum_{i=0}^{j-1} \langle x_j(z), \xi_i(z)\rangle \xi_i(z)\|_{\mathbb{C}^n} \\ &= \|x_j(z)\|_{\mathbb{C}^n} \text{ almost everywhere on } \mathbb{T}. \end{aligned} \tag{3.36}$$

Thus

$$\|\xi_j(z)\|_{\mathbb{C}^n} = \frac{\|x_j(z)\|_{\mathbb{C}^n}}{|h_j(z)|} = 1$$

almost everywhere on $\mathbb{T}$, and hence, by induction on j and for all integers $j = 0, \ldots, r-1$, $\{\xi_0(z), \cdots, \xi_{j-1}(z), \xi_j(z)\}$ is an orthonormal set in $\mathbb{C}^n$ for almost all $z \in \mathbb{T}$.

Next, we will prove that the set $\{\bar\eta_i(z)\}_{i=0}^{j}$, defined in equations (3.28), is orthonormal. For $i = 0$, by equation (3.29), we have

$$\|\bar\eta_0(z)\|_{\mathbb{C}^m} = 1 \text{ almost everywhere on } \mathbb{T}. \tag{3.37}$$

Hence (iii) of Proposition 3.2.1 holds for $\{\bar\eta_i\}_{i=0}^{j}$ in the case $j = 0$. Let T_1 be given by equation (3.15). T_1 is assumed to be a compact operator, and if $T_1 \neq 0$, $v_1 \in H^2(\mathbb{D}, \mathbb{C}^n)$ and $w_1 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ are such that $(\xi_0 \dot\wedge v_1, \bar\eta_0 \dot\wedge w_1)$ is a Schmidt pair corresponding to $\|T_1\| = t_1$. Suppose h_1 is the scalar outer factor of $\xi_0 \dot\wedge v_1$. Let

$$y_1 = (I_{\mathbb{C}^m} - \bar\eta_0 \eta_0^T) w_1 = w_1 - \bar\eta_0 \eta_0^T w_1$$

and let

$$\eta_1(z) = \frac{\bar z \bar y_1(z)}{h_1(z)} \quad \text{almost everywhere on } \mathbb{T}.$$

Then

$$\bar\eta_1(z) = \frac{z y_0(z)}{\bar h_1(z)} = \frac{z w_1(z)}{\bar h_1(z)} - \frac{z \bar\eta_0(z) \eta_0^T(z) w_1(z)}{\bar h_1(z)} \quad \text{almost everywhere on } \mathbb{T}.$$

By equation (3.37), $\|\bar\eta_0(z)\|_{\mathbb{C}^m} = 1$ almost everywhere on $\mathbb{T}$. Hence

$$\begin{aligned}\langle \bar\eta_1(z), \bar\eta_0(z)\rangle_{\mathbb{C}^m} &= \eta_0^T(z)\bar\eta_1(z)\\ &= \frac{z\eta_0^T(z)w_1(z)}{\bar h_1(z)} - \frac{z\eta_0^T(z)\bar\eta_0(z)\eta_0^T(z)w_1(z)}{\bar h_1(z)}\\ &= \frac{z\eta_0^T(z)w_1(z)}{\bar h_1(z)} - \frac{z\langle\bar\eta_0(z),\bar\eta_0(z)\rangle_{\mathbb{C}^m}\eta_0^T(z)w_1(z)}{\bar h_1(z)}\\ &= \frac{z\eta_0^T(z)w_1(z)}{\bar h_1(z)} - \frac{z\eta_0^T(z)w_1(z)}{\bar h_1(z)}\\ &= 0 \quad \text{almost everywhere on } \mathbb{T}.\end{aligned}$$

Recall that h_1 is the scalar outer factor of $\xi_0\dot\wedge v_1$. By equation (3.33) and Proposition 3.2.24,

$$\|x_1(z)\|_{\mathbb{C}^n} = \|y_1(z)\|_{\mathbb{C}^m} = |h_1(z)|$$

almost everywhere on $\mathbb{T}$, thus

$$\|\bar\eta_1(z)\|_{\mathbb{C}^m} = \frac{\|zy_1(z)\|_{\mathbb{C}^m}}{|\bar h_1(z)|} = 1 \quad \text{almost everywhere on } \mathbb{T}.$$

Consequently, $\{\bar\eta_0(z), \bar\eta_1(z)\}$ is an orthonormal set in $\mathbb{C}^m$ for almost every $z \in \mathbb{T}$. Hence (iii) of Proposition 3.2.1 holds for $\{\bar\eta_i\}_{i=0}^{j}$ in the case $j = 1$.

Recursive step: Suppose the entities in equations (3.21) have been constructed and have the stated properties, for $i = 0, \dots, j-1$, and that $\{\bar\eta_i(z)\}_{i=0}^{j-1}$ is an orthonormal set almost everywhere on $\mathbb{T}$. Since by the inductive hypothesis T_j is a compact operator, there exist

$$v_j \in H^2(\mathbb{D}, \mathbb{C}^n), \quad w_j \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$$

such that

$$(\xi_1\dot\wedge\xi_2\dot\wedge\cdots\dot\wedge\xi_{j-1}\dot\wedge v_j, \bar\eta_0\dot\wedge\bar\eta_1\dot\wedge\cdots\dot\wedge\bar\eta_{j-1}\dot\wedge w_j)$$

is a Schmidt pair for T_j corresponding to $\|T_j\| = t_j$. By Proposition 3.2.2,

$$\xi_0\dot\wedge\cdots\dot\wedge\xi_{j-1}\dot\wedge v_j \in H^2(\mathbb{D}, \wedge^{j+1}\mathbb{C}^n).$$

Let h_j be the scalar outer factor of $\xi_0\dot\wedge\xi_1\dot\wedge\cdots\dot\wedge\xi_{j-1}\dot\wedge v_j$. We define

$$y_j = (I_{\mathbb{C}^m} - \bar\eta_0\eta_0^T - \cdots - \bar\eta_{j-1}\eta_{j-1}^T)w_j$$

and

$$\bar\eta_j = \frac{zy_j}{\bar h_j}.$$

Let us show that $\{\bar\eta_0(z), \dots, \bar\eta_j(z)\}$ is an orthonormal set in $\mathbb{C}^m$ almost everywhere on $\mathbb{T}$. We have

$$\bar\eta_j = \frac{zw_j}{\bar h_j} - \cdots - \frac{z\bar\eta_{j-1}\eta_{j-1}^T w_j}{\bar h_j}$$

and so, for $i = 0, \dots, j-1$,

$$\begin{aligned}\langle \bar\eta_j(z), \bar\eta_i(z)\rangle_{\mathbb{C}^m} &= \eta_i^T(z)\bar\eta_j(z)\\ &= \frac{z\eta_i^T(z)w_j(z)}{\bar h_j(z)} - \cdots - \frac{z\eta_i^T(z)\bar\eta_j(z)\eta_j^T(z)w_j(z)}{\bar h_j(z)}\end{aligned}$$

almost everywhere on $\mathbb{T}$.

Notice that, by the inductive hypothesis, for all $i, k = 0, \dots, j-1$ and for almost all $z \in \mathbb{T}$,

$$\eta_i^T(z)\bar\eta_k(z) = \begin{cases} 0, & \text{for } i \neq k \\ 1, & \text{for } i = k \end{cases}.$$

Hence, for all $i = 0, \dots, j-1$,

$$\langle \bar\eta_j(z), \bar\eta_i(z)\rangle_{\mathbb{C}^m} = \frac{z\eta_i^T(z)w_j(z)}{\bar h_j(z)} - \frac{z\eta_i^T(z)w_j(z)}{\bar h_j(z)} = 0$$

almost everywhere on $\mathbb{T}$. Thus by induction on j and for all integers $j = 0, \dots, r-1$, $\{\bar\eta_0(z), \dots, \bar\eta_j(z)\}$ is an orthogonal set in $\mathbb{C}^m$ almost everywhere on $\mathbb{T}$.

To complete the proof, we have to prove that $\|\bar\eta_j(z)\|_{\mathbb{C}^m} = 1$ for almost all $z \in \mathbb{T}$. Recall that h_j is the scalar outer factor of $\xi_0\dot\wedge\xi_1\dot\wedge\cdots\dot\wedge\xi_{j-1}\dot\wedge v_j$. By Proposition 3.2.57,

$$|h_j(z)| = \|x_j(z)\|_{\mathbb{C}^n} = \|y_j(z)\|_{\mathbb{C}^m}$$

almost everywhere on $\mathbb{T}$, thus

$$\|\bar\eta_j(z)\|_{\mathbb{C}^m} = \|\frac{zy_j(z)}{\bar h_j(z)}\|_{\mathbb{C}^m} = 1$$

almost everywhere on $\mathbb{T}$, and hence, by induction on j and for all integers $j = 0, \cdots, r-1$. $\{\bar\eta_0(z), \dots, \bar\eta_j(z)\}$ is an orthonormal set in $\mathbb{C}^m$ almost everywhere on $\mathbb{T}$.

Note that, for $j = 1, \dots, r-1$,

$$\begin{aligned}\bar\eta_0\dot\wedge\cdots\dot\wedge\bar\eta_{j-1}\dot\wedge y_j &= \bar\eta_0\dot\wedge\cdots\dot\wedge\bar\eta_{j-1}\dot\wedge(I_{\mathbb{C}^m} - \bar\eta_0\eta_0^T - \cdots - \bar\eta_{j-1}\eta_{j-1}^T)w_j\\ &= \bar\eta_0\dot\wedge\cdots\dot\wedge\bar\eta_{j-1}\dot\wedge w_j - \sum_{k=0}^{j-1}\bar\eta_0\dot\wedge\cdots\dot\wedge\bar\eta_{j-1}\dot\wedge\bar\eta_k\eta_k^T w_j\\ &= \bar\eta_0\dot\wedge\cdots\dot\wedge\bar\eta_{j-1}\dot\wedge w_j \qquad (3.38)\end{aligned}$$

on account of the pointwise linear dependence of $\bar\eta_k$ and $z \mapsto \bar\eta_k(z)\langle w_j(z), \bar\eta_k(z)\rangle_{\mathbb{C}^m}$ almost everywhere on $\mathbb{T}$. □

3.2.3 The closed subspace X_{j+1} of $H^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^n)$

Notice that, although $x_0 \in H^2(\mathbb{D}, \mathbb{C}^n)$ and ξ_0 is inner, x_i and ξ_i might not be in $H^2(\mathbb{D}, \mathbb{C}^n)$ in general for $i = 1, \cdots, \min(m,n)-2$. However, for every $x \in H^2(\mathbb{D}, \mathbb{C}^n)$, the pointwise wedge product

$$\xi_0\dot{\wedge}\cdots\dot{\wedge}\xi_j\dot{\wedge}x$$

is an element of $H^2(\mathbb{D},\wedge^{j+2}\mathbb{C}^n)$ as the following proposition asserts.

Proposition 3.2.2. *Let $G\in H^\infty(\mathbb{D},\mathbb{C}^{m\times n})+C(\mathbb{T},\mathbb{C}^{m\times n})$, and let $j\leq n-2$. Let the vector-valued functions $\xi_0,\xi_1,\cdots,\xi_j$ be constructed after applying steps $0,\dots,j$ of the algorithm above and be given by equations (3.28). Then*

$$\xi_0\dot{\wedge}\dots\dot{\wedge}\xi_j\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^n)$$

is a subset of $H^2(\mathbb{D},\wedge^{j+2}\mathbb{C}^n)$.

Proof. For $j=0$, since $G\in H^\infty(\mathbb{D},\mathbb{C}^{m\times n})+C(\mathbb{T},\mathbb{C}^{m\times n})$, the Hankel operator H_G is compact. There exist $x_0\in H^2(\mathbb{D},\mathbb{C}^n), y_0\in H^2(\mathbb{D},\mathbb{C}^m)^\perp$ such that (x_0,y_0) is a Schmidt pair for the Hankel operator H_G corresponding to the singular value $\|H_G\|$. By Lemma 3.1.12, x_0,y_0 admit the inner-outer factorisations

$$x_0=\xi_0h_0,\quad \bar{z}\bar{y}_0=\eta_0h_0$$

for some inner $\xi_0\in H^\infty(\mathbb{D},\mathbb{C}^n),\eta_0\in H^\infty(\mathbb{D},\mathbb{C}^m)$ and some scalar outer $h_0\in H^2(\mathbb{D},\mathbb{C})$. Then, by Proposition 2.2.13, $\xi_0\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^n)\subset H^2(\mathbb{D},\wedge^2\mathbb{C}^n)$.

Let us now consider the case where $j=1$. By definition,

$$X_1=\xi_0\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^n),\quad Y_1=\bar{\eta}_0\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^m)^\perp$$

and, by the inductive hypothesis, $T_1\colon X_1\to Y_1$ given by equation (3.15) is a compact operator. Suppose $\|T_1\|\neq 0$ and let $(\xi_0\dot{\wedge}v_1,\bar{\eta}_0\dot{\wedge}w_1)$ be a Schmidt pair corresponding to $\|T_1\|$, where $v_1\in H^2(\mathbb{D},\mathbb{C}^n)$ and $w_1\in H^2(\mathbb{D},\mathbb{C}^m)^\perp$. We define

$$x_1=(I_{\mathbb{C}^n}-\xi_0\xi_0^*)v_1.$$

Note that, by Proposition 2.2.13, $\xi_0\dot{\wedge}v_1\in H^2(\mathbb{D},\wedge^2\mathbb{C}^n)$. Let $h_1\in H^2(\mathbb{D},\mathbb{C})$ be the scalar outer factor of $\xi_0\dot{\wedge}v_1\in H^2(\mathbb{D},\wedge^2\mathbb{C}^n)$. Then we define

$$\xi_1=\frac{x_1}{h_1}.$$

Note that ξ_0 and $\xi_0\xi_0^*v_1$ are pointwise linearly dependent on $\mathbb{D}$, since $\xi_0^*v_1$ is a mapping from $\mathbb{D}$ to $\mathbb{C}$. Thus, for all $x\in H^2(\mathbb{D},\mathbb{C}^n)$ and $z\in\mathbb{D}$, we have

$$(\xi_0\dot{\wedge}\xi_1\dot{\wedge}x)(z)=\xi_0(z)\wedge\xi_1(z)\wedge x(z)=\xi_0(z)\wedge\frac{x_1(z)}{h_1(z)}\wedge x(z),$$

and by substituting the value of x_1, we get

$$\begin{aligned}\xi_0(z) \wedge \frac{x_1(z)}{h_1(z)} \wedge x(z) &= \frac{1}{h_1(z)}\xi_0(z) \wedge (v_1(z) - \xi_0(z)\xi_0(z)^* v_1(z)) \wedge x(z)\\ &= \frac{1}{h_1(z)}\xi_0(z) \wedge v_1(z) \wedge x(z) - \frac{1}{h_1(z)}\xi_0(z) \wedge \xi_0(z)\xi_0(z)^* v_1(z) \wedge x(z)\\ &= \left(\frac{1}{h_1}\xi_0 \dot{\wedge} v_1 \dot{\wedge} x\right)(z).\end{aligned}$$

Note that $v_1 \in H^2(\mathbb{D}, \mathbb{C}^n)$, $\xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$ and $h_1 \in H^2(\mathbb{D}, \mathbb{C})$ is the scalar outer factor of $\xi_0 \dot{\wedge} v_1$. By Proposition 2.2.8, for every $x \in H^2(\mathbb{D}, \mathbb{C}^n)$,

$$\frac{1}{h_1}\xi_0 \dot{\wedge} v_1 \dot{\wedge} x$$

is analytic on $\mathbb{D}$. By Proposition 2.2.14, since ξ_0 and ξ_1 are pointwise orthogonal on $\mathbb{T}$,

$$\|\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} x\|_{L^2(\mathbb{T}, \wedge^3\mathbb{C}^n)} < \infty.$$

Hence

$$\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} x \in \frac{1}{h_1}\xi_0 \dot{\wedge} v_1 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n) \subset H^2(\mathbb{D}, \wedge^3\mathbb{C}^n).$$

Recursive step: suppose we have constructed vector-valued functions $\xi_0, \dots, \xi_{j-1}$, $\eta_0, \dots, \eta_{j-1}$, spaces X_j, Y_j and a compact operator $T_j \colon X_j \to Y_j$ after applying steps $0, \dots, j$ of the algorithm from Section 3.2.1 satisfying

$$\xi_0 \dot{\wedge} \cdots \dot{\wedge} \xi_{j-1} \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n) \subset H^2(\mathbb{D}, \wedge^{j+1}\mathbb{C}^n). \tag{3.39}$$

Since T_j is a compact operator, there exist vector-valued functions $v_j \in H^2(\mathbb{D}, \mathbb{C}^n)$, $w_j \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ such that

$$(\xi_0 \dot{\wedge} \cdots \dot{\wedge} \xi_{j-1} \dot{\wedge} v_j, \bar{\eta}_0 \dot{\wedge} \dots \dot{\wedge} \bar{\eta}_{j-1} \dot{\wedge} w_j)$$

is a Schmidt pair for T_j corresponding to $\|T_j\|$. Define

$$x_j = (I_n - \xi_0\xi_0^* - \cdots - \xi_{j-1}\xi_{j-1}^*)v_j. \tag{3.40}$$

By assumption, $\xi_0 \dot{\wedge} \cdots \dot{\wedge} \xi_{j-1} \dot{\wedge} v_j$ lies in $H^2(\mathbb{D}, \wedge^{j+1}\mathbb{C}^n)$. Let $h_j \in H^2(\mathbb{D}, \mathbb{C})$ be the scalar outer factor of $\xi_0 \dot{\wedge} \cdots \dot{\wedge} \xi_{j-1} \dot{\wedge} v_j$. Define $\xi_j = \frac{x_j}{h_j}$. Note that ξ_i and $z \mapsto \xi_i(z)\langle v_j(z), \xi_i(z)\rangle_{\mathbb{C}^n}$ are pointwise linearly dependent almost everywhere on $\mathbb{T}$ for $i = 0, \dots, j-1$. Thus, for all $x \in H^2(\mathbb{D}, \mathbb{C}^n)$ and all $z \in \mathbb{D}$,

$$
\begin{aligned}
(\xi_0\dot\wedge\cdots\dot\wedge\xi_{j-1}\dot\wedge\xi_j\dot\wedge x)(z) &= (\xi_0\dot\wedge\cdots\dot\wedge\xi_{j-1}\dot\wedge\frac{x_j}{h_j}\dot\wedge x)(z)\\
&= \xi_0(z)\wedge\cdots\wedge\xi_{j-1}(z)\wedge\frac{1}{h_j(z)}\bigg(v_j(z)-\xi_0(z)\xi_0^*(z)v_j(z)-\cdots\\
&\qquad\qquad -\xi_{j-1}(z)\xi_{j-1}^*(z)v_j(z)\bigg)\wedge x(z)\\
&= \xi_0(z)\wedge\cdots\wedge\xi_{j-1}(z)\wedge\frac{1}{h_j(z)}v_j(z)\wedge x(z)\\
&-\frac{1}{h_j(z)}\sum_{i=0}^{j-1}\xi_0(z)\wedge\ldots\wedge\xi_{j-1}(z)\wedge\xi_i(z)\xi_i^*(z)v_j(z)\wedge x(z)\\
&= \left(\frac{1}{h_j}\xi_0\dot\wedge\cdots\dot\wedge\xi_{j-1}\dot\wedge v_j\dot\wedge x\right)(z). \qquad (3.41)
\end{aligned}
$$

Recall that, for $i=0,\ldots,j-1$, by the algorithm from Section 3.2.1,

$$x_i=(I_n-\xi_0\xi_0^*-\cdots-\xi_{i-1}\xi_{i-1}^*)v_i$$

and

$$\xi_i=\frac{x_i}{h_i}.$$

By equation (3.41), for all $z\in\mathbb{D}$,

$$(\xi_0\dot\wedge\cdots\dot\wedge\xi_{j-1}\dot\wedge\xi_j\dot\wedge x)(z)=\left(\frac{1}{h_j}\xi_0\dot\wedge\cdots\dot\wedge\xi_{j-1}\dot\wedge v_j\dot\wedge x\right)(z).$$

Substituting $\frac{x_i}{h_i}$ for ξ_i in the latter equation, where x_i are given by equation (3.40) for all $i=1,\ldots,j-1$, we obtain

$$(\xi_0\dot\wedge\cdots\dot\wedge\xi_{j-1}\dot\wedge\xi_j\dot\wedge x)(z)=\left(\frac{1}{h_1}\frac{1}{h_2}\cdots\frac{1}{h_j}\xi_0\dot\wedge v_1\dot\wedge\cdots\dot\wedge v_{j-1}\dot\wedge v_j\dot\wedge x\right)(z),\ z\in\mathbb{D}$$

on account of the pointwise linear dependence of ξ_k and $z\mapsto\xi_k(z)\langle v_k(z),\xi_i(z)\rangle_{\mathbb{C}^n}$ almost everywhere on $\mathbb{T}$ for $k=0,\ldots,i$. By Proposition 2.2.8, for every $x\in H^2(\mathbb{D},\mathbb{C}^n)$,

$$\frac{1}{h_1}\frac{1}{h_2}\cdots\frac{1}{h_j}\xi_0\dot\wedge v_1\dot\wedge\cdots\dot\wedge v_j\dot\wedge x$$

is analytic on $\mathbb{D}$. By Proposition 2.2.14, since $\xi_0,\xi_1,\ldots,\xi_j$ are pointwise orthogonal on $\mathbb{T}$,

$$\|\xi_0\dot\wedge\xi_1\dot\wedge\cdots\dot\wedge\xi_j\dot\wedge x\|_{L^2(\mathbb{T},\wedge^{j+2}\mathbb{C}^n)}<\infty.$$

Thus, for every $x\in H^2(\mathbb{D},\mathbb{C}^n)$,

$$\xi_0\dot\wedge\xi_1\dot\wedge\cdots\dot\wedge\xi_j\dot\wedge x\in H^2(\mathbb{D},\wedge^{j+2}\mathbb{C}^n)$$

and the claim has been proved. □

Proposition 3.2.3. *In the notation of Proposition 3.2.2,*

$$\xi_0 \dot{\wedge} \dots \dot{\wedge} \xi_j \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n)$$

is a closed subspace of $H^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^n)$.

Proof. Let us first show that $\xi_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n)$ is a closed subspace of $H^2(\mathbb{D}, \wedge^2\mathbb{C}^n)$. Observe that, by Proposition 2.2.13, $\xi_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n) \subset H^2(\mathbb{D}, \wedge^2\mathbb{C}^n)$. Let

$$\Xi_0 = \{f \in H^2(\mathbb{D}, \mathbb{C}^n) : \langle f(z), \xi_0(z)\rangle_{\mathbb{C}^n} = 0 \quad \text{almost everywhere on } \mathbb{T}\}.$$

Consider a vector-valued function $w \in H^2(\mathbb{D}, \mathbb{C}^n)$. For all $z \in \mathbb{D}$, we may write w as

$$w(z) = w(z) - \langle w(z), \xi_0(z)\rangle_{\mathbb{C}^n}\xi_0(z) + \langle w(z), \xi_0(z)\rangle_{\mathbb{C}^n}\xi_0(z).$$

Then, for all $w \in H^2(\mathbb{D}, \mathbb{C}^n)$ and for all $z \in \mathbb{D}$,

$$\begin{aligned}(\xi_0 \dot{\wedge} w)(z) \quad &= \xi_0(z) \wedge \big(w(z) - \langle w(z), \xi_0(z)\rangle_{\mathbb{C}^n}\xi_0(z) + \langle w(z), \xi_0(z)\rangle_{\mathbb{C}^n}\xi_0(z)\big) \\ &= \xi_0(z) \wedge \big(w(z) - \langle w(z), \xi_0(z)\rangle_{\mathbb{C}^n}\xi_0(z)\big)\end{aligned}$$

due to the pointwise linear dependence of ξ_0 and $z \mapsto \langle w(z), \xi_0(z)\rangle_{\mathbb{C}^n}\xi_0(z)$ almost everywhere on $\mathbb{T}$. Note that

$$w(z) - \langle w(z), \xi_0(z)\rangle_{\mathbb{C}^n}\xi_0(z) \in \Xi_0,$$

thus

$$\xi_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n) \subset \xi_0 \dot{\wedge} \Xi_0.$$

By Corollary 2.2.29, Ξ_0 is a closed subspace of $H^2(\mathbb{D}, \mathbb{C}^n)$, hence

$$\xi_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n) \supset \xi_0 \dot{\wedge} \Xi_0,$$

and so,

$$\xi_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n) = \xi_0 \dot{\wedge} \Xi_0.$$

Consider the mapping

$$C_{\xi_0} : \Xi_0 \to \xi_0 \dot{\wedge} \Xi_0$$

given by

$$C_{\xi_0} w = \xi_0 \dot{\wedge} w$$

for all $w \in \Xi_0$. Notice that, by Proposition 3.2.1, $\|\xi_0(e^{i\theta})\|^2_{\mathbb{C}^n} = 1$ for almost every $e^{i\theta} \in \mathbb{T}$. Therefore, for any $w \in \Xi_0$, we have

$$\begin{aligned}
\|\xi_0\dot{\wedge}w\|^2_{L^2(\mathbb{T},\wedge^2\mathbb{C}^n)} &= \frac{1}{2\pi}\int_0^{2\pi}\langle \xi_0\dot{\wedge}w, \xi_0\dot{\wedge}w\rangle(e^{i\theta})d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(\|\xi_0(e^{i\theta})\|^2_{\mathbb{C}^n}\|w(e^{i\theta})\|^2_{\mathbb{C}^n} - |\langle w(e^{i\theta}), \xi_0(e^{i\theta})\rangle|^2\right)\, d\theta \\
&= \|w\|^2_{L^2(\mathbb{T},\mathbb{C}^n)},
\end{aligned}$$

since w is pointwise orthogonal to ξ_0 almost everywhere on $\mathbb{T}$. Thus the mapping

$$C_{\xi_0} \colon \Xi_0 \to \xi_0\dot{\wedge}\Xi_0$$

is an isometry. Furthermore, $C_{\xi_0} \colon \Xi_0 \to \xi_0\dot{\wedge}\Xi_0$ is a surjective mapping, thus Ξ_0 and $\xi_0\dot{\wedge}\Xi_0$ are isometrically isomorphic. Since Ξ_0 is a closed subspace of $H^2(\mathbb{D},\mathbb{C}^n)$, hence complete, the space $\xi_0\dot{\wedge}\Xi_0$ is complete. Therefore $\xi_0\dot{\wedge}\Xi_0$ is a closed subspace of $H^2(\mathbb{D},\wedge^2\mathbb{C}^n)$, and thus $\xi_0\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^n)$ is a closed subspace of $H^2(\mathbb{D},\wedge^2\mathbb{C}^n)$.

To prove that $\xi_0\dot{\wedge}\dots\dot{\wedge}\xi_j\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^n)$ is a closed subspace of $H^2(\mathbb{D},\wedge^{j+2}\mathbb{C}^n)$, let us consider

$$\Xi_j = \{f \in H^2(\mathbb{D},\mathbb{C}^n) : \langle f(z), \xi_i(z)\rangle_{\mathbb{C}^n} = 0, \text{ for } i = 0, \cdots, j\}$$

which is the pointwise orthogonal complement of $\xi_0, \dots, \xi_j$ in $H^2(\mathbb{D},\mathbb{C}^n)$. Let $\psi \in H^2(\mathbb{D},\mathbb{C}^n)$. We may write ψ as

$$\psi(z) = \psi(z) - \sum_{i=0}^{j}\langle\psi(z), \xi_i(z)\rangle_{\mathbb{C}^n}\xi_i(z) + \sum_{i=0}^{j}\langle\psi(z), \xi_i(z)\rangle_{\mathbb{C}^n}\xi_i(z).$$

Then, for all $\psi \in H^2(\mathbb{D},\mathbb{C}^n)$ and for almost all $z \in \mathbb{T}$,

$$(\xi_0\dot{\wedge}\cdots\dot{\wedge}\xi_j\dot{\wedge}\psi)(z) = \xi_0(z)\wedge\cdots\wedge\left(\psi(z) - \sum_{i=0}^{j}\langle\psi(z), \xi_i(z)\rangle_{\mathbb{C}^n}\xi_i(z)\right)$$

due to the pointwise linear dependence of ξ_k and $z \mapsto \xi_k(z)\langle\psi(z), \xi_k(z)\rangle_{\mathbb{C}^n}$ almost everywhere on $\mathbb{T}$. Notice that $\left(\psi(z) - \sum_{i=0}^{j}\langle\psi(z), \xi_i(z)\rangle_{\mathbb{C}^n}\xi_i(z)\right)$ lies in Ξ_j, thus

$$\xi_0\dot{\wedge}\cdots\dot{\wedge}\xi_j\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^n) \subseteq \xi_0\dot{\wedge}\cdots\dot{\wedge}\xi_j\dot{\wedge}\Xi_j.$$

The reverse inclusion holds by the definition of Ξ_j, hence

$$\xi_0\dot{\wedge}\cdots\dot{\wedge}\xi_j\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^n) = \xi_0\dot{\wedge}\cdots\dot{\wedge}\xi_j\dot{\wedge}\Xi_j.$$

Consequently, in order to prove the proposition it suffices to show that $\xi_0\dot{\wedge}\cdots\dot{\wedge}\xi_j\dot{\wedge}\Xi_j$ is a closed subspace of $H^2(\mathbb{D},\wedge^{j+2}\mathbb{C}^n)$. By Corollary 2.2.29, Ξ_j is a closed subspace of $H^2(\mathbb{D},\mathbb{C}^n)$, being a finite intersection of closed subspaces. For any $f \in \Xi_j$,

$$\|\xi_0\dot\wedge\xi_1\dot\wedge\cdots\dot\wedge\xi_j\dot\wedge f\|^2_{L^2(\mathbb{T},\wedge^{j+2}\mathbb{C}^n)}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \det\begin{pmatrix} \|\xi_0(e^{i\theta})\|^2_{\mathbb{C}^n} & \cdots & \langle \xi_0(e^{i\theta}), f(e^{i\theta})\rangle_{\mathbb{C}^n} \\ \langle \xi_1(e^{i\theta}), \xi_0(e^{i\theta})\rangle_{\mathbb{C}^n} & \|\xi_1(e^{i\theta})\|^2_{\mathbb{C}^n} & \cdots \\ \vdots & & \ddots \\ \langle f(e^{i\theta}), \xi_0(e^{i\theta})\rangle_{\mathbb{C}^n} & \cdots & \|f(e^{i\theta})\|^2_{\mathbb{C}^n} \end{pmatrix} d\theta.$$

Note that f and ξ_i are pointwise orthogonal almost everywhere on $\mathbb{T}$, and, by Proposition 3.2.1, $\{\xi_0(z),\dots,\xi_j(z)\}$ is an orthonormal set for almost every $z\in\mathbb{T}$. Hence

$$\begin{aligned}\|\xi_0\dot\wedge\xi_1\dot\wedge\cdots\dot\wedge\xi_j\dot\wedge f\|^2_{L^2(\mathbb{T},\wedge^{j+2}\mathbb{C}^n)} &= \frac{1}{2\pi}\int_0^{2\pi}\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & & \ddots \\ 0 & 0 & \|f(e^{i\theta})\|^2_{\mathbb{C}^n}\end{pmatrix} d\theta \\ &= \|f\|^2_{L^2(\mathbb{T},\mathbb{C}^n)}.\end{aligned}$$

Thus

$$\xi_0\dot\wedge\xi_1\dot\wedge\cdots\dot\wedge\xi_j\dot\wedge\cdot\colon \Xi_j \to \xi_0\dot\wedge\xi_1\dot\wedge\cdots\dot\wedge\xi_j\dot\wedge\Xi_j$$

is an isometry. Furthermore

$$(\xi_0\dot\wedge\xi_1\dot\wedge\cdots\dot\wedge\xi_j\dot\wedge\cdot)\colon \Xi_j \to \xi_0\dot\wedge\xi_1\dot\wedge\cdots\dot\wedge\xi_j\dot\wedge\Xi_j$$

is a surjective mapping, thus Ξ_j and $\xi_0\dot\wedge\cdots\dot\wedge\xi_j\dot\wedge\Xi_j$ are isometrically isomorphic. Therefore, since Ξ_j is a closed subspace of $H^2(\mathbb{D},\mathbb{C}^n)$, the space $\xi_0\dot\wedge\cdots\dot\wedge\xi_j\dot\wedge\Xi_j$ is a closed subspace of $H^2(\mathbb{D},\wedge^{j+2}\mathbb{C}^n)$. Hence

$$\xi_0\dot\wedge\cdots\dot\wedge\xi_j\dot\wedge H^2(\mathbb{D},\mathbb{C}^n)$$

is a closed subspace of $H^2(\mathbb{D},\wedge^{j+2}\mathbb{C}^n)$. □

3.2.4 The closed subspace Y_{j+1} of $H^2(\mathbb{D},\wedge^{j+2}\mathbb{C}^m)^\perp$.

Proposition 3.2.4. *Given $\bar\eta_0 = \frac{\bar z \bar y_0}{\bar h_0}$ as constructed in the algorithm in Section* 3.2.1, *the space $\bar\eta_0\dot\wedge H^2(\mathbb{D},\mathbb{C}^m)^\perp$ is a closed subspace of $H^2(\mathbb{D},\wedge^2\mathbb{C}^m)^\perp$.*

Proof. As in Proposition 3.2.2, one can show that

$$\eta_0\dot\wedge zH^2(\mathbb{D},\mathbb{C}^m)\subset zH^2(\mathbb{D},\wedge^2\mathbb{C}^m)$$

and therefore

$$\bar\eta_0\dot\wedge\bar z\overline{H^2(\mathbb{D},\mathbb{C}^m)}\subset\bar z\overline{H^2(\mathbb{D},\wedge^2\mathbb{C}^m)}.$$

Hence

$$\bar\eta_0\dot\wedge H^2(\mathbb{D},\mathbb{C}^m)^\perp\subset H^2(\mathbb{D},\wedge^2\mathbb{C}^m)^\perp.$$

By virtue of the fact that complex conjugation is a unitary operator on $L^2(\mathbb{T}, \mathbb{C}^m)$, an equivalent statement to Proposition 3.2.4 is that $\eta_0 \dot{\wedge} z H^2(\mathbb{D}, \mathbb{C}^m)$ is a closed subspace of $zH^2(\mathbb{D}, \wedge^2\mathbb{C}^m)$.

Let

$$V = \{f \in zH^2(\mathbb{D}, \mathbb{C}^m) \ : \ \langle f(z), \eta_0(z)\rangle_{\mathbb{C}^m} = 0 \quad \text{for almost all } z \in \mathbb{T}\}$$

be the pointwise orthogonal complement of η_0 in $zH^2(\mathbb{D}, \mathbb{C}^m)$.

Consider $g \in zH^2(\mathbb{D}, \mathbb{C}^m)$. We may write g as

$$g(z) = g(z) - \langle g(z), \eta_0(z)\rangle_{\mathbb{C}^m} \cdot \eta_0(z) + \langle g(z), \eta_0(z)\rangle_{\mathbb{C}^m} \cdot \eta_0(z)$$

for every $z \in \mathbb{D}$. Then, for all $g \in zH^2(\mathbb{D}, \mathbb{C}^m)$ and for all $z \in \mathbb{D}$,

$$(\eta_0 \dot{\wedge} g)(z) = \eta_0(z) \wedge [g(z) - \langle g(z), \eta_0(z)\rangle_{\mathbb{C}^m} \eta_0(z)]$$

on account of the pointwise linear dependence of η_0 and $\langle g, \eta_0\rangle_{H^2(\mathbb{D},\mathbb{C}^m)}\eta_0$ on $\mathbb{D}$.

Note that $g(z) - \langle g(z), \eta_0(z)\rangle_{\mathbb{C}^m}\eta_0(z) \in V$, thus

$$\eta_0 \dot{\wedge} z H^2(\mathbb{D}, \mathbb{C}^m) \subset \eta_0 \dot{\wedge} V.$$

The reverse inclusion is obvious, hence

$$\eta_0 \dot{\wedge} z H^2(\mathbb{D}, \mathbb{C}^m) = \eta_0 \dot{\wedge} V.$$

To prove the proposition, it suffices to show that

$$\eta_0 \dot{\wedge} V$$

is a closed subspace of

$$zH^2(\mathbb{D}, \wedge^2\mathbb{C}^m).$$

Consider the mapping

$$C_{\eta_0} \colon V \to \eta_0 \dot{\wedge} V$$

defined by

$$C_{\eta_0}\nu = \eta_0 \dot{\wedge} \nu$$

for all $\nu \in V$. Notice that, by Proposition 3.2.1, $\|\eta_0(e^{i\theta})\|^2_{\mathbb{C}^m} = 1$ for almost every $e^{i\theta} \in \mathbb{T}$. Then, for any $\upsilon \in V$, we have

$$\|\eta_0 \dot{\wedge} \upsilon\|^2_{L^2(\mathbb{T}, \wedge^2\mathbb{C}^m)} = \frac{1}{2\pi} \int_0^{2\pi} \langle \eta_0 \dot{\wedge} \upsilon, \eta_0 \dot{\wedge} \upsilon\rangle(e^{i\theta}) d\theta,$$

which is in turn equal to

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\|\eta_0(e^{i\theta})\|^2_{\mathbb{C}^m}\|\upsilon(e^{i\theta})\|^2_{\mathbb{C}^m}-|\langle\upsilon(e^{i\theta}),\eta_0(e^{i\theta})\rangle|^2\right)\,d\theta=\|\upsilon\|^2_{L^2(\mathbb{T},\mathbb{C}^m)},$$

since υ is pointwise orthogonal to η_0 almost everywhere on $\mathbb{T}$. Thus the mapping $C_{\eta_0}\colon V\to\eta_0\dot{\wedge}V$ is an isometry.
Note that by Corollary 2.2.29, V is a closed subspace of $zH^2(\mathbb{D},\mathbb{C}^m)$. Furthermore,

$$C_{\eta_0}\colon V\to\eta_0\dot{\wedge}V$$

is a surjective mapping, thus V and $\eta_0\dot{\wedge}V$ are isometrically isomorphic. Since V is a closed subspace of $zH^2(\mathbb{D},\mathbb{C}^m)$, the space $\eta_0\dot{\wedge}V$ is complete and therefore a closed subspace of $zH^2(\mathbb{D},\wedge^2\mathbb{C}^m)$. Hence $\bar{\eta}_0\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^m)^\perp$ is a closed subspace of $H^2(\mathbb{D},\wedge^2\mathbb{C}^m)^\perp$. □

Corollary 3.2.5. *The orthogonal projection P_{Y_1} from $L^2(\mathbb{T},\wedge^2\mathbb{C}^m)$ onto $\bar{\eta}_0\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^m)^\perp$ is well-defined.*

Proof. By Proposition 2.2.25, $H^2(\mathbb{D},\wedge^2\mathbb{C}^m)$ can be identified with a closed subspace of $L^2(\mathbb{T},\wedge^2\mathbb{C}^m)$, thus we have

$$H^2(\mathbb{D},\wedge^2\mathbb{C}^m)^\perp=L^2(\mathbb{T},\wedge^2\mathbb{C}^m)\ominus H^2(\mathbb{D},\wedge^2\mathbb{C}^m).$$

Now the assertion follows immediately from Proposition 3.2.4. □

Proposition 3.2.6. *Let $0\le j\le m-2$. Let the functions $\bar{\eta}_i$ be given by equations* (3.28) *in the algorithm from Section* 3.2.1, *that is, $\bar{\eta}_i=\dfrac{zy_i}{\bar{h}_i}$ for all $i=0,\cdots,j$. Then, the space*

$$\bar{\eta}_0\dot{\wedge}\bar{\eta}_1\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_j\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^m)^\perp$$

is a closed linear subspace of $H^2(\mathbb{D},\wedge^{j+2}\mathbb{C}^m)^\perp$.

Proof. First let us show that, for every $x\in H^2(\mathbb{D},\mathbb{C}^m)$,

$$\eta_0\dot{\wedge}\eta_1\dot{\wedge}\cdots\dot{\wedge}\eta_j\dot{\wedge}zx\in zH^2(\mathbb{D},\wedge^{j+2}\mathbb{C}^m).$$

Recall that

$$y_j=(I_m-\bar{\eta}_0\eta_0^T-\cdots-\bar{\eta}_{j-1}\eta_{j-1}^T)w_j$$

and

$$\eta_0\dot{\wedge}\cdots\dot{\wedge}\eta_{j-1}\dot{\wedge}\bar{z}\bar{y}_j=\eta_0\dot{\wedge}\cdots\dot{\wedge}\eta_{j-1}\dot{\wedge}(\bar{z}\bar{w}_j-\sum_{i=0}^{j-1}\eta_i\eta_i^*\bar{z}\bar{w}_j)=\eta_0\dot{\wedge}\cdots\dot{\wedge}\eta_{j-1}\dot{\wedge}\bar{z}\bar{w}_j \tag{3.42}$$

because of the pointwise linear dependence of η_i and $\eta_i\eta_i^*\bar{z}\bar{w}_{j+1}$ on $\mathbb{D}$.
By Proposition 3.2.57,

$$|h_i(z)|=\|y_i(z)\|_{\mathbb{C}^m}$$

almost everywhere on $\mathbb{T}$.
Substituting $\eta_i = \frac{\bar{z}\bar{y}_i}{h_i}$ for all $i = 0, \dots, j-1$ in equation (3.42), we obtain

$$\eta_0 \dot\wedge \cdots \dot\wedge \eta_{j-1} \dot\wedge \bar{z}\bar{y}_j = \frac{1}{h_0}\frac{1}{h_1}\cdots\frac{1}{h_j}\eta_0 \dot\wedge \bar{z}\bar{w}_1 \dot\wedge \cdots \dot\wedge \bar{z}\bar{w}_j.$$

Observe that, by Proposition 2.2.8, for every $x \in H^2(\mathbb{D}, \mathbb{C}^m)$,

$$\frac{1}{h_0}\frac{1}{h_1}\cdots\frac{1}{h_j}\eta_0 \dot\wedge \bar{z}\bar{w}_1 \dot\wedge \cdots \dot\wedge \bar{z}\bar{w}_j \dot\wedge zx$$

is analytic on $\mathbb{D}$. By Proposition 2.2.14, for all $x \in H^2(\mathbb{D}, \mathbb{C}^m)$, since $\eta_0, \cdots, \eta_j$ are pointwise orthogonal on $\mathbb{T}$,

$$\|\eta_0 \dot\wedge \eta_1 \dot\wedge \cdots \dot\wedge \eta_j \dot\wedge zx\|_{L^2(\mathbb{T}, \wedge^{j+2}\mathbb{C}^m)} < \infty.$$

Hence, for every $x \in H^2(\mathbb{D}, \mathbb{C}^m)$,

$$\eta_0 \dot\wedge \eta_1 \dot\wedge \cdots \dot\wedge \eta_j \dot\wedge zx = z\frac{1}{h_0}\frac{1}{h_1}\cdots\frac{1}{h_j}\eta_0 \dot\wedge \bar{z}\bar{w}_1 \dot\wedge \cdots \dot\wedge \bar{z}\bar{w}_j \dot\wedge x$$

is in $zH^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^m)$.
Taking complex conjugates, we infer that

$$Y_{j+1} \stackrel{\text{def}}{=} \bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_{j-1} \dot\wedge \bar\eta_j \dot\wedge H^2(\mathbb{D}, \mathbb{C}^m)^\perp \subset H^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^m)^\perp.$$

Let us prove that Y_{j+1} is a closed linear subspace of $H^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^m)^\perp$. Since complex conjugation is a unitary operator on $L^2(\mathbb{T}, \mathbb{C}^m)$, an equivalent statement to the above is that

$$\eta_0 \dot\wedge \eta_1 \dot\wedge \cdots \dot\wedge \eta_j \dot\wedge zH^2(\mathbb{D}, \mathbb{C}^m)$$

is a closed linear subspace of $zH^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^m)$.
Let

$$V_j = \{\varphi \in zH^2(\mathbb{D}, \mathbb{C}^m) \ : \ \langle \varphi(z), \eta_i(z)\rangle_{\mathbb{C}^m} = 0, \text{ for } i = 0, \cdots, j\}$$

be the pointwise orthogonal complement of $\eta_0, \cdots, \eta_j$ in $zH^2(\mathbb{D}, \mathbb{C}^m)$. Consider $f \in zH^2(\mathbb{D}, \mathbb{C}^m)$. We may write f as

$$f(z) = f(z) - \sum_{i=0}^{j} \langle f(z), \eta_i(z)\rangle \eta_i(z) + \sum_{i=0}^{j} \langle f(z), \eta_i(z)\rangle \eta_i(z).$$

Then, for all $f \in zH^2(\mathbb{D}, \mathbb{C}^m)$ and for almost all $z \in \mathbb{T}$,

$$(\eta_0 \dot\wedge \eta_1 \dot\wedge \cdots \dot\wedge \eta_j \dot\wedge f)(z) = \eta_0(z) \wedge \eta_1(z) \wedge \cdots \wedge \eta_j(z) \wedge \left(f(z) - \sum_{i=0}^{j} \langle f(z), \eta_i(z)\rangle \eta_i(z)\right).$$

Notice that $\left(f(z) - \sum_{i=0}^{j} \langle f(z), \eta_i(z) \rangle \eta_i(z) \right) \in V_j$, thus

$$\eta_0 \dot{\wedge} \eta_1 \dot{\wedge} \cdots \dot{\wedge} \eta_j \dot{\wedge} z H^2(\mathbb{D}, \mathbb{C}^m) \subset \eta_0 \dot{\wedge} \eta_1 \dot{\wedge} \cdots \dot{\wedge} \eta_j \dot{\wedge} V_j.$$

The reverse inclusion holds by the definition of V_j, hence

$$\eta_0 \dot{\wedge} \eta_1 \dot{\wedge} \cdots \dot{\wedge} \eta_j \dot{\wedge} z H^2(\mathbb{D}, \mathbb{C}^m) = \eta_0 \dot{\wedge} \eta_1 \dot{\wedge} \cdots \dot{\wedge} \eta_j \dot{\wedge} V_j.$$

Consequently, in order to prove the proposition it suffices to show that $\eta_0 \dot{\wedge} \eta_1 \dot{\wedge} \cdots \dot{\wedge} \eta_j \dot{\wedge} V_j$ is a closed subspace of $zH^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^m)$. By Corollary 2.2.29, V_j is a closed subspace of $zH^2(\mathbb{D}, \mathbb{C}^m)$, being an intersection of closed subspaces. For any $f \in V_j$, we get

$$\begin{aligned} &\|\eta_0 \dot{\wedge} \eta_1 \dot{\wedge} \cdots \dot{\wedge} \eta_j \dot{\wedge} f\|^2_{L^2(\mathbb{T}, \wedge^{j+2}\mathbb{C}^m)} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix} \|\eta_0(e^{i\theta})\|^2_{\mathbb{C}^m} & \cdots & \langle \eta_0(e^{i\theta}), f(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ \langle \eta_1(e^{i\theta}), \eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \|\eta_1(e^{i\theta})\|^2_{\mathbb{C}^m} & \cdots \\ \vdots & \ddots & \\ \langle f(e^{i\theta}), \eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \cdots & \|f(e^{i\theta})\|^2_{\mathbb{C}^m} \end{pmatrix} d\theta. \end{aligned}$$

Note that f and η_i are pointwise orthogonal almost everywhere on $\mathbb{T}$ and, by Proposition 3.2.1,$\{\eta_0(z), \ldots, \eta_j(z)\}$ is an orthonormal set for almost every $z \in \mathbb{T}$. Hence

$$\begin{aligned} \|\eta_0 \dot{\wedge} \eta_1 \dot{\wedge} \cdots \dot{\wedge} \eta_j \dot{\wedge} f\|^2_{L^2(\mathbb{T}, \wedge^{j+2}\mathbb{C}^m)} &= \frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|f(e^{i\theta})\|^2_{\mathbb{C}^m} \end{pmatrix} d\theta \\ &= \|f\|^2_{L^2(\mathbb{T}, \mathbb{C}^m)}. \end{aligned}$$

Thus

$$\eta_0 \dot{\wedge} \eta_1 \dot{\wedge} \cdots \dot{\wedge} \eta_j \dot{\wedge} \cdot \colon V_j \to \eta_0 \dot{\wedge} \eta_1 \dot{\wedge} \cdots \dot{\wedge} \eta_j \dot{\wedge} V_j$$

is an isometry. Furthermore

$$(\eta_0 \dot{\wedge} \eta_1 \dot{\wedge} \cdots \dot{\wedge} \eta_j \dot{\wedge} \cdot) \colon V_j \to \eta_0 \dot{\wedge} \eta_1 \dot{\wedge} \cdots \dot{\wedge} \eta_j \dot{\wedge} V_j$$

is a surjective mapping, thus V_j and $\eta_0 \dot{\wedge} \cdots \dot{\wedge} \eta_j \dot{\wedge} V_j$ are isometrically isomorphic. Therefore, since V_j is a closed subspace of $zH^2(\mathbb{D}, \mathbb{C}^m)$, the space $\eta_0 \dot{\wedge} \cdots \dot{\wedge} \eta_j \dot{\wedge} V_j$ is a closed subspace of $zH^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^m)$. Hence

$$\bar{\eta}_0 \dot{\wedge} \cdots \dot{\wedge} \bar{\eta}_j \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp$$

is a closed subspace of $H^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^m)^\perp$. □

Corollary 3.2.7. *Let $0 \leq j \leq m-2$. The orthogonal projection*

$$P_{Y_j} \colon L^2(\mathbb{T}, \wedge^{j+2}\mathbb{C}^m) \to Y_j$$

is well-defined.

Proof. By Proposition 2.2.25, $H^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^m)$ can be identified with a closed subspace of $L^2(\mathbb{T}, \wedge^{j+2}\mathbb{C}^m)$, thus we have

$$H^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^m)^\perp = L^2(\mathbb{T}, \wedge^{j+2}\mathbb{C}^m) \ominus H^2(\mathbb{D}, \wedge^{j+2}\mathbb{C}^m).$$

Now the assertion follows immediately from Proposition 3.2.6. □

3.2.5 T_j is a well-defined operator

Proposition 3.2.8. *Let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$ and let $0 \leq j \leq \min(m,n) - 2$. Let the functions ξ_i, η_i be defined by equations (3.28), that is,*

$$\xi_i = \frac{x_i}{h_i}, \quad \eta_i = \frac{\bar{z}\bar{\eta}_i}{h_i} \tag{3.43}$$

for $i = 0, \cdots, j$ and let

$$X_i = \xi_0 \dot\wedge \xi_1 \dot\wedge \cdots \dot\wedge \xi_{i-1} \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n) \subset H^2(\mathbb{D}, \wedge^{i+1}\mathbb{C}^n), \quad i = 0, \cdots, j,$$

$$Y_i = \bar{\eta}_0 \dot\wedge \bar{\eta}_1 \dot\wedge \cdots \dot\wedge \bar{\eta}_{i-1} \dot\wedge H^2(\mathbb{D}, \mathbb{C}^m)^\perp \subset H^2(\mathbb{D}, \wedge^{i+1}\mathbb{C}^m)^\perp, \quad i = 0, \cdots, j.$$

Let $Q_i \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ satisfy

$$(G - Q_i)x_k = t_k y_k, \quad (G - Q_i)^* y_k = t_k x_k \tag{3.44}$$

for all $k = 0, \ldots, i-1$.
Then, the operators $T_i \colon X_i \to Y_i$, $i = 0, \cdots, j$, given by

$$T_i(\xi_0 \dot\wedge \xi_1 \dot\wedge \cdots \dot\wedge \xi_{i-1} \dot\wedge x) = P_{Y_i}\left(\bar{\eta}_0 \dot\wedge \bar{\eta}_1 \dot\wedge \cdots \dot\wedge \bar{\eta}_{i-1} \dot\wedge (G - Q_i)x\right) \tag{3.45}$$

are well-defined and are independent of the choice of $Q_i \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ satisfying equations (3.44).

Proof. By Corollary 3.2.7, the projections P_{Y_i} are well-defined for all $i = 0, \cdots, j$. Hence it suffices to show that, for all $i = 0, 1, \cdots, j$, T_i maps a zero from its domain to a zero in its range and that T_i does not depend on the choice of Q_i, which satisfies equations (3.44).

For $i = 0$, the operator T_0 is the Hankel operator H_G. If $f_0 \equiv 0$, then $H_G f_0 = 0$ and, moreover, H_G is independent of the choice of any $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ as $H_{G-Q} = H_G$. Thus, T_0 is well-defined.

For $i = 1$, let (x_0, y_0) be a Schmidt pair for the compact operator H_G corresponding to $t_0 = \|H_G\|$, where $x_0 \in H^2(\mathbb{D}, \mathbb{C}^n)$ and $y_0 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$. By Lemma 3.1.12, $x_0, \bar{z}\bar{y}_0$ admit the inner-outer factorisations $x_0 = \xi_0 h_0, \quad \bar{z}\bar{y}_0 = \eta_0 h_0$, where $\xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$, $\eta_0 \in H^\infty(\mathbb{D}, \mathbb{C}^m)$ are inner vector-valued functions and $h_0 \in H^2(\mathbb{D}, \mathbb{C})$ is scalar outer. The spaces X_1 and Y_1 are given by the formulas

$$X_1 = \xi_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n), \quad Y_1 = \bar{\eta}_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp.$$

The operator $T_1 \colon X_1 \to Y_1$ is given by

$$T_1(\xi_0 \dot{\wedge} x) = P_{Y_1}(\bar{\eta}_0 \dot{\wedge} (G - Q_1)x)$$

for all $x \in H^2(\mathbb{D}, \mathbb{C}^n)$, where $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ satisfies equations (3.44).

Lemma 3.2.9. *Let $\xi_0 \dot{\wedge} u = \xi_0 \dot{\wedge} v$ for some $u, v \in H^2(\mathbb{D}, \mathbb{C}^n)$. Then*

$$\bar{\eta}_0 \dot{\wedge} (G - Q_1)u = \bar{\eta}_0 \dot{\wedge} (G - Q_1)v.$$

Proof. Suppose that $\xi_0 \dot{\wedge} u = \xi_0 \dot{\wedge} v$ for some $u, v \in H^2(\mathbb{D}, \mathbb{C}^n)$. Let $x = u - v$, then $\xi_0 \dot{\wedge} x = 0$, and so x and ξ_0 are pointwise linearly dependent in $\mathbb{C}^n$ on $\mathbb{D}$. Therefore there exist maps $\beta, \lambda \colon \mathbb{D} \to \mathbb{C}$, having no common zero in $\mathbb{D}$, such that

$$\beta(z)\xi_0(z) = \lambda(z)x(z) \text{ in } \mathbb{C}^n, \tag{3.46}$$

for all $z \in \mathbb{D}$. By assumption, $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ satisfies equations (3.44). Thus, for all $z \in \mathbb{D}$,

$$t_0 y_0(z) = (G - Q_1)(z) x_0(z). \tag{3.47}$$

By equations (3.43) and (3.46),

$$\beta(z)x_0(z) = \beta(z)h_0(z)\xi_0(z) = h_0(z)\lambda(z)x(z) \tag{3.48}$$

for all $z \in \mathbb{D}$. By equations (3.47) and (3.48), for all $z \in \mathbb{D}$,

$$\begin{aligned} t_0 y_0(z) \quad &= (G - Q_1)(z) x_0(z), \\ \beta(z) t_0 z \frac{y_0(z)}{\bar{h}_0(z)} \quad &= (G - Q_1)(z) h_0(z) \lambda(z) x(z) \frac{z}{\bar{h}_0(z)}. \end{aligned}$$

Therefore, by equations (3.43), for all $z \in \mathbb{D}$,

$$t_0 \beta(z) \bar{\eta}_0(z) = (G - Q_1)(z) x(z) \mu(z) \text{ in } \mathbb{C}^m,$$

where

$$\mu(z) = \frac{z h_0(z) \lambda(z)}{\bar{h}_0(z)}, \text{ for all } z \in \mathbb{D}.$$

Hence, by Definition 2.2.2, $\bar{\eta}_0$ and $(G - Q_1)x$ are pointwise linearly dependent in $\mathbb{C}^m$ on $\mathbb{D}$, and so

$$\bar{\eta}_0 \dot{\wedge} (G - Q_1)x = 0.$$

Consequently,

$$\bar{\eta}_0 \dot{\wedge} (G - Q_1)u = \bar{\eta}_0 \dot{\wedge} (G - Q_1)v. \qquad \square$$

Therefore the formula (3.45) (with $i = 1$) does uniquely define $T_1 u \in Y_1$. Next, we show that the operator T_1 is independent of the choice of $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$, which satisfies equations (3.44).
By Theorem D.2.4, there exist $Q_1, Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ which satisfy

$$(G - Q_1)x_0 = t_0 y_0 \ , \ y_0^*(G - Q_1) = t_0 x_0^* \tag{3.49}$$

and

$$(G - Q_2)x_0 = t_0 y_0 \ , \ y_0^*(G - Q_2) = t_0 x_0^*. \tag{3.50}$$

Then, we would like to prove that, for all $x \in H^2(\mathbb{D}, \mathbb{C}^n)$,

$$P_{Y_1}(\bar{\eta}_0 \dot{\wedge} (G - Q_1)x) = P_{Y_1}(\bar{\eta}_0 \dot{\wedge} (G - Q_2)x),$$

that is,

$$P_{Y_1}(\bar{\eta}_0 \dot{\wedge} (Q_1 - Q_2)x) = 0.$$

The latter is equivalent to the property that $\bar{\eta}_0 \dot{\wedge} (Q_2 - Q_1)x$ is orthogonal to $\bar{\eta}_0 \dot{\wedge} \varrho$ for all $x \in H^2(\mathbb{D}, \mathbb{C}^n)$ and for all $\varrho \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$. As a matter of convenience, set

$$Ax = (Q_2 - Q_1)x, \quad x \in H^2(\mathbb{D}, \mathbb{C}^n).$$

We have to prove that

$$\langle \bar{\eta}_0 \dot{\wedge} Ax, \bar{\eta}_0 \dot{\wedge} \varrho \rangle_{L^2(\mathbb{T}, \wedge^2\mathbb{C}^m)} = 0$$

for all $x \in H^2(\mathbb{D}, \mathbb{C}^n)$ and all $\varrho \in H^2(\mathbb{C}^m)^\perp$. Note that

$$\langle \bar{\eta}_0 \dot{\wedge} Ax, \bar{\eta}_0 \dot{\wedge} \varrho \rangle^2_{L^2(\mathbb{T}, \wedge^2\mathbb{C}^m)} = \frac{1}{2\pi} \int_0^{2\pi} \langle \bar{\eta}_0(e^{i\theta}) \dot{\wedge} A(e^{i\theta}) x(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \dot{\wedge} \varrho(e^{i\theta}) \rangle_{\wedge^2\mathbb{C}^m} \, d\theta,$$

which by Proposition 2.1.19 yields

$$\frac{1}{2\pi}\int_0^{2\pi} \det \begin{pmatrix} \langle \bar{\eta}_0(e^{i\theta}), \bar{\eta}_0(e^{i\theta})\rangle_{\mathbb{C}^m} & \langle \bar{\eta}_0(e^{i\theta}), \varrho(e^{i\theta})\rangle_{\mathbb{C}^m} \\ \langle A(e^{i\theta})x(e^{i\theta}), \bar{\eta}_0(e^{i\theta})\rangle_{\mathbb{C}^m} & \langle A(e^{i\theta})x(e^{i\theta}), \varrho(e^{i\theta})\rangle_{\mathbb{C}^m} \end{pmatrix} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \|\bar{\eta}_0(e^{i\theta})\|^2_{\mathbb{C}^m} \langle A(e^{i\theta})x(e^{i\theta}), \varrho(e^{i\theta})\rangle_{\mathbb{C}^m}\, d\theta$$

$$- \frac{1}{2\pi}\int_0^{2\pi} \langle A(e^{i\theta})x(e^{i\theta}), \bar{\eta}_0(e^{i\theta})\rangle_{\mathbb{C}^m} \langle \bar{\eta}_0(e^{i\theta}), \varrho(e^{i\theta})\rangle_{\mathbb{C}^m}\, d\theta.$$

By Proposition 3.2.1, $\|\bar{\eta}_0(e^{i\theta})\|_{\mathbb{C}^m} = 1$ for almost every $e^{i\theta} \in \mathbb{T}$. Since $Ax \in H^2(\mathbb{D}, \mathbb{C}^m)$ and $\varrho \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$,

$$\frac{1}{2\pi}\int_0^{2\pi} \langle A(e^{i\theta})x(e^{i\theta}), \varrho(e^{i\theta}\rangle_{\mathbb{C}^m} d\theta = \langle Ax, \varrho\rangle_{L^2(\mathbb{T},\mathbb{C}^m)} = 0.$$

Thus

$$\begin{aligned}\langle \bar{\eta}_0 \dot{\wedge} Ax, \bar{\eta}_0 \dot{\wedge} \varrho\rangle_{L^2(\mathbb{T},\wedge^2\mathbb{C}^m)} &= \frac{1}{2\pi}\int_0^{2\pi} \langle A(e^{i\theta})x(e^{i\theta}), \bar{\eta}_0(e^{i\theta})\rangle_{\mathbb{C}^m} \langle \bar{\eta}_0(e^{i\theta}), \varrho(e^{i\theta})\rangle_{\mathbb{C}^m} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \bar{\eta}_0^*(e^{i\theta})A(e^{i\theta})x(e^{i\theta}) \langle \bar{\eta}_0(e^{i\theta}), \varrho(e^{i\theta})\rangle_{\mathbb{C}^m} d\theta.\end{aligned}$$

Recall that by equation (3.9), $\bar{\eta}_0(z) = \dfrac{z y_0(z)}{\bar{h}_0(z)}$, $z \in \mathbb{T}$, so that

$$\bar{\eta}_0^*(e^{i\theta}) = \left(\frac{e^{i\theta} y_0(e^{i\theta})}{\bar{h}_0(e^{i\theta})}\right)^* = \frac{e^{-i\theta} y_0^*(e^{i\theta})}{h_0(e^{i\theta})}.$$

Therefore

$$\langle \bar{\eta}_0 \dot{\wedge} Ax, \bar{\eta}_0 \dot{\wedge} \varrho\rangle_{L^2(\mathbb{T},\wedge^2\mathbb{C}^m)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-i\theta} y_0^*(e^{i\theta})}{h_0(e^{i\theta})} A(e^{i\theta})x(e^{i\theta}) \langle \bar{\eta}_0(e^{i\theta}), \varrho(e^{i\theta})\rangle_{\mathbb{C}^m}) d\theta.$$

Recall our initial assumption was that Q_1, Q_2 satisfy equations (3.49) and (3.50), consequently,

$$y_0^*(G - Q_i) = t_0 x_0^*, \text{ for } i = 1, 2.$$

Hence, for $z \in \mathbb{T}$,

$$\begin{aligned} y_0^*(z)A(z)x(z) &= y_0^*(z)(G - Q_1)(z)x(z) - y_0^*(z)(G - Q_2)(z)x(z) \\ &= (t_0 x_0^* x - t_0 x_0^* x)(z) \\ &= 0. \end{aligned}$$

We deduce that

$$\frac{1}{2\pi}\int_0^{2\pi} \bar{\eta}_0^*(e^{i\theta})A(e^{i\theta})x(e^{i\theta})\langle \bar{\eta}_0(e^{i\theta}), \varrho(e^{i\theta})\rangle_{\mathbb{C}^m})d\theta = 0.$$

To conclude, we have proved that, if $Q_1, Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ satisfy equations (3.49) and (3.50), then

$$P_{Y_1}(\bar{\eta}_0\dot{\wedge}(G-Q_1)x) = P_{Y_1}(\bar{\eta}_0\dot{\wedge}(G-Q_2)x),$$

that is, T_1 is independent of the choice of Q_1. Thus T_1 is a well-defined operator.

