# Harmonic Analysis, Hecke Algebra and Cohomology on Groups of Trees and Buildings


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**Abstract**

The PhD project consists of two parts. The first part is about finite trees, realizations and random walks. The second part is about the Hecke algebras of infinite trees and buildings and the cohomology groups. We note that some examples of finite trees can be generalized to the infinite cases. The words of finite length in the example of ultrametric rooted trees with finite depth can be extended to doubly infinite chains in the infinite homogeneous trees thus defining a Banach algebra.

As the background of the project, we study the topics of finite phylogenetic trees by understanding the combinatorial and geometrical structure of rooted and unrooted discrete BHV tree space of $n$ taxa. Certain types of random walks on the space of trees can be used to model the evolution process. As a method to improve the computation of such random walks, we realize some tree spaces into polytopes in Euclidean space where the vertices, edges and faces indicate trees of different degenerate levels. In particular, we study the links between the permuto-associahedra and the BHV tree-space. One specific realization is called the secondary polytope, which is used to construct the associahedra, and we will generalize this construction into more complicated examples and compare with the BME polytopes of the BHV trees.

In order to study the random walks on tree space, we apply several classic methods such as the eigensolutions of Markov chains, Gelfand pairs and spherical functions to decompose the functions on tree space. We present some classic examples where these methods solve the random walk explicitly. We consider biinvariant subalgebras of group algebra which are commutative under convolution. These arise from Gelfand pairs where spherical functions can be used to produce the eigenvectors of the transition matrix of the random walk. We note that an example of the $q$-homogeneous rooted tree of a finite depth is a good link to generalize the study from finite to infinite cases where the space is still discrete.

The first example in the second part of the thesis is the infinite homogeneous trees and we study the invariant subalgebra under the $\ell^1$ norm. The space can be discretized to $\mathbb{Z}_+$ and we show that it is isomorphic to a Hecke algebra with single generator, the Hecke operator which corresponds to the random walk generator. It is natural to consider some key properties of the algebra, i.e. the spherical functions, character space, derivations and b.a.i.. The main example we study is the Gelfand pair given by projective general linear groups over $p$-adic numbers and the subgroup corresponding to the the $p$-adic integers, where the example of the smallest dimension corresponds to infinite homogeneous trees and examples of higher dimensions correspond to the Bruhat-Tits buildings of type $\tilde{A}$.

We claim that the Hecke algebras of these Gelfand pairs are isomorphic to the invariant subalgebras of functions on the $\tilde{A}$ lattice subject to weight conditions determined by $p$.

Based on the isomorphic algebras on the type $\tilde{A}$ lattices, we consider the examples of types $\tilde{A}$ and $\tilde{B}$, with and without the invariance conditions under the Weyl group action on the lattices. We show that the above examples are all finitely generated and the number of generators in each case are equal to the dimension of the lattice in the Euclidean space. We then compute cohomology groups of the algebras of functions on the weighted lattices. We build up from the methods introduced in the examples of those similar to $\mathbb{Z}_+$ and $\mathbb{Z}_+^k$ . The general idea is to calculate the approximate formulae from the precise ones in $\mathbb{Z}_+$ and $\mathbb{Z}_+^k$ and iterate the process with an induction by reducing the degree of the leading terms. We also expect this method can be generalized to the Hecke algebras of other Gelfand pairs with corresponding weighted lattices.

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# Contents

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# Chapter 1

# Overview

In this chapter, we briefly outline the materials discussed in the contents. This PhD project is motivated by finite phylogenetic trees and the aim is to investigate specific problems related to the combinatorial, geometrical and algebraic structure of trees. The first part of the thesis focused on the problems of finite dimensional examples, mainly on the combinatorial properties and random walks. We review some background in phylogenetics and give our own realizations of certain tree spaces in terms of convex polytopes in Euclidean space. The second part of thesis move on to the problems in infinite dimensional examples, namely the infinite homogeneous trees and buildings. We study the Banach algebra of functions on the vertices of trees and buildings and state the link to the Iwahori-Hecke algebras with the $\ell^1$ norm. We also study some cohomology groups of algebras of functions on lattices which corresponds to the subalgebras of functions on trees and buildings via an isomorphism.

Chapter 2 starts with a brief introduction to the topic of phylogenetic trees. We fix the number of leaves to be $n$ and consider the *semi-labeled* trees. We define a tree as a connected graph with no cycles and specify the leaf vertices of degree 1 to be labeled and the internal vertices to be unlabeled. For $n \geq 3$, the space of fully resolved semi-labeled trees, denoted by $\mathcal{S}_n$ in Definition 2.11, consists of all trees with all $n$ leaf vertices labeled by the set of $n$ species and unlabeled $(n-2)$ internal vertices of degree 3. The discrete phylogenetic tree space, denoted by $\mathcal{S}_n^*$ in Definition 2.12, consist of all semi-labeled trees with $n$ leaves, including the trees in $\mathcal{S}_n$ and the semi-labeled trees with internal edges of degree greater than 3. Both $\mathcal{S}_n$ and $\mathcal{S}_n^*$ are finite sets thus we consider them to be discrete. By defining the adjacencies between the trees, e.g. by the *nearest neighbour interchange* in Section 2.2.2, we can define a graph and study the simple random walk on the corresponding graph.

We also define the continuous tree space $\mathcal{T}_n$ by specifying the internal edge lengths for the trees in the discrete space $\mathcal{S}_n^*$, as in Definition 2.16. We embed the space $\mathcal{T}_n$ into the Euclidean space $\mathbb{R}^{2^{n-1}-n-2}$ whose basis corresponds to the non-trivial splits of the $n$ leaf labels.

The new results are presented in Section 2.3. We present the realizations of some tree spaces which preserve the nearest neighbour interchange adjacencies and combinatorial structures. We realize the permutations of $n$ leaf labels on the permutohedron $P_n$ as an $(n-1)$-dimensional convex polytope in Section 2.3.1. We then realize the space of fully resolved trees with leaf labels in a certain permutation on the associahedron $K_n$ as an $(n-2)$-dimensional convex polytope in Section 2.3.2. We apply the secondary polytope construction to produce an explicit half-space presentation of $K_n$, with the proof using an original idea of the folding process in Theorem 2.39. We then combine the realizations of $P_n$ and $K_n$ to construct the $(n-1)$-dimensional convex polytope permuto-associahedron $KP_n$ which consists of all fully resolved trees in the discrete space $\mathcal{S}_n$ in Section 2.3.3. Finally we generalize the secondary polytope construction to construct a polytope of higher dimension for the discrete space $\mathcal{S}_n$ as a generalized associahedron.

Chapter 3 is motivated by the topic of simple random walks on discrete tree spaces. We consider the algebra of functions on a group $G$ and on the homogeneous space $X = G/K$ where $K$ is a subgroup. We present the definitions of finite Gelfand pairs and study some examples where the bi-$K$-invariant subalgebra is commutative under convolution. The spherical functions are applied to analyse the characters of the subalgebra. We will clarify the relation between the lumpable random walk on the partitions generated from the double cosets of a Gelfand pair $(G, K)$ and the bi-$K$-invariant subalgebra of funtions on the finite group $G$. The eigensolutions to the transition matrix of the simple random walk on the homogeneous space $G/K$ can be obtained from the spherical functions for the Gelfand pair $(G, K)$. In particular, we study the random walk on the vertices of the Petersen graph, which corresponds to the discrete tree space $\mathcal{S}_5^*$. We also briefly outline how the random walks on other discrete tree space can be lumped to reduce the dimensions of the transition matrices for the simple random walks.

Chapter 4 starts with the infinite homogeneous tree $\mathbb{T}_q$. We study the Gelfand pair given by the automorphism group acting on $\mathbb{T}_q$ and the subgroup which stabilizes a fixed vertex $x_0$. The Iwahori-Hecke algebra defined from the Gelfand pair is singly generated and can be explicitly considered as an algebra on $\mathbb{Z}_+$. We study the spherical functions, characters, the existence of point derivations and bounded approximate identities of these algebras. We introduce a new method of the shift matrix to compute spherical functions and characters. The main result is Theorem 4.25 which proves that the bi-$K$-invariant subalgebras of functions on the $p$-adic infinite trees with $\ell^1$ norm are isomorphic to the $S_2$-invariant subalgebras on integers where the multiplications satisfy the rules for multiplications between Laurent polynomials with an $\omega_R$-weighted $\ell^1$ norm.

Chapter 5 is a generalization from the example of the automorphism group acting on $\mathbb{T}_q$. We consider the Bruhat-Tits buildings defined from the homogeneous space $PGL_n\left(\mathbb{Q}_p\right)/PGL_n\left(\mathbb{Z}_p\right)$, the projective general linear groups over $p$-adic numbers and integers. In particular, we set the group $G = PGL_n\left(\mathbb{Q}_p\right)$ and the subgroup $K = PGL_n\left(\mathbb{Z}_p\right)$ and show that the pair $(G, K)$ is a weakly symmetric Gelfand pair. Some results still need further justification so we present them as conjectures. The bi-$K$-invariant subalgebra of functions on the building correspond to the algebra of functions on a Weyl chamber of type $\tilde{A}$ lattices which is isomorphic to $\mathbb{Z}_+^{n-1}$, satisfying a multiplication rule $*_p$ determined by the value of the prime number $p$.

In Chapter 6, we study the algebra of functions on Type $\tilde{A}$ and Type $\tilde{B}$ lattices with the $\omega_R$-weighted $\ell^1$ norm. We also study the invariant subalgebra under the group actions of $S_n$ and $B_n$ on the corresponding lattices, namely $\mathcal{A}_{n,\omega_R}$ and $\mathcal{B}_{n,\omega_R}$ . We describe the multiplication rules as the multiplications between the Laurent polynomials, the generators and point derivations of the two types of algebras.

For the higher cohomology groups of singly generated algebras, we review some well known methods to compute the cohomology groups of $\ell^1\left(\mathbb{Z}_+\right)$ on the point modules and dual modules, with and without the weight condition $\omega_R$ on the $\ell^1$ norm. For the simplicial and cyclic cohomology, we study an explicit construction of coboundaries for the algebra $\ell^1\left(\mathbb{Z}_+\right)$ in [20]. The method is applied to derive an approximate construction of coboundaries with finite inductive steps to compute the simplicial and cyclic cohomology groups of the algebra $\mathcal{A}_{2,\omega_R}$ which is isomorphic to the Iwahori-Hecke algebra generated from the infinite homogeneous tree in Chapter 4. The main statement is presented in Theorem 6.45. We finish with a conjecture of the higher cohomology groups of the invariant subalgeras of functions on the weighted type $\tilde{A}$ and type $\tilde{B}$ lattices under the Weyl group actions.

# Part I

# Finite trees, realizations and random walks

# Chapter 2

# Background and finite tree spaces

## 2.1 Introduction

Biologists use phylogenetic trees to analyse the evolution and structure of genes. Phylogenetic trees can be built by comparing DNA sequences using specific parts of the genome [53], [61]. Trees play a very important role in many fields of biology such as bioinformatics, systematics, and comparative phylogenetics [53]. Information can be obtained from a phylogenetic tree so that we give ourselves a better knowledge of evolutionary relationships between species.

Trees have mathematical structures based on graph theory and the space of phylogenetic trees can be studied from combinatoric and geometric view points [9], [62] . Scientists are keen to use mathematical tools to describe the space of evolutionary trees, which are widely used by biologists and statisticians to analyze data numerically [4]. In a specific version of tree space, the candidates for species on the leaves are fixed therefore we consider the space to be finite. Within the space, the divergent patterns and their network of paths give us numerous possibilities to infer the most likely evolutionary paths.

The space of phylogenetic trees is mostly studied by pure mathematicians analytically [37], [26], [27]. Originally the finite dimensional problem was approached by the triangulations of convex polygons, which are the dual graphs of a binary tree with fixed cyclic order of the labels [28]. In the last decade, the study of tree spaces has exploded and attracted different approaches from various directions [8], [9], [56], [23], [19], [57]. Therefore we review some important results in this research subject and develop it into more general cases.

The aim of this thesis is to investigate some pure mathematics developed for the study of tree spaces in both finite and infinite dimensions. We start by sketching a classification of tree spaces from different properties of geometry and combinatorics. We will then move on to the relations and adjacencies within the same tree space (the random walk) and compare our analysis with others. There are also published results from different perspectives and we would like to clarify the relationships between these approaches after the analytic work [59], [52].

We would like to produce a result on the decomposition of functions on tree spaces which use the Laplace operators from the adjacencies. The solution will be the decomposition into eigenvectors giving harmonic analysis on tree spaces. Meanwhile, the random walk on continuous spaces can be interpreted as the solutions to the heat equation and diffusion processes in applied mathematics. Similarly, the eigensolutions from the adjacency matrices will give us information including processes and paths of evolution by likelihood in statistics.

In graph theory, a tree is defined to be an undirected graph $T = (E, V)$ which is connected and without any simple cycles. Every pair of distinct vertices on a tree is joined by a unique path which is a set of edges. We consider trees with no trivial internal vertices of degree 2 in the unrooted cases. The only vertex we allow to have degree 2 is the vertex that corresponds to the root in a rooted tree. A *leaf vertex* is a vertex of degree 1 and a *leaf edge* is the edge which is connected to a leaf vertex. A tree space $\mathcal{T}$ is defined to be the collection of all trees that satisfy certain properties. We present some classic examples of trees, their related expressions and embeddings.

**Definition 2.1.** *For $n \geq 3$, a star $s_n$ is a tree with $n$ edges and $(n+1)$ vertices, the only internal vertex $i_0$ and the leaf vertices $\{v_1, \cdots, v_n\}$.*

The internal vertex $i_0$ has degree $n$ and is connected with all leaf vertices and every leaf vertex has degree 1 is only joined to the internal vertex $i_0$. There are no internal edges in this graph. We may consider the star tree as the fully degenerate phylogenetic tree of $n$ species. In the realization and the computation of metrics on tree spaces, the star tree is often considered as the origin in the corresponding Euclidean space [5].

Note that the star tree is *bipartite* therefore there exists an eigensolution which corresponds to the alternating eigenvector with eigenvalue $-1$ in the simple random walk on the graph in addition to the constant solution with eigenvalue 1. In this case, the simple random walk does not converge to a stationary distribution.

**Definition 2.2.** *A tree is called* semi-labeled *if the leaf vertices are labeled and the internal vertices are not labeled.*

For a semi-labeled unrooted tree with $n$ leaves, we choose $X = \{t_1, t_2, \ldots, t_n\}$ to be set of leaf labels which correspond to the species. To simplify the labeling, we might replace them with the set of numbers $\mathcal{N} = \{1, 2, \ldots, n\}$. For a semi-labeled rooted tree, one of the leaf labels is chosen to be the root. Note that in a binary rooted phylogenetic tree, the leaf edge for the root has positive length therefore the internal vertex adjacent to the root vertex has degree at least 3.

**Definition 2.3.** *A* permutation *of $X$ is a sequence: $i_{\pi(1)}, i_{\pi(2)}, \ldots, i_{\pi(n)}$ where $\{i_1, i_2, \ldots, i_n\} = X$ and the group element $\pi \in S_n$.*

We start from a string given by a permutation of $X$. Sometimes we write the root label $R$ or 0 in front of the string so that it has length $(n + 1)$. We then add brackets on the string by the following rules. We pick a subsequence of consecutive elements of length at least 2 in the string, i.e. the subsequence $j_1, \ldots, j_k$ where there exists $t$ such that $j_l = i_{l+t}$ for all $l = 1, \ldots, k$. Then we put a left bracket to the left of $j_1$ and a right bracket to the right of $j_k$ on the string so that the subsequence $j_1, \ldots, j_k$ is bracketed.

**Definition 2.4.** *The* partition *given by a bracketed subsequence $j_1, \ldots, j_k$ of the sequence $i_{\pi(1)}, i_{\pi(2)}, \ldots, i_{\pi(n)}$ is a pair of disjoint subsets $\{j_1, \ldots, j_k\}$ and $\{X \cup R\} \setminus \{j_1, \ldots, j_k\}$ of the set $X \cup R$.*

We then repeat the process by bracketing more subsequences of consecutive elements of the string by choosing a subsequence $X_{jk}$ that satisfies that following conditions:

- the length of $X_{jk}$ must be at least 2;
- the subsequence $X_{jk}$ must not be identical to any subsequence that has been already bracketed;
- given any previously bracketed sequence $X_0$, $X_{jk}$ and $X_0$ can either be disjoint, or one of them is completely included in the other i.e. the two brackets for $X_0$ and $X_{jk}$ are *compatible*.

**Remark 2.5.** *The first two conditions make sure that all bracketings are nontrivial and the third condition defines the pairwise* compatible *relations of the set $X \cup R$ for any two distinct brackets.*

The string is fully bracketed if there are not any nontrivial brackets to be added by the rules above. Note that a single label in the string is also fully bracketed. In a fully bracketed string, every bracketed subsequence consists of a unique partition of two disjoint bracketed subsequences. Every single label in the string is also one of the two disjoint sets of the partition for a unique larger bracketed subsequence.

**Definition 2.6.** *A semi-labeled* Newick string *of $X$ without edge lengths is a string of $X$ with compatible nontrivial bracketings.*

We will now construct the semi-labeled rooted binary tree from the semi-labeled Newick string which is fully bracketed. The semi-labeled rooted binary tree will be constructed by defining the vertices and edges which are (computer-)readable from the Newick string. For the binary tree to be nontrivial we require $n \geq 3$. We start with the star tree $s_3$ and label one leaf vertex by the root label $R$. Then we find two subsequences $A_1$ and $A_2$ whose union is the full sequence $A$ on the Newick string such that $A_1$ and $A_2$ are both bracketed or just consist of a single label. We then define both subsequences to be the *children* of the root $R$ and label them on the other two vertices of the star tree respectively. If a subsequence $X_{jk}$ consists of more than one labels, we label two vertices by the two disjoint bracketed subsequences of its partition as its *children* and join each of these two vertices to the vertex labeled $X_{jk}$ by an edge. We repeat this process on all subsequences until vertices of children all correspond to single labels. We obtain a connected graph which is a tree with $(n+1)$ leaves labeled by the set $X$ and the root $R$. The bracketed subsequences on the internal vertices not only define the partition of splits, but also define the subtrees as they are all in the semi-labeled *Newick string* format. An example of semi-labeled rooted binary tree is given by the figure below.

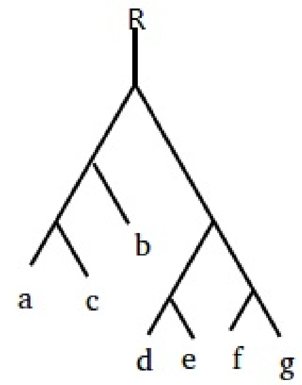


Figure 2.1: The rooted binary tree constructed from the Newick string $\big((a,c)\,,b\big)\,,\big((d,e)\,,(f,g)\big)$.

Every fully bracketed semi-labeled Newick string and the rooted binary tree has a planar embedding to a triangulated convex polygon and these two constructions have many key properties in common.

**Definition 2.7.** *A* triangulation *of a convex polygon with labeled edges is a cutting of the polygon into triangles by connecting vertices with non-crossing line segments.*

For a semi-labeled *Newick string* of $X$ where $|X| = n$, we set the convex polygon to have $(n+1)$ edges and label the top edge as the root label $R$. Then we label the other edges anticlockwise from $R$ by the string of $X$. Every non-crossing line segment inside the convex polygon splits the edge labels into a partition of two disjoint strings with length at least 2. The string without the root label $R$ is a subsequence which can be bracketed. The non-crossing condition for the line segments correspond to the property that all partitions of the bracketings are compatible. When the convex polygons are cut into triangles, we cannot add any other line segments which connect vertices and do not cross the existing line segments. This property corresponds to the condition of the *fully bracketed* string. Hence we construct the corresponding triangulation of the convex polygon from a fully bracketed Newick string. The embedding of the rooted binary tree is by putting the vertices inside the corresponding triangles of partitions and outside the labeled edges of the convex polygon. We then join two vertices by an edge if their regions share a common edge.

**Definition 2.8.** *A triangulated convex polygon is the* dual polygon *of a semi-labeled Newick string and the corresponding rooted binary tree if it satisfies two conditions:*

1. *the anticlockwise permutation of the edge labels apart from the root label $R$ is the same to the permutation for the sequence of the Newick string;*
2. *every partition of the edge labels given by a line segment inside the polygon is a partition defined by a bracketed subsequence as in Definition 2.4.*

An example of the triangulation with the embedded binary tree of the above Newick string is given by the figure below.

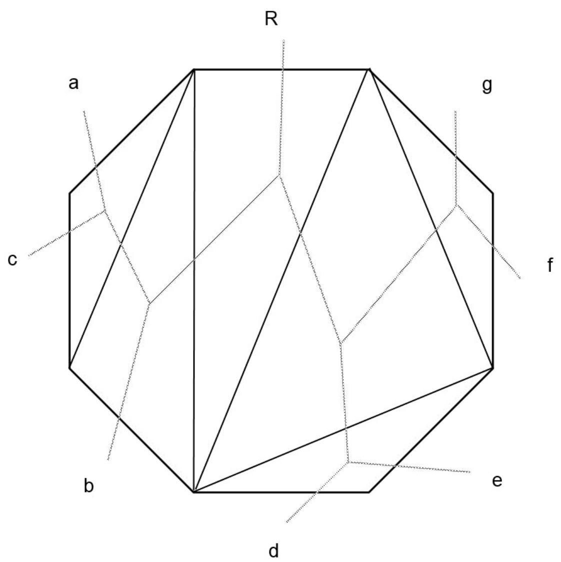


Figure 2.2: The triangulation of the dual polygon to the Newick string $\big((a,c)\,,b\big)\,,\big((d,e)\,,(f,g)\big)$.

As a triangulation defines a unique set of compatible partitions of $X \cup R$, it gives a full bracketing of the semi-labeled Newick string of $X$ which shows that it is also a one-to-one correspondence. There are $C_{n-1}$ ways to triangulate a convex polygon where $C_m$ is the $m$-th Catalan number $\frac{1}{m+1}\binom{2m}{m}$.

Conversely, it is also possible to construct a triangulated convex polygon from a binary tree as both graphs are planar and we can place all leaves across the polygon's edges and connect the vertices through the partitions of leaf labels given by the internal edges of the tree. The triangulated polygon which is constructed from the binary tree is not unique as there are more than one planar embeddings given by the valid cyclic orders of the leaves. We will then discuss the groups that act on the Newick strings and triangulated polygons, thus find the invariance conditions of the embedded binary trees.

If we fix the root label $R$ and the positions of the brackets, the symmetric group $S_n$ acts on the permutation of elements of $X$. This group action preserves the shape of the triangulation but changes the semi-labeling of the corresponding convex polygon.

We will now define a equivalence relation on the set of triangulated polygons and their corresponding Newick strings which give the same binary trees.

**Definition 2.9.** *Two semi-labeled* Newick strings *are said to be* BHV-equivalent *if all partitions read from their corresponding bracketings are the same.*

This implies the same equivalence relations on the triangulations for the *dual polygon* due to the one-to-one correspondence.

If the polygon is regular, the dihedral group $D_{2(n+1)}$ acts on the semi-labeled triangulated convex through rotations and reflections. The cyclic order of the elements in the string $X$ and the embedded binary tree is preserved under this action but the top edge for the root label may change and the bracketing of the Newick string might be different. Note that a rooted tree can also be unrooted if we do not specify the labeled leaf $R$ as its root. Given an internal edge in the triangulation that cuts the polygon into two small semi-labeled polygons, the reflection of one of the two small polygons about the perpendicular bisector of that internal edge, together with the labels, also fix the partitions, but is likely to change the permutation of the labels. In the next section, we will enumerate the number of fully resolved phylogenetic trees through the constructions of semi-labeled Newick strings and triangulations and calculate the number of *BHV-equivalent* binary trees by the invariance conditions given by the actions described above.

**Example 2.10.** *The $q$-homogeneous rooted tree of depth $n$.*

Let $\Sigma = \{0, 1, \dots, q-1\}$ be the *alphabet* where $q \in \mathbb{Z}_+$. A finite *word* $\omega$ over $\Sigma$ is a sequence $\omega = \sigma_1 \dots \sigma_k$ of length $k$ where $\sigma_j \in \Sigma$ for all $j = 1, \dots, k$. We denote by $\Sigma^k$ the set of words of length $k$ and $\Sigma^0 = \emptyset$ the *empty word*.

We define the graph $T = (V, E)$ of the finite *$q$-homogeneous tree of depth $n$* $\mathbb{T}_{q,n}$ where $n \geq 2$. First we define the vertex set $V$ to be $\bigcup_{k=0}^{n} \Sigma^k$, the set of words with length

0 to $n$. Then we define the edge set $E$ by the adjacency relations between the vertices. Two vertices are connected by a single edge if and only if one word can be obtained from the other by adding or deleting the final letter. Two vertices are adjacent if they are connected. We can see that there are not any cycles in the graph therefore it is a tree. The empty word $\emptyset$ is adjacent to the $q$ words of length 1 and every word of length $n$, $\omega = \sigma_1 \dots \sigma_n \in \Sigma^n$ is adjacent to one word of length $n-1$, $\omega' = \sigma_1 \dots \sigma_{n-1}$. Every internal vertex corresponds to a word of length $k$ where $1 \leq k \leq n-1$ and is adjacent to $q$ words of length $k+1$ and one word of length $k-1$. The leaves are given by $\Sigma^n$, the set of words of length $n$. All internal vertices have degree $q+1$, therefore the tree is homogeneous. Given an internal vertex $v = \sigma_1 \dots \sigma_k \in \Sigma^k$, we define $\mathbb{T}_v$ to be the *subtree rooted at* $v$. The subtree $\mathbb{T}_v$ is a subgraph of $\mathbb{T}_{q,n}$ which consists of vertices of words of the form $vv'$ where $v' \in \bigcup_{j=0}^{n-k} \Sigma^j$ and the edges are given by the adjacency relations defined above.

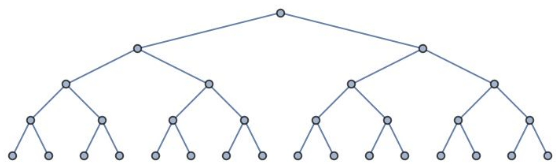

Figure 2.3: $\mathbb{T}_{2,4}$, the binary tree of depth 4.

The symmetric group $S_q$ acts at every internal vertex and the automorphism group $Aut\left(\mathbb{T}_{q,n}\right)$ acts on the entire tree. We define $G_v \simeq S_q$ to be the group that acts at the internal vertex of the word $v$ and there exists an isomorphism between $G_v$ and $S_q$ given by $\theta_v(\pi) = \theta_{v,\pi} \in Aut\left(\mathbb{T}_{q,n}\right)$ where $G \in S_q$. The group $G_v \simeq S_q$ stabilizes all vertices which are not on the subtree $\mathbb{T}_v$. For a word $v\sigma_{k+1}v' \in \mathbb{T}_v$ where the word $v$ has length $k$ and $v' \in \bigcup_{j=0}^{n-k-1} \Sigma^j$, given a group element $\pi \in S_q$ and the corresponding element $\theta_{v,\pi} \in Aut\left(\mathbb{T}_{q,n}\right)$, we have $\theta_{v,\pi}\left(v\sigma_{k+1}v'\right) = v\pi\left(\sigma_{k+1}\right)v'$. In the next chapter, we will define the ultrametric on this space and identify a subgroup $K$ which is the stabilizer of a fixed word $00\dots0$. We also consider the algebra of functions on the group and compute the spherical functions of the commutative bi-invariant subalgebras arising from the Gelfand pair of the automorphism group and the subgroup $K$.

## 2.2 Phylogenetic trees

In this section, we present various spaces of phylogenetic trees with $n$ labeled leaves. We will first define the discrete spaces given by the set of trees as graphs by considering the case where all internal vertices have degree 3 and the case where some internal vertex has degree at least 4. We will also define the continuous space of semi-labeled phylogenetic trees where the internal edges are assigned with lengths.

We start by enumerating the tree shapes in the discrete semi-labeled tree space where all internal vertices have degree 3 and introduce different representations of a semi-labeled tree including the subtree splits, the quartet display and the trees and matchings of $2n$ numbers [17]. We will then describe the adjacency relations between the trees in the space as an essential setting for a random walk e.g. the nearest neighbour interchange (NNI). The settings will be applied in the next section where we realize the discrete tree spaces to construct the polytope from the adjacency relations.

### 2.2.1 Combinatorics

In terms of a graph, for $n \geq 3$, a *fully resolved* semi-labeled phylogenetic tree with $n$ species on the leaves consists of $n$ leaves, $(n-3)$ internal edges with positive lengths and $(n-2)$ internal vertices of degree 3. A tree is said to be *semi-labeled* by the set of species $X$ if the leaves are labeled by distinct elements of the set $X$ and the internal vertices are not labeled. Normally we just define the set of $n$ species $X$ to be the set of numbers $\mathcal{N} = \{1, 2, \ldots, n\}$. As a graph, a tree can be rooted with a root label $R$ or unrooted if we do not specify a root label. If there is a root, we may define the root label to be the number 0 as it satisfies the condition of a leaf on the graph of the tree.

A tree is either *fully resolved* if every internal vertex has degree 3 or *degenerate* if there exists an internal vertex with degree at least 4. We start from the *discrete* tree space to consider the trees as graphs and do not assign lengths to the edges. We will now define the sets of semi-labeled trees for only the fully resolved case and the set which consists of both fully resolved and degenerate trees.

**Definition 2.11.** *The* fully resolved discrete phylogenetic tree space $\mathcal{S}_n$ *is the set of fully resolved semi-labeled trees with n leaves [9].*

We will describe the structure of the adjacencies between the all trees in the set $\mathcal{S}_n$ in the next subsection 2.2.2.

**Definition 2.12.** *The* discrete phylogenetic tree space $\mathcal{S}_n^*$ *is the set of semi-labeled trees with n leaves which consists of both fully resolved and degenerate trees.*

If we use the same set $\mathcal{N}$ for the leaf labels, we can see that $\mathcal{S}_n$ is a subset of $\mathcal{S}_n^*$. We will first examine the combinatorial structure of $\mathcal{S}_n$ and compare with the Newick strings introduced in the previous section 2.1.

**Definition 2.13.** *A* split *of a set $X$ is a partition of $X$ into two disjoint subsets $\left\{E, E^C\right\}$.*

**Definition 2.14.** *A split $\left\{E, E^C\right\}$ of $X$ is* nontrivial *if $|X| \geq 4$ and $2 \leq |E| \leq |X| - 2$.*

**Definition 2.15.** *Two splits $\left\{E, E^C\right\}$ and $\left\{F, F^C\right\}$ are* compatible *if $E \cap F \in \left\{E, F, \emptyset\right\}$.*

This compatibility relation is also described for the bracketing of the Newick string in Definition 2.4 and Remark 2.5 in the previous section 2.1, where $E$ and $F$ can only be disjoint or one of them is a subset of the other. For a fully resolved semi-labeled tree, every internal edge generates a split of the leaf labels. This is because when we remove an internal edge, the graph becomes disconnected and consists of two semi-labeled trees with disjoint sets of leaf labels. Therefore every fully resolved tree determines $(n-3)$ compatible nontrivial partitions. Conversely, given $(n-3)$ compatible nontrivial partitions of the set $X$, we can reconstruct the corresponding fully resolved tree [55].

To be precise, every internal edge is adjacent to four other edges which gives the partition of four disjoint subsets of the leaf labels. The *quartet display* of an internal edge of a semi-labeled tree $T$ with leaf labels $X$ is given by $Q(T) = (T_1, T_2|T_3, T_4)$ of the subtrees where the disjoint sets $T_1, T_2, T_3, T_4 \in X$ with $T_1 \cup T_2 \cup T_3 \cup T_4 = X$ and the notation $|$ indicates the split $\left\{\{T_1 \cup T_2\}, \{T_3 \cup T_4\}\right\}$.

For $\mathcal{S}_n^*$, in a degenerate tree with $n$ leaves, there are less than $(n-3)$ internal edges. However we can still produce the set of compatible splits from the existing internal edges. The star tree $s_n$ is the fully degenerate tree as the internal edges do not exist therefore there are not any nontrivial splits of the leaf labels.

Given a tree $T \in \mathcal{S}_n$, every partition $\left\{E, E^C\right\}$ generated from an internal edge subdivides the tree into two subtrees. Each one of the two subtrees has a special internal vertex of degree 2 if we remove that internal edge. The subtree with the subset $E$ for the leaf labels is defined to be a *cherry* if $|E| = 2$. For $n \geq 4$, a tree has least 2 cherries. A tree is called a *caterpillar tree* if it only has 2 cherries [9].

There are various ways to count the number of trees in the space of $\mathcal{S}_n$ and we will present three of them. The three methods are given by (I) the induction of building up from the star tree $s_3$ [21]; (II) the counting of $BHV$-equivalent triangulations of semi-labeled polygon; (III) the matching algorithm that sets the one-to-one correspondence between $\mathcal{S}_n$ and the disjoint unordered 2-subsets of $(2n-4)$ numbers [17].

Method (I): we verify that for $n = 3$, there is only one possible tree which is the star tree with leaf labels $\{1, 2, 3\}$ and there is no internal edge for a star tree. For $n = 4$, there is only one internal edge and the three partitions for the three trees in $\mathcal{S}_4$ are $\left\{\{1,2\}, \{3,4\}\right\}$, $\left\{\{1,3\}, \{2,4\}\right\}$ and $\left\{\{1,4\}, \{2,3\}\right\}$. Equivalently, the three splits correspond to the three edges of the star tree $s_3$ on which we insert the leaf edge labeled 4. Note that there are 4 leaf edges and 1 internal edge for the trees in $\mathcal{S}_4$. To construct a tree in $\mathcal{S}_5$, we may insert the leaf edge labeled 5 on any one of the 5 existing edges in a tree $T \in \mathcal{S}_4$. We then repeat this inserting process, which means that to construct a tree in $\mathcal{S}_k$ from a tree in $\mathcal{S}_{k-1}$, we may insert the leaf edge labeled $k$ to any one of the $(2k-5)$ existing edges on a tree $T \in \mathcal{S}_{k-1}$ for $k \geq 5$ [21]. Every single tree $T$ in $\mathcal{S}_n^*$ can be constructed by this process. The construction can be reversed by removing the labeled leaves, from $n$ down to 4. Then we will know the positions for the inserting process for all labeled leaves. By induction, the enumeration result of the number of trees in $\mathcal{S}_n$ is given by the falling factorial number $(2n-5)!!$.

Method (II): we can count the number of distinct trees from the fully triangulated semi-labeled regular polygon with $n$ edges and then quotient out the corresponding binary trees which are BHV-equivalent given in the previous section 2.1. Every permutation for the labels and a triangulation gives a dual graph of a fully resolved tree in $\mathcal{S}_n$ as in Definition 2.8. The dihedral group $D_{2n}$ acts on the $n$-gon through rotations and reflections. Two trees are also BHV-equivalent under the twisting of internal edges which corresponds to the reflection of one of the two small polygons about the perpendicular bisector of the corresponding internal line segment in the triangulation. The twisting does not allow both small polygons to be reflected as this is one of the reflections generated from $D_{2n}$. Therefore every twist of an internal edge gives a two-fold symmetry which cannot be generated from the dihedral group action as a twist only reverses the edge labels on one of the two small polygons. These two actions, which are the dihedral group actions and the $(n-3)$ twists of the internal line segments, preserve the set of splits thus preserves the tree in $\mathcal{S}_n$. All of the $(n-3)$ twists are independent from each other and also independent from the dihedral group action $D_{2n}$ [9].

We enumerate the number of triangulated regular $n$-gons with $n$ labeled edges, which is given by $|S_n|\,C_{n-2}$ where $C_{n-2}$ is the Catalan number and equal to the number of triangulations of a convex $n$-gon. We also calculate the number of $BHV$-equivalent trees which are embedded to these $|S_n|\,C_{n-2}$ labeled triangulated polygons. For a triangulated regular $n$-gon with $n$ labeled edges, any actions of the $(n-3)$ twists and $D_{2n}$ give a another triangulated regular $n$-gon which is BHV-equivalent to the original one. Hence the number of BHV-equivalent triangulated regular $n$-gons can be obtained by the product of the sizes of the independent twists and dihedral group action [9], which is given by $|D_{2n}|\,2^{n-3}$. Therefore the size of the $\mathcal{S}_n$ can be calculated by

$$|\mathcal{S}_n| = \frac{|S_n|\,C_{n-2}}{|D_{2n}|\,2^{n-3}} = \frac{n!\,(2n-4)!}{2^{n-2}n(n-1)\left((n-2)!\right)^2} = (2n-5)!!.$$

Method (III): We note that the number $(2n-5)!!$ is equal to the number of $(n-2)$ unordered disjoint 2-subsets of $(2n-4)$ numbers. There indeed exists such matchings of $n$ pairs for a rooted tree of Newick string format with $(n-1)$ leaves [17]. We will state the one-to-one correspondence between the matchings and trees. First we relabel the leaf labeled $n$ as the root $R$, alternatively denoted by 0. The idea is to view the graph as hanging the tree up by its root. Then we express the tree in the Newick String format, without the length of edges. We will then label the internal vertices from the original semi-labeled tree to complete the labeling of the entire tree. As there exist at least two cherries in a fully resolved tree, we can spot them as the brackets $(i_1, i_2), (i_3, i_4), \ldots, (i_{2k-1}, i_{2k})$ which all consist of two leaf labels. We pick the bracket $(j_1, j_2)$ if $j_1$ or $j_2$ is the smallest in the set $\{i_1, i_2, \ldots, i_{2k-1}, i_{2k}\}$ and write down the 2-subset with the two numbers from this bracket $\{j_1, j_2\}$. Then we replace this bracket by the number $n$ in the Newick string and obtain another Newick string. This process labels the corresponding internal vertex on the rooted tree.

This process can be repeated by using the Newick string obtained from the previous step, which equivalently chops off a cherry with the smallest number and then labels the new leaf label with the next unused natural number. We repeat the process until all internal vertices are labeled and the $(n-2)$ 2-subsets are written down. For example, the matching process of the Newick string $\Big(((5,2),4),(3,1)\Big)$ are given by

$$\begin{aligned}
&\Big(((5,2),4),(3,1)\Big) : \big\{\{1,3\}\big\},\\
\longrightarrow &\Big(((5,2),4),6\Big) : \big\{\{1,3\},\{2,5\}\big\},\\
\longrightarrow &((7,4),6) : \big\{\{1,3\},\{2,5\}\big\},\\
\longrightarrow &(8,6) : \big\{\{1,3\},\{2,5\},\{4,7\}\big\},\\
\longrightarrow &: \big\{\{1,3\},\{2,5\},\{4,7\},\{6,8\}\big\}.
\end{aligned}$$

Two Newick strings are BHV-equivalent if their corresponding splits defined by the bracketings are identical. Therefore two Newick strings return the same matchings if they are BHV-equivalent as they correspond to the same tree in $\mathcal{S}_n$ given by the same splits.

Conversely, given $(n-2)$ unordered disjoint 2-subsets of a set of $(2n-4)$ numbers, we are able to reconstruct the Newick string through the same process of spotting the cherry with the smallest number. At the start, there exists a 2-subset where both numbers are less than or equal to $(n-1)$. Every 2-subsets which satisfy this condition is a cherry. The internal vertex of the cherry with the smallest number will be labeled by the number $n$. We then remove the 2-subset and also remove those two numbers in that 2-subset from the big set of $(2n-4)$ numbers. Equivalently we remove that semi-labeled cherry from the semi-labeled tree and label the previous internal vertex of that cherry with number $n$ as that vertex is now a labeled leaf vertex. We obtain another semi-labeled rooted binary tree and then repeat the process of spotting the cherry with the smallest number. This process gives a unique semi-labeled binary tree in the Newick string format which can be reversed to obtain the matchings. Therefore the one-to-one correspondence gives us the number of fully resolved trees in $\mathcal{S}_n$, which is the number of $(n-2)$ unordered disjoint 2-subsets of $(2n-4)$ numbers, $\frac{(2(n-2))!}{2^{n-2}(n-2)!} = (2n-5)!!$.

We will now define the space of phylogenetic trees with $n$ taxa with specified lengths for the internal edges. We always assume that the lengths of the leaf edges are strictly positive. We do not include the data of the lengths of the leaf edges as the splits for leaf labels are determined by the internal edges [9].

**Definition 2.16.** *The* continuous phylogenetic tree space $\mathcal{T}_n$ *is given by the set of semi-labeled trees of* $n$ *leaves* $\mathcal{S}_n^*$ *with specified internal edge lengths.*

Let the space $\mathcal{S}_n$ be the tree space with leaf labels $\{i_1, \ldots, i_n\}$. The number of nontrivial splits is the number of subsets with $i_n$ of the set of the $n$ leaf labels with size of every subset not equal to 0, 1, $n-1$ or $n$. For $n \geq 4$, the number of nontrivial splits is therefore $2^{n-1} - n - 2$. We define the standard basis of the vector space $\mathbb{R}^{2^{n-1}-n-2}$ to be

$$\left\{ \mathbf{1}_E : i_n \in E, \left\{ E, E^C \right\} \text{ is a nontrivial split of leaf labels of } \mathcal{S}_n^* \right\}.$$

Every fully resolved tree $T \in \mathcal{S}_n$ is determined by $(n-3)$ compatible splits which correspond to the $(n-3)$ internal edges. We write $E \tilde{\in} T$ if $\{E, E^c\}$ is a nontrivial split of leaf labels in $T$. In $\mathcal{T}_n$, every internal edge which corresponds to a split $\{E, E^c\}$ in the fully resolved tree $T \in \mathcal{S}_n$ has a positive length $l_E$. We explicitly write a tree $T$ with internal edge lengths as a vector $\sum_{E \tilde{\in} T} l_E \mathbf{1}_E \in \mathbb{R}^{2^{n-1}-n-2}$ where $l_E > 0$ for all $E \tilde{\in} T$. Therefore in $\mathcal{T}_n$, the trees with the same set of compatible splits can be identified in a Euclidean region isomorphic to $\mathbb{R}_+^{n-3}$ with standard basis $\left\{ \mathbf{1}_E : E \tilde{\in} T \right\}$.

In $\mathcal{T}_n$, every fully resolved tree corresponds to a vector in $\mathbb{R}^{2^{n-1}-n-2}$, which is a unique linear sum of the $(n-3)$ unit vectors for the $(n-3)$ compatible splits in the standard basis of $\mathbb{R}^{2^{n-1}-n-2}$ with strictly positive coefficients given by the internal edge lengths. Every degenerate tree corresponds to a vector in $\mathbb{R}^{2^{n-1}-n-2}$, which is the linear sum of $(n-3)$ unit vectors for the $(n-3)$ compatible splits in the standard basis of $\mathbb{R}^{2^{n-1}-n-2}$ with non-negative coefficients. A degenerate tree in $\mathcal{S}_n^*$ has less than $(n-3)$ internal edges which generate less than $(n-3)$ compatible splits therefore the vector as a linear sum of the $(n-3)$ unit vectors will have strictly positive coefficients for the splits which correspond to the existing internal edges, and coefficients 0 for the unit vectors of the non-existing internal edges. Hence every semi-labeled tree with $n$ leaves with specified internal edge lengths can be uniquely written as a linear sum of no more than $(n-3)$ unit vectors of the compatible splits in the standard basis of $\mathbb{R}^{2^{n-1}-n-2}$ with non-negative coefficients. Therefore the space $\mathcal{T}_n$ can be identified as a subset of $\mathbb{R}_+^{2^{n-1}-n-2}$ where the axes are labeled by the nontrivial splits of the leaf labels.

The space of a specific fully resolved tree with positive internal edge lengths is then isomorphic to an $\mathbb{R}_+^{n-3}$ space as the $(n-3)$ positive entries in the corresponding vectors in $\mathbb{R}_+^{2^{n-1}-n-2}$ are given by the lengths of the internal edges for the $(n-3)$ compatible splits.

**Definition 2.17.** *The space of a specific fully resolved tree $T$ is the* orthant *of $T$ in the space of $\mathcal{T}_n$.*

The faces of an orthant consist of trees which can be written by less than $(n-3)$ unit vectors in the standard basis of its corresponding compatible splits. The space $\mathcal{T}_n$ consists of all orthants of the fully resolved trees isomorphic to $\mathbb{R}_+^{n-3}$. The interior of these orthants are all disjoint as their corresponding fully resolved trees do not have the same set of splits thus their corresponding vectors in $\mathbb{R}_+^{2^{n-1}-n-2}$ have nonzero entries in different positions. These orthants of the fully resolved trees are glued together in $\mathcal{T}_n$ which is isomorphic to a subset of $\mathbb{R}^{2^{n-1}-n-2}$. Two orthants share a boundary face which is isomorphic to $\mathbb{R}_+^k$ if they have $k$ splits in common and face is given by the vectors of those splits.

In the space of $\mathcal{T}_n$, the star tree $s_n$ corresponds to the origin point which is the zero vector. The star tree is also considered to be *fully degenerate* as there are not any internal edges.

We can define a natural metric on $\mathcal{T}_n$ by the distance between the trees. If two trees are in the same orthant, the distance will be given by the Euclidean distance between their corresponding vectors. If two trees are in different orthants, we can define paths between them which only go through points in the subset of $\mathbb{R}^{2^{n-1}-n-2}$ which corresponds to $\mathcal{T}_n$. The length of the shortest path is defined to be the *geodesic distance* and does not necessarily go through the origin for two trees not in the same orthant [6], [50].

We can see that the space $\mathcal{T}_n$ can be embedded in an $\left(2^{n-1}-n-2\right)$-dimensional Euclidean space but some other constructions have shown that the trees can be fit in a much lower dimension of the Euclidean space [36].

Some stochastic process on $\mathcal{T}_n$ are computed by changing the internal edge lengths and going across the faces of orthants with certain rules. This can be set as a movement of a point with some rules in the space of $\mathcal{T}_n$. Within an orthant of a fixed tree in $\mathcal{S}_n$, the change of internal edge lengths can be set as the diffusion on the space of $\mathbb{R}_+^{n-3}$. We may also define other spaces which have similar structures to $\mathcal{T}_n$. For example, we can restrict the lengths to be integers by setting a lattice of mesh points of $\mathbb{Z}_+^{n-3}$ in $\mathbb{R}_+^{n-3}$. Then the stochastic process which corresponds to the diffusion on the space $R_+^{n-3}$ is replaced by a another process on $\mathbb{Z}_+^{n-3}$. The stochastic process will be given in terms of the random walk and we seek the eigensolutions to the adjacency operators. We can also define another space $\mathcal{T}_n^L$ which consists of all trees in $\mathcal{S}_n^*$ with a fixed positive total internal edge lengths $L$. In this case, the star tree $s_n$ does not exist and within each orthant, the dimension goes one lower as the vectors are in a subspace.

Our aim is to understand the adjacency relations between the trees in $\mathcal{S}_n$ and between the orthants in $\mathcal{T}_n$. We will then apply the adjacency relations to realize the space of trees into a graph in a Euclidean space of lower dimension to simplify the numerical computations of changing edge lengths and going across the orthants. The adjacency relations and realizations will give us better understandings to the property of tree spaces thus simplify the numerical work to analyse the diffusions and stochastic process by reducing the dimensions through the symmetry properties in the representations of $\mathcal{S}_n$.

### 2.2.2 The nearest neighbour interchange and random walk

The random walk on the space of discrete semi-labeled phylogenetic trees $\mathcal{S}_n$ is generated by the *nearest neighbour interchange*, or NNI [63]. Given an internal edge with internal vertices $u$ and $v$ in a fully resolved tree, we may identify its four adjacent edges thus the four subtrees $A$,$B$,$C$ and $D$ with the disjoint set of leaf labels $\{T_A, T_B|T_C, T_D\}$ as in the quartet display. The NNI process regroup the four subtrees into one of the possible two partitions apart from the original structure. If the original split is given by $\{T_A \cup T_B, T_C \cup T_D\}$, there will be two possible splits after the NNI process, which are $\{T_A \cup T_C, T_B \cup T_D\}$ and $\{T_A \cup T_D, T_B \cup T_C\}$ as in the diagram below.

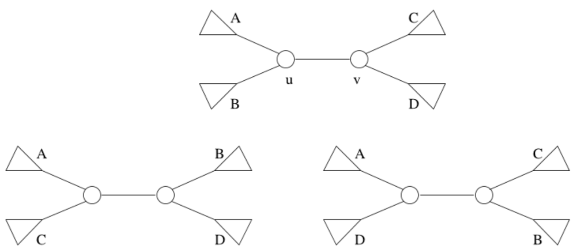


Figure 2.4: The nearest neighbour interchange at an internal edge with 4 subtrees $A$,$B$,$C$,$D$.

Note that the splits given by any other internal edges are preserved. When the cyclic order of the leaf labels are fixed by $T_A, T_B, T_C, T_D$, the split $\{T_D \cup T_A, T_B \cup T_C\}$ is the only possible outcome of the NNI process from the split $\{T_A \cup T_B, T_C \cup T_D\}$.

In the NNI process between trees in $\mathcal{S}_n$, we skip the degenerate tree with an internal vertex of degree 4 for the case of the internal edge $uv$ in the fully resolved tree shrinks to a vertex. In $\mathcal{S}_n^*$, the three fully resolved trees in the diagram are all neighbours of this degenerate tree. In the NNI process, we are not allowed to stay in the same tree although there exist three possible fully resolved trees from this degenerate tree. We also note the $(n-2)$ compatible splits for the degenerate tree are included in each of these three fully resolved trees.

Two fully resolved trees are *BHV-adjacent* if they differ by one nearest neighbour interchange. Every fully resolved tree has $(n-3)$ internal edges which means that it is adjacent to other $2(n-3)$ fully resolved trees in $\mathcal{S}_n$. Therefore we can define a graph whose vertices are the fully resolved trees in $\mathcal{S}_n$ and the edges are obtained from the *BHV-adjacencies* [38], [60]. In the simple random walk on this graph, the probabilities of moving from a vertex to all of its $2(n-3)$ neighbours are equal to $\frac{1}{2(n-3)}$.

The four subsets in a quartet display on an internal edge are preserved under the NNI process of that specific edge. But the quartet display on other internal edges will change.

In the continuous phylogenetic tree space $\mathcal{T}_n$ where the data of internal edge lengths are isomorphic to $\mathbb{R}_+^{n-3}$, we regard the NNI process as one of the internal edge length shrinks to zero and becomes a vertex of degree 4 and can still identify the 4 subtrees from this vertex. The tree is now degenerate and on the topological boundary and on an $(n-4)$ dimensional face of an orthant. There are three ways to 2-2-split the 4 subtrees to make it fully resolved again and one of them is back to the orthant of the previous fully resolved tree. The problem of computing the geodesic distance applies the NNI process to find the shortest path between two trees in the continuous space $\mathcal{T}_n$ [5].

Within an orthant where the splits of labels are fixed, the random walk is isomorphic to the diffusion on $\mathbb{R}_+^{n-3}$ for the continuous space or isomorphic to the simple random walk on $\mathbb{Z}_+^{n-3}$ lattice if we restrict the edge lengths to be integers [49]. On the continuous space $\mathcal{T}_n$, when the random walk goes across the topological boundary of the orthants, the probability are often to be considered as equal to the three possible directions given by the nearest neighbour interchange. We study the random walk by finding the eigensolutions of the transition matrix on the vector space of probability measures which can be applied to simulate the stochastic process and random walks. The dimensions for both of the spaces $\mathcal{S}_n$ and $\mathcal{T}_n$ are huge therefore we seek other methods to simplify the calculations. In Section 2.3, we consider the realization of tree spaces into polytopes which reduce the dimensions to $O(n)$ or $O(n^2)$. And in Chapter 3, we seek analytic solutions to the random walk under the invariance conditions of the group that acts on the tree space.

The space $\mathcal{S}_4$ only consists of three fully resolved trees and they are all BHV-adjacent to each other. The smallest nontrivial example to consider is $\mathcal{S}_5$.

**Example 2.18.** *The space $\mathcal{S}_5^*$ and the Petersen graph.*

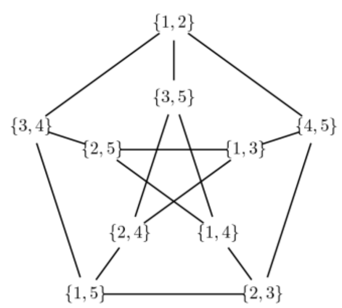


Figure 2.5: The Petersen graph with compatible partitions for $\mathcal{S}_5$.

The Petersen graph is a regular graph with 15 edges and 10 vertices of degree 3. The vertices can be labeled as above where the every edge consists of two disjoint subsets of the set $\{1, 2, 3, 4, 5\}$. Every 2-subsets of the full set can be extended to a 2-3-split, which is the only possible size of nontrivial partitions in $\mathcal{S}_5$ and $\mathcal{S}_5^*$. We can see that the two partitions for the two vertices of every edge are compatible. Therefore the 15 edges of the Petersen graph and the 15 fully resolved trees of $\mathcal{S}_5$ are in one-to-one correspondence. The 10 vertices of the Petersen graph and the 10 degenerate trees with one internal edge of $\mathcal{S}_5^*$ are also in one-to-one correspondence. We also notice that two edges share one common vertex if and only if they have a common split and correspond to the BHV-adjacency conditions. Therefore the vertices and edges of the Petersen graph represent the degenerate and fully resolved trees respectively in the space of $\mathcal{S}_5^*$. The Petersen graph is also the dual graph for the graph with vertices of $\mathcal{S}_n$ with the adjacency relations where the 15 fully resolved trees are given as the vertices instead of edges and the 10 degenerate trees are given as the triangles. If the sum of the lengths of the two internal edges are fixed, then every point on the vertices can be identified as a Newick string with internal edge lengths.