Recursive step: suppose that functions $x_{i-1} \in L^2(\mathbb{T}, \mathbb{C}^n)$, $y_{i-1} \in L^2(\mathbb{T}, \mathbb{C}^m)$, outer functions $h_{i-1} \in H^2(\mathbb{D}, \mathbb{C})$, positive numbers t_i, matrix-valued functions $Q_i \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$, spaces X_i, Y_i and compact operators $T_i \colon X_i \to Y_i$ are constructed inductively by the algorithm for all $i = 0, \dots, j$.

Let us prove that $T_j \colon X_j \to Y_j$, given by equation (3.25), is well-defined for all $0 \le j \le \min(m,n) - 2$. Note, by Corollary 3.2.7, the projection P_{Y_j} is well-defined. We will prove that T_j maps zeros from its domain to zeros to its range and T_j is independent of the choice of Q_j that satisfies equations (3.44).
Suppose $\xi_0\dot{\wedge}\xi_1\dot{\wedge}\cdots\dot{\wedge}\xi_{j-1}\dot{\wedge}x = 0$. Then $x(z)$ is pointwise linearly dependent on $\xi_0(z), \xi_1(z), \cdots, \xi_{j-1}(z)$ in $\mathbb{C}^n$ for almost all $z \in \mathbb{T}$. This means there exist maps

$$\lambda_i, \nu \colon \mathbb{T} \to \mathbb{C}\ ,\ i = 0, \cdots, j-1$$

which are non-zero almost everywhere on $\mathbb{T}$ and are such that

$$\nu(z)x(z) = \sum_{i=0}^{j-1} \lambda_i(z)\xi_i(z).$$

By Theorem 1.1.4, there exists a function $Q_j \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ that lexicographically minimises

$$(s_0^\infty(G-Q), s_1^\infty(G-Q), \dots, s_j^\infty(G-Q))$$

over all $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$. By Proposition 3.2.47, any such function Q_j necessarily satisfies

$$(G-Q_j)x_i = t_i y_i, \quad y_i^*(G-Q_j) = t_i x_i, \quad \text{for all } i = 0, \cdots, j-1. \tag{3.51}$$

By equations (3.43),

$$\xi_i = \frac{x_i}{h_i}, \quad \eta_i = \frac{\bar{z}\bar{y}_i}{h_i},\ i = 0\cdots, j-1.$$

Then, for almost all $z \in \mathbb{T}$,

$$
\begin{aligned}
(G-Q_j)(z)x_i(z) &= t_i y_i(z) \\
(G-Q_j)(z)\lambda_i(z)x_i(z)\frac{1}{h_i(z)} &= t_i\lambda_i(z)\frac{1}{h_i(z)}y_i(z) \\
(G-Q_j)(z)\lambda_i(z)x_i(z)\frac{1}{h_i(z)} &= t_i\lambda_i(z)\frac{1}{h_i(z)}y_i(z)\frac{z\,\bar{h}_i(z)}{z\,\bar{h}_i(z)} \\
(G-Q_j)(z)\lambda_i(z)x_i(z)\frac{1}{h_i(z)} &= t_i\lambda_i(z)\frac{1}{zh_i(z)}\bar{\eta}_i(z) \\
(G-Q_j)(z)\sum_{i=1}^{j-1}\lambda_i(z)\xi_i(z) &= \sum_{i=1}^{j-1} t_i\lambda_i(z)\frac{1}{zh_i(z)}\bar{\eta}_i(z) \\
(G-Q_j)(z)\nu(z)x(z) &= \sum_{i=1}^{j-1}\mu_i(z)\bar{\eta}_i(z)
\end{aligned}
$$

where

$$t_i\lambda_i(z)\frac{1}{zh_i(z)} = \mu_i(z).$$

Therefore for all $x \in H^2(\mathbb{D},\mathbb{C}^n)$, $\bar{\eta}_0(z),\cdots,\bar{\eta}_{j-1}(z)$ and $((G-Q_j)x)(z)$ are pointwise linearly dependent in $\mathbb{C}^m$ almost everywhere on $\mathbb{T}$. Hence

$$\bar{\eta}_0\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_{j-1}\dot{\wedge}(G-Q_j)x = 0.$$

Consequently, T_j maps a zero from its domain to a zero in its range.

For the operator T_j to be well-defined, it remains to prove T_j is independent of the choice of $Q_j \in H^\infty(\mathbb{D},\mathbb{C}^{m\times n})$ which satisfies equations (3.51). Let $\tilde{Q}_j, \hat{Q}_j \in H^\infty(\mathbb{D},\mathbb{C}^{m\times n})$ satisfy

$$(G-\tilde{Q}_j)x_i = t_i y_i, \quad (G-\hat{Q}_j)x_i = t_i y_i, \quad y_i^*(G-\tilde{Q}_j) = t_i x_i^*, \quad y_i^*(G-\hat{Q}_j) = t_i x_i^* \tag{3.52}$$

for $i = 0,\cdots,j-1$.
We would like to prove that, for all $x \in H^2(\mathbb{D},\mathbb{C}^n)$,

$$P_{Y_j}(\bar{\eta}_0\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_{j-1}\dot{\wedge}(G-\tilde{Q}_j)x) = P_{Y_j}(\bar{\eta}_0\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_{j-1}\dot{\wedge}(G-\hat{Q}_j)x).$$

The latter equality holds if and only if, for all $x \in H^2(\mathbb{D},\mathbb{C}^n)$,

$$P_{Y_j}(\bar{\eta}_0\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_{j-1}\dot{\wedge}(\hat{Q}_j-\tilde{Q}_j)x) = 0$$

which is equivalent to the assertion that $\bar{\eta}_0\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_{j-1}\dot{\wedge}(\hat{Q}_j-\tilde{Q}_j)x$ is orthogonal to $\bar{\eta}_0\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_{j-1}\dot{\wedge}q$ for all $x \in H^2(\mathbb{D},\mathbb{C}^n)$ and for all $q \in H^2(\mathbb{D},\mathbb{C}^m)^\perp$.

Equivalently

$$\langle \bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_{j-1} \dot\wedge (\hat Q_j - \tilde Q_j)x, \bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_{j-1} \dot\wedge q \rangle_{L^2(\mathbb{T}, \wedge^{j+1}\mathbb{C}^m)} = 0$$

for all $x \in H^2(\mathbb{D}, \mathbb{C}^n)$ and for all $q \in H^2(\mathbb{C}^m)^\perp$. Set $Ax = (\hat Q_j - \tilde Q_j)x$, $x \in H^2(\mathbb{D}, \mathbb{C}^n)$. By Proposition 2.1.19,

$$\langle \bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_{j-1} \dot\wedge (\hat Q_j - \tilde Q_j)x, \bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_{j-1} \dot\wedge q \rangle_{L^2(\mathbb{T}, \wedge^{j+1}\mathbb{C}^m)}$$

is equal to

$$\frac{1}{2\pi}\int_0^{2\pi} \det \begin{pmatrix} \langle \bar\eta_0(e^{i\theta}), \bar\eta_0(e^{i\theta})\rangle_{\mathbb{C}^m} & \cdots & \langle \bar\eta_0(e^{i\theta}), \bar\eta_{j-1}(e^{i\theta})\rangle_{\mathbb{C}^m} & \langle \bar\eta_0(e^{i\theta}), q(e^{i\theta})\rangle_{\mathbb{C}^m} \\ \vdots & \ddots & \vdots & \vdots \\ \langle \bar\eta_{j-1}(e^{i\theta}), \bar\eta_0(e^{i\theta})\rangle_{\mathbb{C}^m} & \cdots & \langle \bar\eta_{j-1}(e^{i\theta}), \bar\eta_{j-1}(e^{i\theta})\rangle_{\mathbb{C}^m} & \langle \bar\eta_{j-1}(e^{i\theta}), q(e^{i\theta})\rangle_{\mathbb{C}^m} \\ \langle A(e^{i\theta})x(e^{i\theta}), \bar\eta_0(e^{i\theta})\rangle_{\mathbb{C}^m} & \cdots & \langle A(e^{i\theta})x(e^{i\theta}), \bar\eta_{j-1}(e^{i\theta})\rangle_{\mathbb{C}^m} & \langle A(e^{i\theta})x(e^{i\theta}), q(e^{i\theta})\rangle_{\mathbb{C}^m} \end{pmatrix} d\theta.$$

Notice that Ax and q are orthogonal in $L^2(\mathbb{T}, \mathbb{C}^m)$ and, by Proposition 3.2.1, $\{\bar\eta_i(z)\}_{i=0}^{j-1}$ is an orthonormal set in $\mathbb{C}^m$ almost everywhere on $\mathbb{T}$. Also, for all $i = 0, \cdots, j-1$, by equations (3.52),

$$\begin{aligned} \langle A(e^{i\theta})x(e^{i\theta}), \bar\eta_i(e^{i\theta})\rangle\rangle_{\mathbb{C}^m} &= \eta_i^T(e^{i\theta})A(e^{i\theta})x(e^{i\theta}) \\ &= \frac{e^{-i\theta}y_i^*(e^{i\theta})}{h_i(e^{i\theta})}A(e^{i\theta})x(e^{i\theta}) \\ &= \frac{e^{-i\theta}}{h_i(e^{i\theta})}\Big(y_i^*(e^{i\theta})(G - \tilde Q_j)(z)x(z) - y_i^*(e^{i\theta})(G - \hat Q_j)(z)x(z)\Big) \\ &= \frac{e^{-i\theta}}{h_i(e^{i\theta})}(t_i x_i^* x - t_i x_i^* x) \\ &= 0. \end{aligned}$$

Thus

$$\begin{aligned} &\langle \bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge (\hat Q_j - \tilde Q_j)x, \bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge q \rangle_{L^2(\mathbb{T}, \wedge^{j+1}\mathbb{C}^m)} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \det \begin{pmatrix} 1 & 0 & \cdots & \langle \bar\eta_0(e^{i\theta}), q(e^{i\theta})\rangle_{\mathbb{C}^m} \\ 0 & 1 & \cdots & \langle \bar\eta_2(e^{i\theta}), q(e^{i\theta})\rangle_{\mathbb{C}^m} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \langle A(e^{i\theta})x(e^{i\theta}), q(e^{i\theta})\rangle_{\mathbb{C}^m} \end{pmatrix} d\theta \\ &= \langle Ax, q\rangle_{L^2(\mathbb{T}, \mathbb{C}^m)} = 0. \end{aligned}$$

Consequently

$$P_{Y_j}(\bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_{j-1} \dot\wedge (G - \tilde Q_j)x) = P_{Y_j}(\bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_{j-1} \dot\wedge (G - \hat Q_j)x),$$

and so T_j is independent of the choice of Q_j that satisfies equations (3.51). Thus we have proven that the operator T_j is well-defined. □

3.2.6 Compactness of the operators T_1 and T_2

In this section, we use notations from the algorithm from Section 3.2.1 to prove the compactness of the operators T_1, T_2 given by equations (3.15) and (3.20) respectively. To this end, several auxiliary results are required.

Recall that since $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$, by Hartman's theorem, the operator $T_0 = H_G$ is compact and hence there exist $x_0 \in H^2(\mathbb{D}, \mathbb{C}^n)$ and $y_0 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ such that (x_0, y_0) is a Schmidt pair for H_G corresponding to the singular value $\|H_G\| = t_0$.

By Lemma 3.1.12, $x_0, \bar{z}\bar{y}_0$ admit the inner-outer factorisations

$$x_0 = \xi_0 h_0, \quad \bar{z}\bar{y}_0 = \eta_0 h_0, \tag{3.53}$$

where $\xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$, $\eta_0 \in H^\infty(\mathbb{D}, \mathbb{C}^m)$ are vector-valued inner functions and $h_0 \in H^2(\mathbb{D}, \mathbb{C})$ is a scalar outer function. Moreover there exist unitary-valued functions of types $n \times n, m \times m$ respectively, of the form

$$V_0 = \begin{pmatrix} \xi_0 & \bar{\alpha}_0 \end{pmatrix}, \quad W_0 = \begin{pmatrix} \eta_0 & \bar{\beta}_0 \end{pmatrix}^T, \tag{3.54}$$

where α_0, β_0 are inner, co-outer, quasi-continuous functions of types $n \times (n-1)$, $m \times (m-1)$ respectively and all minors on the first columns of V_0, W_0^T are in H^∞. Furthermore every $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ which is at minimal distance from G satisfies

$$W_0(G - Q_1)V_0 = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & F_1 \end{pmatrix}$$

for some

$$F_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1)\times(n-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-1)\times(n-1)})$$

and some quasi-continuous function u_0 with $|u_0(z)| = 1$ almost everywhere on $\mathbb{T}$.
Recall that

$$X_1 = \xi_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n), \quad Y_1 = \bar{\eta}_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp$$

and $T_1 \colon X_1 \to Y_1$ is given by

$$T_1(\xi_0 \dot{\wedge} x) = P_{Y_1}[\bar{\eta}_0 \dot{\wedge} (G - Q_1)x] \quad \text{for all } x \in H^2(\mathbb{D}, \mathbb{C}^n).$$

Our first endeavour in this subsection is to prove the following theorem.

Theorem 3.2.10. *Let*

$$\mathcal{K}_1 \stackrel{\text{def}}{=} V_0 \begin{pmatrix} 0 \\ H^2(\mathbb{D}, \mathbb{C}^{n-1}) \end{pmatrix}, \quad \mathcal{L}_1 \stackrel{\text{def}}{=} W_0^* \begin{pmatrix} 0 \\ H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp \end{pmatrix}, \tag{3.55}$$

and let the maps

$$U_1 \colon H^2(\mathbb{D}, \mathbb{C}^{n-1}) \to \mathcal{K}_1,$$

$$U_2 \colon H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp \to \mathcal{L}_1$$

be given by

$$U_1 x = V_0 \begin{pmatrix} 0 \\ x \end{pmatrix}, \quad U_2 y = W_0^* \begin{pmatrix} 0 \\ y \end{pmatrix}$$

for all $x \in H^2(\mathbb{D}, \mathbb{C}^{n-1})$, $y \in H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp$. *Consider the operator* $\Gamma_1 = P_{\mathcal{L}_1} M_{G-Q_1}|_{\mathcal{K}_1}$. *Then*

(i) *The maps* U_1, U_2 *are unitaries.*

(ii) *The maps* $(\xi_0 \dot\wedge \cdot)\colon \mathcal{K}_1 \to H^2(\mathbb{D}, \wedge^2\mathbb{C}^n)$ *and* $(\bar\eta_0 \dot\wedge \cdot)\colon \mathcal{L}_1 \to H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ *are unitaries.*

(iii) *The following diagram is commutative:*

$$\begin{array}{ccccc} H^2(\mathbb{D}, \mathbb{C}^{n-1}) & \xrightarrow{U_1} & \mathcal{K}_1 & \xrightarrow{\xi_0 \dot\wedge \cdot} & \xi_0 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n) = X_1 \\ \Big\downarrow H_{F_1} & & \Big\downarrow \Gamma_1 & & \Big\downarrow T_1 \\ H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp & \xrightarrow{U_2} & \mathcal{L}_1 & \xrightarrow{\bar\eta_0 \dot\wedge \cdot} & \bar\eta_0 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^m)^\perp = Y_1. \end{array} \tag{3.56}$$

(iv) T_1 *is a compact operator.*

(v) $\|T_1\| = \|\Gamma_1\| = t_1$.

Proof. Statement (i) follows from Lemma 3.1.17. Statement (ii) follows from Propositions 3.2.17 and 3.2.21, which are consequences of Beurling's theorem and the lemmas that follow.

Theorem 3.2.11 (Beurling's Theorem, [38], p. 99)**.** *Let* S *be a non-zero closed subspace of* $H^2(\mathbb{D}, \mathbb{C})$. *Then* S *is invariant under multiplication by* z *if and only if* $S = \theta H^2(\mathbb{D}, \mathbb{C})$, *where* θ *is an inner function.*

Lemma 3.2.12. *In the notation of Theorem 3.2.10, the Hankel operator* H_G *has a maximizing vector* x_0 *of unit norm such that* ξ_0, *which is defined by* $\xi_0 = \frac{x_0}{h_0}$, *is a co-outer function.*

Proof. Choose any maximizing vector x_0. By Lemma 3.1.12, x_0 has the inner-outer factorisation $x_0 = \xi_0 h_0$, where h_0 is a scalar outer factor. Then, the closure of $\xi_0^T H^2(\mathbb{D}, \mathbb{C}^n)$, denoted by $\mathrm{clos}(\xi_0^T H^2(\mathbb{D}, \mathbb{C}^n))$, is a closed shift-invariant subspace of $H^2(\mathbb{D}, \mathbb{C})$, so, by Beurling's theorem,

$$\mathrm{clos}(\xi_0^T H^2(\mathbb{D}, \mathbb{C}^n)) = \phi H^2(\mathbb{D}, \mathbb{C})$$

for some scalar inner function ϕ. Hence

$$\bar\phi \xi_0^T H^2(\mathbb{D}, \mathbb{C}^n) \subset H^2(\mathbb{D}, \mathbb{C}).$$

Thus, if $\xi_0^T = (\xi_{01}, \cdots, \xi_{0n})$, we have $\bar\phi \xi_{0j} \in H^\infty(\mathbb{D}, \mathbb{C})$ for $j = 1, \cdots, n$, and so,

$$\overline{\phi} \xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n).$$

Hence

$$\overline{\phi}x_0 = \overline{\phi}\xi_0 h_0 \in H^2(\mathbb{D}, \mathbb{C}^n).$$

Let Q be a best H^∞ approximation to G. Since x_0 is a maximizing vector for H_G, by Theorem D.2.4,

$$(G - Q)x_0 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$$

and

$$\|(G - Q)(z)x_0(z)\|_{\mathbb{C}^m} = \|H_G\|\|x_0(z)\|_{\mathbb{C}^n}$$

for almost all $z \in \mathbb{T}$. Thus

$$(G - Q)\overline{\phi}x_0 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$$

and

$$\|(G - Q)\overline{\phi}x_0(z)\|_{\mathbb{C}^m} = \|H_G\|\|\overline{\phi}x_0(z)\|_{\mathbb{C}^n}$$

for almost all $z \in \mathbb{T}$.

Hence $\overline{\phi}x_0 \in H^2(\mathbb{D}, \mathbb{C}^n)$ is a maximizing vector for H_G, and $\overline{\phi}x_0$ is co-outer. Then $\frac{\bar{\phi}x_0}{\|x_0\|}$ is a co-outer maximizing vector of unit norm for H_G. □

Remark 3.2.13. *Lemma 3.2.12 asserts that in the scalar case one can* always *choose an* outer *eigenvector corresponding to the largest eigenvalue of the Hankel operator.*

Lemma 3.2.14. *Let x_0 be a co-outer maximizing vector of unit norm for H_G, and let $x_0 = \xi_0 h_0$ be the inner-outer factorisation of x_0. Then*

(i) ξ_0 is a quasi-continuous function and

(ii) there exists a function $A \in H^\infty(\mathbb{D}, \mathbb{C}^n)$ such that

$$A^T \xi_0 = 1.$$

Proof. Let us first show that

$$\xi_0 \in (H^\infty(\mathbb{D}, \mathbb{C}^n) + C(\mathbb{T}, \mathbb{C}^n)) \cap \overline{H^\infty(\mathbb{D}, \mathbb{C}^n) + C(\mathbb{T}, \mathbb{C}^n)}.$$

Let Q be a best H^∞ approximation to G. Then, by Theorem D.2.4, the function Q satisfies the equation

$$(G - Q)^* y_0 = t_0 x_0.$$

Taking complex conjugates in equations (3.53), we get

$$(G - Q)^T \bar{y}_0 = t_0 \overline{x_0}.$$

Hence, for $z \in \mathbb{T}$,

$$(G - Q)^T z h_0 \eta_0 = t_0 \overline{h_0 \xi_0},$$

and therefore

$$\frac{(G-Q)^T z h_0 \eta_0}{t_0 \overline{h_0}} = \overline{\xi_0}.$$

Recall that, by equation (3.5) (with $\phi = 1$), $u_0 = \frac{\bar{z}\bar{h}_0}{h_0}$. By Lemma 3.1.12, $u_0 \in QC$, hence $\overline{u_0} \in H^\infty + C$. Note $\overline{u_0} = \frac{z h_0}{\overline{h_0}}$, and hence

$$\overline{\xi_0} = \frac{(G-Q)^T \overline{u_0} \eta_0}{t_0}.$$

Since $H^\infty + C$ is an algebra and $(G-Q)^T,\ \eta_0 \in H^\infty + C$, it follows that $\overline{\xi_0} \in H^\infty + C$, thus

$$\xi_0 \in (H^\infty(\mathbb{D}, \mathbb{C}^n) + C(\mathbb{T}, \mathbb{C}^n)) \cap \overline{H^\infty(\mathbb{D}, \mathbb{C}^n) + C(\mathbb{T}, \mathbb{C}^n)}.$$

The conclusion that there exists a function $A \in H^\infty(\mathbb{D}, \mathbb{C}^n)$ such that $A^T \xi_0 = 1$ now follows directly from Lemma 3.1.18. □

Lemma 3.2.15. *In the notation of Theorem 3.2.10, let $\xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$ be a vector-valued inner, co-outer, quasi-continuous function and let*

$$V_0 = \begin{pmatrix} \xi_0 & \bar{\alpha}_0 \end{pmatrix}$$

be a thematic completion of ξ_0 as described in Lemma 3.1.12, where α_0 is an inner, co-outer, quasi-continuous function of order $n \times (n-1)$ and all minors on the first column of V_0 are analytic. Then,

$$\alpha_0^T H^2(\mathbb{D}, \mathbb{C}^n) = H^2(\mathbb{D}, \mathbb{C}^{n-1}).$$

Proof. By Lemma 3.1.18, for the given α_0, there exists $A_0 \in H^\infty(\mathbb{D}, \mathbb{C}^{(n-1)\times n})$ such that $A_0 \alpha_0 = I_{n-1}$. Equivalently, $\alpha_0^T A_0^T = I_{n-1}$.

Let $g \in H^2(\mathbb{D}, \mathbb{C}^{n-1})$. Then $g = (\alpha_0^T A_0^T) g \in \alpha_0^T A_0^T H^2(\mathbb{D}, \mathbb{C}^{n-1})$, which implies that $g \in \alpha_0^T H^2(\mathbb{D}, \mathbb{C}^n)$. Hence $H^2(\mathbb{D}, \mathbb{C}^{n-1}) \subseteq \alpha_0^T H^2(\mathbb{D}, \mathbb{C}^n)$.

For the reverse inclusion, note that since α_0 is in $H^\infty(\mathbb{D}, \mathbb{C}^{n\times(n-1)})$, we have $\alpha_0^T H^2(\mathbb{D}, \mathbb{C}^n) \subseteq H^2(\mathbb{D}, \mathbb{C}^{n-1})$. Thus

$$\alpha_0^T H^2(\mathbb{D}, \mathbb{C}^n) = H^2(\mathbb{D}, \mathbb{C}^{n-1}).$$ □

Proposition 3.2.16. *Let ξ_0, α_0 and V_0 be as in Lemma 3.2.15. Then*

$$V_0^* \operatorname{POC}(\{\xi_0\}, L^2(\mathbb{T}, \mathbb{C}^n)) = \begin{pmatrix} 0 \\ L^2(\mathbb{T}, \mathbb{C}^{n-1}) \end{pmatrix}.$$

Proof. Let $g \in V_0^* \operatorname{POC}(\{\xi_0\}, L^2(\mathbb{T}, \mathbb{C}^n))$. Equivalently, g can be written as $g = V_0^* f$ for some $f \in L^2(\mathbb{T}, \mathbb{C}^n)$ such that $f(z) \perp \xi_0(z)$ for almost all $z \in \mathbb{T}$. This in turn is equivalent to the assertion that $g = V_0^* f$ for some $f \in L^2(\mathbb{T}, \mathbb{C}^n)$ such that $(V_0^* f)(z) \perp (V_0^* \xi_0)(z)$ for almost all $z \in \mathbb{T}$, since $V_0(z)$ is unitary for almost all $z \in \mathbb{T}$.

Note that, by the fact that V_0 is unitary-valued almost everywhere on $\mathbb{T}$, we get

$$\begin{aligned} I_n &= V_0^*(z)V_0(z) \\ &= \begin{pmatrix} \xi_0^*(z) \\ \alpha_0^T(z) \end{pmatrix} \begin{pmatrix} \xi_0(z) & \bar{\alpha}_0(z) \end{pmatrix} \\ &= \begin{pmatrix} \xi_0^*(z)\xi_0(z) & \xi_0^*(z)\bar{\alpha}_0(z) \\ \alpha_0^T(z)\xi_0(z) & \alpha_0^T(z)\bar{\alpha}_0(z) \end{pmatrix} \quad \text{almost everywhere on } \mathbb{T}, \end{aligned} \tag{3.57}$$

and so

$$V_0^*\xi_0 = \begin{pmatrix} \xi_0^* \\ \alpha_0^T \end{pmatrix} \xi_0 = \begin{pmatrix} 1 \\ 0_{(n-1)\times 1} \end{pmatrix},$$

where $0_{(n-1)\times 1}$ denotes the zero vector in $\mathbb{C}^{n-1}$.
Hence $g = V_0^* f$ with $(V_0^* f)(z)$ orthogonal to $(V_0^*\xi_0)(z)$ for almost every $z \in \mathbb{T}$, is equivalent to the statement $g \in L^2(\mathbb{T}, \mathbb{C}^n)$ and

$$g(z) \perp \begin{pmatrix} 1 \\ 0_{(n-1)\times 1} \end{pmatrix}$$

for almost all $z \in \mathbb{T}$, or equivalently, $g \in \begin{pmatrix} 0 \\ L^2(\mathbb{T}, \mathbb{C}^{n-1}) \end{pmatrix}$. □

Proposition 3.2.17. *Under the assumptions of Theorem 3.2.10, where x_0 is a co-outer maximizing vector of unit norm for H_G, $\xi_0 \in H^\infty(\mathbb{D}, \mathbb{C}^n)$ is a vector-valued inner function given by $\xi_0 = \frac{x_0}{h_0}$, $V_0 = \begin{pmatrix} \xi_0 & \bar{\alpha}_0 \end{pmatrix}$ is a thematic completion of ξ_0 and $\mathcal{K}_1$ is defined by*

$$\mathcal{K}_1 = V_0 \begin{pmatrix} 0 \\ H^2(\mathbb{D}, \mathbb{C}^{n-1}) \end{pmatrix} \subseteq L^2(\mathbb{T}, \mathbb{C}^n),$$

we have

$$\xi_0 \dot\wedge \mathcal{K}_1 = \xi_0 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n)$$

and the operator

$$(\xi_0 \dot\wedge \cdot)\colon \mathcal{K}_1 \to \xi_0 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n)$$

is unitary.

Proof. Let us first prove $\xi_0 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n) \subset \xi_0 \dot\wedge \mathcal{K}_1$. Let $\xi_0 \dot\wedge \varphi \in \xi_0 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n)$, for some $\varphi \in H^2(\mathbb{D}, \mathbb{C}^n)$. Since V_0 is unitary-valued, we get

$$\xi_0 \xi_0^* + \bar{\alpha}_0 \alpha_0^T = I_n.$$

Thus

$$\begin{aligned}\xi_0 \dot\wedge \varphi &= \xi_0 \dot\wedge (\xi_0 \xi_0^* \varphi + \bar\alpha_0 \alpha_0^T \varphi)\\ &= \xi_0 \dot\wedge \xi_0 (\xi_0^* \varphi) + \xi_0 \dot\wedge \bar\alpha_0 (\alpha_0^T \varphi)\\ &= 0 + \xi_0 \dot\wedge \bar\alpha_0 (\alpha_0^T \varphi)\end{aligned}$$

on account of the pointwise linear dependence of ξ_0 and $\xi_0 \xi_0^* \varphi$ on $\mathbb{D}$. Recall that, by Lemma 3.2.15, $\alpha_0^T \varphi \in H^2(\mathbb{D}, \mathbb{C}^{n-1})$ and, by the definition of $\mathcal{K}_1$,

$$\mathcal{K}_1 = \bar\alpha_0 H^2(\mathbb{D}, \mathbb{C}^{n-1}).$$

Hence, for $\varphi \in H^2(\mathbb{D}, \mathbb{C}^n)$,

$$\xi_0 \dot\wedge \varphi = \xi_0 \dot\wedge \bar\alpha_0 \alpha_0^T \varphi \in \xi_0 \dot\wedge \bar\alpha_0 H^2(\mathbb{D}, \mathbb{C}^{n-1}),$$

and thus

$$\xi_0 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n) \subseteq \xi_0 \dot\wedge \mathcal{K}_1. \tag{3.58}$$

Let us now show that $\xi_0 \dot\wedge \mathcal{K}_1 \subseteq \xi_0 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n)$. Since $\mathcal{K}_1 = \bar\alpha_0 H^2(\mathbb{D}, \mathbb{C}^{n-1})$, an arbitrary element $u \in \xi_0 \dot\wedge \mathcal{K}_1$ is of the form

$$u = \xi_0 \dot\wedge \bar\alpha_0 g,$$

for some $g \in H^2(\mathbb{D}, \mathbb{C}^{n-1})$. Note that, by Lemma 3.2.15, there exists a function $f \in H^2(\mathbb{D}, \mathbb{C}^n)$ such that $g = \alpha_0^T f$. Hence $u = \xi_0 \dot\wedge \bar\alpha_0 \alpha_0^T f$. By equation (3.57), $\xi_0 \xi_0^* + \bar\alpha_0 \alpha_0^T = I_n$. Thus

$$u = \xi_0 \dot\wedge (I_{\mathbb{C}^n} - \xi_0 \xi_0^*) f = \xi_0 \dot\wedge f - \xi_0 \dot\wedge \xi_0 \xi_0^* f = \xi_0 \dot\wedge f \in \xi_0 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n),$$

and so, $\xi_0 \dot\wedge \mathcal{K}_1 \subseteq \xi_0 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n)$. Combining the latter inclusion with relation (3.58), we have

$$\xi_0 \dot\wedge \mathcal{K}_1 = \xi_0 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n).$$

Now, let us show that the operator $(\xi_0 \dot\wedge \cdot)\colon \mathcal{K}_1 \to \xi_0 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n)$ is unitary. As we have shown above, the operator is surjective. We will show it is also an isometry.
Let $f \in \mathcal{K}_1$. Then,

$$\begin{aligned}\|\xi_0 \dot\wedge f\|^2_{L^2(\mathbb{T}, \wedge^2 \mathbb{C}^n)} &= \langle \xi_0 \dot\wedge f, \xi_0 \dot\wedge f \rangle_{L^2(\mathbb{T}, \wedge^2 \mathbb{C}^n)}\\ &= \frac{1}{2\pi} \int_0^{2\pi} \langle \xi_0(e^{i\theta}) \wedge f(e^{i\theta}), \xi_0(e^{i\theta}) \wedge f(e^{i\theta}) \rangle_{\wedge^2 \mathbb{C}^n} d\theta\end{aligned}.$$

By Proposition 2.1.19, the latter integral is equal to

$$\frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix} \langle \xi_0(e^{i\theta}), \xi_0(e^{i\theta}) \rangle_{\mathbb{C}^n} & \langle \xi_0(e^{i\theta}), f(e^{i\theta}) \rangle_{\mathbb{C}^n} \\ \langle f(e^{i\theta}), \xi_0(e^{i\theta}) \rangle_{\mathbb{C}^n} & \langle f(e^{i\theta}), f(e^{i\theta}) \rangle_{\mathbb{C}^n} \end{pmatrix} \, d\theta,$$

which equals

$$\frac{1}{2\pi}\int_0^{2\pi} \|\xi_0(e^{i\theta})\|^2_{\mathbb{C}^n}\langle f(e^{i\theta}), f(e^{i\theta})\rangle_{\mathbb{C}^n} - |\langle \xi_0(e^{i\theta}), f(e^{i\theta})\rangle_{\mathbb{C}^n}|^2 \, d\theta.$$

Note that, by Proposition 3.2.1, $\|\xi_0(e^{i\theta})\|_{\mathbb{C}^n} = 1$ for almost all $e^{i\theta}$ on $\mathbb{T}$. Moreover, since

$$\mathcal{K}_1 = \bar{\alpha}_0 H^2(\mathbb{D}, \mathbb{C}^{n-1}),$$

$f = \bar{\alpha}_0 g$ for some $g \in H^2(\mathbb{D}, \mathbb{C}^{n-1})$. Hence

$$\langle \xi_0(e^{i\theta}), f(e^{i\theta})\rangle_{\mathbb{C}^n} = \langle \xi_0(e^{i\theta}), \bar{\alpha}_0(e^{i\theta}) g(e^{i\theta})\rangle_{\mathbb{C}^n} = \langle \alpha_0^T(e^{i\theta})\xi_0(e^{i\theta}), g(e^{i\theta})\rangle_{\mathbb{C}^{n-1}} = 0$$

almost everywhere on $\mathbb{T}$, since $V_0 = \begin{pmatrix} \xi_0 & \bar{\alpha}_0 \end{pmatrix}$ is unitary-valued. Thus

$$\|\xi_0 \dot\wedge f\|^2_{L^2(\mathbb{T}, \wedge^2\mathbb{C}^n)} = \|f\|^2_{L^2(\mathbb{T}, \mathbb{C}^n)},$$

that is, the operator $(\xi_0 \dot\wedge \cdot)\colon \mathcal{K}_1 \to \xi_0 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n)$ is an isometry. Therefore, by Theorem A.2.4, the operator $(\xi_0 \dot\wedge \cdot)$ is unitary. □

Lemma 3.2.18. *Let $u \in L^2(\mathbb{T}, \mathbb{C}^m)$ and let $\eta_0 \in H^\infty(\mathbb{D}, \mathbb{C}^m)$ be a vector-valued inner function. Then*

$$\langle \bar{\eta}_0 \dot\wedge u, \bar{\eta}_0 \dot\wedge \bar{z}\bar{f}\rangle_{L^2(\mathbb{T}, \wedge^2\mathbb{C}^m)} = 0 \quad \text{for all} \quad f \in H^2(\mathbb{D}, \mathbb{C}^m) \tag{3.59}$$

if and only if the function

$$z \mapsto u(z) - \langle u(z), \bar{\eta}_0(z)\rangle_{\mathbb{C}^m} \bar{\eta}_0(z)$$

belongs to $H^2(\mathbb{D}, \mathbb{C}^m)$.

Proof. The statement that $\bar{\eta}_0 \dot\wedge u$ is orthogonal to $\bar{\eta}_0 \dot\wedge \bar{z}\bar{f}$ in $L^2(\mathbb{T}, \wedge^2\mathbb{C}^m)$ is equivalent to the equation $I = 0$, where

$$I = \frac{1}{2\pi}\int_0^{2\pi} \langle \bar{\eta}_0(e^{i\theta}) \dot\wedge u(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \dot\wedge e^{-i\theta}\bar{f}(e^{i\theta})\rangle_{\wedge^2\mathbb{C}^m} \, d\theta.$$

By Proposition 2.1.19,

$$I = \frac{1}{2\pi}\int_0^{2\pi} \det\begin{pmatrix} \langle \bar{\eta}_0(e^{i\theta}), \bar{\eta}_0(e^{i\theta})\rangle_{\mathbb{C}^m} & \langle \bar{\eta}_0(e^{i\theta}), e^{-i\theta}\bar{f}(e^{i\theta})\rangle_{\mathbb{C}^m} \\ \langle u(e^{i\theta}), \overline{\eta_0}(e^{i\theta})\rangle_{\mathbb{C}^m} & \langle u(e^{i\theta}), e^{-i\theta}\bar{f}(e^{i\theta})\rangle_{\mathbb{C}^m} \end{pmatrix} d\theta.$$

Notice that, since η_0 is an inner function, $\|\bar{\eta}_0(e^{i\theta})\|_{\mathbb{C}^m} = 1$ almost everywhere on $\mathbb{T}$, and hence

$$I = \frac{1}{2\pi}\int_0^{2\pi} \det\begin{pmatrix} 1 & \langle \bar{\eta}_0(e^{i\theta}), e^{-i\theta}\bar{f}(e^{i\theta})\rangle_{\mathbb{C}^m} \\ \langle u(e^{i\theta}), \overline{\eta_0}(e^{i\theta})\rangle_{\mathbb{C}^m} & \langle u(e^{i\theta}), e^{-i\theta}\bar{f}(e^{i\theta})\rangle_{\mathbb{C}^m} \end{pmatrix} d\theta.$$

Calculations yield

$$\begin{aligned} I &= \frac{1}{2\pi}\int_0^{2\pi} \langle u(e^{i\theta}), e^{-i\theta}\bar{f}(e^{i\theta})\rangle_{\mathbb{C}^m} \\ &\qquad -\langle \bar{\eta}_0(e^{i\theta}), e^{-i\theta}\bar{f}(e^{i\theta})\rangle_{\mathbb{C}^m}\langle u(e^{i\theta}), \bar{\eta}_0(e^{i\theta})\rangle_{\mathbb{C}^m} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \langle u(e^{i\theta}), e^{-i\theta}\bar{f}(e^{i\theta})\rangle_{\mathbb{C}^m} \\ &\qquad - \left\langle \langle u(e^{i\theta}), \bar{\eta}_0(e^{i\theta})\rangle_{\mathbb{C}^m}\bar{\eta}_0(e^{i\theta}), e^{-i\theta}\bar{f}(e^{i\theta})\right\rangle_{\mathbb{C}^m} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\langle u(e^{i\theta}) - \langle u(e^{i\theta}), \bar{\eta}_0(e^{i\theta})\rangle_{\mathbb{C}^m}\bar{\eta}_0(e^{i\theta}), e^{-i\theta}\bar{f}(e^{i\theta})\right\rangle_{\mathbb{C}^m} d\theta. \end{aligned}$$

Thus condition (3.59) holds if and only if

$$\frac{1}{2\pi}\int_0^{2\pi} \langle \bar{\eta}_0(e^{i\theta})\dot{\wedge} u(e^{i\theta}), \bar{\eta}_0(e^{i\theta})\dot{\wedge} e^{-i\theta}\bar{f}(e^{i\theta})\rangle_{\wedge^2\mathbb{C}^m}\, d\theta = 0 \quad \text{for all} \quad f \in H^2(\mathbb{D},\mathbb{C}^m)$$

if and only if

$$\frac{1}{2\pi}\int_0^{2\pi} \left\langle u(e^{i\theta}) - \langle u(e^{i\theta}), \bar{\eta}_0(e^{i\theta})\rangle_{\mathbb{C}^m}\bar{\eta}_0(e^{i\theta}), e^{-i\theta}\bar{f}(e^{i\theta})\right\rangle_{\mathbb{C}^m} d\theta = 0$$

for all $f \in H^2(\mathbb{D},\mathbb{C}^m)$, and the latter equation holds if and only if

$$u(e^{i\theta}) - \langle u(e^{i\theta}), \bar{\eta}_0(e^{i\theta})\rangle_{\mathbb{C}^m}\bar{\eta}_0(e^{i\theta})$$

belongs to $H^2(\mathbb{D},\mathbb{C}^m)$. □

Lemma 3.2.19. *In the notation of Theorem 3.2.10,*

$$\mathcal{L}_1^\perp = \{f \in L^2(\mathbb{T},\mathbb{C}^m)\ : \beta_0^* f \in H^2(\mathbb{D},\mathbb{C}^{m-1})\}.$$

Proof. It is easy to see that $\mathcal{L}_1 = \beta_0 H^2(\mathbb{D},\mathbb{C}^{m-1})^\perp$. A typical element of $\mathcal{L}_1$ is $\beta_0\bar{z}\bar{g}$, for some $g \in H^2(\mathbb{D},\mathbb{C}^{m-1})$. A function $f \in L^2(\mathbb{T},\mathbb{C}^m)$ lies in $\mathcal{L}_1^\perp$ if and only if

$$\langle f, \beta_0\bar{z}\bar{g}\rangle_{L^2(\mathbb{T},\mathbb{C}^m)} = 0 \quad \text{for all} \quad g \in H^2(\mathbb{D},\mathbb{C}^{m-1}).$$

Equivalently, $f \in \mathcal{L}_1^\perp$ if and only if

$$\frac{1}{2\pi}\int_0^{2\pi} \langle f(e^{i\theta}), \beta_0(e^{i\theta})e^{-i\theta}\bar{g}(e^{i\theta})\rangle_{\mathbb{C}^m} d\theta = 0 \quad \text{for all} \quad g \in H^2(\mathbb{D},\mathbb{C}^{m-1})$$

if and only if

$$\frac{1}{2\pi}\int_0^{2\pi} \langle \beta_0(e^{i\theta})^* f(e^{i\theta}), e^{-i\theta}\bar{g}(e^{i\theta})\rangle_{\mathbb{C}^{m-1}} d\theta = 0 \quad \text{for all} \quad g \in H^2(\mathbb{D},\mathbb{C}^{m-1}).$$

The latter statement is equivalent to the assertion that $\beta_0^* f$ is orthogonal to $H^2(\mathbb{D},\mathbb{C}^{m-1})^\perp$

in $L^2(\mathbb{T}, \mathbb{C}^{m-1})$, which holds if and only if $\beta_0^* f$ belongs to $H^2(\mathbb{D}, \mathbb{C}^{m-1})$.
Hence

$$\mathcal{L}_1^{\perp} = \{f \in L^2(\mathbb{T}, \mathbb{C}^m) \ : \beta_0^* f \in H^2(\mathbb{D}, \mathbb{C}^{m-1})\}$$

as required. □

Proposition 3.2.20. *Under the assumptions of Theorem 3.2.10, let η_0 be defined by equation* (3.53) *and let $W_0^T = \begin{pmatrix} \eta_0 & \bar{\beta}_0 \end{pmatrix}$ be a thematic completion of η_0, where β_0 is an inner, co-outer, quasi-continuous function of type $m \times (m-1)$. Then,*

$$\beta_0^* H^2(\mathbb{D}, \mathbb{C}^m)^{\perp} = H^2(\mathbb{D}, \mathbb{C}^{m-1})^{\perp}.$$

Proof. By virtue of the fact that complex conjugation is a unitary operator on $L^2(\mathbb{T}, \mathbb{C}^m)$, an equivalent statement is that $\beta_0^T zH^2(\mathbb{D}, \mathbb{C}^m) = zH^2(\mathbb{D}, \mathbb{C}^{m-1})$. By Lemma 3.1.18, since β_0 is an inner, co-outer and quasi-continuous function, there exists a matrix-valued function $B_0 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1)\times m})$ such that

$$B_0 \beta_0 = I_{m-1}$$

or, equivalently,

$$\beta_0^T B_0^T = I_{m-1}.$$

Let $g \in zH^2(\mathbb{D}, \mathbb{C}^{m-1})$. Then,

$$g = (\beta_0^T B_0^T) g \in \beta_0^T B_0^T zH^2(\mathbb{D}, \mathbb{C}^{m-1}) \subseteq \beta_0^T zH^2(\mathbb{D}, \mathbb{C}^m).$$

Hence

$$zH^2(\mathbb{D}, \mathbb{C}^{m-1}) \subseteq \beta_0^T zH^2(\mathbb{D}, \mathbb{C}^m).$$

Note that, since $\beta_0 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times(m-1)})$, $\beta_0^T zH^2(\mathbb{D}, \mathbb{C}^{m-1}) \subseteq zH^2(\mathbb{D}, \mathbb{C}^{m-1})$, and so,

$$zH^2(\mathbb{D}, \mathbb{C}^{m-1}) \subseteq \beta_0^T zH^2(\mathbb{D}, \mathbb{C}^m) \subseteq zH^2(\mathbb{D}, \mathbb{C}^{m-1}).$$

Thus

$$\beta_0^T zH^2(\mathbb{D}, \mathbb{C}^m) = zH^2(\mathbb{D}, \mathbb{C}^{m-1}).$$

□

Proposition 3.2.21. *In the notation of Theorem 3.2.10, let $\eta_0 \in H^\infty(\mathbb{D}, \mathbb{C}^m)$ be a vector-valued inner function given by equation* (3.53)*, let $W_0^T = \begin{pmatrix} \eta_0 & \bar{\beta}_0 \end{pmatrix}$ be a thematic completion of η_0 given by equation* (3.54)*, and let*

$$\mathcal{L}_1 = W_0^* \begin{pmatrix} 0 \\ H^2(\mathbb{D}, \mathbb{C}^{m-1})^{\perp} \end{pmatrix}.$$

Then,

$$\bar{\eta}_0 \dot{\wedge} \mathcal{L}_1 = \bar{\eta}_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^{\perp}$$

and the operator

$$(\bar{\eta}_0 \dot{\wedge} \cdot) \colon \mathcal{L}_1 \to \bar{\eta}_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp$$

is unitary.

Proof. Let us first prove that $\bar{\eta}_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp \subseteq \bar{\eta}_0 \dot{\wedge} \mathcal{L}_1$. Consider an element

$$\bar{\eta}_0 \dot{\wedge} f \in \bar{\eta}_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp,$$

where $f \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$. Note that, since W_0^T is unitary valued, we have

$$\bar{\eta}_0 \eta_0^T + \beta_0 \beta_0^* = I_m. \tag{3.60}$$

Thus

$$\begin{aligned} \bar{\eta}_0 \dot{\wedge} f &= \bar{\eta}_0 \dot{\wedge} (\bar{\eta}_0 \eta_0^T + \beta_0 \beta_0^*) f \\ &= \bar{\eta}_0 \dot{\wedge} \bar{\eta}_0 \eta_0^T f + \bar{\eta}_0 \dot{\wedge} \beta_0 \beta_0^* f \\ &= 0 + \bar{\eta}_0 \dot{\wedge} \beta_0 \beta_0^* f, \end{aligned}$$

the last equality following by the pointwise linear dependence of $\bar{\eta}_0$ and $\bar{\eta}_0(\eta_0^T f)$ on $\mathbb{D}$. By Proposition 3.2.20,

$$\beta_0^* H^2(\mathbb{D}, \mathbb{C}^m)^\perp = H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp,$$

and, by the definition of $\mathcal{L}_1$, we have

$$\mathcal{L}_1 = \beta_0 H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp.$$

Hence, for $f \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$,

$$\bar{\eta}_0 \dot{\wedge} f = \bar{\eta}_0 \dot{\wedge} \beta_0 \beta_0^* f \in \bar{\eta}_0 \dot{\wedge} \beta_0 H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp,$$

and thus

$$\bar{\eta}_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp \subseteq \bar{\eta}_0 \dot{\wedge} \mathcal{L}_1.$$

Let us show

$$\bar{\eta}_0 \dot{\wedge} \mathcal{L}_1 \subseteq \bar{\eta}_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp.$$

A typical element in $\bar{\eta}_0 \dot{\wedge} \mathcal{L}_1$ is of the form

$$\bar{\eta}_0 \dot{\wedge} \beta_0 g,$$

for some $g \in H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp$. By Proposition 3.2.20, there exists a $\phi \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ such that $\beta_0^* \phi = g$. Then

$$\bar{\eta}_0 \dot{\wedge} \beta_0 g = \bar{\eta}_0 \dot{\wedge} \beta_0 \beta_0^* \phi.$$

By equation (3.60), we get

$$\bar{\eta}_0 \dot{\wedge} \beta_0 g = \bar{\eta}_0 \dot{\wedge} (I_{\mathbb{C}^m} - \bar{\eta}_0 \eta_0^T)\phi = \bar{\eta}_0 \dot{\wedge} \phi,$$

the last equality following by pointwise linear dependence of $\bar{\eta}_0$ and $\bar{\eta}_0(\eta_0^T \phi)$ on $\mathbb{D}$. Thus

$$\bar{\eta}_0 \dot{\wedge} \beta_0 g \in \bar{\eta}_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp,$$

and so $\bar{\eta}_0 \dot{\wedge} \mathcal{L}_1 \subseteq \bar{\eta}_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp$. Consequently

$$\bar{\eta}_0 \dot{\wedge} \mathcal{L}_1 = \bar{\eta}_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp.$$

To prove the operator

$$(\bar{\eta}_0 \dot{\wedge} \cdot) \colon \mathcal{L}_1 \to \bar{\eta}_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp$$

is unitary, it suffices to show that it is an isometry, since the preceding discussion asserts it is surjective. To this end, let $s \in \mathcal{L}_1$. Then,

$$\begin{aligned}
\|\bar{\eta}_0 \dot{\wedge} s\|^2_{L^2(\mathbb{T}, \wedge^2 \mathbb{C}^m)} &= \langle \bar{\eta}_0 \dot{\wedge} s, \bar{\eta}_0 \dot{\wedge} s \rangle_{L^2(\mathbb{T}, \wedge^2 \mathbb{C}^m)} \\
&= \frac{1}{2\pi} \int_0^{2\pi} \langle \bar{\eta}_0(e^{i\theta}) \dot{\wedge} s(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \dot{\wedge} s(e^{i\theta}) \rangle_{\wedge^2 \mathbb{C}^m} \, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix} \langle \bar{\eta}_0(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \bar{\eta}_0(e^{i\theta}), s(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ \langle s(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle s(e^{i\theta}), s(e^{i\theta}) \rangle_{\mathbb{C}^m} \end{pmatrix} d\theta.
\end{aligned}$$

By Proposition 3.2.1, $\|\bar{\eta}_0(z)\|_{\mathbb{C}^m} = 1$ almost everywhere on $\mathbb{T}$. Moreover, since $s \in \mathcal{L}_1$, there exists a function $\psi \in H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp$ such that $s = \beta_0 \psi$. Then

$$\langle \bar{\eta}_0(e^{i\theta}), s(e^{i\theta}) \rangle_{\mathbb{C}^m} = \langle \bar{\eta}_0(e^{i\theta}), \beta_0(e^{i\theta}) \psi(e^{i\theta}) \rangle_{\mathbb{C}^m} = \langle \beta_0^*(e^{i\theta}) \bar{\eta}_0(e^{i\theta}), \psi(e^{i\theta}) \rangle_{\mathbb{C}^m} = 0$$

almost everywhere on $\mathbb{T}$, which follows by the fact that W_0 is unitary-valued, and so

$$(W_0 W_0^*)(z) = \begin{pmatrix} \eta_0^T(z) \\ \beta_0^*(z) \end{pmatrix} \begin{pmatrix} \bar{\eta}_0(z) & \beta_0(z) \end{pmatrix} = \begin{pmatrix} \eta_0^T(z) \bar{\eta}_0(z) & \eta_0^T(z) \beta_0^T(z) \\ \beta_0^*(z) \bar{\eta}_0(z) & \beta_0^*(z) \beta_0(z) \end{pmatrix} = I_m$$

almost everywhere on $\mathbb{T}$.

Thus, for all $s \in \mathcal{L}_1$,

$$\|\bar{\eta}_0 \dot{\wedge} s\|^2_{L^2(\mathbb{T}, \wedge^2 \mathbb{C}^m)} = \|s\|^2_{L^2(\mathbb{T}, \mathbb{C}^m)},$$

which shows that the operator

$$(\bar{\eta}_0 \dot{\wedge} \cdot) \colon \mathcal{L}_1 \to \bar{\eta}_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp$$

is an isometry. We have proved it is also surjective, hence, by Theorem A.2.4, the operator is unitary. □

Continuation of the proof of Theorem 3.2.10.