The 10 vertices of the Petersen graph also represent the 10 degenerate trees where one of the internal edge has length zero. The symmetric group $S_5$ acts on the 5-set and the stabilizer of a fixed partition is isomorphic to $S_2 \times S_3$. In the stochastic process for the discrete random walk, we will fix the size of splits and seek the eigensolutions on a set of degenerate trees for the adjacency operators. The group $S_5$ and the subgroup $S_2 \times S_3$ form a Gelfand pair. In Chapter 3, we shall use the properties of the Gelfand pair and further reduce the dimension for the vector space of the random walk to find the eigenfunctions which decompose the functions on the discrete space.

## 2.3 Realizations of different tree spaces

In this section, we seek better realizations of tree spaces as polytopes. As every binary tree has a planar embedding in a semi-labeled triangulated polygon, we construct a simplicial complex where the vertices correspond to the semi-labeled triangulated polygons. The aim is to realize the simplicial complex of a tree space as a polytope in the Euclidean space.

Throughout this section, we use the word "facet" for all lower dimensional boundaries. We use the word "face" if a facet is $(n-1)$-dimensional on an $n$-dimensional polytope.

We first construct the polytope in the Euclidean space where the vertices correspond to the permutations of the $(n+1)$ leaf labels, namely the permutohedron $P_n$ which is $n$-dimensional; i.e., given $n$ unit vectors from the standard basis of the Euclidean space, every point on the surface and inside $P_n$ can be uniquely written as a linear sum of these $n$ vectors. Then we fix a permutation; i.e., the labels on the edges of the polygon, and construct the polytope where the vertices correspond to the triangulations, namely the associahedron $K_n$. We apply the Dorman Luke construction [16] to construct the dual polytopes $P_n^*$ and $K_n^*$ of the permutohedron and the associahedron. We prove that the dimension of $K_n$ is $(n-1)$-dimensional, thus always one lower than the dimension of $P_n$.

Finally we obtain the associahedra with different permutations of the edge labels and identify them on the faces of a dual permutohedron, namely the permuto-associahedron $KP_n$, where the vertices correspond to the fully bracketed Newick strings of $n$ leaf labels with all possible permutations and bracketings (triangulations). We also outline a method to realize the the space of $\mathcal{S}_n$ using the secondary polytope construction and compare with the balanced minimal evolution (BME) polytope [36] of the same dimension.

### 2.3.1 Realizations of the permutohedra

When we fix a root label in the discrete tree space $\mathcal{S}_{n+2}$, we consider the binary trees written in the Newick string format. The Newick string is a fully bracketed sequence which corresponds to a permutation of the $(n+1)$ leaf labels. Our first aim is to find the realization of all permutations of the leaf labels in a Euclidean space.

**Definition 2.19.** *The $n$-permutohedron is the convex hull of all permutations of the vector $(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$ where $x_j \neq x_k$ for all $j \neq k$.*

**Definition 2.20.** *The standard $n$-permutohedron $P_n$ is the convex hull of all permutations of the vector $(1, 2, \ldots, n+1) \in \mathbb{R}^{n+1}$.*

The symmetric group $S_{n+1}$ acts on the entries of the vector and total number of points is $(n+1)!$. We can verify that the all points of $P_n$, $\underline{v} = (v_1, \ldots, v_{n+1}) \in \mathbb{R}^{n+1}$ lie on a $n$-dimensional subspace satisfying the affine condition $\langle \underline{v}, \underline{1} \rangle = \sum_{i=1}^{n+1} v_i = \frac{1}{2}(n+1)(n+2)$. The subspace is given by the orthogonal vector $(1, \ldots, 1)$ therefore the permutohedron lies in an $n$-dimensional subspace of $\mathbb{R}^{n+1}$.

Let $E = \{j_1, j_2, \ldots, j_l\}$ be a nontrivial subset of $\{1, 2, \ldots, n+1\}$ with $1 \leq |E| \leq n$. Let $\underline{1_E} \in \mathbb{R}^{n+1}$ be the vector with value 1 in the $j_1$-th, $j_2$-th,...,$j_l$-th entries and 0 elsewhere. We define a hyperplane in $\mathbb{R}^{n+1}$ from the normal vector $\underline{1_E}$ as the set of points

$$\left\{ \underline{x} \in \mathbb{R}^{n+1} : \langle \underline{x}, \underline{1_E} \rangle = \sum_{j \in E} v_j = \frac{|E| \big( |E| + 1 \big)}{2} \right\}.$$

**Definition 2.21.** *The* half-space $H_{P_n}(E)$ *of the nonempty subset $E \subsetneq \{1, 2, \ldots, n+1\}$ is the set of points given by*

$$H_{P_n}(E) := \left\{ \underline{v} \in \mathbb{R}^{n+1} : \sum_{j \in E, E \subsetneq \{1, \ldots, n+1\}} v_j = \langle \underline{v}, \underline{1_E} \rangle \geq \frac{|E| \big( |E| + 1 \big)}{2} \right\},$$

*where $\underline{1_E}$ is the normal vector that defines the hyperplane for the half-space $H_{P_n}(E)$.*

The boundary of the half-space $H_{P_n}(E)$ is the hyperplane where the points are obtained by the equality relation in Definition 2.21. The points which are not in the half-space $H_{P_n}(E)$ are all on the same side of $\mathbb{R}^{n+1}$ which is split by the hyperplane. The points in the half-space $H_{P_n}(E)$ which are not on the boundary correspond to the strict inequality in the definition.

**Lemma 2.22.** *The permutohedron $P_n$ is in an $n$-dimensional subspace with half-space representation given by the intersection of all half-spaces*

$$P_n := \bigcap_{E \neq \emptyset, E \subsetneq \{1,\ldots,n+1\}} H_{P_n}(E).$$

*Proof.* Given all permutations of the vector $(1, 2, \ldots, n+1) \in \mathbb{R}^{n+1}$, the equality in the condition of the half-space $H_{P_n}(E)$ holds for a vector $\underline{v}$ on the boundary. If a vector is given by a permutation of $(1, 2, \ldots, n+1)$ and on the boundary of $H_{P_n}(E)$, then the first $l$ numbers $\{1, \ldots, l\}$ are in the $j_1$-th, $j_2$-th,...,$j_l$-th entries of the vector for the nonempty subset $E = \{j_1, j_2, \ldots, j_l\} \subsetneq \{1, \ldots, n+1\}$.

We get the strict inequality if any number greater than $l$ appears to be in one of the $j_1$-th, $j_2$-th,...,$j_l$-th entries as the sum of the entries $\sum_{j \in E} v_j$ will be greater than $\frac{(l+1)l}{2}$. The left hand side of the inequality in Definition 2.21 can be considered as an inner product between the vector $\underline{v}$ and another vector $\underline{1_E} \in \mathbb{R}^{n+1}$ where the entries are either 1 or 0. All of the hyperplanes that define the half-spaces are $(n-1)$-dimensional in the $n$-dimensional subspace of $\mathbb{R}^{n+1}$.

We have a natural *maximal flag* of the set $\{1, 2, \ldots, n+1\}$ given by

$$\emptyset \subsetneq \{1\} \subsetneq \{1, 2\} \subsetneq \cdots \subsetneq \{1, 2, \ldots, n\} \subsetneq \{1, 2, \ldots, n, n+1\}.$$

Every vertex $\underline{v}$ of the convex hull corresponds to a permutation of $(1, 2, \ldots, n+1)$ thus corresponds to another unique maximal flag of the set $\{1, 2, \ldots, n+1\}$, given by the positions in $\underline{v}$ for the elements in the natural maximal flag defined above, explicitly given by

$$\emptyset \subsetneq \{j_1\} \subsetneq \{j_1, j_2\} \subsetneq \cdots \subsetneq \{j_1, j_2, \ldots, j_n\} \subsetneq \{j_1, j_2, \ldots, j_n, j_{n+1}\},$$

where $j_k$ is the position of entry with number $k$ in $\underline{v}$, i.e. $\underline{v}_{j_k} = k$.

Apart from the full set and the empty set, every set in the middle of a maximal flag is a set $E$ which corresponds to a half-space $H_{P_n}(E)$ and a defining hyperplane. Every vertex which is a permutation of the vector $(1, 2, \ldots, n+1)$ is then well defined by the intersection of the $n$ hyperplanes which correspond to the subsets in the maximal flag for the positions of the entries in the $n$-dimensional subspace of $\mathbb{R}^{n+1}$ given by the set of points $\left\{\underline{x} : \sum_{i=1}^{n+1} x_i = \frac{(n+1)(n+2)}{2}\right\}$. The coordinates of a vector on the convex hull can always be obtained by solving the $(n+1)$ linear equations, which are the $n$ equations for the half-spaces and the one equation $\sum_{i=1}^{n+1} x_i = \frac{(n+1)(n+2)}{2}$ for the fixed sum of the entries. $\square$

**Remark 2.23.** *Note that during the proof, the maximal flag for the vector* $\underline{v}$*,*

$$\emptyset \subsetneq \{j_1\} \subsetneq \{j_1, j_2\} \subsetneq \cdots \subsetneq \{j_1, j_2, \ldots, j_n\} \subsetneq \{j_1, j_2, \ldots, j_n, j_{n+1}\},$$

*where* $j_k$ *is the position of entry with number* $k$ *in* $\underline{v}$*, i.e.* $\underline{v}_{j_k} = k$*, also defines a permutation* $(j_1, j_2, \ldots, j_n, j_{n+1})$*. If* $\underline{v} = g(1, 2, \ldots n+1)$ *for* $g \in S_{n+1}$*, then we have* $(j_1, j_2, \ldots, j_n, j_{n+1}) = g^{-1}(1, 2, \ldots n+1)$ *and we define the permutation* $(j_1, j_2, \ldots, j_n, j_{n+1})$ *to be the* inverse permutation *of* $\underline{v}$*.*

The lemma proves that $P_n$ is an $n$-dimensional convex polytope with a half-space representation. We will describe the $(n-1)$-dimensional faces and lower dimensional facets of $P_n$. Every $(n-1)$-dimensional face of $P_n$ is on the boundary, i.e. the hyperplane of a half-space that defines $P_n$ when the equality is achieved in the definition. Two faces on the boundaries of the two distinct half-spaces $H_{P_n}(E_1)$ and $H_{P_n}(E_2)$ intersect on the surface of $P_n$ if and only if one of the two sets, $E_1$ or $E_2$, is the subset of the other one.

There are also lower-dimensional facets on $P_n$ given by the intersections of more than two $(n-1)$-dimensional faces. A 1-dimensional facet, which is a line segment on the polytope, is given by the intersection of $(n-1)$ faces which correspond to $(n-1)$ nonempty subsets of $\{1, \ldots, n+1\}$ in the natural maximal flag. The $(n-1)$ half-spaces determine the entries of $(n-1)$ numbers and there are only two positions left to put the two adjacent numbers on. These two vectors differ by a swap of the two undetermined entries. If we take the 1-skeleton of the polytope which is isomorphic to a graph, the two vectors which correspond to the two vertices are connected by an edge and they are *adjacent* on the graph. Moreover, if we relabel the vertices of $P_n$ by the *inverse permutation* of the vectors for the coordinates, the edges of the 1-skeleton will be generated by a swap of two adjacent entries in the relabeled permutations.

In particular, when $|E| = 1$ or $n$, the $(n-1)$-dimensional faces of $P_n$ given by the boundaries of the half-spaces $H_{P_n}(E)$ are isomorphic to the permutohedron $P_{n-1}$. Faces with such properties fix the position of the number 1 and there exists an isomorphism from the vectors on a face to the vectors of $P_{n-1}$ by shifting the vectors by the constant vector $(1, 1, \ldots, 1)$. The two half-spaces $H_{P_n}(E)$ and $H_{P_n}\left(E^C\right)$ are parallel as they are determined by opposite normal vectors $\underline{1_E}$ and $\underline{1_{E^C}}$ that define their hyperplanes.

**Example 2.24.** *The permutohedron* $P_1$*.*

The first nontrivial example of a permutohedron is the convex hull of two points $(1, 2)$ and $(2, 1)$ in $\mathbb{R}^2$ and the line segment is clearly 1-dimensional.

**Example 2.25.** *The permutohedron* $P_2$*.*

The permutohedron $P_2$ consists of 6 points in $\mathbb{R}^3$ which are the permutations of the vector $(1, 2, 3)$. They live in a 2-dimensional subspace given by the vectors $\underline{v} = (v_1, v_2, v_3)$ where $v_1 + v_2 + v_3 = 6$. The convex hull of the 6 points is a hexagon and the 1-dimensional faces, which are the edges of the hexagon, are given by the inequality of half-spaces $H_{P_2}(E)$ for $|E| = 1$ or 2, with the coordinates satisfying $v_j = 1$ and $v_j + v_k = 1 + 2$ where $j \neq k$ on the faces.

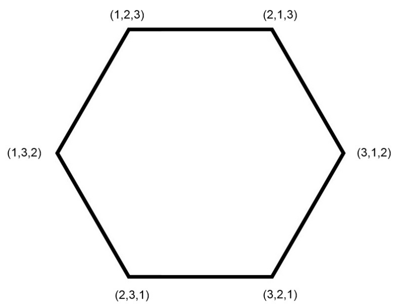


Figure 2.6: The permutohedron $P_2$.

**Example 2.26.** *The permutohedron $P_3$.*

The permutohedron $P_3$ consists of 24 points in $\mathbb{R}^4$ which are the permutations of the vector $(1, 2, 3, 4)$ thus in the subspace where the sum of entries are equal to 10. The convex hull is a 3-dimensional polytope and the shapes of the 2-dimensional faces are hexagons and squares. The hexagons are given by the hyperplanes for the half-spaces $H_{P_3}(E)$ for $|E| = 1$ or 3 and they are isomorphic to $P_2$. The squares are given by the hyperplanes for the half-spaces $H_{P_3}(E)$ for $|E| = 2$. This polytope can also be considered as a truncated octahedron and we can recover the octahedron by sticking the pyramids back to the 6 square faces.

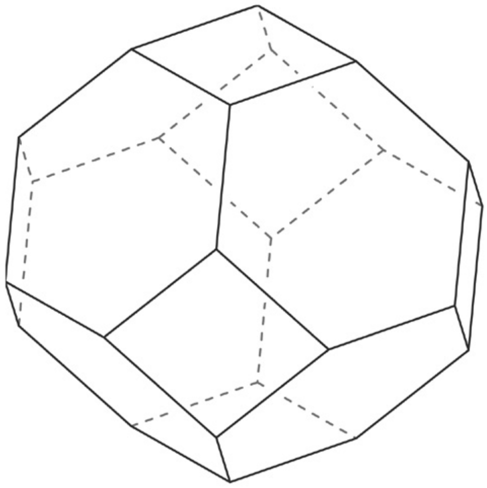

Figure 2.7: The permutohedron $P_3$.

**Definition 2.27.** *Given two $n$-dimensional convex polytopes $P$ and $P^*$, one is the dual polytope [64] of the other if they satisfy the following conditions:*

1. *the vertices of one correspond to the faces of the other;*
2. *the $k$-dimensional facets given by the intersection of the faces of one polytope correspond to the $(n-1-k)$-dimensional facets given by the corresponding vertices of the other polytope for $1 \leq k \leq n-2$ where the corresponding vertices are given by condition 1.*

Given an $n$-dimensional convex polytope $P$, we apply the Dorman Luke construction [16] to obtain the the dual polytope $P^*$. The process is to obtain the convex hull from a set of points on a sphere which are given by the positive multiples of the corresponding vectors on the faces of $P$. We give a detailed construction for the dual polytope of the permutohedron $P_n$ from the Dorman Luke construction.

The centre of the polytope $P_n$ is given by the average position of all vectors which are the permutations of $(1, 2, \ldots, n+1)$. Therefore the centre $\underline{O}_{P_n}$ has coordinates $\left(\frac{n}{2}+1, \ldots, \frac{n}{2}+1\right)$. We can also verify that every permutation of $(1, 2, \ldots, n+1)$ have the same Euclidean distance to the centre $\underline{O}_{P_n}$ thus all vertices of the permutohedron $P_n$ are on the $S^{n-1}$ sphere centered at $\underline{O}_{P_n}$.

Recall the $(n-1)$-dimensional faces of $P_n$ given by the boundary of the half-spaces $H_{P_n}(E)$. We define the centre of the face to the average position of all vertices on the face. For a face given by the boundary of the half-space $H_{P_n}(E)$, the coordinates of the centre of the face $\underline{E} \in \mathbb{R}^{n+1}$ has value $\frac{|E|+1}{2}$ at the $k$-th entry for all $k \in E$ and value $\frac{|E|+n+2}{2}$ at the $j$-th entry for all $j \in E^C$. Note that the point $\underline{E}$ is inside the sphere as the corresponding $(n-1)$-dimensional face is inside the ball.

We take the $S^{n-1}$ sphere centered at $\underline{O}_{P_n}$ which consists of all vertices of the permutohedron $P_n$, in the $n$-dimensional subspace of $\mathbb{R}^{n+1}$, where the coordinates of the points satisfy $\sum_{i=1}^{n+1} x_i = \frac{1}{2}(n+1)(n+2)$ for $\underline{x} \in \mathbb{R}^{n+1}$. We define the vector $\underline{E^*}$ which is a point on the sphere such that the vector $\left(\underline{E^*} - \underline{O}_{P_n}\right)$ is a positive multiple of the vector $\left(\underline{E^*} - \underline{O}_{P_n}\right)$. This point $\underline{E^*}$ is unique as only one of the two intersections between the sphere and the line given by the scalar multiples of $\left(\underline{E^*} - \underline{O}_{P_n}\right)$ is positive.

**Definition 2.28.** *We define the dual permutohedron $P_n^*$ to be the convex hull of the vectors $\underline{E^*}$*

$$P_n^* := conv\left\{\underline{E^*} : |E| \geq 1, E \subsetneq \{1, \ldots n+1\}\right\}.$$

By this construction, the dual permutohedron $P_n^*$ is an $n$-dimensional convex polytope. We can verify that both sets of vertices of $P_n$ and $P_n^*$ are on the same sphere. There are $(n+1)!$ distinct $(n-1)$-dimensional faces on $P_n^*$ where each $(n-1)$-dimensional face consists of $n$ vertices. Every $(n-1)$-dimensional face of the dual permutohedron $P_n^*$ is an $(n-1)$-dimensional simplex where the $n$ vertices correspond to $n$ distinct subsets of the set of $(n+1)$ numbers. The $n$ distinct subsets all have different sizes and generate a maximal flag for the set of $(n+1)$ numbers as the face corresponds to a permutation of the $(n+1)$ numbers.

Every $k$-dimensional facet of $P_n^*$ is given by the convex hull of $(k+1)$ vertices which correspond to $(k+1)$ subsets in a maximal flag of the set of $(n+1)$ numbers. Every $(n-k)$-dimensional facet of $P_n$ is given by the intersection of $k$ $(n-1)$-dimensional faces which also correspond to $(k+1)$ subsets in a maximal flag of the set of $(n+1)$ numbers. We can set the one-to-one correspondence between the $k$-dimensional facets of $P_n^*$ and the $(n-k)$-dimensional facets of $P_n$ as both sets of facets are generated from $(k+1)$ subsets from the maximal flags of the set of $(n+1)$ numbers. Therefore $P_n^*$ can be considered as the dual polytope to $P_n$ as the correspondence between the facets in all dimensions are satisfied.

The 2-dimensional dual permutohedron $P_2^*$ is still a hexagon as the dual graph of a hexagon is also a hexagon. The 3-dimensional dual permutohedron $P_3^*$ is a polytope with 24 triangular faces which correspond to the 4! permutations of 4 numbers. The dual permutohedron $P_3^*$ consists of 14 points which correspond to the 14 half-spaces of $P_3$. This polytope can also be considered as sticking 6 square faces of 6 pyramids to the 6 square faces of a cube as in the diagram below.

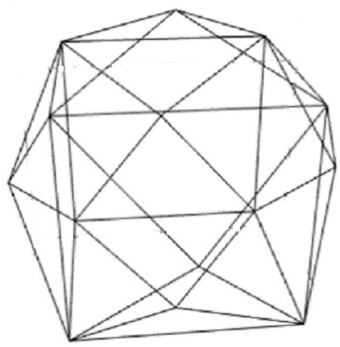

Figure 2.8: The dual permutohedron $P_3^*$.

### 2.3.2 Realizations of the associahedra

Recall the fully bracketed Newick string for a sequence of $n$ letters and the corresponding dual polygon of the embedded binary tree of the Newick string in Definition 2.6, Definition 2.7 and Definition 2.8 in section 2.1. Given a permutation of $n$ leaf labels and a specified root label, the binary trees of the fully bracketed Newick strings of these labels in the permutation have planar embeddings as convex polygons whose edges have the same permutation as the leaf labels.

**Definition 2.29.** *The associahedron $K_n$, also known as the Stasheff polytope [58], is a convex polytope whose vertices correspond to the triangulations of a convex $(n+1)$-gon with $n$ edges labeled by the sequence apart from a root edge. Two vertices are connected by an edge if their corresponding triangulated polygons can be obtained from each other by removing a single diagonal and replacing by a different diagonal.*

In this subsection, we will fix a permutation of the leaf labels and give an explicit realization of the space of triangulations as a convex polytope, namely the associahedron where the adjacency relation between the vertices of the polytope is restricted by the nearest neighbour interchange in the space of discrete fully resolved trees. We use the method of the secondary polytope [31] and give the half-space representation for the faces.

We obtain a new method of describing the faces of the associahedron in a folding process which is described in detail in Theorem 2.39. We also apply this idea of the generalized secondary polytope of higher dimensions to realize the actual space of $\mathcal{S}_n$ and $\mathcal{T}_n$. We will also produce a realization for the dual associahedron and show that it is isomorphic to a subset of the corresponding trees in the continuous space $\mathcal{T}_n$.

It is well known that the associahedron $K_n$ is an $(n-2)$-dimensional convex polytope whose boundary is homeomorphic to the sphere $S^{n-3}$ [58]. The vertices on the 1-skeleton of $K_n$ and the triangulations of a convex $(n+1)$-gon with fixed labels on the edges are in one-to-one correspondence.

Two vertices of $K_n$ are joined by an edge if their corresponding triangulations have $(n-3)$ diagonals in common, i.e. their corresponding embedded fully resolved trees have $(n-3)$ splits in common out of the $(n-2)$ splits of the leaf labels as the edge labels are the same with leaf labels. Therefore the two fully resolved trees which correspond to the two vertices on an edge of $K_n$ differ by precisely one split of the $(n-2)$ splits of the leaf labels, which means that the two trees are $BHV$-adjacent and can be obtained from each other through a nearest neighbour interchange.

In the secondary polytope realization, we will show that all triangulations which correspond to the vertices on an $(n-3)$-dimensional face of the $(n-2)$-dimensional polytope $K_n$ have one diagonal in common and their corresponding fully resolved trees have one split in common. Every $(n-2-j)$-dimensional facets of $K_n$, which is the intersection of $j$ distinct $(n-3)$-dimensional faces, consist of vertices whose corresponding triangulations have $j$ diagonals in common; i.e., their corresponding fully resolved trees have $j$ splits in common.

As a graph, the sets of vertices and edges of adjacencies only generate the combinatorial relations. The realization of the associahedron is to assign the vertices with vectors in the Euclidean space such that the polytope constructed from the convex hull of these vectors have clearer interpretation of the combinatorial relation from the geometrical structure. There are many classic realizations of the associahedra [35], [42], [31], [54], [43]. We choose the method of the secondary polytope construction as it has the symmetry of the dihedral group $D_{2(n+1)}$ and gives the explicit half-space representation for both the associahedron $K_n$ and the dual associahedron $K_n^*$.

We assume $n \geq 4$ for the convex $n$-gon. We give a natural labeling on the vertices around a convex $n$-gon by the sequence $(1, 2, \ldots, n)$. A *diagonal* $ij$ is a line segment joining the vertices $i$ and $j$ inside the $n$-gon with $2 \leq |i-j| \leq n-2$.

**Definition 2.30.** *Two distinct diagonals are* non-crossing *if the two line segments do not intersect inside the $n$-gon.*

Two distinct diagonals $i_1 j_1$, $i_2 j_2$ are always non-crossing if they have a common vertex on the polygon i.e.$\big|\{i_1, j_1\} \cap \{i_2, j_2\}\big| = 1$.

Every triangulation $\Delta$ of a convex $n$-gon can also be uniquely written as a set of $(n-3)$ non-crossing diagonals $\Delta := \{i_1 j_1, \ldots, i_{n-3} j_{n-3}\}$, where $i_k$ and $j_k$ are the labels on the vertices. Alternatively we can express the triangulation $\Delta$ as the set of $(n-2)$ triangles which constitute the full polygon

$$\varDelta := \left\{ \varDelta_{ijk} : ijk \text{ is a triangle inside the } n\text{-gon} \right\}.$$

If we place the triangulated $n$-gon on the $\mathbb{R}^2$ plane, we can calculate the area of every triangle area $\left(\varDelta_{ijk}\right)$ as well as the area of the $n$-gon given by the sum of the areas of all triangles. Let $V_{\varDelta_j}$ be the sum of the areas of the triangles with vertex $j$.

We define a vector space $\mathbb{R}^n$ with standard basis $\{\mathbf{e}_i\}_{i=1}^n$ labeled by the vertices of the $n$-gon.

**Definition 2.31.** *The vector $\mathbf{V}_\varDelta \in \mathbb{R}^n$ of the triangulation $\varDelta$ is given by*

$$\mathbf{V}_\varDelta = \sum_{j=1}^{n} V_{\varDelta_j} \mathbf{e}_j = \left(V_{\varDelta_1}, V_{\varDelta_2}, \ldots, V_{\varDelta_n}\right).$$

**Remark 2.32.** *The vector $\mathbf{V}_\varDelta \in \mathbb{R}^n$ of the triangulation $\varDelta$ can also be obtained by*

$$\mathbf{V}_\varDelta = \sum_{\varDelta_{ijk} \in \varDelta} \mathit{area}\left(\varDelta_{ijk}\right) \left(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k\right).$$

**Definition 2.33.** *Given a convex polygon $\mathcal{P}$ with $n$ labeled vertices $\{1, 2, \ldots, n\}$, the* secondary polytope $\mathcal{Q}$ *of $\mathcal{P}$ is given by the convex hull in $\mathbb{R}^n$, explicitly given by*

$$\mathcal{Q}\left(\mathcal{P}\right) := \mathit{conv}\{\mathbf{V}_\varDelta : \varDelta \text{ is a triangulation of } \mathcal{P}\}.$$

To preserve the symmetries of the dihedral group, we choose the polygon $\mathcal{P}$ to be a regular $n$-gon which can be put on the unit circle centered at the origin of the $\mathbb{R}^2$ plane. Every labeled vertex $j$ on $\mathcal{P}$ has coordinates $\underline{v_j} = \left(\cos\frac{2\pi j}{n}, \sin\frac{2\pi j}{n}\right)$.