(iii). We have to prove that diagram (3.56) commutes. Recall that, by Lemma 3.1.17, the left hand square commutes, so it suffices to show that that the right hand square, namely

$$\begin{array}{ccc} \mathcal{K}_1 & \xrightarrow{\xi_0\dot{\wedge}\cdot} & \xi_0\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^n) = X_1 \\ \Big\downarrow{\scriptstyle \Gamma_1} & & \Big\downarrow{\scriptstyle T_1} \\ \mathcal{L}_1 & \xrightarrow{\bar{\eta}_0\dot{\wedge}\cdot} & \bar{\eta}_0\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^m)^\perp = Y_1, \end{array} \tag{3.61}$$

also commutes. That is, we would like to prove that, for all $x \in \mathcal{K}_1$,

$$T_1(\xi_0\dot{\wedge}x) = \bar{\eta}_0\dot{\wedge}\Gamma_1(x),$$

where $\Gamma_1(x) = P_{\mathcal{L}_1}((G-Q_1)x)$ for any function $Q_1 \in H^\infty(\mathbb{D},\mathbb{C}^{m\times n})$ that satisfies the following equations

$$(G-Q_1)x_0 = t_0y_0, \quad y_0^*(G-Q_1) = t_0x_0^*.$$

By Proposition 3.2.17,

$$\xi_0\dot{\wedge}\mathcal{K}_1 = \xi_0\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^n),$$

and so, for every $x \in \mathcal{K}_1$, there exists $\tilde{x} \in H^2(\mathbb{D},\mathbb{C}^n)$ such that

$$\xi_0\dot{\wedge}x = \xi_0\dot{\wedge}\tilde{x}.$$

Thus, for $x \in \mathcal{K}_1$,

$$T_1(\xi_0\dot{\wedge}x) = T_1(\xi_0\dot{\wedge}\tilde{x}) = P_{Y_1}(\bar{\eta}_0\dot{\wedge}(G-Q_1)\tilde{x}),$$

and

$$\bar{\eta}_0\dot{\wedge}\Gamma_1(x) = \bar{\eta}_0\dot{\wedge}P_{\mathcal{L}_1}(G-Q_1)x.$$

Hence to prove the commutativity of diagram (3.61), it suffices to show that, for all $x \in \mathcal{K}_1$,

$$P_{Y_1}[\bar{\eta}_0\dot{\wedge}(G-Q_1)\tilde{x})] = \bar{\eta}_0\dot{\wedge}P_{\mathcal{L}_1}(G-Q_1)x$$

in Y_1, where $\xi_0\dot{\wedge}(x-\tilde{x}) = 0$. By Proposition 3.2.21,

$$\bar{\eta}_0\dot{\wedge}\mathcal{L}_1 = \bar{\eta}_0\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^m)^\perp = Y_1,$$

and so, for all $x \in \mathcal{K}_1$, $\bar{\eta}_0\dot{\wedge}P_{\mathcal{L}_1}(G-Q_1)x \in Y_1$. Let us show that, for $x \in \mathcal{K}_1$,

$$\bar{\eta}_0\dot{\wedge}(G-Q_1)\tilde{x} - \bar{\eta}_0\dot{\wedge}P_{\mathcal{L}_1}(G-Q_1)x$$

is orthogonal to Y_1 in $L^2(\mathbb{T},\wedge^2\mathbb{C}^m)$, or equivalently, that for every $f \in H^2(\mathbb{D},\mathbb{C}^m)$,

$$\left\langle \bar{\eta}_0\dot{\wedge}[(G-Q_1)\tilde{x} - P_{\mathcal{L}_1}(G-Q_1)x], \bar{\eta}_0\dot{\wedge}\bar{z}\bar{f}\right\rangle_{L^2(\mathbb{T},\wedge^2\mathbb{C}^m)} = 0 \tag{3.62}$$

for $x \in \mathcal{K}_1$ and for any $\tilde{x} \in H^2(\mathbb{D}, \mathbb{C}^n)$ such that $\xi_0 \dot{\wedge} \tilde{x} = \xi_0 \dot{\wedge} x$. By Lemma 3.2.9,

$$\bar{\eta}_0 \dot{\wedge} (G - Q_1)x = \bar{\eta}_0 \dot{\wedge} (G - Q_1)\tilde{x}.$$

Then equation (3.62) is equivalent to the equation

$$\langle \bar{\eta}_0 \dot{\wedge} P_{\mathcal{L}_1^\perp}(G - Q_1)x, \bar{\eta}_0 \dot{\wedge} \bar{z}\bar{f} \rangle_{L^2(\mathbb{T}, \wedge^2\mathbb{C}^m)} = 0 \tag{3.63}$$

for any $x \in \mathcal{K}_1$. By Lemma 3.2.18, equation (3.63) holds if and only if the function

$$z \mapsto [P_{\mathcal{L}_1^\perp}(G - Q_1)x](z) - \langle [P_{\mathcal{L}_1^\perp}(G - Q_1)x](z), \bar{\eta}_0(z) \rangle_{\mathbb{C}^m} \bar{\eta}_0(z) \tag{3.64}$$

belongs to $H^2(\mathbb{D}, \mathbb{C}^m)$. By Lemma 3.2.19, there exists a function $\psi \in L^2(\mathbb{T}, \mathbb{C}^m)$ such that

$$P_{\mathcal{L}_1^\perp}(G - Q_1)x = \psi, \tag{3.65}$$

$$\beta_0^* \psi \in H^2(\mathbb{D}, \mathbb{C}^{m-1}).$$

Equation (3.65) implies

$$(G - Q_1)x - \psi \in \mathcal{L}_1 = \beta_0 H^2(\mathbb{D}, \mathbb{C}^{m-1})^\perp.$$

Hence, to prove that the function defined by equation (3.64) belongs to $H^2(\mathbb{D}, \mathbb{C}^m)$, we have to show that

$$\psi - (\eta_0^T \psi)\bar{\eta}_0 \in H^2(\mathbb{D}, \mathbb{C}^m).$$

Since $W_0 = \begin{pmatrix} \eta_0 & \overline{\beta_0} \end{pmatrix}^T$ is a unitary-valued function,

$$\bar{\eta}_0(z)\eta_0^T(z) + \beta_0(z)\beta_0^*(z) = I_m$$

almost everywhere on $\mathbb{T}$. Since $\eta_0^T \psi$ is a scalar-valued function,

$$\begin{aligned} \psi - \eta_0^T \psi \bar{\eta}_0 &= (I_m - \bar{\eta}_0 \eta_0^T)\psi \\ &= \beta_0 \beta_0^* \psi \in \beta_0 H^2(\mathbb{D}, \mathbb{C}^{m-1}) \subset H^2(\mathbb{D}, \mathbb{C}^m). \end{aligned}$$

Recall that $\beta_0^* \psi \in H^2(\mathbb{D}, \mathbb{C}^{m-1})$, and so $\beta_0 \beta_0^* \psi \in H^2(\mathbb{D}, \mathbb{C}^m)$. Thus diagram (3.61) commutes.
(iv). By Lemma 3.1.12,

$$F_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1)\times(n-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-1)\times(n-1)}).$$

Then, by Hartman's Theorem 3.1.2, the Hankel operator H_{F_1} is compact, and by (iii),

$$(\bar{\eta}_0 \dot{\wedge} \cdot) \circ (U_2 H_{F_1} U_1^*) \circ (\xi_0 \dot{\wedge} \cdot)^* = T_1.$$

By (i) and (ii), the operators U_1, U_2, $(\xi_0 \dot\wedge \cdot)$ and $(\bar\eta_0 \dot\wedge \cdot)$ are unitary. Hence T_1 is a compact operator.
(v). Since diagram (3.56) is commutative and $U_1, U_2, (\xi_0 \dot\wedge \cdot)$ and $(\eta_0 \dot\wedge \cdot)$ are unitaries,

$$\|T_1\| = \|\Gamma_1\| = \|H_{F_1}\|. \qquad \square$$

In what follows, we will prove an analogous statement to Theorem 3.2.10 for T_2. To this end, we need the following results.

Lemma 3.2.22. *In the notation of Theorem 3.2.10, $v_1 \in H^2(\mathbb{D}, \mathbb{C}^n)$ and $w_1 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ are such that $(\xi_0 \dot\wedge v_1, \bar\eta_0 \dot\wedge w_1)$ is a Schmidt pair for the operator T_1 corresponding to $\|T_1\|$. Then* (i) *there exist $x_1 \in \mathcal{K}_1$ and $y_1 \in \mathcal{L}_1$ such that (x_1, y_1) is a Schmidt pair for the operator Γ_1;* (ii) *for any $x_1 \in \mathcal{K}_1$ and $y_1 \in \mathcal{L}_1$ such that*

$$\xi_0 \dot\wedge x_1 = \xi_0 \dot\wedge v_1, \quad \bar\eta_0 \dot\wedge y_1 = \bar\eta_0 \dot\wedge w_1,$$

the pair (x_1, y_1) is a Schmidt pair for Γ_1 corresponding to $\|\Gamma_1\|$.

Proof. (i). By Theorem 3.2.10, the diagram (3.56) commutes, $(\xi_0 \dot\wedge \cdot)$ is unitary from $\mathcal{K}_1$ to X_1, and $(\bar\eta_0 \dot\wedge \cdot)$ is unitary from $\mathcal{L}_1$ to Y_1. Thus $\|\Gamma_1\| = \|T_1\| = t_1$. Moreover, by Lemma 3.1.17, the operator $\Gamma_1 \colon \mathcal{K}_1 \to \mathcal{L}_1$ is compact, hence there exist $x_1 \in \mathcal{K}_1$, $y_1 \in \mathcal{L}_1$ such that (x_1, y_1) is a Schmidt pair for Γ_1 corresponding to $\|\Gamma_1\| = t_1$.
(ii). Suppose that $x_1 \in \mathcal{K}_1, y_1 \in \mathcal{L}_1$ satisfy

$$\xi_0 \dot\wedge x_1 = \xi_0 \dot\wedge v_1, \tag{3.66}$$

$$\bar\eta_0 \dot\wedge y_1 = \bar\eta_0 \dot\wedge w_1. \tag{3.67}$$

Let us show that (x_1, y_1) is a Schmidt pair for Γ_1 corresponding to t_1, that is,

$$\Gamma_1 x_1 = t_1 y_1, \quad \Gamma_1^* y_1 = t_1 x_1.$$

Since diagram (3.61) commutes,

$$T_1 \circ (\xi_0 \dot\wedge \cdot) = (\bar\eta_0 \dot\wedge \cdot) \circ \Gamma_1, \quad (\xi_0 \dot\wedge \cdot)^* \circ T_1^* = \Gamma_1^* \circ (\bar\eta_0 \dot\wedge \cdot)^*. \tag{3.68}$$

By hypothesis,

$$T_1(\xi_0 \dot\wedge v_1) = t_1(\bar\eta_0 \dot\wedge w_1), \quad T_1^*(\bar\eta_0 \dot\wedge w_1) = t_1(\xi_0 \dot\wedge v_1). \tag{3.69}$$

Thus, by equations (3.67), (3.68) and (3.69),

$$\begin{aligned} \Gamma_1 x_1 &= (\bar\eta_0 \dot\wedge \cdot)^* T_1(\xi_0 \dot\wedge v_1) \\ &= (\bar\eta_0 \dot\wedge \cdot)^* t_1(\bar\eta_0 \dot\wedge w_1) \\ &= t_1(\bar\eta_0 \dot\wedge \cdot)^*(\bar\eta_0 \dot\wedge y_1). \end{aligned}$$

Hence

$$\Gamma_1 x_1 = t_1(\bar\eta_0 \dot\wedge \cdot)^*(\bar\eta_0 \dot\wedge \cdot) y_1 = t_1 y_1.$$

By equation (3.66),

$$x_1 = (\xi_0 \dot\wedge \cdot)^*(\xi_0 \dot\wedge v_1),$$

and, by equation (3.67),

$$(\bar\eta_0 \dot\wedge \cdot)^*(\bar\eta_0 \dot\wedge w_1) = y_1.$$

Thus

$$\begin{aligned}\Gamma_1^* y_1 &= \Gamma_1^*(\bar\eta_0 \dot\wedge \cdot)^*(\bar\eta_0 \dot\wedge w_1)\\ &= (\xi_0 \dot\wedge \cdot)^* T_1^*(\bar\eta_0 \dot\wedge w_1),\end{aligned}$$

the last equality following by the second equation of (3.68). By equations (3.66) and (3.69), we get

$$T_1^*(\bar\eta_0 \dot\wedge w_1) = t_1(\xi_0 \dot\wedge v_1) = t_1(\xi_0 \dot\wedge x_1),$$

and so,

$$\Gamma_1^* y_1 = t_1 x_1.$$

Therefore (x_1, y_1) is a Schmidt pair for Γ_1 corresponding to $\|\Gamma_1\| = \|T_1\| = t_1$. □

Lemma 3.2.23. *Suppose $(\xi_0 \dot\wedge v_1, \bar\eta_0 \dot\wedge w_1)$ is a Schmidt pair for T_1 corresponding to t_1. Let*

$$x_1 = (I_n - \xi_0 \xi_0^*) v_1, \quad y_1 = (I_m - \bar\eta_0 \eta_0^T) w_1,$$

and let

$$\hat x_1 = \alpha_0^T x_1, \quad \hat y_1 = \beta_0^* y_1.$$

Then
(i)

$$x_1 = \bar\alpha_0 \alpha_0^T x_1, \quad y_1 = \beta_0 \beta_0^* y_1. \tag{3.70}$$

(ii) The pair $(\hat x_1, \hat y_1)$ is a Schmidt pair for H_{F_1} corresponding to $\|H_{F_1}\| = t_1$.

Proof. (i). Since $V_0 = \begin{pmatrix}\xi_0 & \bar\alpha_0\end{pmatrix}$ is unitary-valued, $I_n - \xi_0\xi_0^* = \bar\alpha_0 \alpha_0^T$, and so

$$\begin{aligned}\bar\alpha_0 \alpha_0^T x_1 &= (I_n - \xi_0\xi_0^*)(I_n - \xi_0\xi_0^*) v_1\\ &= (I_n - 2\xi_0\xi_0^* + \xi_0\xi_0^*\xi_0\xi_0^*) v_1\\ &= (I_n - \xi_0\xi_0^*) v_1 = x_1.\end{aligned} \tag{3.71}$$

Similarly, since $W_0^T = \begin{pmatrix}\eta_0 & \bar\beta_0\end{pmatrix}$ is unitary valued, $I_m - \bar\eta_0\eta_0^T = \beta_0\beta_0^*$, and so

$$\begin{aligned}\beta_0\beta_0^* y_1 &= (I_m - \bar\eta_0\eta_0^T)(I_m - \bar\eta_0\eta_0^T) w_1\\ &= (I_m - 2\bar\eta_0\eta_0^T + \bar\eta_0\eta_0^T\bar\eta_0\eta_0^T) w_1\\ &= (I_m - \bar\eta_0\eta_0^T) w_1 = y_1.\end{aligned} \tag{3.72}$$

(ii) Recall that, by Lemma 3.1.17, the maps

$$U_1\colon H^2(\mathbb{D},\mathbb{C}^{n-1})\to\mathcal{K}_1,\quad U_2\colon H^2(\mathbb{D},\mathbb{C}^{m-1})^\perp\to\mathcal{L}_1,$$

defined by

$$U_1\chi=V_0\begin{pmatrix}0\\ \chi\end{pmatrix}=\bar{\alpha}_0\chi,\quad U_2\psi=W_0^*\begin{pmatrix}0\\ \psi\end{pmatrix}=\beta_0\psi$$

for all $\chi\in H^2(\mathbb{D},\mathbb{C}^{n-1})$ and all $\psi\in H^2(\mathbb{D},\mathbb{C}^{m-1})^\perp$, are unitaries. By the commutativity of the diagram (3.56),

$$H_{F_1}=U_2^*\Gamma_1 U_1. \tag{3.73}$$

By Part (i), $x_1\in\mathcal{K}_1$ and $y_1\in\mathcal{L}_1$ and, by Proposition 3.2.1,

$$\xi_0\dot{\wedge}x_1=\xi_0\dot{\wedge}v_1,\quad \bar{\eta}_0\dot{\wedge}y_1=\bar{\eta}_0\dot{\wedge}w_1.$$

Thus, by Lemma 3.2.22, (x_1,y_1) is a Schmidt pair for the operator Γ_1 corresponding to $t_1=\|\Gamma_1\|$, that is,

$$\Gamma_1x_1=t_1y_1,\quad \Gamma_1^*y_1=t_1x_1. \tag{3.74}$$

To prove that the pair $(\hat{x}_1,\hat{y}_1)$ is a Schmidt pair for H_{F_1} corresponding to $\|H_{F_1}\|=t_1$, we need to show that

$$H_{F_1}\hat{x}_1=t_1\hat{y}_1,\quad H_{F_1}^*\hat{y}_1=t_1\hat{x}_1.$$

By equations (3.73) and (3.70), we have

$$\begin{aligned}H_{F_1}\hat{x}_1&=H_{F_1}\alpha_0^T\hat{x}_1\\ &=U_2^*\Gamma_1U_1\alpha_0^Tx_1=U_2^*\Gamma_1\bar{\alpha}_0\alpha_0^Tx_1\\ &=U_2^*\Gamma_1x_1=t_1\beta_0^*y_1=t_1\hat{y}_1.\end{aligned} \tag{3.75}$$

Let us show that $H_{F_1}^*\hat{y}_1=t_1\hat{x}_1$. By equations (3.73) and (3.70), we have

$$\begin{aligned}H_{F_1}^*\hat{y}_1&=H_{F_1}^*\beta_0^*y_1\\ &=U_1^*\Gamma_1^*U_2\beta_0^*y_1=U_1^*\Gamma_1^*\beta_0\beta_0^*y_1\\ &=U_1^*\Gamma_1^*y_1=t_1U_1^*x_1=t_1\alpha_0^Tx_1=t_1\hat{x}_1.\end{aligned} \tag{3.76}$$

Therefore $(\hat{x}_1,\hat{y}_1)$ is a Schmidt pair for H_{F_1} corresponding to $\|H_{F_1}\|=t_1$. □

Proposition 3.2.24. *Let $(\xi_0\dot{\wedge}v_1,\bar{\eta}_0\dot{\wedge}w_1)$ be a Schmidt pair for T_1 corresponding to t_1 for some $v_1\in H^2(\mathbb{D},\mathbb{C}^n), w_1\in H^2(\mathbb{D},\mathbb{C}^m)^\perp$, let $h_1\in H^2(\mathbb{D},\mathbb{C})$ be the scalar outer factor of $\xi_0\dot{\wedge}v_1$, let*

$$x_1=(I_n-\xi_0\xi_0^*)v_1,\quad y_1=(I_m-\bar{\eta}_0\eta_0^T)w_1,$$

and let

$$\hat{x}_1=\alpha_0^Tx_1,\quad \hat{y}_1=\beta_0^*y_1.$$

Then

$$\|\hat{x}_1(z)\|_{\mathbb{C}^{n-1}} = \|\hat{y}_1(z)\|_{\mathbb{C}^{m-1}} = |h_1(z)|,$$

$$\|x_1(z)\|_{\mathbb{C}^n} = \|y_1(z)\|_{\mathbb{C}^m} = |h_1(z)|$$

and

$$\|\xi_0(z) \wedge v_1(z)\|_{\wedge^2\mathbb{C}^n} = \|\bar{\eta}_0(z) \wedge w_1(z)\|_{\wedge^2\mathbb{C}^m} = |h_1(z)|$$

almost everywhere on $\mathbb{T}$.

Proof. By Lemma 3.2.23, $(\hat{x}_1, \hat{y}_1)$ is a Schmidt pair for H_{F_1} corresponding to $\|H_{F_1}\| = t_1$. Hence

$$H_{F_1}\hat{x}_1 = t_1\hat{y}_1 \quad \text{and} \quad H_{F_1}^*\hat{y}_i = t_1\hat{x}_1.$$

By Theorem D.2.4, for the Hankel operator H_{F_1} and the Schmidt pair $(\hat{x}_1, \hat{y}_1)$, we have

$$\|\hat{y}_1(z)\|_{\mathbb{C}^{m-1}} = \|\hat{x}_1(z)\|_{\mathbb{C}^{n-1}} \tag{3.77}$$

almost everywhere on $\mathbb{T}$.

By equations (3.70),

$$x_1 = \bar{\alpha}_0\alpha_0^T x_1 = \bar{\alpha}_0\hat{x}_1, \quad y_1 = \beta_0\beta_0^* y_1 = \beta_0\hat{y}_1.$$

Since $\bar{\alpha}_0(z)$ and $\beta_0(z)$ are isometric for almost every $z \in \mathbb{T}$,

$$\|x_1(z)\|_{\mathbb{C}^n} = \|\hat{x}_1(z)\|_{\mathbb{C}^{n-1}} \text{ and } \|y_1(z)\|_{\mathbb{C}^m} = \|\hat{y}_1(z)\|_{\mathbb{C}^{m-1}}$$

almost everywhere on $\mathbb{T}$. By equations (3.77), we deduce

$$\|x_1(z)\|_{\mathbb{C}^n} = \|y_1(z)\|_{\mathbb{C}^m} \tag{3.78}$$

almost everywhere on $\mathbb{T}$.

By Theorem 3.2.10, $(\xi_0\dot{\wedge}\cdot)$ is an isometry from $\mathcal{K}_1$ to X_1, and $(\bar{\eta}_0\dot{\wedge}\cdot)$ is an isometry from $\mathcal{L}_1$ to Y_1. By Proposition 3.2.1,

$$\xi_0\dot{\wedge}x_1 = \xi_0\dot{\wedge}v_1, \quad \bar{\eta}_0\dot{\wedge}y_1 = \bar{\eta}_0\dot{\wedge}w_1.$$

Hence

$$\|\xi_0(z) \wedge v_1(z)\|_{\wedge^2\mathbb{C}^n} = \|\xi_0(z) \wedge x_1(z)\|_{\wedge^2\mathbb{C}^n} = \|x_1(z)\|_{\mathbb{C}^n}$$

almost everywhere on $\mathbb{T}$. Also

$$\|\bar{\eta}_0(z) \wedge w_1(z)\|_{\wedge^2\mathbb{C}^m} = \|\bar{\eta}_0(z) \wedge y_1(z)\|_{\wedge^2\mathbb{C}^m} = \|y_1(z)\|_{\mathbb{C}^m}$$

almost everywhere on $\mathbb{T}$. Thus, by equation (3.78),

$$\|\xi_0(z) \wedge v_1(z)\|_{\wedge^2\mathbb{C}^n} = \|\bar{\eta}_0(z) \wedge w_1(z)\|_{\wedge^2\mathbb{C}^m} \quad \text{almost everywhere on } \mathbb{T}.$$

Recall that h_1 is the scalar outer factor of $\xi_0\dot{\wedge}v_1$. Hence

$$\|\xi_0(z) \wedge v_1(z)\|_{\wedge^2\mathbb{C}^n} = \|\bar{\eta}_0(z) \wedge w_1(z)\|_{\wedge^2\mathbb{C}^m} = |h_1(z)|,$$

$$\|x_1(z)\|_{\mathbb{C}^n} = \|y_1(z)\|_{\mathbb{C}^m} = |h_1(z)|$$

and

$$\|\hat{x}_1(z)\|_{\mathbb{C}^{n-1}} = \|\hat{y}_1(z)\|_{\mathbb{C}^{m-1}} = |h_1(z)|$$

almost everywhere on $\mathbb{T}$. □

Definition 3.2.25. *Given $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$ and $0 \leq j \leq \min(m, n)$, define Ω_j to be* the set of level j superoptimal analytic approximants to G*, that is, the set of $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ which minimise the tuple*

$$\left(s_0^\infty(G - Q), s_1^\infty(G - Q), \ldots, s_j^\infty(G - Q)\right)$$

with respect to the lexicographic ordering over $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$. For $Q \in \Omega_j$ we call $G - Q$ a level j superoptimal error function*, and we denote by $\mathcal{E}_j$ the* set of all level j superoptimal error functions*, that is*

$$\mathcal{E}_j = \{G - Q \ : \ Q \in \Omega_j\}.$$

Proposition 3.2.26. *Let m, n be positive integers such that $\min(m, n) \geq 2$. Let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$. In line with the algorithm from Section 3.2.1, let $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ satisfy*

$$(G - Q_1)x_0 = t_0 y_0, \quad (G - Q_1)^* y_0 = t_0 x_0.$$

Let the spaces X_1, Y_1 be given by

$$X_1 = \xi_0\dot{\wedge}H^2(\mathbb{D}, \mathbb{C}^n) \subset H^2(\mathbb{D}, \wedge^2\mathbb{C}^n), \quad Y_1 = \bar{\eta}_0\dot{\wedge}H^2(\mathbb{D}, \mathbb{C}^m)^\perp \subset H^2(\mathbb{D}, \wedge^2\mathbb{C}^m)^\perp,$$

and consider the compact operator $T_1 \colon X_1 \to Y_1$ given by

$$T_1(\xi_0\dot{\wedge}x) = P_{Y_1}(\bar{\eta}_0\dot{\wedge}(G - Q_1)x)$$

for all $x \in H^2(\mathbb{D}, \mathbb{C}^n)$. Let $(\xi_0\dot{\wedge}v_1, \bar{\eta}_0\dot{\wedge}w_1)$ be a Schmidt pair for the operator T_1 corresponding to $t_1 = \|T_1\|$, let $h_1 \in H^2(\mathbb{D}, \mathbb{C})$ be the scalar outer factor of $\xi_0\dot{\wedge}v_1$, let

$$x_1 = (I_{\mathbb{C}^n} - \xi_0\xi_0^*)v_1, \quad y_1 = (I_{\mathbb{C}^m} - \bar{\eta}_0\eta_0^T)w_1$$

and let

$$\xi_1 = \frac{x_1}{h_1}, \quad \eta_1 = \frac{\bar{z}\bar{y}_1}{h_1}.$$

Then, there exist unitary-valued functions $\tilde{V}_1, \tilde{W}_1$ of types $(n-1) \times (n-1)$,

$(m-1)\times(m-1)$ respectively of the form

$$\tilde{V}_1 \stackrel{\text{def}}{=} \begin{pmatrix} \alpha_0^T\xi_1 & \overline{\alpha}_1 \end{pmatrix} \tag{3.79}$$

and

$$\tilde{W}_1^T \stackrel{\text{def}}{=} \begin{pmatrix} \beta_0^T\eta_1 & \overline{\beta}_1 \end{pmatrix}, \tag{3.80}$$

where α_1, β_1 are inner, co-outer, quasi-continuous functions of types $(n-1)\times(n-2)$, $(m-1)\times(m-2)$ respectively, and all minors on the first columns of $\tilde{V}_1, \tilde{W}_1^T$ are in H^∞.

Furthermore, the set of all level 1 superoptimal error functions $\mathcal{E}_1$ satisfies

$$\mathcal{E}_1 = W_0^* \begin{pmatrix} 1 & 0 \\ 0 & \tilde{W}_1^* \end{pmatrix} \begin{pmatrix} t_0u_0 & 0 & 0 \\ 0 & t_1u_1 & 0 \\ 0 & 0 & F_2 + H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)}) \cap B(t_1) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{V}_1^* \end{pmatrix} V_0^*, \tag{3.81}$$

where $F_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-2)\times(n-2)})$, $u_1 = \frac{\bar{z}\bar{h}_1}{h_1}$ is a quasi-continuous unimodular function and V_0, W_0^T are as in Theorem 3.2.10, and $B(t_1)$ is the closed ball of radius t_1 in $L^\infty(\mathbb{T}, \mathbb{C}^{(m-2)\times(n-2)})$.

Proof. By Theorem 3.2.10, the following diagram commutes

$$\begin{array}{ccccc} H^2(\mathbb{D},\mathbb{C}^{n-1}) & \xrightarrow{U_1} & \mathcal{K}_1 & \xrightarrow{\xi_0\dot{\wedge}\cdot} & \xi_0\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^n) = X_1 \\ \Big\downarrow{\scriptstyle H_{F_1}} & & \Big\downarrow{\scriptstyle \Gamma_1} & & \Big\downarrow{\scriptstyle T_1} \\ H^2(\mathbb{D},\mathbb{C}^{m-1})^\perp & \xrightarrow{U_2} & \mathcal{L}_1 & \xrightarrow{\bar{\eta}_0\dot{\wedge}\cdot} & \bar{\eta}_0\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^m)^\perp = Y_1. \end{array} \tag{3.82}$$

Let $\hat{x}_1 = \alpha_0^T x_1,\ \hat{y}_1 = \beta_0^* y_1$. By Lemma 3.2.23, $(\hat{x}_1, \hat{y}_1)$ is a Schmidt pair for H_{F_1} corresponding to t_1. By equations (3.70),

$$x_1 = \bar{\alpha}_0\alpha_0^T x_1 = \bar{\alpha}_0\hat{x}_1 \text{ and } y_1 = \beta_0\beta_0^* y_1 = \beta_0\hat{y}_1.$$

We want to apply Lemma 3.1.12 to H_{F_1} and the Schmidt pair $(\hat{x}_1, \hat{y}_1)$ to find unitary-valued functions $\tilde{V}_1, \tilde{W}_1$ such that, for any function $\tilde{Q}_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1)\times(n-1)})$ which is at minimal distance from F_1, the following equation holds

$$F_1 - \tilde{Q}_1 = \tilde{W}_1^* \begin{pmatrix} t_1u_1 & 0 \\ 0 & F_2 \end{pmatrix} \tilde{V}_1^*,$$

for some $F_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-2)\times(n-2)})$. For this purpose we find the inner-outer factorisations $\hat{x}_1$ and $\bar{z}\bar{\hat{y}}_1$. By Proposition 3.2.24,

$$\begin{aligned} &\|\hat{x}_1(z)\|_{\mathbb{C}^{n-1}} = \|x_1(z)\|_{\mathbb{C}^n} = \|\xi_0(z)\dot{\wedge}v_1(z)\|_{\wedge^2\mathbb{C}^n} = |h_1(z)| \\ \text{and } &\|\hat{y}_1(z)\|_{\mathbb{C}^{m-1}} = \|y_1(z)\|_{\mathbb{C}^m} = \|\bar{\eta}_0(z)\dot{\wedge}w_1(z)\|_{\wedge^2\mathbb{C}^m} = |h_1(z)| \end{aligned} \tag{3.83}$$

almost everywhere on $\mathbb{T}$. Equations (3.83) imply that $h_1 \in H^2(\mathbb{D}, \mathbb{C})$ is the scalar outer factor of both $\hat{x}_1$ and $\bar{z}\bar{\hat{y}}_1$. By Lemma 3.1.12, $\hat{x}_1, \bar{z}\bar{\hat{y}}_1$ admit the inner-outer factorisations

$$\hat{x}_1 = \hat{\xi}_1 h_1, \quad \bar{z}\bar{\hat{y}}_1 = \hat{\eta}_1 h_1,$$

for some inner vector-valued $\hat{\xi}_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{n-1})$ and $\hat{\eta}_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{m-1})$. Recall that

$$\hat{x}_1 = \alpha_0^T x_1 = \alpha_0^T \xi_1 h_1, \quad \bar{z}\bar{\hat{y}}_1 = \bar{z}\beta_0^T \bar{y}_1 = \beta_0^T \eta_1 h_1,$$

which imply

$$\hat{\xi}_1 = \alpha_0^T \xi_1 \quad \text{and} \quad \hat{\eta}_1 = \beta_0^T \eta_1.$$

Let us show that $\alpha_0^T \xi_1$, $\beta_0^T \eta_1$ are inner in order to apply Lemma 3.1.12. Recall that, since V_0, W_0^T are unitary-valued, we have

$$I_n - \xi_0 \xi_0^* = \bar{\alpha}_0 \alpha_0^T, \quad I_m - \bar{\eta}_0 \eta_0^T = \beta_0 \beta_0^*.$$

Therefore

$$x_1 = (I_{\mathbb{C}^n} - \xi_0 \xi_0^*) v_1 = \bar{\alpha}_0 \alpha_0^T v_1, \quad y_1 = (I_{\mathbb{C}^m} - \bar{\eta}_0 \eta_0^T) w_1 = \beta_0 \beta_0^* w_1.$$

Then,

$$\alpha_0^T x_1 = \alpha_0^T v_1, \quad \beta_0^T \bar{y}_1 = \beta_0^T \bar{w}_1, \tag{3.84}$$

and since

$$\xi_1 = \frac{x_1}{h_1}, \quad \eta_1 = \frac{\bar{z}\bar{y}_1}{h_1},$$

we find that the functions

$$\alpha_0^T \xi_1 = \frac{\alpha_0^T v_1}{h_1}, \quad \beta_0^T \eta_1 = \frac{\beta_0^T \bar{z}\bar{w}_1}{h_1}$$

are analytic. Furthermore, by Proposition 3.2.24,

$$\|x_1(z)\|_{\mathbb{C}^n} = \|y_1(z)\|_{\mathbb{C}^m} = |h_1(z)| = \|\hat{x}_1(z)\|_{\mathbb{C}^{n-1}} = \|\hat{y}_1(z)\|_{\mathbb{C}^{m-1}}$$

almost everywhere on $\mathbb{T}$. Thus

$$\|\alpha_0^T(z) x_1(z)\|_{\mathbb{C}^{n-1}} = \|\alpha_0^T(z) v_1(z)\|_{\mathbb{C}^{n-1}} = |h_1(z)|$$

and

$$\|\beta_0^T(z) \bar{z}\bar{y}_1(z)\|_{\mathbb{C}^{m-1}} = \|\beta_0^T(z) \bar{z}\bar{w}_1(z)\|_{\mathbb{C}^{m-1}} = |h_1(z)|$$

almost everywhere on $\mathbb{T}$. Hence

$$\|\alpha_0^T(z) \xi_1(z)\|_{\mathbb{C}^{n-1}} = 1, \quad \|\beta_0^T(z) \eta_1(z)\|_{\mathbb{C}^{m-1}} = 1$$

almost everywhere on $\mathbb{T}$. Therefore $\alpha_0^T\xi_1$, $\beta_0^T\eta_1$ are inner functions. By Lemma 3.1.12, there exist inner, co-outer, quasi-continuous functions α_1, β_1 of types $(n-1)\times(n-2)$ and $(m-1)\times(m-2)$ respectively such that

$$\tilde{V}_1 = \begin{pmatrix} \alpha_0^T\xi_1 & \overline{\alpha}_1 \end{pmatrix}, \quad \tilde{W}_1^T = \begin{pmatrix} \beta_0^T\eta_1 & \overline{\beta}_1 \end{pmatrix}$$

are unitary-valued and all minors on the first columns are in H^∞. Furthermore, by Lemma 3.1.12, every $\hat{Q}_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1)\times(n-1)})$ which is at minimal distance from F_1 satisfies

$$F_1 - \hat{Q}_1 = \tilde{W}_1^* \begin{pmatrix} t_1u_1 & 0 \\ 0 & F_2 \end{pmatrix} \tilde{V}_1^*,$$

for some $F_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-2)\times(n-2)})$ and u_1 quasi-continuous unimodular function given by $u_1 = \frac{\bar{z}\bar{h}_1}{h_1}$.

By Lemma 3.1.15, the set

$$\tilde{\mathcal{E}}_0 = \{F_1 - \hat{Q} : \hat{Q} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1)\times(n-1)}), \|F_1 - \hat{Q}\|_{L^\infty} = t_1\}$$

satisfies

$$\tilde{\mathcal{E}}_0 = \tilde{W}_1^* \begin{pmatrix} t_1u_1 & 0 \\ 0 & (F_2 + H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)})) \cap B(t_1) \end{pmatrix} V_1^*,$$

for some F_2 as described above and for the closed ball of radius t_1 in $L^\infty(\mathbb{T}, \mathbb{C}^{(m-2)\times(n-2)})$ denoted by $B(t_1)$. Thus, by Lemma 3.1.15, $\mathcal{E}_1$ admits the factorisation (3.81) as claimed. □

Proposition 3.2.27. *Suppose the function $Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ minimises*

$$(s_0^\infty(G-Q), s_1^\infty(G-Q)).$$

Then Q_2 satisfies

$$(G-Q_2)x_0 = t_0y_0, \quad (G-Q_2)^*y_0 = t_0x_0$$

and

$$(G-Q_2)x_1 = t_1y_1, \quad (G-Q_2)^*y_1 = t_1x_1,$$

where $x_0, x_1, y_0, y_1, t_0, t_1$ are as in Theorem 3.2.10.

Proof. Let (x_0, y_0) be a Schmidt pair for the Hankel operator H_G corresponding to $\|H_G\| = t_0$. Then, by Theorem D.2.4, every $Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ which is at minimal distance from G satisfies

$$(G-Q_2)x_0 = t_0y_0, \quad (G-Q_2)^*y_0 = t_0x_0,$$

and, by Lemma 3.1.12,

$$W_0(G-Q_2)V_0 = \begin{pmatrix} t_0u_0 & 0 \\ 0 & F_1 \end{pmatrix},$$

where $F_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1)\times(n-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-1)\times(n-1)})$.

Moreover, by Lemma 3.1.15, the set $\mathcal{E}_0 = \{G - Q : Q \in \Omega_0\}$ of all level 0 superoptimal error functions satisfies

$$W_0 \mathcal{E}_0 V_0 = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & F_1 + H^\infty(\mathbb{D}, \mathbb{C}^{m-1 \times n-1}) \end{pmatrix} \cap B(t_0). \tag{3.85}$$

Suppose $Q_2 \in \Omega_0$. Then

$$W_0(G - Q_2)V_0 = \begin{pmatrix} \eta_0^T \\ \beta_0^* \end{pmatrix} (G - Q_2) \begin{pmatrix} \xi_0 & \bar{\alpha}_0 \end{pmatrix} = \begin{pmatrix} \eta_0^T (G - Q_2)\xi_0 & \eta_0^T (G - Q_2)\bar{\alpha}_0 \\ \beta_0^*(G - Q_2)\bar{\alpha}_0 & \beta_0^*(G - Q_2)\bar{\alpha}_0 \end{pmatrix}.$$

By equation (3.85), for $\tilde{Q}_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1)\times(n-1)})$ at minimal distance from F_1,

$$\begin{pmatrix} \eta_0^T (G - Q_2)\xi_0 & \eta_0^T (G - Q_2)\bar{\alpha}_0 \\ \beta_0^*(G - Q_2)\bar{\alpha}_0 & \beta_0^*(G - Q_2)\bar{\alpha}_0 \end{pmatrix} = \begin{pmatrix} t_0 u_0 & 0 \\ 0 & F_1 - \tilde{Q}_1 \end{pmatrix} \tag{3.86}$$

Note that, by Theorem D.2.3,

$$\|F_1 - \tilde{Q}_1\|_\infty = \|H_{F_1}\|,$$

and, by Theorem 3.2.10 (part (v)), $\|H_{F_1}\| = t_1$.

Consideration of the $(2,2)$ entries of equation (3.86) yields

$$F_1 - \tilde{Q}_1 = \beta_0^*(G - Q_2)\bar{\alpha}_0. \tag{3.87}$$

Note that, if $(\hat{x}_1, \hat{y}_1)$ is a Schmidt pair for H_{F_1} corresponding to $t_1 = \|H_{F_1}\|$, then, by Theorem D.2.4,

$$(F_1 - \tilde{Q}_1)\hat{x}_1 = t_1 \hat{y}_1, \quad (F - \hat{Q}_1)^* \hat{y}_1 = t_1 \hat{x}_1.$$

In view of equation (3.87), the latter equations imply

$$\beta_0^*(G - Q_2)\bar{\alpha}_0 \hat{x}_1 = t_1 \hat{y}_1, \tag{3.88}$$

and

$$\alpha_0^T (G - Q_2)^* \beta_0 \hat{y}_1 = t_1 \hat{x}_1. \tag{3.89}$$

By Lemma 3.2.23, we may choose the Schmidt pair for H_{F_1} corresponding to $\|H_{F_1}\|$ to be

$$\hat{x}_1 = \alpha_0^T x_1, \quad \hat{y}_1 = \beta_0^* y_1. \tag{3.90}$$

Recall that, by equations (3.70),

$$x_1 = \bar{\alpha}_0 \alpha_0^T x_1 \tag{3.91}$$

and

$$y_1 = \beta_0 \beta_0^* y_1. \tag{3.92}$$

In view of equations (3.88) and (3.90), we obtain

$$\beta_0^*(G-Q_2)\bar{\alpha}_0\alpha_0^T x_1 = t_1\beta_0^* y_1.$$

Multiplying both sides of the latter equation by β_0, we get

$$\beta_0\beta_0^*(G-Q_2)\bar{\alpha}_0\alpha_0^T x_1 = t_1\beta_0\beta_0^* y_1,$$

which, by equation (3.91), implies

$$\beta_0\beta_0^*(G-Q_2)x_1 = t_1\beta_0\beta_0^* y_1,$$

or equivalently,

$$\beta_0\beta_0^*\Big((G-Q_2)x_1 - t_1 y_1\Big) = 0.$$

Since, by Theorem 3.2.10, $U_2^* = M_{\beta_0\beta_0^*}$ is unitary, the latter equation yields

$$(G-Q_2)x_1 = t_1 y_1.$$

Moreover, by equations (3.89) and (3.90), we obtain

$$\alpha_0^T(G-Q_2)^*\beta_0\beta_0^* y_1 = t_1\alpha_0^T x_1.$$

Multiplying both sides of the latter equation by $\bar{\alpha}_0$, we get

$$\bar{\alpha}_0\alpha_0^T(G-Q_2)^*\beta_0\beta_0^* y_1 = t_1\bar{\alpha}_0\alpha_0^T x_1.$$

In view of equation (3.92), the latter expression is equivalent to the equation

$$\bar{\alpha}_0\alpha_0^T(G-Q_2)^* y_1 = t_1\bar{\alpha}_0\alpha_0^T x_1,$$

or equivalently,

$$\bar{\alpha}_0\alpha_0^T\Big((G-Q_2)^* y_1 - t_1 x_1\Big) = 0.$$

Since, by Theorem 3.2.10, $U_1^* = M_{\bar{\alpha}_0\alpha_0^T}$ is unitary, the latter equation yields

$$(G-Q_2)^* y_1 = t_1 x_1.$$

Therefore Q_2 satisfies the required equations. □

The next few propositions are in preparation for Theorem 3.2.37 on the compactness of T_2.

Proposition 3.2.28. *For a thematic completion of the inner matrix-valued function $\beta_0^T\eta_1$ of the form $\tilde{W}_1^T = \begin{pmatrix}\beta_0^T\eta_1 & \bar{\beta}_1\end{pmatrix}$, where β_1 is an inner, co-outer, quasi-continuous function of* *type* $(m-1)\times(m-2)$, *the following equation holds*

$$\beta_1^* H^2(\mathbb{D},\mathbb{C}^{m-1})^\perp = H^2(\mathbb{D},\mathbb{C}^{m-2})^\perp.$$

Proof. By virtue of the fact that complex conjugation is a unitary operator on $L^2(\mathbb{T},\mathbb{C}^m)$, an equivalent statement to Proposition 3.2.28 is that $\beta_1^T zH^2(\mathbb{D},\mathbb{C}^{m-1}) = zH^2(\mathbb{D},\mathbb{C}^{m-2})$. By Lemma 3.1.18, there exists a matrix-valued function $B_1 \in H^\infty(\mathbb{D},\mathbb{C}^{(m-2)\times(m-1)})$ such that

$$B_1\beta_1 = I_{m-2}$$

or, equivalently,

$$\beta_1^T B_1^T = I_{m-2}.$$

Let $f \in zH^2(\mathbb{D},\mathbb{C}^{m-2})$. Then,

$$f = (\beta_1^T B_1^T)f \in \beta_1^T B_1^T zH^2(\mathbb{D},\mathbb{C}^{m-2}) = \beta_1^T zH^2(\mathbb{D},\mathbb{C}^{m-1}).$$

Hence

$$zH^2(\mathbb{D},\mathbb{C}^{m-2}) \subseteq \beta_1^T zH^2(\mathbb{D},\mathbb{C}^{m-1}).$$

Note that, since $\beta_1 \in H^\infty(\mathbb{D},\mathbb{C}^{(m-1)\times(m-2)})$, we have

$$\beta_1^T zH^2(\mathbb{D},\mathbb{C}^{m-1}) \subseteq zH^2(\mathbb{D},\mathbb{C}^{m-2}).$$

Thus

$$\beta_1^T zH^2(\mathbb{D},\mathbb{C}^{m-1}) = zH^2(\mathbb{D},\mathbb{C}^{m-2}).$$ □

Lemma 3.2.29. *For a thematic completion of the inner matrix-valued function* $\alpha_0^T\xi_1$ *of the form* $\tilde{V}_1 = \begin{pmatrix}\alpha_0^T\xi_1 & \bar{\alpha}_1\end{pmatrix}$, *where* α_1 *is an inner, co-outer, quasi-continuous function of type* $(n-1)\times(n-2)$, *the following equation holds*

$$\alpha_1^T H^2(\mathbb{D},\mathbb{C}^{n-1}) = H^2(\mathbb{D},\mathbb{C}^{n-2}).$$

Proof. By Lemma 3.1.18, for the given α_1, there exists $A_1 \in H^\infty(\mathbb{D},\mathbb{C}^{(n-2)\times(n-1)})$ such that $A_1\alpha_1 = I_{n-2}$. Equivalently, $\alpha_1^T A_1^T = I_{n-2}$.

Let $g \in H^2(\mathbb{D},\mathbb{C}^{n-2})$. Then $g = (\alpha_1^T A_1^T)g \in \alpha_1^T A_1^T H^2(\mathbb{D},\mathbb{C}^{n-2})$, which implies that $g \in \alpha_1^T H^2(\mathbb{D},\mathbb{C}^{n-1})$. Hence $H^2(\mathbb{D},\mathbb{C}^{n-2}) \subseteq \alpha_1^T H^2(\mathbb{D},\mathbb{C}^{n-1})$.
For the reverse inclusion, note that since $\alpha_1 \in H^\infty(\mathbb{D},\mathbb{C}^{(n-1)\times(n-2)})$,

$$\alpha_1^T H^2(\mathbb{D},\mathbb{C}^{n-1}) \subseteq H^2(\mathbb{D},\mathbb{C}^{n-2})$$

Thus

$$\alpha_1^T H^2(\mathbb{D},\mathbb{C}^{n-1}) = H^2(\mathbb{D},\mathbb{C}^{n-2}).$$ □

Proposition 3.2.30. *With the notation of Proposition 3.2.26, let unitary completions of* ξ_0 *and* $\alpha_0^T\xi_1$ *be given by*

$$V_0 = \begin{pmatrix} \xi_0 & \bar{\alpha_0} \end{pmatrix}, \quad \tilde{V_1} = \begin{pmatrix} \alpha_0^T\xi_1 & \bar{\alpha_1} \end{pmatrix},$$

where α_0, α_1 *are inner, co-outer, quasi-continuous matrix-valued functions of types* $n \times (n-1)$ *and* $(n-1) \times (n-2)$ *respectively. Let*

$$V_1 = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{V_1} \end{pmatrix}$$

and let

$$\mathcal{K}_2 = V_0 V_1 \begin{pmatrix} 0_{2\times 1} \\ H^2(\mathbb{D}, \mathbb{C}^{n-2}) \end{pmatrix}.$$

Then

$$\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n) = \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \mathcal{K}_2$$

and the operator $(\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \cdot) \colon \mathcal{K}_2 \to \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n)$ *is unitary.*

Proof. Let us first show that

$$\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n) \subseteq \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \mathcal{K}_2.$$

Notice that, since V_0, V_1 are unitary-valued functions, V_0V_1 is unitary-valued, that is,

$$V_0 V_1 V_1^* V_0^* = I_n,$$

which is equivalent to the equation

$$\xi_0 \xi_0^* + \bar{\alpha}_0 \alpha_0^T \xi_1 \xi_1^* \bar{\alpha}_0 \alpha_0^T + \bar{\alpha}_0 \bar{\alpha}_1 \alpha_1^T \alpha_0^T = I_n. \tag{3.93}$$

Let $\omega \in \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n)$ be given by $\omega = \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} f$, for some $f \in H^2(\mathbb{D}, \mathbb{C}^n)$. Then, by equation (3.93),

$$\begin{aligned} \omega &= \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} I_n f \\ &= \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} (\xi_0 \xi_0^* + \bar{\alpha}_0 \alpha_0^T \xi_1 \xi_1^* \bar{\alpha}_0 \alpha_0^T + \bar{\alpha}_0 \bar{\alpha}_1 \alpha_1^T \alpha_0^T) f \\ &= \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \xi_0 \xi_0^* + \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 \alpha_0^T \xi_1 \xi_1^* \bar{\alpha}_0 \alpha_0^T f + \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 \bar{\alpha}_1 \alpha_1^T \alpha_0^T f \\ &= 0 + \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 \alpha_0^T \xi_1 \xi_1^* \bar{\alpha}_0 \alpha_0^T f + \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 \bar{\alpha}_1 \alpha_1^T \alpha_0^T f. \end{aligned}$$

Note that since V_0 is unitary-valued, $\xi_0 \xi_0^* + \bar{\alpha}_0 \alpha_0^T = I_n$. Moreover, ξ_k and each column of $\xi_k \xi_k^*$ are pointwise linearly dependent on $\mathbb{D}$ for $k = 0, 1$, and hence,

$$\begin{aligned} \omega &= \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} (I_{\mathbb{C}^n} - \xi_0 \xi_0^*) \xi_1 \xi_1^* \bar{\alpha}_0 \alpha_0^T f + \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 \bar{\alpha}_1 \alpha_1^T \alpha_0^T f \\ &= \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \xi_1 \xi_1^* \bar{\alpha}_0 \alpha_0^T f - \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \xi_0 \xi_0^* \xi_1 \xi_1^* \bar{\alpha}_0 \alpha_0^T f + \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 \bar{\alpha}_1 \alpha_1^T \alpha_0^T f, \end{aligned}$$

thus

$$\omega = \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 \bar{\alpha}_1 \alpha_1^T \alpha_0^T f.$$

By Lemma 3.2.15,

$$\alpha_0^T H^2(\mathbb{D}, \mathbb{C}^n) = H^2(\mathbb{D}, \mathbb{C}^{n-1}),$$

and, by Lemma 3.2.29,

$$\alpha_1^T H^2(\mathbb{D}, \mathbb{C}^{n-1}) = H^2(\mathbb{D}, \mathbb{C}^{n-2}).$$

Observe that, by definition, $\mathcal{K}_2 = \bar{\alpha}_0 \bar{\alpha}_1 H^2(\mathbb{D}, \mathbb{C}^{n-2})$. Thus $\omega \in \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \mathcal{K}_2$, and so,

$$\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n) \subseteq \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \mathcal{K}_2.$$

For the reverse inclusion, note that a typical element of $\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \mathcal{K}_2$ is of the form $\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 \bar{\alpha}_1 g$, for some $g \in H^2(\mathbb{D}, \mathbb{C}^{n-2})$. By Lemma 3.2.29, there exists a vector-valued function $q \in H^2(\mathbb{D}, \mathbb{C}^{n-1})$ such that $\alpha_1^T q = g$. Then,

$$\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 \bar{\alpha}_1 g = \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 \bar{\alpha}_1 \alpha_1^T q.$$

Since $\tilde{V}_1 = \begin{pmatrix} \alpha_0^T \xi_1 & \bar{\alpha}_1 \end{pmatrix}$ is unitary-valued, $\alpha_0^T \xi_1 \xi_1^* \bar{\alpha}_0 + \bar{\alpha}_1 \alpha_1^T = I_{n-1}$. Hence

$$\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 \bar{\alpha}_1 \alpha_1^T q = \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 (I_{n-1} - \alpha_0^T \xi_1 \xi_1^* \bar{\alpha}_0) q = \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 q - \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 \alpha_0^T \xi_1 \xi_1^* \bar{\alpha}_0 q.$$

Furthermore, since V_0 is unitary-valued, $\xi_0 \xi_0^* + \bar{\alpha}_0 \alpha_0^T = I_n$. Thus

$$\begin{aligned} &\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 q - \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} (I_{\mathbb{C}^n} - \xi_0 \xi_0^*) \xi_1 \xi_1^* \bar{\alpha}_0 q \\ &= \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 q - \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \xi_1 \xi_1^* \bar{\alpha}_0 q + \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \xi_0 \xi_0^* \xi_1 \xi_1^* \bar{\alpha}_0 q \\ &= \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 q + 0 \end{aligned}$$

because of pointwise linear dependence.
By Lemma 3.2.15, there exists $\rho \in H^2(\mathbb{D}, \mathbb{C}^n)$ such that $\alpha_0^T \rho = q$. Hence

$$\begin{aligned} \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 q &= \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \bar{\alpha}_0 \alpha_0^T \rho \\ &= \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} (I_{\mathbb{C}^n} - \xi_0 \xi_0^*) \rho \\ &= \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \rho + 0 \end{aligned}$$

on account of the pointwise linear dependence. Clearly,

$$\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \rho \in \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n).$$

Consequently,

$$\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \mathcal{K}_2 \subseteq \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n),$$

and thus

$$\xi_0\dot{\wedge}\xi_1\dot{\wedge}\mathcal{K}_2 = \xi_0\dot{\wedge}\xi_1\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^n).$$

Let us show that the operator $(\xi_0\dot{\wedge}\xi_1\dot{\wedge}\cdot)\colon \mathcal{K}_2 \to \xi_0\dot{\wedge}\xi_1\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^n)$ is unitary. The foregoing paragraph asserts that the operator is surjective. It remains to be shown that it is an isometry. To this end, let $f \in \mathcal{K}_2$. Then

$$\begin{aligned}\|\xi_0\dot{\wedge}\xi_1\dot{\wedge}f\|^2_{L^2(\mathbb{T},\wedge^3\mathbb{C}^n)} &= \langle \xi_0\dot{\wedge}\xi_1\dot{\wedge}f, \xi_0\dot{\wedge}\xi_1\dot{\wedge}f\rangle_{L^2(\mathbb{T},\wedge^3\mathbb{C}^n)}\\ &= \frac{1}{2\pi}\int_0^{2\pi} \langle \xi_0(e^{i\theta})\dot{\wedge}\xi_1(e^{i\theta})\dot{\wedge}f(e^{i\theta}), \xi_0(e^{i\theta})\dot{\wedge}\xi_1(e^{i\theta})\dot{\wedge}f(e^{i\theta})\rangle_{\wedge^3\mathbb{C}^n}\, d\theta.\end{aligned}$$

By Proposition 2.1.19, the latter integral is equal to

$$\frac{1}{2\pi}\int_0^{2\pi} \det\begin{pmatrix} \langle \xi_0(e^{i\theta}),\xi_0(e^{i\theta})\rangle_{\mathbb{C}^n} & \langle \xi_0(e^{i\theta}),\xi_1(e^{i\theta})\rangle_{\mathbb{C}^n} & \langle \xi_0(e^{i\theta}),f(e^{i\theta})\rangle_{\mathbb{C}^n}\\ \langle \xi_1(e^{i\theta}),\xi_0(e^{i\theta})\rangle_{\mathbb{C}^n} & \langle \xi_1(e^{i\theta}),\xi_1(e^{i\theta})\rangle_{\mathbb{C}^n} & \langle \xi_1(e^{i\theta}),f(e^{i\theta})\rangle_{\mathbb{C}^n}\\ \langle f(e^{i\theta}),\xi_0(e^{i\theta})\rangle_{\mathbb{C}^n} & \langle f(e^{i\theta}),\xi_1(e^{i\theta})\rangle_{\mathbb{C}^n} & \langle f(e^{i\theta}),f(e^{i\theta})\rangle_{\mathbb{C}^n}\end{pmatrix} d\theta.$$

Note that, by Proposition 3.2.1, $\{\xi_0(e^{i\theta}),\xi_1(e^{i\theta})\}$ is an orthonormal set for almost all $e^{i\theta}$ on $\mathbb{T}$. Moreover, since $\mathcal{K}_2 = \bar{\alpha}_0\bar{\alpha}_1 H^2(\mathbb{D},\mathbb{C}^{n-2})$, then $f = \bar{\alpha}_0\bar{\alpha}_1\varphi$ for some $\varphi \in H^2(\mathbb{D},\mathbb{C}^{n-2})$. Hence

$$\begin{aligned}\langle \xi_0(e^{i\theta}), f(e^{i\theta})\rangle_{\mathbb{C}^n} &= \langle \xi_0(e^{i\theta}), \bar{\alpha}_0(e^{i\theta})\bar{\alpha}_1(e^{i\theta})\varphi(e^{i\theta})\rangle_{\mathbb{C}^n}\\ &= \langle \alpha_0^T(e^{i\theta})\xi_0(e^{i\theta}), \bar{\alpha}_1(e^{i\theta})\varphi(e^{i\theta})\rangle_{\mathbb{C}^{n-1}} = 0\end{aligned}$$

almost everywhere on $\mathbb{T}$, since V_0 is unitary-valued. Similarly, since $\tilde{V}_1$ is unitary valued, we deduce that

$$\langle \xi_1(e^{i\theta}), f(e^{i\theta})\rangle_{\mathbb{C}^n} = \langle \alpha_1^T(e^{i\theta})\alpha_0^T(e^{i\theta})\xi_1(e^{i\theta}), \varphi(e^{i\theta})\rangle_{\mathbb{C}^{n-2}} = 0$$

almost everywhere on $\mathbb{T}$. Therefore

$$\|\xi_0\dot{\wedge}\xi_1\dot{\wedge}f\|^2_{L^2(\mathbb{T},\wedge^3\mathbb{C}^n)} = \frac{1}{2\pi}\int_0^{2\pi} \det\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \|f(e^{i\theta})\|^2_{\mathbb{C}^n}\end{pmatrix} d\theta = \|f\|^2_{L^2(\mathbb{T},\mathbb{C}^n)},$$

that is, $(\xi_0\dot{\wedge}\xi_1\dot{\wedge}\cdot)\colon \mathcal{K}_2 \to \xi_0\dot{\wedge}\xi_1\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^n)$ is an isometric operator. Thus, by Theorem A.2.4, the operator $(\xi_0\dot{\wedge}\xi_1\dot{\wedge}\cdot)\colon \mathcal{K}_2 \to \xi_0\dot{\wedge}\xi_1\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^n)$ is unitary. □

Remark 3.2.31. *Let V_0 and $\tilde{V}_1$ be given by equations (3.7) and (3.79) respectively and let $V_1 = \begin{pmatrix} 1 & 0\\ 0 & \tilde{V}_1\end{pmatrix}$. Since V_0, $\tilde{V}_1$ and V_1 are unitary-valued, we have*

$$I_n = V_0V_0^* = \xi_0\xi_0^* + \bar{\alpha}_0\alpha_0^T, \tag{3.94}$$

$$I_{n-1} = \tilde{V}_1\tilde{V}_1^* = \alpha_0^T\xi_1\xi_1^*\bar{\alpha}_0 + \bar{\alpha}_1\alpha_1^T. \tag{3.95}$$

Lemma 3.2.32. *Let V_0 and $\tilde{V}_1$ be given by equations (3.7) and (3.79) respectively. Then*

$$I_n - \xi_0\xi_0^* - \xi_1\xi_1^* = \bar{\alpha}_0\bar{\alpha}_1\alpha_1^T\alpha_0^T \tag{3.96}$$

almost everywhere on $\mathbb{T}$.

Proof. By equation (3.95)

$$\bar{\alpha}_1\alpha_1^T = I_{n-1} - \alpha_0^T\xi_1\xi_1^*\bar{\alpha}_0,$$

thus

$$\bar{\alpha}_0\bar{\alpha}_1\alpha_1^T\alpha_0^T = \bar{\alpha}_0(I_{n-1} - \alpha_0^T\xi_1\xi_1^*\bar{\alpha}_0)\alpha_0^T.$$

By equation (3.94),

$$\bar{\alpha}_0\alpha_0^T = I_n - \xi_0\xi_0^*.$$

Hence

$$\bar{\alpha}_0\bar{\alpha}_1\alpha_1^T\alpha_0^T = (I_n - \xi_0\xi_0^*) - (I_n - \xi_0\xi_0^*)\xi_1\xi_1^*(I_n - \xi_0\xi_0^*).$$

Since, by Proposition 3.2.1, the set $\{\xi_0(z), \xi_1(z)\}$ is orthonormal in $\mathbb{C}^m$ for almost every $z \in \mathbb{T}$,

$$\bar{\alpha}_0\bar{\alpha}_1\alpha_1^T\alpha_0^T = I_n - \xi_0\xi_0^* - \xi_1\xi_1^*$$

almost everywhere on $\mathbb{T}$. □

Let us state certain identities that are useful for the next statements.

Remark 3.2.33. *Let W_0 and $\tilde{W}_1$ be given by equations (3.7) and (3.80) respectively and let $W_1 = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{W}_1 \end{pmatrix}$. Since W_0, $\tilde{W}_1$ and W_1 are unitary-valued almost everywhere on $\mathbb{T}$, we have*

$$I_m = W_0^*W_0 = \bar{\eta}_0\eta_0^T + \beta_0\beta_0^*, \tag{3.97}$$

$$I_{m-1} = \tilde{W}_1^*\tilde{W}_1 = \beta_0^*\bar{\eta}_1\eta_1^T\beta_0 + \beta_1\beta_1^*, \tag{3.98}$$

and

$$\begin{aligned} W_0^*\begin{pmatrix} 1 & 0 \\ 0 & \tilde{W}_1^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \tilde{W}_1 \end{pmatrix}W_0 &= \begin{pmatrix} \bar{\eta}_0 & \beta_0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \begin{pmatrix} \beta_0^*\bar{\eta}_1 & \beta_1 \end{pmatrix} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \begin{pmatrix} \eta_1^T\beta_0 \\ \beta_1^* \end{pmatrix} \end{pmatrix}\begin{pmatrix} \eta_0^T \\ \beta_0^* \end{pmatrix} \\ &= \begin{pmatrix} \bar{\eta}_0 & \beta_0\beta_0^*\bar{\eta}_1 & \beta_0\beta_1 \end{pmatrix}\begin{pmatrix} \eta_0^T \\ \eta_1^T\beta_0\beta_0^* \\ \beta_1^*\beta_0^* \end{pmatrix} \\ &= \bar{\eta}_0\eta_0^T + \beta_0\beta_0^*\bar{\eta}_1\eta_1^T\beta_0\beta_0^* + \beta_0\beta_1\beta_1^*\beta_0^*. \end{aligned}$$

Also, by equations (3.97) and (3.98),

$$\bar{\eta}_0\eta_0^T + \beta_0\beta_0^*\bar{\eta}_1\eta_1^T\beta_0\beta_0^* + \beta_0\beta_1\beta_1^*\beta_0^* = \bar{\eta}_0\eta_0^T + \beta_0(I_{m-1} - \beta_1\beta_1^* + \beta_1\beta_1^*)\beta_0^* = I_m. \tag{3.99}$$

Lemma 3.2.34. *Let W_0 and $\tilde{W}_1$ be given by equations (3.7) and (3.80) respectively. Then*

$$I_m - \bar{\eta}_0\eta_0^T - \bar{\eta}_1\eta_1^T = \beta_0\beta_1\beta_1^*\beta_0^* \tag{3.100}$$

almost everywhere on $\mathbb{T}$.

Proof. By equation (3.98)

$$\beta_1\beta_1^* = I_{m-1} - \beta_0^*\bar{\eta}_1\eta_1^T\beta_0,$$

thus

$$\beta_0\beta_1\beta_1^*\beta_0^* = \beta_0(I_{m-1} - \beta_0^*\bar{\eta}_1\eta_1^T\beta_0)\beta_0^*.$$

By equation (3.97),

$$\beta_0\beta_0^* = I_m - \bar{\eta}_0\eta_0^T.$$

Hence

$$\beta_0\beta_1\beta_1^*\beta_0^* = (I_m - \bar{\eta}_0\eta_0^T) - (I_m - \bar{\eta}_0\eta_0^T)\bar{\eta}_1\eta_1^T(I_m - \bar{\eta}_0\eta_0^T).$$

Since, by Proposition 3.2.1, the set $\{\bar{\eta}_0(z), \bar{\eta}_1(z)\}$ is orthonormal in $\mathbb{C}^m$ for almost every $z \in \mathbb{T}$,

$$\beta_0\beta_1\beta_1^*\beta_0^* = I_m - \bar{\eta}_0\eta_0^T - \bar{\eta}_1\eta_1^T$$

almost everywhere on $\mathbb{T}$. □

Proposition 3.2.35. *Let η_0, η_1 be defined by equations (3.9) and (3.19) respectively, and let β_0, β_1 be inner, co-outer, quasi-continuous functions of types $m\times(m-1)$ and $(m-1)\times(m-2)$ respectively, such that the functions*

$$W_0^T = \begin{pmatrix} \eta_0 & \bar{\beta}_0 \end{pmatrix}, \quad \tilde{W}_1^T = \begin{pmatrix} \beta_0^T\eta_1 & \bar{\beta}_1 \end{pmatrix}$$

are unitary-valued. Let

$$W_1^T = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{W}_1^T \end{pmatrix}$$

and let

$$\mathcal{L}_2 = W_0^*W_1^* \begin{pmatrix} 0_{2\times 1} \\ H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp \end{pmatrix}.$$

Then

$$\bar{\eta}_0\dot{\wedge}\bar{\eta}_1\dot{\wedge}\mathcal{L}_2 = \bar{\eta}_0\dot{\wedge}\bar{\eta}_1\dot{\wedge}H^2(\mathbb{D}, \mathbb{C}^m)^\perp \tag{3.101}$$

and the operator $(\bar{\eta}_0\dot{\wedge}\bar{\eta}_1\dot{\wedge}\cdot)\colon \mathcal{L}_2 \to \bar{\eta}_0\dot{\wedge}\bar{\eta}_1\dot{\wedge}H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ is unitary.

Proof. Observe that $\mathcal{L}_2 = \beta_0\beta_0 H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp$. By virtue of the fact that complex conjugation is a unitary operator on $L^2(\mathbb{T}, \mathbb{C}^m)$, an equivalent statement to (3.101) is that

$$\eta_0\dot{\wedge}\eta_1\dot{\wedge}\bar{\beta}_0\bar{\beta}_1 zH^2(\mathbb{D}, \mathbb{C}^{m-2}) = \eta_0\dot{\wedge}\eta_1\dot{\wedge}zH^2(\mathbb{D}, \mathbb{C}^m).$$

Let us first show that $\eta_0\dot{\wedge}\eta_1\dot{\wedge}zH^2(\mathbb{D}, \mathbb{C}^m) \subseteq \eta_1\dot{\wedge}\bar{\beta}_0\bar{\beta}_1 zH^2(\mathbb{D}, \mathbb{C}^{m-2})$.

Let $f \in zH^2(\mathbb{D}, \mathbb{C}^m)$. Taking complex conjugates in equation (3.99), we have

$$I_m = \eta_0\eta_0^* + \bar{\beta}_0\beta_0^T\eta_1\eta_1^*\bar{\beta}_0\beta_0^T + \bar{\beta}_0\bar{\beta}_1\beta_1^T\beta_0^T,$$

and so,

$$\begin{aligned}\eta_0\dot{\wedge}\eta_1\dot{\wedge}f &= \eta_0\dot{\wedge}\eta_1\dot{\wedge}(\eta_0\eta_0^* + \bar{\beta}_0\beta_0^T\eta_1\eta_1^*\bar{\beta}_0\beta_0^T + \bar{\beta}_0\bar{\beta}_1\beta_1^T\beta_0^T)f \\ &= \eta_0\dot{\wedge}\eta_1\dot{\wedge}\eta_0\eta_0^*f + \eta_0\dot{\wedge}\eta_1\dot{\wedge}\bar{\beta}_0\beta_0^T\eta_1\eta_1^*\bar{\beta}_0\beta_0^Tf + \eta_0\dot{\wedge}\eta_1\dot{\wedge}\bar{\beta}_0\bar{\beta}_1\beta_1^T\beta_0^Tf \\ &= \eta_0\dot{\wedge}\eta_1\dot{\wedge}\bar{\beta}_0\beta_0^T\eta_1\eta_1^*\bar{\beta}_0\beta_0^Tf + \eta_0\dot{\wedge}\eta_1\dot{\wedge}\bar{\beta}_0\bar{\beta}_1\beta_1^T\beta_0^Tf, \end{aligned} \tag{3.102}$$

the last equality following by the pointwise linear dependence of η_0 and $\eta_0\eta_0^*f$ on $\mathbb{D}$. Taking complex conjugates in equation (3.97), we have $\bar{\beta}_0\beta_0^T = I_{\mathbb{C}^m} - \eta_0\eta_0^*$. Hence equation (3.102) yields

$$\begin{aligned}\eta_0\dot{\wedge}\eta_1\dot{\wedge}f &= \eta_0\dot{\wedge}\eta_1\dot{\wedge}(I_{\mathbb{C}^m} - \eta_0\eta_0^*)\eta_1\eta_1^*\bar{\beta}_0\beta_0^Tf + \eta_0\dot{\wedge}\eta_1\dot{\wedge}\bar{\beta}_0\bar{\beta}_1\beta_1^T\beta_0^Tf \\ &= \eta_0\dot{\wedge}\eta_1\dot{\wedge}\eta_1\eta_1^*\bar{\beta}_0\beta_0^Tf - \eta_0\dot{\wedge}\eta_1\dot{\wedge}\eta_0\eta_0^*\eta_1\eta_1^*\bar{\beta}_0\beta_0^Tf + \eta_0\dot{\wedge}\eta_1\dot{\wedge}\bar{\beta}_0\bar{\beta}_1\beta_1^T\beta_0^Tf \\ &= \eta_0\dot{\wedge}\eta_1\dot{\wedge}\bar{\beta}_0\bar{\beta}_1\beta_1^T\beta_0^Tf\end{aligned}$$

on account of the pointwise linear dependence.
By Proposition 3.2.20, there exists a vector-valued function $g \in H^2(\mathbb{D}, \mathbb{C}^{m-1})$ such that $\beta_0^Tf = g$. By Proposition 3.2.28, there exists a vector-valued function $\omega \in zH^2(\mathbb{D}, \mathbb{C}^{m-2})$ such that $\beta_1^Tg = \omega$. Thus

$$\eta_0\dot{\wedge}\eta_1\dot{\wedge}\bar{\beta}_0\bar{\beta}_1\omega \in \eta_0\dot{\wedge}\eta_1\dot{\wedge}\bar{\beta}_0\bar{\beta}_1zH^2(\mathbb{D}, \mathbb{C}^{m-2}),$$

and hence

$$\eta_0\dot{\wedge}\eta_1\dot{\wedge}zH^2(\mathbb{D}, \mathbb{C}^m) \subseteq \eta_0\dot{\wedge}\eta_1\dot{\wedge}\bar{\beta}_0\bar{\beta}_1zH^2(\mathbb{D}, \mathbb{C}^{m-2}).$$

For the reverse inclusion, let

$$u = \eta_0\dot{\wedge}\eta_1\dot{\wedge}\bar{\beta}_0\bar{\beta}_1q \in \eta_0\dot{\wedge}\eta_1\dot{\wedge}\bar{\beta}_0\bar{\beta}_1zH^2(\mathbb{D}, \mathbb{C}^{m-2})$$

for a vector-valued function $q \in zH^2(\mathbb{D}, \mathbb{C}^{m-2})$. By Proposition 3.2.28, there exists a vector-valued function $\phi \in H^2(\mathbb{D}, \mathbb{C}^{m-1})$ such that $q = \beta_1^Tz\phi$. Then,

$$u = \eta_0\dot{\wedge}\eta_1\dot{\wedge}\bar{\beta}_0\bar{\beta}_1\beta_1^Tz\phi.$$

Taking complex conjugates in equations (3.97) and (3.98), we get

$$\bar{\beta}_0\beta_0^T = I_{\mathbb{C}^m} - \eta_0\eta_0^* \quad \text{and} \quad \bar{\beta}_1\beta_1^T = I_{m-1} - \beta_0^T\eta_1\eta_1^*\bar{\beta}_0.$$

Hence

$$u = \eta_0 \dot\wedge \eta_1 \dot\wedge \bar\beta_0 (I_{m-1} - \beta_0^T \eta_1 \eta_1^* \bar\beta_0) z\phi = \eta_0 \dot\wedge \eta_1 \dot\wedge \bar\beta_0 z\phi - \eta_0 \dot\wedge \eta_1 \dot\wedge \bar\beta_0 \beta_0^T \eta_1 \eta_1^* \bar\beta_0 z\phi.$$

By equation (3.97), $\bar\beta_0 \beta_0^T = I_{\mathbb{C}^m} - \eta_0 \eta_0^*$, thus

$$\begin{aligned} \eta_0 \dot\wedge \eta_1 \dot\wedge \bar\beta_0 \beta_0^T \eta_1 \eta_1^* \bar\beta_0 z\phi &= \eta_0 \dot\wedge \eta_1 \dot\wedge (I_{\mathbb{C}^m} - \eta_0 \eta_0^*) \eta_1 \eta_1^* \bar\beta_0 z\phi \\ &= \eta_0 \dot\wedge \eta_1 \dot\wedge \eta_1 \eta_1^* \bar\beta_0 z\phi + \eta_0 \dot\wedge \eta_1 \dot\wedge \eta_0 \eta_0^* \eta_1 \eta_1^* \bar\beta_0 z\phi = 0, \end{aligned}$$

because of pointwise linear dependence, and hence

$$u = \eta_0 \dot\wedge \eta_1 \dot\wedge \bar\beta_0 z\phi.$$

By Proposition 3.2.20, there exists a vector-valued function $\psi \in H^2(\mathbb{D}, \mathbb{C}^m)$ such that $\phi = \beta_0^T z\psi$. Hence

$$\begin{aligned} \eta_0 \dot\wedge \eta_1 \dot\wedge \bar\beta_0 z\phi &= \eta_0 \dot\wedge \eta_1 \dot\wedge \bar\beta_0 \beta_0^T z\psi \\ &= \eta_0 \dot\wedge \eta_1 \dot\wedge (I_{\mathbb{C}^m} - \eta_0 \eta_0^*) z\psi \\ &= \eta_0 \dot\wedge \eta_1 \dot\wedge z\psi \end{aligned}$$

by pointwise linear dependence. Therefore,

$$\eta_0 \dot\wedge \eta_1 \dot\wedge \bar\beta_0 \bar\beta_1 z H^2(\mathbb{D}, \mathbb{C}^{m-2}) \subseteq \eta_0 \dot\wedge \eta_1 \dot\wedge z H^2(\mathbb{D}, \mathbb{C}^m),$$

and thus

$$\eta_0 \dot\wedge \eta_1 \dot\wedge z H^2(\mathbb{D}, \mathbb{C}^m) = \eta_0 \dot\wedge \eta_1 \dot\wedge \bar\beta_0 \bar\beta_1 z H^2(\mathbb{D}, \mathbb{C}^{m-2}).$$

To complete the proof, let us show that the operator

$$(\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \cdot) \colon \mathcal{L}_2 \to \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^m)^\perp$$

is unitary. Observe that the foregoing paragraph asserts the operator is surjective. Hence it suffices to prove that it is an isometry. To this end, let $\upsilon \in \mathcal{L}_2$. Then

$$\|\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \upsilon\|^2_{L^2(\mathbb{T}, \wedge^3 \mathbb{C}^m)} = \langle \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \upsilon, \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \upsilon \rangle_{L^2(\mathbb{T}, \wedge^3 \mathbb{C}^m)},$$

and, by Proposition 2.1.19,

$$\begin{aligned} &\langle \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \upsilon, \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \upsilon \rangle_{L^2(\mathbb{T}, \wedge^3 \mathbb{C}^m)} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix} \langle \bar\eta_0(e^{i\theta}), \bar\eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \bar\eta_0(e^{i\theta}), \bar\eta_1(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \bar\eta_0(e^{i\theta}), \upsilon(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ \langle \bar\eta_1(e^{i\theta}), \bar\eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \bar\eta_1(e^{i\theta}), \bar\eta_1(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \bar\eta_1(e^{i\theta}), \upsilon(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ \langle \upsilon(e^{i\theta}), \bar\eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \upsilon(e^{i\theta}), \bar\eta_1(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \upsilon(e^{i\theta}), \upsilon(e^{i\theta}) \rangle_{\mathbb{C}^m} \end{pmatrix} d\theta. \end{aligned}$$

Notice that, by Proposition 3.2.1, $\{\bar{\eta}_0(e^{i\theta}), \bar{\eta}_1(e^{i\theta})\}$ is an orthonormal set almost everywhere on $\mathbb{T}$. Further, since $\mathcal{L}_2 = \beta_0\beta_1 H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp$, $\upsilon = \beta_0\beta_1\varphi$ for some $\varphi \in H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp$. Hence

$$\begin{aligned}\langle \bar{\eta}_0(e^{i\theta}), \upsilon(e^{i\theta})\rangle_{\mathbb{C}^m} &= \langle \bar{\eta}_0(e^{i\theta}), \beta_0(e^{i\theta})\beta_1(e^{i\theta})\varphi(e^{i\theta})\rangle_{\mathbb{C}^m} \\ &= \langle \beta_0^*(e^{i\theta})\bar{\eta}_0(e^{i\theta}), \beta_1(e^{i\theta})\varphi(e^{i\theta})\rangle_{\mathbb{C}^{m-1}} = 0,\end{aligned}$$

since W_0^T is unitary-valued almost everywhere on $\mathbb{T}$. Similarly, since, by Proposition 3.2.26, $\tilde{W}_1^T$ is unitary-valued almost everywhere on $\mathbb{T}$, we obtain

$$\langle \bar{\eta}_1(e^{i\theta}), \upsilon(e^{i\theta})\rangle_{\mathbb{C}^m} = \langle \beta_1^*(e^{i\theta})\beta_0^*(e^{i\theta})\bar{\eta}_1(e^{i\theta}), \varphi(e^{i\theta})\rangle_{\mathbb{C}^{m-2}} = 0.$$

Therefore

$$\|\bar{\eta}_0\dot{\wedge}\bar{\eta}_1\dot{\wedge}\upsilon\|^2_{L^2(\mathbb{T},\wedge^3\mathbb{C}^m)} = \frac{1}{2\pi}\int_0^{2\pi} \det\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \|\upsilon(e^{i\theta})\|^2_{\mathbb{C}^m}\end{pmatrix} d\theta = \|\upsilon\|^2_{L^2(\mathbb{T},\mathbb{C}^m)},$$

that is, the operator $(\bar{\eta}_0\dot{\wedge}\bar{\eta}_1\dot{\wedge}\cdot)\colon \mathcal{L}_2 \to \bar{\eta}_0\dot{\wedge}\bar{\eta}_1\dot{\wedge}H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ is an isometry. Thus, by Theorem A.2.4, the operator is unitary. □

Proposition 3.2.36. *Let η_0, η_1 be defined by equations (3.9) and (3.19) respectively and let β_0, β_1 be inner, co-outer, quasi-continuous functions of types $m\times(m-1)$ and $(m-1)\times(m-2)$ respectively, such that the functions*

$$W_0^T = \begin{pmatrix}\eta_0 & \bar{\beta}_0\end{pmatrix}, \quad \tilde{W}_1^T = \begin{pmatrix}\beta_0^T\eta_1 & \bar{\beta}_1\end{pmatrix}$$

are unitary-valued. Let

$$\mathcal{L}_2 = W_0^*\begin{pmatrix}1 & 0\\ 0 & \tilde{W}_1^*\end{pmatrix}\begin{pmatrix}0_{2\times 1}\\ H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp\end{pmatrix}.$$

Then

$$\mathcal{L}_2^\perp = \{f \in L^2(\mathbb{T}, \mathbb{C}^m) : \beta_1^*\beta_0^* f \in H^2(\mathbb{D}, \mathbb{C}^{m-2})\}.$$

Proof. Clearly $\mathcal{L}_2 = \beta_0\beta_1 H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp$. The general element of $\beta_0\beta_1 H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp$ is $\beta_0\beta_1\bar{z}\bar{g}$ for $g \in H^2(\mathbb{D}, \mathbb{C}^{m-2})$. A function $f \in L^2(\mathbb{T}, \mathbb{C}^m)$ belongs to $\mathcal{L}_2^\perp$ if and only if

$$\langle f, \beta_0\beta_1\bar{z}\bar{g}\rangle_{L^2(\mathbb{T},\mathbb{C}^m)} = 0 \quad \text{for all} \quad g \in H^2(\mathbb{D}, \mathbb{C}^{m-2})$$

if and only if

$$\frac{1}{2\pi}\int_0^{2\pi} \langle f(e^{i\theta}), \beta_0(e^{i\theta})\beta_1(e^{i\theta})e^{-i\theta}\bar{g}(e^{i\theta})\rangle_{\mathbb{C}^m} d\theta = 0 \quad \text{for all} \quad g \in H^2(\mathbb{D}, \mathbb{C}^{m-2})$$

if and only if

$$\frac{1}{2\pi}\int_0^{2\pi} \langle \beta_1^*(e^{i\theta})\beta_0^*(e^{i\theta})f(e^{i\theta}), e^{-i\theta}\bar{g}(e^{i\theta})\rangle_{\mathbb{C}^{m-2}} d\theta = 0 \quad \text{for all} \quad g \in H^2(\mathbb{D},\mathbb{C}^{m-2}),$$

which in turn is equivalent to the assertion that $\beta_1^*\beta_0^* f$ is orthogonal to $H^2(\mathbb{D},\mathbb{C}^{m-2})^\perp$ in $L^2(\mathbb{T},\mathbb{C}^{m-2})$, which holds if and only if $\beta_1^*\beta_0^* f$ belongs to $H^2(\mathbb{D},\mathbb{C}^{m-2})$. Thus

$$\mathcal{L}_2^\perp = \{f \in L^2(\mathbb{T},\mathbb{C}^m) : \beta_1^*\beta_0^* f \in H^2(\mathbb{D},\mathbb{C}^{m-2})\}$$

as required. □

Theorem 3.2.37. *Let m, n be positive integers such that $\min(m,n) \geq 2$. Let G be in $H^\infty(\mathbb{D},\mathbb{C}^{m\times n}) + C(\mathbb{T},\mathbb{C}^{m\times n})$. Let $(\xi_0 \dot\wedge v_1, \bar\eta_0 \dot\wedge w_1)$ be a Schmidt pair for the operator T_1, as given in equation* (3.15), *corresponding to $t_1 = \|T_1\| \neq 0$, let $h_1 \in H^2(\mathbb{D},\mathbb{C})$ be the scalar outer factor of $\xi_0 \dot\wedge v_1$, let*

$$x_1 = (I_{\mathbb{C}^n} - \xi_0\xi_0^*)v_1, \quad y_1 = (I_m - \bar\eta_0\eta_0^T)w_1,$$

and let

$$\xi_1 = \frac{x_1}{h_1}, \quad \bar\eta_1 = \frac{zy_1}{\bar h_1}.$$

Let

$$V_0 = \begin{pmatrix}\xi_0 & \bar\alpha_0\end{pmatrix}, \quad W_0^T = \begin{pmatrix}\eta_0 & \bar\beta_0\end{pmatrix}$$

be given by equations (3.7), and let

$$\tilde V_1 = \begin{pmatrix}\alpha_0^T\xi_1 & \bar\alpha_1\end{pmatrix}, \quad \tilde W_1^T = \begin{pmatrix}\beta_0^T\eta_1 & \bar\beta_1\end{pmatrix}$$

be given by equations (3.79) and (3.80) respectively. Let

$$X_2 = \xi_0 \dot\wedge \xi_1 \dot\wedge H^2(\mathbb{D},\mathbb{C}^n), \quad Y_2 = \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge H^2(\mathbb{D},\mathbb{C}^m)^\perp,$$

let

$$\mathcal{K}_2 = V_0\begin{pmatrix}1 & 0\\ 0 & \tilde V_1\end{pmatrix}\begin{pmatrix}0_{2\times1}\\ H^2(\mathbb{D},\mathbb{C}^{n-2})\end{pmatrix}, \quad \mathcal{L}_2 = W_0^*\begin{pmatrix}1 & 0\\ 0 & \tilde W_1^*\end{pmatrix}\begin{pmatrix}0_{2\times1}\\ H^2(\mathbb{D},\mathbb{C}^{m-2})^\perp\end{pmatrix}. \tag{3.103}$$

Consider the operator $T_2\colon X_2 \to Y_2$ given by

$$T_2(\xi_0\dot\wedge\xi_1\dot\wedge x) = P_{Y_2}(\bar\eta_0\dot\wedge\bar\eta_1\dot\wedge(G - Q_2)x), \tag{3.104}$$

where $Q_2 \in H^\infty(\mathbb{D},\mathbb{C}^{m\times n})$ satisfies

$$(G - Q_2)x_i = t_i y_i, \quad (G - Q_2)^* y_i = t_i x_i \quad \textit{for } i = 0, 1. \tag{3.105}$$

Let the operator $\Gamma_2\colon \mathcal{K}_2 \to \mathcal{L}_2$ *be given by* $\Gamma_2 = P_{\mathcal{L}_2} M_{G-Q_2}|_{\mathcal{K}_2}$. *Then*
(i) The maps $M_{\bar\alpha_0\bar\alpha_1}$, $M_{\beta_0\beta_1}$ *are unitaries.*
(ii) The maps $(\xi_0\dot\wedge\xi_1\dot\wedge\cdot)\colon \mathcal{K}_2 \to X_2$, $(\bar\eta_0\dot\wedge\bar\eta_1\dot\wedge\cdot)\colon \mathcal{L}_2 \to Y_2$ *are unitaries.*
(iii) The following diagram commutes

$$\begin{array}{ccccc} H^2(\mathbb{D},\mathbb{C}^{n-2}) & \xrightarrow{M_{\bar\alpha_0\bar\alpha_1}} & \mathcal{K}_2 & \xrightarrow{\xi_0\dot\wedge\xi_1\dot\wedge\cdot} & \xi_0\dot\wedge\xi_1\dot\wedge H^2(\mathbb{D},\mathbb{C}^n) = X_2 \\ \Big\downarrow H_{F_2} & & \Big\downarrow \Gamma_2 & & \Big\downarrow T_2 \\ H^2(\mathbb{D},\mathbb{C}^{m-2})^\perp & \xrightarrow{M_{\beta_0\beta_1}} & \mathcal{L}_2 & \xrightarrow{\bar\eta_0\dot\wedge\bar\eta_1\dot\wedge\cdot} & \bar\eta_0\dot\wedge\bar\eta_1\dot\wedge H^2(\mathbb{D},\mathbb{C}^m)^\perp = Y_2, \end{array} \tag{3.106}$$

where $F_2 \in H^\infty(\mathbb{D},\mathbb{C}^{(m-2)\times(n-2)}) + C(\mathbb{T},\mathbb{C}^{(m-2)\times(n-2)})$ *is the function defined in Proposition 3.2.26. (iv)* T_2 *is a compact operator. (v)* $\|T_2\| = \|\Gamma_2\| = \|H_{F_2}\| = t_2$, *where* $t_2 = \|T_2\|$.

Proof. (i) follows from Lemma 3.1.16.
(ii) follows from Propositions 3.2.30 and 3.2.35.
(iii). By Proposition 3.2.8, T_2 is well-defined and is independent of the choice of $Q_2 \in H^\infty(\mathbb{D},\mathbb{C}^{m\times n})$ satisfying equations (3.105). We can choose Q_2 which minimises $(s_0^\infty(G-Q), s_1^\infty(G-Q))$, and therefore satisfies equations (3.105). By Lemma 3.1.17 and Theorem D.2.4, the left hand side of diagram (3.106) commutes. Let us show the right hand side also commutes. A typical element of $\mathcal{K}_2$ is of the form $\bar\alpha_0\bar\alpha_1 x$ where $x \in H^2(\mathbb{D},\mathbb{C}^{n-2})$. Then, by equation (3.104),

$$T_2(\xi_0\dot\wedge\xi_1\dot\wedge\bar\alpha_0\bar\alpha_1 x) = P_{Y_2}\left(\bar\eta_0\dot\wedge\bar\eta_1\dot\wedge(G-Q_2)\bar\alpha_0\bar\alpha_1 x\right).$$

By Proposition (3.2.26), every $Q_2 \in H^\infty(\mathbb{D},\mathbb{C}^{m\times n})$ which minimises $(s_0^\infty(G-Q), s_1^\infty(G-Q))$ satisfies the following equation (see equation (3.81)),

$$(G-Q_2)V_0\begin{pmatrix}1 & 0\\ 0 & \tilde V_1\end{pmatrix}\begin{pmatrix}0\\0\\x\end{pmatrix} = W_0^*\begin{pmatrix}1 & 0\\ 0 & \tilde W_1^*\end{pmatrix}\begin{pmatrix}0\\0\\F_2x\end{pmatrix}, \tag{3.107}$$

for some $F_2 \in H^\infty(\mathbb{D},\mathbb{C}^{(m-2)\times(n-2)}) + C(\mathbb{T},\mathbb{C}^{(m-2)\times(n-2)})$. This implies that

$$(G-Q_2)\bar\alpha_0\bar\alpha_1 x = \beta_0\beta_1 F_2 x, \tag{3.108}$$

for $x \in H^2(\mathbb{D},\mathbb{C}^{n-2})$. Hence

$$T_2(\xi_0\dot\wedge\xi_1\dot\wedge\bar\alpha_0\bar\alpha_1 x) = P_{Y_2}(\bar\eta_0\dot\wedge\bar\eta_1\dot\wedge\beta_0\beta_1 F_2 x). \tag{3.109}$$

Furthermore,

$$(\bar\eta_0\dot\wedge\bar\eta_1\dot\wedge\cdot)\Gamma_2(\bar\alpha_0\bar\alpha_1 x) \;= \bar\eta_0\dot\wedge\bar\eta_1\dot\wedge P_{\mathcal{L}_2}[(G-Q_2)\bar\alpha_0\bar\alpha_1 x].$$

Hence, by equation (3.108),

$$(\bar\eta_0\dot\wedge\bar\eta_1\dot\wedge\cdot)\Gamma_2\,(\bar\alpha_0\bar\alpha_1 x) = \bar\eta_0\dot\wedge\bar\eta_1\dot\wedge P_{\mathcal{L}_2}(\beta_0\beta_1 F_2 x). \tag{3.110}$$

To show commutativity of the right hand square in the diagram (3.106), we need to prove that, for every $x \in H^2(\mathbb{D}, \mathbb{C}^{n-2})$,

$$T_2(\xi_0 \dot\wedge \xi_1 \dot\wedge \bar\alpha_0 \bar\alpha_1 x) = (\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \cdot)\Gamma_2(\bar\alpha_0 \bar\alpha_1 x). \tag{3.111}$$

By equations (3.109) and (3.110), it is equivalent to show that

$$P_{Y_2}(\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \beta_0 \beta_1 F_2 x) = \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge P_{\mathcal{L}_2}(\beta_0 \beta_1 F_2 x). \tag{3.112}$$

Therefore, we need to show that

$$\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge P_{\mathcal{L}_2}(\beta_0 \beta_1 F_2 x) \in Y_2$$

and that

$$\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \beta_0 \beta_1 F_2 x - \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge P_{\mathcal{L}_2}(\beta_0 \beta_1 F_2 x)$$

is orthogonal to Y_2. By Proposition 3.2.35, $\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge P_{\mathcal{L}_2}(\beta_0 \beta_1 F_2 x)$ is indeed an element of Y_2. Furthermore,

$$\begin{aligned} \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \beta_0 \beta_1 F_2 x - \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge P_{\mathcal{L}_2}(\beta_0 \beta_1 F_2 x) &= \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge [\beta_0 \beta_1 F_2 x - P_{\mathcal{L}_2}(\beta_0 \beta_1 F_2 x)] \\ &= \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge P_{\mathcal{L}_2^\perp}(\beta_0 \beta_1 F_2 x). \end{aligned}$$

Let us show that $\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge P_{\mathcal{L}_2^\perp}(\beta_0 \beta_1 F_2 x)$ is orthogonal to Y_2.

It is so if and only if

$$\left\langle \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge P_{\mathcal{L}_2^\perp}(\beta_0 \beta_1 F_2 x), \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge g \right\rangle_{L^2(\mathbb{T}, \wedge^3 \mathbb{C}^m)} = 0 \quad \text{for every } g \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp. \tag{3.113}$$

Let $\Phi = P_{\mathcal{L}_2^\perp}(\beta_0 \beta_1 F_2 x) \in L^2(\mathbb{T}, \mathbb{C}^m)$. By Proposition 3.2.36,

$$\beta_1^* \beta_0^* \Phi \in H^2(\mathbb{D}, \mathbb{C}^{m-2}). \tag{3.114}$$

Then, by Proposition 2.1.19, assertion (3.113) is equivalent to the following assertion

$$\frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix} \langle \bar\eta_0(e^{i\theta}), \bar\eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \bar\eta_0(e^{i\theta}), \bar\eta_1(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \bar\eta_0(e^{i\theta}), g(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ \bar\eta_1(e^{i\theta}), \bar\eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \bar\eta_1(e^{i\theta}), \bar\eta_1(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \bar\eta_1(e^{i\theta}), g(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ \langle \Phi(e^{i\theta}), \bar\eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \Phi(e^{i\theta}), \bar\eta_1(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \Phi(e^{i\theta}), g(e^{i\theta}) \rangle_{\mathbb{C}^m} \end{pmatrix} d\theta = 0$$

for every $g \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$, which in turn, by Proposition 3.2.1, is equivalent to the assertion

$$\frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix} 1 & 0 & \langle \bar\eta_0(e^{i\theta}), g(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ 0 & 1 & \langle \bar\eta_1(e^{i\theta}), g(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ \langle \Phi(e^{i\theta}), \bar\eta_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \Phi(e^{i\theta}), \bar\eta_1(e^{i\theta}) \rangle_{\mathbb{C}^m} & \langle \Phi(e^{i\theta}), g(e^{i\theta}) \rangle_{\mathbb{C}^m} \end{pmatrix} d\theta = 0$$

for every $g \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$. The latter statement is equivalent to the assertion

$$\frac{1}{2\pi}\int_0^{2\pi} \langle \Phi(e^{i\theta}), g(e^{i\theta})\rangle_{\mathbb{C}^m} \quad -\langle \Phi(e^{i\theta}), \bar{\eta}_1(e^{i\theta})\rangle_{\mathbb{C}^m}\langle \bar{\eta}_1(e^{i\theta}), g(e^{i\theta})\rangle_{\mathbb{C}^m}$$
$$-\langle \Phi(e^{i\theta}), \bar{\eta}_0(e^{i\theta})\rangle_{\mathbb{C}^m}\langle \bar{\eta}_0(e^{i\theta}), g(e^{i\theta})\rangle_{\mathbb{C}^m}\, d\theta = 0$$

for every $g \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$, which in turn is equivalent to the statement that

$$\frac{1}{2\pi}\int_0^{2\pi} g^*(e^{i\theta})\Phi(e^{i\theta}) \quad -g^*(e^{i\theta})\bar{\eta}_0\eta_0^T(e^{i\theta})\Phi(e^{i\theta})(e^{i\theta})$$
$$-g^*(e^{i\theta})\bar{\eta}_1(e^{i\theta})\eta_1^T(e^{i\theta})\Phi(e^{i\theta})\, d\theta = 0$$

for every $g \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$. Equivalently

$$\frac{1}{2\pi}\int_0^{2\pi} g^*(e^{i\theta})\Big(I_m - \bar{\eta}_0(e^{i\theta})\eta_0^T(e^{i\theta}) - \bar{\eta}_1(e^{i\theta})\eta_1^T(e^{i\theta})\Big)\Phi(e^{i\theta})\, d\theta = 0$$

for every $g \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ if and only if

$$\Big(I_m - \bar{\eta}_0(e^{i\theta})\eta_0^T(e^{i\theta}) - \bar{\eta}_1(e^{i\theta})\eta_1^T(e^{i\theta})\Big)\Phi(e^{i\theta})$$

is orthogonal to $H^2(\mathbb{D}, \mathbb{C}^m)^\perp$, which occurs if and only if

$$\left(I_m - \bar{\eta}_0\eta_0^T - \bar{\eta}_1\eta_1^T\right)\Phi \in H^2(\mathbb{D}, \mathbb{C}^m).$$

By Lemma 3.2.34,

$$\left(I_m - \bar{\eta}_0\eta_0^T - \bar{\eta}_1\eta_1^T\right)\Phi = \beta_0\beta_1\beta_1^*\beta_0^*\Phi.$$

Recall that, by assertions (3.114), $\beta_1^*\beta_0^*\Phi \in H^2(\mathbb{D}, \mathbb{C}^{m-2})$, and so

$$\beta_0\beta_1\beta_1^*\beta_0^*\Phi \in H^2(\mathbb{D}, \mathbb{C}^m).$$

Thus the right hand square in the diagram (3.106) commutes, and so the diagram (3.106) commutes.

(iv). By Proposition 3.2.26,

$$F_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-2)\times(n-2)}).$$

Thus, by Hartman's Theorem, the Hankel operator H_{F_2} is compact. By (iii),

$$(\bar{\eta}_0 \dot{\wedge} \bar{\eta}_1 \dot{\wedge} \cdot) \circ (M_{\beta_0\beta_1} H_{F_2} M_{\alpha_0^T\alpha_1^T}) \circ (\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \cdot)^* = T_2.$$

By (i) and (ii), the operators $M_{\bar{\alpha}_0\bar{\alpha}_1}$, $M_{\beta_0\beta_1}$, $(\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \cdot)$ and $(\bar{\eta}_0 \dot{\wedge} \bar{\eta}_1 \dot{\wedge} \cdot)$ are unitaries. Hence T_2 is a compact operator.

(v). Since diagram (3.106) commutes and the operators $M_{\bar{\alpha}_0\bar{\alpha}_1}$, $M_{\beta_0\beta_1}$, $(\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \cdot)$ and $(\bar{\eta}_0 \dot{\wedge} \bar{\eta}_1 \dot{\wedge} \cdot)$ are unitaries, $\|T_2\| = \|\Gamma_2\| = \|H_{F_2}\| = t_2$. □

Lemma 3.2.38. *In the notation of Theorem 3.2.37, let* $v_2 \in H^2(\mathbb{D}, \mathbb{C}^n)$ *and* $w_2 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ *be such that* $(\xi_0 \dot\wedge \xi_1 \dot\wedge v_2, \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge w_2)$ *is a Schmidt pair for the operator* T_2 *corresponding to* $\|T_2\|$*. Then*
(i) There exist $x_2 \in \mathcal{K}_2$ *and* $y_2 \in \mathcal{L}_2$ *such that* (x_2, y_2) *is a Schmidt pair for the operator* Γ_2*.*
(ii) For any $x_2 \in \mathcal{K}_2$ *and* $y_2 \in \mathcal{L}_2$ *such that*

$$\xi_0 \dot\wedge \xi_1 \dot\wedge x_2 = \xi_0 \dot\wedge \xi_1 \dot\wedge v_2, \quad \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge y_2 = \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge w_2,$$

the pair (x_2, y_2) *is a Schmidt pair for* Γ_2 *corresponding to* $\|\Gamma_2\|$*.*

Proof. (i). By Theorem 3.2.37, the diagram (3.106) commutes, $(\xi_0 \dot\wedge \xi_1 \dot\wedge \cdot)$ is unitary from $\mathcal{K}_2$ to X_2, $(\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \cdot)$ is unitary from $\mathcal{L}_2$ to Y_2 and $\|\Gamma_2\| = \|T_2\| = t_2$. Moreover, by the commutativity of diagram (3.106), the operator $\Gamma_2 \colon \mathcal{K}_2 \to \mathcal{L}_2$ is compact, hence there exist $x_2 \in \mathcal{K}_2$, $y_2 \in \mathcal{L}_2$ such that (x_2, y_2) is a Schmidt pair for Γ_2 corresponding to $\|\Gamma_2\| = t_2$.
(ii). Suppose that $x_2 \in \mathcal{K}_2, y_2 \in \mathcal{L}_2$ satisfy

$$\xi_0 \dot\wedge \xi_1 \dot\wedge x_2 = \xi_0 \dot\wedge \xi_1 \dot\wedge v_2 \tag{3.115}$$

and

$$\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge y_2 = \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge w_2. \tag{3.116}$$

Let us show that (x_2, y_2) is a Schmidt pair for Γ_2, that is,

$$\Gamma_2 x_2 = t_2 y_2, \quad \Gamma_2^* y_2 = t_2 x_2.$$

Since diagram (3.106) commutes,

$$T_2 \circ (\xi_0 \dot\wedge \xi_1 \dot\wedge \cdot) = (\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \cdot) \circ \Gamma_2, \quad (\xi_0 \dot\wedge \xi_1 \dot\wedge \cdot)^* \circ T_2^* = \Gamma_2^* \circ (\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \cdot)^*. \tag{3.117}$$

By hypothesis,

$$T_2(\xi_0 \dot\wedge \xi_1 \dot\wedge v_2) = t_2(\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge w_2), \quad T_2^*(\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge w_2) = t_2(\xi_0 \dot\wedge \xi_1 \dot\wedge v_2). \tag{3.118}$$

Thus, by equations (3.115), (3.116) and (3.118),

$$\begin{aligned}
\Gamma_2 x_2 &= (\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \cdot)^* T_2(\xi_0 \dot\wedge \xi_1 \dot\wedge v_2) \\
&= (\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \cdot)^* t_2(\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge w_2) \\
&= t_2(\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \cdot)^* (\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge y_2).
\end{aligned}$$

Hence

$$\Gamma_2 x_2 = t_2(\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \cdot)^* (\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \cdot) y_2 = t_2 y_2.$$

By equation (3.115),

$$x_2 = (\xi_0 \dot\wedge \xi_1 \dot\wedge \cdot)^* (\xi_0 \dot\wedge \xi_1 \dot\wedge v_2),$$

and, by equation (3.116),

$$(\bar{\eta}_0\dot{\wedge}\bar{\eta}_1\dot{\wedge}\cdot)^*(\bar{\eta}_0\dot{\wedge}\bar{\eta}_1\dot{\wedge}w_2) = y_2.$$

Thus

$$\begin{aligned}\Gamma_2^* y_2 &= \Gamma_2^*(\bar{\eta}_0\dot{\wedge}\bar{\eta}_1\dot{\wedge}\cdot)^*(\bar{\eta}_0\dot{\wedge}\bar{\eta}_1\dot{\wedge}w_2)\\ &= (\xi_0\dot{\wedge}\xi_1\dot{\wedge}\cdot)^* T_2^*(\bar{\eta}_0\dot{\wedge}\bar{\eta}_1\dot{\wedge}w_2),\end{aligned}$$

last equality following by the second equation of (3.117). By equations (3.115) and (3.118), we get

$$T_2^*(\bar{\eta}_0\dot{\wedge}\bar{\eta}_1\dot{\wedge}w_2) = t_2(\xi_0\dot{\wedge}\xi_1\dot{\wedge}v_2) = t_2(\xi_0\dot{\wedge}\xi_1\dot{\wedge}x_2),$$

and so,

$$\Gamma_2^* y_2 = t_2 x_2.$$

Therefore (x_2, y_2) is a Schmidt pair for Γ_2 corresponding to $\|\Gamma_2\|$. □

Lemma 3.2.39. *Suppose* $(\xi_0\dot{\wedge}\xi_1\dot{\wedge}v_2, \bar{\eta}_0\dot{\wedge}\bar{\eta}_1\dot{\wedge}w_2)$ *is a Schmidt pair for* T_2 *corresponding to* t_2*. Let*

$$x_2 = (I_{\mathbb{C}^n} - \xi_0\xi_0^* - \xi_1\xi_1^*)v_2, \quad y_2 = (I_{\mathbb{C}^m} - \bar{\eta}_0\eta_0^T - \bar{\eta}_1\eta_1^T)w_2,$$

and let

$$\hat{x}_2 = \alpha_1^T\alpha_0^T x_2, \quad \hat{y}_2 = \beta_1^*\beta_0^* y_2.$$

Then the pair $(\hat{x}_2, \hat{y}_2)$ *is a Schmidt pair for* H_{F_2} *corresponding to* $\|H_{F_2}\| = t_2$*.*

Proof. Let us first show that $\hat{x}_2 \in H^2(\mathbb{D}, \mathbb{C}^{n-2})$ and $x_2 \in \mathcal{K}_2$. Recall that $V_0 = \begin{pmatrix}\xi_0 & \bar{\alpha}_0\end{pmatrix}$ and $\tilde{V}_1 = \begin{pmatrix}\alpha_0^T\xi_1 & \bar{\alpha}_1\end{pmatrix}$ are unitary-valued, that is, $\alpha_0^T\xi_0 = 0$, $\alpha_1^T\alpha_0^T\xi_1 = 0$,

$$I_n - \xi_0\xi_0^* = \bar{\alpha}_0\alpha_0^T, \tag{3.119}$$

and

$$I_{n-1} - \alpha_0^T\xi_1\xi_1^*\bar{\alpha}_0 = \bar{\alpha}_1\alpha_1^T. \tag{3.120}$$

Then

$$\begin{aligned}\hat{x}_2 &= \alpha_1^T\alpha_0^T x_2\\ &= \alpha_1^T\alpha_0^T(I_{\mathbb{C}^n} - \xi_0\xi_0^* - \xi_1\xi_1^*)v_2\\ &= \alpha_1^T\alpha_0^T v_2 - \alpha_1^T\alpha_0^T\xi_0\xi_0^* v_2 - \alpha_1^T\alpha_0^T\xi_1\xi_1^* v_2\\ &= \alpha_1^T\alpha_0^T v_2,\end{aligned} \tag{3.121}$$

which, by Propositions 3.2.15 and 3.2.29, implies that $\hat{x}_2 \in H^2(\mathbb{D}, \mathbb{C}^{n-2})$. Moreover, by Lemma 3.2.32, we obtain

$$\begin{aligned}\bar{\alpha}_0\bar{\alpha}_1\hat{x}_2 &= \bar{\alpha}_0\bar{\alpha}_1\alpha_1^T\alpha_0^T v_2\\ &= (I_n - \xi_0\xi_0^* - \xi_1\xi_1^*)v_2 = x_2.\end{aligned}$$

Hence

$$x_2 = \bar{\alpha}_0 \bar{\alpha}_1 \alpha_1^T \alpha_0^T v_2 = \bar{\alpha}_0 \bar{\alpha}_1 \hat{x}_2, \tag{3.122}$$

and thus $x_2 \in \mathcal{K}_2$.

Next, we shall show that $\hat{y}_2 \in H^2(\mathbb{D}, \mathbb{C}^{n-2})^\perp$ and $y_2 \in \mathcal{L}_2$. Notice that since $\tilde{W}_1^T = \begin{pmatrix} \beta_0^T \eta_1 & \bar{\beta}_1 \end{pmatrix}$ and $W_0^T = \begin{pmatrix} \eta_0 & \bar{\beta}_0 \end{pmatrix}$ are unitary-valued, $\beta_0^* \bar{\eta}_0 = 0$, $\beta_1^* \beta_0^* \bar{\eta}_1 = 0$,

$$\left(I_{m-1} - \beta_0^* \bar{\eta}_1 \eta_1^T \beta_0\right) = \beta_1 \beta_1^* \tag{3.123}$$

and

$$\left(I_m - \bar{\eta}_0 \eta_0^T\right) = \beta_0 \beta_0^*. \tag{3.124}$$

We have

$$\begin{aligned} \hat{y}_2 &= \beta_1^* \beta_0^* y_2 \\ &= \beta_1^* \beta_0^* (I_{\mathbb{C}^m} - \bar{\eta}_0 \eta_0^T - \bar{\eta}_1 \eta_1^T) w_2 \\ &= \beta_1^* \beta_0^* w_2 - \beta_1^* \beta_0^* \bar{\eta}_0 \eta_0^T w_2 - \beta_1^* \beta_0^* \bar{\eta}_1 \eta_1^T w_2 \\ &= \beta_1^* \beta_0^* w_2, \end{aligned} \tag{3.125}$$

which, by Propositions 3.2.20 and 3.2.28, implies that $\hat{y}_2 \in H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp$. By Lemma 3.2.34, we have

$$\begin{aligned} \beta_0 \beta_1 \hat{y}_2 \ &= \beta_0 \beta_1 \beta_1^* \beta_0^* w_2 \\ &= (I_m - \bar{\eta}_0 \eta_0^T - \bar{\eta}_1 \eta_1^T) w_2 = y_2. \end{aligned}$$

Hence

$$y_2 = \beta_0 \beta_1 \beta_1^* \beta_0^* w_2 = \beta_0 \beta_1 \hat{y}_2, \tag{3.126}$$

and therefore $y_2 \in \mathcal{L}_2$.

By Theorem 3.2.37, the maps

$$M_{\bar{\alpha}_0 \bar{\alpha}_1} \colon H^2(\mathbb{D}, \mathbb{C}^{n-2}) \to \mathcal{K}_2, \quad M_{\beta_0 \beta_1} \colon H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp \to \mathcal{L}_2,$$

are unitaries and

$$H_{F_2} = M_{\beta_0 \beta_1}^* \circ \Gamma_2 \circ M_{\bar{\alpha}_0 \bar{\alpha}_1}. \tag{3.127}$$

We need to show that

$$H_{F_2} \hat{x}_2 = t_2 \hat{y}_2, \quad H_{F_2}^* \hat{y}_2 = t_2 \hat{x}_2.$$

By equations (3.121) and (3.122),

$$x_2 = \bar{\alpha}_0 \bar{\alpha}_1 \alpha_1^T \alpha_0^T x_2. \tag{3.128}$$

Hence equation (3.127) yields

$$H_{F_2} \hat{x}_2 = \beta_1^* \beta_0^* \Gamma_2 \bar{\alpha}_0 \bar{\alpha}_1 \hat{x}_2 = \beta_1^* \beta_0^* \Gamma_2 x_2. \tag{3.129}$$

By Proposition 3.2.1(ii),

$$\xi_0 \dot\wedge \xi_1 \dot\wedge x_2 = \xi_0 \dot\wedge \xi_1 \dot\wedge v_2, \quad \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge y_2 = \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge w_2.$$

Thus, by Lemma 3.2.38, (x_2, y_2) is a Schmidt pair for the operator Γ_2 corresponding to $t_2 = \|\Gamma_2\|$, that is,

$$\Gamma_2 x_2 = t_2 y_2, \quad \Gamma_2^* y_2 = t_2 x_2. \tag{3.130}$$

Thus equation (3.129) yields

$$H_{F_2} \hat{x}_2 = \beta_1^* \beta_0^* \Gamma_2 x_2 = \beta_1^* \beta_0^* t_2 y_2 = t_2 \hat{y}_2$$

as required. Let us show that $H_{F_2}^* \hat{y}_2 = t_2 \hat{x}_2$. By equations (3.125) and (3.126),

$$y_2 = \beta_0 \beta_1 \beta_1^* \beta_0^* y_2. \tag{3.131}$$

Hence, by equations (3.127) and (3.131),

$$H_{F_2}^* \beta_1^* \beta_0^* y_2 = \alpha_1^T \alpha_0^T \Gamma_2^* y_2, \tag{3.132}$$

and, by equations (3.130) and (3.132),

$$H_{F_2}^* \hat{y}_2 = \alpha_1^T \alpha_0^T \Gamma_2^* y_2 = \alpha_1^T \alpha_0^T t_2 x_2 = t_2 \hat{x}_2.$$

Therefore $(\hat{x}_2, \hat{y}_2)$ is a Schmidt pair for H_{F_2} corresponding to $\|H_{F_2}\| = t_2$. □

Proposition 3.2.40. *Let $(\xi_0 \dot\wedge \xi_1 \dot\wedge v_2, \bar\eta_0 \dot\wedge \bar\eta_0 \dot\wedge w_2)$ be a Schmidt pair for T_2 corresponding to t_2 for some $v_2 \in H^2(\mathbb{D}, \mathbb{C}^n), w_2 \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$, let $h_2 \in H^2(\mathbb{D}, \mathbb{C})$ be the scalar outer factor of $\xi_0 \dot\wedge \xi_1 \dot\wedge v_2$, let*

$$x_2 = (I_n - \xi_0 \xi_0^* - \xi_1 \xi_1^*) v_2, \quad y_2 = (I_m - \bar\eta_0 \eta_0^T - \bar\eta_1 \eta_1^T) w_2,$$

and let

$$\hat{x}_2 = \alpha_1^T \alpha_0^T x_2 \quad \text{and} \quad \hat{y}_2 = \beta_1^* \beta_0^* y_2. \tag{3.133}$$

Then

$$\|\hat{x}_2(z)\|_{\mathbb{C}^{n-2}} = \|\hat{y}_2(z)\|_{\mathbb{C}^{m-2}} = |h_2(z)|,$$

$$\|x_2(z)\|_{\mathbb{C}^n} = \|y_2(z)\|_{\mathbb{C}^m} = |h_2(z)|$$

and

$$\|\xi_0(z) \wedge \xi_1(z) \wedge v_2(z)\|_{\wedge^3 \mathbb{C}^n} = \|\bar\eta_0(z) \wedge \bar\eta_1(z) \wedge w_2(z)\|_{\wedge^3 \mathbb{C}^m} = |h_2(z)|$$

almost everywhere on $\mathbb{T}$.

Proof. By Lemma 3.2.39, $(\hat{x}_2, \hat{y}_2)$ is a Schmidt pair for H_{F_2} corresponding to $\|H_{F_2}\| = t_2$

(see Theorem 3.2.37 (v)). Hence

$$H_{F_2}\hat{x}_2 = t_2\hat{y}_2 \quad \text{and} \quad H^*_{F_2}\hat{y}_2 = t_2\hat{x}_2.$$

Then, by Theorem D.2.4,

$$\|\hat{y}_2(z)\|_{\mathbb{C}^{m-2}} = \|\hat{x}_2(z)\|_{\mathbb{C}^{n-2}} \tag{3.134}$$

almost everywhere on $\mathbb{T}$. Notice that, by equations (3.133),

$$x_2 = \bar{\alpha}_0\bar{\alpha}_1\hat{x}_2,$$

and since $\bar{\alpha}_0(z), \bar{\alpha}_1(z)$ are isometric for almost every $z \in \mathbb{T}$, we obtain

$$\|x_2(z)\|_{\mathbb{C}^n} = \|\hat{x}_2(z)\|_{\mathbb{C}^{n-2}}.$$

Furthermore, by equations (3.133),

$$y_2 = \beta_0\beta_1\hat{y}_2,$$

and since $\beta_0(z), \beta_1(z)$ are isometries almost everywhere on $\mathbb{T}$, we get

$$\|y_2(z)\|_{\mathbb{C}^m} = \|\hat{y}_2(z)\|_{\mathbb{C}^{m-2}}$$

almost everywhere on $\mathbb{T}$. By equation (3.134), we deduce that

$$\|x_2(z)\|_{\mathbb{C}^n} = \|\hat{x}_2(z)\|_{\mathbb{C}^{n-2}} = \|\hat{y}_2(z)\|_{\mathbb{C}^{m-2}} = \|y_2(z)\|_{\mathbb{C}^m} \tag{3.135}$$

almost everywhere on $\mathbb{T}$.

By Proposition 3.2.1,

$$\xi_0 \wedge \xi_1 \dot{\wedge} v_2 = \xi_0 \wedge \xi_1 \dot{\wedge} x_2, \quad \bar{\eta}_0 \wedge \bar{\eta}_1 \wedge w_2 = \bar{\eta}_0 \wedge \bar{\eta}_1 \wedge y_2.$$

By Theorem 3.2.37, $(\xi_0\dot{\wedge}\xi_1\dot{\wedge}\cdot)$ is an isometry from $\mathcal{K}_2$ to X_2, and $(\bar{\eta}_0\dot{\wedge}\bar{\eta}_1\dot{\wedge}\cdot)$ is an isometry from $\mathcal{L}_2$ to Y_2. Hence

$$\|\xi_0(z) \wedge \xi_1(z)\dot{\wedge}v_2(z)\|_{\wedge^3\mathbb{C}^n} = \|\xi_0(z) \wedge \xi_1(z) \wedge x_2(z)\|_{\wedge^3\mathbb{C}^n} = \|x_2(z)\|_{\mathbb{C}^n}$$

almost everywhere on $\mathbb{T}$. Furthermore

$$\|\bar{\eta}_0(z) \wedge \bar{\eta}_1(z) \wedge w_2(z)\|_{\wedge^3\mathbb{C}^m} = \|\bar{\eta}_0(z) \wedge \bar{\eta}_1(z) \wedge y_2(z)\|_{\wedge^3\mathbb{C}^m} = \|y_2(z)\|_{\mathbb{C}^m}$$

almost everywhere on $\mathbb{T}$. Thus, by equation (3.135),

$$\|\xi_0(z) \wedge \xi_1(z) \wedge v_2(z)\|_{\wedge^3\mathbb{C}^n} = \|\bar{\eta}_0(z) \wedge \bar{\eta}_1(z) \wedge w_2(z)\|_{\wedge^3\mathbb{C}^m}$$

almost everywhere on $\mathbb{T}$.

Recall that h_2 is the scalar outer factor of $\xi_0\dot\wedge\xi_1\dot\wedge v_2$. Hence

$$\|\hat{x}_2(z)\|_{\mathbb{C}^{n-2}} = \|\hat{y}_2(z)\|_{\mathbb{C}^{m-2}} = |h_2(z)|,$$

$$\|x_2(z)\|_{\mathbb{C}^n} = \|y_2(z)\|_{\mathbb{C}^m} = |h_2(z)|$$

and

$$\|\xi_0(z) \wedge \xi_1(z) \wedge v_2(z)\|_{\wedge^3\mathbb{C}^n} = \|\bar\eta_0(z) \wedge \bar\eta_1(z) \wedge w_2(z)\|_{\wedge^3\mathbb{C}^m} = |h_2(z)|$$

almost everywhere on $\mathbb{T}$. □

Proposition 3.2.41. *Let m, n be positive integers such that $\min(m,n) \geq 2$. Let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$. In line with the algorithm from Section 3.2.1, let $Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ satisfy*

$$\begin{aligned} (G - Q_2)x_0 &= t_0 y_0, & (G - Q_2)^* y_0 &= t_0 x_0, \\ (G - Q_2)x_1 &= t_1 y_1, & (G - Q_2)^* y_1 &= t_1 x_1. \end{aligned} \tag{3.136}$$

Let the spaces X_2, Y_2 be given by

$$X_2 = \xi_0\dot\wedge\xi_1\dot\wedge H^2(\mathbb{D}, \mathbb{C}^n), \quad Y_2 = \bar\eta_0\dot\wedge\bar\eta_1\dot\wedge H^2(\mathbb{D}, \mathbb{C}^m)^\perp,$$

and consider the compact operator $T_2\colon X_2 \to Y_2$ given by

$$T_2(\xi_0\dot\wedge\xi_1\dot\wedge x) = P_{Y_2}(\bar\eta_0\dot\wedge\bar\eta_1\dot\wedge(G - Q_2)x)$$

for all $x \in H^2(\mathbb{D}, \mathbb{C}^n)$. Let $(\xi_0\dot\wedge\xi_1\dot\wedge v_2, \bar\eta_0\dot\wedge\bar\eta_1\dot\wedge w_2)$ be a Schmidt pair for the operator T_2 corresponding to $t_2 = \|T_2\|$, let $h_2 \in H^2(\mathbb{D}, \mathbb{C})$ be the scalar outer factor of $\xi_0\dot\wedge\xi_1\dot\wedge v_2$, let

$$x_2 = (I_{\mathbb{C}^n} - \xi_0\xi_0^* - \xi_1\xi_1^*)v_2, \quad y_2 = (I_{\mathbb{C}^m} - \bar\eta_0\eta_0^T - \bar\eta_1\eta_1^T)w_2$$

and let

$$\xi_2 = \frac{x_2}{h_2}, \quad \eta_2 = \frac{\bar z \bar y_2}{h_2}.$$

Then there exist unitary-valued functions $\tilde V_2, \tilde W_2$ of types $(n-2)\times(n-2)$ and $(m-2)\times(m-2)$ respectively of the form

$$\tilde V_2 = \begin{pmatrix} \alpha_1^T\alpha_0^T\xi_2 & \bar\alpha_2 \end{pmatrix}, \quad \tilde W_2^T = \begin{pmatrix} \beta_1^T\beta_0^T\eta_2 & \bar\beta_2 \end{pmatrix},$$

where α_2, β_2 are inner, co-outer, quasi-continuous and all minors on the first columns of $\tilde V_2, \tilde W_2^T$ are in H^∞. Furthermore, the set $\mathcal{E}_2$ of all level 2 superoptimal error functions for G satisfies

$$\mathcal{E}_2 = W_0^* W_1^* \begin{pmatrix} I_2 & 0 \\ 0 & \tilde{W}_2^* \end{pmatrix} \begin{pmatrix} t_0u_0 & 0 & 0 & 0 \\ 0 & t_1u_1 & 0 & 0 \\ 0 & 0 & t_2u_2 & 0 \\ 0 & 0 & 0 & (F_3 + H^\infty) \cap B(t_2) \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & \tilde{V}_2^* \end{pmatrix} V_1^* V_0^*,$$

for some $F_3 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-3)\times(n-3)}) + C(\mathbb{T}, \mathbb{C}^{(m-3)\times(n-3)})$, *where* $u_3 = \frac{\bar{z}\bar{h}_3}{h_3}$ *is a quasi-continuous unimodular function and* $B(t_2)$ *is the closed ball of radius* t_2 *in* $L^\infty(\mathbb{T}, \mathbb{C}^{(m-3)\times(n-3)})$.

Proof. By Theorem 3.2.37, the following diagram commutes

$$\begin{array}{ccccc} H^2(\mathbb{D}, \mathbb{C}^{n-2}) & \xrightarrow{M_{\bar{\alpha}_0\bar{\alpha}_1}} & \mathcal{K}_2 & \xrightarrow{\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \cdot} & \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n) = X_2 \\ \Big\downarrow H_{F_2} & & \Big\downarrow \Gamma_2 & & \Big\downarrow T_2 \\ H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp & \xrightarrow{M_{\beta_0\beta_1}} & \mathcal{L}_2 & \xrightarrow{\bar{\eta}_0 \dot{\wedge} \bar{\eta}_1 \dot{\wedge} \cdot} & \bar{\eta}_0 \dot{\wedge} \bar{\eta}_1 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp = Y_2. \end{array} \tag{3.137}$$

Recall that the operators $M_{\bar{\alpha}_0\bar{\alpha}_1}$, $M_{\beta_0\beta_1}$, $(\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \cdot)$ and $(\bar{\eta}_0 \dot{\wedge} \bar{\eta}_1 \dot{\wedge} \cdot)$ are unitaries.