**Lemma 2.34.** *All $\mathbf{V}_\varDelta \in \mathbb{R}^n$ satisfy the following three affine conditions given in the form of the inner product:*

$$\langle \mathbf{V}_\varDelta, \mathbf{1} \rangle = \sum_{j=1}^{n} V_{\varDelta_j} = 3 \ \mathit{area}\left(\mathcal{P}\right),$$

$$\langle \mathbf{V}_\varDelta, \mathbf{v}_{\cos} \rangle = \sum_{j=1}^{n} V_{\varDelta_j} \cos\frac{2\pi j}{n} = 0,$$

$$\langle \mathbf{V}_\varDelta, \mathbf{v}_{\sin} \rangle = \sum_{j=1}^{n} V_{\varDelta_j} \sin\frac{2\pi j}{n} = 0,$$

*Proof.* As the sum of the areas of all triangles in a triangulation is equal to the area of the polygon $\mathcal{P}$, we can see that the sum of the components in the vector $\mathbf{V}_\varDelta$ is equal to three times of the area of $\mathcal{P}$, which gives us the first affine condition

$$\langle \mathbf{V}_{\Delta}, \mathbf{1} \rangle = \sum_{j=1}^{n} V_{\Delta_j} = 3 \text{ area}(\mathcal{P})$$

We choose the $\mathcal{P}$ to be a regular $n$-gon which can be put on the unit circle centered at the origin of the $\mathbb{R}^2$ plane. Every labeled vertex $j$ on $\mathcal{P}$ has coordinates $\underline{v_j} = \left(\cos\frac{2\pi j}{n}, \sin\frac{2\pi j}{n}\right)$. The barycentre of the triangle $ijk$ is given by

$$\underline{v_{ijk}} = \frac{1}{3}\left(\underline{v_i} + \underline{v_j} + \underline{v_k}\right),$$

which is also the centre of mass of the triangle if we distribute the mass uniformly in the interior of $\mathcal{P}$. The position of the centre of mass $\underline{v_{\mathcal{P}}} \in \mathbb{R}^2$, which is also the barycentre of $\mathcal{P}$, can be calculated by the equation

$$\text{area}(\mathcal{P})\,\underline{v_{\mathcal{P}}} = \sum_{\Delta_{ijk}\in\Delta} \text{area}\left(\Delta_{ijk}\right)\underline{v_{ijk}} = \frac{1}{3}\sum_{j=1}^{n} V_{\Delta_j}\underline{v_j}.$$

As the centre of the regular polygon $\mathcal{P}$ is at the origin $(0,0)$ of $\mathbb{R}^2$, we can obtain two affine conditions for the components of $\mathbf{V}_{\Delta}$ for all triangulations $\Delta$ of $\mathcal{P}$. The fixed $x$-coordinates gives

$$\langle \mathbf{V}_{\Delta}, \mathbf{v}_{\cos} \rangle = \sum_{j=1}^{n} V_{\Delta_j} \cos\frac{2\pi j}{n} = 0.$$

The fixed $y$-coordinates gives

$$\langle \mathbf{V}_{\Delta}, \mathbf{v}_{\sin} \rangle = \sum_{j=1}^{n} V_{\Delta_j} \sin\frac{2\pi j}{n} = 0.$$

□

**Remark 2.35.** *The following three methods of distributing the mass $m$ over a triangle give the same position for the centre of mass at the barycentre of the triangle:*

1. *put the entire mass $m$ at the barycentre of the triangle;*
2. *put mass $\frac{m}{3}$ at each one of the three vertices of the triangle;*
3. *distribute the mass $m$ uniformly in the interior of the triangle.*

In the vector $\mathbf{V}_{\Delta}$ for the triangulation $\Delta$, the sum of the entries is equal to $3$ area$(\mathcal{P})$, which can also be considered as the mass uniformly distributed inside the $\mathcal{P}$ with constant density 3.

We will now derive the half-space representation for the secondary polytope $\mathcal{Q}(\mathcal{P})$. Let $ij$ be a diagonal in the triangulated polygon $\mathcal{P}$ and we always assume that $1 \leq i < j \leq n$ and $2 \leq j - i \leq n-2$. Set $\theta_{ij} = \frac{(n-i-j)\pi}{n}$. Let $\mathbf{v}_{ij} \in \mathbb{R}^n$ be a vector with value $\cos\left(\frac{2\pi k}{n} + \theta_{ij}\right)$ at the $k$-th entry if $1 \leq k \leq i-1$ or $j+1 \leq k \leq n$ and with value $2\cos\theta_{ij} - \cos\left(\frac{2\pi k}{n} + \theta_{ij}\right)$ at the $k$-th entry if $i < k < j$.

**Definition 2.36.** *The inner product $\langle \mathbf{x}, \mathbf{v}_{ij}\rangle$ is evaluated as*

$$\langle \mathbf{x}, \mathbf{v}_{ij}\rangle = \sum_{1\leq k\leq i, j\leq k\leq n} x_k \cos\left(\frac{2\pi k}{n} + \theta_{ij}\right) + \sum_{k=i}^{j} x_k \left(2\cos\theta_{ij} - \cos\left(\frac{2\pi k}{n} + \theta_{ij}\right)\right).$$

**Remark 2.37.** *Note that all entries in $\mathbf{v}_{ij}$ are greater than or equal to $\cos\theta_{ij}$.*

**Definition 2.38.** *The half-space $H_{K_n}(i,j)$ for the diagonal $ij$ is given by the set of points*

$$H_{K_n}(i,j) := \left\{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{v}_{ij}\rangle \leq 3 \ \mathit{area}(\mathcal{P})\, x_{ij}\right\},$$

*where* $x_{ij} = \frac{1}{3n}\left((6j-6i-4)\cos\theta_{ij} - 4\sum_{k=i+1}^{j-1}\cos\left(\frac{2\pi k}{n} + \theta_{ij}\right) - \frac{2\cos\theta_{ij}\sin\frac{2\pi(j-i)}{n}}{\sin\frac{2\pi}{n}}\right)$.

**Theorem 2.39.** *The secondary polytope $\mathcal{Q}(\mathcal{P})$ is an $(n-3)$-dimensional polytope in the $(n-3)$-dimensional subspace of $\mathbb{R}^n$ given in Lemma 2.34 with half-space representation*

$$\mathcal{Q}(\mathcal{P}) := \bigcap_{i,j:1\leq i<j\leq n, 2\leq j-i\leq n-2} H_{K_n}(i,j).$$

*Proof.* Originally, the regular $n$-gon $\mathcal{P}$ is placed on the unit circle centered at the origin of the $\mathbb{R}^2$ plane with labeled vertices $k$ at $\left(\cos\frac{2\pi k}{n}, \sin\frac{2\pi k}{n}\right)$ for all $k = 1, \ldots, n$. We rotate $\mathcal{P}$ anticlockwise by $\theta_{ij} = \frac{(n-i-j)\pi}{n}$ such that the line connecting $ij$ is perpendicular to the $x$-axis with the vertex $i$ above the $x$-axis. After the rotation, the vertex $k$ will have coordinates $\left(\cos\left(\frac{2\pi k}{n} + \theta_{ij}\right), \sin\left(\frac{2\pi k}{n} + \theta_{ij}\right)\right)$.

First we consider the case where $ij$ is a diagonal of the triangulation $\Delta$. The diagonal $ij$ splits the triangulated polygon $\mathcal{P}$ into two triangulated polygons: the left polygon $\mathcal{P}_L$ with vertices $\{i, i+1, \ldots, j\}$, and the right polygon $\mathcal{P}_R$ with vertices $\{j, j+1, \ldots, n, 1, 2, \ldots, i\}$. The triangles in $\mathcal{P}_L$ and the triangles in $\mathcal{P}_R$ do not overlap and the two sets of triangles constitute the set of all triangles in the triangulated polygon $\mathcal{P}$ thus cover the entire interior of the polygon $\mathcal{P}$. Regardless of how the two polygons $\mathcal{P}_L$ and $\mathcal{P}_R$ are triangulated, their areas are both fixed by area$(\mathcal{P}_L)$ and area$(\mathcal{P}_R)$.

The centre of mass for the left polygon $\mathcal{P}_L$ and the centre of mass for the right polygon $\mathcal{P}_R$ are both fixed on the $\mathbb{R}^2$ plane if we distribute the mass uniformly by a constant density 3 in the interior of all triangles inside $\mathcal{P}$. The two centers of mass of the two polygons $\mathcal{P}_L$

and $\mathcal{P}_R$ can also be calculated by a weighted sum of the centers of mass of the triangles in each side. By changing the distribution of the mass as described in Remark 2.35, for every triangle $\varDelta_{k_1k_2k_3}$, we put the mass of value area $\left(\varDelta_{k_1k_2k_3}\right)$ at the three vertices $k_1$, $k_2$ and $k_3$ respectively and the centre of mass will stay the same if the mass is uniformly distributed in the interior of the triangles. Therefore the centers of mass for both polygons $\mathcal{P}_L$ and $\mathcal{P}_R$ will also stay the same at the centre of the mass if the mass is uniformly distributed in the interior of the entire polygon $\mathcal{P}$.

The above method for distributing the mass is equivalent to assigning the vertex $k$ with the value of the sum of the areas of all triangles with vertex $k$, which is equal to the $k$-th entry of the vector $\mathbf{V}_\varDelta$ by Definition 2.31. We can obtain the centre of mass of the right polygon $\mathcal{P}_R$ explicitly on the $\mathbb{R}^2$ plane. The $y$-coordinate is always 0 as the centre of mass of $\mathcal{P}_R$ is always on the $x$-axis. The $x$-coordinates $x_{ij,R}$ is given by evaluating the weighted sum

$$3 \text{ area}\left(\mathcal{P}_R\right) x_{ij,R} = \sum_{\varDelta_{k_1k_2k_3} \text{ is a triangle in } \mathcal{P}_R} \left( \text{area}\left(\varDelta_{k_1k_2k_3}\right) \left( \sum_{l=1}^{3} \cos\left( \frac{2\pi k_l}{n} + \theta_{ij} \right) \right) \right).$$

Similarly, we can explicitly work out the coordinates $\left(x_{ij,L}, 0\right)$ for the centre of mass of the left polygon $\mathcal{P}_L$ where $x_{ij,L}$ is given by

$$3 \text{ area}\left(\mathcal{P}_L\right) x_{ij,L} = \sum_{\varDelta_{k_1k_2k_3} \text{ is a triangle in } \mathcal{P}_L} \left( \text{area}\left(\varDelta_{k_1k_2k_3}\right) \left( \sum_{l=1}^{3} \cos\left( \frac{2\pi k_l}{n} + \theta_{ij} \right) \right) \right).$$

The centre of mass of $\mathcal{P}$ is at $(0,0)$ and can also be obtained from the weighted sum

$$3 \text{ area}\left(\mathcal{P}_L\right) \left(x_{ij,L}, 0\right) + 3 \text{ area}\left(\mathcal{P}_R\right) \left(x_{ij,R}, 0\right) = 3 \text{ area}\left(\mathcal{P}\right) (0,0).$$

We then reflect the left polygon $\mathcal{P}_L$ about the diagonal $ij$, i.e. fold the polygon $\mathcal{P}$ along the line $ij$ so that the folded left polygon $\mathcal{P}'_L$ is to the right of $ij$. After the folding process, every vertex $k$ where $i < k < j$ have coordinates

$$\left( 2\cos\theta_{ij} - \cos\left( \frac{2\pi k}{n} + \theta_{ij} \right), \sin\left( \frac{2\pi k}{n} + \theta_{ij} \right) \right).$$

The area of $\mathcal{P}'_L$ is equal to the area of $\mathcal{P}_L$. The centre of mass $\left(x'_{ij,L}, 0\right)$ of $\mathcal{P}'_L$ can be obtained from the reflected centre of mass of $\mathcal{P}_L$ where

$$x'_{ij,L} = 2\cos\theta_{ij} - x_{ij,L}.$$

Note that $x'_{ij,L}$ is fixed as $x_{ij,L}$ is fixed.

The new folded object which consists of both $\mathcal{P}'_L$ and $\mathcal{P}_R$, has a fixed centre of mass $\left(x'_{ij}, 0\right)$ on the $x$-axis where the value $x'_{ij}$ can be obtained from the weighted sum

$$x'_{ij} = \frac{3 \text{ area}\left(\mathcal{P}'_L\right) x'_{ij,L} + 3 \text{ area}\left(\mathcal{P}_R\right) x_{ij,R}}{3 \text{ area}\left(\mathcal{P}'_L\right) + 3 \text{ area}\left(\mathcal{P}_R\right)} = \frac{3 \text{ area}\left(\mathcal{P}_L\right) x'_{ij,L} + 3 \text{ area}\left(\mathcal{P}_R\right) x_{ij,R}}{3 \text{ area}\left(\mathcal{P}\right)}.$$

Recall the two vectors $\mathbf{V}_\Delta$ defined in Definition 2.31 and the vector $\mathbf{v}_{ij}$ in Definition 2.36. We can also obtain the fixed $x$-coordinate for the centre of mass of the folded object by the weighted sum of mass of all vertices given as the inner product between the vector of areas assigned to the vertices $\mathbf{V}_\Delta$ and the vector of the $x$-coordinates $\mathbf{v}_{ij}$

$$x'_{ij} = \frac{\langle \mathbf{V}_\Delta, \mathbf{v}_{ij} \rangle}{\sum_{k=1}^{n} V_{\Delta_k}} = \frac{\langle \mathbf{V}_\Delta, \mathbf{v}_{ij} \rangle}{3 \text{ area}\left(\mathcal{P}\right)} = x_{ij}.$$

Therefore the equality of the half-space $H_{K_n}(i,j)$ as in Definition 2.38 is achieved for the vector $\mathbf{V}_\Delta$ if $ij$ is a diagonal in the triangulation $\Delta$; i.e., $\mathbf{V}_\Delta$ is on the hyperplane which is the boundary of the half-space $H_{K_n}(i,j)$. Note that $x_{ij} > \cos\theta_{ij}$ for all diagonals $ij$ inside $\mathcal{P}$.

Since every triangulation $\Delta$ has $(n-3)$ distinct diagonals inside $\mathcal{P}$, we can identify the $(n-3)$ hyperplanes which are the boundaries of $(n-3)$ half-spaces correspond to the $(n-3)$ diagonals. Therefore the corresponding vector $\mathbf{V}_\Delta$ is given by the intersection of the $(n-3)$ hyperplanes in the $(n-3)$-dimensional subspace of $\mathbb{R}^n$.

We will now consider the case where $ij$ is not a diagonal in the triangulation $\Delta$ and show that the vector $\mathbf{V}_\Delta$ is indeed in the half-space $H_{K_n}(i,j)$ but not on the boundary, i.e. the strict inequality of $\langle \mathbf{V}_\Delta, \mathbf{v}_{ij} \rangle > 3 \text{ area}\left(\mathcal{P}\right) x_{ij}$ is achieved.

If $ij$ is not a diagonal in the triangulation $\Delta$, we draw a line segment that connects the two vertices $i$ and $j$. The line segment $ij$ crosses with other diagonals inside $\mathcal{P}$. We label the crossing points inside $\mathcal{P}$ by the set $\left\{\alpha_1, \ldots, \alpha_p\right\}$. This additional line segment $ij$ will separate some triangles in the triangulation $\Delta$ into several small polygons. Note that sum of all separated regions of triangles and small polygons are still fixed by $3 \text{ area}\left(\mathcal{P}\right)$.

We apply the *additional triangulation*: for the small polygons which are not triangles inside $\mathcal{P}$, we triangulate them individually by drawing non-crossing line segments joining the vertices within the regions without creating any other crossing points apart from $\left\{\alpha_1, \ldots, \alpha_p\right\}$ inside $\mathcal{P}$. After the additional triangulation, we obtain a vector $\mathbf{V}'_\Delta = \left(V'_{\Delta_1}, \ldots, V'_{\Delta_n}\right) \in \mathbb{R}^n$ where the component $V'_{\Delta_k}$ is the mass assigned to vertex $k$, given by the sum of the areas of all triangles with vertex $k$. For the entries of the two vectors $\mathbf{V}'_\Delta$ and $\mathbf{V}'_\Delta$, we have $V'_{\Delta_k} \leq V_{\Delta_k}$ for all $k = 1, \ldots, n$, and some strict inequalities are achieved as some of triangles with vertex $k$ are cut to smaller regions. We also assign mass $V_{\alpha_l}$ to the vertex $\alpha_l$ inside $\mathcal{P}$, which is given by the sum of the areas of all triangles with vertex $\alpha_l$ for all $l = 1, \ldots, p$. We have the equation obtained by the fixed sum of the mass and areas of all triangles

$$\sum_{k=1}^{n} V'_{\Delta_k} + \sum_{l=1}^{p} V_{\alpha_l} = 3 \text{ area} (\mathcal{P}) .$$

We then repeat the process folding along the line segment $ij$ and obtain the fixed centre of mass of the folded object, with coordinates $\left(x_{ij}, 0\right)$. The value of $x_{ij}$ is obtained as a equation of the weighted sum given by

$$3 \text{ area} (\mathcal{P})\, x_{ij} = \langle \mathbf{V}'_{\Delta}, \mathbf{v}_{ij} \rangle + \sum_{l=1}^{p} V_{\alpha_l} \cos \theta_{ij}.$$

If we move the mass on the vertices $\left\{\alpha_1, \ldots, \alpha_p\right\}$ to the vertices $\{1, \ldots, n\}$ so that every vertex $k$ on the folded object has mass $V_{\Delta_k}$, we eventually shift the centre of mass of the folded object to the right of $\left(x_{ij}, 0\right)$. Hence the strict inequality $\langle \mathbf{V}_{\Delta}, \mathbf{v}_{ij} \rangle > 3 \text{ area} (\mathcal{P})\, x_{ij}$ is achieved.

Therefore we obtain the half-space representation of the polytope $K_n$ in the $(n-3)$-dimensional of $\mathbb{R}^n$ as the intersection of half-spaces

$$\mathcal{Q}(\mathcal{P}) := \bigcap_{i,j:1 \leq i < j \leq n, 2 \leq j-i \leq n-2} H_{K_n}(i, j) .$$

where a vector $\mathbf{V}_{\Delta}$ is on the boundary of the half-space $H_{K_n}(i, j)$ if the diagonal $ij$ is in the triangulation $\Delta$. Every point $\mathbf{V}_{\Delta}$ on the secondary polytope is the intersection of $(n-3)$ hyperplanes of the half-spaces given by the $(n-3)$ diagonals of the triangulation $\Delta$. $\square$

Every $(n-4)$-dimensional face which is on the boundary of a half-space $H_{K_n}(i, j)$ of the $(n-3)$-dimensional polytope $\mathcal{Q}(\mathcal{P})$ is given by the convex hull of the vectors $\mathbf{V}_{\Delta}$ diagonal $ij$ is in the triangulation $\Delta$

$$\text{conv} \{\mathbf{V}_{\Delta} : \Delta \text{ is a triangulation with diagonal } ij\} .$$

Two $(n-4)$-dimensional faces intersect on the surface of $\mathcal{Q}(\mathcal{P})$ if their half-spaces correspond to two distinct non-crossing diagonals. Every $(n-3-k)$-dimensional facet of $\mathcal{Q}(\mathcal{P})$ is given by the intersection of the $k$ hyperplanes which are the boundaries half-spaces of the corresponding $k$ non-crossing diagonals inside $\mathcal{P}$.

Every edge of the 1-skeleton of $\mathcal{Q}(\mathcal{P})$ can be considered as a 1-dimensional facet where the two vertices of the edge correspond to the two triangulations of the two embedded fully resolved trees which only differ by one split of the edge labels. Therefore the two embedded fully resolved trees can be obtained from each other through a nearest neighbour interchange (NNI). Hence the construction of $\mathcal{Q}(\mathcal{P})$ preserves the adjacency relation for both triangulations and the embedded trees.

We apply the Dorman Luke construction to obtain the dual polytope of $\mathcal{Q}(\mathcal{P})$ for the triangulations of the regular polygon $\mathcal{P}$, namely the dual associahedron $K^*_{n-1}$. The dual polytope $\mathcal{Q}^*(\mathcal{P})$ obtained from the Dorman Luke construction is also an $(n-3)$-dimensional polytope in the same $(n-3)$-dimensional subspace of $\mathbb{R}^n$ with the secondary polytope $\mathcal{Q}(\mathcal{P})$.

Define the centre of the secondary polytope $\mathcal{Q}(\mathcal{P})$ to be the average vector of all vectors that constitute the set of vertices of the convex hull

$$\mathbf{O}_{K_n} = \frac{1}{C_{n-2}} \sum_{\Delta \text{ is a triangulation of } \mathcal{P}} \mathbf{V}_{\Delta}.$$

Let $\mathcal{P}_{ij}$ be set of triangulations of $\mathcal{P}$ with diagonal $ij$.

**Definition 2.40.** *The centre of the $(n-4)$-dimensional face $\mathbf{H}_{ij} \in \mathbb{R}^n$ on the hyperplane for the half-space $H_{K_n}(i,j)$ is given by the average vector of all vectors $\mathbf{V}_\Delta$ with diagonal $ij$ in the triangulation $\Delta$:*

$$\mathbf{H}_{ij} := \frac{1}{\left|\mathcal{P}_{ij}\right|} \sum_{\Delta \in \mathcal{P}_{ij}} \mathbf{V}_{\Delta}.$$

Consider the $(n-3)$-dimensional subspace defined in Lemma 2.34. Let $\mathbf{H}^*_{ij} \in \mathbb{R}^n$ be the vector on the $S^{n-4}$ unit sphere centered at $\mathbf{O}_{K_n}$ with $\left(\mathbf{H}^*_{ij} - \mathbf{O}_{K_n}\right) = \nu_{ij}\left(\mathbf{H}_{ij} - \mathbf{O}_{K_n}\right)$ where $\nu_{ij} > 0$.

**Definition 2.41.** *The polytope $\mathcal{Q}^*(\mathcal{P})$ is defined to be the convex hull of the set of vectors $\mathbf{H}^*_{ij}$*

$$\mathcal{Q}^*(\mathcal{P}) := conv\left\{\mathbf{H}^*_{ij} : ij \text{ is a diagonal in } \mathcal{P}\right\}.$$

The $(n-4)$-dimensional faces of $\mathcal{Q}^*(\mathcal{P})$ correspond to the vertices of $\mathcal{Q}(\mathcal{P})$ thus correspond to the triangulations of the convex polygon $\mathcal{P}$. Every $(n-4)$-dimensional face of $\mathcal{Q}^*(\mathcal{P})$ consists of $(n-3)$ vertices for the $(n-3)$ diagonals in the corresponding triangulation. Therefore every $(n-4)$-dimensional face is an $(n-4)$-simplex. Every point on a face of $\mathcal{Q}^*(\mathcal{P})$ can be uniquely written as a weighted sum of the $(n-3)$ vectors of the $(n-3)$ vertices of that simplex $\sum_{ij\in\Delta} \lambda_{ij}\mathbf{H}^*_{ij}$ with $\sum_{ij\in\Delta} \lambda_{ij} = 1$. The boundary of $\mathcal{Q}^*(\mathcal{P})$ is given by the union of all faces of $\mathcal{Q}^*(\mathcal{P})$, which is homeomorphic to the sphere $S^{n-4}$ as the $(n-3)$-dimensional polytope is homeomorphic to the $(n-3)$-ball. Therefore the set of points on the surface of $\mathcal{Q}^*(\mathcal{P})$ and the set of points on the sphere $S^{n-4}$ are in one-to-one correspondence.

Recall that every diagonal inside the polygon generates a split of the edge labels and the leaf labels of the corresponding tree written in the Newick string format with a fixed permutation $\underline{v}$. We pick the trees which can be written in the Newick string format with permutation $\underline{v}$ from the continuous space $\mathcal{T}_n$ and fix the sum of the internal edge lengths to be 1 to obtain a subset $\mathcal{T}^1_{n,\underline{v}}$ of $\mathcal{T}_n$. Then every tree in the subset $\mathcal{T}^1_{n,\underline{v}}$ can be first identified on a specific $(n-4)$-dimensional face of $\mathcal{Q}^*(\mathcal{P})$, and each one of the $(n-3)$ vertices of the face corresponds to a diagonal in the triangulated $n$-gon, which also corresponds to a split of leaf labels of that tree. Therefore the trees in the subset $\mathcal{T}^1_{n,\underline{v}}$ of $\mathcal{T}_n$ and all points on the surface of the polytope $\mathcal{Q}^*(\mathcal{P})$, are in one-to-one correspondence.

Consider a triangulation $\Delta$ with edges labeled by the leaf labels of permutation $\underline{v}$ and the embedded fully resolved tree $T$. The internal edge given by the split from the diagonal $ij$ has length $\lambda_{ij}$ in the unique weighted sum for the point $\sum_{ij\in\Delta}\lambda_{ij}\mathbf{H}^*_{ij}$ with $\sum_{ij\in\Delta}\lambda_{ij}=1$ on the simplex. Therefore the space $\mathcal{T}^1_{n,\underline{v}}$ is homeomorphic to the surface of the polytope $\mathcal{Q}^*(\mathcal{P})$.

**Corollary 2.42.** *Let $\mathcal{T}_{n,\underline{v}}$ be the subset of $\mathcal{T}_n$ where all trees can be written in the Newick string format with permutation $\underline{v}$ for the leaf labels. The space $\mathcal{T}_{n,\underline{v}}$ is homeomorphic to the vector space $\mathbb{R}^{n-3}$.*

*Proof.* The space $\mathcal{T}_{n,\underline{v}}$ is the union of the subsets given by

$$\mathcal{T}_{n,\underline{v}} = \bigcup_{L\in\mathbb{R}_+}\mathcal{T}^L_{n,\underline{v}},$$

where $L$ indicates the total lengths of the internal edges.

The dual associahedron $\mathcal{Q}^*(\mathcal{P})$ is homeomorphic to both the space $\mathcal{T}^1_{n,\underline{v}}$ and the sphere $S^{n-4}$. Therefore the sphere $S^{n-4}$ is homeomorphic to the space $\mathcal{T}^L_{n,\underline{v}}$ for all positive $L$.

The dual associahedron $\mathcal{Q}^*(\mathcal{P})$ is in an $(n-3)$-dimensional subspace of $\mathbb{R}^n$ which is isomorphic to $\mathbb{R}^{n-3}$. We shift the polytope $\mathcal{Q}^*(\mathcal{P})$ so that the centre $\mathbf{O}_{K_n}$ is at the origin and the vertices $\mathbf{H}^*_{ij}$ are shifted to $\mathbf{H}'_{ij}$ on the unit sphere centered at the origin. We define the cone of the triangulation $\Delta$ to be the set of vectors

$$\left\{\sum\lambda_{ij}\mathbf{H}'_{ij} : ij \text{ is a diagonal in the triangulation } \Delta, \lambda_{ij}\geq 0\right\}.$$

Let $\lambda_{ij}$ be the length of internal edge of the embedded tree given by the corresponding diagonal $\lambda_{ij}$. As in Definition 2.17, the cone of the triangulation $\Delta$ isomorphic to the orthant of a tree $T$ in the space $\mathcal{T}_n$, therefore isomorphic to $\mathbb{R}^{n-3}_+$.