By Proposition 3.2.8, T_2 is well-defined and is independent of the choice of $Q_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ satisfying conditions (3.136). Hence we may choose Q_2 to minimise $(s_0^\infty(G-Q), s_1^\infty(G-Q))$, and then, by Proposition 3.2.27, the conditions (3.136) hold.

By Lemma 3.2.38, (x_2, y_2) defined above is a Schmidt pair for Γ_2 corresponding to t_2. By Lemma 3.2.39, $(\hat{x}_2, \hat{y}_2)$ is a Schmidt pair for H_{F_2} corresponding to t_2, where

$$\hat{x}_2 = \alpha_1^T \alpha_0^T x_2, \quad \hat{y}_2 = \beta_1^* \beta_0^* y_2.$$

We would like to apply Lemma 3.1.12 to H_{F_2} and the Schmidt pair $(\hat{x}_2, \hat{y}_2)$ to find unitary-valued functions $\tilde{V}_2, \tilde{W}_2$ such that, for every $\tilde{Q}_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)})$ which is at minimal distance from F_2, a factorisation of the form

$$F_2 - \tilde{Q}_2 = \tilde{W}_2^* \begin{pmatrix} t_2u_2 & 0 \\ 0 & F_3 \end{pmatrix} \tilde{V}_2^*$$

is obtained, for some $F_3 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-2)\times(n-2)})$. For this purpose we find the inner-outer factorisations of $\hat{x}_2$ and $\bar{z}\bar{\hat{y}}_2$. By Lemma 3.2.40

$$\|\hat{x}_2(z)\|_{\mathbb{C}^{n-2}} = |h_2(z)| \text{ and } \|\hat{y}_2(z)\|_{\mathbb{C}^{m-2}} = |h_2(z)| \tag{3.138}$$

almost everywhere on $\mathbb{T}$. Equations (3.138) imply that $h_2 \in H^2(\mathbb{D}, \mathbb{C})$ is the scalar outer factor of both $\hat{x}_2$ and $\bar{z}\bar{\hat{y}}_2$. By Lemma 3.1.12, $\hat{x}_2, \bar{z}\bar{\hat{y}}_2$ admit the inner outer factorisations

$$\hat{x}_2 = \hat{\xi}_2 h_2, \quad \bar{z}\bar{\hat{y}}_2 = \hat{\eta}_2 h_2,$$

for some inner $\hat{\xi}_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{n-2}), \hat{\eta}_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{m-2})$. Then

$$\hat{x}_2 = \hat{\xi}_2 h_2 = \alpha_1^T \alpha_0^T x_2, \quad \bar{z}\bar{\hat{y}}_2 = \hat{\eta}_2 h_2 = \bar{z}\beta_1^T \beta_0^T \bar{y}_2,$$

from which we obtain

$$\hat{\xi}_2 = \alpha_1^T \alpha_0^T \xi_2, \quad \hat{\eta}_2 = \beta_1^T \beta_0^T \eta_2.$$

We would like to show that $\alpha_1^T \alpha_0^T \xi_2,\ \beta_1^T \beta_0^T \eta_2$ are inner in order to apply Lemma 3.1.12 and obtain $\tilde{V}_2$ and $\tilde{W}_2$. Recall that, by Lemma 3.2.39,

$$x_2 = (I_{\mathbb{C}^n} - \xi_0\xi_0^* - \xi_1\xi_1^*)v_2 = \bar{\alpha}_0\bar{\alpha}_1\alpha_1^T\alpha_0^T v_2, \quad y_2 = (I_{\mathbb{C}^m} - \bar{\eta}_0\eta_0^T - \bar{\eta}_1\eta_1^T)w_2 = \beta_0\beta_1\beta_1^*\beta_0^* w_2.$$

Then,

$$\alpha_1^T\alpha_0^T x_2 = \alpha_1^T\alpha_0^T v_2, \quad \beta_1^T\beta_0^T \bar{y}_2 = \beta_1^T\beta_0^T \bar{w}_2,$$

and since

$$\xi_2 = \frac{x_2}{h_2}, \quad \eta_2 = \frac{\bar{z}\bar{y}_2}{h_2},$$

we deduce that the functions

$$\alpha_1^T\alpha_0^T\xi_2 = \frac{\alpha_1^T\alpha_0^T v_2}{h_2}, \quad \beta_1^T\beta_0^T\eta_2 = \frac{\beta_1^T\beta_0^T\bar{z}\bar{w}_2}{h_2}$$

are analytic. Furthermore, $\|\xi_2(z)\|_{\mathbb{C}^n} = 1$ and $\|\eta_2(z)\|_{\mathbb{C}^m} = 1$ almost everywhere on $\mathbb{T}$, and, by equations (3.138),

$$\|\alpha_1^T(z)\alpha_0^T(z)x_2(z)\|_{\mathbb{C}^{n-2}} = \|\alpha_1^T(z)\alpha_0^T(z)v_2(z)\|_{\mathbb{C}^{n-2}} = |h_2(z)|$$

and

$$\|\beta_1^T(z)\beta_0^T(z)\bar{y}_2(z)\|_{\mathbb{C}^{m-2}} = \|\beta_1^T(z)\beta_0^T(z)\bar{w}_2(z)\|_{\mathbb{C}^{m-2}} = |h_2(z)|$$

almost everywhere on $\mathbb{T}$. Hence

$$\|\alpha_1^T(z)\alpha_0^T(z)\xi_2(z)\|_{\mathbb{C}^{n-2}} = 1, \quad \|\beta_1^T(z)\beta_0^T(z)\eta_2(z)\|_{\mathbb{C}^{m-2}} = 1$$

almost everywhere on $\mathbb{T}$. Thus $\alpha_1^T\alpha_0^T\xi_2,\ \beta_1^T\beta_0^T\eta_2$ are inner functions.

By Lemma 3.1.12, there exist inner, co-outer, quasi-continuous functions α_2, β_2 of types $(n-2)\times(n-3), (m-2)\times(m-3)$ respectively such that the functions

$$\tilde{V}_2 = \begin{pmatrix} \alpha_1^T\alpha_0^T\xi_2 & \bar{\alpha}_2 \end{pmatrix}, \quad \tilde{W}_2^T = \begin{pmatrix} \beta_1^T\beta_0^T\eta_2 & \bar{\beta}_2 \end{pmatrix}$$

are unitary-valued with all minors on the first columns in H^∞.
Furthermore, by Lemma 3.1.12, every $\hat{Q}_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)})$ which is at minimal distance from F_2 satisfies

$$F_2 - \hat{Q}_2 = \tilde{W}_2^* \begin{pmatrix} t_2u_2 & 0 \\ 0 & F_3 \end{pmatrix} \tilde{V}_2^*,$$

for some $F_3 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-3)\times(n-3)}) + C(\mathbb{T}, \mathbb{C}^{(m-3)\times(n-3)})$, and for the quasi-continuous unimodular function u_2 given by $u_2 = \frac{\bar{z}\bar{h}_2}{h_2}$. By Lemma 3.1.15, the set

$$\tilde{\mathcal{E}}_2 = \{F_2 - \hat{Q} : \hat{Q} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)}), \|F_2 - \hat{Q}\|_{L^\infty} = t_2\}$$

satisfies

$$\tilde{\mathcal{E}}_2 = \tilde{W}_2^* \begin{pmatrix} t_2 u_2 & 0 \\ 0 & (F_3 + H^\infty) \cap B(t_2) \end{pmatrix} V_2^*,$$

where $B(t_2)$ is the closed ball of radius t_2 in $L^\infty(\mathbb{T}, \mathbb{C}^{(m-3)\times(n-3)})$. Thus, by Proposition 3.2.26, $\mathcal{E}_2$ admits the factorisation claimed. □

Proposition 3.2.42. *Every $Q_3 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ which minimises*

$$(s_0^\infty(G-Q), s_1^\infty(G-Q), s_2^\infty(G-Q))$$

satisfies

$$(G-Q_3)x_i = t_i y_i, \quad (G-Q_3)^* y_i = t_i x_i \quad \textit{for} \quad i = 0,1,2.$$

Proof. By Proposition 3.2.27, every $Q_3 \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ that minimises

$$(s_0^\infty(G-Q), s_1^\infty(G-Q))$$

satisfies

$$(G-Q_3)x_i = t_i y_i, \quad (G-Q_3)^* y_i = t_i x_i \quad \text{for} \quad i = 0,1.$$

Hence it suffices to show that Q_3 satisfies

$$(G-Q_3)x_2 = t_2 y_2, \quad (G-Q_3)^* y_2 = t_2 x_2.$$

By Theorem 3.2.37, the following diagram commutes

$$\begin{array}{ccccc} H^2(\mathbb{D}, \mathbb{C}^{n-2}) & \xrightarrow{M_{\bar\alpha_0\bar\alpha_1}} & \mathcal{K}_2 & \xrightarrow{\xi_0 \dot\wedge \xi_1 \dot\wedge \cdot} & \xi_0 \dot\wedge \xi_1 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n) = X_2 \\ \Big\downarrow H_{F_2} & & \Big\downarrow \Gamma_2 & & \Big\downarrow T_2 \\ H^2(\mathbb{D}, \mathbb{C}^{m-2})^\perp & \xrightarrow{M_{\beta_0\beta_1}} & \mathcal{L}_2 & \xrightarrow{\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \cdot} & \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^m)^\perp = Y_2, \end{array}$$

where the operator $\Gamma_2 \colon \mathcal{K}_2 \to \mathcal{L}_2$ is given by $\Gamma_2 = P_{\mathcal{L}_2} M_{G-Q_2}|_{\mathcal{K}_2}$ and $F_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-2)\times(n-2)})$ is constructed as follows.
By Lemma 3.1.12 and Proposition 3.2.26, there exist unitary-valued functions

$$\tilde{V}_1 = \begin{pmatrix} \alpha_0^T \xi_1 & \bar\alpha_1 \end{pmatrix}, \quad \tilde{W}_1^T = \begin{pmatrix} \beta_0^T \eta_1 & \bar\beta_1 \end{pmatrix},$$

where α_1, β_1 are inner, co-outer, quasi-continuous functions of types $(n-1)\times(n-2)$ and $(m-1)\times(m-2)$ respectively, and all minors on the first columns of $\tilde{V}_1, \tilde{W}_1^T$ are in H^∞. Furthermore, the set of all level 1 superoptimal functions $\mathcal{E}_1 = \{G - Q : Q \in \Omega_1\}$ satisfies

$$\mathcal{E}_1 = W_0^* \begin{pmatrix} 1 & 0 \\ 0 & \tilde{W}_1^* \end{pmatrix} \begin{pmatrix} t_0 u_0 & 0 & 0 \\ 0 & t_1 u_1 & 0 \\ 0 & 0 & (F_2 + H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)})) \cap B(t_1) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{V}_1^* \end{pmatrix} V_0^*, \tag{3.139}$$

for some $F_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-2)\times(n-2)})$, for the quasi-continuous unimodular function $u_1 = \frac{\bar{z}\bar{h}_1}{h_1}$, where $B(t_1)$ is the closed ball of radius t_1 in $L^\infty(\mathbb{T}, \mathbb{C}^{(m-2)\times(n-2)})$.

Consider some $Q_3 \in \Omega_1$, so that, according to equation (3.139),

$$\begin{pmatrix} 1 & 0 \\ 0 & \tilde{W}_1 \end{pmatrix} W_0(G-Q_3)V_0 \begin{pmatrix} 1 & 0 \\ 0 & \tilde{V}_1 \end{pmatrix} = \begin{pmatrix} t_0u_0 & 0 & 0 \\ 0 & t_1u_1 & 0 \\ 0 & 0 & F_2 - \tilde{Q}_2 \end{pmatrix},$$

for some $\tilde{Q}_2 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)})$, that is,

$$\begin{pmatrix} 1 & 0 \\ 0 & \begin{pmatrix} \eta_1^T\beta_0 \\ \beta_1^* \end{pmatrix} \end{pmatrix} \begin{pmatrix} \eta_0^T \\ \beta_0^* \end{pmatrix} (G-Q_3)\begin{pmatrix} \xi_0 & \bar{\alpha}_0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha_0^T\xi_1 & \bar{\alpha}_1 \end{pmatrix} = \begin{pmatrix} t_0u_0 & 0 & 0 \\ 0 & t_1u_1 & 0 \\ 0 & 0 & F_2 - \tilde{Q}_2 \end{pmatrix}. \tag{3.140}$$

Observe

$$\begin{pmatrix} \eta_0^T \\ \beta_0^* \end{pmatrix} (G-Q_3)\begin{pmatrix} \xi_0 & \bar{\alpha}_0 \end{pmatrix} = \begin{pmatrix} t_0u_0 & 0 \\ 0 & \beta_0^*(G-Q_3)\bar{\alpha}_0 \end{pmatrix},$$

hence

$$\begin{pmatrix} 1 & 0 \\ 0 & \begin{pmatrix} \eta_1^T\beta_0 \\ \beta_1^* \end{pmatrix} \end{pmatrix} \begin{pmatrix} t_0u_0 & 0 \\ 0 & \beta_0^*(G-Q_3)\bar{\alpha}_0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha_0^T\xi_1 & \bar{\alpha}_1 \end{pmatrix}$$

is equal to

$$\begin{pmatrix} t_0u_0 & 0 & 0 \\ 0 & \eta_1^T\beta_0\beta_0^*(G-Q_3)\bar{\alpha}_0\alpha_0^T\xi_1 & \eta_1^T\beta_0\beta_0^*(G-Q_3)\bar{\alpha}_0\bar{\alpha}_1 \\ 0 & \beta_1^*\beta_0^*(G-Q_3)\bar{\alpha}_0\alpha_0^T\xi_1 & \beta_1^*\beta_0^*(G-Q_3)\bar{\alpha}_0\bar{\alpha}_1 \end{pmatrix},$$

and so equation (3.140) yields

$$\begin{pmatrix} t_0u_0 & 0 & 0 \\ 0 & \eta_1^T\beta_0\beta_0^*(G-Q_3)\bar{\alpha}_0\alpha_0^T\xi_1 & \eta_1^T\beta_0\beta_0^*(G-Q_3)\bar{\alpha}_0\bar{\alpha}_1 \\ 0 & \beta_1^*\beta_0^*(G-Q_3)\bar{\alpha}_0\alpha_0^T\xi_1 & \beta_1^*\beta_0^*(G-Q_3)\bar{\alpha}_0\bar{\alpha}_1 \end{pmatrix} = \begin{pmatrix} t_0u_0 & 0 & 0 \\ 0 & t_1u_1 & 0 \\ 0 & 0 & F_2 - \tilde{Q}_2 \end{pmatrix},$$

which is equivalent to the following equations

$$\eta_1^T\beta_0\beta_0^*(G-Q_3)\bar{\alpha}_0\alpha_0^T\xi_1 = t_1u_1,$$

$$\eta_1^T\beta_0\beta_0^*(G-Q_3)\bar{\alpha}_0\bar{\alpha}_1 = 0,$$

$$\beta_1^*\beta_0^*(G-Q_3)\bar{\alpha}_0\alpha_0^T\xi_1 = 0,$$

and

$$\beta_1^* \beta_0^* (G - Q_3) \bar{\alpha}_0 \bar{\alpha}_1 = F_2 - \tilde{Q}_2. \tag{3.141}$$

By Theorem D.2.4 applied to H_{F_2}, if $(\hat{x}_2, \hat{y}_2)$ is a Schmidt pair for H_{F_2} corresponding to $t_2 = \|H_{F_2}\|$, then, for any $\tilde{Q}_2$ which is at minimal distance from F_2, we have

$$(F_2 - \tilde{Q}_2)\hat{x}_2 = t_2 \hat{y}_2, \quad (F_2 - \tilde{Q}_2)^* \hat{y}_2 = t_2 \hat{x}_2. \tag{3.142}$$

By equations (3.141) and (3.142),

$$\beta_1^* \beta_0^* (G - Q_3) \bar{\alpha}_0 \bar{\alpha}_1 \hat{x}_2 = t_2 \hat{y}_2 \tag{3.143}$$

and

$$\alpha_1^T \alpha_0^T (G - Q_3)^* \beta_0 \beta_1 \hat{y}_2 = t_2 \hat{x}_2. \tag{3.144}$$

Recall that, by equations (3.122) and (3.126),

$$\bar{\alpha}_0 \bar{\alpha}_1 \hat{x}_2 = x_2 \quad \text{and} \quad \hat{y}_2 = \beta_1^* \beta_0^* y_2. \tag{3.145}$$

Hence, by equation (3.143), we obtain

$$\beta_1^* \beta_0^* (G - Q_3) x_2 = t_2 \beta_1^* \beta_0^* y_2,$$

or equivalently,

$$\beta_1^* \beta_0^* \Big((G - Q_3) x_2 - t_2 y_2 \Big) = 0.$$

Since, by Theorem 3.2.37, $M_{\beta_0 \beta_1}$ is unitary, the latter equation yields

$$(G - Q_3) x_2 = t_2 y_2.$$

Moreover, in view of equations (3.141), (3.142) and (3.145), equation (3.144) implies

$$\alpha_1^T \alpha_0^T (G - Q_3)^* y_2 = t_2 \alpha_1^T \alpha_0^T x_2,$$

which in turn is equivalent to the equation

$$\alpha_1^T \alpha_0^T \Big((G - Q_3)^* y_2 - t_2 x_2 \Big) = 0.$$

By Theorem 3.2.37, $M_{\bar{\alpha}_0 \bar{\alpha}_1}$ is unitary, hence the latter equation yields

$$(G - Q_3)^* y_2 = t_2 x_2$$

and therefore the assertion has been proved. □

3.2.7 Compactness of the operator T_{j+1}

At this point, the reader is able to distinguish the method of proving the compactness of the operators T_1 and T_2. Suppose we have applied steps $0, \dots, j$ of the superoptimal analytic approximation algorithm from Section 3.2.1 to G, we have constructed

$$\begin{aligned}
& t_0 \geq t_1 \geq \cdots \geq t_j > 0 \\
& x_0, x_1, \cdots, x_j \in L^2(\mathbb{T}, \mathbb{C}^n) \\
& y_0, y_1, \cdots, y_j \in L^2(\mathbb{T}, \mathbb{C}^m) \\
& h_0, h_1, \cdots, h_j \in H^2(\mathbb{D}, \mathbb{C}) \text{ outer} \\
& \xi_0, \xi_1, \cdots, \xi_j \in L^2(\mathbb{T}, \mathbb{C}^n) \text{ pointwise orthonormal on } \mathbb{T} \\
& \eta_0, \eta_1, \cdots, \eta_j \in L^2(\mathbb{T}, \mathbb{C}^m) \text{ pointwise orthonormal on } \mathbb{T} \\
& X_0 = H^2(\mathbb{D}, \mathbb{C}^n), X_1, \cdots, X_j \\
& Y_0 = H^2(\mathbb{D}, \mathbb{C}^m)^\perp, Y_1, \cdots, Y_j \\
& T_0, T_1, \cdots, T_j \text{ compact operators,}
\end{aligned}$$

and all the claimed properties hold. We shall apply a similar method to show that the operator T_{j+1} as given in equation (3.25) is compact.

Proposition 3.2.43. *Let m, n be positive integers such that $\min(m, n) \geq 2$. Let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$. In line with the algorithm from Section 3.2.1, let $Q_j \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ satisfy*

$$(G - Q_j)x_i = t_i y_i, \quad (G - Q_j)^* y_i = t_i x_i \quad \text{for} \quad i = 0, 1, \dots, j-1. \tag{3.146}$$

Let the spaces X_j, Y_j be given by

$$X_j = \xi_0 \dot\wedge \xi_1 \dot\wedge \dots \dot\wedge \xi_{j-1} \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n), \quad Y_j = \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \dots \dot\wedge \bar\eta_{j-1} \dot\wedge H^2(\mathbb{D}, \mathbb{C}^m)^\perp,$$

and consider the compact operator $T_j \colon X_j \to Y_j$ given by

$$T_j(\xi_0 \dot\wedge \xi_1 \dot\wedge \dots \dot\wedge \xi_{j-1} \dot\wedge x) = P_{Y_j}(\bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \dots \dot\wedge \bar\eta_{j-1} \dot\wedge (G - Q_j)x)$$

for all $x \in H^2(\mathbb{D}, \mathbb{C}^n)$. Let $(\xi_0 \dot\wedge \xi_1 \dot\wedge \dots \dot\wedge \xi_{j-1} \dot\wedge v_j, \bar\eta_0 \dot\wedge \bar\eta_1 \dot\wedge \dots \dot\wedge \bar\eta_{j-1} \dot\wedge w_j)$ be a Schmidt pair for the operator T_j corresponding to $t_j = \|T_j\|$, let $h_j \in H^2(\mathbb{D}, \mathbb{C})$ be the scalar outer factor of $\xi_0 \dot\wedge \xi_1 \dot\wedge \dots \dot\wedge \xi_{j-1} \dot\wedge v_j$, let

$$x_j = (I_{\mathbb{C}^n} - \xi_0 \xi_0^* \cdots - \xi_{j-1}\xi_{j-1}^*)v_j, \quad y_j = (I_{\mathbb{C}^m} - \bar\eta_0 \eta_0^T - \cdots - \bar\eta_{j-1}\eta_{j-1}^T)w_j$$

and let

$$\xi_j = \frac{x_j}{h_j}, \quad \eta_j = \frac{\bar z \bar y_j}{h_j}. \tag{3.147}$$

Let, for $i = 0, 1, \dots, j-1$,

$$\tilde V_i = \begin{pmatrix} \alpha_{i-1}^T \cdots \alpha_0^T \xi_i & \bar\alpha_i \end{pmatrix}, \quad \tilde W_i^T = \begin{pmatrix} \beta_{i-1}^T \cdots \beta_0^T \eta_i & \bar\beta_i \end{pmatrix} \tag{3.148}$$

be unitary-valued functions, as described in Lemma 3.1.12 (see also Proposition 3.2.41 for $\tilde{V}_2$ *and* $\tilde{W}_2^T$*),* $u_i = \frac{\bar{z}\bar{h}_i}{h_i}$ *are quasi-continuous unimodular functions, and*

$$V_i = \begin{pmatrix} I_i & 0 \\ 0 & \tilde{V}_i \end{pmatrix}, \quad W_i = \begin{pmatrix} I_i & 0 \\ 0 & \tilde{W}_i \end{pmatrix}.$$

There exist unitary-valued functions $\tilde{V}_j, \tilde{W}_j$ *of the form*

$$\tilde{V}_j = \begin{pmatrix} \alpha_{j-1}^T \cdots \alpha_0^T \xi_j & \bar{\alpha}_j \end{pmatrix}, \quad \tilde{W}_j^T = \begin{pmatrix} \beta_{j-1}^T \cdots \beta_0^T \eta_j & \bar{\beta}_j \end{pmatrix}, \tag{3.149}$$

where $\alpha_0, \ldots, \alpha_{j-1}$ *and* $\beta_0, \ldots, \beta_{j-1}$ *are of types* $n \times (n-1), \ldots, (n-j-1) \times (n-j-2)$ *and* $m \times (m-1), \ldots, (m-j-1) \times (m-j-2)$ *respectively, and are inner, co-outer and quasi-continuous.*

Furthermore, the set of all level j *superoptimal error functions* $\mathcal{E}_j$ *satisfies*

$$\mathcal{E}_j = W_0^* W_1^* \cdots W_j^* \begin{pmatrix} t_0u_0 & 0 & \cdots & 0 & 0_{1\times(n-j-1)} \\ 0 & t_1u_1 & \ldots & 0 & 0_{1\times(n-j-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_ju_j & 0 \\ 0_{(m-j-1)\times 1} & 0_{(m-j-1)\times 1} & \ldots & \ldots & (F_{j+1} + H^\infty) \cap B(t_j) \end{pmatrix} V_j^* \cdots V_0^*, \tag{3.150}$$

for some $F_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-1)\times(n-j-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-j-1)\times(n-j-1)})$*, for the quasi-continuous unimodular functions* $u_i = \frac{\bar{z}\bar{h}_i}{h_i}$*, for all* $i = 0, \ldots, j$*, for the closed ball* $B(t_j)$ *of radius* t_j *in* $L^\infty(\mathbb{T}, \mathbb{C}^{(m-j-1)\times(n-j-1)})$*, and*

$$V_j = \begin{pmatrix} I_j & 0 \\ 0 & \tilde{V}_j \end{pmatrix}, \quad W_j = \begin{pmatrix} I_j & 0 \\ 0 & \tilde{W}_j \end{pmatrix}$$

are unitary valued functions.

Proof. Suppose we have applied steps $0, \ldots, j$ of the algorithm from Section 3.2.1 and the following diagram commutes

$$\begin{array}{ccccc} H^2(\mathbb{D}, \mathbb{C}^{n-j}) & \xrightarrow{M_{\bar{\alpha}_0 \cdots \bar{\alpha}_{j-1}}} & \mathcal{K}_j & \xrightarrow{\xi_{(j-1)} \dot{\wedge} \cdot} & \xi_{(j-1)} \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n) = X_j \\ \big\downarrow H_{F_j} & & \big\downarrow \Gamma_j & & \big\downarrow T_j \\ H^2(\mathbb{D}, \mathbb{C}^{m-j})^\perp & \xrightarrow{M_{\beta_0 \cdots \beta_{j-1}}} & \mathcal{L}_j & \xrightarrow{\bar{\eta}_{(j-1)} \dot{\wedge} \cdot} & \bar{\eta}_{(j-1)} \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp = Y_j, \end{array} \tag{3.151}$$

where the maps

$$M_{\bar{\alpha}_0 \cdots \bar{\alpha}_{j-1}} \colon H^2(\mathbb{D}, \mathbb{C}^{n-j}) \to \mathcal{K}_j \colon x \mapsto \bar{\alpha}_0 \cdots \bar{\alpha}_{j-1} x,$$

$$M_{\beta_0 \cdots \beta_{j-1}} \colon H^2(\mathbb{D}, \mathbb{C}^{m-j})^\perp \to \mathcal{L}_j \colon y \mapsto \beta_0 \cdots \beta_{j-1} y,$$

$$(\xi_{(j-1)} \dot{\wedge} \cdot) \colon \mathcal{K}_j \to X_j \ \text{ and } (\bar{\eta}_{(j-1)} \dot{\wedge} \cdot) \colon \mathcal{L}_j \to Y_j$$

are unitaries.

Let $(\xi_{(j-1)}\dot{\wedge}v_j, \bar{\eta}_{(j-1)}\dot{\wedge}w_j)$ be a Schmidt pair for the compact operator T_j. Then $x_j \in \mathcal{K}_j$, $y_j \in \mathcal{L}_j$ are such that (x_j, y_j) is a Schmidt pair for Γ_j corresponding to $t_j = \|\Gamma_j\|$, and $(\hat{x}_j, \hat{y}_j)$ is a Schmidt pair for H_{F_j} corresponding to $t_j = \|H_{F_j}\|$, where

$$\hat{x}_j = \alpha_{j-1}^T \cdots \alpha_0^T x_j, \quad \hat{y}_j = \beta_{j-1}^* \cdots \beta_0^* y_j. \tag{3.152}$$

We would like to apply Lemma 3.1.12 to H_{F_j} and the Schmidt pair $(\hat{x}_j, \hat{y}_j)$ to find unitary-valued functions $\tilde{V}_j, \tilde{W}_j$ such that, for every $\tilde{Q}_j \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j)\times(n-j)})$ which is at minimal distance from F_j, a factorisation of the form

$$F_j - \tilde{Q}_j = \tilde{W}_j^* \begin{pmatrix} t_j u_j & 0 \\ 0 & F_{j+1} \end{pmatrix} \tilde{V}_j^*$$

is obtained, for some $F_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-2)\times(n-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-2)\times(n-2)})$. For this purpose we find the inner-outer factorisations of $\hat{x}_j$ and $\bar{z}\bar{\hat{y}}_j$.

By the inductive hypothesis (see Lemma 3.2.40 for $j = 2$), we have

$$\begin{aligned} |h_j(z)| &= \|\xi_0(z) \wedge \ldots \wedge \xi_{j-1}(z) \wedge v_j(z)\|_{\wedge^{j+1}\mathbb{C}^n} = \|\bar{\eta}_0(z) \wedge \ldots \wedge \bar{\eta}_{j-1}(z) \wedge w_j(z)\|_{\wedge^{j+1}\mathbb{C}^m}, \\ \|\hat{x}_j(z)\|_{\mathbb{C}^{n-j}} &= \|\hat{y}_j(z)\|_{\mathbb{C}^{m-j}} = |h_j(z)|, \text{ and} \\ \|x_j(z)\|_{\mathbb{C}^n} &= \|y_j(z)\|_{\mathbb{C}^m} = |h_j(z)|, \end{aligned} \tag{3.153}$$

almost everywhere on $\mathbb{T}$. Equations (3.153) imply that $h_j \in H^2(\mathbb{D}, \mathbb{C})$ is the scalar outer factor of both $\hat{x}_j$ and $\bar{z}\bar{\hat{y}}_j$.

By Lemma 3.1.12, $\hat{x}_j, \bar{z}\bar{\hat{y}}_j$ admit the inner-outer factorisations

$$\hat{x}_j = \hat{\xi}_j h_j, \quad \bar{z}\bar{\hat{y}}_j = \hat{\eta}_j h_j, \tag{3.154}$$

where $\hat{\xi}_j \in H^\infty(\mathbb{D}, \mathbb{C}^{n-j})$ and $\hat{\eta}_j \in H^\infty(\mathbb{D}, \mathbb{C}^{m-j})$ are vector-valued inner functions.

By equations (3.152) and (3.154), we deduce that

$$\hat{\xi}_j = \alpha_{j-1}^T \cdots \alpha_0^T \xi_j, \quad \hat{\eta}_j = \beta_{j-1}^T \cdots \beta_0^T \eta_j.$$

We would like to show that $\alpha_{j-1}^T \cdots \alpha_0^T \xi_j$, $\beta_{j-1}^T \cdots \beta_0^T \eta_j$ are inner in order to apply Lemma 3.1.12 and obtain $\tilde{V}_j$ and $\tilde{W}_j$ as required. We have

$$\begin{aligned} \hat{x}_j &= \alpha_{j-1}^T \cdots \alpha_0^T x_j \\ &= \alpha_{j-1}^T \cdots \alpha_0^T (I_{\mathbb{C}^n} - \xi_0\xi_0^* - \cdots - \xi_{j-1}\xi_{j-1}^*) v_j \\ &= \alpha_{j-1}^T \cdots \alpha_0^T v_j - \alpha_{j-1}^T \cdots \alpha_0^T \xi_0\xi_0^* v_j - \cdots - \alpha_{j-1}^T \cdots \alpha_0^T \xi_{j-1}\xi_{j-1}^* v_j. \end{aligned}$$

Recall that, by the inductive hypothesis, for $i = 0, \ldots, j-1$, each

$$\tilde{V}_i = \begin{pmatrix} \alpha_{i-1}^T \cdots \alpha_0^T \xi_i & \bar{\alpha}_i \end{pmatrix}$$

is unitary-valued, and so $\alpha_i^T \alpha_{i-1}^T \cdots \alpha_0^T \xi_i = 0$. Hence, if $0 \le i \le j-1$, we have

$$\alpha_{j-1}^T \cdots \alpha_{i+1}^T \alpha_i^T \cdots \alpha_0^T \xi_i = 0.$$

Thus

$$\hat{x}_j = \alpha_{j-1}^T \cdots \alpha_0^T x_j = \alpha_{j-1}^T \cdots \alpha_0^T v_j,$$

that is, $\hat{x}_j \in H^2(\mathbb{D}, \mathbb{C}^{n-j})$ and

$$\alpha_{j-1}^T \cdots \alpha_0^T \xi_j = \frac{1}{h_j} \alpha_{j-1}^T \cdots \alpha_0^T x_j = \frac{1}{h_j} \alpha_{j-1}^T \cdots \alpha_0^T v_j$$

is analytic. Moreover, by equations (3.153),

$$\|\alpha_{j-1}^T(z) \cdots \alpha_0^T(z) x_j(z)\|_{\mathbb{C}^{n-j}} = \|\alpha_{j-1}^T(z) \cdots \alpha_0^T(z) v_j(z)\|_{\mathbb{C}^{n-j}} = |h_j(z)|$$

almost everywhere on $\mathbb{T}$, and hence

$$\|\alpha_{j-1}^T(z) \cdots \alpha_0^T(z) \xi_j(z)\|_{\mathbb{C}^{n-j}} = 1$$

almost everywhere on $\mathbb{T}$. Therefore $\alpha_{j-1}^T \cdots \alpha_0^T \xi_j$ is inner.

Furthermore

$$\begin{aligned} \hat{y}_j &= \beta_{j-1}^* \cdots \beta_0^* y_j \\ &= \beta_{j-1}^* \cdots \beta_0^* (I_{\mathbb{C}^m} - \bar{\eta}_0 \eta_0^T - \cdots - \bar{\eta}_{j-1} \eta_{j-1}^T) w_j \\ &= \beta_{j-1}^* \cdots \beta_0^* w_j - \beta_{j-1}^* \cdots \beta_0^* \bar{\eta}_0 \eta_0^T w_j - \cdots - \beta_{j-1}^* \cdots \beta_0^* \bar{\eta}_{j-1} \eta_{j-1}^T w_j. \end{aligned}$$

Notice that, by the inductive hypothesis, for $i = 0, \dots, j-1$, each

$$\tilde{W}_i^T = \begin{pmatrix} \beta_{i-1}^T \cdots \beta_0^T \eta_i & \bar{\beta}_i \end{pmatrix}$$

is unitary-valued, and so $\beta_i^* \cdots \beta_0^* \bar{\eta}_i = 0$. Hence, if $0 \le i \le j-1$, we have

$$\beta_{j-1}^* \cdots \beta_{i+1}^* \beta_i^* \cdots \beta_0^* \bar{\eta}_i = 0.$$

Thus

$$\hat{y}_j = \beta_{j-1}^* \cdots \beta_0^* y_j = \beta_{j-1}^* \cdots \beta_0^* w_j,$$

that is, $\hat{y}_j \in H^2(\mathbb{D}, \mathbb{C}^{m-j})^\perp$ and

$$\beta_{j-1}^T \cdots \beta_0^T \eta_j = \frac{1}{h_j} \beta_{j-1}^T \cdots \beta_0^T \bar{z} \bar{y}_j = \frac{1}{h_j} \beta_{j-1}^T \cdots \beta_0^T \bar{z} \bar{w}_j$$

is analytic. Further, by equations (3.153),

$$\|\beta_{j-1}^T(z)\cdots\beta_0^T(z)\bar{z}\bar{y}_j(z)\|_{\mathbb{C}^{m-j}} = \|\beta_{j-1}^T(z)\cdots\beta_0^T(z)\bar{z}\bar{w}_j(z)\|_{\mathbb{C}^{m-j}} = |h_j(z)|$$

almost everywhere on $\mathbb{T}$, and therefore

$$\|\beta_{j-1}^T(z)\cdots\beta_0^T(z)\eta_j(z)\|_{\mathbb{C}^m} = 1$$

almost everywhere on $\mathbb{T}$, that is, $\beta_{j-1}^T\cdots\beta_0^T\eta_j$ is inner.

We apply Lemma 3.1.12 to the Hankel operator H_{F_j} and the Schmidt pair $(\hat{x}_j,\hat{y}_j)$ to deduce that there exist inner, co-outer, quasi-continuous functions α_j,β_j of types $(n-j)\times(n-j-1)$, $(m-j)\times(m-j-1)$ respectively such that

$$\tilde{V}_j = \begin{pmatrix}\alpha_{j-1}^T\cdots\alpha_0^T\xi_j & \bar{\alpha}_j\end{pmatrix}, \quad \tilde{W}_j^T = \begin{pmatrix}\beta_{j-1}^T\cdots\beta_0^T\eta_j & \bar{\beta}_j\end{pmatrix}$$

are unitary-valued and all minors on the first columns of $\tilde{V}_j,\tilde{W}_j$ are in H^∞. Moreover, every function $\hat{Q}_j\in H^\infty(\mathbb{D},\mathbb{C}^{(m-j)\times(n-j)})$, which is at minimal distance from F_j, satisfies

$$F_j-\hat{Q}_j = \tilde{W}_j^*\begin{pmatrix}t_ju_j & 0\\ 0 & F_{j+1}\end{pmatrix}\tilde{V}_j^*,$$

for some $F_{j+1}\in H^\infty(\mathbb{D},\mathbb{C}^{(m-j-1)\times(n-j-1)})+C(\mathbb{T},\mathbb{C}^{(m-j-1)\times(n-j-1)})$ and for the quasi-continuous unimodular function $u_j=\frac{\bar{z}\bar{h}_j}{h_j}$.

By Lemma 3.1.15, the set

$$\tilde{\mathcal{E}}_j = \{F_j-\hat{Q} : \hat{Q}\in H^\infty(\mathbb{D},\mathbb{C}^{(m-j)\times(n-j)}), \|F_j-\hat{Q}\|_{L^\infty}=t_j\}$$

satisfies

$$\tilde{\mathcal{E}}_j = \tilde{W}_j^*\begin{pmatrix}t_ju_j & 0\\ 0 & (F_{j+1}+H^\infty)\cap B(t_j)\end{pmatrix}\tilde{V}_j^*,$$

where $B(t_j)$ is the closed ball of radius t_j in $L^\infty(\mathbb{T},\mathbb{C}^{(m-j-1)\times(n-j-1)})$.

By the inductive hypothesis, the set of all level j superoptimal error functions $\mathcal{E}_j$ satisfies

$$\mathcal{E}_{j-1} = W_0^*W_1^*\cdots W_{j-1}^*\begin{pmatrix} t_0u_0 & 0 & \cdots & 0 & 0_{1\times(n-j)}\\ 0 & t_1u_1 & \dots & 0 & 0_{1\times(n-j)}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & t_{j-1}u_{j-1} & 0\\ 0_{(m-j)\times 1} & 0_{(m-j)\times 1} & \dots & \dots & (F_j+H^\infty)\cap B(t_{j-1})\end{pmatrix} V_{j-1}^*\cdots V_0^*, \tag{3.155}$$

for some $F_j\in H^\infty(\mathbb{D},\mathbb{C}^{(m-j)\times(n-j)})+C(\mathbb{T},\mathbb{C}^{(m-j)\times(n-j)})$, $u_i=\frac{\bar{z}\bar{h}_i}{h_i}$ are quasi-continuous unimodular functions for all $i=0,\dots,j-1$, and for the closed ball $B(t_{j-1})$ of radius t_{j-1} in $L^\infty(\mathbb{T},\mathbb{C}^{(m-j)\times(n-j)})$.

Thus, by equation (3.155), $\mathcal{E}_j$ admits the factorisation (3.150) as claimed. □

Remark 3.2.44. *Let, for $i = 0, 1, \ldots, j$,*

$$\tilde{V}_i = \begin{pmatrix} \alpha_{i-1}^T \cdots \alpha_0^T \xi_i & \bar{\alpha}_i \end{pmatrix}, \quad \tilde{W}_i^T = \begin{pmatrix} \beta_{i-1}^T \cdots \beta_0^T \eta_i & \bar{\beta}_i \end{pmatrix} \tag{3.156}$$

be unitary-valued functions, as described in Lemma 3.1.12. Let

$$V_j = \begin{pmatrix} I_j & 0 \\ 0 & \tilde{V}_j \end{pmatrix}, \quad W_j = \begin{pmatrix} I_j & 0 \\ 0 & \tilde{W}_j \end{pmatrix}.$$

Let $A_j = \alpha_0 \alpha_1 \ldots \alpha_j$, $A_{-1} = I_n$, $B_j = \beta_0 \beta_1 \ldots \beta_j$ and $B_{-1} = I_m$.

Note

$$W_1 W_0 = \begin{pmatrix} 1 & 0 \\ 0 & \eta_1^T \beta_0 \\ 0 & \beta_1^* \end{pmatrix} \begin{pmatrix} \eta_0^T \\ \beta_0^* \end{pmatrix} = \begin{pmatrix} \eta_0^T \\ \eta_1^T B_0 B_0^* \\ B_1^* \end{pmatrix}$$

and

$$W_2 W_1 W_0 = \begin{pmatrix} I_2 & 0 \\ 0 & \eta_2^T B_1 \\ 0 & \beta_2^* \end{pmatrix} \begin{pmatrix} \eta_0^T \\ \eta_1^T B_0 B_0^* \\ B_1^* \end{pmatrix} = \begin{pmatrix} \eta_0^T \\ \eta_1^T B_0 B_0^* \\ \eta_2^T B_1 B_1^* \\ B_2^* \end{pmatrix}.$$

Similarly one obtains

$$W_j W_{j-1} \cdots W_0 = \begin{pmatrix} \eta_0^T \\ \eta_1^T B_0 B_0^* \\ \vdots \\ \eta_j^T B_{j-1} B_{j-1}^* \\ B_j^* \end{pmatrix}. \tag{3.157}$$

Therefore

$$W_0^* W_1^* \cdots W_j^* = \begin{pmatrix} \bar{\eta}_0 & B_0 B_0^* \bar{\eta}_1 & \ldots & B_{j-1} B_{j-1}^* \bar{\eta}_j & B_j \end{pmatrix}.$$

Thus

$$I_m = W_0^* W_1^* \cdots W_j^* W_j \cdots W_1 W_0 = \sum_{i=0}^{j} B_{i-1} B_{i-1}^* \bar{\eta}_i \eta_i^T B_{i-1} B_{i-1}^* + B_j B_j^*. \tag{3.158}$$

Furthermore

$$V_0 V_1 = \begin{pmatrix} \xi_0 & \bar{\alpha}_0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha_0^T \xi_1 & \bar{\alpha}_1 \end{pmatrix} = \begin{pmatrix} \xi_0 & A_0^T \xi_1 & \bar{A}_1 \end{pmatrix}$$

and

$$V_0 V_1 V_2 = \begin{pmatrix} \xi_0 & \bar{A}_0 A_0^T \xi_1 & \bar{A}_1 \end{pmatrix} \begin{pmatrix} I_2 & 0 & 0 \\ 0 & A_1^T \xi_2 & \bar{\alpha}_2 \end{pmatrix} = \begin{pmatrix} \xi_0 & \bar{A}_0 A_0^T \xi_1 & \bar{A}_1 A_1^T \xi_2 & \bar{A}_2 \end{pmatrix}.$$

One can show that

$$V_0 \cdots V_j = \begin{pmatrix} \xi_0 & \bar{A}_0 A_0^T \xi_1 & \bar{A}_1 A_1^T \xi_2 & \ldots & \bar{A}_{j-1} A_{j-1}^T \xi_j & \bar{A}_j \end{pmatrix}. \tag{3.159}$$

Therefore,

$$I_n = V_0 \cdots V_j V_j^* \cdots V_0^* = \xi_0 \xi_0^* + \bar{A}_0 A_0^T \xi_1 \xi_1^* \bar{A}_0 A_0^T + \ldots \bar{A}_{j-1} A_{j-1}^T \xi_j \xi_j^* \bar{A}_{j-1} A_{j-1}^T + \bar{A}_j A_j^T. \tag{3.160}$$

Lemma 3.2.45. *Let*

$$\tilde{V}_i = \begin{pmatrix} \alpha_{i-1}^T \cdots \alpha_0^T \xi_i & \bar{\alpha}_i \end{pmatrix} \tag{3.161}$$

be unitary-valued functions, for $i = 0, 1, \ldots, j$, as described in Lemma 3.1.12 Let, for $i = 0, 1, \ldots, j$, $A_i = \alpha_0 \alpha_1 \ldots \alpha_i$ and $A_{-1} = I_n$. Then, for $i = 0, 1, \ldots, j$,

$$\bar{A}_i A_i^T = I_n - \sum_{k=0}^{i} \xi_k \xi_k^* \tag{3.162}$$

almost everywhere on $\mathbb{T}$.

Proof. By equation (3.160), for $k = 0, \ldots, j$,

$$\bar{A}_k A_k^T = I_n - \sum_{i=0}^{k} \bar{A}_{i-1} A_{i-1}^T \xi_i \xi_i^* \bar{A}_{i-1} A_{i-1}^T. \tag{3.163}$$

Thus to prove condition (3.162) it suffices to show that, for $k = 0, \ldots, j$,

$$\bar{A}_{k-1} A_{k-1}^T \xi_k \xi_k^* \bar{A}_{k-1} A_{k-1}^T = \xi_k \xi_k^*.$$

For $k = 0$,

$$\bar{A}_{-1} A_{-1}^T \xi_0 \xi_0^* \bar{A}_{-1} A_{-1}^T = \xi_0 \xi_0^*,$$

and so, equation (3.163) yields

$$\bar{A}_0 A_0^T = I_n - \xi_0 \xi_0^*.$$

For $k = 1$,

$$\bar{A}_0 A_0^T \xi_1 \xi_1^* \bar{A}_0 A_0^T = (I_n - \xi_0 \xi_0^*) \xi_1 \xi_1^* (I_n - \xi_0 \xi_0^*)$$

By Proposition 3.2.1, ξ_1 and ξ_0 are pointwise orthogonal almost everywhere on $\mathbb{T}$, hence

$$\bar{A}_0 A_0^T \xi_1 \xi_1^* \bar{A}_0 A_0^T = \xi_1 \xi_1^*,$$

and in view of equation (3.163), we get

$$\bar{A}_1 A_1^T = I_n - \xi_0 \xi_0^* - \xi_1 \xi_1^*.$$

Suppose

$$\bar{A}_{\ell-1}A_{\ell-1}^T\xi_\ell\xi_\ell^*\bar{A}_{\ell-1}A_{\ell-1}^T = \xi_\ell\xi_\ell^* \tag{3.164}$$

holds for every $\ell \leq k$, where $0 \leq k \leq j$. By equations (3.163) and (3.164), this implies

$$\bar{A}_k A_k^T = I_n - \sum_{i=0}^{k} \xi_i\xi_i^*.$$

Let us show that

$$\bar{A}_k A_k^T \xi_{k+1}\xi_{k+1}^* \bar{A}_k A_k^T = \xi_{k+1}\xi_{k+1}^*.$$

Note that

$$\bar{A}_k A_k^T \xi_{k+1}\xi_{k+1}^* \bar{A}_k A_k^T = (I_n - \sum_{i=0}^{k} \xi_i\xi_i^*)\xi_{k+1}\xi_{k+1}^*(I_n - \sum_{i=0}^{k} \xi_i\xi_i^*).$$

By Proposition 3.2.1, the set $\{\xi_i(z)\}_{i=0}^{k+1}$ is pointwise orthogonal almost everywhere on $\mathbb{T}$, and therefore

$$\bar{A}_k A_k^T \xi_{k+1}\xi_{k+1}^* \bar{A}_k A_k^T = \xi_{k+1}\xi_{k+1}^*.$$

Thus, by equation (3.163),

$$\bar{A}_{k+1} A_{k+1}^T = I_n - \sum_{i=0}^{k+1} \xi_i\xi_i^*,$$

and the assertion has been proved. □

Lemma 3.2.46. *Let*

$$\tilde{W}_i^T = \begin{pmatrix} \beta_{i-1}^T \cdots \beta_0^T \eta_i & \bar{\beta}_i \end{pmatrix} \tag{3.165}$$

be unitary-valued functions, for $i = 0, 1, \dots, j$, as described in Lemma 3.1.12. Let, for $i = 0, 1, \dots, j$, $B_i = \beta_0\beta_1 \dots \beta_i$ and $B_{-1} = I_m$. Then, for $k = 0, 1, \dots, j$,

$$B_k B_k^* = I_m - \sum_{i=0}^{k} \bar{\eta}_i \eta_i^T \tag{3.166}$$

almost everywhere on $\mathbb{T}$.

Proof. By equation (3.158), for $k = 0, \dots, j$,

$$B_k B_k^* = I_m - \sum_{i=0}^{k} B_{i-1}B_{i-1}^* \bar{\eta}_i \eta_i^T B_{i-1}B_{i-1}^*. \tag{3.167}$$

Thus to prove condition (3.166) it suffices to show that, for $k = 0, \dots, j$,

$$B_{k-1}B_{k-1}^* \bar{\eta}_k \eta_k^T B_{k-1}B_{k-1}^* = \bar{\eta}_k \eta_k^T.$$

For $k = 0$,

$$B_{-1}B_{-1}^* \bar{\eta}_0 \eta_0^T B_{-1}B_{-1}^* = I_m \bar{\eta}_0 \eta_0^T I_m = \bar{\eta}_0 \eta_0^T,$$

and so, equation (3.167) yields

$$B_0 B_0^* = I_m - \bar{\eta}_0 \eta_0^T.$$

For $k = 1$,

$$B_0 B_0^* \bar{\eta}_1 \eta_1^T B_0 B_0^* = (I_m - \bar{\eta}_0 \eta_0^T) \bar{\eta}_1 \eta_1^T (I_m - \bar{\eta}_0 \eta_0^T).$$

By Proposition 3.2.1, η_1 and η_0 are pointwise orthogonal almost everywhere on $\mathbb{T}$, hence

$$B_0 B_0^* \bar{\eta}_1 \eta_1^T B_0 B_0^* = \bar{\eta}_1 \eta_1^T,$$

and in view of equation (3.167), we get

$$B_1 B_1^* = I_m - \bar{\eta}_0 \eta_0^T - \bar{\eta}_1 \eta_1^T.$$

Suppose

$$B_{\ell-1} B_{\ell-1}^* \bar{\eta}_\ell \eta_\ell^T B_{\ell-1} B_{\ell-1}^* = \bar{\eta}_\ell \eta_\ell^T \tag{3.168}$$

holds for every $\ell \leq k$, where $0 \leq k \leq j$. By equations (3.167) and (3.168), this implies

$$B_k B_k^* = I_m - \sum_{i=0}^{k} \bar{\eta}_i \eta_i^T.$$

Let us show that

$$B_k B_k^* \bar{\eta}_{k+1} \eta_{k+1}^T B_k B_k^* = \bar{\eta}_{k+1} \eta_{k+1}^T.$$

Note that

$$B_k B_k^* \bar{\eta}_{k+1} \eta_{k+1}^T B_k B_k^* = (I_m - \sum_{i=0}^{k} \bar{\eta}_i \eta_i^T) \bar{\eta}_{k+1} \eta_{k+1}^T (I_m - \sum_{i=0}^{k} \bar{\eta}_i \eta_i^T).$$

By Proposition 3.2.1, the set $\{\bar{\eta}_i(z)\}_{i=0}^{k+1}$ is pointwise orthogonal almost everywhere on $\mathbb{T}$, and therefore

$$B_k B_k^* \bar{\eta}_{k+1} \eta_{k+1}^T B_k B_k^* = \bar{\eta}_{k+1} \eta_{k+1}^T.$$

Thus, by equation (3.167),

$$B_{k+1} B_{k+1}^* = I_m - \sum_{i=0}^{k+1} \bar{\eta}_i \eta_i^T$$

and the assertion has been proved. □

The following statement asserts that any function $Q_{j+1} \in \Omega_j$ necessarily satisfies equations (3.22).

Proposition 3.2.47. *Every $Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m \times n})$ which minimises*

$$(s_0^\infty(G - Q), s_1^\infty(G - Q), \dots, s_j^\infty(G - Q))$$

satisfies

$$(G - Q_{j+1}) x_i = t_i y_i, (G - Q_{j+1})^* y_i = t_i x_i, \quad \textit{for } i = 0, 1, \dots, j.$$

Proof. By the recursive step of the algorithm from Section 3.2.1, every $Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ that minimises

$$(s_0^\infty(G-Q), \dots, s_{j-1}^\infty(G-Q))$$

satisfies

$$(G-Q_{j+1})x_i = t_i y_i, \quad (G-Q_{j+1})^* y_i = t_i x_i \quad \text{for} \quad i = 0, 1, \dots, j-1.$$

Hence it suffices to show that Q_{j+1} satisfies

$$(G-Q_{j+1})x_j = t_j y_j, (G-Q_{j+1})^* y_j = t_j x_j.$$

Notice that, by the inductive step, the following diagram commutes

$$\begin{array}{ccccc} H^2(\mathbb{D}, \mathbb{C}^{n-j}) & \xrightarrow{M_{\bar{\alpha}_0 \cdots \bar{\alpha}_{j-1}}} & \mathcal{K}_j & \xrightarrow{\xi_{(j-1)}\dot{\wedge}\cdot} & \xi_{(j-1)}\dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n) = X_j \\ \big\downarrow H_{F_j} & & \big\downarrow \Gamma_j & & \big\downarrow T_j \\ H^2(\mathbb{D}, \mathbb{C}^{m-j})^\perp & \xrightarrow{M_{\beta_0 \cdots \beta_{j-1}}} & \mathcal{L}_j & \xrightarrow{\bar{\eta}_{(j-1)}\dot{\wedge}\cdot} & \bar{\eta}_{(j-1)}\dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp = Y_j, \end{array} \tag{3.169}$$

where the maps $M_{\bar{\alpha}_0 \cdots \bar{\alpha}_{j-1}}$, $M_{\beta_0 \cdots \beta_{j-1}}$, $(\xi_{(j-1)}\dot{\wedge}\cdot)\colon \mathcal{K}_j \to X_j$ and $(\bar{\eta}_{(j-1)}\dot{\wedge}\cdot)\colon \mathcal{L}_j \to Y_j$ are unitaries, and $F_j \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j)\times(n-j)}) + C(\mathbb{T}, \mathbb{C}^{(m-j)\times(n-j)})$.

By equation (3.155), the set of all level $j-1$ superoptimal error functions

$$\mathcal{E}_{j-1} = \{G - Q : Q \in \Omega_{j-1}\}$$

satisfies

$$\mathcal{E}_{j-1} = W_0^* W_1^* \cdots W_{j-1}^* \begin{pmatrix} t_0 u_0 & 0 & \cdots & 0 & 0_{1\times(n-j)} \\ 0 & t_1 u_1 & \dots & 0 & 0_{1\times(n-j)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_{j-1}u_{j-1} & 0 \\ 0_{(m-j)\times 1} & 0_{(m-j)\times 1} & \dots & \dots & (F_j + H^\infty) \cap B(t_{j-1}) \end{pmatrix} V_{j-1}^* \cdots V_0^*, \tag{3.170}$$

for some $F_j \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j)\times(n-j)}) + C(\mathbb{T}, \mathbb{C}^{(m-j)\times(n-j)})$, where $u_i = \frac{\bar{z}\bar{h}_i}{h_i}$ are quasi-continuous unimodular functions for $i = 0, \dots, j-1$, and $B(t_{j-1})$ is the closed ball of radius t_{j-1} in $L^\infty(\mathbb{T}, \mathbb{C}^{(m-j)\times(n-j)})$. Consider some $Q_{j+1} \in \Omega_{j-1}$, so that, according to equation (3.170),

$$\begin{pmatrix} I_{j-1} & 0 \\ 0 & \tilde{W}_{j-1} \end{pmatrix} \cdots W_0 (G - Q_{j+1}) V_0 \cdots \begin{pmatrix} I_{n-j-1} & 0 \\ 0 & \tilde{V}_{j-1} \end{pmatrix} = \begin{pmatrix} t_0 u_0 & 0 & \dots & 0 \\ 0 & t_1 u_1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & t_{j-1}u_{j-1} & 0 \\ 0 & \dots & \dots & F_j - \tilde{Q}_j \end{pmatrix}, \tag{3.171}$$

where $\tilde{Q}_j \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j)\times(n-j)})$ is at minimal distance from F_j. Let $B_j = \beta_0 \cdots \beta_j$ and let $A_j = \alpha_0 \cdots \alpha_j$. By equations (3.148), we have

$$\begin{pmatrix} I_{j-1} & 0 \\ 0 & \tilde{W}_{j-1} \end{pmatrix} \cdots W_0(G-Q_{j+1})V_0 \cdots \begin{pmatrix} I_{n-j-1} & 0 \\ 0 & \tilde{V}_{j-1} \end{pmatrix}$$
$$= \begin{pmatrix} t_0u_0 & 0 & \dots & 0 \\ \vdots & \ddots & \dots & \vdots \\ 0 & \dots & \eta_{j-1}^T B_{j-2}B_{j-2}^*(G-Q_{j+1})\bar{A}_{j-2}A_{j-2}^T\xi_{j-1} & \eta_{j-1}^T B_{j-2}B_{j-2}^*(G-Q_{j+1})\bar{A}_{j-1} \\ 0 & \dots & B_{j-1}^*(G-Q_{j+1})\bar{A}_{j-2}A_{j-2}^T\xi_{j-1} & B_{j-1}^*(G-Q_{j+1})\bar{A}_{j-1} \end{pmatrix},$$

which, combined with equation (3.171), yields

$$B_{j-1}^*(G-Q_{j+1})\bar{A}_{j-1} = F_j - \tilde{Q}_j. \tag{3.172}$$

Since $\tilde{Q}_j$ is at minimal distance from F_j,

$$\|F_j - \tilde{Q}_j\|_\infty = \|H_{F_j}\| = t_j.$$

Note that, if $(\hat{x}_j, \hat{y}_j)$ is a Schmidt pair for H_{F_j} corresponding to t_j, then, by Theorem D.2.4,

$$(F_j - \tilde{Q}_j)\hat{x}_j = t_j\hat{y}_j, \quad (F_j - \tilde{Q}_j)^*\hat{y}_j = t_j\hat{x}_j.$$

In view of equation (3.172), the latter equations imply

$$B_{j-1}^*(G-Q_{j+1})\bar{A}_{j-1}\hat{x}_j = t_j\hat{y}_j, \quad A_{j-1}^T(G-Q_{j+1})^*B_{j-1}\hat{y}_j = t_j\hat{x}_j.$$

By equation (3.152),

$$\hat{x}_j = A_{j-1}^T x_j, \quad \hat{y}_j = B_{j-1}^* y_j$$

Thus

$$B_{j-1}^*(G-Q_{j+1})\bar{A}_{j-1}\hat{x}_j = B_{j-1}^*(G-Q_{j+1})x_j = t_jB_{j-1}^*y_j,$$

or equivalently,

$$B_{j-1}^*\big((G-Q_{j+1})x_j - t_jy_j\big) = 0,$$

and since, by the inductive hypothesis, $M_{B_{j-1}}$ is a unitary map, we have

$$(G-Q_{j+1})x_j = t_jy_j.$$

Furthermore

$$A_{j-1}^T(G-Q_{j+1})^*B_{j-1}\hat{y}_j = A_{j-1}^T(G-Q_{j+1})^*y_j = t_jA_{j-1}^Tx_j,$$

or equivalently,

$$A_{j-1}^T\big((G-Q_{j+1})^*y_j - t_jx_j\big) = 0.$$

By the inductive hypothesis, $M_{\bar{A}_{j-1}}$ is a unitary map, hence

$$(G - Q_{j+1})^* y_j = t_j x_j,$$

and therefore Q_{j+1} satisfies the required equations. □

Lemma 3.2.48. *Let*

$$\tilde{V}_i = \begin{pmatrix} \alpha_{i-1}^T \cdots \alpha_0^T \xi_i & \bar{\alpha}_i \end{pmatrix}, \quad \tilde{W}_i^T = \begin{pmatrix} \beta_{i-1}^T \cdots \beta_0^T \eta_i & \bar{\beta}_i \end{pmatrix}, \quad i = 0, 1, \ldots, j, \tag{3.173}$$

be unitary-valued functions, as described in Lemma 3.1.12. Then

$$\alpha_l^T H^2(\mathbb{D}, \mathbb{C}^{n-l}) = H^2(\mathbb{D}, \mathbb{C}^{n-l-1})$$

and

$$\beta_l^* (H^2(\mathbb{D}, \mathbb{C}^{m-l})^\perp = H^2(\mathbb{D}, \mathbb{C}^{m-l-1})^\perp,$$

for all $l = 0, \ldots, j$.

Proof. Recall that, by Lemma 3.1.18, for all $l = 0, \ldots, j$, the inner, co-outer, quasi-continuous functions α_l, β_l of types $(n-l) \times (n-l-1)$ and $(m-l) \times (m-l-1)$ respectively, are left invertible. The rest of the proof is similar to Lemmas 3.2.15 and 3.2.20. □

As a preparation for proof of the main inductive step we prove several propositions.

Lemma 3.2.49. *Let* $\tilde{V}_i$ *be unitary-valued functions as given in equations (3.149), for* $i = 0, 1, \ldots, j$. *Let* $A_i = \alpha_0 \cdots \alpha_i$, *for* $i = 0, 1, \ldots, j$ *and* $A_{-1} = I_n$. *Let*

$$V_i = \begin{pmatrix} I_i & 0 \\ 0 & \tilde{V}_i \end{pmatrix}, \quad \textit{for } i = 0, 1, \ldots, j$$

and let

$$\mathcal{K}_{j+1} = V_0 \cdots V_j \begin{pmatrix} 0_{(j+1)\times 1} \\ H^2(\mathbb{D}, \mathbb{C}^{n-j-1}) \end{pmatrix}. \tag{3.174}$$

Let $\xi_{(j)} = \xi_0 \dot{\wedge} \ldots \dot{\wedge} \xi_j$. *Then, for all* $0 \le j \le \min(m,n) - 2$ *and every* $f \in L^2(\mathbb{T}, \mathbb{C}^n)$,

$$\xi_{(j)} \dot{\wedge} \bar{A}_{j-1} A_{j-1}^T \xi_j \xi_j^* f = 0.$$

Proof. For $j = 0, 1, 2$, by Propositions 3.2.17 and 3.2.30, the assertion has been proved. Suppose the assertion holds for all $j = 0, \ldots, \ell$. Then the entities constructed by the recursion step of the algorithm from Section 3.2.1 satisfy

$$\xi_{(\ell-1)} \dot{\wedge} \bar{A}_{\ell-2} A_{\ell-2}^T \xi_{\ell-1} \xi_{\ell-1}^* f = 0 \quad \text{for all } \ell,$$

where

$$A_{\ell-2} = \alpha_0 \cdots \alpha_{\ell-2}$$

and $\alpha_0, \dots, \alpha_{\ell-2}$ are inner, co-outer, quasi-continuous functions of types $n \times (n-1), \dots, (n-\ell+1) \times (n-\ell+2)$ respectively.

We will show the assertion holds for $j = 0, \dots, \ell+1$. Since T_ℓ is a compact operator, there exist functions $v_\ell \in H^2(\mathbb{D}, \mathbb{C}^n), w_\ell \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ such that $(\xi_{(\ell-1)} \dot{\wedge} v_\ell, \bar{\eta}_{(\ell-1)} \dot{\wedge} w_\ell)$ is a Schmidt pair for operator T_ℓ corresponding to $t_\ell = \|T_\ell\|$. By Proposition 3.2.2, $\xi_{(\ell-1)} \dot{\wedge} v_\ell$ is an element of $H^2(\mathbb{D}, \wedge^{\ell+1}\mathbb{C}^n)$. Let $h_\ell \in H^2(\mathbb{D}, \mathbb{C})$ be the scalar outer factor of $\xi_{(\ell-1)} \dot{\wedge} v_\ell$. Define

$$x_\ell = (I_{\mathbb{C}^n} - \xi_0 \xi_0^* \cdots - \xi_{\ell-1} \xi_{\ell-1}^*) v_\ell$$

and

$$\xi_\ell = \frac{x_\ell}{h_\ell}.$$

Then

$$\begin{aligned} \alpha_{\ell-1}^T \cdots \alpha_0^T x_\ell &= A_{\ell-1}^T (I_{\mathbb{C}^n} - \xi_0 \xi_0^* - \cdots - \xi_{\ell-1} \xi_{\ell-1}^*) v_\ell \\ &= A_{\ell-1}^T \cdots v_\ell - A_{\ell-1}^T \xi_0 \xi_0^* v_\ell - \cdots - A_{\ell-1}^T \xi_{\ell-1} \xi_{\ell-1}^* v_\ell. \end{aligned}$$

Recall that, by the inductive hypothesis, for $j = 0, \dots, \ell-1$, each V_i is unitary-valued, hence $\alpha_{i-1}^T \cdots \alpha_0^T \xi_i = 0$. Thus

$$\alpha_{\ell-1}^T \cdots \alpha_0^T x_\ell = \alpha_{\ell-1}^T \cdots \alpha_0^T v_\ell,$$

that is,

$$\alpha_{\ell-1}^T \cdots \alpha_0^T \xi_\ell = \frac{1}{h_\ell} \alpha_{\ell-1}^T \cdots \alpha_0^T x_\ell$$

is analytic. Moreover, since $\alpha_j(z)$ are isometries for all $j = 0, \cdots, \ell-1$,

$$\|\alpha_{\ell-1}^T(z) \cdots \alpha_0^T(z)(z) x_\ell(z)\|_{\mathbb{C}^{n-\ell}} = \|\alpha_{\ell-1}^T(z) \cdots \alpha_0^T(z)(z) v_\ell(z)\|_{\mathbb{C}^{n-\ell}}\| = |h_\ell(z)|$$

almost everywhere on $\mathbb{T}$, and hence

$$\|\alpha_{\ell-1}^T(z) \cdots \alpha_0^T(z) \xi_\ell(z)\|_{\mathbb{C}^{n-\ell}} = 1$$

almost everywhere on $\mathbb{T}$. Therefore $\alpha_{\ell-1}^T \cdots \alpha_0^T \xi_\ell$ is inner.

Then, by Theorem 3.1.10, there exists an inner, co-outer, quasi-continuous α_ℓ of size $(n-\ell) \times (n-\ell+1)$ such that

$$\tilde{V}_\ell = \begin{pmatrix} A_{\ell-1}^T \xi_\ell & \bar{\alpha}_\ell \end{pmatrix}$$

is unitary-valued and all minors on the first column of $\tilde{V}_\ell$ are in H^∞. Let $V_\ell = \begin{pmatrix} I_\ell & 0 \\ 0 & \tilde{V}_\ell \end{pmatrix}$.

Recall that

$$V_0 \cdots V_{\ell-1} V_{\ell-1}^* \cdots V_0 = I_n,$$

which is equivalent to the equation

$$\sum_{j=0}^{\ell-1} \bar{A}_{j-1} A_{j-1}^T \xi_j \xi_j^* \bar{A}_{j-1} A_{j-1}^T + \bar{A}_{\ell-1} A_{\ell-1}^T = I_n. \tag{3.175}$$

We have that, for every $f \in L^2(\mathbb{T}, \mathbb{C}^n)$,

$$\begin{aligned}
\xi_{(\ell)} \dot\wedge \bar{A}_{\ell-1} A_{\ell-1}^T \xi_\ell \xi_\ell^* f &= \xi_{(\ell)} \dot\wedge (I_n - \sum_{j=0}^{\ell-1} \bar{A}_{j-1} A_{j-1}^T \xi_j \xi_j^* \bar{A}_{j-1} A_{j-1}^T) \xi_\ell \xi_\ell^* f \\
&= \xi_{(\ell)} \dot\wedge \xi_\ell \xi_\ell^* f - \xi_{(\ell)} \dot\wedge \sum_{j=0}^{\ell-1} \bar{A}_{j-1} A_{j-1}^T \xi_j \xi_j^* \bar{A}_{j-1} A_{j-1}^T \xi_\ell \xi_\ell^* f \\
&= 0 - \xi_{(\ell)} \dot\wedge \sum_{j=0}^{\ell-1} \bar{A}_{j-1} A_{j-1}^T \xi_j \xi_j^* \bar{A}_{j-1} A_{j-1}^T \xi_\ell \xi_\ell^* f
\end{aligned}$$

because of pointwise linear dependence. We set $\bar{A}_{j-1} A_{j-1}^T \xi_\ell \xi_\ell^* f = g \in L^2(\mathbb{T}, \mathbb{C}^n)$. By the inductive hypothesis, we get that, for every $g \in L^2(\mathbb{T}, \mathbb{C}^n)$,

$$\xi_{(\ell-1)} \dot\wedge \xi_\ell \dot\wedge \sum_{j=0}^{\ell-1} \bar{A}_{j-1} A_{\ell-1}^T \xi_j \xi_j^* g = 0.$$

Thus we have proved that for every $0 \le j \le \min(m, n) - 2$ and every $f \in L^2(\mathbb{T}, \mathbb{C}^n)$,

$$\xi_{(j)} \dot\wedge \bar{A}_{j-1} A_{j-1}^T \xi_j \xi_j^* f = 0. \qquad \square$$

Proposition 3.2.50. *Let $\tilde{V}_i$ be unitary-valued functions as given in equations (3.149), for $i = 0, 1, \dots, j$. Let $A_i = \alpha_0 \cdots \alpha_i$, for $i = 0, 1, \dots, j$ and $A_{-1} = I_n$. Let*

$$V_i = \begin{pmatrix} I_i & 0 \\ 0 & \tilde{V}_i \end{pmatrix}, \quad \text{for } i = 0, 1, \dots, j$$

and let

$$\mathcal{K}_{j+1} = V_0 \cdots V_j \begin{pmatrix} 0_{(j+1)\times 1} \\ H^2(\mathbb{D}, \mathbb{C}^{n-j-1}) \end{pmatrix}. \tag{3.176}$$

Let $\xi_{(j)} = \xi_0 \dot\wedge \dots \dot\wedge \xi_j$. Then, for every j,

$$\xi_{(j)} \dot\wedge \mathcal{K}_{j+1} = \xi_{(j)} \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n)$$

and the operator $(\xi_{(j)} \dot\wedge \cdot) \colon \mathcal{K}_{j+1} \to \xi_{(j)} \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n)$ is unitary.

Proof. Let us first prove the inclusion

$$\xi_{(j)} \dot\wedge \mathcal{K}_{j+1} \subseteq \xi_{(j)} \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n).$$

A typical element $\rho \in \xi_{(j)} \dot\wedge \mathcal{K}_{j+1}$ is of the form $\rho = \xi_{(j)} \dot\wedge \bar{A}_{j-1} \psi$ for $\psi \in H^2(\mathbb{D}, \mathbb{C}^{n-j-1})$. By Lemma 3.2.48, there exists a function $\varphi \in H^2(\mathbb{D}, \mathbb{C}^{n-j-2})$ such that

$$\alpha_{j-1}^T \phi = \psi.$$

Then

$$\rho = \xi_{(j)}\dot{\wedge}\bar{A}_{j-1}\alpha_{j-1}^T\phi = \xi_{(j)}\dot{\wedge}\bar{A}_{j-2}\bar{\alpha}_{j-1}\alpha_{j-1}^T\phi.$$

Since $\tilde{V}_{j-1}$ is unitary-valued,

$$A_{j-2}^T\xi_{j-1}\xi_{j-1}^*\bar{A}_{j-2} + \bar{\alpha}_{j-1}\alpha_{j-1}^T = I_{n-j+1}.$$

Hence

$$\begin{aligned}\rho &= \xi_{(j)}\dot{\wedge}\bar{A}_{j-2}\bar{\alpha}_{j-1}\alpha_{j-1}^T\phi \\ &= \xi_{(j)}\dot{\wedge}\bar{A}_{j-2}(I_{n-j+1} - A_{j-2}^T\xi_{j-1}\xi_{j-1}^*\bar{A}_{j-2})\phi \\ &= \xi_{(j)}\dot{\wedge}\bar{A}_{j-2}\phi - \xi_{(j)}\dot{\wedge}\bar{A}_{j-2}A_{j-2}^T\xi_{j-1}\xi_{j-1}^*\bar{A}_{j-2}\phi \\ &= \xi_{(j)}\dot{\wedge}\bar{A}_{j-2}\phi\end{aligned},$$

last equality being obtained by Lemma 3.2.49. It is evident that, by continuing in a similar way, we get

$$\rho = \xi_{(j)}\dot{\wedge}\varphi$$

for $\varphi \in H^2(\mathbb{D}, \mathbb{C}^n)$, and so,

$$\rho \in \xi_{(j)}\dot{\wedge}H^2(\mathbb{D}, \mathbb{C}^n).$$

Hence

$$\xi_{(l)}\dot{\wedge}\mathcal{K}_{l+1} \subseteq \xi_{(l)}\dot{\wedge}H^2(\mathbb{D}, \mathbb{C}^n). \tag{3.177}$$

Let us show that $\xi_{(j)}\dot{\wedge}\mathcal{K}_{j+1} \supseteq \xi_{(j)}\dot{\wedge}H^2(\mathbb{D}, \mathbb{C}^n)$. Let $f \in H^2(\mathbb{D}, \mathbb{C}^n)$. Since $V_0, \dots, V_j$ are unitary-valued, we have

$$\begin{aligned}\xi_{(j)}\dot{\wedge}f &= \xi_{(j)}\dot{\wedge}I_n f \\ &= \xi_{(j)}\dot{\wedge}\left(\sum_{k=0}^{j}\bar{A}_{k-1}A_{k-1}^T\xi_k\xi_k^*\bar{A}_{k-1}A_{k-1}^T + \bar{A}_jA_j^T\right)f \\ &= \xi_{(j)}\dot{\wedge}\sum_{k=0}^{j}\bar{A}_{k-1}A_{k-1}^T\xi_k\xi_k^*\bar{A}_{k-1}A_{k-1}^Tf + \xi_{(j)}\dot{\wedge}\bar{A}_jA_j^Tf.\end{aligned}$$

By Lemma 3.2.49,

$$\xi_{(j)}\dot{\wedge}f = \xi_{(j)}\dot{\wedge}\sum_{k=0}^{j}\bar{A}_{k-1}A_{k-1}^T\xi_k\xi_k^*\bar{A}_{k-1}A_{k-1}^Tf + \xi_{(j)}\dot{\wedge}\bar{A}_jA_j^Tf = \xi_{(j)}\dot{\wedge}\bar{A}_jA_j^Tf.$$

By Lemma 3.2.48, $A_j^TH^2(\mathbb{D}, \mathbb{C}^{n-j}) = H^2(\mathbb{D}, \mathbb{C}^{n-j-1})$, thus

$$\xi_{(j)}\dot{\wedge}f \in \xi_{(j)}\dot{\wedge}\bar{A}_jH^2(\mathbb{D}, \mathbb{C}^{n-j-1}) = \xi_{(j)}\dot{\wedge}\mathcal{K}_{j+1},$$

proving that

$$\xi_{(j)}\dot{\wedge}H^2(\mathbb{D}, \mathbb{C}^n) \subseteq \xi_{(j)}\dot{\wedge}\mathcal{K}_{j+1}.$$

Combining the latter inclusion with inclusion (3.177), we deduce that

$$\xi_{(j)}\dot{\wedge}\mathcal{K}_{j+1} = \xi_{(j)}\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^n).$$

To show that the operator $(\xi_{(j)}\dot{\wedge}\cdot)\colon \mathcal{K}_{j+1} \to \xi_{(j)}\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^n)$ is unitary, it suffices to prove that, for every $\vartheta \in \mathcal{K}_{j+1}$,

$$\|\xi_{(j)}\dot{\wedge}\vartheta\|_{L^2(\mathbb{T},\wedge^{j+2}\mathbb{C}^n)} = \|\vartheta\|_{L^2(\mathbb{T},\mathbb{C}^n)}.$$

Let $\vartheta \in \mathcal{K}_{j+1}$. Then, by Proposition 2.1.19, we get

$$\begin{aligned}
\|\xi_{(j)}\dot{\wedge}\vartheta\|^2_{L^2(\mathbb{T},\wedge^{j+2}\mathbb{C}^n)} &= \langle \xi_{(j)}\dot{\wedge}\vartheta, \xi_{(j)}\dot{\wedge}\vartheta\rangle_{L^2(\mathbb{T},\wedge^{j+2}\mathbb{C}^n)} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \langle \xi_{(j)}(e^{i\theta})\dot{\wedge}\vartheta(e^{i\theta}), \xi_{(j)}(e^{i\theta})\dot{\wedge}\vartheta(e^{i\theta})\rangle_{\wedge^{j+2}\mathbb{C}^n} d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \det\begin{pmatrix} \langle \xi_0(e^{i\theta}), \xi_0(e^{i\theta})\rangle_{\mathbb{C}^n} & \dots & \langle \xi_0(e^{i\theta}), \vartheta(e^{i\theta})\rangle_{\mathbb{C}^n} \\ \langle \xi_1(e^{i\theta}), \xi_0(e^{i\theta})\rangle_{\mathbb{C}^n} & \dots & \langle \xi_1(e^{i\theta}), \vartheta(e^{i\theta})\rangle_{\mathbb{C}^n} \\ \vdots & \ddots & \vdots \\ \langle \vartheta(e^{i\theta}), \xi_0(e^{i\theta})\rangle_{\mathbb{C}^n} & \dots & \langle \vartheta(e^{i\theta}), \vartheta(e^{i\theta})\rangle_{\mathbb{C}^n} \end{pmatrix} d\theta.
\end{aligned}$$

By Proposition 3.2.1, $\{\xi_i(z)\}_{i=0}^j$ is an orthonormal set in $\mathbb{C}^n$ for almost every $z \in \mathbb{T}$. Thus the latter integral is equal to

$$\frac{1}{2\pi}\int_0^{2\pi} \det\begin{pmatrix} 1 & 0 & \dots & \langle \xi_0(e^{i\theta}), \vartheta(e^{i\theta})\rangle_{\mathbb{C}^n} \\ 0 & 1 & \dots & \langle \xi_1(e^{i\theta}), \vartheta(e^{i\theta})\rangle_{\mathbb{C}^n} \\ \vdots & & \ddots & \vdots \\ \langle \vartheta(e^{i\theta}), \xi_0(e^{i\theta})\rangle_{\mathbb{C}^n} & & \dots & \langle \vartheta(e^{i\theta}), \vartheta(e^{i\theta})\rangle_{\mathbb{C}^n} \end{pmatrix} d\theta.$$

Note that since $\vartheta \in \mathcal{K}_{j+1}$, then $\vartheta = \bar{A}_j\psi$ for some $\psi \in H^2(\mathbb{D},\mathbb{C}^{n-j-1})$. Also, since each $\tilde{V}_k$ is unitary valued for all $k = 0,\dots,j$, then, for almost every $e^{i\theta} \in \mathbb{T}$,

$$\langle \xi_k(e^{i\theta}), \vartheta(e^{i\theta})\rangle_{\mathbb{C}^n} = \langle \xi_k(e^{i\theta}), \bar{A}_j(e^{i\theta})\psi(e^{i\theta})\rangle_{\mathbb{C}^n} = 0.$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi} \det\begin{pmatrix} 1 & 0 & \dots & \langle \xi_0(e^{i\theta}), \vartheta(e^{i\theta})\rangle_{\mathbb{C}^n} \\ 0 & 1 & \dots & \langle \xi_1(e^{i\theta}), \vartheta(e^{i\theta})\rangle_{\mathbb{C}^n} \\ \vdots & & \ddots & \vdots \\ \langle \vartheta(e^{i\theta}), \xi_0(e^{i\theta})\rangle_{\mathbb{C}^n} & & \dots & \langle \vartheta(e^{i\theta}), \vartheta(e^{i\theta})\rangle_{\mathbb{C}^n} \end{pmatrix} d\theta$$

is equal to

$$\frac{1}{2\pi}\int_0^{2\pi} \det\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & \langle \vartheta(e^{i\theta}), \vartheta(e^{i\theta})\rangle_{\mathbb{C}^n} \end{pmatrix} d\theta,$$

which yields

$$\frac{1}{2\pi}\int_0^{2\pi} \|\vartheta(e^{i\theta})\|^2_{\mathbb{C}^n} d\theta = \|\vartheta\|^2_{L^2(\mathbb{T},\mathbb{C}^n)}.$$

Therefore, by Theorem A.2.4, the operator $(\xi_{(j)}\dot{\wedge}\cdot)\colon \mathcal{K}_{j+1} \to \xi_{(j)}\dot{\wedge}H^2(\mathbb{D},\mathbb{C}^n)$ is unitary. □

Lemma 3.2.51. *Let $\tilde{W}_j^T$ be given by equations (3.149), let $W_j^T = \begin{pmatrix} I_j & 0 \\ 0 & \tilde{W}_j^T \end{pmatrix}$, let*

$$\mathcal{L}_{j+1} = W_0^* \cdots W_j^* \begin{pmatrix} 0_{(j+1)\times 1} \\ H^2(\mathbb{D},\mathbb{C}^{m-j-1})^\perp \end{pmatrix},$$

and let $\bar{\eta}_{(j)} = \bar{\eta}_0\dot{\wedge}\ldots\dot{\wedge}\bar{\eta}_j$. Let $B_j = \beta_0\cdots\beta_j$.

Then, for every $u \in H^2(\mathbb{D},\mathbb{C}^m)^\perp$ and all $0 \le j \le \min(m,n)-2$,

$$\bar{\eta}_{(j)}\dot{\wedge}B_{j-1}B_{j-1}^*\bar{\eta}_j\eta_j^T B_{j-1}u = 0.$$

Proof. Note that by Propositions 3.2.21 and 3.2.35, the assertion holds for $j = 0,1,2$. Suppose it holds for $j = 0,\cdots,l-1$ and for every $u \in H^2(\mathbb{D},\mathbb{C}^m)^\perp$. This means that, for all l,

$$\bar{\eta}_{(l-1)}\dot{\wedge}B_{l-2}B_{l-2}^*\bar{\eta}_{l-1}\eta_{l-1}^T B_{l-2}B_{l-2}^* f = 0 \quad \text{for all } f \in L^2(\mathbb{T},\mathbb{C}^m), \tag{3.178}$$

where $\beta_0,\ldots,\beta_{l-2}$ are inner, co-outer, quasi-continuous functions of types $m\times(m-1),\ldots,(m-l+2)\times(m-l+1)$ respectively.

We will show that assertion (3.178) holds for $j = 0,\ldots,l$. By the inductive hypothesis, T_l is a compact operator, so there exist functions $v_l \in H^2(\mathbb{D},\mathbb{C}^n)$, $w_l \in H^2(\mathbb{D},\mathbb{C}^m)^\perp$ such that $(\xi_{(l-1)}\dot{\wedge}v_l, \bar{\eta}_{(l-1)}\dot{\wedge}w_l)$ is a Schmidt pair for operator T_l corresponding to $t_l = \|T_l\|$. By Proposition 3.2.2, $\xi_{(l-1)}\dot{\wedge}v_l$ is an element of $H^2(\mathbb{D},\wedge^{l+1}\mathbb{C}^n)$. Let $h_l \in H^2(\mathbb{D},\mathbb{C})$ be the scalar outer factor of $\xi_{(l-1)}\dot{\wedge}v_l$. Define

$$y_l = (I_m - \bar{\eta}_0\eta_0^T - \cdots - \bar{\eta}_{l-1}\eta_{l-1}^T)w_l$$

and

$$\eta_l = \frac{\bar{z}\bar{y}_l}{h_l}.$$

We have

$$\begin{aligned} \beta_{l-1}^*\cdots\beta_0^* y_l &= B_{l-1}^*(I_m - \bar{\eta}_0\eta_0^T - \cdots - \bar{\eta}_{l-1}\eta_{l-1}^T)w_l \\ &= B_{l-1}^* w_l - B_{l-1}^*\bar{\eta}_0\eta_0^T w_l - \cdots - B_{l-1}^*\bar{\eta}_{l-1}\eta_{l-1}^T w_l. \end{aligned}$$

Notice that, by the inductive hypothesis, for $i = 0,\ldots,l-1$, each W_i is unitary-valued, hence $\beta_{i-1}^*\cdots\beta_0^*\bar{\eta}_{i-1} = 0$. Thus

$$B_{l-1}^* y_l = B_{l-1}^* w_l$$

that is,

$$B_{l-1}^T \eta_l = \frac{1}{h_l} B_{l-1}^T \bar{z} \bar{y}_l = \frac{1}{h_l} B_{l-1}^T \bar{z} \bar{w}_l$$

is analytic. Further, since $\beta_i(z)$ are isometries for all $i = 0, \ldots, l-1$,

$$\|B_{l-1}^T(z)\bar{z}\bar{y}_l(z)\|_{\mathbb{C}^{m-l}} = \|B_{l-1}^T(z)\bar{z}\bar{w}_l(z)\|_{\mathbb{C}^{m-l}} = |h_l(z)|$$

almost everywhere on $\mathbb{T}$, and therefore

$$\|B_{l-1}^T(z)\eta_l(z)\|_{\mathbb{C}^m} = 1$$

almost everywhere on $\mathbb{T}$, that is, $B_{l-1}^T \eta_l$ is inner.

By Theorem 3.1.10, there exists an inner, co-outer, quasi-continuous function β_l of size $(m-l) \times (m-l-1)$ such that

$$\tilde{W}_l^T = \begin{pmatrix} B_{l-1}^T \eta_l & \bar{\beta}_l \end{pmatrix}$$

is a thematic completion of $B_{l-1}^T \eta_l$. Then

$$\beta_l \beta_l^* + B_{l-1}^* \bar{\eta}_l \eta_l^T B_{l-1} = I_{m-l}. \tag{3.179}$$

Let $W_l^T = \begin{pmatrix} I_l & 0 \\ 0 & \tilde{W}_l^T \end{pmatrix}$ and let

$$\mathcal{L}_{l+1} = W_0^* \cdots W_l^* \begin{pmatrix} 0_{(l+1)\times 1} \\ H^2(\mathbb{D}, \mathbb{C}^{m-l-1})^\perp \end{pmatrix}.$$

Since $W_0, \cdots, W_{l-1}$ are unitary-valued, $W_0^* \cdots W_{l-1}^* W_{l-1} \cdots W_0 = I_m$, or equivalently,

$$B_{l-1}B_{l-1}^* + \sum_{k=0}^{l-1} B_{k-1}B_{k-1}^* \bar{\eta}_k \eta_k^T B_{k-1} B_{k-1}^* = I_m.$$

Then, for any $u \in H^2(\mathbb{D}, \mathbb{C}^{m-l-2})^\perp$,

$$\bar{\eta}_{(l)} \dot{\wedge} B_{l-1}B_{l-1}^* \bar{\eta}_l \eta_l^T B_{l-1} u = \bar{\eta}_{(l)} \dot{\wedge} \left(I_m - \sum_{k=0}^{l-1} B_{k-1}B_{k-1}^* \bar{\eta}_k \eta_k^T B_{k-1} B_{k-1}^* \right) \bar{\eta}_l \eta_l^T B_{l-1} u,$$

which is equal to

$$\bar{\eta}_{(l)} \dot{\wedge} \bar{\eta}_l \eta_l^T B_{l-1} u - \bar{\eta}_{(l)} \dot{\wedge} \sum_{k=0}^{l-1} B_{k-1}B_{k-1}^* \bar{\eta}_k \eta_k^T B_{k-1} B_{k-1}^* \bar{\eta}_l \eta_l^T B_{l-1} u.$$

Hence

$$\bar{\eta}_{(l)} \dot{\wedge} B_{l-1}B_{l-1}^* \bar{\eta}_l \eta_l^T B_{l-1} u = -\bar{\eta}_{(l)} \dot{\wedge} \sum_{k=0}^{l-1} B_{k-1}B_{k-1}^* \bar{\eta}_k \eta_k^T B_{l-1} B_{l-1}^* \bar{\eta}_l \eta_l^T B_{l-1} u,$$

last equality following by the pointwise linear dependence of $\bar{\eta}_l$ and $\bar{\eta}_l\eta_l^T B_{l-1}u$ on $\mathbb{D}$. If we set $B_{l-1}^*\bar{\eta}_l\eta_l^T B_l u = f \in L^2(\mathbb{T}, \mathbb{C}^{m-l-2})$, then, by the inductive hypothesis (3.178),

$$\bar{\eta}_{(l)}\dot{\wedge}\sum_{k=0}^{l-1} B_{k-1}B_{k-1}^*\bar{\eta}_k\eta_k^T B_{k-1}b_{k-1}^* f = 0.$$

Hence, for all $f \in L^2(\mathbb{T}, \mathbb{C}^{m-j-2})$ and all j,

$$\bar{\eta}_{(j)}\dot{\wedge}B_{j-1}B_{j-1}^*\bar{\eta}_j\eta_j^T B_{j-1}f = 0 \tag{3.180}$$

and the assertion has been proved. □

Proposition 3.2.52. *Let $\tilde{W}_j^T$ be given by equations (3.149), let $W_j^T = \begin{pmatrix} I_j & 0 \\ 0 & \tilde{W}_j^T \end{pmatrix}$, let*

$$\mathcal{L}_{j+1} = W_0^* \cdots W_j^* \begin{pmatrix} 0_{(j+1)\times 1} \\ H^2(\mathbb{D}, \mathbb{C}^{m-j-1})^\perp \end{pmatrix}, \tag{3.181}$$

and let $\bar{\eta}_{(j)} = \bar{\eta}_0\dot{\wedge}\ldots\dot{\wedge}\bar{\eta}_j$. Then,

$$\bar{\eta}_{(j)}\dot{\wedge}\mathcal{L}_{j+1} = \bar{\eta}_{(j)}\dot{\wedge}H^2(\mathbb{D}, \mathbb{C}^m)^\perp$$

and the operator $(\bar{\eta}_{(j)}\dot{\wedge}\cdot)\colon \mathcal{L}_{j+1} \to \bar{\eta}_{(j)}\dot{\wedge}H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ is unitary.

Proof. Let us first show $\bar{\eta}_{(j)}\dot{\wedge}\mathcal{L}_{j+1} \subseteq \bar{\eta}_{(j)}\dot{\wedge}H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ A typical element $q \in \bar{\eta}_{(j)}\dot{\wedge}\mathcal{L}_{j+1}$ is of the form

$$q = \bar{\eta}_{(j)}\dot{\wedge}B_j\upsilon$$

for some $\upsilon \in H^2(\mathbb{D}, \mathbb{C}^{m-l-1})^\perp$. By Lemma 3.2.48, there exists a vector-valued function $u \in H^2(\mathbb{D}, \mathbb{C}^{m-j})^\perp$ such that $\beta_j^* u = \upsilon$. Then, $q = \bar{\eta}_{(j)}\dot{\wedge}B_j\beta_j^* u$, and by equation (3.179),

$$\begin{aligned} q &= \bar{\eta}_{(j)}\dot{\wedge}B_j\beta_j^* u \\ &= \bar{\eta}_{(j)}\dot{\wedge}B_{j-1}\beta_j\beta_j^* u \\ &= \bar{\eta}_{(j)}\dot{\wedge}B_{j-1}(I_{m-j} - B_{j-1}^*\bar{\eta}_j\eta_j^T B_{j-1})u \\ &= \bar{\eta}_{(j)}\dot{\wedge}B_{j-1}u - \bar{\eta}_{(j)}\dot{\wedge}B_{j-1}B_{j-1}^*\bar{\eta}_j\eta_j^T B_{j-1}u \\ &= \bar{\eta}_{(j)}\dot{\wedge}B_{j-1}u \end{aligned}$$

last equality following by Lemma 3.2.51. It is obvious that continuing in a similar way, we obtain $q = \bar{\eta}_{(j)}\dot{\wedge}\psi$ for $\psi \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$. Thus

$$\bar{\eta}_{(j)}\dot{\wedge}\mathcal{L}_{j+1} \subseteq \bar{\eta}_{(j)}\dot{\wedge}H^2(\mathbb{D}, \mathbb{C}^m)^\perp. \tag{3.182}$$

Let us now prove $\bar{\eta}_{(j)}\dot{\wedge}\mathcal{L}_{j+1} \supseteq \bar{\eta}_{(j)}\dot{\wedge}H^2(\mathbb{D}, \mathbb{C}^m)^\perp$. Let $\rho = \bar{\eta}_{(j)}\dot{\wedge}\tau \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ for some

$\tau \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$. Note that, since $W_0, \dots, W_j$ are unitary-valued, we get

$$W_0^* \cdots W_j^* W_j \cdots W_0 = I_m,$$

which is equivalent to the equation

$$B_j B_j^* + \sum_{k=0}^{j} B_{k-1} B_{k-1}^* \bar{\eta}_k \eta_k^T B_k B_k^* = I_m.$$

We, then, obtain

$$\begin{aligned} \bar{\eta}_{(j)} \dot{\wedge} \tau &= \bar{\eta}_{(j)} \dot{\wedge} I_m \tau \\ &= \bar{\eta}_{(j)} \dot{\wedge} \left(B_j B_j^* + \sum_{k=0}^{j} B_{k-1} B_{k-1}^* \bar{\eta}_k \eta_k^T B_{k-1} B_{k-1}^* \right) \tau \\ &= \bar{\eta}_{(j)} \dot{\wedge} B_j B_j^* \tau + \bar{\eta}_{(j)} \dot{\wedge} \sum_{k=0}^{j} B_{k-1} B_{k-1}^* \bar{\eta}_k \eta_k^T B_{k-1} B_{k-1}^* \tau. \end{aligned}$$

By Lemma 3.2.51,

$$\bar{\eta}_{(j)} \dot{\wedge} \sum_{k=0}^{j} B_{k-1} B_{k-1}^* \bar{\eta}_k \eta_k^T B_{k-1} B_{k-1}^* \tau = 0,$$

hence

$$\bar{\eta}_{(j)} \dot{\wedge} \tau = \bar{\eta}_{(j)} \dot{\wedge} B_j B_j^* \tau.$$

By Lemma 3.2.48, $B_j^* H^2(\mathbb{D}, \mathbb{C}^m)^\perp = H^2(\mathbb{D}, \mathbb{C}^{m-j+1})^\perp$, thus

$$\bar{\eta}_{(j)} \dot{\wedge} \tau \in \bar{\eta}_{(j)} \dot{\wedge} B_j H^2(\mathbb{D}, \mathbb{C}^{m-j+1})^\perp = \bar{\eta}_{(j)} \dot{\wedge} \mathcal{L}_{j+1},$$

and so

$$\bar{\eta}_{(j)} \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp \subseteq \bar{\eta}_{(j)} \dot{\wedge} \mathcal{L}_{j+1}.$$

Combining the latter inclusion with inclusion (3.182), we deduce that

$$\bar{\eta}_{(j)} \dot{\wedge} \mathcal{L}_{j+1} = \bar{\eta}_{(j)} \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp.$$

To show that the operator $(\bar{\eta}_{(j)} \dot{\wedge} \cdot) \colon \mathcal{L}_{j+1} \to \bar{\eta}_{(j)} \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ is unitary, it suffices to prove that, for every $\varphi \in \mathcal{L}_{j+1}$, $\|\bar{\eta}_{(j)} \dot{\wedge} \varphi\|_{L^2(\mathbb{T}, \wedge^{j+2}\mathbb{C}^m)} = \|\varphi\|_{L^2(\mathbb{T}, \mathbb{C}^m)}$. By Proposition 2.1.19, we get

$$\begin{aligned} \|\bar{\eta}_{(j)} \dot{\wedge} \varphi\|^2_{L^2(\mathbb{T}, \wedge^{j+2}\mathbb{C}^m)} &= \langle \bar{\eta}_{(j)} \dot{\wedge} \varphi, \bar{\eta}_{(j)} \dot{\wedge} \varphi \rangle_{L^2(\mathbb{T}, \wedge^{j+2}\mathbb{C}^m)} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix} \langle \bar{\eta}_0(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \dots & \langle \bar{\eta}_0(e^{i\theta}), \varphi(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ \langle \bar{\eta}_1(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \dots & \langle \bar{\eta}_1(e^{i\theta}), \varphi(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ \vdots & \ddots & \vdots \\ \langle \varphi(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \dots & \langle \varphi(e^{i\theta}), \varphi(e^{i\theta}) \rangle_{\mathbb{C}^m} \end{pmatrix} d\theta. \end{aligned}$$

Recall that, by Proposition 3.2.1, the set $\{\bar{\eta}_i(z)\}_{i=0}^{j}$ is orthonormal in $\mathbb{C}^n$ for almost every $z \in \mathbb{T}$. Then the latter integral is equal to

$$\frac{1}{2\pi}\int_0^{2\pi} \det \begin{pmatrix} 1 & 0 & \dots & \langle \bar{\eta}_0(e^{i\theta}), \varphi(e^{i\theta})\rangle_{\mathbb{C}^m} \\ 0 & 1 & \dots & \langle \bar{\eta}_1(e^{i\theta}), \varphi(e^{i\theta})\rangle_{\mathbb{C}^m} \\ \vdots & & \ddots & \vdots \\ \langle \varphi(e^{i\theta}), \bar{\eta}_0(e^{i\theta})\rangle_{\mathbb{C}^m} & \dots & \dots & \langle \varphi(e^{i\theta}), \varphi(e^{i\theta})\rangle_{\mathbb{C}^m} \end{pmatrix} d\theta.$$

Observe that since $\varphi \in \mathcal{L}_{j+1}$, there exists a $\rho \in H^2(\mathbb{D}, \mathbb{C}^{m-j-1})^\perp$ such that $\varphi = B_j\rho$. Also, since each $\tilde{W}_k$ is unitary valued for all $k = 0, \dots, j$, then, for almost every $e^{i\theta} \in \mathbb{T}$,

$$\langle \bar{\eta}_k(e^{i\theta}), \varphi(e^{i\theta})\rangle_{\mathbb{C}^m} = \langle \bar{\eta}_k(e^{i\theta}), \beta_0(e^{i\theta}) \cdots \beta_j(e^{i\theta})\rho(e^{i\theta})\rangle_{\mathbb{C}^m} = 0.$$

Thus the latter integral equals

$$\frac{1}{2\pi}\int_0^{2\pi} \det \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \langle \varphi(e^{i\theta}), \varphi(e^{i\theta})\rangle_{\mathbb{C}^m} \end{pmatrix} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \langle \varphi(e^{i\theta}), \varphi(e^{i\theta})\rangle_{\mathbb{C}^m} d\theta = \|\varphi\|^2_{L^2(\mathbb{T},\mathbb{C}^m)}.$$

Hence by Theorem A.2.4, the operator $(\bar{\eta}_{(j)} \dot{\wedge} \cdot) \colon \mathcal{L}_{j+1} \to \bar{\eta}_{(j)} \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ is unitary. ☐

Proposition 3.2.53. *With the notation of Proposition 3.2.52*

$$\mathcal{L}_{j+1}^\perp = \{f \in L^2(\mathbb{T}, \mathbb{C}^m) : \beta_j^* \cdots \beta_0^* f \in H^2(\mathbb{D}, \mathbb{C}^{m-j-1})\}.$$

Proof. Clearly $\mathcal{L}_{j+1} = \beta_0 \cdots \beta_j H^2(\mathbb{D}, \mathbb{C}^{m-j-1})$. A function $f \in L^2(\mathbb{T}, \mathbb{C}^m)$ belongs to $\mathcal{L}_{j+1}^\perp$ if and only if

$$\langle f, \beta_0 \cdots \beta_j e^{-i\theta}\bar{g}\rangle_{L^2(\mathbb{T},\mathbb{C}^m)} = 0 \quad \text{for all} \quad g \in H^2(\mathbb{D}, \mathbb{C}^{m-j-1}).$$

Equivalently,

$$\frac{1}{2\pi}\int_0^{2\pi} \langle f(e^{i\theta}), \beta_0(e^{i\theta}) \cdots \beta_j(e^{i\theta}) e^{-i\theta}\bar{g}(e^{i\theta})\rangle_{\mathbb{C}^m} d\theta = 0 \quad \text{for all} \quad g \in H^2(\mathbb{D}, \mathbb{C}^{m-j-1})$$

if and only if

$$\frac{1}{2\pi}\int_0^{2\pi} \langle \beta_j^*(e^{i\theta}) \cdots \beta_0^*(e^{i\theta}) f(e^{i\theta}), e^{-i\theta}\bar{g}(e^{i\theta})\rangle_{\mathbb{C}^{m-2}} d\theta = 0 \quad \text{for all} \quad g \in H^2(\mathbb{D}, \mathbb{C}^{m-j-1}).$$

The latter statement is equivalent to the assertion that $\beta_j^* \cdots \beta_0^* f$ is orthogonal to $H^2(\mathbb{D}, \mathbb{C}^{m-j-1})^\perp$ in $L^2(\mathbb{T}, \mathbb{C}^{m-j-1})$, which holds if and only if

$$\beta_j^* \cdots \beta_0^* f \in H^2(\mathbb{D}, \mathbb{C}^{m-j-1}).$$

Thus

$$\mathcal{L}_{j+1}^{\perp} = \{f \in L^2(\mathbb{T}, \mathbb{C}^m) : \beta_j^* \cdots \beta_0^* f \in H^2(\mathbb{D}, \mathbb{C}^{m-j-1})\}$$

as required. □

Let us proceed to the main theorem of this section.

Theorem 3.2.54. *Let m, n be positive integers such that $\min(m, n) \geq 2$. Let G be in $H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$. In the notation of the algorithm from Section 3.2.1, let*

$$(\xi_0 \dot{\wedge} \dots \dot{\wedge} \xi_{j-1} \dot{\wedge} v_j, \bar{\eta}_0 \dot{\wedge} \dots \dot{\wedge} \bar{\eta}_{j-1} \dot{\wedge} w_j)$$

be a Schmidt pair for T_j corresponding to $t_j = \|T_j\| \neq 0$. Let $h_j \in H^2(\mathbb{D}, \mathbb{C})$ be the scalar outer factor of

$$\xi_0 \dot{\wedge} \dots \dot{\wedge} \xi_{j-1} \dot{\wedge} v_j.$$

Let

$$x_j = (I_n - \xi_0 \xi_0^* - \cdots - \xi_{j-1} \xi_{j-1}^*) v_j,$$
$$y_j = (I_m - \bar{\eta}_0 \eta_0^T - \cdots - \bar{\eta}_{j-1} \eta_{j-1}^T) w_j$$

and

$$\xi_j = \frac{x_j}{h_j}, \quad \bar{\eta}_j = \frac{z y_j}{\bar{h}_j}.$$

For $i = 0, 1, \dots, j$, let

$$\tilde{V}_i = \begin{pmatrix} \alpha_{i-1}^T \cdots \alpha_0^T \xi_i & \bar{\alpha}_i \end{pmatrix}, \quad \tilde{W}_i^T = \begin{pmatrix} \beta_{i-1}^T \cdots \beta_0^T \eta_i & \bar{\beta}_i \end{pmatrix} \tag{3.183}$$

be unitary-valued functions, as described in Lemma 3.1.12. Let

$$V_j = \begin{pmatrix} I_j & 0 \\ 0 & \tilde{V}_j \end{pmatrix}, \quad W_j = \begin{pmatrix} I_j & 0 \\ 0 & \tilde{W}_j \end{pmatrix}.$$

Let $A_j = \alpha_0 \alpha_1 \dots \alpha_j$, $A_{-1} = I_n$, $B_j = \beta_0 \beta_1 \dots \beta_j$ and $B_{-1} = I_m$. Let

$$X_{j+1} = \xi_0 \dot{\wedge} \dots \dot{\wedge} \xi_j \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^n) \subset H^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^n),$$

and let

$$Y_{j+1} = \bar{\eta}_0 \dot{\wedge} \dots \dot{\wedge} \bar{\eta}_j \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^m)^\perp \subset H^2(\mathbb{D}, \wedge^{j+2} \mathbb{C}^m)^\perp.$$

Let

$$T_{j+1}(\xi_0 \dot{\wedge} \dots \dot{\wedge} \xi_j \dot{\wedge} x) = P_{Y_{j+1}}(\bar{\eta}_0 \dot{\wedge} \dots \dot{\wedge} \bar{\eta}_j \dot{\wedge} (G - Q_{j+1}) x)$$

for all $x \in H^2(\mathbb{D}, \mathbb{C}^n)$, where Q_{j+1} satisfies

$$(G - Q_{j+1}) x_i = t_i y_i, \text{ and } (G - Q_{j+1})^* y_i = t_i x_i, \quad \text{for } i = 0, 1, \dots, j. \tag{3.184}$$

Let

$$\mathcal{K}_{j+1} = V_0 \cdots V_j \begin{pmatrix} 0_{(j+1)\times 1} \\ H^2(\mathbb{D}, \mathbb{C}^{n-j-1}) \end{pmatrix}, \quad \mathcal{L}_{j+1} = W_0^* \cdots W_j^* \begin{pmatrix} 0_{(j+1)\times 1} \\ H^2(\mathbb{D}, \mathbb{C}^{m-j-1})^\perp \end{pmatrix}. \tag{3.185}$$

Let the operator $\Gamma_{j+1} \colon \mathcal{K}_{j+1} \to \mathcal{L}_{j+1}$ *be given by*

$$\Gamma_{j+1} = P_{\mathcal{L}_{j+1}} M_{G-Q_{j+1}}|_{\mathcal{K}_{j+1}}.$$

Then (i) The maps

$$M_{\bar{A}_j} \colon H^2(\mathbb{D}, \mathbb{C}^{n-j-1}) \to \mathcal{K}_{j+1} \colon x \mapsto \bar{A}_j x, \quad \text{and} \quad M_{B_j} \colon H^2(\mathbb{D}, \mathbb{C}^{m-j-1})^\perp \to \mathcal{L}_{j+1} \colon y \mapsto B_j y$$

are unitaries.
(ii) The maps $(\xi_0 \dot\wedge \ldots \dot\wedge \xi_j \dot\wedge \cdot) \colon \mathcal{K}_{j+1} \to X_{j+1}$, $(\bar\eta_0 \dot\wedge \ldots \dot\wedge \bar\eta_j \dot\wedge \cdot) \colon \mathcal{L}_{j+1} \to Y_{j+1}$ *are unitaries.*
(iii) The following diagram commutes

$$\begin{array}{ccccc} H^2(\mathbb{D}, \mathbb{C}^{n-j}) & \xrightarrow{M_{\bar\alpha_0 \cdots \bar\alpha_j}} & \mathcal{K}_{j+1} & \xrightarrow{\xi_{(j)} \dot\wedge \cdot} & \xi_{(j)} \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n) = X_{j+1} \\ \Big\downarrow H_{F_{j+1}} & & \Big\downarrow \Gamma_{j+1} & & \Big\downarrow T_{j+1} \\ H^2(\mathbb{D}, \mathbb{C}^{m-j})^\perp & \xrightarrow{M_{\beta_0 \cdots \beta_j}} & \mathcal{L}_{j+1} & \xrightarrow{\bar\eta_{(j)} \dot\wedge \cdot} & \bar\eta_{(j)} \dot\wedge H^2(\mathbb{D}, \mathbb{C}^m)^\perp = Y_{j+1}, \end{array} \tag{3.186}$$

where $F_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-1)\times(n-j-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-j-1)\times(n-j-1)})$ *is the function defined in Proposition 3.2.43.*
(iv) Γ_{j+1} *and* T_{j+1} *are compact operators.*
(v) $\|T_{j+1}\| = \|\Gamma_{j+1}\| = \|H_{F_{j+1}}\| = t_{j+1}$.

Proof. (i). It follows from Lemma 3.1.16.
(ii). Follows from Propositions 3.2.50 and 3.2.52.
(iii). By Theorem 1.1.4, there exists a function $Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ such that the sequence

$$\left(s_0^\infty(G - Q_{j+1}), s_1^\infty(G - Q_{j+1}), \ldots, s_{j+1}^\infty(G - Q_{j+1})\right)$$

is lexicographically minimised. By Proposition 3.2.47, any such Q_{j+1} satisfies

$$(G - Q_{j+1})x_i = t_i y_i, (G - Q_{j+1})^* y_i = t_i x_i, \quad \text{for } i = 0, 1, \ldots, j. \tag{3.187}$$

By Proposition 3.2.8, T_{j+1} is well-defined and is independent of the choice of $Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ satisfying equations (3.187). We can choose Q_{j+1} which minimises $\left(s_0^\infty(G - Q_{j+1}), s_1^\infty(G - Q_{j+1}), \ldots, s_{j+1}^\infty(G - Q_{j+1})\right)$, and therefore satisfies equations (3.187). Consider the following diagram.

$$\begin{array}{ccc} \mathcal{K}_{j+1} & \xrightarrow{\xi_0 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge \cdot} & \xi_0 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n) = X_{j+1} \\ \Big\downarrow \Gamma_{j+1} & & \Big\downarrow T_{j+1} \\ \mathcal{L}_{j+1} & \xrightarrow{\bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge \cdot} & \bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge H^2(\mathbb{D}, \mathbb{C}^m)^\perp = Y_{j+1}. \end{array} \tag{3.188}$$

Let us prove first that diagram (3.188) commutes. By Theorem 1.1.4, there exists a function $Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ such that the sequence

$$\left(s_0^\infty(G - Q_{j+1}), s_1^\infty(G - Q_{j+1}), \dots, s_{j+1}^\infty(G - Q_{j+1})\right)$$

is lexicographically minimised. By Proposition 3.2.47, such Q_{j+1} satisfies

$$(G - Q_{j+1})x_i = t_i y_i, (G - Q_{j+1})^* y_i = t_i x_i, \quad \text{for } i = 0, 1, \dots, j. \tag{3.189}$$

By Proposition 3.2.8, T_{j+1} is well-defined and is independent of the choice of $Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ satisfying equations (3.189). We can choose Q_{j+1} which minimises $\left(s_0^\infty(G - Q_{j+1}), s_1^\infty(G - Q_{j+1}), \dots, s_{j+1}^\infty(G - Q_{j+1})\right)$, and therefore satisfies equations (3.189). Consider the following diagram.

$$\begin{array}{ccc} \mathcal{K}_{j+1} & \xrightarrow{\xi_0 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge \cdot} & \xi_0 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge H^2(\mathbb{D}, \mathbb{C}^n) = X_{j+1} \\ \Big\downarrow \Gamma_{j+1} & & \Big\downarrow T_{j+1} \\ \mathcal{L}_{j+1} & \xrightarrow{\bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_{j+1} \dot\wedge \cdot} & \bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge H^2(\mathbb{D}, \mathbb{C}^m)^\perp = Y_{j+1}. \end{array} \tag{3.190}$$

Let us prove first that diagram (3.190) commutes.

By Proposition 3.2.43, every $Q_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$, which minimises

$$\left(s_0^\infty(G - Q_{j+1}), s_1^\infty(G - Q_{j+1}), \dots, s_{j+1}^\infty(G - Q_{j+1})\right),$$

satisfies the following equation (see equation (3.150)).

$$G - Q_{j+1} = W_0^* W_1^* \cdots W_j^* \begin{pmatrix} t_0 u_0 & 0 & \cdots & 0 & 0_{1\times(n-j-1)} \\ 0 & t_1 u_1 & \dots & 0 & 0_{1\times(n-j-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_j u_j & 0 \\ 0_{(m-j-1)\times 1} & 0_{(m-j-1)\times 1} & \dots & \dots & (F_{j+1} + H^\infty) \cap B(t_j) \end{pmatrix} V_j^* \cdots V_0^*, \tag{3.191}$$

Thus, for every $\chi \in H^2(\mathbb{D}, \mathbb{C}^{n-j-1})$,

$$\begin{aligned} &(G - Q_{j+1}) V_0 \cdots V_j \begin{pmatrix} 0_{(j+1)\times 1} \\ H^2(\mathbb{D}, \mathbb{C}^{m-j-1})^\perp \end{pmatrix} \\ &= W_0^* W_1^* \cdots W_j^* \begin{pmatrix} t_0 u_0 & 0 & \cdots & 0 & 0_{1\times(n-j-1)} \\ 0 & t_1 u_1 & \dots & 0 & 0_{1\times(n-j-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_j u_j & 0 \\ 0_{(m-j-1)\times 1} & 0_{(m-j-1)\times 1} & \dots & \dots & (F_{j+1} + H^\infty) \cap B(t_j) \end{pmatrix} \begin{pmatrix} 0_{(j+1)\times 1} \\ H^2(\mathbb{D}, \mathbb{C}^{m-j-1})^\perp \end{pmatrix}, \end{aligned} \tag{3.192}$$

for some $F_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-1)\times(n-j-1)}) + C(\mathbb{T}, \mathbb{C}^{(m-j-1)\times(n-j-1)})$, for the quasi-continuous unimodular functions $u_i = \frac{\bar{z}\bar{h}_i}{h_i}$, for all $i = 0, \dots, j$, for the closed ball $B(t_j)$ of radius t_j in $L^\infty(\mathbb{T}, \mathbb{C}^{(m-j-1)\times(n-j-1)})$. By equation (3.157),

$$W_0^* W_1^* \cdots W_j^* = \begin{pmatrix} \bar{\eta}_0 & B_0 B_0^* \bar{\eta}_1 & \dots & B_{j-1} B_{j-1}^* \bar{\eta}_j & B_j \end{pmatrix}.$$

By equation (3.159),

$$V_0 \cdots V_j = \begin{pmatrix} \xi_0 & \bar{A}_0 A_0^T \xi_1 & \bar{A}_1 A_1^T \xi_2 & \dots & \bar{A}_{j-1} A_{j-1}^T \xi_{j-1} & \bar{A}_j \end{pmatrix}. \tag{3.193}$$

Therefore, by equation (3.192), for every $\chi \in H^2(\mathbb{D}, \mathbb{C}^{n-j-1})$,

$$(G - Q_{j+1})\bar{A}_j \chi = B_j F_{j+1} \chi. \tag{3.194}$$

A typical element $x \in \mathcal{K}_{j+1}$ is of the form $x = \bar{A}_j \chi$, for some $\chi \in H^2(\mathbb{D}, \mathbb{C}^{n-j+1})$. Then, by Proposition 3.2.50,

$$(\xi_0 \dot{\wedge} \dots \dot{\wedge} \xi_j \dot{\wedge} \cdot)\bar{A}_j \chi = \xi_0 \dot{\wedge} \dots \dot{\wedge} \xi_j \dot{\wedge} \bar{A}_j \chi \in X_{j+1}.$$

Therefore, by the definition of T_{j+1} and by equation (3.194),

$$\begin{aligned} T_{j+1}(\xi_0 \dot{\wedge} \dots \dot{\wedge} \xi_j \dot{\wedge} \bar{A}_j \chi) &= P_{Y_{j+1}}(\bar{\eta}_0 \dot{\wedge} \cdots \dot{\wedge} \bar{\eta}_j \dot{\wedge} (G - Q_{j+1})\bar{A}_j \chi) \\ &= P_{Y_{j+1}}(\bar{\eta}_0 \dot{\wedge} \cdots \dot{\wedge} \bar{\eta}_j \dot{\wedge} B_j F_{j+1} \chi). \end{aligned}$$

Furthermore, by the definition of Γ_{j+1} and by equation (3.194),

$$(\bar{\eta}_0 \dot{\wedge} \cdots \dot{\wedge} \bar{\eta}_j \dot{\wedge} \cdot)\Gamma_{j+1}(\bar{A}_j \chi) = \bar{\eta}_0 \dot{\wedge} \cdots \dot{\wedge} \bar{\eta}_j \dot{\wedge} P_{\mathcal{L}_{j+1}} B_j F_{j+1} \chi.$$

In order to prove the commutativity of diagram (3.190), we need to show that

$$\bar{\eta}_0 \dot{\wedge} \cdots \dot{\wedge} \bar{\eta}_j \dot{\wedge} B_j F_{j+1} \chi \in Y_{j+1}$$

and that

$$\bar{\eta}_0 \dot{\wedge} \cdots \dot{\wedge} \bar{\eta}_j \dot{\wedge} \left(B_j F_{j+1} \chi - P_{\mathcal{L}_{j+1}} B_j F_{j+1} \chi\right) = \bar{\eta}_0 \dot{\wedge} \cdots \dot{\wedge} \bar{\eta}_j \dot{\wedge} P_{\mathcal{L}_{j+1}^\perp} B_j F_{j+1} \chi$$

is orthogonal to Y_{j+1}, for any $\chi \in H^2(\mathbb{D}, \mathbb{C}^{n-j})$. Observe that, by Proposition 3.2.52, $\bar{\eta}_0 \dot{\wedge} \cdots \dot{\wedge} \bar{\eta}_j \dot{\wedge} B_j F_{j+1} \chi$ is indeed an element of Y_{j+1}. To prove the latter assertion, first notice that, by Proposition 3.2.53, there exists a $\Phi \in L^2(\mathbb{T}, \mathbb{C}^m)$ such that

$$\Phi = P_{\mathcal{L}_{j+1}^\perp} B_j F_{j+1} \chi \quad \text{and} \quad B_j^* \Phi \in H^2(\mathbb{D}, \mathbb{C}^{m-j-1}).$$

Let

$$\bar{\eta}_{(j)} = \bar{\eta}_0 \dot{\wedge} \cdots \dot{\wedge} \bar{\eta}_j.$$

It suffices to prove that

$$\langle \bar{\eta}_{(j)} \dot{\wedge} \Phi, \bar{\eta}_{(j)} \dot{\wedge} \psi \rangle_{L^2(\mathbb{T}, \wedge^{j+2}\mathbb{C}^m)} = 0$$

for all $\psi \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$. By Proposition 2.1.19,

$$\langle \bar{\eta}_{(j)} \dot{\wedge} \Phi, \bar{\eta}_{(j)} \dot{\wedge} \psi \rangle_{L^2(\mathbb{T}, \wedge^{j+2}\mathbb{C}^m)} = \frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix} \langle \bar{\eta}_0(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \dots & \langle \bar{\eta}_0(e^{i\theta}), \psi(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ \langle \bar{\eta}_1(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \dots & \langle \bar{\eta}_1(e^{i\theta}), \psi(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ \vdots & \ddots & \vdots \\ \langle \Phi(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & \dots & \langle \Phi(e^{i\theta}), \psi(e^{i\theta}) \rangle_{\mathbb{C}^m} \end{pmatrix} d\theta$$

for all $\psi \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$. Recall that, by Proposition 3.2.1, the set $\{\eta_i\}_{i=0}^j$ is an orthonormal set in $\mathbb{C}^m$ almost everywhere on $\mathbb{T}$. Hence

$$\langle \bar{\eta}_{(j)} \dot{\wedge} \Phi, \bar{\eta}_{(j)} \dot{\wedge} \psi \rangle_{L^2(\mathbb{T}, \wedge^{j+2}\mathbb{C}^m)} = \frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix} 1 & 0 & \dots & \langle \bar{\eta}_0(e^{i\theta}), \psi(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ 0 & 1 & \dots & \langle \bar{\eta}_1(e^{i\theta}), \psi(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ \vdots & & \ddots & \vdots \\ \langle \Phi(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & & \dots & \langle \Phi(e^{i\theta}), \psi(e^{i\theta}) \rangle_{\mathbb{C}^m} \end{pmatrix} d\theta.$$

Multiplying the k-th column with $\langle \bar{\eta}_k(e^{i\theta}), \psi(e^{i\theta}) \rangle_{\mathbb{C}^m}$ and adding it to the last column of the determinant above, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ \langle \Phi(e^{i\theta}), \bar{\eta}_0(e^{i\theta}) \rangle_{\mathbb{C}^m} & & \dots & \begin{matrix} \langle \Phi(e^{i\theta}), \psi(e^{i\theta}) \rangle_{\mathbb{C}^m} \\ -\sum_{i=0}^j \langle \Phi(e^{i\theta}), \bar{\eta}_i(e^{i\theta}) \rangle_{\mathbb{C}^m} \langle \bar{\eta}_i(e^{i\theta}), \psi(e^{i\theta}) \rangle_{\mathbb{C}^m} \end{matrix} \end{pmatrix} d\theta,$$

which is equal to

$$\frac{1}{2\pi} \int_0^{2\pi} \psi^*(e^{i\theta}) \Phi(e^{i\theta}) - \sum_{i=0}^j \psi^*(e^{i\theta}) \bar{\eta}_i(e^{i\theta}) \eta_i^T(e^{i\theta}) \Phi(e^{i\theta}) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \psi^*(e^{i\theta}) \left(I_m - \sum_{i=0}^j \bar{\eta}_i(e^{i\theta}) \eta_i^T(e^{i\theta}) \right) \Phi(e^{i\theta}) d\theta.$$

Then

$$\langle \bar{\eta}_{(j)} \dot{\wedge} \Phi, \bar{\eta}_{(j)} \dot{\wedge} \psi \rangle_{L^2(\mathbb{T}, \wedge^{j+2}\mathbb{C}^m)} = 0$$

for all $\psi \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ if and only if

$$\frac{1}{2\pi} \int_0^{2\pi} \left\langle \left(I_m - \sum_{i=0}^j \bar{\eta}_i(e^{i\theta}) \eta_i^T(e^{i\theta}) \right) \Phi(e^{i\theta}), \psi(e^{i\theta}) \right\rangle_{\mathbb{C}^m} = 0$$

for all $\psi \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$, which holds if and only if

$$\left(I_m - \sum_{i=0}^j \bar{\eta}_i \eta_i^T \right) \Phi \in H^2(\mathbb{D}, \mathbb{C}^m).$$

Notice that $W_0, \cdots, W_j$ being unitary-valued, implies $W_0^* \cdots W_j^* W_j \cdots W_0 = I_m$, or equivalently,

$$B_j B_j^* + \sum_{i=0}^{j} B_{i-1} B_{i-1}^* \bar{\eta}_i \eta_i^T B_{i-1} B_{i-1}^* = I_m,$$

which, by Lemma 3.2.46, is equivalent to the following equation

$$B_j B_j^* = I_m - \sum_{i=0}^{j} \bar{\eta}_i \eta_i^T.$$

Thus

$$\langle \bar{\eta}_{(j)} \dot{\wedge} \Phi, \bar{\eta}_{(j)} \dot{\wedge} \psi \rangle_{L^2(\mathbb{T}, \wedge^{j+2}\mathbb{C}^m)} = 0$$

for all $\psi \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ if and only if

$$\frac{1}{2\pi} \int_0^{2\pi} \langle B_j(e^{i\theta}) B_j^*(e^{i\theta}) \Phi(e^{i\theta}), \psi(e^{i\theta}) \rangle_{\mathbb{C}^m} = 0,$$

which holds if and only if $B_j B_j^* \Phi \in H^2(\mathbb{D}, \mathbb{C}^m)$, which is true by Proposition 3.2.53. Hence diagram (3.190) commutes.