Then every face of the shifted polytope $\mathcal{Q}^*(\mathcal{P})$, which is given by $(n-3)$ vertices on the unit sphere, can be identified in the corresponding cone of of the triangulation $\Delta$. When we restrict the sum of the internal edges of the embedded trees to be bounded by

$L_0$ for the triangulations, i.e., take the union

$$\mathcal{T}_{n,\underline{v}} = \bigcup_{L \leq L_0} \mathcal{T}^L_{n,\underline{v}},$$

the corresponding vectors in the cones of all triangulations is a set of points given by the convex hull

$$\operatorname{conv}\left\{L_0 \mathbf{H}'_{ij} : ij \text{ is a diagonal in } \mathcal{P}\right\}.$$

By the definition of $\mathcal{Q}^*(\mathcal{P})$, the convex hull defined above is the scaled dual associahedron constructed inside the $(n-3)$-ball centered at the origin with radius $L_0$.

The vector space $\mathbb{R}^{n-3}$ is homeomorphic to the union of the zero vector and the $S^{n-4}$ spheres of all positive sizes, which is homeomorphic to union of the boundaries of the scaled dual associahedra of all positive sizes. □

**Remark 2.43.** *The homeomorphism between $\mathcal{T}_{n,\underline{v}}$ and $\mathbb{R}^{n-3}$ suggests that we can simplify the numerical work of $\mathcal{T}_n$ to the space of $\mathbb{R}^{n-3}$ for a set of specific trees which can be written in the Newick string format with permutation $\underline{v}$ for the leaf labels.*

**Example 2.44.** *The associahedron $K_3$.*

The associahedron $K_3$ consists of two vertices which are connected by an edge. The two vertices correspond to the two triangulations of a square.

**Example 2.45.** *The associahedron $K_4$.*

The associahedron $K_4$ is given by the five triangulations of a pentagon, which can be constructed in $\mathbb{R}^2$ and homeomorphic to the circle $S^1$. Every edge in $K_4$ corresponds to a pentagon with one diagonal, which is the common diagonal of the two triangulations which correspond to the two vertices of that edge in $K_4$. The dual graph of $K_4$ is also a pentagon as there is no higher dimensional faces in $K_4$.

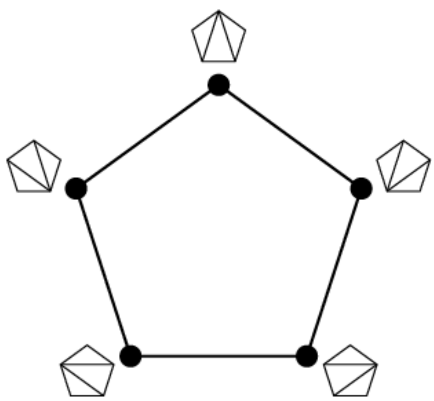

Figure 2.9: The associahedron $K_4$

**Example 2.46.** *The associahedron $K_5$.*

The associahedron $K_5$ can be constructed as a polytope in $\mathbb{R}^3$ whose 14 vertices correspond to the 14 triangulations of a labeled hexagon. Every vertex has degree 3 and is the intersection of 3 faces. The 9 faces of $K_5$ correspond to the 9 possible diagonals inside the hexagon. If a diagonal cuts the hexagon into a triangle and a pentagon, the corresponding face of the diagonal will be a pentagon which is isomorphic to $K_4$.

If a diagonal cuts the hexagon into two quadrilaterals with separated regions, the triangulation to each quadrilateral is a given by the associahedron $K_3$ which is a line segment. The triangulations of the two quadrilaterals are independent therefore the corresponding face is a quadrilateral homeomorphic to the product $K_3 \times K_3$.

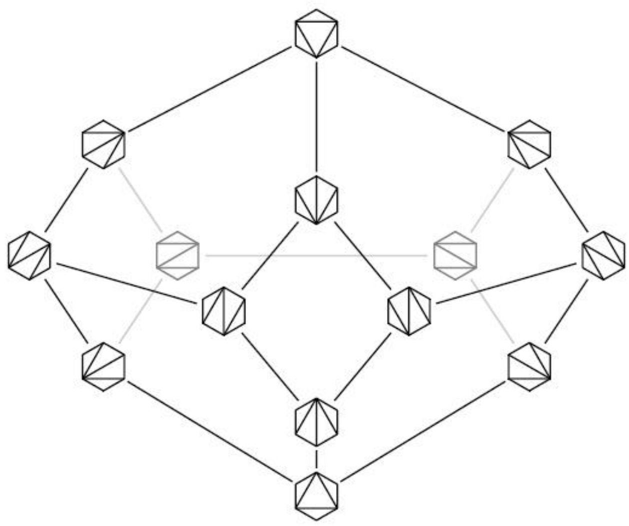

Figure 2.10: The associahedron $K_5$

The dual associahedron $K_5^*$ is the dual polytope of $K_5$, with 9 vertices and 14 triangular faces. The dual associahedron $K_5^*$ is also called the triaugmented triangular prism and can be constructed by sticking three pyramids to the three square faces of a triangular prism. Every triangle corresponds to a triangulation and is adjacent to the other 3 triangles. Every vertex corresponds to a diagonal inside the hexagon and there are six vertices of degree 5 and three vertices of degree 4.

By Corollary 2.42 and Remark 2.43, we can identify the trees in the subset $\mathcal{T}_{n,\underline{v}}$ where $\underline{v}$ is a fixed permutation of the leaf labels for the Newick string format of some trees in $\mathcal{T}_6$. The numerical work can be simplified to $\mathbb{R}^3$ for the trees in the subset, which is an improvement compared to the space of $\mathcal{T}_6$ defined as a subset of $\mathbb{R}^{15}$.

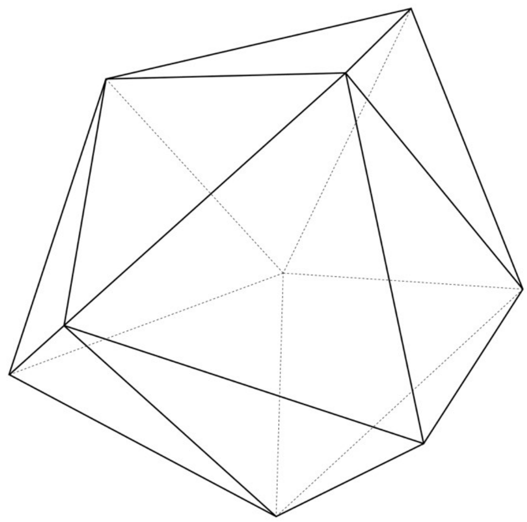

Figure 2.11: The dual associahedron $K_5^*$

### 2.3.3 Realizations of the permuto-associahedra

We have shown that the permutohedron $P_{n-1}$ and its dual polytope $P_{n-1}^*$ are realizable in the Euclidean space $\mathbb{R}^{n-1}$ and homeomorphic to the sphere $S^{n-2}$. Each one of the $n!$ distinct $(n-2)$-dimensional faces of $P_n^*$ corresponds to a permutation of $n$ elements. The associahedron $K_n$ and its dual polytope $K_n^*$ are realizable in $\mathbb{R}^{n-2}$ and homeomorphic to the sphere $S^{n-3}$. Every vertex on the associahedron $K_n$ corresponds to a triangulation of a convex polygon with fixed labels on the its $(n+1)$ edges.

We will give a brief description of the realization of the permuto-associahedron, the polytope denoted by $KP_n$, where the vertices correspond to the triangulations of a convex polygon with all permutations [40] of edge labels. We also present how the vertices of $KP_n$ link to the semi-labeled fully resolved trees with $(n+1)$ leaves in the discrete phylogenetic tree space $\mathcal{S}_{n+1}$ and show that the adjacencies on $KP_n$ are restricted by the nearest neighbour interchange in $\mathcal{S}_{n+1}$.

**Definition 2.47.** *The permuto-associahedron $KP_n$, also known as the* Type A Coxeter-associahedron *[40], is the convex polytope whose vertices correspond to the fully bracketed strings of n letters of all permutations. Two vertices are connected by an edge of the* 1-*skeleton of $KP_n$ if their corresponding fully bracketed strings can be obtained from each other by one of the two actions:*

1. *a swap of two letters in a bracket which only consists of the two letters;*

2. *replacing one bracket with another bracket so that the string is still fully bracket i.e. given three subsequences $A_1$, $A_2$ and $A_3$ which form a bracketed subsequence in the*

*Newick string, the two bracketings* $\big(A_1, (A_2, A_3)\big)$ *and* $\big((A_1, A_2), A_3\big)$ *can be obtained from each other by this action.*

Recall the permutohedron $P_{n-1}$ whose vertices are the vectors $\underline{v} \in \mathbb{R}^n$ which are permutations of $(1, \ldots, n)$. Given $(n+1)$ leaf labels $\{i_0, i_1, \ldots, i_n\}$, we assign every vertex $(v_1, \ldots, v_n)$ of $P_{n-1}$ a permutation of leaf labels $\big(i_0, i_{j_1}, \ldots, i_{j_n}\big)$ where the subscripts in the permutation of the leaf labels are given by the *inverse permutation* of $\underline{v}$ apart from $i_0$.

We also assign the same permutations of leaf labels to the corresponding faces of the dual permutohedron $P^*_{n-1}$. Note that $i_0$ is always fixed and corresponds to the specified root label on the tree. Two faces on the dual permutohedron $P^*_{n-1}$ are adjacent, i.e. their corresponding simplexes have $(n-2)$ vertices in common, if their corresponding permutations of the leaf labels differ by a swap of two adjacent labels.

Loday's explicit realization of the associahedron $K_n$ [43] proves that the vertices of the polytope can be constructed as a subset of an $(n-2)$-simplex. The faces of $K_n$ which correspond to the diagonal $j, (j+2)$ can be set parallel to the faces of the $(n-2)$-simplex and the vertices in Loday's realization which is constructed by truncating the simplex. We define this realization of the associahedron as the polytope $\mathcal{K}_n$ where the coordinates of its vertices satisfy the above properties.

Similar to Kapranov's realization by putting the associahedra around the vertices of the permutohedron $P_{n-1}$ [40], we construct the permuto-associahedron $KP_n$ by the convex hull of points of the associahedra $\mathcal{K}_n$ identically on all $n!$ faces of the dual permutohedron $P^*_{n-1}$.

**Definition 2.48.** *Let* $\mathcal{K}_{n,\underline{v}}$ *be the Loday's realization of the associahedron on the face with of the dual permutohedron* $P^*_n$*, constructed by the triangulations of the convex* $(n+1)$*-gon with edges labeled by the permutation* $\underline{v}$ *with fixed* $i_0$ *[24]. The polytope* $\mathcal{KP}_n$ *is given by the convex hull*

$$conv\left\{\mathcal{K}_{n,\underline{i}} : \ \underline{i} \text{ is a permutation of the leaf labels } \{i_0, i_1, \ldots, i_n\} \text{ with fixed } i_0\right\}.$$

The polytope $\mathcal{KP}_n$ given by the convex hull is an $(n-1)$-dimensional convex polytope in the Euclidean space. As the vertices are placed in the interior of the $(n-2)$-dimensional faces of the dual permutohedron $P^*_{n-1}$, the convex hull can also be considered as a truncated dual permutohedron. The $n!$ associahedra on the polytope $\mathcal{KP}_n$ can be identified on the $n!$ faces of the dual permutohedron $P^*_{n-1}$. There are also additional $(n-2)$-dimensional faces on the polytope $\mathcal{KP}_n$ apart from the faces of the dual permutohedron $P^*_{n-1}$. The additional faces are given by the edges whose pairs of vertices correspond to the same triangulation in the polygons with different permutation of edge labels which only differ by a swap of two adjacent edge labels if these two labels are in the same triangle.

We consider the adjacencies between the vertices on the polytope $\mathcal{KP}_n$ which are given by the edges of the 1-skeleton of $\mathcal{KP}_n$ . Two vertices are joined by an edge in an associahedron $\mathcal{K}_{n,(\underline{i})}$ if their corresponding sets of splits leaf labels differ by only one split, which means that their corresponding embedded trees in $\mathcal{S}_{n+1}$ are adjacent under the nearest neighbour interchange.

Two vertices are joined by an edge in one of those additional faces if their corresponding trees in $\mathcal{S}_{n+1}$ differ by a twist of a cherry. A twist of only one cherry corresponds to a swap of two adjacent edge labels not including $i_0$. The swapping of two adjacent edge labels changes the permutation of the edge labels and correspond to the two adjacent faces of $P_n^*$. The swap does not change any diagonals inside the $(n+1)$-gon thus preserve all splits of the embedded tree. Therefore every pair of vertices which are joined by an edge on the polytope is either a correspondence of a nearest neighbour interchange, or preserves the splits of a tree in the space of $\mathcal{S}_{n+1}$.

The conditions of the permuto-associahedron are satisfied in the construction of $\mathcal{KP}_n$, which shows that the realization is valid and the $S_n$ symmetry of the $(n-1)$ dimensional polytope is preserved.

As the polytope $KP_n$ consists of points of all possible permutations and compatible splits of the leaf labels, all trees in the discrete space $\mathcal{S}_{n+1}$ can be identified by $2^{n-2}$ points. The BHV-adjacencies in $\mathcal{S}_{n+1}$, i.e. the set of all nearest neighbour interchanges, can be identified by the edges in the $n!$ associahedra on the surface of $KP_n$.

**Example 2.49.** *The permuto-associahedron $KP_3$.*

As shown in Figure 2.12, the permuto-associahedron $KP_3$ is a dodecagon, which is a 2-dimensional convex polytope. We can identify the 6 faces of the dual permutohedron $P_2^*$, which are labeled by the permutations inside the 6 edges of the dodecagon. Every face which is labeled by a permutation is an edge and an associahedron $K_3$, where the vertices are given in the bracketings of the Newick string format labeled on the 12 vertices of $KP_3$. The 6 unlabeled edges correspond to a swap of two adjacent labels in the same bracket, which is a cherry in the embedded tree.

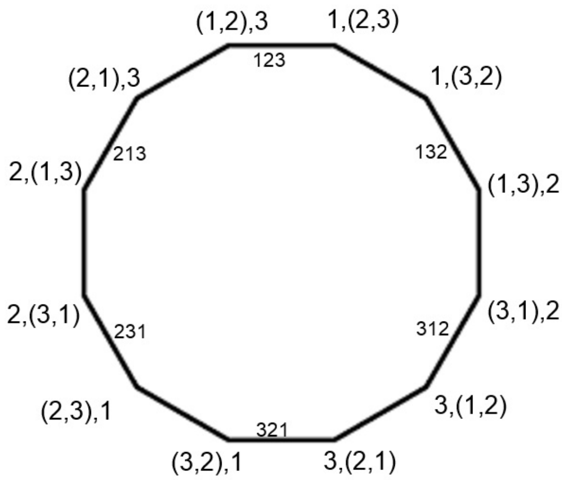


Figure 2.12: The permuto-associahedron $KP_3$

**Example 2.50.** *The permuto-associahedron* $KP_4$

The permuto-associahedron $KP_4$ is a 3-dimensional polytope with 24 pentagon faces, 8 dodecagon faces, 6 square faces and 24 rectangle faces as shown in the figure below.

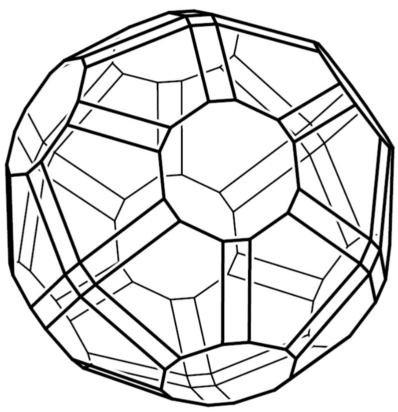

Figure 2.13: The permuto-associahedron $KP_4$

Each one of the 120 vertices can be identified in one of the 24 pentagons. Every pentagon can be identified as an associahedron $K_4$ with 5 vertices correspond to the triangulations of a pentagon with specific permutations of edge labels which correspond to the faces of the dual permutohedron $P_3^*$. The edges which do not belong to any associahedron connect to vertices which correspond to the same bracketing of two Newick strings with the permutations differ by a swap of two labels in the same bracket which only consist of those two labels, e.g. the two vertices which correspond to the Newick strings

$\left((i_1,(i_3,i_4)),i_2\right)$ and $\left((i_1,(i_4,i_3)),i_2\right)$ as they differ by a swap of the leaf labels $i_3$ and $i_4$ inside the bracket with two labels. The vertices on the square faces all correspond to the bracketing $((i_1,i_2),(i_3,i_4))$ and all of its permutations of the 4 leaf labels. A vertex is on a dodecagon face if it is not on a square face. Every dodecagon face is isomorphic to the permuto-associahedron $KP_3$. We can identify the isomorphism by fixing the first or the fourth leaf label of the Newick string with 4 leaf labels, the Newick strings with the other three labels correspond to the permuto-associahedron $KP_3$.

### 2.3.4 Realizations of the space of finite phylogenetic trees

In this subsection, we first briefly present the balanced minimal evolution (BME) polytope constructed as the convex hull of vectors in $\mathbb{R}^{\binom{n}{2}}$ which correspond to the fully resolved trees in the discrete tree space $\mathcal{S}_n$. Then we outline another idea of the realization of the space $\mathcal{S}_n$ from the method of realizing the associahedron as the secondary polytope. These two realizations give two different families of polytopes of the same dimension in the Euclidean space which may lead to further applications and numerical work.

Given a fully resolved tree $T \in \mathcal{S}_n$ with leaf labels $\{1,2,\ldots,n\}$, we define $l_{ij}$ to be the number of internal vertices in the path from leaf $i$ to leaf $j$. We also have $l_{ii}=0$ and $l_{ji}=l_{ij}$. The values for $l_{ij}$ are integers with $1 \le i,j \le n-2$ for all pairs $i \neq j$ [9]. We define a vector $\mathbf{d} \in \mathbb{R}^{\binom{n}{2}}$ with subscripts of the components in the lexicographic order $\mathbf{d}=(d_{12},d_{13},\ldots,d_{1n},d_{23},d_{24},\ldots,d_{n-1\ n})$. The vector $\mathbf{l}^T \in \mathbb{R}^{\binom{n}{2}}$ is given by $\mathbf{l}=(l_{12},l_{13},\ldots,l_{1n},l_{23},l_{24},\ldots,l_{n-1\ n})$. Let $\mathbf{c}^T \in \mathbb{R}^{\binom{n}{2}}$ be the vector with components in the lexicographic order which have values $c^T_{ij}=2^{-l_{ij}}$. We may choose to multiply all entries by $2^{n-2}$ so that components in the vector $\mathbf{x}^T=2^{n-2}\mathbf{c}^T$ are all integers. Then every fully resolved tree $T \in \mathcal{S}_n$ corresponds to a different vector $\mathbf{x}^T \in \mathbb{R}^{\binom{n}{2}}$ [36].

**Definition 2.51.** *The* balanced minimum evolution *(BME) polytope $BME_n$ for the discrete fully resolved tree space $\mathcal{S}_n$ is defined as*

$$BME_n := conv\left\{\mathbf{x}^T : T \in \mathcal{S}_n\right\},$$

*the convex hull in $\mathbb{R}^{\binom{n}{2}}$ given by the vectors determined by the number of internal vertices between any pair of leaves.*

Given any leaf label $i$ for a vector $\mathbf{x}^T$ on the BME polytope $BME_n$, we have a Kraft equality [36] given by

$$\sum_{i:i\neq j} x^T_{ij} = 2^{n-2}.$$

These $n$ equalities for the $n$ leaves give the dimension of the BME polytope $BME_n$, $\dim(BME_n)=\binom{n}{2}-n$.

For example, the three fully resolved trees in $BHV_4$, given in the quartet display $\{1,2|3,4\}$,$\{1,3|2,4\}$ and $\{1,4|2,3\}$, correspond to the vectors $(2,1,1,1,1,2)$, $(1,2,1,1,2,1)$ and $(1,1,2,2,1,1)$. The three points given by the three vectors form a triangle in $\mathbb{R}^6$, which is a 2-dimensional convex polytope.

Recent publications have described some of the facets on the BME polytopes and outlined an idea to describe all faces of the maximum dimension and the adjacencies between the vertices which correspond to the nearest neighbour interchange in $\mathcal{S}_n$ [25], [24].

However, we would like to have a realization of the discrete space $\mathcal{S}_n$ as a convex polytope which can be further applied to construct the dual polytope to realize the continuous space $\mathcal{T}_n$. When we have a fixed cyclic order of the leaf labels, the embedded fully resolved trees can be put in the dual polygon with fixed leaf labels on the edges in $\mathbb{R}^2$. If we do not fix the cyclic order of the leaf labels of $\mathcal{S}_n$, it is natural to consider them on the $n$ vertices of a standard $(n-1)$-simplex. We apply the similar idea of the secondary polytope construction for the associahedra and generalize to higher dimensions; i.e., the generalized secondary polytope for the triangulation of an $(n-1)$-dimensional polytope. We outline the idea of the realization which can be extended to further analysis.

**Definition 2.52.** *Two hyperplanes are* non-crossing *inside a polytope $\mathcal{P}$ if the intersection of the two hyperplanes are not in the interior of the polytope $\mathcal{P}$.*

**Definition 2.53.** *A cutting hyperplane of the $n$-dimensional convex polytope $\mathcal{P}$ is an $(n-1)$-dimensional hyperplane which is an $(n-1)$-dimensional convex hull of some vertices of the polytope $\mathcal{P}$.*

**Definition 2.54.** *A triangulation of an $n$-dimensional convex polytope $\mathcal{P}$ is a cutting of the polytope into disjoint regions with the maximum number of non-crossing $(n-1)$-dimensional cutting hyperplanes inside $\mathcal{P}$.*

Note that the disjoint regions after the triangulation are still convex or non-concave.

We identify the vertices for every disjoint region $\mathit{\Delta}_{j_1\ldots j_k}$ inside $\mathcal{P}$, given as vectors in Euclidean space $\{\mathbf{j}_k,\ldots,\mathbf{j}_k\}$. We will assign mass $\mathrm{vol}\left(\mathit{\Delta}_{j_1\ldots j_k}, j_l\right)$ to every vertex of $\mathit{\Delta}_{j_1\ldots j_k}$ by the *barycentric distribution* which has the following properties. The sum of the mass assigned to all vertices $\sum_{l=1}^{k}\mathrm{vol}\left(\mathit{\Delta}_{j_1\ldots j_k}, j_l\right)$, are equal to its volume $\mathrm{vol}\left(\mathit{\Delta}_{j_1\ldots j_k}\right)$. The centre of mass $\underline{\mathit{\Delta}_{j_1\ldots j_k}}$, which can be evaluated by the weighted sum of the vertices' coordinates

$$\mathrm{vol}\left(\mathit{\Delta}_{j_1\ldots j_k}\right)\underline{\mathit{\Delta}_{j_1\ldots j_k}} = \sum_{l=1}^{k}\mathrm{vol}\left(\mathit{\Delta}_{j_1\ldots j_k}, j_l\right)\mathbf{j}_l,$$

have the same coordinates as the fixed barycenter of the region, for the case where the mass is uniformly distributed within the interior of the region. This distribution of mass to the vertices can always be achieved as the region is $n$-dimensional in the Euclidean space, equal to the dimension of $\mathcal{P}$, thus consists of at least $(n+1)$ vertices which are not on an $(n-1)$-dimensional hyperplane. This can be done by setting the mass at the vertices as unknown variables and we can obtain precisely $(n+1)$ equations. One of these equations corresponds to the fixed sum of mass, which is equal to the volume of the region. The other $n$ equations correspond to the $n$ components for the coordinates of the centre of mass in the Euclidean space, which are equal to the entries for the barycenter of the region.

**Definition 2.55.** *For convex polytope $\mathcal{P}$ with $m$ vertices, let $\{\mathbf{e}_i\}_{i=1}^m$ be the standard basis of the vector space $\mathbb{R}^m$ with subscripts labeled by the vertices of $\mathcal{P}$. Let $\Delta$ be a triangulation of $\mathcal{P}$ and $vol_i$ be the sum of all mass assigned to the vertex $i$. The vector $\mathbf{V}_\Delta \in \mathbb{R}^m$ is given by*

$$\mathbf{V}_\Delta = \sum_{i=1}^{m} vol_i \mathbf{e}_i.$$

**Remark 2.56.** *The vector $\mathbf{V}_\Delta$ can also be expressed as*

$$\mathbf{V}_\Delta = \sum_{\Delta_{j_1 \cdots j_k} \text{ is a truncated region in } \mathcal{P}} \sum_{l=1}^{k} vol\left(\Delta_{j_1 \ldots j_k}, j_l\right) \mathbf{e}_{j_l},$$

*where $vol\left(\Delta_{j_1 \ldots j_k}, j_l\right)$ is the mass given by the volume of the region $\Delta_{j_1 \ldots j_k}$ assigned to the vertex $j_l$ by the* barycentric distribution*.*

**Definition 2.57.** *The amended* generalized secondary polytope *$\mathcal{Q}$ for the convex polytope $\mathcal{P}$ with $m$ vertices is given by the convex hull*

$$\mathcal{Q} := conv\{\mathbf{V}_\Delta : \Delta \text{ is a triangulation of } \mathcal{P}\}.$$

We define the polytope $\Sigma'_n$ to be the convex hull of the middle points of all edges of the standard $(n-1)$-simplex in $\mathbb{R}^n$. The $n$ vertices of the standard $(n-1)$-simplex have coordinates $\mathbf{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$ where the only nonzero component 1 is at the $j$-th entry. In $\mathbb{R}^n$, the standard $n$-simplex is explicitly given by the set of points

$$\left\{ \mathbf{x} = (x_1, \ldots, x_n) \middle| \sum_{i=1}^{n} x_i = 1, 0 \le x_i \le 1 \text{ for } i = 1, \ldots, n \right\}.$$

The middle points of the edges of this standard $(n-1)$-simplex have coordinates $\mathbf{e}_{ij} = \mathbf{e}_{ji} = \left(0, \ldots, \frac{1}{2}, \ldots, \frac{1}{2}, \ldots, 0\right)$ where the only two nonzero components $\frac{1}{2}$ are at the $i$-th and $j$-th entries. The polytope $\Sigma'_n$ can be considered as a truncated $n$-simplex, which is the union of the standard $(n-1)$-simplex and the $n$ half spaces defined by $x_j \le \frac{1}{2}$ for

$j = 1, \dots, n$; i.e., the set of points whose components are all between 0 and $\frac{1}{2}$ inside the standard $n$-simplex. Therefore in $\mathbb{R}^n$, the polytope $\Sigma'_n$ is explicitly given by the set of points

$$\Sigma'_n := \left\{ \mathbf{x} = (x_1, \dots, x_n) \middle| \sum_{i=1}^{n} x_i = 1, 0 \leq x_i \leq \frac{1}{2} \text{ for } i = 1, \dots, n \right\}.$$

There are $n$ faces of $\Sigma'_n$ given by the hyperplanes $x_j = \frac{1}{2}$ for $j = 1, \dots, n$ in the subspace where the components satisfy $\sum_{i=1}^{n} x_i = 1$. We will label these $n$ faces by the $n$ leaf labels in the discrete tree space $\mathcal{S}_n$ and consider the triangulations of $\Sigma'_n$. Note that every pair of labeled faces $i$ and $j$ do not have any common edge, but only have one point in common, the vertex $\mathbf{e}_{ij}$. The other unlabeled $n$ faces of $\Sigma'_n$ are part of the original faces of the $n$-simplex before the truncation, given by the hyperplanes $x_j = 0$ for $j = 1, \dots, n$ where the components satisfy $\sum_{i=1}^{n} x_i = 1$.

We will consider the hyperplanes that cut $\Sigma'_n$ into disjoint regions which satisfy the conditions of triangulating an $(n-1)$-dimensional convex polytope. For $n \geq 4$, let $E$ be a subset of the set of leaf labels with $2 \leq |E| \leq n-2$. Therefore the two sets $E$ and $E^C$ constitute a nontrivial split of the leaf labels. We may always assume that the leaf label $n$ is in the subset $E$.

**Definition 2.58.** *Let $\mathbf{v}_E \in \mathbb{R}^n$ be the vector with values 1 at the entries with subscripts in the subset $E$ and $-1$ at the entries with subscripts in the subset $E^C$. The cutting hyperplane $\underline{E}$ inside $\Sigma'_n$ is given by the set of points*

$$\underline{E} := \left\{ \mathbf{x} \in \Sigma'_n : \langle \mathbf{x}, \mathbf{v}_E \rangle = \sum_{i \in E} x_i - \sum_{j \in E^C} x_j = 0 \right\}.$$

We can see that a vertex $\mathbf{e}_{ij}$ of the polytope $\Sigma'_n$ is on the hyperplane $\underline{E}$ if $i$ and $j$ are not in the same subset $E$ or $E^C$. The hyperplane $\underline{E}$ also defines the half-spaces inside $\Sigma'_n$; i.e., we obtain the inequalities for the vertices which are not on the hyperplane from the split:

$$\langle \mathbf{e}_{ij}, \mathbf{v}_E \rangle > 0 \text{ if } i, j \in E,$$
$$\langle \mathbf{e}_{ij}, \mathbf{v}_E \rangle < 0 \text{ if } i, j \in E^C.$$

Two distinct vertices $\mathbf{e}_{i_1 j_1}$ and $\mathbf{e}_{i_2 j_2}$ are connected by an edge of $\Sigma'_n$ if and only if $\left|\{i_1, j_1\} \cap \{i_2, j_2\}\right| = 1$. Therefore every edge given by two adjacent vertices $e_{i_1 j_1}$ and $e_{i_2 j_2}$ satisfies precisely one of the three following conditions.

- the edge is parallel to the hyperplane $\underline{E}$, if the three numbers $i, j_1, j_2$ are in the same subset $E$ or $E^C$ as it is on the parallel hyperplane defined by $\langle \mathbf{e}_{ij}, \mathbf{v}_E \rangle = 1$ or $-1$;

- the edge is on the hyperplane $\underline{E}$, if $i \in E, j_1, j_2 \in E^C$ or $i \in E^C, j_1, j_2 \in E$;
- the edge crosses the hyperplane at $e_{i_1 j_1}$ or $e_{i_2 j_2}$ if $j_1$ and $j_2$ are not in the same subset $E$ or $E^C$.

Therefore the hyperplane $\underline{E}$ cuts the polytope $\Sigma'_n$ into two disjoint regions without creating additional vertices on the surface of $\Sigma'_n$. Both regions are also convex polytopes and consist of at least two labeled faces which correspond to the nontrivial split of the leaf labels.

Given two compatible nontrivial splits of the leaf labels $E$ and $F$, the two hyperplanes $\underline{E}$ and $\underline{F}$ do not cross in the interior of the polytope $\Sigma'_n$. The entries for the coordinates of any interior points of the $n$-simplex are all nonzero and there does not exists any nonzero solutions for the coordinates $\mathbf{x}$ that satisfy both equations below of the hyperplane if their splits are compatible.

$$\langle \mathbf{x}, \mathbf{v}_E \rangle = \sum_{i \in E} x_i - \sum_{j \in E^C} x_j = 0,$$
$$\langle \mathbf{x}, \mathbf{v}_F \rangle = \sum_{i \in F} x_i - \sum_{j \in F^C} x_j = 0.$$

Moreover, the two hyperplanes $\underline{E}$ and $\underline{F}$ cross on the boundary of the polytope $\Sigma'_n$, defined by the facets with vertices $\mathbf{e}_{ij}$ where $i$ and $j$ are not in same subset $E$ or $E^C$, and not in same subset $F$ or $F^C$.

Therefore the $(n-3)$ compatible nontrivial splits of a fully resolved tree $T \in \mathcal{S}_n$ correspond to $(n-3)$ non-crossing hyperplanes inside the convex polytope $\Sigma'_n$ which gives a triangulation of $\Sigma'_n$, as every hyperplane $\underline{E}$ of a split generates the same split of the labeled faces of $\Sigma'_n$. We will now define the amended secondary polytope of $\Sigma'_n$ given by the convex hull of the vectors define by the triangulations of $\Sigma'_n$.

**Definition 2.59.** *Let $\Delta$ be a triangulation of $\Sigma'_n$ and $\tilde{vol}\left(\Delta_{j_1 \dots j_k}, j_l\right)$ be the mass of from the region $\Delta_{j_1 \dots j_k}$ assigned to the vertex $j_l$ by the* barycentric distribution. *The* barycentric distribution *is a* symmetric distribution *if the value $\tilde{vol}\left(\Delta_{j_1 \dots j_k}, j_l\right)$ is invariant under the group action that stabilizes the region $\Delta_{j_1 \dots j_k}$ by permuting the $n$ labels in the subscripts of the vertices on $\Sigma'_n$.*

**Definition 2.60.** *Let $\tilde{\mathbf{V}}_\Delta \in \mathbb{R}^{\binom{n}{2}}$ be the vector of the triangulation $\Delta$ of the convex polytope $\Sigma'_n$ where the mass is assigned to the vertices by the* symmetric distribution.

By definition we have

$$\tilde{\mathbf{V}}_\Delta = \sum_{\Delta_{j_1 \cdots j_k} \text{ is a truncated region in } \Sigma'_n} \sum_{l=1}^{k} \tilde{\mathrm{vol}}\left(\Delta_{j_1 \dots j_k}, j_l\right) \mathbf{e}_{j_l} = \sum_{\alpha \text{ is a vertex of } \Sigma'_n} \tilde{\mathrm{vol}}_\alpha \mathbf{e}_\alpha,$$

**Definition 2.61.** *The amended* generalized secondary polytope $\mathcal{Q}_n$ *for the convex polytope* $\Sigma'_n$ *is given by the convex hull*

$$\mathcal{Q}_n := conv\left\{\tilde{\mathbf{V}}_{\Delta} : \Delta \text{ is a triangulation of } \Sigma'_n\right\},$$

The dimension of the secondary polytope $\mathcal{Q}_n$ is $\left(\binom{n}{2} - n\right)$ where $\binom{n}{2}$ corresponds to the number of vertices on $\Sigma'_n$. The points on the secondary polytope $\mathcal{Q}_n$, given in the form of vectors in $\mathbb{R}^{\binom{n}{2}}$, are in the subspace generated from the $n$ affine conditions. One affine condition is given by the fixed sum of mass and the other $(n-1)$ conditions are given by the fixed position of the barycentre of the convex polytope $\Sigma'_n$.

The secondary polytope $\mathcal{Q}_n$ and the BME polytope $BME_n$ have the same dimension in the Euclidean space. Both realizations have reduced the dimension of the graph of the discrete space $\mathcal{S}_n$ to the order of $n^2$, which suggests a construction of the continuous space $\mathcal{T}_n$ in the same dimension.

# Chapter 3

# Random walks, finite Gelfand pairs and spherical functions

In this chapter, we study the definitions and properties of *Gelfand pairs* and *spherical functions* and seek methods to solve the random walks given by nearest neighbour interchange on the discrete tree spaces $\mathcal{S}_n$ and $\mathcal{S}_n^*$ defined in Chapter 2.

Some examples of tree spaces we consider are relatively large and cost more than exponential time to simulate numerically [53], [61]. We can produce the Markov chain from the nearest neighbour interchange (NNI) process that generates the simple random walk in $\mathcal{S}_n$ but the dimension of the transition matrix is $(2n-5)!!$. Our aim is to find the decomposed eigensolutions under the invariance conditions of the symmetric groups acting on the tree spaces $\mathcal{S}_n$ and $\mathcal{S}_n^*$.

We also note that some tree spaces can be parametrized by a large group $G$ and a non-trivial subgroup $K$ which form a Gelfand pair. Thus we apply the connection between the Gelfand pairs and the simple random walks to find the characters and spherical functions of the bi-$K$-invariant subalgebra $\ell^1\left(K\backslash G/K\right)$ which can be interpreted as the eigensolutions of the transition matrix for the simple random walks.

The Gelfand pair $(S_n, S_m \times S_{n-m})$ is used to solve the random walk on the split of the leaf labels of the trees in $\mathcal{S}_n$ and $\mathcal{S}_n^*$. We outline a method to solve the spherical functions from the Johnson scheme and the Bernoulli-Laplace diffusion model. In particular, we present the solution for the random walk on the vertices and edges of the Petersen graph which corresponds to $\mathcal{S}_5$ and $\mathcal{S}_5^*$.

The trees and matchings of $2n$ elements into unordered $n$ pairs as defined in Method (II) of enumerating the number of trees in $\mathcal{S}_{n+2}$ in Section 2.2 correspond to the Gelfand pair $(S_{2n}, S_2 \wr S_n)$. The matchings only correspond to the set of fully resolved trees and do not have the same NNI adjacencies and symmetry properties as the space of phylogenetic trees [17].

The automorphism group $Aut\left(\mathbb{T}_{q,n}\right)$ acts on the rooted $q$-ary ultrametric trees of depth $n$. The automorphism group $Aut\left(\mathbb{T}_{q,n}\right)$ and the stabilizer of a fixed leaf $K_{q,n}$ form a Gelfand pair and is a model which can be generalized to the infinite case; i.e., Banach algebras in the second part of the thesis.

## 3.1 Finite Gelfand pairs

Throughout this chapter we assume that the group $G$ is finite and consider the $\ell^1$ norm on the group algebras and the invariant subalgebras. We mainly use the definitions and notations from [15].

Let $G$ be a finite group and $\ell^1(G)$ be the algebra of summable complex-valued functions on the group. For all $f \in \ell^1(G)$, we have $\sum_{g \in G} \left|f(g)\right| < \infty$.

Let $\delta_g \in \ell^1(G)$ be the characteristic function on the group element $g \in G$. We have $\delta_g(g) = 1$ and $\delta_g(h) = 0$ for $h \in G$ with $h \neq g$. The algebra $\ell^1(G)$ has a basis $\left\{\delta_g\right\}_{g \in G}$ given by the characteristic functions on the group elements.

**Definition 3.1.** *Given a group $G$ and a subgroup $K \subseteq G$, a function $f \in \ell^1(G) : G \longrightarrow \mathbb{C}$ is right-$K$-invariant if for all $g \in G$ and for all $k \in K$, we have $f(g) = f(gk)$ and is left-$K$-invariant if for all $g \in G$ and for all $k \in K$, we have $f(g) = f(kg)$.*

The set of summable right-$K$-invariant functions $\ell^1\left(G/K\right)$ is a subspace of $\ell^1(G)$. A function $f \in \ell^1\left(G/K\right)$ satisfies $f(h_1) = f(h_2)$ for all $h_1, h_2$ in the same right coset $gK$. Set $\delta_{gK} \in \ell^1(G) = \sum_{h \in gK} \delta_h$ to be the sum of characteristic functions of the group elements $h$ in the right coset $gK$. The right-$K$-invariant subspace $\ell^1\left(G/K\right)$ has a basis $\left\{\delta_{gK}\right\}_{gK \subset G}$ given by the characteristic functions of the right cosets $\delta_{gK} = \sum_{h \in gK} \delta_h$.

Let $G$ acts on the space $X$ and $K$ be the stabilizer of $x_0 \in X$. Then $X$ is the homogeneous space of $G/K$ and for all $x \in X$ we have $x = gx_0$ for some $g \in G$. The set of right cosets $\{gK\}_{g \in G}$ is isomorphic to the homogeneous space $X$.

We can then define $f_X \in \ell^1(X) : X \longrightarrow \mathbb{C}$ as the complex valued functions on $X$. The values of $f_X$ are given by the right-$K$-invariant function $f \in \ell^1(G)$ where $f_X(x_0) = f(e)$ and $f_X(gx_0) = f(g)$. Therefore the right-$K$-invariant subspace $\ell^1\left(G/K\right)$ has a basis $\{\delta_x\}_{x \subset X}$ for all $x = gx_0$. The right-$K$-invariant subspace $\ell^1\left(G/K\right)$ can also be considered as the algebra of functions on the homogeneous space $X$. When $X$ is finite, the dimension of $\ell^1(X)$ is equal to the cardinality $|X|$. The basis is given by the set of characteristic functions $\{\delta_x : x \in X\}$. For all $y \in X$, we have

$$\delta_x(y) = \begin{cases} 1 \text{ if } y = x \\ 0 \text{ if } y \neq x \end{cases} .$$

**Definition 3.2.** *Given a group $G$ and a subgroup $K \subseteq G$, we define that a function $f \in \ell^1(G)$ is bi-$K$-invariant if $f$ is both left-$K$-invariant and right-$K$-invariant; i.e., for all $g \in G$ and for all $k_1, k_2 \in K$, we have $f(g) = f(k_1 g k_2)$.*

**Definition 3.3.** *Let $G$ be a finite group and $K$ be a subgroup of $G$. The pair $(G, K)$ is called a* Gelfand pair *if the algebra $\ell^1\left(K \backslash G / K\right)$ of bi-$K$-invariant functions under the convolution multiplication $*$ is commutative [15].*

**Lemma 3.4.** *Let $\ell^1\left(K \backslash G / K\right)$ be the subalgebra of bi-$K$-invariant functions under the convolution multiplication $*$. Then $\ell^1\left(K \backslash G / K\right)$ closed is with respect to convolution. That is, for all $f_1, f_2 \in \ell^1\left(K \backslash G / K\right)$, $f_1 * f_2$ is bi-$K$-invariant.*

*Proof.* For all $g' \in KgK$, there exists $k_1, k_2 \in K$ such that $g' = k_1 g k_2$.

$$
\begin{aligned}
[f_1 * f_2]\left(g'\right) &= \sum_{h \in G} f_1(g'h) f_2(h^{-1}) = \sum_{h \in G} f_1(k_1 g k_2 h) f_2(h^{-1}) \\
&= \sum_{h \in G} f_1(g k_2 h) f_2(h^{-1}) = \sum_{h \in G} f_1(g k_2 h) f_2(h^{-1} k_2^{-1}) \\
\text{setting } h' = k_2 h & \\
&= \sum_{h' \in G} f_1(g h') f_2(h'^{-1}) = [f_1 * f_2](g).
\end{aligned}
$$

□

**Lemma 3.5.** *Given a group $G$ and a subgroup $K \subseteq G$, Suppose for any $g \in G$ we have $g^{-1} \in KgK$. Then $(G, K)$ is a Gelfand pair (Example 4.3.2 of [15]).*

*Proof.* We assume that $f(g) = f(g^{-1})$ for all $f \in \ell^1\left(K \backslash G / K\right)$. Then for $f_1, f_2$ in this algebra we get

$$
\begin{aligned}
[f_1 * f_2](g) &= \sum_{h \in G} f_1(gh) f_2(h^{-1}) \\
&= \sum_{h \in G} f_1(gh) f_2(h) \\
\text{setting } t = gh & \\
&= \sum_{t \in G} f_1(t) f_2(g^{-1} t) \\
&= \sum_{t \in G} f_1(t^{-1}) f_2(g^{-1} t) \\
&= [f_2 * f_1](g^{-1}).
\end{aligned}
$$

By Lemma 3.4, $f_2 * f_1$ is also bi-$K$-invariant. Hence we have $[f_2 * f_1](g^{-1}) = [f_2 * f_1](g)$. Therefore $[f_1 * f_2](g) = [f_2 * f_1](g)$,which shows that $(G, K)$ is a Gelfand pair. □

**Remark 3.6.** *If $(G, K)$ is a Gelfand pair and $g^{-1} \in KgK$ for all $g \in G$, then $(G, K)$ is called a* symmetric Gelfand pair*.*

**Definition 3.7.** *Let $(X, d)$ be a finite metric space and $G$ be a group acting on $X$ by isometries i.e. $d(gx, gy) = d(x, y)$ for all $x, y \in X$. The action is* 2-point homogeneous *(or* distance-transitive*) if for all $(x_1, y_1), (x_2, y_2) \in X \times X$ such that $d(x_1, y_1) = d(x_2, y_2)$, there exists $g \in G$ such that $gx_1 = x_2$ and $gy_1 = y_2$ [15].*

Let $X$ be the homogeneous space $G/K$. Let $G$ be the finite group that acts isometrically and 2-point homogeneously on the metric space $(X, d)$. The condition of the metric $d(x_0, y) = d(y, x_0)$ implies that there exists $g \in G$ such that $y = gx_0$ and $x_0 = gy$ for all $x_0, y \in X$ which further implies $x_0 = gy = g^{-1}y$ and $d(x_0, gx_0) = d\left(x_0, g^{-1}x_0\right)$. If $K$ is the stabilizer of $x_0$, then $x_0 = kx_0$ and $d\left(kx_0, kg^{-1}x_0\right) = d\left(x_0, kg^{-1}x_0\right) = d\left(x_0, gk^{-1}x_0\right) = d(x_0, gx_0)$ for all $k \in K$, The equality $d\left(x_0, kg^{-1}x_0\right) = d(x_0, gx_0)$ shows that there exists $k' \in G$ such that $x_0 = k'x_0$ and $k'kg^{-1}x_0 = gx_0$. Therefore we have $k' \in K$ and $g^{-1} \in KgK$ thus $(G, K)$ is a symmetric Gelfand pair.

We identify the following examples of groups and subgroups and show that they form symmetric Gelfand pairs and present the links to the corresponding tree spaces.

**Example 3.8.** *We consider the group that acts on the $m, (n-m)$ splits of the leaf labels on trees in the discrete tree space $\mathcal{S}_n^*$.*

Let $G$ be the symmetric group $S_n$ acting on the $m$-subsets of the set $\{1, \ldots, n\}$ by permuting the numbers. Let $K$ be the subgroup $S_m \times S_{n-m}$ which is isomorphic to the stabilizer of the $m$-subset $x_0 = \{1, \ldots, m\}$. The homogeneous space $X$, which is the set of all $m$-subsets of $\{1, \ldots, n\}$, is isomorphic to $G/K$.

Let $\mathcal{S}_n^*$ be the tree space with leaf labels $\{1, \ldots, n\}$ . Every nontrivial split of a tree $T \in \mathcal{S}_n^*$ is given by the partition of a subset of $\{1, \ldots, n\}$ and the complement of the subset. In this example, we fix the size of the subset to be $m$ and consider the splits correspond to the $m$-subsets of the set of leaf labels $\{1, \ldots, n\}$.

**Lemma 3.9.** *The pair $(S_n, S_m \times S_{n-m})$ is a symmetric Gelfand pair.*

*Proof.* First we identify the double cosets. The group $G$ acts transitively on the $m$-subset of $\{i_1, \ldots, i_n\}$ by

$$\pi\{i_1, \ldots, i_n\} = \left\{\pi(i_1), \ldots, \pi(i_n)\right\}, \text{ for all } \pi \in G.$$

We have $kx_0 = k$ and $|kx \cap x_0| = |kx \cap x_0|$ for all $k \in K$. For all $k_1, k_2 \in K$, we have

$$|k_1 g k_2 x_0 \cap x_0| = |k_1 g x_0 \cap x_0| = |g x_0 \cap x_0|,$$

which shows that the double cosets of $g$ are determined by the number of unmoved elements in $x_0$ after the group action $g$:

$$KgK = \left\{\pi \in G \big| |\pi x_0 \cap x_0| = |gx_0 \cap x_0|\right\}.$$

Given a group element $g \in G$ and a number $i_k \in x_0$, we have either $gi_k \in x_0$ or $gi_k \notin x_0$. The group $S_n$ also acts on all subsets of $\{1,\ldots,n\}$. We define the set $x_1 = \{i_k \in x_0 | gi_k \in x_0\}$ and the set $x_2 = \left\{i_k \in x_0 \big| gi_k \notin x_0\right\}$. Then we have a partition $x_0 = x_1 \cup x_2$ and $gx_0 = gx_1 \cup gx_2$. Note that $x_1 \cap x_2 = \phi$, $|gx_2 \cap x_0| = \left|x_2 \cap g^{-1}x_0\right| = 0$ and $|gx_0 \cap x_0| = \left|x_0 \cap g^{-1}x_0\right| = |gx_1| = |x_1|$.

We have $g^{-1}x_0 \cap x_0 = \left(g^{-1}\left(x_1 \cup x_2\right)\right) \cap x_0 = \left(g^{-1}x_1 \cap x_0\right) \cup \left(g^{-1}x_2 \cap x_0\right)$. Then $\left|g^{-1}x_0 \cap x_0\right| = |x_1 \cap x_0| = |x_1|$, i.e. if $g$ stabilizes $j$ elements in $x_0$, then $g^{-1}$ also stabilizes $j$ elements in $x_0$ which implies $|gx_0 \cap x_0| = \left|g^{-1}x_0 \cap x_0\right|$. Therefore $g^{-1} \in KgK$ and $(G,K)$ is a symmetric Gelfand pair. □

**Remark 3.10.** *Let $d : X \times X \longrightarrow \mathbb{R}_+$ be the metric defined as $d(x,y) = \min\{m, n-m\} - |x \cap y|$ for all $x, y$ which are the $m$-subsets of the set of $n$ numbers. The action of $S_n$ on the metric space $(X,d)$ is 2-point homogeneous.*

In the next section, we consider the random walk on the $m$-subsets of the set of $n$ numbers and the random walk on the double cosets given by the subalgebra of bi-$K$-invariant functions. We may use this setting as the model of the random walks on the trees in $\mathcal{S}_n^*$ and only consider the degenerate trees when the splits correspond to the $m$-subsets. The eigensolutions to both random walks can be obtained from each other. When $n$ is small, the number of possible choices of $m$ is also small and we will be able to use this Gelfand pair to solve the random walks quickly. For $n = 5$, we have only one nontrivial choice of $m = 2$ which corresponds to the random walk on the Petersen graph as described in Example 2.18. When $n$ is large, the number of tree topologies given by the partitions of the internal edges are not easy to compute [9].

**Example 3.11.** *We consider trees in the discrete space $\mathcal{S}_n$ and the matchings of the $(n-2)$ unordered disjoint 2-subsets of $(2n-4)$ numbers.*

As described in Method (II) of enumerating the number of trees in $\mathcal{S}_n$ in section 2.2.1, we set up a one-to-one correspondence between the semi-labeled fully resolved trees and the $(n-2)$ unordered disjoint 2-subsets of $(2n-4)$ numbers [17].

Let $X$ be the family of all partitions of $\{1,\ldots,2n\}$ consisting of 2-subsets. The elements $x \in X$ are of the form $\left\{\{i_1,i_2\},\{i_3,i_4\},\ldots,\{i_{2n-1},i_{2n}\}\right\}$. The symmetric group $S_{2n}$ acts on these $n$ unordered pairs. For all $\pi \in S_{2n}$,

$$\pi x = \left\{\{\pi i_1, \pi i_2\},\{\pi i_3, \pi i_4\},\ldots,\{\pi i_{2n-1}, \pi i_{2n}\}\right\}.$$

Set $x_0 = \big\{\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}\big\}$ and let the subgroup $K$ be the stabilizer of $x_0$. We express $K$ as the wreath product $S_2 \wr S_n$ and we can compute that $|X| = \frac{(2n)!}{2^n n!}$.

The pair $(S_{2n}, S_2 \wr S_n)$ is a symmetric Gelfand pair [17]. However the elements of $X$ which belong to a double coset $KgK$ do not have the $S_{n+2}$ symmetry we expect to have for $\mathcal{S}_n$. The adjacencies are different from the nearest neighbour interchange [17] hence the corresponding homogeneous space and double cosets from the Gelfand pair $(S_{2n}, S_2 \wr S_n)$ do not correspond to the discrete metric structure of the space $\mathcal{S}_n$ [9].

**Example 3.12.** *We consider the $q$-homogeneous rooted tree of depth $n$, the ultrametric space, the automorphism group $Aut(\mathbb{T}_{q,n})$ and the stabilizer of the word $0\ldots0$.*

As defined in Example 2.10 in section 2.1, the *$q$-homogeneous rooted tree of depth $n$* is a graph whose vertices correspond to the words from the alphabet $\Sigma = \{0,1,\ldots,q-1\}$ with length at most $n$. The automorphism group $Aut\left(\mathbb{T}_{q,n}\right)$ is isomorphic to the $n$-iterated wreath product $S_q \wr \cdots \wr S_q$ [7] and we define the subgroup $K_{q,n}$ to be the stabilizer of the word $0\ldots0$ of length $n$.

**Definition 3.13.** *Given two leaves of $\mathbb{T}_{q,n}$ which correspond to the two words $x = x_1\cdots x_n \in \Sigma^n$ and $y = y_1\cdots y_n \in \Sigma^n$, we define the function $d : \Sigma^n \times \Sigma^n \longrightarrow \mathbb{R}_+$ as*

$$d\,(x,y) = n - \max\left\{k : x_i = y_i \text{ for all } i \le k\right\}.$$

We see that $d\,(x,y) = d\,(y,x) \ge 0$ for all $x,y \in \Sigma^n$ and the equality holds if and only if $x = y$. Note that for any three words $x,y,z \in \Sigma^n$, the triangle inequality $d\,(x,y) \le d\,(x,z) + d\,(z,y)$ can be obtained from the relation

$$d\,(x,y) \le \max\left\{d\,(x,z) + d\,(z,y)\right\},$$

which is called the *ultrametric inequality*. Therefore the function $d$ is a metric. This metric $d$ is called the *ultrametric distance* and $(\Sigma^n, d)$ is called the *ultrametric space*. For $x,y \in \Sigma^n$ we have $d\,(x,y) \in \{0,1,\ldots,n\}$.

We denote the stabilizer of the word $x_0 = 00\cdots0 \in \Sigma^n$ by

$$K_{q,n} = \left\{g \in Aut\left(\mathbb{T}_{q,n}\right) : g\,(x_0) = x_0\right\}.$$

If the subgroup $K_{q,n}$ stabilizes $x_0$, then $K_{q,n}$ also stabilizes all words of the form $00\cdots0$ of any length less than $n$ including the empty word $\phi$.