Our quest now is to associate the operator Γ_{j+1} with a compact operator in order to reach the conclusion that T_{j+1} is also compact. Recall that, by Lemma 3.1.17, the following diagram also commutes

$$\begin{array}{ccc} H^2(\mathbb{D}, \mathbb{C}^{n-j-1}) & \xrightarrow{M_{\bar{A}_j}} & \mathcal{K}_{j+1} \\ \Big\downarrow H_{F_{j+1}} & & \Big\downarrow \Gamma_{j+1} \\ H^2(\mathbb{D}, \mathbb{C}^{m-j-1})^\perp & \xrightarrow{M_{B_j}} & \mathcal{L}_{j+1}. \end{array} \tag{3.195}$$

(iv). Since $F_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{(n-j-1)\times(m-j-1)}) + C(\mathbb{T}, \mathbb{C}^{(n-j-1)\times(m-j-1)})$, by Hartman's Theorem, the Hankel operator $H_{F_{j+1}}$ is compact, hence the operator Γ_{j+1} is compact. Since diagram (3.195) commutes and the operators $M_{\bar{A}_j}$ and M_{B_j} are unitaries, Γ_{j+1} is compact. By (iii),

$$(\bar{\eta}_0 \dot{\wedge} \ldots \dot{\wedge} \bar{\eta}_j \dot{\wedge} \cdot) \circ (M_{B_j} \circ H_{F_{j+1}} \circ M_{\bar{A}_j}^*) \circ (\xi_0 \dot{\wedge} \ldots \dot{\wedge} \xi_j \dot{\wedge} \cdot)^* = T_{j+1}.$$

By (i) and (ii), the operators $M_{\bar{A}_j}$, M_{B_j}, $(\xi_0 \dot{\wedge} \ldots \dot{\wedge} \xi_j \dot{\wedge} \cdot)$ and $(\bar{\eta}_0 \dot{\wedge} \ldots \dot{\wedge} \bar{\eta}_j \dot{\wedge} \cdot)$ are unitaries, Hence T_{j+1} is a compact operator.

(v). Since diagram (3.186) commutes and the operators $M_{\bar{A}_j}$, M_{B_j}, $(\xi_0 \dot{\wedge} \ldots \dot{\wedge} \xi_j \dot{\wedge} \cdot)$ and $(\bar{\eta}_0 \dot{\wedge} \ldots \dot{\wedge} \bar{\eta}_j \dot{\wedge} \cdot)$ are unitaries,

$$\|T_{j+1}\| = \|\Gamma_{j+1}\| = \|H_{F_{j+1}}\| = t_{j+1}.$$ □

Lemma 3.2.55. *Let $v_{j+1} \in H^2(\mathbb{D}, \mathbb{C}^n)$ and $w_{j+1} \in H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ be such that*

$$(\xi_0 \dot{\wedge} \cdots \dot{\wedge} \xi_j \dot{\wedge} v_{j+1}, \bar{\eta}_0 \dot{\wedge} \cdots \dot{\wedge} \bar{\eta}_j \dot{\wedge} w_{j+1})$$

is a Schmidt pair for the operator T_{j+1} *corresponding to* $\|T_{j+1}\|$. *Then*
(i) there exist $x_{j+1} \in \mathcal{K}_{j+1}$ *and* $y_{j+1} \in \mathcal{L}_{j+1}$ *such that* (x_{j+1}, y_{j+1}) *is a Schmidt pair for the operator* Γ_{j+1}.
(ii). For any $x_{j+1} \in \mathcal{K}_{j+1}$ *and* $y_{j+1} \in \mathcal{L}_{j+1}$ *such that*

$$\xi_0 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge x_{j+1} = \xi_0 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge v_{j+1}, \quad \bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge y_{j+1} = \bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge w_{j+1},$$

the pair (x_{j+1}, y_{j+1}) *is a Schmidt pair for* Γ_{j+1} *corresponding to* $\|\Gamma_{j+1}\|$.

Proof. (i). By Theorem 3.2.54, the diagram (3.190) commutes, $(\xi_0 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge \cdot)$ is unitary from $\mathcal{K}_{j+1}$ to X_{j+1}, and $(\bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge \cdot)$ is unitary from $\mathcal{L}_{j+1}$ to Y_{j+1}. Thus

$$\|\Gamma_{j+1}\| = \|T_{j+1}\| = t_{j+1}.$$

Moreover, the operator $\Gamma_{j+1} \colon \mathcal{K}_{j+1} \to \mathcal{L}_{j+1}$ is compact, hence there exist $x_{j+1} \in \mathcal{K}_{j+1}$, $y_{j+1} \in \mathcal{L}_{j+1}$ such that (x_{j+1}, y_{j+1}) is a Schmidt pair for Γ_{j+1} corresponding to $\|\Gamma_{j+1}\| = t_{j+1}$.
(ii). Suppose that $x_{j+1} \in \mathcal{K}_{j+1}, y_{j+1} \in \mathcal{L}_{j+1}$ satisfy

$$\xi_0 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge x_{j+1} = \xi_0 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge v_{j+1}, \tag{3.196}$$

$$\bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge y_{j+1} = \bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge w_{j+1}. \tag{3.197}$$

Let us show that (x_{j+1}, y_{j+1}) is a Schmidt pair for Γ_{j+1}, that is,

$$\Gamma_{j+1} x_{j+1} = t_{j+1} y_{j+1}, \quad \Gamma^*_{j+1} y_{j+1} = t_{j+1} x_{j+1}.$$

Since diagram (3.190) commutes,

$$T_{j+1} \circ (\xi_0 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge \cdot) = (\bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge \cdot) \circ \Gamma_{j+1} \tag{3.198}$$

and

$$(\xi_0 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge \cdot)^* \circ T^*_{j+1} = \Gamma^*_{j+1} \circ (\bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge \cdot)^*. \tag{3.199}$$

By hypothesis,

$$T_{j+1}(\xi_0 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge v_{j+1}) = t_{j+1}(\bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge w_{j+1}) \tag{3.200}$$

and

$$T^*_{j+1}(\bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge w_{j+1}) = t_{j+1}(\xi_0 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge v_{j+1}). \tag{3.201}$$

Thus, by equations (3.196), (3.197) and (3.200),

$$\begin{aligned}
\Gamma_{j+1} x_{j+1} &= (\bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge \cdot)^* T_{j+1}(\xi_0 \dot\wedge \cdots \dot\wedge \xi_j \dot\wedge v_{j+1}) \\
&= (\bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge \cdot)^* \circ t_{j+1}(\bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge w_{j+1}) \\
&= t_{j+1}(\bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge \cdot)^* \circ (\bar\eta_0 \dot\wedge \cdots \dot\wedge \bar\eta_j \dot\wedge y_{j+1}).
\end{aligned}$$

Hence

$$\Gamma_{j+1}x_{j+1} = t_{j+1}(\bar{\eta}_0\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_j\dot{\wedge}\cdot)^*(\bar{\eta}_0\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_j\dot{\wedge}\cdot)y_{j+1} = t_{j+1}y_{j+1}.$$

By equation (3.196),

$$x_{j+1} = (\xi_0\dot{\wedge}\cdots\dot{\wedge}\xi_j\dot{\wedge}\cdot)^*(\xi_0\dot{\wedge}\cdots\dot{\wedge}\xi_j\dot{\wedge}v_{j+1}),$$

and, by equation (3.197),

$$(\bar{\eta}_0\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_j\dot{\wedge}\cdot)^*(\bar{\eta}_0\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_j\dot{\wedge}w_{j+1}) = y_{j+1}.$$

Thus

$$\begin{aligned}\Gamma^*_{j+1}y_{j+1} &= \Gamma^*_{j+1}\circ(\bar{\eta}_0\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_j\dot{\wedge}\cdot)^*(\bar{\eta}_0\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_j\dot{\wedge}w_{j+1})\\ &= (\xi_0\dot{\wedge}\cdots\dot{\wedge}\xi_j\cdot)^*\circ T^*_{j+1}(\bar{\eta}_0\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_j\dot{\wedge}w_{j+1}),\end{aligned}$$

last equality following by equation (3.199). By equations (3.196) and (3.200), we get

$$T^*_{j+1}(\bar{\eta}_0\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_j\dot{\wedge}w_{j+1}) = t_{j+1}(\xi_0\dot{\wedge}\cdots\dot{\wedge}\xi_j\dot{\wedge}v_{j+1}) = t_{j+1}(\xi_0\dot{\wedge}\cdots\dot{\wedge}\xi_j\dot{\wedge}x_{j+1}),$$

and so,

$$\Gamma^*_{j+1}y_{j+1} = t_{j+1}x_{j+1}.$$

Therefore (x_{j+1}, y_{j+1}) is a Schmidt pair for Γ_{j+1} corresponding to $\|\Gamma_{j+1}\| = t_{j+1}$. □

Lemma 3.2.56. *Suppose that*

$$(\xi_0\dot{\wedge}\cdots\dot{\wedge}\xi_j\dot{\wedge}v_{j+1}, \bar{\eta}_0\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_j\dot{\wedge}w_{j+1})$$

is a Schmidt pair for the operator T_{j+1} corresponding to $\|T_{j+1}\| = t_{j+1}$. Let

$$x_{j+1} = (I_{\mathbb{C}^n} - \xi_0\xi_0^* - \cdots - \xi_j\xi_j^*)v_{j+1},$$

$$y_{j+1} = (I_{\mathbb{C}^m} - \bar{\eta}_0\eta_0^T - \cdots - \bar{\eta}_j\eta_j^T)w_{j+1},$$

and let

$$\hat{x}_{j+1} = A_j^T x_{j+1}, \quad \hat{y}_{j+1} = B_j^* y_{j+1}.$$

Then the pair $(\hat{x}_{j+1}, \hat{y}_{j+1})$ is a Schmidt pair for $H_{F_{j+1}}$ corresponding to $\|H_{F_{j+1}}\| = t_{j+1}$.

Proof. Let us first show that $\hat{x}_{j+1} \in H^2(\mathbb{D}, \mathbb{C}^{n-j-1})$ and $x_{j+1} \in \mathcal{K}_{j+1}$. Recall that, for $i = 0, \cdots, j$, V_0 and $\tilde{V}_i$ are unitary-valued, that is $\alpha_0^T\xi_0 = 0$ and $A_i^T\xi_i = 0$. Hence we have

$$\begin{aligned}\hat{x}_{j+1} &= A_j^T x_{j+1}\\ &= A_j^T(I_n - \xi_0\xi_0^* - \cdots - \xi_j\xi_j^*)v_{j+1}\\ &= A_j^T v_{j+1} - A_j^T\xi_0\xi_0^* v_{j+1} - \cdots - A_j^T\xi_j\xi_j^* v_{j+1}\\ &= A_j^T v_{j+1}, \end{aligned} \tag{3.202}$$

which implies that $\hat{x}_2 \in H^2(\mathbb{D}, \mathbb{C}^{n-2})$. By Lemma 3.2.45,

$$\bar{A}_j \hat{x}_{j+1} = \bar{A}_j A_j^T v_{j+1} = x_{j+1}, \tag{3.203}$$

and thus $x_{j+1} \in \mathcal{K}_{j+1}$.

Next, we shall show that and $\hat{y}_{j+1} \in H^2(\mathbb{D}, \mathbb{C}^{n-j-1})^\perp$ and $y_{j+1} \in \mathcal{L}_{j+1}$. Notice that, for all $i = 1, \cdots, j$, W_0 and $\tilde{W}_i$ are unitary valued, that is $\beta_0^* \bar{\eta}_0 = 0$ and $B_i^* \bar{\eta}_i = 0$. Then we have

$$\begin{aligned} \hat{y}_{j+1} &= B_j^* y_{j+1} \\ &= B_j^* (I_m - \bar{\eta}_0 \eta_0^T - \cdots - \bar{\eta}_j \eta_j^T) w_{j+1} \\ &= B_j^* w_{j+1} - B_j^* \bar{\eta}_0 \eta_0^T w_{j+1} - \cdots - B_j^* \bar{\eta}_j \eta_j^T w_{j+1} \\ &= B_j^* w_{j+1}, \end{aligned} \tag{3.204}$$

which implies that $\hat{y}_{j+1} \in H^2(\mathbb{D}, \mathbb{C}^{m-j-1})^\perp$.

By Lemma 3.2.46,

$$B_j \hat{y}_{j+1} = B_j B_j^* w_{j+1} = y_{j+1}, \tag{3.205}$$

and hence $y_{j+1} \in \mathcal{L}_{j+1}$.

Recall that, by Theorem 3.2.54, the maps

$$M_{\bar{A}_j} \colon H^2(\mathbb{D}, \mathbb{C}^{n-j-1}) \to \mathcal{K}_{j+1}, \quad M_{B_j} \colon H^2(\mathbb{D}, \mathbb{C}^{m-j-1})^\perp \to \mathcal{L}_{j+1},$$

are unitaries and

$$H_{F_{j+1}} = M_{B_j}^* \circ \Gamma_{j+1} \circ M_{\bar{A}_j}. \tag{3.206}$$

Furthermore, by Proposition 3.2.55,

$$\Gamma_{j+1} x_{j+1} = t_{j+1} y_{j+1}, \quad \Gamma_{j+1}^* y_{j+1} = t_{j+1} x_{j+1}. \tag{3.207}$$

We need to show that

$$H_{F_{j+1}} \hat{x}_{j+1} = t_{j+1} \hat{y}_{j+1}, \quad H_{F_{j+1}}^* \hat{y}_{j+1} = t_{j+1} \hat{x}_{j+1}.$$

By equation (3.206), we have

$$\begin{aligned} H_{F_{j+1}} \hat{x}_{j+1} &= H_{F_{j+1}} A_j^T x_{j+1} \\ &= B_j^* \Gamma_{j+1} \bar{A}_j A_j^T x_{j+1} \end{aligned} \tag{3.208}$$

Notice that, by equations (3.202) and (3.203),

$$x_{j+1} = \bar{A}_j A_j^T x_{j+1}. \tag{3.209}$$

Hence, by equations (3.207) and (3.208), we obtain

$$H_{F_{j+1}}\hat{x}_{j+1} = B_j^*\Gamma_{j+1}x_{j+1} = t_{j+1}B_j^*y_{j+1} = t_{j+1}\hat{y}_{j+1}.$$

Let us show that $H^*_{F_{j+1}}\hat{y}_{j+1} = t_{j+1}\hat{x}_{j+1}$. For $\hat{y}_{j+1} = B_j^*y_{j+1}$ and by equation (3.207), we get

$$\begin{aligned} H^*_{F_{j+1}}\hat{y}_{j+1} &= H^*_{F_{j+1}}B_j^*y_{j+1} \\ &= A^T_{j+1}\Gamma^*_{j+1}B_jB_j^*y_{j+1} \end{aligned} \tag{3.210}$$

Observe that, in view of equations (3.204) and (3.205), we have

$$y_{j+1} = B_jB_j^*y_{j+1}. \tag{3.211}$$

Hence, by equations (3.207) and (3.210), we obtain

$$H^*_{F_{j+1}}\hat{y}_{j+1} = A^T_{j+1}\Gamma^*_{j+1}y_{j+1} = t_{j+1}A^T_{j+1}x_{j+1} = t_{j+1}\hat{x}_{j+1}.$$

Therefore $(\hat{x}_{j+1}, \hat{y}_{j+1})$ is a Schmidt pair for the Hankel operator $H_{F_{j+1}}$ corresponding to $\|H_{F_{j+1}}\| = t_{j+1}$. □

Proposition 3.2.57. *Let*

$$(\xi_0\dot\wedge\cdots\dot\wedge\xi_j\dot\wedge v_{j+1}, \bar{\eta}_0\dot\wedge\cdots\dot\wedge\bar{\eta}_j\dot\wedge w_{j+1})$$

be a Schmidt pair for T_{j+1} corresponding to t_{j+1} for some $v_{j+1} \in H^2(\mathbb{D},\mathbb{C}^n)$, $w_{j+1} \in H^2(\mathbb{D},\mathbb{C}^m)^\perp$. Let

$$x_{j+1} = (I_n - \xi_0\xi_0^* - \cdots - \xi_j\xi_j^*)v_{j+1}, \quad y_{j+1} = (I_m - \bar{\eta}_0\eta_0^T - \cdots - \bar{\eta}_j\eta_j^T)w_{j+1},$$

and let

$$\hat{x}_{j+1} = A_j^Tx_{j+1} \quad \textit{and} \quad \hat{y}_{j+1} = B_j^*y_{j+1}. \tag{3.212}$$

Then

$$\begin{aligned} \|\xi_0(z)\wedge\ldots\wedge\xi_j(z)\wedge v_{j+1}(z)\|_{\wedge^{j+2}\mathbb{C}^n} &= \|\bar{\eta}_0(z)\wedge\ldots\wedge\bar{\eta}_j(z)\wedge w_{j+1}(z)\|_{\wedge^{j+2}\mathbb{C}^m} = |h_{j+i}(z)|, \\ \|\hat{x}_{j+1}(z)\|_{\mathbb{C}^{n-j-1}} &= \|\hat{y}_{j+1}(z)\|_{\mathbb{C}^{m-j-1}} = |h_{j+1}(z)|, \text{ and} \\ \|x_{j+1}(z)\|_{\mathbb{C}^n} &= \|y_{j+1}(z)\|_{\mathbb{C}^m} = |h_{j+1}(z)|, \end{aligned} \tag{3.213}$$

almost everywhere on $\mathbb{T}$.

Proof. By Lemma 3.2.56, $(\hat{x}_{j+1}, \hat{y}_{j+1})$ is a Schmidt pair for $H_{F_{j+1}}$ corresponding to $\|H_{F_{j+1}}\| = t_{j+1}$. Hence

$$H_{F_{j+1}}\hat{x}_{j+1} = t_{j+1}\hat{y}_{j+1} \quad \text{and} \quad H^*_{F_{j+1}}\hat{y}_{j+1} = t_{j+1}\hat{x}_{j+1}.$$

By Theorem D.2.4,

$$t_{j+1}\|\hat{y}_{j+1}(z)\|_{\mathbb{C}^{m-j-1}} = \|H_{F_{j+1}}\|\|\hat{x}_{j+1}(z)\|_{\mathbb{C}^{n-j-1}}$$

almost everywhere on $\mathbb{T}$. Thus

$$\|\hat{y}_{j+1}(z)\|_{\mathbb{C}^{m-j-1}} = \|\hat{x}_{j+1}(z)\|_{\mathbb{C}^{n-j-1}} \tag{3.214}$$

almost everywhere on $\mathbb{T}$.

Notice that, $\bar{A}_j(z)$ are isometric for almost every $z \in \mathbb{T}$, and therefore, by equations (3.212), we obtain

$$\|x_{j+1}(z)\|_{\mathbb{C}^n} = \|\hat{x}_{j+1}(z)\|_{\mathbb{C}^{n-j-1}}.$$

Moreover, since $B_j(z)$ are isometries almost everywhere on $\mathbb{T}$, by equations (3.212), we get

$$\|y_{j+1}(z)\|_{\mathbb{C}^m} = \|\hat{y}_{j+1}(z)\|_{\mathbb{C}^{m-j-1}}$$

almost everywhere on $\mathbb{T}$. By equations (3.214), we deduce

$$\|x_{j+1}(z)\|_{\mathbb{C}^n} = \|y_{j+1}(z)\|_{\mathbb{C}^m} \tag{3.215}$$

almost everywhere on $\mathbb{T}$.

By Proposition 3.2.1,

$$\xi_0\dot{\wedge}\cdots\dot{\wedge}\xi_j\dot{\wedge}x_{j+1} = \xi_0\dot{\wedge}\cdots\dot{\wedge}\xi_j\dot{\wedge}v_{j+1} \tag{3.216}$$

and

$$\bar{\eta}_0\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_j\dot{\wedge}y_{j+1} = \bar{\eta}_0\dot{\wedge}\cdots\dot{\wedge}\bar{\eta}_j\dot{\wedge}w_{j+1}. \tag{3.217}$$

Hence, by Proposition 2.1.22,

$$\begin{aligned}
&\|\xi_0(z)\wedge\cdots\wedge\xi_j(z)\wedge v_{j+1}(z)\|_{\wedge^{j+2}\mathbb{C}^n}\\
&= \|\xi_0(z)\wedge\cdots\wedge\xi_j(z)\wedge x_{j+1}(z)\|_{\wedge^{j+2}\mathbb{C}^n},\\
&= \|x_{j+1}(z) - \sum_{i=0}^{j}\langle x_{j+1}(z),\xi_i(z)\rangle\xi_i(z)\|_{\mathbb{C}^n} = \|x_{j+1}(z)\|_{\mathbb{C}^n},
\end{aligned}$$

almost everywhere on $\mathbb{T}$. Furthermore

$$\begin{aligned}
\|\bar{\eta}_0(z)\wedge\cdots\wedge\bar{\eta}_j(z)\wedge w_{j+1}(z)\|_{\wedge^{j+2}\mathbb{C}^m} &= \|\bar{\eta}_0(z)\wedge\cdots\wedge\bar{\eta}_j(z)\wedge y_{j+1}(z)\|_{\wedge^{j+2}\mathbb{C}^m}\\
&= \|y_{j+1}(z) - \sum_{i=0}^{j}\langle y_{j+1}(z),\bar{\eta}_i(z)\rangle\bar{\eta}_i(z)\|_{\mathbb{C}^m} = \|y_{j+1}(z)\|_{\mathbb{C}^m}
\end{aligned}$$

almost everywhere on $\mathbb{T}$. Thus, by equation (3.215),

$$\|\bar{\eta}_0(z)\wedge\cdots\wedge\bar{\eta}_j(z)\wedge w_{j+1}(z)\|_{\wedge^{j+2}\mathbb{C}^m} = \|\xi_0(z)\wedge\cdots\wedge\xi_j(z)\wedge v_{j+1}(z)\|_{\wedge^{j+2}\mathbb{C}^n}$$

almost everywhere on $\mathbb{T}$.

Recall that h_{j+1} is the scalar outer factor of $\xi_0 \wedge \cdots \wedge \xi_j \wedge v_{j+1}$. Hence

$$\|\hat{x}_{j+1}(z)\|_{\mathbb{C}^{n-j-1}} = \|\hat{y}_{j+1}(z)\|_{\mathbb{C}^{m-j-1}} = |h_{j+1}(z)|,$$

$$\|x_{j+1}(z)\|_{\mathbb{C}^n} = \|y_{j+1}(z)\|_{\mathbb{C}^m} = |h_{j+1}(z)|,$$

and

$$\|\xi_0(z) \wedge \cdots \wedge \xi_j(z) \wedge v_{j+1}(z)\|_{\wedge^{j+2}\mathbb{C}^n} = \|\bar{\eta}_0(z) \wedge \cdots \wedge \bar{\eta}_j(z) \wedge w_{j+1}(z)\|_{\wedge^{j+2}\mathbb{C}^m}\| = |h_{j+1}(z)|,$$

almost everywhere on $\mathbb{T}$. □

Proposition 3.2.58. *In the notation of Theorem 3.2.54, there exist unitary-valued functions $\tilde{V}_{j+1}, \tilde{W}_{j+1}$ of types $(n-j-1) \times (n-j-2)$ and $(m-j-1) \times (m-j-2)$ respectively of the form*

$$\tilde{V}_{j+1} = \begin{pmatrix} A_j \xi_{j+1} & \bar{\alpha}_{j+1} \end{pmatrix}, \quad \tilde{W}^T_{j+1} = \begin{pmatrix} B_j^T \eta_{j+1} & \bar{\beta}_{j+1} \end{pmatrix},$$

where $\alpha_{j+1}, \beta_{j+1}$ are inner, co-outer, quasi-continuous and all minors on the first columns of $\tilde{V}_{j+1}, \tilde{W}^T_{j+1}$ are in H^∞. Furthermore, the set $\mathcal{E}_{j+1}$ of all level $j+1$ superoptimal error functions for G is equal to the following set

$$W_0^* \cdots \begin{pmatrix} I_{j+1} & 0 \\ 0 & \tilde{W}^*_{j+1} \end{pmatrix} \begin{pmatrix} t_0 u_0 & 0 & 0 & 0 \\ 0 & t_1 u_1 & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & t_{j+1} u_{j+1} & 0 \\ 0 & 0 & 0 & (F_{j+2} + H^\infty) \cap B(t_{j+1}) \end{pmatrix} \begin{pmatrix} I_{j+1} & 0 \\ 0 & \tilde{V}^*_{j+1} \end{pmatrix} \cdots V_0^*,$$

where $F_{j+2} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-2)\times(n-j-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-j-2)\times(n-j-2)})$, $u_{j+1} = \frac{\bar{z}\bar{h}_{j+1}}{h_{j+1}}$ is a quasi-continuous unimodular function and $B(t_{j+1})$ is the closed ball of radius t_{j+1} in $L^\infty(\mathbb{T}, \mathbb{C}^{(m-j-2)\times(n-j-2)})$.

Proof. Recall that, in diagrams (3.190) and (3.195), the operators $M_{\bar{A}_j}$, M_{B_j}, $(\xi_0 \dot{\wedge} \cdots \dot{\wedge} \xi_j \cdot)$ and $(\bar{\eta}_0 \dot{\wedge} \cdots \dot{\wedge} \bar{\eta}_j \dot{\wedge} \cdot)$ are unitaries. Since both diagrams commute and (x_{j+1}, y_{j+1}) defined above is a Schmidt pair for Γ_{j+1} corresponding to t_{j+1}, by Lemma 3.2.56, $(\hat{x}_{j+1}, \hat{y}_{j+1})$ is a Schmidt pair for $H_{F_{j+1}}$ corresponding to t_{j+1}, where

$$\hat{x}_{j+1} = A_j x_{j+1}, \quad \hat{y}_{j+1} = B_j^* y_{j+1}.$$

We would like to apply Lemma 3.1.12 to $H_{F_{j+1}}$ and the Schmidt pair $(\hat{x}_{j+1}, \hat{y}_{j+1})$ to find unitary-valued functions $\tilde{V}_{j+1}, \tilde{W}_{j+1}$ such that, for every $\tilde{Q}_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-1)\times(n-j-1)})$ which is at minimal distance from F_{j+1}, we obtain a factorisation of the form

$$F_{j+1} - \tilde{Q}_{j+1} = \tilde{W}^*_{j+1} \begin{pmatrix} t_{j+1} u_{j+1} & 0 \\ 0 & F_{j+2} \end{pmatrix} \tilde{V}^*_{j+1},$$

for some $F_{j+2} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-2)\times(n-j-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-j-2)\times(n-j-2)})$. For this purpose we find the inner-outer factorisations of $\hat{x}_{j+1}$ and $\bar{z}\bar{\hat{y}}_{j+1}$.

By Proposition 3.2.57,

$$\|\hat{x}_{j+1}(z)\|_{\mathbb{C}^{n-j-1}} = |h_{j+1}(z)| \tag{3.218}$$

and

$$\|\hat{y}_{j+1}(z)\|_{\mathbb{C}^{m-j-1}} = |h_{j+1}(z)| \tag{3.219}$$

almost everywhere on $\mathbb{T}$. Equations (3.218) and (3.219) imply that $h_{j+1} \in H^2(\mathbb{D}, \mathbb{C})$ is the scalar outer factor of both $\hat{x}_{j+1}$ and $\bar{z}\bar{\hat{y}}_{j+1}$.

Hence, by Lemma 3.1.12, $\hat{x}_{j+1}, \bar{z}\bar{\hat{y}}_{j+1}$ admit the inner outer factorisations

$$\hat{x}_{j+1} = \hat{\xi}_{j+1} h_{j+1}, \quad \bar{z}\bar{\hat{y}}_{j+1} = \hat{\eta}_{j+1} h_{j+1},$$

for some inner $\hat{\xi}_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{n-j-1}), \hat{\eta}_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{m-j-1})$. Then

$$\hat{x}_{j+1} = \hat{\xi}_{j+1} h_{j+1} = A_j^T x_{j+1}, \quad \bar{z}\bar{\hat{y}}_{j+1} = \hat{\eta}_{j+1} h_{j+1} = \bar{z} B_j^T \bar{y}_{j+1},$$

from which we obtain

$$\hat{\xi}_{j+1} = A_j^T \xi_{j+1}, \quad \hat{\eta}_{j+1} = B_j^T \eta_{j+1}.$$

We would like to show that $A_j^T \xi_{j+1},\ B_j^T \eta_{j+1}$ are inner functions in order to apply Lemma 3.1.12 and obtain $\tilde{V}_{j+1}$ and $\tilde{W}_{j+1}$. Observe that, by equations (3.203), (3.205), (3.209) and (3.211),

$$x_{j+1} = \bar{A}_j A_j^T v_{j+1}, \quad y_{j+1} = B_j B_j^* w_{j+1}.$$

Then

$$A_j^T x_{j+1} = A_j^T v_{j+1}, \quad B_j^T \bar{y}_{j+1} = B_j^T \bar{w}_{j+1},$$

and since

$$\xi_{j+1} = \frac{x_{j+1}}{h_{j+1}}, \quad \eta_{j+1} = \frac{\bar{z}\bar{y}_{j+1}}{h_{j+1}},$$

we get that the functions

$$A_j^T \xi_{j+1} = \frac{A_j^T v_{j+1}}{h_{j+1}}, \quad B_j^T \eta_{j+1} = \frac{B_j^* w_{j+1}}{h_2}$$

are analytic. Furthermore, by Proposition 3.2.1 $\|\xi_{j+1}(z)\|_{\mathbb{C}^n} = 1$ and $\|\eta_{j+1}(z)\|_{\mathbb{C}^m} = 1$ almost everywhere on $\mathbb{T}$, and, by equations (3.218),

$$\|A_j^T(z)\xi_{j+1}(z)\|_{\mathbb{C}^{n-j-1}} = 1, \quad \|B_j^T(z)\eta_{j+1}(z)\|_{\mathbb{C}^{m-j-1}} = 1$$

almost everywhere on $\mathbb{T}$. Thus $A_j^T \xi_{j+1},\ B_j^T \eta_{j+1}$ are inner functions.

By Lemma 3.1.12, there exist inner, co-outer, quasi-continuous functions $\alpha_{j+1}, \beta_{j+1}$ of types $(n-j-1)\times(n-j-2), (m-j-1)\times(m-j-2)$ respectively such that the functions

$$\tilde{V}_{j+1} = \begin{pmatrix} A_j^T \xi_{j+1} & \bar{\alpha}_{j+1} \end{pmatrix}, \quad \tilde{W}_{j+1}^T = \begin{pmatrix} B_j^T \eta_{j+1} & \bar{\beta}_{j+1} \end{pmatrix}$$

are unitary-valued with all minors on the first columns in H^∞.
Furthermore, by Lemma 3.1.12, every $\hat{Q}_{j+1} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-1)\times(n-j-1)})$ which is at minimal distance from F_{j+1} satisfies

$$F_{j+1} - \hat{Q}_{j+1} = \tilde{W}_{j+1}^* \begin{pmatrix} t_{j+1}u_{j+1} & 0 \\ 0 & F_{j+2} \end{pmatrix} \tilde{V}_{j+1}^*,$$

for some $F_{j+2} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-2)\times(n-j-2)}) + C(\mathbb{T}, \mathbb{C}^{(m-j-2)\times(n-j-2)})$ and u_{j+1} a quasi-continuous unimodular function given by $u_{j+1} = \frac{\bar{z}\bar{h}_{j+1}}{h_{j+1}}$.

By Lemma 3.1.15, the set

$$\tilde{\mathcal{E}}_{j+1} = \{F_{j+1} - \hat{Q} : \hat{Q} \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j-1)\times(n-j-1)}), \|F_{j+1} - \hat{Q}\|_{L^\infty} = t_{j+1}\}$$

satisfies

$$\tilde{\mathcal{E}}_{j+1} = \tilde{W}_{j+1}^* \begin{pmatrix} t_{j+1}u_{j+1} & 0 \\ 0 & (F_{j+2} + H^\infty) \cap B(t_{j+1}) \end{pmatrix} V_{j+1}^*,$$

where $B(t_{j+1})$ is the closed ball of radius t_{j+1} in $L^\infty(\mathbb{T}, \mathbb{C}^{(m-j-2)\times(n-j-2)})$. Thus, by Proposition 3.2.43, $\mathcal{E}_{j+1}$ admits the factorisation claimed. □

Theorem 3.2.59. *Let $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n}) + C(\mathbb{T}, \mathbb{C}^{m\times n})$, where m, n are positive integers with $\min(m,n) \geq 2$. Let T_i, t_i, x_i, y_i, h_i, for $i \geq 0$, be defined by the algorithm from Section 3.2.1. Let r be the least index $j \geq 0$ such that $T_j = 0$. Then the superoptimal analytic approximant $\mathcal{A}G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ is equal to*

$$\mathcal{A}G = G - \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2}. \tag{3.220}$$

Proof. First observe that, if $T_0 = H_G = 0$, then this implies $G \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$, and so

$$\mathcal{A}G = G.$$

Otherwise, let $t_0 = \|H_G\| > 0$. If $T_1 = 0$, by Theorem 3.2.10, $H_{F_1} = 0$, that is,

$$F_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-1)\times(n-1)}).$$

Then, by Lemma 3.1.15, we get

$$W_0(G - \mathcal{A}G)V_0 = \begin{pmatrix} t_0u_0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Equivalently

$$\begin{aligned} G - \mathcal{A}G &= W_0^* \begin{pmatrix} t_0 u_0 & 0 \\ 0 & 0 \end{pmatrix} V_0^* \\ &= \begin{pmatrix} \bar{\eta}_0 & \beta_0 \end{pmatrix} \begin{pmatrix} t_0 u_0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_0^* \\ \alpha_0^T \end{pmatrix} \\ &= \begin{pmatrix} \bar{\eta}_0 t_0 u_0 & 0 \end{pmatrix} \begin{pmatrix} \xi_0^* \\ \alpha_0^T \end{pmatrix} \\ &= \bar{\eta}_0 t_0 u_0 \xi_0^* = t_0 \frac{z y_0}{\bar{h}_0} \frac{\bar{z}\bar{h}_0}{h_0} \frac{x_0^*}{\bar{h}_0} \\ &= \frac{t_0 y_0 x_0^*}{|h_0|^2}. \end{aligned}$$

Let j be a non-negative integer such that $T_j = 0$ and $T_i \neq 0$ for $1 \le i < j$. By the commutativity of the diagrams (3.190) and (3.195), $H_{F_j} = 0$, and therefore $F_j \in H^\infty(\mathbb{D}, \mathbb{C}^{(m-j)\times(n-j)})$. By Proposition 3.2.58, the superoptimal analytic approximant $\mathcal{A}G$ satisfies equation (3.150), that is,

$$G - \mathcal{A}G = W_0^* W_1^* \cdots W_{j-1}^* \begin{pmatrix} t_0 u_0 & 0 & & \dots & 0 \\ 0 & t_1 u_1 & & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & t_{j-1} u_{j-1} & 0 \\ 0 & \dots & & \dots & 0 \end{pmatrix} V_{j-1}^* \cdots V_1^* V_0^*, \tag{3.221}$$

where, for $i = 0, 1, \dots, j-1$,

$$\tilde{V}_i = \begin{pmatrix} \alpha_{i-1}^T \cdots \alpha_0^T \xi_i & \bar{\alpha}_i \end{pmatrix}, \quad \tilde{W}_i^T = \begin{pmatrix} \beta_{i-1}^T \cdots \beta_0^T \eta_i & \bar{\beta}_i \end{pmatrix}$$

are unitary-valued functions, as described in Proposition 3.149, $u_i = \frac{\bar{z}\bar{h}_i}{h_i}$ are quasi-continuous unimodular functions, and

$$V_i = \begin{pmatrix} I_i & 0 \\ 0 & \tilde{V}_i \end{pmatrix}, \quad W_i = \begin{pmatrix} I_i & 0 \\ 0 & \tilde{W}_i \end{pmatrix}.$$

Recall that, by equations (3.28), for $i = 0, \dots, j-1$,

$$\xi_i = \frac{x_i}{h_i}, \quad \eta_i = \frac{\bar{z}\bar{y}_i}{h_i}. \tag{3.222}$$

By Proposition 3.2.57, for $i = 0, \dots, j-1$,

$$|h_i(z)| = \|x_i(z)\|_{\mathbb{C}^n} = \|y_i(z)\|_{\mathbb{C}^m} \quad \text{almost everywhere on } \mathbb{T}.$$

Multiplication of matrices in (3.221) gives the following formula

$$G - \mathcal{A}G = \frac{t_0 y_0 x_0^*}{|h_0|^2} + t_1 \frac{1}{|h_1|^2} \beta_0 \beta_0^* y_1 x_1^* \bar{\alpha}_0 \alpha_0^T + \dots$$
$$+ t_{j-1} \frac{1}{|h_{j-1}|^2} \beta_0 \beta_1 \dots \beta_{j-1} \beta_{j-1}^* \dots \beta_1^* \beta_0^* y_{j-1} x_{j-1}^* \bar{\alpha}_0 \bar{\alpha}_1 \dots \bar{\alpha}_{j-1} \alpha_{j-1}^T \dots \alpha_1^T \alpha_0^T. \tag{3.223}$$

By equations (3.209) and (3.211), for $i = 0, \dots, j-1$,

$$x_i = \bar{\alpha}_0 \bar{\alpha}_1 \dots \bar{\alpha}_{i-1} \alpha_{i-1}^T \dots \alpha_0^T x_i \quad \text{and} \quad y_i = \beta_0 \beta_1 \dots \beta_{i-1} \beta_{i-1}^* \dots \beta_1^* \beta_0^* y_i.$$

Thus

$$G - \mathcal{A}G = \sum_{i=0}^{r-1} \frac{t_i y_i x_i^*}{|h_i|^2}$$

and the assertion has been proved. □

Chapter 4

Application of the algorithm

In this chapter we present the application of the algorithm from Section 3.2.1 to two concrete examples. The first is a trivial example and has been briefly explained in Chapter 1, however its simplicity provides the reader with the opportunity to understand how the steps we describe in Section 3.2.1 work. For our second example we choose the matrix-function that appeared in [25]. The solution we provide gives to a substantial illustration of the similarities and differences to the method used in [25].

Problem 4.0.1. *Find the superoptimal analytic approximant of*

$$G(z) = \begin{pmatrix} 2/z & 0 \\ 0 & 1/z \end{pmatrix}, \quad z \in \mathbb{T}.$$

Solution: Step 0. First, we find the Hankel operator with symbol G. This is

$$H_G\colon H^2(\mathbb{D}, \mathbb{C}^2) \to H^2(\mathbb{D}, \mathbb{C}^2)^\perp$$

and its matrix representation with respect to the orthonormal bases of $H^2(\mathbb{D}, \mathbb{C}^2)$ and $H^2(\mathbb{D}, \mathbb{C}^2)^\perp$, namely

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} z \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z \end{pmatrix}, \dots \right\}$$

and

$$\left\{ \begin{pmatrix} \bar{z} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \bar{z} \end{pmatrix}, \begin{pmatrix} \bar{z}^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \bar{z}^2 \end{pmatrix}, \dots \right\}$$

respectively, is

$$H_G = \begin{pmatrix} 2 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

all other entries being zero.

Thus

$$t_0 = \|H_G\| = \left\| \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right\| = 2.$$

The maximizing vectors $x_0 \in H^2(\mathbb{D}, \mathbb{C}^2)$ for H_G are those which satisfy

$$x_0 \neq 0 \quad \text{and} \quad \|H_G x_0\|_{H^2(\mathbb{D},\mathbb{C}^2)^\perp} = 2\|x_0\|_{H^2(\mathbb{D},\mathbb{C}^2)}.$$

Thus, a maximizing vector is $x_0(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $z \in \mathbb{D}$. Following that, we can find a vector $y_0 \in H^2(\mathbb{D}, \mathbb{C}^2)^\perp$ such that (x_0, y_0) is a Schmidt pair for H_G corresponding to the singular value $\|H_G\| = 2$, by solving the equations

$$H_G x_0 = 2y_0 \quad \text{and} \quad H_G^* y_0 = 2x_0. \tag{4.1}$$

Then,

$$\begin{aligned} H_G x_0 &= P_-(Gx_0) \\ &= P_- \begin{pmatrix} 2/z & 0 \\ 0 & 1/z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= P_- \begin{pmatrix} 2/z \\ 0 \end{pmatrix}, \end{aligned}$$

hence

$$y_0(z) = \begin{pmatrix} 1/z \\ 0 \end{pmatrix} \quad \text{for all} \quad z \in \mathbb{T}.$$

We can see that for these x_0, y_0 the other condition of (4.1) is also satisfied since

$$H_G^* y_0 = P_+ G^* y_0 = P_+ \left(\begin{pmatrix} \frac{2}{\bar{z}} & 0 \\ 0 & \frac{1}{\bar{z}} \end{pmatrix} \begin{pmatrix} \frac{1}{z} \\ 0 \end{pmatrix} \right) = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2x_0.$$

Also,

$$\|x_0(z)\|_{\mathbb{C}^2} = \|y_0(z)\|_{\mathbb{C}^2} = 1 \quad \text{almost everywhere on} \quad \mathbb{T}.$$

By Lemma 3.1.12, x_0 and y_0 admit the inner-outer factorisations

$$x_0 = \xi_0 h_0, \quad \bar{z}\bar{y}_0 = \eta_0 h_0, \tag{4.2}$$

for some inner $\xi_0, \eta_0 \in H^\infty(\mathbb{D}, \mathbb{C}^2)$ and some scalar outer $h_0 \in H^2(\mathbb{D}, \mathbb{C})$. Clearly, for almost all $z \in \mathbb{T}$

$$x_0(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \xi_0(z)h_0(z) \quad \text{and} \quad \bar{z}\bar{y}_0(z) = \bar{z}\begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \eta_0(z)h_0(z),$$

where

$$h_0(z) = 1, \quad \xi_0(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \eta_0(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Let us find a function $Q_0 \in H^\infty(\mathbb{D}, \mathbb{C}^{2\times 2})$ at minimal distance from G. Such a function, by Theorem D.2.4, necessarily satisfies

$$(G - Q_0)x_0 = 2y_0 \tag{4.3}$$

and

$$y_0^*(G - Q_0) = 2x_0^*. \tag{4.4}$$

Let us work each equation separately. Let

$$Q_0 = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \in H^\infty(\mathbb{D}, \mathbb{C}^{2\times 2}).$$

Equation (4.3) is equivalent to

$$Q_0 x_0 = Gx_0 - 2y_0.$$

Substituting from above, we get, for all $z \in \mathbb{D}$,

$$Q_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2/z & 0 \\ 0 & 1/z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 1/z \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives us

$$q_{11}(z) = q_{21}(z) = 0,$$

for all $z \in \mathbb{D}$. By equation (4.4), we get

$$y_0^* Q_0 = y_0^* G - 2x_0^*.$$

Substituting from above,

$$\begin{pmatrix} z & 0 \end{pmatrix} Q_0 = \begin{pmatrix} z & 0 \end{pmatrix} \begin{pmatrix} 2/z & 0 \\ 0 & 1/z \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix},$$

for all $z \in \mathbb{D}$. This yields $q_{12}(z) = 0$. Thus Q_0 is of the form

$$Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & q_{22} \end{pmatrix}, \quad \text{for some } q_{22} \in H^\infty(\mathbb{D}, \mathbb{C}).$$

For any such Q_0,

$$(G - Q_0)(z) = \begin{pmatrix} 2\bar{z} & 0 \\ 0 & \bar{z} - q_{22}(z) \end{pmatrix} \quad \text{for all } z \in \mathbb{T},$$

where $\bar{z} - q_{22}(z)$ must satisfy $\|\bar{z} - q_{22}\|_{L^\infty} \leq 2$, for Q_0 to be at minimal distance from G. It suffices to choose $q_{22} = 1$, since

$$\|G - Q_0\|_{L^\infty} = \left\| \begin{pmatrix} 2/z & 0 \\ 0 & \frac{1-z}{z} \end{pmatrix} \right\|_{L^\infty} = \max\left\{2, \operatorname*{ess\,sup}_{z\in\mathbb{T}} \left|\frac{1-z}{z}\right|\right\} = 2.$$

Thus we take

$$Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in H^\infty(\mathbb{D}, \mathbb{C}^{2\times 2}).$$

Step 1. Let

$$X_1 \stackrel{def}{=} \xi_0 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^2), \quad Y_1 \stackrel{def}{=} \bar\eta_0 \dot\wedge H^2(\mathbb{D}, \mathbb{C}^2)^\perp,$$

where ξ_0, η_0 are introduced in equation (4.2) and

$$\xi_0(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \eta_0(z) \quad \text{for all } z \in \mathbb{D}.$$

Define $T_1 \colon X_1 \to Y_1$ by

$$T_1\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \dot\wedge x\right) = P_{Y_1}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \dot\wedge (G - Q_0)x\right), \tag{4.5}$$

for all $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in H^2(\mathbb{D}, \mathbb{C}^2)$. We have

$$\begin{aligned} T_1\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \dot\wedge x\right) &= P_{Y_1}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \dot\wedge (G - Q_0)x\right) \\ &= P_{Y_1}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \dot\wedge \begin{pmatrix} \frac{2}{z} & 0 \\ 0 & \frac{1}{z} - q_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \\ &= P_{Y_1}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \dot\wedge \begin{pmatrix} 2\bar z x_1 \\ (\bar z - q_{22})x_2 \end{pmatrix}\right). \end{aligned}$$

Note that

$$\begin{aligned} X_1 &= \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \dot\wedge \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : f_1, f_2 \in H^2(\mathbb{D}, \mathbb{C}) \right\} \\ &\cong \{f_2 : f_2 \in H^2(\mathbb{D}, \mathbb{C})\} \\ &= H^2(\mathbb{D}, \mathbb{C}), \\ Y_1 &= \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \dot\wedge \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} : \phi_1, \phi_2 \in H^2(\mathbb{D}, \mathbb{C})^\perp \right\} \\ &\cong \{\phi_2 : \phi_2 \in H^2(\mathbb{D}, \mathbb{C})^\perp\} \\ &= H^2(\mathbb{D}, \mathbb{C})^\perp, \end{aligned}$$

where the isomorphisms are following from Proposition 2.1.26. Thus

$$X_1 \cong H^2(\mathbb{D},\mathbb{C}), \quad Y_1 \cong H^2(\mathbb{D},\mathbb{C})^\perp.$$

Hence $T_1 \colon H^2(\mathbb{D},\mathbb{C}) \to H^2(\mathbb{D},\mathbb{C})^\perp$ is defined by

$$T_1(x) = P_{H^2(\mathbb{D},\mathbb{C})^\perp}\left(\frac{1}{z} - q_{22}\right)x = P_{H^2(\mathbb{D},\mathbb{C})^\perp}\left(\frac{1}{z} - 1\right)x.$$

Therefore

$$T_0(x) = H_g x \quad \text{for all} \quad x \in H^2(\mathbb{D},\mathbb{C}),$$

where $H_g \colon H^2(\mathbb{D},\mathbb{C}) \to H^2(\mathbb{D},\mathbb{C})^\perp$ is the Hankel operator with symbol $g(z) = \frac{1}{z} - 1$.
If $T_1 = 0$, then $H_g x = 0$ for all $x \in H^2(\mathbb{D},\mathbb{C})$. Note that $H_g x = 0$ means

$$H_{\frac{1}{z}-1}x = H_{\frac{1}{z}}x = 0 \text{ for all } x \in H^2(\mathbb{D},\mathbb{C}).$$

This a contradiction, since $1/z \notin H^\infty(\mathbb{D},\mathbb{C})$. Thus $T_1 \neq 0$.
Let us now calculate $\|T_1\|$. By equation (3.15), for all $x \in H^2(\mathbb{D},\mathbb{C})$,

$$\|T_1(x)\| = \|H_{1/z}x\|,$$

and

$$\|H_{1/z}\| = \text{dist}(1/z, H^\infty) = 1.$$

If we take $x(z) = 1$ for all $z \in \mathbb{D}$, then $H_{1/z}x = 1/z$ and

$$\|H_{1/z}x\| = 1.$$

Thus $\|T_1\| = 1$. Since $g \in H^\infty(\mathbb{D},\mathbb{C}) + C(\mathbb{T},\mathbb{C})$, H_g is a compact operator, as a result, T_0 is a compact operator. This means that there exist functions $v \in H^2(\mathbb{D},\mathbb{C}^2), w \in L^2(\mathbb{T},\mathbb{C}^2)$, say

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

such that $(\xi_0 \dot{\wedge} v, \bar{\eta}_0 \dot{\wedge} w)$ is a Schmidt pair corresponding to $\|T_1\|$, or equivalently, the following equations hold

$$T_1(\xi_0 \dot{\wedge} v) = \|T_1\|(\bar{\eta}_0 \dot{\wedge} w) \tag{4.6}$$

and

$$T_1^*(\bar{\eta}_0 \dot{\wedge} w) = \|T_1\|(\xi_0 \dot{\wedge} v). \tag{4.7}$$

Let us find v and w. Equation (4.6) yields

$$T_1(v_2) = H_{1/z}v_2 = w_2.$$

Also, equation (4.7) yields

$$T_1^*(w_2) = v_2.$$

The adjoint operator T_1^* will be $H_{\frac{1}{z}}^* \colon H^2(\mathbb{D},\mathbb{C})^\perp \to H^2(\mathbb{D},\mathbb{C})$. By equation (4.7),

$$H_{1/z}^* w_2 = v_2.$$

If we take $v_2(z) = 1$ for all $z \in \mathbb{D}$, we get $w_2(z) = 1/z$ for all $z \in \mathbb{T}$. Thus we may choose

$$v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in H^2(\mathbb{D},\mathbb{C}^2), \quad w = \begin{pmatrix} 0 \\ 1/z \end{pmatrix} \in L^2(\mathbb{T},\mathbb{C}^2).$$

Next, let

$$x_1(z) = (I_{\mathbb{C}^2} - \xi_0(z)\xi_0^*(z))v(z), \quad y_1(z) = (I_{\mathbb{C}^2} - \overline{\eta_0(z)}\eta_0^T(z))w(z),$$

for all $z \in \mathbb{D}$. We have

$$x_1(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad y_1(z) = \begin{pmatrix} 0 \\ 1/z \end{pmatrix} \text{ for all } z \in \mathbb{T}.$$

Set

$$\xi_1 = \frac{x_1}{h_1}, \quad \eta_1 = \frac{zy_1}{\bar{h}_1}.$$

Clearly, for all $z \in \mathbb{D}$

$$h_1(z) = 1,\ \xi_1(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \eta_1(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let us find a function $Q_1 \in H^\infty(\mathbb{D},\mathbb{C}^{2\times 2})$ which satisfies the equations

$$(G - Q_1)x_0 = \|T_0\|y_0, \quad y_0^*(G - Q_1) = \|T_0\|x_0^*,$$

$$(G - Q_1)x_1 = \|T_1\|y_1, \quad y_1^*(G - Q_1) = \|T_1\|x_1^*.$$

Those equations yield

$$Q_1 = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{q}_{22} \end{pmatrix}, \quad \text{for some } \tilde{q}_{22} \in H^\infty(\mathbb{D},\mathbb{C})$$

and

$$Gx_1 - y_1 = Q_1x_1 \quad \text{and} \quad y_1^*G - x_1^* = y_1^*Q_1.$$

Substituting from above we have

$$\begin{pmatrix} 2/z & 0 \\ 0 & 1/z \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{q_{22}} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & \frac{1}{z} \end{pmatrix} \begin{pmatrix} 2/z & 0 \\ 0 & 1/z \end{pmatrix} - \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{z} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \tilde{q_{22}} \end{pmatrix}.$$

The first equality gives

$$\tilde{q}_{22} = 0.$$

For Q_1 to satisfy the above equations, it suffices to choose

$$Q_1 = \mathbb{O}_{\mathbb{C}^{2\times 2}} \in H^\infty(\mathbb{D}, \mathbb{C}^{2\times 2}).$$

Step 2. Let

$$X_2 = \xi_0 \dot{\wedge} \xi_1 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^2), \quad Y_2 = \bar{\eta}_0 \dot{\wedge} \bar{\eta}_1 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^2)^\perp.$$

Note that for all $z \in \mathbb{D}$ and for every $x \in H^2(\mathbb{D}, \mathbb{C}^2)$, $x(z)$ will be in $\operatorname{span}\{\xi_0(z), \xi_1(z)\}$. Also, for all $z \in \mathbb{D}$ and for every $y \in H^2(\mathbb{D}, \mathbb{C}^2)^\perp$, $y(z)$ will be in $\operatorname{span}\{\bar{\eta}_0(z), \bar{\eta}_1(z)\}$. Therefore $X_2 = Y_2 = \{0\}$, and so $T_2 = 0$. Thus the algorithm terminates. The solution is given by

$$\begin{aligned} G - \mathcal{A}G &= \sum_{i=0}^{1} \frac{t_i x_i y_i^*}{|h_i|^2} \\ &= \frac{t_0 y_0 x_0^*}{|h_0|^2} + \frac{t_1 y_1 x_1^*}{|h_1|^2} \\ &= 2 \begin{pmatrix} 1/z \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot 1 + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1/z \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot 1 \\ &= \begin{pmatrix} 2/z & 0 \\ 0 & 1/z \end{pmatrix} = G. \end{aligned}$$

Thus,

$$\mathcal{A}G = G - G = \mathbb{O}_{\mathbb{C}^{2\times 2}}.$$

Therefore G is a very badly approximable function. □

Let us now consider the example Peller and Young studied in [25].

Problem 4.0.2. *Let $G = B^{-1}A \in L^\infty(\mathbb{D}, \mathbb{C}^{2\times 2})$ where*

$$A(z) = \begin{pmatrix} \sqrt{3} + 2z & 0 \\ 0 & 1 \end{pmatrix}, \quad B(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} z^2 & z \\ z & -1 \end{pmatrix}, \quad \textit{for all } z \in \mathbb{T}.$$

Find the superoptimal singular values of G and its superoptimal approximant $\mathcal{A}G \in H^\infty$, that is, the unique $\mathcal{A}G$ such that the sequence

$$s^\infty(G - \mathcal{A}G) = (s_0^\infty(G - \mathcal{A}G), s_1^\infty(G - \mathcal{A}G))$$

is lexicographically minimised.

We will illustrate how the algorithm from Section 3.2.1 is used to determine the superoptimal analytic approximant $\mathcal{A}G$ of the G that was studied in [25]. It should be emphasised that, in line with Theorem 1.1.4, we obtain exactly the same $\mathcal{A}G$ as Peller and Young in [25].

Solution: On $\mathbb{T}$ it is

$$G(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{3}\bar{z}^2 + 2\bar{z} & \bar{z} \\ \sqrt{3}\bar{z} + 2 & -1 \end{pmatrix}.$$

The operator $H_G^* H_G$ with respect to the orthonormal basis

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} z \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ z \end{pmatrix} \right\}$$

of $(z^2 H^2)^\perp$, has matrix representation

$$H_G^* H_G \sim \frac{1}{\sqrt{2}} \begin{pmatrix} 10 & 2\sqrt{3} & 2 & 0 \\ 2\sqrt{3} & 3 & \sqrt{3} & 0 \\ 2 & \sqrt{3} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 0: In this case $t_0 = \|H_G\| = \sqrt{6}$ and a non-zero vector $x_0 \in H^2(\mathbb{D}, \mathbb{C}^2)$ such that

$$\|H_G x_0\|_{H^2(\mathbb{D},\mathbb{C}^2)^\perp} = \|H_G\| \|x_0\|_{H^2(\mathbb{D},\mathbb{C}^2)}$$

is

$$x_0(z) = \begin{pmatrix} 4 + \sqrt{3}z \\ 1 \end{pmatrix}.$$

For (x_0, y_0) to be a Schmidt pair for H_G corresponding to $\|H_G\|$, the vector $y_0 \in H^2(\mathbb{D}, \mathbb{C}^2)^\perp$ can be calculated by

$$y_0(z) = \frac{H_G x_0(z)}{\|H_G\|} = 2\bar{z} \begin{pmatrix} \bar{z} + \sqrt{3} \\ 1 \end{pmatrix} \in H^2(\mathbb{D}, \mathbb{C}^2)^\perp.$$

Next, we perform the inner-outer factorisations

$$x_0 = \xi_0 h_0, \quad \bar{z}\bar{y}_0 = \eta_0 h_0$$

for some inner $\xi_0, \eta_0 \in H^\infty(\mathbb{D}, \mathbb{C}^2)$ and some scalar outer $h_0 \in H^2(\mathbb{D}, \mathbb{C})$. In this example

$$\xi_0(z) = \frac{x_0}{h_0} = \frac{a}{4\sqrt{3}(1 - \gamma z)} \begin{pmatrix} 4 + \sqrt{3}z \\ 1 \end{pmatrix},$$

$$\bar{\eta}_0(z) = \frac{zy_0}{\bar{h}_0} = \frac{2a}{4\sqrt{3}(1-\gamma\bar{z})}\begin{pmatrix}\bar{z}+\sqrt{3}\\ 1\end{pmatrix},$$

where

$$h_0(z) = \frac{4\sqrt{3}}{a}(1-\gamma z),$$

$a = \sqrt{10-2\sqrt{13}}$ and $\gamma = -\frac{a^2}{4\sqrt{3}}$.