We identify the double cosets from the orbits of the leaves determined by the ultrametric $d$. For $j \in \{0,1,\ldots,n\}$, define the set of leaves $\Omega_{n,j}$ as the set of words $\left\{x \in \Sigma^n : d\,(x_0,x) = j\right\}$. The total numbers of words in each orbits are given by $\left|\Omega_{n,0}\right| = 1$ and $\left|\Omega_{n,j}\right| = (q-1)\,q^{j-1}$ for $1 \le j \le n$. Given $g_{n,j} \in Aut\left(\mathbb{T}_{q,n}\right)$ where $d\left(x_0, g_{n,j}x_0\right) = j$, the double coset $Kg_{n,j}K$ is given by

$$Kg_{n,j}K = \left\{ g \in Aut\left(\mathbb{T}_{q,n}\right) \middle| gx_0 \in \Omega_{n,j} \right\}.$$

**Lemma 3.14.** *The pair* $\left(Aut\left(\mathbb{T}_{q,n}\right), K_{q,n}\right)$ *is a symmetric Gelfand pair.*

*Proof.* For all $k \in K_{q,n}$ we have $k^{-1} \in K_{q,n}$.

If $g \notin K$, let $x = gx_0 = 0 \cdots 0x_j \cdots x_n$ where the first nonzero letter $x_j \in \Sigma \backslash \{0\}$ appears in the $j$-th position. Then we have $y = g^{-1}x_0 = 0 \cdots 0y_j \cdots y_n$ where the first nonzero letter $y_j \in \Sigma \backslash \{0\}$ also appears in the $j$-th position. Therefore $d(x, x_0) = d(y, x_0) = n - j$. We can see that $g^{-1} \in KgK$ which implies that $\left(Aut\left(\mathbb{T}_{q,n}\right), K_{q,n}\right)$ is a symmetric Gelfand pair. □

**Remark 3.15.** *The action of* $Aut\left(\mathbb{T}_{q,n}\right)$ *on the metric space* $(\Sigma^n, d)$ *is* 2*-point homogeneous [15].*

In the next chapter, we study the automorphism group $Aut\left(\mathbb{T}_q\right)$ acting on the infinite homogeneous tree $\mathbb{T}_q$ and the stabilizer $K$ of a fixed vertex $x_0$. We use the 2-point homogeneous property to show that $\left(Aut\left(\mathbb{T}_q\right), K\right)$ is a Gelfand pair and study the properties of the corresponding Iwahori-Hecke Algebra with $\ell^1$ norm.

## 3.2 Spherical functions

In this section, we introduce the definitions and some key properties of the spherical functions for a commutative bi-$K$-invariant subalgebra of the group algebra $\ell^1(G)$. We also present how the spherical functions link to the minimal idempotents of the bi-$K$-invariant subalgebra for finite dimensions.

We consider the the bi-$K$-invariant subalgebra $\ell^1\left(K \backslash G / K\right)$. The subalgebra $\ell^1\left(K \backslash G / K\right)$ has a natural basis given by the set of characteristic functions on the double cosets $\left\{Y_{KgK}\right\}_{g \in G}$ given by

$$Y_{KgK}(h) = \begin{cases} 1 \text{ if } h \in KgK \\ 0 \text{ if } h \notin KgK \end{cases}.$$

**Remark 3.16.** *There exists a natural normalization for the set of bases* $\left\{Y_{KgK}\right\}_{g \in G}$ *as* $\left\{\frac{Y_{KgK}}{|KgK|}\right\}_{g \in G}$ *such that all elements in the set of bases have norm* 1.

**Definition 3.17.** *Let* $(G, K)$ *be a Gelfand pair. A bi-*$K$*-invariant function* $\phi$ *on* $G$ *is a* spherical function *if it satisfies the following conditions*

1. *For all* $f \in \ell^1\left(K \backslash G/K\right)$ *there exists* $\lambda_f \in \mathbb{C}$ *such that* $\phi * f = \lambda_f \phi$.

2. $\phi\left(1_G\right) = 1$.

Note that the constant function $\phi(g) = 1$ is always a spherical function as for all $g \in G$, we have

$$\phi * f(g) = \sum_{t \in G} \phi\left(gt\right) f\left(t^{-1}\right) = \left(\sum_{h \in G} f\left(h\right)\right) \phi\left(g\right).$$

The following statement is proved in Theorem 4.5.3 in [15].

**Theorem 3.18.** *A nonzero bi-*$K$*-invariant function* $\phi$ *is a spherical function if and only if*

$$\frac{1}{|K|} \sum_{k \in K} \phi\left(gkh\right) = \phi(g)\phi(h)$$

*for all* $g, h \in G$.

**Corollary 3.19.** *Let* $Y_{KgK} \in \ell^1\left(K \backslash G/K\right)$ *be the characteristic function of the double coset* $KgK$*. Let* $\delta_\phi\left(Y_{KgK}\right) = \phi(g)$*. Then* $\delta_\phi$ *is a character on the algebra* $\ell^1\left(K \backslash G/K\right)$ *given by the spherical function* $\phi$*.*

*Proof.* The convolution product of two characteristic functions is given by

$$\begin{aligned}
Y_{Kg_1K} * Y_{Kg_2K}(h) &= \sum_{t \in G} Y_{Kg_1K}\left(ht^{-1}\right) Y_{Kg_2K}\left(t\right) \\
&= \sum_{t \in Kg_2K} Y_{Kg_1K}\left(ht^{-1}\right) Y_{Kg_2K}\left(t\right) \\
&= \frac{1}{|K|^2} \sum_{k_1, k_2 \in K} Y_{Kg_1K}\left(hk_1 g_2^{-1} k_2\right) \\
&= \frac{1}{|K|} \sum_{k \in K} Y_{Kg_1kg_2K}\left(h\right).
\end{aligned}$$

The identity element of $\ell^1\left(K \backslash G/K\right)$ for the convolution multiplication is the characteristic function $Y_{K1_GK}$. We verify that the character corresponds to the characteristic function of $K1_GK$ is $\chi_\phi\left(Y_{K1_GK}\right) = \phi\left(1_G\right) = 1$. Therefore the character is nontrivial.

Given two characteristic functions of $\ell^1\left(K\backslash G/K\right)$, by Theorem 3.18 we have

$$\begin{aligned}\chi_\phi\left(Y_{Kg_1K} * Y_{Kg_2K}\right) &= \chi_\phi\left(\frac{1}{|K|}\sum_{k\in K} Y_{Kg_1kg_2K}\right)\\ &= \frac{1}{|K|}\sum_{k\in K}\phi\left(g_1kg_2\right)\\ &= \phi\left(g_1\right)\phi\left(g_2\right)\\ &= \chi_\phi\left(Y_{Kg_1K}\right)\chi_\phi\left(Y_{Kg_2K}\right),\end{aligned}$$

which shows that $\chi_\phi$ is a character. □

**Remark 3.20.** *Note that the characters are in the dual of $\ell^1\left(K\backslash G/K\right)$. The dual of the space $\ell^1(G)$ is given by $\ell^\infty(G)$.*

Let $G$ be a finite group and $(G,K)$ be a Gelfand pair with the number of orbits of $K$ on the corresponding homogeneous space $X \simeq G/K$ equal to $N+1$. We apply the following statements from [15] to present the connection between the idempotents of the subalgebra $\ell^1\left(K\backslash G/K\right)$ and the spherical functions:

1. There exist $N+1$ pairwise orthogonal spherical functions $\left\{\phi_j\right\}_{j=0}^N$, including the constant spherical function, $\phi_0(g)=1$ for all $g\in G$, by Corollary 4.5.6, Proposition 4.5.7 in [15].

2. $\ell^1(X)$ decomposes into $N+1$ distinct irreducible subrepresentations, by Theorem 4.6.1 in [15].

3. Every spherical function corresponds to a 1-dimensional irreducible subrepresentation in $\ell^1(X)$, Theorem 4.6.2 in [15].

The invariant subalgebra $\ell^1\left(K\backslash G/K\right)$ is $(N+1)$-dimensional and has a set of $(N+1)$ distinct minimal idempotents $\{\mathfrak{e}_i\}_{i=0}^N$ which correspond to the natural basis of $\mathbb{C}^{N+1}$. The minimal idempotents satisfy $\mathfrak{e}_i\mathfrak{e}_i=\mathfrak{e}_i$ and $\mathfrak{e}_i\mathfrak{e}_j=0$ for $i\neq j$. Every minimal idempotent $\mathfrak{e}_i$ corresponds to a character $\chi_i$ where $\chi_i(\mathfrak{e}_i)=1$ and $\chi_i(\mathfrak{e}_i)=0$ for $i\neq j$.

Every function $f\in\ell^1\left(K\backslash G/K\right)$ can be written as a linear sum of the minimal idempotents, given by $f=\sum_{j=0}^N\alpha_j\mathfrak{e}_j$. We have $f\mathfrak{e}_j=\alpha_j\mathfrak{e}_j$ and $\chi_j\left(f\mathfrak{e}_j\right)=\chi_j(f)\,\chi_j\left(\mathfrak{e}_j\right)=\alpha_j$ for all $j=0,1,\ldots,N$.

Let $(G,K)$ be a Gelfand pair and $\left\{Kg_jK\right\}_{j=0}^N$ be the set of distinct double cosets which constitute the finite group $G$ as a disjoint union. Let $\left\{\phi_j\right\}_{j=0}^N$ be the set of spherical functions for the subalgebra $\ell^1\left(K\backslash G/K\right)$. We find the set of minimal idempotents $\{\mathfrak{e}_i\}_{i=0}^N$ from the set of spherical functions.

If $\phi_j$ is a spherical function, then there exists $\lambda_j \in \mathbb{C}$ such that $\phi_j * \phi_j = \lambda_j \phi_j$. Let $\mathfrak{f}_j = \frac{1}{\lambda_j}\phi_j$ be the scalar multiple of the spherical function $\phi_j$. We have $\mathfrak{f}_j * \mathfrak{f}_j = \mathfrak{f}_j$, which shows that $\mathfrak{f}_j$ is an idempotent. Since all spherical functions are orthogonal, i.e. $\phi_j * \phi_l = 0$ when $j \neq l$, the set of scaled spherical functions $\{\mathfrak{f}_j\}_{j=0}^{N}$ constitute the set of minimal idempotents of $\ell^1(K\backslash G/K)$.

## 3.3 Random walk on tree spaces

Random walks are widely used in simulating the data of discrete tree spaces [9]. Statisticians require the probability distribution to analyze the stochastic process which simulates the evolutions as a random walk with a transition matrix [9], [4]. The dimension of the transition matrix is normally given by the size of the corresponding set which consists of elements in $\mathcal{S}_n^*$. The aim is to find the eigensolutions for the random walk which are given in the form of eigenvectors and eigenvalues which predict the probability distribution after certain steps without expanding the multiplications of the transition matrix.

The data given by the trees can be large but there might exist an invariance condition between the trees under certain group actions on the labels. Thus we seek better methods to simplify the random walk on a set of trees; e.g., to reduce the dimension of the transition matrix by defining a partition of the set of the trees. The partition is normally given by a lumping process as in Definition 3.24 and the eigensolutions of the random walk on the partition are preserved from the eigensolutions of the random walk on the original set through an invariant group action.

In some particular examples, given a symmetric Gelfand pair $(G, K)$, the random walk is on the homogeneous space $X \simeq G/K$ and is lumpable (Definition 3.24) to the random walk on the partition of $X$ given by the double cosets. We show that the adjacency operator on the vector space of the partition is equivalent to the convolution multiplication of the bi-$K$-invariant subalgebra of the group algebra thus find the correspondence between the eigensolutions of the random walk and spherical functions of the subalgebra.

One obvious example to study is the random walk on the $m$-subsets of a set of $n$ numbers which correspond to the random walk on the $m,(n-m)$-splits of trees in $\mathcal{S}_n^*$ and $\mathcal{S}_n$. We describe the lumpable random walk on the space of the $m$-subsets of a set of $n$ numbers and link to the Gelfand pair $(S_n, S_m \times S_{n-m})$. We also present that the eigensolutions of the random walk can be interpreted as the spherical functions of the commutative bi-invariant subalgebra defined from the Gelfand pair.

### 3.3.1 Lumpable random walk and eigensolutions

When we consider the random walks on the set $X$, we construct a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as a realization of the space $X$. The set of vertices $\mathcal{V}$ correspond to the elements in $X$ and the edges $\mathcal{E}$ correspond to the adjacency relations in the random walks.

For $x, y \in X$, let $p(x,y)$ be the probability of moving from $x$ to $y$ in one step. The transition matrix $P = \big(p(x,y)\big)_{x,y\in X}$ is a square matrix of dimension $|X|$ with nonnegative entries and satisfies $\sum_{y\in X} p(x,y) = 1$ for all $x \in X$.

**Definition 3.21.** *Let $X$ be a finite set and $\nu_0$ be a probability distribution on $X$. A (homogeneous)* Markov chain *with* state space $X$, *initial distribution $\nu_0$ and* transition matrix $P$ *is a finite sequence $\nu_0, \nu_1, \ldots, \nu_n, \ldots$ where $\nu_j = P^j \nu_0$ for all $j \in \mathbb{Z}_+$.*

Note that the distribution on $X$ is normally given as a column vector so that the left multiplication with the transition matrix $P$ is still a column vector.

A distribution $\nu_T \in \ell^1(X)$ is a *stationary distribution* if $P\nu_T = \nu_T$. When the initial distribution $\nu_0$ is given by a *Dirac measure* $\delta_x$, we define that the Markov chain *starts* at the point $x$. We choose a starting position $x_0$ in the space and compute the probability distribution after $n$ steps of diffusion. In the discrete phylogenetic tree space $\mathcal{S}_n^*$, a possible initial distribution is the Dirac measure from the star tree which is fully degenerate.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph where there is at most one edge between two connected vertices. We denote an edge by $x \sim y$ if two vertices $x, y \in \mathcal{V}$ are connected by an edge. The probabilities of moving from a vertex $x$ to its adjacent vertices are all equal. The corresponding transition matrix $P$ for the simple random walk is given by

$$p(x,y) = \begin{cases} \frac{1}{\deg x} \text{ if } x \sim y \\ 0 \text{ otherwise} \end{cases} .$$

Let $P_{\mathcal{G}}$ be a $|\mathcal{V}|$ by $|\mathcal{V}|$ matrix with entries given by $p(x,y)$ as above. We verify that $\sum_{y\in\mathcal{V}} p(x,y) = 1$ for all $x \in \mathcal{V}$. Hence $P_{\mathcal{G}}$ is a transition matrix.

**Definition 3.22.** *A* simple random walk *on the vertices $\mathcal{V}$ of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a particular case of Markov chain (Definition 3.21)with state space $\mathcal{V} \in \ell^1(\mathcal{V})$ and the transition matrix $P_{\mathcal{G}}$.*

**Remark 3.23.** *Multiple edges and self loops are allowed in the graph that defines the simple random walk.*

Given the discrete tree space $\mathcal{S}_n^*$, we may define the graph as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where the vertices $\mathcal{V}$ correspond to the trees in $\mathcal{S}_n^*$ and the edges correspond to the adjacencies, e.g. the nearest neighbour interchange. We can also define a subgraph where the vertices are given by a subset of $\mathcal{S}_n^*$. The aim is to find the eigensolutions of the random walk on the graph $\mathcal{G}$ given as the eigenvectors and eigenvalues.

Let $X' = \bigcup_{j=0}^{N} X_j$ be a partition of the finite set $X$ by the disjoint subsets $\{X_j\}_{j=0}^{N}$. Let $\{\mathbf{e}_x\}_{x\in X}$ be the natural basis of $\ell^1(X)$. Given a probability distribution $v = \sum_{x\in X} v(x)\, \mathbf{e}_x \in \ell^1(X)$, we obtain the corresponding probability distribution on the partition $X'$ as the vector $v' = \sum_{j=0}^{N} v'\left(X_j\right)\mathbf{e}_j \in \ell^1\left(X'\right)$ where $v'\left(X_j\right) = \sum_{x\in X_j} v\left(x\right)$. Let $P_X$ be the transition matrix of a random walk on $X$ and the sequence $v_0, v_1, \ldots$ be the Markov chain with initial distribution $v_0$.

**Definition 3.24.** *The random walk on $X$ given by the transition matrix $P_X$ is* lumpable *with respect to the partition $X'$ if for any initial distribution $v_0 \in \ell^1\left(X\right)$, the sequence of the corresponding probability distributions on the partition $X'$, $v_0', v_1', \ldots$ is also a Markov chain whose transition matrix does not depend on the choice of $v_0$.*

There exists a transition matrix $P_X'$ for the partition $X'$ from the lumpable random walk given by the transition matrix $P_X = \left(p\left(x,y\right)\right)_{x,y\in X}$. By Proposition 1.10.2 in [15], for all $x \in X_j$, $y \in X_k$ and $X_j, X_k \in X'$, we require the transition matrix $P_X$ to satisfy $\sum_{y\in X_k} p\left(x,y\right)$ is a constant $p\left(X_j, X_k\right)$; i.e., the probability of moving from subset $X_j$ to the subset $X_k$ does not depend on the choice of $x \in X_j$. Therefore the transition matrix $P_X'$ is an $(N+1)$ by $(N+1)$ square matrix given by

$$P_X' = \Big(p\left(X_j, X_k\right)\Big)_{X_j, X_k \in X'}.$$

Let $\mathcal{G} = (X, \mathcal{E})$ be a finite, connected graph without self loops. Let $d(x,y)$ be the number of edges in the shortest path joining the two vertices $x, y \in X$. We verify that $d(x,y)$ is always a nonnegative integer satisfying

1. $d(x,x) = 0$ for all $x \in X$,
2. $d(x,y) = d(y,x)$ for all $x, y \in X$,
3. $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$ where equality holds if the vertex $z$ is in the shortest path joining $x$ and $y$.

Therefore $(X, d)$ is a metric space and there exists a smallest $N \in \mathbb{Z}_+$ such that $d(x,y) \in \{0, 1, \ldots, N\}$ for all $x, y \in X$. We denote $\operatorname{diam}(X) = N = \max\left\{d(x,y) : x, y \in X\right\}$ by the *diameter* of the graph. Note that a vertex $y$ is a *neighbour* of a vertex $x$ if $d(x,y) = 1$ and the degree of a vertex $\deg(x)$ is the number of the neighbours of $x$. A graph is *regular* if the degrees on all vertices are equal.

**Definition 3.25.** *Let $\Delta_0, \Delta_1, \ldots, \Delta_N : \ell^1\left(X\right) \longrightarrow \ell^1\left(X\right)$ be the linear operators on the vector space $\ell^1\left(X\right)$ given by*

$$\left(\Delta_j f\right)(x) = \sum_{y\in X, d(x,y)=j} f(y).$$

For $j \leq -1$ or $j \geq N+1$, we define the other linear operators $\Delta_j : \ell^1\left(X\right) \longrightarrow \ell^1\left(X\right)$ to be the zero operators.

Consider the simple random walk on the regular graph $\mathcal{G} = (X, \mathcal{E})$ where the probability of moving to one neighbours is $\frac{1}{\deg(x)}$. The operator $\Delta_0$ is the identity operator. The multiplication between the transition matrix $P_X$ and a probability distribution column vector $v \in \ell^1(X)$ can be obtained from the normalized operator $\Delta_1$ as

$$P_X v = \frac{1}{\deg(x)} \Delta_1 v.$$

The operator $\Delta_1$ is called the *Laplace operator* of the graph and normalized operator $\frac{1}{\deg(x)}\Delta_1$ is called the *random walk operator*.

Let $\mathcal{G} = (X, \mathcal{E})$ be a *distance-regular* graph; that is, there exists a set of constants $\{a_0, b_0, c_0, a_1, b_1, c_1, \ldots, a_N, b_N, c_N\}$ for all $j = 0, 1, \ldots, N$, one has

$$\Delta_j \Delta_1 = a_j \Delta_{j-1} + b_j \Delta_j + a_j \Delta_{j+1},$$

which implies $\Delta_j = p_j(\Delta_1)$ is a polynomial of $\Delta_1$ with degree $j$. Therefore all distance operators $\Delta_0, \Delta_1, \ldots, \Delta_N$ can be generated from the single generator $\Delta_1$. The algebra of functions on the vertices of the finite graph $\mathcal{G}$ with operators $\Delta_0, \Delta_1, \ldots, \Delta_N$ is also called the *Bose-Mesner algebra* associated with the set of vertices $X$ [41], [18], [39].

Let $X_j = \{x \in X : d(x, x_0) = j\}$ be the subset of $X$ where all vertices have distance $j$ away from the vertex $x_0$. Then $X' = \bigcup_{j=0}^{N} X_j$ is a partition of the finite set $X$. Let $\{\mathbf{e}_x\}_{x \in X}$ be the natural basis of $\ell^1(X)$. We have $\Delta_j(\mathbf{e}_{x_0}) = p_j(\Delta_1)(\mathbf{e}_{x_0}) = \sum_{x \in X_j} \mathbf{e}_x$ for all linear operators $\Delta_0, \Delta_1, \ldots, \Delta_N$ as defined in Definition 3.25.

Consider the simple random walk on the distance-regular graph $\mathcal{G} = (X, \mathcal{E})$ which is lumpable with respect to the partition $X'$. Let the probability distribution on $X$, $v \in \ell^1(X)$ be invariant on the vertices which belong to the same subset $X_j$. The basis of the probability distribution which are constant on the subsets $\{X_j\}_{j=0}^{N}$ on $X$ are given by the vectors $\mathbf{e}_{X_j} = \sum_{x \in X_j} \mathbf{e}_x \in \ell^1(X)$ for all $j = 0, 1, \ldots, N$. The basis $\left\{\mathbf{e}_{X_j}\right\}_{j=0}^{N}$ correspond to the natural basis $\{\mathbf{e}_j\}_{j=0}^{N}$ the vector space $\ell^1(X')$. The random walk on the partition $X'$ from the lumpable random walk on $X$ is also generated by a single random walk operator which corresponds to the Laplace operator $\Delta_1$ for $\ell^1(X)$ [15].

Let $G$ be a finite group acting on the metric space $(X, d)$ transitively and 2-point-homogeneously where $d(x, y) \in \{0, 1, \ldots, N\}$ for all $x, y \in X$. Let $K$ be the stabilizer of $x_0 \in X$ where the homogeneous space $X$ is given by $G/K$. Then $(G, K)$ is a symmetric Gelfand pair. We choose $X'$ to be the partition of the double cosets $KgK$ where every double coset is a union of right cosets which correspond to a subset of $X$. Let $X' = \bigcup_{j=0}^{N} X_j$ be the partition of the orbits determined by the distance $j$ from the element $x_0$. The bi-$K$-invariant subalgebra $\ell^1(K \backslash G / K)$ has a basis $\left\{Y_{Kg_jK}\right\}_{j=0}^{N}$ given by the characteristic functions of the double cosets.

Define a graph with the set of vertices $X$ with no self loops. Two vertices $x$ and $y$ are connected by an edge if there exists $g \in Kg_1K$ such that $gx = y$. The characteristic function $Y_{Kg_0K}$ is the identity in the convolution multiplication in $\ell^1\left(K\backslash G/K\right)$ and can be considered as the zero-th power of any other characteristic functions.

**Definition 3.26.** *The bi-$K$-invariant subalgebra $\ell^1\left(K\backslash G/K\right)$ is singly generated if there exists a function $\mathfrak{f} \in \ell^1\left(K\backslash G/K\right)$ such that all functions in $\ell^1\left(K\backslash G/K\right)$ can be uniquely written as a polynomial of $\mathfrak{f}$.*

**Remark 3.27.** *If the bi-$K$-invariant subalgebra $\ell^1\left(K\backslash G/K\right)$ is singly generated by the characteristic function $Y_{Kg_1K}$, then the characteristic functions of other double cosets can be written as polynomial of $Y_{Kg_1K}$.*

Assume that the bi-$K$-invariant subalgebra $\ell^1\left(K\backslash G/K\right)$ is singly generated by $Y_{Kg_1K}$. The convolution multiplications between the characteristic functions are given by

$$Y_{Kg_1K} * Y_{Kg_jK} = \sum_{l=0}^{N} m_{j,l} Y_{Kg_lK}.$$

Let $P'_X = \left(p(j,l)\right)_{j,l\in\{0,1,\ldots,N\}}$ be a square matrix labeled by the double cosets of the Gelfand pair $(G,K)$ where

$$p(j,l) = \frac{|Kg_lK|}{\left|Kg_jK\right| |Kg_1K|} m_{j,l}.$$

We verify that $0 \leq p(j,l) \leq 1$ for all $j,l \in \{0,1,\ldots,N\}$ and $\sum_{l=0}^{N} p(j,l) = 1$ for all $j = 0,1,\ldots,N$. Therefore $P'_X$ is a transition matrix thus define a random walk on the double cosets. Note that the scaled operator $\frac{Y_{Kg_1K}}{|Kg_1K|}$ can be considered as the corresponding random walk operator for the vector space $\ell^1\left(X'\right)$ with scaled characteristic functions as the basis, given as

$$\frac{Y_{Kg_1K}}{|Kg_1K|} * \frac{Y_{Kg_jK}}{\left|Kg_jK\right|} = \sum_{l=0}^{N} p(j,l) \frac{Y_{Kg_lK}}{|Kg_lK|}.$$

**Lemma 3.28.** *The eigenvectors of the transition matrix $P'_X$ are given by the spherical functions of the bi-$K$-invariant subalgebra $\ell^1\left(K\backslash G/K\right)$.*

*Proof.* Let $X'$ be the partition of the homogeneous space $X = G/K$ given by the corresponding double cosets. Let $\left\{\mathbf{e}_j\right\}_{j=0}^{N}$ be the natural basis of $\ell^1\left(X'\right) \simeq \mathbb{C}^{N+1}$. For $f = \sum_{j=0}^{N} f\left(g_j\right) Y_{Kg_jK} \in \ell^1\left(K\backslash G/K\right)$, we define the column vector $v \in \ell^1\left(X'\right)$ as

$$v_f = \sum_{j=0}^{N} \frac{1}{\left|Y_{Kg_jK}\right|} f\left(g_j\right) \mathbf{e}_j.$$

Note that $\left\{\mathbf{e}_j\right\}_{j=0}^{N}$ can be considered as another basis of $\ell^1\left(K\backslash G/K\right)$ as $\mathbf{e}_j$ is given by a scalar multiple of the characteristic function $Y_{Kg_jK}$ for $\ell^1(X)$. This implies the one-to-one correspondence between vectors $v_f$ and the functions $f \in \ell^1\left(K\backslash G/K\right)$.

Let $\phi$ be a spherical function of $\ell^1\left(K\backslash G/K\right)$. By Definition 3.17, there exists $\lambda_\phi \in \mathbb{C}$ such that

$$\frac{1}{|Kg_1K|}Y_{Kg_jK} * \phi = \lambda_\phi \phi.$$

Therefore the vector $v_\phi$ is an eigenvector of the transition matrix $P'_X$ with eigenvalue $\lambda_\phi$, i.e. $P'_X v_\phi = \lambda_\phi v_\phi$.