A function $Q_1 \in H^\infty(\mathbb{D}, \mathbb{C}^{2\times 2})$ that satisfies

$$(G-Q_1)x_0 = t_0 y_0, \quad (G-Q_1)^* y_0 = t_0 x_0$$

is

$$Q_1(z) = \begin{pmatrix} 0 & \sqrt{6} \\ 2\sqrt{2} & -\sqrt{6}(z+\sqrt{3})\end{pmatrix}.$$

Step 1: Let $X_1 = \xi_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^2)$ and $Y_1 = \bar{\eta}_0 \dot{\wedge} H^2(\mathbb{D}, \mathbb{C}^2)^\perp$.
Let the compact operator $T_1\colon X_1 \to Y_1$ be given by

$$T_1(\xi_0 \dot{\wedge} x) = P_{Y_1}(\bar{\eta}_0 \dot{\wedge} (G-Q_1)x)$$

for all $x \in H^2(\mathbb{D}, \mathbb{C}^2)$.
Note that

$$\begin{aligned} X_1 &= \left\{ \xi_0 \dot{\wedge} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \ : \ f_i \in H^2(\mathbb{D}, \mathbb{C}) \right\} \\ &= \left\{ \frac{a}{4\sqrt{3}} \frac{(4+\sqrt{3}z)f_2 - f_1}{1-\gamma z} \ : \ f_i \in H^2(\mathbb{D}, \mathbb{C}) \right\}. \end{aligned}$$

If we choose

$$f_1 = -\frac{4\sqrt{3}}{a}(1-\gamma z)g \quad \text{and} \quad f_2 = 0$$

for some $g \in H^2(\mathbb{D}, \mathbb{C})$, we obtain $X_1 = H^2(\mathbb{D}, \mathbb{C})$.

In a similar way, we have

$$\begin{aligned} Y_1 &= \left\{ \bar{\eta}_0 \dot{\wedge} \begin{pmatrix} \bar{z}\bar{\phi}_1 \\ \bar{z}\bar{\phi}_2 \end{pmatrix} \ : \ \phi_i \in H^2(\mathbb{D}, \mathbb{C}) \right\} \\ &= \left\{ \frac{a\bar{z}}{2\sqrt{3}} \frac{(\bar{z}+\sqrt{3})\bar{\phi}_2 - \bar{\phi}_1}{1-\gamma\bar{z}} \ : \ \phi_i \in H^2(\mathbb{D}, \mathbb{C}) \right\}. \end{aligned}$$

If we choose

$$\phi_1 = -\frac{2\sqrt{3}}{a}(1-\gamma z)\psi, \quad \text{and} \quad \phi_2 = 0$$

for some $\psi \in H^2(\mathbb{D}, \mathbb{C})$, we obtain $Y_1 = H^2(\mathbb{D}, \mathbb{C})^\perp$.
We have

$$T_1\left(\xi_0\dot{\wedge}\begin{pmatrix}f_1\\f_2\end{pmatrix}\right)=T_1\left(\xi_0\dot{\wedge}\begin{pmatrix}-\frac{4\sqrt{3}}{a}(1-\gamma z)g\\0\end{pmatrix}\right)=\frac{u(\gamma)}{z-\gamma}$$

where

$$u(\gamma)=\sqrt{2}(1-\gamma^2)(2\sqrt{3}\gamma+1)g(\gamma).$$

Then $t_1=\|T_1\|=\sqrt{2}(4-\sqrt{13})$. Since T_1 is a compact operator, there exist $v_1\in H^2(\mathbb{D},\mathbb{C}^2)$, $w_1\in H^2(\mathbb{D},\mathbb{C}^2)^{\perp}$ such that

$$T_1(\xi_0\dot{\wedge}v_1)=t_1(\bar{\eta}_0\dot{\wedge}w_1),\quad T_1^*(\bar{\eta}_0\dot{\wedge}w_1)=t_1(\xi_0\dot{\wedge}v_1).$$

Here we can choose

$$v_1(z)=\frac{4\sqrt{3}}{a}\begin{pmatrix}-1\\0\end{pmatrix},\quad w_1(z)=\frac{2\sqrt{3}}{a}\bar{z}\begin{pmatrix}1\\0\end{pmatrix}.$$

Perform the inner-outer factorisation of $\xi_0\dot{\wedge}v_1\in H^2(\mathbb{D},\wedge^2\mathbb{C}^2)$. The function $h_1(z)=\frac{1}{1-\gamma z}$ is the scalar outer factor of $\xi_0\dot{\wedge}v_1$.

Let

$$x_1(z)=(I-\xi_0(z)\xi_0^*(z))v_1(z),\quad y_1(z)=(I-\bar{\eta}_0(z)\eta_0^T(z))w_1(z).$$

Then

$$x_1=\frac{\gamma}{\alpha}\frac{1}{(1-\gamma z)(1-\gamma\bar{z})}\begin{pmatrix}\frac{-4\sqrt{3}}{\gamma}(1-\gamma z)(1-\gamma\bar{z})-19-4\sqrt{3}(z+\bar{z})\\-4-\sqrt{3}\bar{z}\end{pmatrix}$$

and

$$y_1=\frac{2\gamma\bar{z}}{\alpha}\frac{1}{(1-\gamma z)(1-\gamma\bar{z})}\begin{pmatrix}\frac{\sqrt{3}}{\gamma}(1-\gamma z)(1-\gamma\bar{z})+4+\sqrt{3}(z+\bar{z})\\z+\sqrt{3}\end{pmatrix}.$$

Calculations yield

$$x_1=\frac{\gamma}{\alpha}\frac{1}{(1-\gamma z)(1-\gamma\bar{z})}\begin{pmatrix}1\\-4-\sqrt{3}\bar{z}\end{pmatrix},\quad y_1=\frac{2\gamma\bar{z}}{\alpha}\frac{1}{(1-\gamma z)(1-\gamma\bar{z})}\begin{pmatrix}-1\\z+\sqrt{3}\end{pmatrix}.$$

Observe that the algorithm stops after at most $\min(m,n)$ steps, hence in this case after 2 steps. Then, by Theorem 3.2.54, the unique analytic superoptimal approximant $\mathcal{A}G$ is given by the formula

$$\mathcal{A}G=G-\frac{t_0y_0x_0^*}{|h_0|^2}-\frac{t_1y_1x_1^*}{|h_1|^2}.$$

Now, all terms can be calculated and

$$\mathcal{A}G=\frac{\sqrt{2}}{1-\gamma z}\begin{pmatrix}-\gamma & \sqrt{3}+4\gamma\\2+\gamma\sqrt{3}-\gamma z & -(\sqrt{3}+4\gamma)(\sqrt{3}+z)\end{pmatrix},$$

which is the *unique superoptimal analytic approximant* for the given G. □

Appendix A

Hilbert tensor product

A.1 Algebraic tensor product

We will consider complex linear spaces. Let E_1, E_2 be linear spaces over $\mathbb{C}$. We present a well-known construction of the algebraic tensor product $E_1 \otimes E_2$, which can be found in [10].

Definition A.1.1. [10, Definition II.1.1.] *Let E_1, E_2 be linear spaces. We say that the pair (Θ, θ), where Θ is a linear space and $\theta\colon E_1 \times E_2 \to \Theta$ is a bilinear operator, has the* universal property *in the category of linear spaces and linear operators if for any linear space G and for any bilinear operator $\mathcal{R}\colon E_1 \times E_2 \to G$, there is a unique linear operator $R\colon \Theta \to G$ such that the following diagram is commutative*

$$\begin{array}{ccc} E_1 \times E_2 & \overset{\mathcal{R}}{\to} & G \\ \big\downarrow \theta & \nearrow R & \\ \Theta & & \end{array}, \tag{A.1}$$

that is, $R \circ \theta = \mathcal{R}$.

Definition A.1.2. *Let E_1, E_2 be linear spaces. The pair (Θ, θ), where Θ is a linear space and $\theta\colon E_1 \times E_2 \to \Theta$ is a bilinear operator, is called the algebraic tensor product of E_1 and E_2 if it has the universal property in the category of linear spaces and linear operators.*

Let us construct the algebraic tensor product. Let $E_1 \circ E_2$ denote the space of formal linear combinations with complex coefficients of the elements of $E_1 \times E_2$. We use the notation $x \circ y$, instead of (x, y) for the elements of $E_1 \circ E_2$ and consider the set $M \subset E_1 \circ E_2$ of elements in any of the following forms:

$$(x_1 + x_2) \circ y - x_1 \circ y - x_2 \circ y;$$

$$x \circ (y_1 + y_2) - x \circ y_1 - x \circ y_2;$$

$$\lambda(x \circ y) - (\lambda x) \circ y;$$

$$x \circ (\lambda y) - \lambda(x \circ y);$$

where $x, x_1, x_2 \in E_1, \quad y, y_1, y_2 \in E_2$ and $\lambda \in \mathbb{C}$.

Let $\operatorname{span}(M)$ be the linear span of the set M. We define $E_1 \otimes E_2$ to be the quotient space $E_1 \circ E_2 / \operatorname{span}(M)$, $x \otimes y$ to be the coset $x \circ y + \operatorname{span}(M)$ and ϑ to be the bilinear operator

$$\vartheta : E_1 \times E_2 \to E_1 \otimes E_2 \text{ , given by } \vartheta(x, y) = x \otimes y.$$

Theorem A.1.3. [10, Theorem II.1.4] *The pair $(E_1 \otimes E_2, \vartheta)$ is the algebraic tensor product of the spaces E_1 and E_2.*

Proof. Let $\mathcal{R}\colon E_1 \times E_2 \to G$ be a bilinear operator. Then the operator $R^\circ\colon E_1 \circ E_2 \to G$ is uniquely defined by $R^\circ(x \circ y) = \mathcal{R}(x, y)$ and maps all the elements of M, and hence $\operatorname{span}(M)$, to zero. Consequently R° generates an operator $R\colon E_1 \otimes E_2 \to G$ such that diagram from Definition A.1.1 with $E_1 \otimes E_2$ instead of Θ and ϑ instead of θ is commutative. Furthermore, since $E_1 \otimes E_2 = \operatorname{span}(\operatorname{Im} \vartheta)$, R is uniquely defined.

□

Proposition A.1.4. [10][II.1.5] *Let E_1, E_2 be linear spaces over $\mathbb{C}$. Every $u \in E_1 \otimes E_2$, $u \neq 0$ can be written as*

$$u = \sum_{k=1}^{n} x_k \otimes y_k,$$

where the vectors $x_k \in E_1$ are linearly independent and $y_1 \neq 0$.

Definition A.1.5 ([10])**.** *Suppose E_k, F_k, $k = 1, 2$, are linear spaces and consider the operators $T_1\colon E_1 \to F_1$, $T_2\colon E_2 \to F_2$. The operator*

$$T_1 \otimes T_2\colon E_1 \otimes E_2 \to F_1 \otimes F_2$$

given by

$$(T_1 \otimes T_2)(x \otimes y) = T_1(x) \otimes T_2(y), \text{ for } x \in E_1, y \in E_2,$$

is called the tensor product of the operators T_1 and T_2.

Proposition A.1.6. *Let E_1, E_2, F_1, F_2 be linear spaces and let*

$$T_1\colon E_1 \to F_1, \quad T_2\colon E_2 \to F_2$$

be linear operators. Then, $T_1 \otimes T_2\colon E_1 \otimes E_2 \to F_1 \otimes F_2$ is a linear operator.

Proof. By the universal property from Definition A.1.1, for every bilinear operator $T_1 \times T_2\colon E_1 \times E_2 \to F_1 \otimes F_2$ there exists a unique linear operator $T_1 \otimes T_2$ such that

$$(T_1 \otimes T_2)(x_1 \otimes x_2) = T_1(x_1) \otimes T_2(x_2).$$

It suffices to show that $T_1 \times T_2$ is a bilinear operator. Let $u, u_1, u_2 \in E_1,\ v, v_1, v_2 \in E_2$ and $\lambda, \mu \in \mathbb{C}$. Then,

$$(T_1 \times T_2)(\lambda u_1 + \mu u_2, v) = T_1(\lambda u_1 + \mu u_2) \otimes T_2(v) = (\lambda T_1(u_1) + \mu T_2(u_2)) \otimes T_2(u_2)$$

and

$$(T_1 \times T_2)(u, \lambda v_1 + \mu v_2) = T_1(u) \otimes T_2(\lambda v_1 + \mu v_2) = T_1(u) \otimes (\lambda T_2(v_1) + \mu T_2(v_2)). \qquad \square$$

A.2 Hilbert tensor product

Let $(H_1, \langle \cdot, \cdot \rangle_{H_1})$, $(H_2, \langle \cdot, \cdot \rangle_{H_2})$ be Hilbert spaces and let $\|x\|_{H_1} = \langle x, x \rangle^{\frac{1}{2}}$, $\|y\|_{H_2} = \langle y, y \rangle^{\frac{1}{2}}$ for $x \in H_1$, $y \in H_2$. Information on Hilbert tensor product can be found in [8].

One can consider an inner product space $(H_1 \otimes H_2, \langle \cdot, \cdot \rangle)$, where the inner product is defined by

$$\langle u, v \rangle = \sum_{k=1}^{n} \sum_{p=1}^{m} \langle a_k, c_p \rangle \langle b_k, d_p \rangle,$$

for

$$u = \sum_{k=1}^{n} a_k \otimes b_k \ , \ v = \sum_{p=1}^{m} c_p \otimes d_p.$$

Definition A.2.1. *The completion of $(H_1 \otimes H_2, \langle \cdot, \cdot \rangle)$ with respect to $\| \cdot \| = \langle \cdot, \cdot \rangle^{\frac{1}{2}}$ is called the Hilbert tensor product of H_1 and H_2 and is denoted by $H_1 \otimes_H H_2$.*

Definition A.2.2. *Let $(E, \| \cdot \|_E), (F, \| \cdot \|_F)$ be Hilbert spaces and let $T \in \mathcal{L}(E, F)$. The linear operator $T^* \colon F^* \to E^*$ which satisfies*

$$\langle Ta, b \rangle_F = \langle a, T^* b \rangle_E,$$

for all $a \in E,\ b \in F$, is called the adjoint operator of T.

Definition A.2.3 ([38], p. 38)**.** *A linear operator $T \colon E \to F$, where $E,\ F$ are Hilbert spaces is a unitary operator if it is bijective and preserves inner products, that is, it satisfies*

$$\langle Tx, Ty \rangle = \langle x, y \rangle, \ \textit{for all}\, x, y \in E.$$

Theorem A.2.4 ([38], p. 38)**.** *Let E, F be Hilbert spaces and $T \colon E \to F$ be a linear and surjective mapping. Then T is unitary if and only if*

$$\|Tx\| = \|x\|, \ \textit{for all}\, x \in E.$$

Definition A.2.5. *Let E, F be Hilbert spaces and let $W \colon H \to K$ be a bounded linear operator. W will be called a partial isometry if W is isometric on the orthogonal complement* *of its kernel. Then,* $M = (\ker W)^{\perp}$ *is called the initial space and* $N = WM$ *the final space of* W.

Theorem A.2.6. *Let* E, F *be Hilbert spaces. A bounded linear operator* $W\colon E \to F$ *is a partial isometry if and only if* W^*W *is a projection operator. In this case,* W^*W *is the projection of* E *on the initial space of* W.

Remark A.2.7 ([8])**.** *Let* E_1, E_2, F_1, F_2 *be Hilbert spaces and let*

$$T_1\colon E_1 \to F_1,\;\; T_2\colon E_2 \to F_2$$

be bounded linear operators. Then

$$T_1 \otimes T_2 : (E_1 \otimes E_2, \|\cdot\|) \to (F_1 \otimes F_2, \|\cdot\|)$$

is bounded. Hence $T_1 \otimes T_2$ *can be extended to the tensor product* $E_1 \otimes_H E_2$ *as follows: we set*

$$u = \lim_{n\to\infty} \sum_{k=1}^{m_n} a_k^n \otimes b_k^n$$

and

$$(T_1 \otimes T_2)(u) = \lim_{n\to\infty} (T_1 \otimes T_2)(\sum_{k=1}^{m_n} a_k^n \otimes b_k^n).$$

Proposition A.2.8 ([8])**.** *Let* $(H_1, \langle\cdot,\cdot\rangle_{H_1})$, $(H_2\langle\cdot,\cdot\rangle_{H_2})$, $(G_1, \langle\cdot,\cdot\rangle_{G_1})$, $(G_2, \langle\cdot,\cdot\rangle_{G_2})$, *be Hilbert spaces and* $T_1\colon H_1 \to G_1$, $T_2\colon H_2 \to G_2$ *be bounded linear operators. Then,*

$$T_1 \otimes T_2\colon H_1 \otimes_H H_2 \to G_1 \otimes_H G_2$$

is a bounded linear operator, and $\|T_1 \otimes T_2\| = \|T_1\| \cdot \|T_2\|$.

Lemma A.2.9. *Let* E_1, E_2, F_1, F_2 *be Hilbert spaces and let*

$$T_1\colon E_1 \to F_1,\;\; T_2\colon E_2 \to F_2$$

be bounded linear operators. Then $(T_1 \otimes T_2)^* = T_1^* \otimes T_2^*\colon F_1 \otimes_H F_2 \to E_1 \otimes_H E_2$.

Definition A.2.10 ([24], p. 301)**.** $\mathbb{C}^{m\times n}$ *is the space of* $m \times n$ *complex matrices. Every* $A \in \mathbb{C}^{m\times n}$ *is a linear operator from* $\mathbb{C}^n$ *to* $\mathbb{C}^m$*, where* $\mathbb{C}^n$, $\mathbb{C}^m$ *are Hilbert spaces with their standard inner products. Also,*

$$\|A\| = \sup_{\|x\|_{\mathbb{C}^n}\le 1} \|Ax\|_{\mathbb{C}^m}.$$

Remark A.2.11. *Let* X_1, X_2, Y_1, Y_2 *be Hilbert spaces of dimensions* n, m, n', m' *with* $m \ge n$*, and* $m' \ge n'$*. Let* $A\colon X_1 \to Y_1$ *and let* $B\colon X_2 \to Y_2$ *be linear transformations. Let* $(e_i)_{i=1}^n$,

$(f_j)_{j=1}^m$, $(e'_i)_{i=1}^{n'}$, $(f'_j)_{j=1}^{m'}$ be orthonormal bases of X_1, Y_1, X_2 and Y_2 respectively. Then $\{e_i \otimes e'_l : 1 \leq i \leq n,\ 1 \leq l \leq n'\}$ is a basis of $X_1 \otimes X_2$ and $\{f_i \otimes f'_l : 1 \leq i \leq m,\ 1 \leq l \leq m'\}$ is a basis of $Y_1 \otimes Y_2$.

Suppose that

$$Ae_i = \lambda_i f_i, \quad 1 \leq i \leq n$$

and

$$Be'_i = \mu_i f'_i, \quad 1 \leq i \leq n'.$$

One can write

$$Ax = \sum_{i=1}^{n} \lambda_i \langle x, e_i \rangle_{X_1} f_i \quad \text{for } x \in X_1$$

and

$$Bx' = \sum_{k=1}^{n'} \mu_k \langle x', e'_k \rangle_{X_2} f'_k \quad \text{for } x' \in X_2.$$

Then $A \otimes B : X_1 \otimes X_2 \to Y_1 \otimes Y_2$ can be presented by the following formula, for $x \in X_1$ and $x' \in X_2$,

$$\begin{aligned}
(A \otimes B)(x \otimes x') &= Ax \otimes Bx' \\
&= \left(\sum_{i=1}^{n} \lambda_i \langle x, e_i \rangle_{X_1} f_i \right) \otimes \left(\sum_{k=1}^{n'} \mu_k \langle x', e'_k \rangle_{X_2} f'_k \right) \\
&= \sum_{i,k=1}^{n,n'} \lambda_i \mu_k \langle x \otimes x', e_i \otimes e'_k \rangle_{X_1 \otimes X_2} f_i \otimes f'_k \\
&= \sum_{i,k=1}^{n,n'} \lambda_i \mu_k \langle x, e_i \rangle_{X_1} \langle x', e'_k \rangle_{X_2} f_i \otimes f'_k \\
&= \textstyle\sum_{i,k=1}^{n,n'} \lambda_i \mu_k \langle x, e_i \rangle_{X_1} \langle x', e'_k \rangle_{X_2} f_i \otimes f'_k.
\end{aligned}$$

Moreover

$$\begin{aligned}
(A \otimes B)(e_i \otimes e'_l) &= \sum_{j,k=1}^{n,n'} \lambda_j \mu_k \langle e_i \otimes e'_l, e_j \otimes e'_k \rangle_{X_1 \otimes X_2} f_j \otimes f'_k \\
&= \sum_{j,k=1}^{n,n'} \lambda_j \mu_k \delta_{ij} \delta_{lk} f_j \otimes f'_k \\
&= \lambda_i \mu_l f_i \otimes f'_l.
\end{aligned}$$

Notation A.2.12. *A matrix of the form*

$$A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

denotes the matrix $A = (a_{ij})$ *where* $a_{ij} = 0$ *for all* $i \neq j$ *and* $a_{ii} = \lambda_i$.

Lemma A.2.13. *Let* $A = (a_{ij})_{i,j=1}^n$, $B = (b_{ij})_{i,j=1}^n$ *and let*

$$A^*A = U^* \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} U, \quad B^*B = V^* \begin{pmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_n \end{pmatrix} V,$$

for some unitary operators U, V. *Then,*

$$A^*A \otimes B^*B = (U \otimes V)^* \begin{pmatrix} \lambda_1\mu_1 & & & & & & & \\ & \lambda_1\mu_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_1\mu_n & & & & \\ & & & & \ddots & & & \\ & & & & & \lambda_n\mu_1 & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_n\mu_n \end{pmatrix} (U \otimes V).$$

Lemma A.2.14. *Suppose* $m \geq n$. *Given* $A, B \in \mathbb{C}^{m\times n}$, *with*

$$A = U_1 \begin{pmatrix} s_1 & \cdots & 0 \\ 0 & \ddots & \\ \vdots & & s_n \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix} V_1, B = U_2 \begin{pmatrix} t_1 & \cdots & 0 \\ 0 & \ddots & \\ \vdots & & t_n \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix} V_2,$$

for unitary matrices $U_1, U_2 \in \mathbb{C}^{m\times m}$ *and for unitary matrices* $V_1, V_2 \in \mathbb{C}^{n\times n}$. *Then*

$$A \otimes B = (U_1 \otimes U_2) \begin{pmatrix} s_1 t_1 & & & & & & & \\ & s_1 t_2 & & & & & & \\ & & \ddots & & & & & \\ & & & s_1 t_n & & & & \\ & & & & \ddots & & & \\ & & & & & s_n t_1 & & \\ & & & & & & \ddots & \\ & & & & & & & s_n t_n \\ 0 & & \cdots & & & & & 0 \\ \vdots & & & & & & & \vdots \\ 0 & & & & & & & 0 \end{pmatrix} (U_2 \otimes V_2).$$

Appendix B

Scalar inner and outer functions

Definition B.0.1 ([11], p. 62). *An* inner function *is an analytic function g in the unit disc $\mathbb{D}$ such that $|g(z)| \leq 1$ and $|g(e^{i\theta})| = 1$ almost everywhere on the unit circle $\mathbb{T}$. A non-constant inner function without zeros which is positive at the origin is called a* singular inner function*.*

Definition B.0.2 ([11], p. 62). *An* outer function *is an analytic function F in the unit disc of the form*

$$F(z) = \lambda \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}k(\theta)d\theta\right] \tag{B.1}$$

where k is a real-valued integrable function on the circle and λ is a complex number of modulus 1*.*

Remark B.0.3 ([11], p. 63). *Such an outer function F is in $H^1(\mathbb{D},\mathbb{C})$ if and only if e^k is also integrable; when F is an outer function in $H^1(\mathbb{D},\mathbb{C})$ we have necessarily*

$$k(\theta) = \log|F(e^{i\theta})| \text{ almost everywhere.}$$

Indeed, applying the logarithmic function to equation (B.1), we get

$$\log|F(e^{i\theta})| = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)k(\theta)d\theta\right]$$

and taking the limit as r goes to 1, we have the result we need. Here $P_r(\theta)$ is the Poisson's kernel defined in equation (C.3).

Theorem B.0.4 ([11],p. 63). *Let F be a non zero function in $H^1(\mathbb{D},\mathbb{C})$. The following are equivalent:*
(i) F is an outer function.
(ii) If f is any function in $H^1(\mathbb{D},\mathbb{C})$ such that $|f| = |F|$ almost everywhere on $\mathbb{T}$, then

$$|F(z)| \geq |f(z)| \text{ for all } z \in \mathbb{T}.$$

(iii) $\log|F(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(e^{i\theta})| d\theta.$

Proof. *(i)⇒ (ii).* Let f be a non-zero function in $H^1(\mathbb{D},\mathbb{C})$. Then, by Fatou's Theorem, f has radial limits

$$f(e^{i\theta}) = \lim_{z\to e^{i\theta}} f(z)$$

almost everywhere on $\mathbb{T}$, and f is given by the Poisson integral

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})P_r(\theta - t)\, dt.$$

Since $f(0) \neq 0$, by a result in [11, p. 51], $\log|f(e^{it})|$ is Lebesgue integrable. Let $F \in H^1(\mathbb{D},\mathbb{C})$ be an outer function given by

$$F(z) = \lambda \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|f(e^{i\theta})| d\theta\right]$$

and without loss of generality assume $|\lambda| = 1$. Notice that $|F| = e^u$, where u is the Poisson integral of $\log|f|$. Hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(re^{i\theta})|\, d\theta \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|\, d\theta.$$

Therefore $|F| = |f|$ almost everywhere on $\mathbb{T}$. Since F is an outer function and $|F| = e^u$, F has no zeros in $\mathbb{D}$ and

$$\log|F(re^{i\theta})| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(e^{it})|P_r(\theta - t) dt.$$

By Jensen's inequality with $dm = \frac{1}{2\pi}P_r(\theta - t)$, we get

$$\log|f(re^{i\theta})| \leq \frac{1}{2\pi}\int_{0}^{2\pi} \log|f(e^{it})|P_r(\theta - t) dt = \log|F(re^{i\theta})|,$$

and we infer that $|F(z)| \geq |f(z)|$ for all $z \in \mathbb{D}$.

(ii)⇒(iii). Suppose (ii) holds and let G be an outer function,

$$G(z) = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|F(e^{i\theta})| d\theta\right].$$

Then $|F(z)| \leq |G(z)| \leq |F(z)|$ on $\mathbb{D}$. Thus F/G is analytic of absolute value 1. So, $F = \lambda G$ with $|\lambda| = 1$ and F is outer.

(iii) ⇒ (i). Suppose (iii) holds and define G as previously. Then F/G is bounded by 1 on $\mathbb{D}$ and has absolute value 1 at $z = 0$. Thus $F/G = \lambda$ with $|\lambda| = 1$. □

Theorem B.0.5 ([11],p. 63)**.** *Let f be a non-zero function in $H^1(\mathbb{D}, \mathbb{C})$. Then one can write*

$$f = gF$$

where g is an inner function and F is an outer function in $H^1(\mathbb{D}, \mathbb{C})$. The factorisation is unique up to a unimodular constant.

Proof. By Theorem B.0.4, if

$$F(z) = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log|f(e^{i\theta})|d\theta\right],$$

then F is an outer function in $H^1(\mathbb{D}, \mathbb{C})$ and $f/F = g$ is an inner function. If F_1 is another outer function in $H^1(\mathbb{D}, \mathbb{C})$ and g_1 another inner function, we have $f = g_1F_1$ and $|F| = |F_1|$ on $\mathbb{T}$. Then, $F = \lambda F_1$ for some λ with $\lambda = 1$. Thus $\lambda g_1 F_1 = g_1 G_1$ and $g_1 = \lambda g$. □

Remark B.0.6. *The preceding results also hold for $1 < p \leq \infty$, as a more detailed view presented in [11] asserts.*

Definition B.0.7 ([11], p. 11)**.** *Let H be an inner-product space and N be any collection of vectors in H. N is called an orthogonal set if any two distinct vectors in N are orthogonal. An orthonormal set is an orthogonal set, each vector of which has norm 1. If $N = \{n_1, \cdots, n_k\}$ is a countable orthonormal set in H, then N will be called a complete orthonormal set if the only vector orthogonal to every n_i is the zero vector.*

Remark B.0.8 ([11], p. 28)**.** *Suppose that f is analytic in $\mathbb{D}$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let $f_r(\theta) = f(re^{i\theta})$. Note that if we restrict the function f to the circle of radius r, we obtain a continuous function on that circle which we can also interpret as a function on the unit circle. Now,*

$$f_r(\theta) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$$

which means that the n-th Fourier coefficient of f_r is $a_n r^n$ for $n \geq 0$ and is zero for $n < 0$. If f is analytic in $\bar{\mathbb{D}}$, the boundary value function f_1 has the Fourier coefficients a_n.

Theorem B.0.9. [28, Theorem 11.20] *Every $f \in H^\infty(\mathbb{D}, \mathbb{C})$ can be extended to a function $f^* \in L^\infty(\mathbb{T}, \mathbb{C})$ defined almost everywhere by*

$$f^*(e^{it}) = \lim_{r\to 1} f(re^{it}) \tag{B.2}$$

Also, $\|f\|_\infty = \|f^*\|_{L^\infty}$. For all $z \in \mathbb{D}$ the Cauchy formula

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f^*(\xi)}{\xi - z}d\xi \tag{B.3}$$

holds, where γ is the positively orientated unit circle, $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$.

The functions $f^* \in L^\infty(\mathbb{T})$ which are obtained in this manner are those which satisfy

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(e^{it})e^{-int}dt = 0 \quad (n = -1, -2, \dots). \tag{B.4}$$

Appendix C

Operator-valued functions and Fatou's theorem

C.1 The scalar case

Definition C.1.1. [11] *A complex valued function u on $\mathbb{D}$ is harmonic if it satisfies Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Proposition C.1.2. [28, p. 232] *Every harmonic function u on $\mathbb{D}$ which satisfies*

$$\sup_{0 \le r < 1} \int_{-\pi}^{\pi} |u(re^{i\theta})| d\theta < \infty,$$

has radial limits at almost all points of $\mathbb{T}$.

Theorem C.1.3 (Fatou's Theorem, *[38, Theorem 13.10]*)**.** *Let $f \in H^2(\mathbb{D}, \mathbb{C})$. For almost all $z \in \mathbb{T}$, the radial limits*

$$\lim_{r \to 1} f(rz)$$

exist almost everywhere on $\mathbb{T}$ and define a function in $L^2(\mathbb{T}, \mathbb{C})$.

C.2 The operator-valued case

The following material is from [14]. For any separable Hilbert space E we denote by $L^2(\mathbb{T}, E)$ the class of functions $v \colon \mathbb{T} \to E$ which are measurable and satisfy

$$\|v\|^2 = \frac{1}{2\pi} \int_0^{2\pi} \|v(e^{it})\|_E^2 dt < \infty.$$

Two functions will be considered equal if they coincide almost everywhere with respect to Lebesgue measure.
Suppose that $v_n(e^{it})_{n\in\mathbb{N}}$ is a sequence converging to $v(e^{it})$ in $L^2(\mathbb{T}, E)$, that is,

$$\frac{1}{2\pi}\int_0^{2\pi} \|v_n(e^{it}) - v(e^{it})\|_E^2 dt \to 0$$

as n tends to ∞.
Then we can choose a subsequence $v_{n_k}(e^{it})$, $k = 1, 2, \cdots$, such that

$$\sum_{k=1}^{\infty}\int_0^{2\pi} \|v_{n_k}(e^{it}) - v(e^{it})\|_E^2 dt < \infty.$$

By the theorem of Beppo Levi we have

$$\sum_{k=1}^{\infty} \|v_{n_k}(e^{it}) - v(e^{it})\|_E^2 < \infty$$

almost everywhere and so

$$\|v_{n_k}(e^{it}) - v(e^{it})\| \to 0$$

almost everywhere as $k \to \infty$.
For any $k \in \mathbb{Z}$, denote by F_k the subspace of $L^2(\mathbb{T}, E)$ which contains the all the functions of the form $e^{ikt}a$, with $a \in E$. Then $F_k \perp F_j$ for $k \neq j$ and

$$L^2(\mathbb{T}, E) = \oplus_{-\infty}^{\infty} F_k.$$

Indeed, let $v \in L^2(\mathbb{T}, A)$ be orthogonal to all F_k, that is,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}\langle v(e^{it}), a\rangle_E dt = 0,\ a \in E,\ k \in \mathbb{Z}.$$

Then, $\langle v(e^{it}), a\rangle_E = 0$ everywhere, except possibly the points t of a set E_a depending on a and of zero measure. Letting a run over a countable, dense subset of E and taking the union of the corresponding sets E_a we obtain a set E of zero measure and $v(e^{it}) = 0\ \forall t \notin E$, so $v = 0$ as an element of $L^2(\mathbb{T}, E)$.
Furthermore

$$\|e^{ikt}a\|_{L^2(\mathbb{T},E)} = \|a\|_E.$$

As a result, there exists a one to one correspondence between the elements v of $L^2(\mathbb{T}, E)$ and the sequences a_k, $a_k \in E$ with $\sum_k \|a_k\|_E^2 < \infty$, in such a way that for corresponding v

and a_k we have

$$v(e^{it}) = \sum_{-\infty}^{\infty} e^{ikt} a_k \tag{C.1}$$

and

$$\|v\|^2 = \sum \|a_k\|_E^2. \tag{C.2}$$

Relation (C.2) follows from the fact that

$$\frac{1}{2\pi}\int_0^{2\pi} \|v(e^{it}) - \sum_{-m}^{n} e^{ikt} a_k\|_E^2 dt \to 0$$

as $m, n \to \infty$.

From relation (C.1)

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} v(e^{it}) dt,\ k \in \mathbb{Z},$$

and so (C.1) is the Fourier series of v.

Definition C.2.1 ([14], p. 184)**.** *We will denote by $L_+^2(\mathbb{T}, E)$ the subspace of $L^2(\mathbb{T}, E)$ consisting of those functions for which $a_k = 0$ for $k < 0$.*

Now, we associate any function

$$v(e^{it}) = \sum_{0}^{\infty} e^{ikt} a_k \in L_+^2(\mathbb{T}, E)$$

with the function

$$u(z) = \sum_{0}^{\infty} z^k a_k$$

of the complex variable z, defined and holomorphic on $\mathbb{D}$ since

$$\|\sum_{m}^{n} z^k a_k\|_E \le \sum_{m}^{n} |z|^k \|a_k\|_E \le (1 - |z|^2)^{-1/2} \left(\sum_{m}^{n} (\|a_k\|_E^2\right)^{1/2} \to 0$$

for $n > m \to \infty$, for $|z| < 1$, uniformly for $|z| \le r_0 < 1$.

One can retrieve $v(e^{it})$ from $u(z)$ as a radial limit in $L^2(\mathbb{T}, E)$

$$\frac{1}{2\pi}\int_0^{2\pi} \|v(e^{it}) - u(re^{it})\|_E^2 dt = \frac{1}{2\pi}\int_0^{2\pi} \|\sum_{0}^{\infty} (1 - r^k) e^{ikt} a_k\|_E^2 dt = \sum_{0}^{\infty} (1 - r^k)^2 \|a_k\|_E^2 \to 0$$

as $r \to 1$. Also, for $0 \le r < 1$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \|u(re^{it})\|_E^2 = \sum_0^\infty r^{2k}\|a_k\|_E^2 \le \sum_0^\infty \|a_k\|_E^2 < \infty.$$

Definition C.2.2 ([14], p. 185)**.** *The class of functions*

$$u(z) = \sum_0^\infty z^k a_k$$

with values in E, holomorphic on $\mathbb{D}$ and such that

$$\frac{1}{2\pi}\int_0^{2\pi} \|u(re^{it})\|_E^2 dt \quad 0 \le r < 1$$

has a bound independent of r, will be denoted by $H^2(\mathbb{D}, E)$.

Remark C.2.3. *Note that*

$$\frac{1}{2\pi}\int_0^{2\pi} \|u(re^{it})\|_E^2 = \sum_0^\infty \|a_k\|_E^2,$$

so the condition mentioned in Definition C.2.2 is equivalent to the condition

$$\sum_0^\infty \|a_k\|_E^2 < \infty.$$

Hence we see that every function $u(z) \in H^2(\mathbb{D}, E)$ can be retrieved from a function $v \in L^2_+(\mathbb{T}, E)$, indeed from $v(e^{it}) = \sum_0^\infty e^{ikt}a_k$. As $v(z)$ and $v(t)$ determine each other, we can identify the classes $H^2(\mathbb{D}, E)$ and $L^2_+(\mathbb{T}, E)$. If we provide $H^2(\mathbb{D}, E)$ with the Hilbert space structure of $L^2_+(\mathbb{T}, E)$, we can then embed $H^2(\mathbb{D}, E)$ in $L^2(\mathbb{T}, E)$ as a subspace.

Remark C.2.4. *We can retrieve $u(z)$ from $v(t)$ using the Poisson formula*

$$u(re^{it}) = \frac{1}{2\pi}\int_o^{2\pi} P_r(t-s)v(s)ds \quad 0 \le r < 1$$

where

$$P_r(t) = \frac{1-r^2}{1-2r\cos t + r^2}. \tag{C.3}$$

Theorem C.2.5 (Generalised Fatou's Theorem, [14, p. 186])**.** *Let E be a separable Hilbert space. Suppose $u \in H^2(\mathbb{D}, E)$ is given by*

$$u(z) = \sum_0^\infty z^k a_k \text{ for all } z \in \mathbb{D}$$

and suppose $v \in L^2(\mathbb{T}, E)$ *is given by*

$$v(e^{it}) = \sum_0^\infty e^{ikt} a_k \textit{ for all } e^{it} \in \mathbb{T}.$$

Then $u(z)$ *tends to* $v(t)$ *with respect to* $\|\cdot\|_E$ *as* z *tends to* e^{it} *along any path that is not tangent to the unit circle, and at every point* t *such that*

$$\frac{1}{2s} \int_{t-s}^{t+s} v(e^{i\tau})\, d\tau \to v(e^{i\tau}) \quad \textit{strongly} \quad (s \to 0),$$

thus almost everywhere.

Now, consider a function $\Theta(z)$ whose values are bounded operators from a separable Hilbert space E to a separable Hilbert space F, and suppose that the function has a power series representation

$$\Theta(z) = \sum_{k=0}^\infty z^k \Theta_k \tag{C.4}$$

with Θ_k being bounded operators from E to F. Suppose also that the series is convergent in $\mathbb{D}$. If, also,

$$\|\Theta(z)\| \le M \text{ on } \mathbb{D},$$

we will call such a function a *bounded analytic function* on $\mathbb{D}$.

For a bounded analytic function we have

$$\frac{1}{2\pi} \int_0^{2\pi} \|\Theta(re^{it})a\|_F^2 dt \le M^2 \|a\|_F^2 \; (0 \le r < 1)$$

and

$$\sum_{k=0}^\infty \|\Theta_k a\|_F^2 \le M^2 \|a\|_E^2$$

for all $a \in E$.

As in the scalar case, the limit

$$\Theta(e^{it}) = \lim_{z \to e^{it}} \Theta(z)$$

exists almost everywhere as a strong limit of operators. Moreover,

$$\Theta(e^{it}) = \lim_{r \to 1} \Theta(re^{it})$$

and $\Theta(re^{it})a$ converges in $L^2(F)$ to $\Theta(e^{it})a$ as $r \to 1$ and this limit has the Fourier expansion

$$\Theta(e^{it})a = \sum_{k=0}^\infty e^{ikt} \Theta_k a.$$

Now, with every bounded analytic function $\Theta(z)$ we associate the operator

$$\Theta\colon L^2(E) \to L^2(F),$$

defined by

$$(\Theta v)(t) = \Theta(e^{it})v(t),\ v \in L^2(E)$$

and the operator

$$\Theta_+\colon H^2(E) \to H^2(F)$$

defined by

$$(\Theta_+ u)(z) = \Theta(z)u(z),\ u \in H^2(E).$$

Definition C.2.6 ([14], p. 190)**.** *The analytic operator-valued function $\Theta(z)$ will be called*

i) inner if $\Theta(e^{it})$ is an isometry from E to F for almost every t.

ii) outer if $\Theta_+ H^2(\mathbb{D}, E)$ is dense in $H^2(\mathbb{D}, F)$.

Appendix D

The Nehari Problem

D.1 The Scalar Nehari Problem

Definition D.1.1 ([38], p. 157)**.** *We denote by* $L^\infty(\mathbb{T}, \mathbb{C})$ *the Banach space of essentially bounded Lebesgue measurable* $\mathbb{C}$*-valued functions on the unit circle* $\mathbb{T}$ *with pointwise algebraic operations and essential supremum norm:*

$$\|f\|_{L^\infty} = \operatorname{ess}\sup_{|z|=1} |f(z)|.$$

A function is said to be *essentially bounded* if it is bounded on the complement of a set of measure zero. In L^p-spaces two functions are identified if they take the same values everywhere except for a set of measure zero. So, a number $M > 0$ is an *essential upper bound* for a function $f\colon \mathbb{T} \to \mathbb{R}$ if the set

$$\{z \in \mathbb{T} : |f(z)| > M\}$$

is a set of measure zero. Then we can define

$$\operatorname{ess}\sup |f(z)| = \inf\{M > 0 : M \text{ is an essential upper bound for } |f(z)| \text{ on } \mathbb{T}\}.$$

Definition D.1.2 ([38], p. 159)**.** $H^\infty(\mathbb{D}, \mathbb{C})$ *denotes the space of bounded analytic functions on the unit disc* $\mathbb{D}$ *with the supremum norm*

$$\|Q\|_{H^\infty} \overset{\text{def}}{=} \|Q\|_{L^\infty} \overset{\text{def}}{=} \sup_{z\in\mathbb{D}} \|Q(z)\|.$$

Definition D.1.3 ([11], p. 13)**.** *We define by* $L^2(\mathbb{T}, \mathbb{C})$ *the space of square integrable functions on the unit circle with the inner product*

$$\langle f, g\rangle_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta.$$

Also, we define by $H^2(\mathbb{D},\mathbb{C})$ the space of holomorphic functions f on the open unit disc such that

$$\lim_{r\to 1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^2 d\theta\right)^{1/2} < \infty.$$

Remark D.1.4. *Let $0 < r < 1$. Suppose $f \in H^2(\mathbb{D},\mathbb{C})$. By Fatou's Theorem C.1.3, the radial limits*

$$\lim_{r\to 1} f(re^{i\theta})$$

exist almost everywhere on $\mathbb{T}$.

Definition D.1.5 ([11], p. 13)**.** *Consider the complete orthonormal set $\phi_n(\theta) = e^{in\theta}$, $n = 1, 2, \cdots$, in $L^2(\mathbb{T},\mathbb{C})$. If $f \in L^2(\mathbb{T},\mathbb{C})$, the numbers*

$$c_n = \langle f, \phi_n\rangle_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta$$

are the Fourier coefficients of f. The series

$$\sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

is the Fourier series for f.

Definition D.1.6 ([11], p. 13)**.** *Suppose that $f(z) = \sum_{-\infty}^{\infty} a_n z^n$, $z \in \mathbb{T}$, is the Fourier expansion of a function f. We denote by $\hat{f}(n)$ the n-th Fourier coefficient a_n of f.*

Definition D.1.7 ([38], p. 39)**.** *The orthogonal complement of a subset E of a Hilbert space H is the set*

$$\{x \in H : \langle x, y\rangle = 0, \textit{for all}\, y \in E\}.$$

It is denoted by $H \ominus E$ or by $E^{\perp}$.

Theorem D.1.8 ([11])**.** *Let E be a closed linear subspace of a Hilbert space H. Then $H = E \oplus E^{\perp}$, that is, every vector x in H is uniquely expressible in the form $x = y + z$ where $y \in E$ and $z \in E^{\perp}$.*

Definition D.1.9 ([38], p. 188)**.** *Let M be a closed linear subspace of a Hilbert space H. The orthogonal projection from H to M is the operator $P\colon H \to M$ defined by*

$$Px = y, \textit{ if } x = y + z, \textit{where}\, y \in M, z \in M^{\perp}.$$

Definition D.1.10 ([38], p. 190)**.** *For $f \in L^2(\mathbb{T},\mathbb{C})$ given by*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,\ z \in \mathbb{T}$$

we define the orthogonal projection $P_-\colon L^2(\mathbb{T},\mathbb{C}) \to L^2(\mathbb{T},\mathbb{C}) \ominus H^2(\mathbb{D},\mathbb{C})$ *by*

$$P_-\left(\sum_{-\infty}^{\infty} a_n z^n\right) = \sum_{-\infty}^{-1} a_n z^n.$$

Remark D.1.11. *By Fatou's Theorem,* $H^2(\mathbb{D},\mathbb{C})$ *can be identified with a closed subspace of* $L^2(\mathbb{T},\mathbb{C})$*. This implies that the above projection is well-defined.*

Definition D.1.12 ([38], p. 190)**.** *Suppose that* $\phi \in L^\infty(\mathbb{T},\mathbb{C})$*. The Hankel operator* H_ϕ *is the operator*

$$P_- \circ M_\phi|_{H^2(\mathbb{D},\mathbb{C})}\colon H^2(\mathbb{D},\mathbb{C}) \to L^2(\mathbb{T},\mathbb{C}) \ominus H^2(\mathbb{D},\mathbb{C}),$$

where M_ϕ *is the operator of multiplication by* ϕ *on* $L^2(\mathbb{T},\mathbb{C})$*.*

Definition D.1.13. *We can write* $L^2(\mathbb{T},\mathbb{C}) \ominus H^2(\mathbb{D},\mathbb{C})$ *as* $H^2(\mathbb{D},\mathbb{C})^\perp$*, where*

$$H^2(\mathbb{D},\mathbb{C})^\perp \stackrel{\text{def}}{=} \{f \in L^2(\mathbb{T},\mathbb{C}) \ : \ \langle f, g\rangle_{L^2} = 0 \ , \ \textit{for all } g \in H^2(\mathbb{D},\mathbb{C})\}.$$

Definition D.1.14. *Given two Hilbert spaces* H_1 *and* H_2*, denote by* $\mathcal{L}(H_1,H_2)$ *the set of all bounded linear operators* $T\colon H_1 \to H_2$*.*

Definition D.1.15. *Let* H_1 *and* H_2 *be Hilbert spaces and let* $T \in \mathcal{L}(H_1,H_2)$*. A maximizing vector for* T *is a non-zero vector* $x \in H_1$ *at which* T *attains its norm, that is, such that*

$$\|Tx\| = \|T\|\|x\|.$$

In general, a maximizing vector need not exist for a bounded linear operator.

Definition D.1.16 ([29], p. 103)**.** *Let* $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$ *be Banach spaces, let* $U = \{x \in E\colon \|x\|_E \le 1\}$ *be the unit ball in* E *and let* $T\colon E \to F$ *be a linear operator.* T *is a* compact operator *if the closure of* $T(U)$ *is a compact set in* $(F, \|\cdot\|_F)$*.*

Definition D.1.17 ([19], p. 25)**.** *Let* E, F *be Hilbert spaces and* $T\colon E \to F$*. The* essential norm *of the operator* T *is defined by*

$$\|T\|_e = \inf\{\|T-K\| : \ K \textit{ is compact }\}.$$

Theorem D.1.18 (Hartman's theorem, [19], p. 27)**.** *Let* $\phi \in L^\infty(\mathbb{T},\mathbb{C})$*. Then*

$$H_\phi \textit{ is compact if and only if } \phi \in H^\infty(\mathbb{D},\mathbb{C}) + C(\mathbb{T},\mathbb{C}).$$

Definition D.1.19 ([19])**.** *Consider the space* $L^2(\mathbb{T},\mathbb{C})$*. The* bilateral shift operator *is defined to be the multiplication by* z *on* $L^2(\mathbb{T},\mathbb{C})$*. Its restriction to* $H^2(\mathbb{D},\mathbb{C})$ *is called the* unilateral shift.

Theorem D.1.20 ([19], p. 26). *Let $\phi \in L^\infty(\mathbb{T}, \mathbb{C})$. Then, the essential norm of the Hankel operator $H_\phi \colon H^2(\mathbb{D}, \mathbb{C}) \to H^2(\mathbb{D}, \mathbb{C})^\perp$ satisfies*

$$\|H_\phi\|_e = \mathrm{dist}_{L^\infty}(\phi, H^\infty(\mathbb{D}, \mathbb{C}) + C(\mathbb{T}, \mathbb{C})).$$

Lemma D.1.21 ([19], p. 26). *Let $K \colon H^2(\mathbb{D}, \mathbb{C}) \to H^2(\mathbb{D}, \mathbb{C})^\perp$ be a compact operator. Then*

$$\lim_{n\to\infty} \|KS^n\| = 0,$$

for the shift operator S on $H^2(\mathbb{D}, \mathbb{C})$.

Problem D.1.22 (The Nehari Problem). [38, Problem 15.6] *Given $\phi \in L^\infty(\mathbb{T}, \mathbb{C})$, find $g \in H^\infty(\mathbb{D}, \mathbb{C})$ such that*

$$\|\phi - g\|_{L^\infty}$$

is minimised.

Theorem D.1.23 (Nehari's Theorem). [38, Theorem 15.14] *Suppose that $\phi \in L^\infty(\mathbb{T}, \mathbb{C})$. Then*

$$\|H_\phi\| = \mathrm{dist}(\phi, H^\infty(\mathbb{D}, \mathbb{C})).$$

Moreover there exists $\psi \in L^\infty(\mathbb{T}, \mathbb{C})$ such that the Hankel operators H_ϕ, H_ψ satisfy

$$H_\phi = H_\psi$$

and

$$\|\psi\|_\infty = \|H_\phi\|.$$

Any function $Q \in H^\infty$ at which the infimum $\inf_{Q \in H^\infty} \|\phi - Q\|_{L^\infty}$ is attained will be called a solution of the Nehari Problem for ϕ. For $\phi \in L^\infty$ there may be a unique solution or infinitely many solutions.

Theorem D.1.24 ([38],p. 196). *Let $\phi \in L^\infty(\mathbb{T}, \mathbb{C})$ and suppose that the Hankel operator H_ϕ has a maximizing vector $v \in H^2(\mathbb{D}, \mathbb{C})$. Then there exists a solution of the Nehari problem and every solution Q satisfies*

$$(\phi - Q)v = H_\phi v$$

and so,

$$Q(z) = \phi(z) - \frac{H_\phi v(z)}{v(z)} \tag{D.1}$$

almost everywhere on $\mathbb{T}$.

Remark D.1.25. *Since $v \in H^2(\mathbb{D})$, and v is not identically zero on $\mathbb{D}$, then v is non-zero almost everywhere on $\mathbb{T}$.*

Remark D.1.26. *It follows from Theorem D.1.24 that if H_ϕ has a maximizing vector, then the Nehari problem for ϕ has a unique solution Q given by equation (D.1).*

D.2 The Matricial Nehari Problem

In this section we present an established generalisation of the results obtained in Appendix B.1 to the matrix-valued setting.

Definition D.2.1. *For any $G \in L^\infty(\mathbb{T}, \mathbb{C}^{m\times n})$, we define the Hankel operator with symbol G to be the operator*

$$H_G \colon H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^m)^\perp$$

given by $H_G x = P_-(Gx)$, where

$$P_- \colon L^2(\mathbb{C}^m) \to H^2(\mathbb{C}^m)^\perp \stackrel{\text{def}}{=} L^2(\mathbb{C}^m) \ominus H^2(\mathbb{C}^m)$$

is the orthogonal projection operator.

The following is the Nehari Problem for matrix-valued functions.

Problem D.2.2. *Given $\phi \in L^\infty(\mathbb{T}, \mathbb{C}^{m\times n})$, find all $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ such that $\|\phi - Q\|_{L^\infty}$ is minimised.*

Theorem D.2.3 ([16])**.** *For any matrix-valued $\phi \in L^\infty(\mathbb{T}, \mathbb{C}^{m\times n})$,*

$$\inf_{Q\in H^\infty(\mathbb{D},\mathbb{C}^{m\times n})} \|\phi - Q\|_\infty = \|H_\phi\|$$

and the infimum is attained.

Theorem D.2.4. [24, Theorem 0.2] *Let $\phi \in L^\infty(\mathbb{T}, \mathbb{C}^{m\times n})$ be such that H_ϕ has a Schmidt pair (v, w) corresponding to the singular value $t = \|H_\phi\|$. Let Q be a function in $H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$ at minimal distance from ϕ. Then*

$$(\phi - Q)v = tw \ \text{ and } \ (\phi - Q)^* w = tv.$$

Moreover

$$\|w(z)\|_{\mathbb{C}^m} = \|v(z)\|_{\mathbb{C}^n} \text{ almost everywhere on } \mathbb{T}$$

and

$$\|\phi(z) - Q(z)\| = t \text{ almost everywhere on } \mathbb{T}.$$

Proof. By Nehari's Theorem, $\|\phi - Q\|_{L^\infty} = t$ and, by hypothesis,

$$H_\phi v = tw, \qquad H_\phi^* w = tv.$$

If $t = 0$ then $\phi \in H^\infty(\mathbb{D}, \mathbb{C}^{m\times n})$, so that $\phi = Q$ and the statement of the theorem is trivially true. We may therefore assume $t > 0$. Thus $H_\phi^* H_\phi v = t^2 v$, and so v is a maximising vector for H_ϕ. We can assume that v is a unit vector in $H^2(\mathbb{D}, \mathbb{C}^n)$, and then w is a unit vector in $H^2(\mathbb{D}, \mathbb{C}^m)^\perp$ and is a maximising vector for H_ϕ^*. We have

$$t = \|H_\phi v\| = \|H_{\phi - Q} v\| = \|P_-(\phi - Q)v\| \le \|(\phi - Q)v\| \le \|\phi - Q\|_{L^\infty} = t.$$

The inequalities must hold with equality throughout, and therefore

$$\|P_-(\phi - Q)v\| = \|(\phi - Q)v\|,$$

which implies that $(\phi - Q)v \perp H^2$ and so

$$H_\phi v = P_-(\phi - Q)v = (\phi - Q)v.$$

Furthermore $\|(\phi - Q)v\| = \|(\phi - Q)\|_{L^\infty}\|v\|$ and since $v(z)$ is therefore a maximizing vector for $\phi(z) - Q(z)$ for almost all z, we have $\|\phi(z) - Q(z)\| = \|H_\phi\|$.

Likewise,

$$\begin{aligned} t = \|H_\phi^*\| = \|H_{\phi-Q}^*\| = \|H_{\phi-Q}^* w\| &= \|P_+(\phi - Q)^* w\|_{L^2} \le \|(\phi - Q)^* w\|_{L^2} \\ &\le \|(\phi - Q)^*\|_{L^\infty}\|w\|_{L^2} = \|(\phi - Q)^*\|_{L^\infty} = t. \end{aligned}$$

Again, the inequalities hold with equality throughout, and in particular

$$\|P_+(\phi - Q)^* w\|_{L^2} = \|(\phi - Q)^* w\|_{L^2},$$

so that $(\phi - Q)^* w \in H^2$ and

$$(\phi - Q)^* w = H_\phi^* w = tv.$$

□

Theorem D.2.5. [12, Theorem 7.3.5] *If $A \in \mathbb{C}^{m\times n}$ has rank k, then A can be written as*

$$A = UWV,$$

for some matrix $W \in \mathbb{C}^{m\times n}$ with non-negative diagonal entries and for some unitary matrices $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$. The matrix $W = (s_{ij}) \in \mathbb{C}^{m\times n}$ has $s_{ij} = 0$ for $i \ne j$, $i = 0, 1, \cdots, m-1$, $j = 0, 1, \cdots, n-1$, and for $i = j$,

$$s_0 \ge s_1 \ge \cdots \ge s_k \ge s_{(k+1)} = \cdots = s_q = 0$$

with $q = \min\{m, n\}$. The numbers s_i are the non-negative square roots of the eigenvalues of

AA^* and are also known as the singular values *of the matrix A.*

The following example encapsulates the necessity of considering the superoptimal analytic approximation in order to obtain a unique best approximant.

Example D.2.6. *Let*

$$G(z) = \begin{pmatrix} \bar{z} & 0 \\ 0 & 0 \end{pmatrix} \in L^\infty(\mathbb{T}, \mathbb{C}^{2\times 2}).$$

Find a function $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{2\times 2})$ such that $\|G - Q\|_{L^\infty(\mathbb{T},\mathbb{C}^{2\times 2})}$ is minimised.

Solution. Firstly,

$$\|H_G\| = \text{dist}_{L^\infty}(\bar{z}, H^\infty) = 1.$$

Consider an arbitrary $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{(2\times 2)})$ of the form

$$Q(z) = \begin{pmatrix} q_{11}(z) & q_{12}(z) \\ q_{21}(z) & q_{22}(z) \end{pmatrix}.$$

The unique $q_{11} \in H^\infty(\mathbb{D}, \mathbb{C})$ such that $\|q_{11} - \bar{z}\|_{L^\infty} \leq 1$ is $q_{11} = 0$.

Hence $Q \in H^\infty(\mathbb{D}, \mathbb{C}^{2\times 2})$ is a best approximant of G if and only if Q is of the form

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & q_{22} \end{pmatrix},$$

where

$$1 = \|G - Q\|_{L^\infty(\mathbb{T},\mathbb{C}^{2\times 2})} = \left\| \begin{pmatrix} \bar{z} & 0 \\ 0 & -q_{22} \end{pmatrix} \right\|_{L^\infty(\mathbb{T},\mathbb{C}^{2\times 2})} = \max\{1, \|q_{22}\|_{H^\infty}\},$$

i.e. $\|q_{22}\|_{H^\infty} \leq 1$.

Thus the set of best analytic approximants of G is

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & q_{22} \end{pmatrix} : \|q_{22}\|_{H^\infty} \leq 1 \right\}.$$

At this point, we encounter a difficulty. The set of all optimal solutions is typically large, and we would like to be able to determine the "very best" among these best approximants. For this reason, we need to impose some additional constraints other than the minimisation of the L^∞ norm of the largest singular value, namely to regard minimizing the L^∞ norm of all the subsequent singular values.

Observe that, in this case, $s_0^\infty(G - Q) = 1$ and that

$$s_1^\infty(G - Q) = \operatorname*{ess\,sup}_{z\in\mathbb{T}} s_1 \begin{pmatrix} 0 & 0 \\ 0 & -q_{22}(z) \end{pmatrix} = \operatorname*{ess\,sup}_{z\in\mathbb{T}} |q_{22}(z)| = \|q_{22}\|_{H^\infty}.$$

Hence the unique best analytic approximant of G for which both $s_0^\infty(G-Q)$ and $s_1^\infty(G-Q)$

are minimised occurs when $q_{22} = 0$, that is,

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \qquad \square$$

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