Conversely, if $v_\phi$ is an eigenvector of the transition matrix $P'_X$ with eigenvalue $\lambda_\phi$, then $\phi$ is a spherical function of $\ell^1\left(K\backslash G/K\right)$. Every function $f \in \ell^1\left(K\backslash G/K\right)$ can be written as a polynomial of the characteristic function $Y_{Kg_1K}$ as $f = p_f\left(Y_{Kg_1K}\right)$. Therefore there exists $\lambda_f \in \mathbb{C}$ such that $p_f\left(Y_{Kg_1K}\right) * \phi = \lambda_f \phi$. □

**Remark 3.29.** *The random walk on $X'$ given by the transition matrix $P'_X$ is the corresponding lumped simple random walk on the graph $\mathcal{G} = (X, \mathcal{E})$ where $x$ and $y$ are joined by an edge if there exists $g \in Kg_1K$ such that $y = gx$.*

Let $v \in \ell^1(X)$ be an eigenvector of the transition matrix for the simple random walk on $\mathcal{G} = (X, \mathcal{E})$. The vector $v^k$ given by the group action $k \in K$ on the vertices $X$ is still an eigenvector if the graph and orbits of vertices given by the double cosets are invariant under $K$. Therefore the average of all eigenvectors under the group action $K$, $\frac{1}{|K|}\sum_{k\in K} v^k$, is also an eigenvector with the same eigenvalue.

The eigenvectors of the random walk on $\mathcal{G} = (X, \mathcal{E})$ which are invariant on the vertices inside the same orbit given by a double coset, correspond to the eigenvectors of the lumped random walk with transition matrix $P'_X$. Therefore by solving the random walk on $X'$, we effectively obtain all eigenvalues and $K$-invariant eigenvectors.

### 3.3.2 Random walks on graphs of phylogenetic tree spaces

In this section, we consider examples of random walks on sets which are related to discrete tree spaces. We show that the simple random walk on the permutohedra and the associahedra are lumpable. We also present the explicit eigensolutions for the lumpable simple random walk on the vertices of the Petersen graph as the eigensolutions corresponds to the random walk on the trees in $\mathcal{S}_5^*$ under the nearest neighbour interchange. The eigenvectors of the transition matrix of the lumped random walk can be interpreted as the spherical functions of the bi-invariant subalgebra defined from the Gelfand pair $(S_5, S_2 \times S_3)$.

**Example 3.30.** *We consider the simple random walk on the permutohedron $P_{n-1}$.*

As introduced in Section 2.3.1, we label the vertices of the standard permutohedron $P_{n-1}$ by the *inverse permutation* of the coordinates of the $n$ numbers in the coordinates in the Euclidean space $\mathbb{R}^n$.

The vertices with the *inverse permutations* correspond to the sequence of the planar embedding of the leaf labels on the binary trees. The 1-skeleton of the permutohedron $P_{n-1}$ can be considered as a regular graph $\mathcal{G}_{P_n}$ which does not contain any multiple edges or self loops. The edges of $\mathcal{G}_{P_n}$ indicate the adjacencies between the sequences of the leaf labels. Two vertices are connected by an edge if their corresponding sequences of the leaf labels differ by a swap of two adjacent labels in the sequence of $n$ elements. Hence we can see that every vertex of $\mathcal{G}_{P_n}$ has degree $(n-1)$.

As defined in Definition 2.21, given a half-space $H_{P_n}(E)$, we can identify the normal vector $\underline{1}_E$ that defines the hyperplane of the half-space. A vertex with coordinates $\underline{v} \in \mathbb{R}^n$ is on the hyperplane if it satisfies

$$\langle \underline{v}, \underline{1}_E \rangle = \sum_{j \in E} v_j = \frac{|E|\left(|E|+1\right)}{2}.$$

Since all entries in the coordinates of vertices on $P_{n-1}$ are integers, the inner product $\langle \underline{v}, \underline{1}_E \rangle = \sum_{j \in E} v_j$ always gives an integer for all vertices $\underline{v}$ of $P_{n-1}$. We can verify that

$$\frac{|E|\left(|E|+1\right)}{2} \leq \langle \underline{v}, \underline{1}_E \rangle \leq \frac{|E|\left(|E|+1\right)}{2} + |E|\left(n-|E|\right),$$

which shows that the values of $\langle \underline{v}, \underline{1}_E \rangle$ is the set of integers

$$\mathcal{L}_E = \left\{ \frac{|E|\left(|E|+1\right)}{2}, \frac{|E|\left(|E|+1\right)}{2}+1, \ldots, \frac{|E|\left(|E|+1\right)}{2} + |E|\left(n-|E|\right) \right\}.$$

Therefore every vertex $\underline{v} \in P_{n-1}$ can be identified in a set

$$\mathcal{V}_{E,l} = \left\{ \langle \underline{v}, \underline{1}_E \rangle = l, l \in \mathcal{L}_E \right\}.$$

Given a vertex with coordinates $\underline{v}$, the $(n-1)$ neighbours of $\underline{v} \in \mathcal{V}_{E,l}$ can only be in one of the three subsets: $\mathcal{V}_{E,l-1}$, $\mathcal{V}_{E,l}$ and $\mathcal{V}_{E,l+1}$ as the adjacencies are generated from a single swap of two adjacent integers. The numbers of neighbours of $\underline{v}$ in all three subsets are also fixed. Hence by Definition 3.24, the simple random walk on $\mathcal{G}_{P_n}$ is lumpable with respect to the partition $\bigcup_{l \in \mathcal{L}_E} \mathcal{V}_{E,l}$ for all subset $E \subsetneq \{1, 2, \ldots, n\}$.

**Example 3.31.** *We consider the simple random walk on the associahedron $K_{n-1}$.*

As introduced in Section 2.3.2, given a permutation $\underline{v} \in \mathbb{R}^n$ of $n$ leaf labels, we obtain a subset of trees in the discrete space $\mathcal{S}_n$ and $\mathcal{S}_n^*$. The vertices of the associahedron $K_{n-1}$

correspond to a subset of $\mathcal{S}_n$ which is a set of fully resolved trees with a fixed permutation of leaf labels in their corresponding Newick string formats.

The adjacencies between the vertices of $K_{n-1}$ are given by one of the two neighbours from the nearest neighbour interchange process which preserve the cyclic order of the leaf labels. Therefore we can define the simple random walk on $K_{n-1}$ which can be considered as the simple random walk on the graph $\mathcal{G}_{K_{n-1}}$ obtained from the 1-skeleton of $K_{n-1}$ with the set of vertices $\{\mathbf{V}_\Delta : \Delta \text{ is a triangulation of the regular } n\text{-gon}\}$. The dihedral group $D_{2n}$ acts on the coordinates of the vertices from the secondary polytope realization of the regular $n$-gon defined in Section 2.3.2.

Let $\mathcal{V}_{\underline{\Delta}}$ be a subset of vertices of $K_{n-1}$ constructed from the secondary polytope realization. A vertex $\mathbf{V}'_\Delta$ is in the subset $\mathcal{V}_{\underline{\Delta}}$ if the corresponding triangulated regular $n$-gon can be obtained from the triangulation $\Delta$ through the $D_{2n}$ action on the labeled coordinates in $\mathbb{R}^n$.

Since the number of triangulations is finite, we obtain a partition $\bigcup \mathcal{V}_{\underline{\Delta}}$ which is a union of disjoint subsets which constitute the set of all vertices on $K_{n-1}$. Every subset $\mathcal{V}_{\underline{\Delta}}$ is invariant under the $D_{2n}$ action thus the simple random walk on $\mathcal{G}_{K_{n-1}}$ is lumpable with respect to the partition $\bigcup \mathcal{V}_{\underline{\Delta}}$.

**Example 3.32.** *We consider the simple random walk on the vertices of the Petersen graph.*

Let the vertices of the Petersen graph be labeled as below.

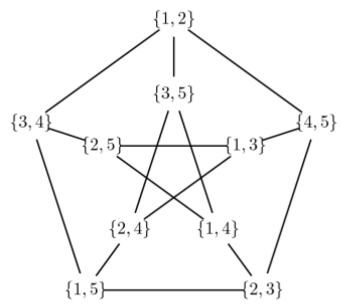


Figure 3.1: The Petersen graph

The 10 vertices correspond to the set of 2-subsets $X$ or the 2-3-splits of the set $\{1, 2, 3, 4, 5\}$. Two vertices are connected by an edge if their corresponding subsets are disjoint; i.e., the two corresponding 2-3-splits for the leaf labels of the discrete tree space $\mathcal{S}_5^*$ are compatible.

The transition matrix $P_X = \Big(p\left(\{i_1, i_2\}, \{j_1, j_2\}\right)\Big)$ is a 10 by 10 square matrix labeled by the 10 subsets. The nonzero entries are given by $p\left(\{i_1, i_2\}, \{j_1, j_2\}\right) = \frac{1}{3}$ if $\{i_1, i_2\}$ and $\{j_1, j_2\}$ are disjoint.

We define three subsets of the set of 10 vertices as follows:

$$
\begin{aligned}
X_0 &= \{\{1,2\}\} \\
X_1 &= \{\{3,4\},\{3,5\},\{4,5\}\}, \\
X_2 &= \{\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\}\}.
\end{aligned}
$$

The partition $X'$ is given by $X = X_0 \cup X_1 \cup X_2$. We can see that the simple random walk on the Petersen graph is lumpable with respect to the partition $X'$ [49]. We obtain the transition matrix $P'_X$ for the lumped random walk on $X'$ as

$$
P'_X = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ 1 & 0 & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix}.
$$

As the partition $X'$ is singly generated by the random walk operator, we can obtain the eigensolutions by solving the equations of the eigenvalue $\lambda$ and express the entries of the corresponding eigenvector as a polynomial of $\lambda$. All eigenvectors of $P'_X$ are of the form $v = \left(1, 3\lambda, 9\lambda^2 - 3\right)$ with eigenvalue $\lambda$. The three eigenvectors of $P'_X$ are $v_0 = (1,3,6)$ with eigenvalue 1, $v_1 = (1,-2,1)$ with eigenvalue $-\frac{2}{3}$ and $v_2 = (1,1,-2)$ with eigenvalue $\frac{1}{3}$.

We also note that the symmetric group $G = S_5$ acts transitively and 2-point-homogeneously on the set of 2-subsets $X$ and the stabilizer of the subset $\{1,2\}$ is isomorphic to the subgroup $K = S_2 \times S_3$. The pair $(G,K)$ is a symmetric Gelfand pair and the partition $X'$ correspond to the partition $G = Kg_0K \cup Kg_1K \cup Kg_2K$ as a union of the three disjoint double cosets. Therefore by Lemma 3.28, we can obtain the spherical functions of the bi-$K$-invariant subalgebra $\ell^1\left(K\backslash G/K\right)$ by normalizing the eigenvectors of the transition matrix $P'_X$ for the lumped random walk on the Petersen graph.

**Example 3.33.** *We consider the simple random walk on the space $X$ of the $m$-subsets of the set $\{1,\dots,n\}$.*

For $n - m \geq 2$, we construct a graph $\mathcal{G}_{n,m}$ where the vertices correspond to the $m$-subsets or $m,(n-m)$-splits of the set $\{1,\dots,n\}$. Two vertices are connected by an edge if their corresponding $m$-subsets have exactly $(m-1)$ elements in common.

The symmetric group $G = S_n$ acts transitively and 2-point-homogeneously on the $m$-subsets $\{1,\dots,n\}$. The stabilizer of $\{1,\dots,m\}$ is isomorphic to the subgroup $K = S_m \times S_{n-m}$. The pair $(G,K)$ is a symmetric Gelfand pair [15] and we can define the metric space $(X,d)$ as in Remark 3.10.

The random walk on $\mathcal{G}_{n,m}$ is lumpable with respect to the partition $X'$ which corresponds to the double cosets from the Gelfand pair $(G,K)$. The bi-$K$-invariant subalgebra $\ell^1\left(K\backslash G/K\right)$ is also singly generated thus we can obtain the spherical functions from the eigenvectors of the lumped random walk.

Note that the $m,(n-m)$-splits of the set $\{1,\ldots,n\}$ correspond to the split of leaf labels in the discrete tree space $\mathcal{S}_n^*$. We may apply the eigensolutions of the lumpable random walk on $\mathcal{G}_{n,m}$ to the random walk on a subset of $\mathcal{S}_n^*$ which only consists of degenerate trees given by a single internal edge that indicates the $m,(n-m)$-splits of the leaf labels.

**Example 3.34.** *We consider the random walk on the tree shapes of the space of fully resolved trees $\mathcal{S}_n$.*

We consider the simple random walk on the discrete tree space $\mathcal{S}_n$ with the nearest neighbour interchange as the adjacencies. The transition matrix is $(2n-5)!!$-dimensional but does not have the symmetries from the symmetric Gelfand pair $(S_{2n}, S_n \wr S_2)$. The symmetric group $S_{2n}$ does not act transitively and 2-point homogeneously on the graph of $\mathcal{S}_n$ where the adjacencies are obtained from the nearest neighbour interchange.

We can still define a partition $X'$ of the space $\mathcal{S}_n$. The symmetric group $S_n$ acts on the leaf labels, thus acts on all splits of the fully resolved trees. Two fully resolved trees $T_1, T_2 \in \mathcal{S}_n$ are of the same *tree-shape* if they can be obtained from each other by permuting the leaf labels in all $(n-3)$ splits with a group element $g \in S_n$. The partition $X'$ is then given by the disjoint union of trees with different tree-shapes.

The simple random walk on $\mathcal{S}_n$ with the nearest neighbour interchange is lumpable with respect to the partition $X'$ of the tree-shapes. The eigensolutions of the lumped random walk may help us obtain an approximate distribution of different tree shapes for small $n$, but the number of tree-shapes grows fast for $n > 10$ [55].

# Part II

# Infinite trees, buildings and the cohomology groups of their Banach algebras

# Chapter 4

# Algebras of automorphism groups of homogeneous trees

In this chapter, we consider some algebras of automorphism groups of homogeneous trees and buildings and the *Iwahori-Hecke algebras* of the corresponding infinite Gelfand pairs [11]. These particular examples of Banach algebras are motivated by those which arise from harmonic analysis on structures related to phylogenetic trees. The method is to apply the properties of Gelfand pairs and spherical functions to solve some specific random walk problems and to check the amenability and other homological properties of those Banach algebra examples.

We start with a group acting on the vertices of the infinite homogeneous tree $\mathbb{T}_q$ and the subgroup which stabilizes a fixed point. We show that these two locally compact groups form a Gelfand pair. We define the algebra of summable functions on the vertices $\mathcal{V}$ of $\mathbb{T}_q$ whose values are determined by the distance from a fixed vertex $x_0$. The Laplace operator which is given by the average adjacency relations, corresponds to the diffusion or the simple random walk to the adjacent vertices [14].

By fixing the Haar measure to normalize the size of the subgroup, the algebra of integrable bi-invariant functions on the locally compact group, is discretized to be isomorphic to the Hecke algebra $A_q$ on $\mathbb{Z}_+$, with $\ell^1$ norm. The Hecke operator can be understood as the single generator of the algebra. The conditions for the eigenfunctions for the Laplace operator can be worked out straightforwardly from the random walk equation, and scaled to be spherical functions. We show that the bounded spherical functions indeed give the values of bounded characters for the isomorphic Hecke algebra $A_q$.

We use a shift matrix to compute the characters on $A_q$. The character space is parametrized by an ellipse arising from a symmetrized bi-disc, generated by the set of unordered pairs of points with fixed product on an annulus on the complex plane. We will prove the existence of point derivations and bounded approximate identities given by the interior and the boundary of the character space.

## 4.1 The groups acting on the infinite homogeneous tree $\mathbb{T}_q$

A tree is a non-directed connected graph without any closed cycles. An edge is an unordered pair of connected vertices $(x_i, x_j)$. The vertices $\mathcal{V}$ in the infinite homogeneous tree $\mathbb{T}_q$ all have degree $q+1$ and there exists a unique path between any pair of vertices $(x, y)$.

A sequence of vertices $X_n = [x_0, x_1, \ldots, x_n]$ is a chain if every adjacent pair of vertices $(x_i, x_{i+1})$ is connected as an edge and $x_i \neq x_j$ for all $0 \leq i < j \leq n$. The length of the chain is $d(x_0, x_n) = d(x_n, x_0) = n$. This sequence of vertices can be extended to be doubly infinite as $[\ldots, x_{-N}, \ldots, x_0, \ldots, x_N, \ldots]$.

A map $g : \mathcal{V} \longrightarrow \mathcal{V}$ is an *automorphism* if it is bijective, and $(x, y)$ is an edge if and only if $\big(g(x), g(y)\big)$ is an edge. Given a chain which is a sequence of vertices, every adjacent pair of vertices form an edge. Under the automorphism, an edge $(x, y)$ is mapped to another edge $(x', y')$. If $X_n$ is a chain then $gX_n$ is also a chain. Therefore the composition of two automorphisms maps one edge to another edge and also maps a chain to another chain. If $g$ is an automorphism, then $g^{-1}$ is also an automorphism. The identity map is an automorphism. Hence the set of automorphisms of the infinite homogeneous tree is a group under composition.

We define $G$ to be the group of automorphisms of $\mathbb{T}_q$. Hence $G$ preserves all chain structures on $\mathbb{T}_q$. The infinite tree $\mathbb{T}_q$ is *locally finite* and *homogeneous* as all vertices have the same degree $q+1$ with $q < \infty$, as the common cardinality; i.e., the graph is $(q+1)$-regular.

When $q = 1$, the vertices of the tree $\mathbb{T}_2$ are isomorphic to the integers $\mathbb{Z}$ and there exists only one doubly infinite chain. In this case, the group algebra is easier to compute and we will discuss this example in a later section.

The group of automorphisms $\mathrm{Aut}\big(\mathbb{T}_q\big)$ of the infinite homogeneous tree can be turned into a topological group [22], [48]. For any $W \subseteq \mathrm{Aut}\big(\mathbb{T}_q\big)$, $W$ is open in the compact-open topology. We also use the fact that $\mathbb{T}_q$ is discrete and $\mathrm{Aut}\big(\mathbb{T}_q\big)$ is a Hausdorff topological group.

In general, we assume that $q \geq 2$. We choose a subgroup $K$ of the locally compact group $G$ where $K$ is the stabilizer of a given vertex $x_0$. Note that $G$ is unimodular and $K$ is compact [51]. When $q$ is a prime number, we will show that $(G, K)$ is a Gelfand pair and the pair of $p$-adic projective general linear groups $\Big(PGL_2\left(\mathbb{Q}_p\right), PGL_2\left(\mathbb{Z}_p\right)\Big)$ is also a Gelfand pair. The two bi-invariant subalgebras under the $\ell^1$-norm from the two Gelfand pairs are isomorphic to each other.

For any $j \in \mathbb{Z}_+$, we define a set of vertices

$$\mathcal{V}_j = \left\{ x \in \mathcal{V} \middle| d(x, x_0) = j \right\}.$$

For every $\mathcal{V}_j$, we define a subset of $G$ as

$$\Omega_j = K g_j K = \left\{ g \in G : g x_0 \in \mathcal{V}_j \right\}.$$

We fix a family $\left(g_j\right)_{j=0}^{\infty}$ such that $g_j \in \Omega_j$. This ensures that $d(g_j(x_0), x_0) = j$ and $\Omega_j = K g_j K$, which shows that the group $G$ is an infinite disjoint union of the double cosets. The size of every orbit can be computed by enumerating the number of vertices with the same distance away from $x_0$, explicitly as

$$|\mathcal{V}_0| = \left|\{x_0\}\right| = 1 \text{ and } \left|\mathcal{V}_j\right| = (q+1)q^{j-1} \text{ for } q \geq 2.$$

We will show that $(G, K)$ is a Gelfand pair and compute the random walks on the vertices of $\mathbb{T}_q$ in the following sections. We define the group algebra and the bi-$K$-invariant subalgebra with $L^1$ norm by fixing a Haar measure. The commutativity from the property of the Gelfand pair allows us to find the isomorphic Iwahori-Hecke algebra in a discretized version.

## 4.2 The algebra of functions on the vertices on $\mathbb{T}_q$

The purpose of this section is to study the algebra of integrable bi-$K$-invariant functions in $L^1(K\backslash G/K)$. We present all bounded spherical functions of the Gelfand pair $(G, K)$. We set up an isomorphism from the algebra of bi-$K$-invariant functions in $L^1(K\backslash G/K)$ to a Banach algebra on the discrete space $\mathbb{Z}_+$ and find the space of characters and describe the topology. It is then natural to determine the existence of b.a.i. and point derivations for the character space.

**Definition 4.1.** *Let $G$ be a locally compact, unimodular group and let $K$ be a compact subgroup. An integrable function $f \in L^1(G)$ is said to be bi-$K$-invariant if $f(k_1 g k_2) = f(g)$ for all $k_1, k_2 \in K$ and $g \in G$, from Chapter II, Section 4 in [22] .*

We write $L^1\left(K\backslash G/K\right)$ for the space of all bi-$K$-invariant integrable functions on $G$. We say that $(G, K)$ is a Gelfand pair if $f_1 * f_2 = f_2 * f_1$ for all $f_1, f_2 \in L^1\left(K\backslash G/K\right)$.

**Lemma 4.2.** *$L^1\left(K\backslash G/K\right)$ is closed with respect to convolution. That is, if $f_1, f_2 \in L^1\left(K\backslash G/K\right)$ then $f_1 * f_2 \in L^1\left(K\backslash G/K\right)$.*

*Proof.* For all $g' \in KgK$, there exists $k_1, k_2 \in K$ such that $g' = k_1 g k_2$.

$$\begin{aligned}
[f_1 * f_2]\left(g'\right) &= \int_{h \in G} f_1(g'h) f_2(h^{-1}) dh = \int_{h \in G} f_1(k_1 g k_2 h) f_2(h^{-1}) dh \\
&= \int_{h \in G} f_1(g k_2 h) f_2(h^{-1}) dh = \int_{h \in G} f_1(g k_2 h) f_2(h^{-1} k_2^{-1}) dh \\
\text{setting } h' = k_2 h & \\
&= \int_{h' \in G} f_1(gh') f_2(h'^{-1}) dh' = [f_1 * f_2](g) .
\end{aligned}$$

□

**Proposition 4.3.** *For all $g \in \Omega_j$, we have $g^{-1} \in \Omega_j$.*

*Proof.* Given any $g \in Aut\left(\mathbb{T}_q\right)$, suppose $g \in \Omega_j$, we have $d(x_0, gx_0) = d(g^{-1}x_0, x_0) = j$. The subgroup $K$ acts transitively on all infinite chains starting at $x_0$ thus $K$ acts on the finite chains with length $d(g^{-1}x_0, x_0)$ and starting at $x_0$. Hence $K$ acts on the all vertices $x$ which have the same distance $n$ from $x_0$, namely $\mathcal{V}_j$. This implies that there exists $k \in K$ such that $(kg^{-1})x_0 = gx_0$ and $h = g^{-1}kg^{-1} \in K$, which shows that $g^{-1} = hgk^{-1}$ and $g^{-1} \in KgK$. □

**Corollary 4.4.** *The pair $(G, K)$ is a Gelfand pair, i.e. $L^1(K\backslash G/K)$ is commutative under convolution.*

*Proof.* By Proposition 4.3, we have $f(g) = f(g^{-1})$ for all $f \in L^1(K\backslash G/K)$. We show that for all $f_1, f_2 \in L^1(K\backslash G/K)$ where $f_1 = f_2$ almost everywhere, the convolution satisfies

$$\begin{aligned}
f_1 * f_2(g) &= \int_{h \in G} f_1(h) f_2\left(h^{-1}g\right) dh \\
&= \int_{h \in G} f_1(h) f_2\left(\left(g^{-1}h\right)^{-1}\right) dh \\
\text{(setting } h = gh_1) & \\
&= \int_{h_1 \in G} f_1(gh_1) f_2\left(h_1^{-1}\right) dh_1 \\
\text{(as } G \text{ is unimodular)} & \\
&= \int_{h_1 \in G} f_1\left(h_1^{-1}g^{-1}\right) f_2\left(h_1\right) dh_1 \\
&= \int_{h_1 \in G} f_2\left(h_1\right) f_1\left(h_1^{-1}g^{-1}\right) dh_1 \\
&= f_2 * f_1\left(g^{-1}\right) .
\end{aligned}$$

By Lemma 4.2, we have $f_2 * f_1(g^{-1}) = f_2 * f_1(g)$. Thus $f_1 * f_2 = f_2 * f_1$ for all $f_1, f_2 \in L^1(K\backslash G/K)$, which shows that the pair $(G, K)$ is a Gelfand pair. □

We will now define a basis for the bi-$K$-invariant functions $L^1(K\backslash G/K)$, the functions on the vertices $\mathcal{V}$ and the functions on $\mathbb{Z}_+$.

The basis of $L^1(K\backslash G/K)$ is given by the characteristic function

$$\delta_{\Omega_j}(g) = \begin{cases} 1 \text{ if } g \in \Omega_j \\ 0 \text{ elsewhere} \end{cases}.$$

For a bi-$K$-invariant function on $G$, it is right invariant i.e. $f(g) = f(h)$ if $h \in gK$. We therefore define the summable functions on the vertices $\mathcal{V}$, $f_{\mathcal{V}} \in \ell^1(\mathcal{V})$ as

$$f_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathbb{C}, \quad f_{\mathcal{V}}(gx_0) = \int_{h \in gK} f(h)dh.$$

We can also define the basis of $\ell^1(\mathcal{V})$ by the characteristic function

$$\delta_{gx_0}(x) = \begin{cases} 1 \text{ if } x = gx_0 \\ 0 \text{ elsewhere} \end{cases}.$$

Since $f$ is bi-$K$-invariant function on $G$, the function on vertices $f_{\mathcal{V}}$ is invariant on every orbit set $\mathcal{V}_j$. Therefore the values of $f_{\mathcal{V}}$ are determined by a *radial function* [29] $\tilde{f}$ on $\mathbb{Z}_+$, which is defined to be

$$\tilde{f} : \mathbb{Z}_+ \longrightarrow \mathbb{C}, \ \tilde{f}(j) = \tilde{f}(d(gx_0, x_0)) = \left|\mathcal{V}_j\right| f_{\mathcal{V}}(x) \text{ for } g \in \Omega_j.$$

The basis for the radial function $\tilde{f} \in \ell^1(\mathbb{Z}_+)$ is given by the characteristic function $Y_j$ where

$$Y_j(k) = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ if } j \neq k \end{cases}.$$

Fix the Haar measure on $G$ by setting the mass of the subgroup $K$ to be 1. By normalizing the characteristic functions, we will show that there exists an isomorphism between $L^1(K\backslash G/K)$ and a Banach algebra on $\mathbb{Z}_+$ with specific multiplication rules $*_q$ given by the properties of $\mathbb{T}_q$.

**Definition 4.5.** *Let $A_q = \ell^1(\mathbb{Z}_+, *_q)$ be a commutative Banach algebra with multiplication $*_q$. Let $\left\{Y_j\right\}_{j=0}^{\infty}$ be the normalized characteristic functions on $\mathbb{Z}_+$. Let $Y_0$ be the identity* *and $Y_1$ be the single generator of $A_q$, given by*

$$\begin{aligned} Y_m *_q Y_0 =& Y_m, \\ Y_m *_q Y_1 =& \frac{q}{q+1} Y_{m+1} + \frac{1}{q+1} Y_{m-1} \quad \text{for} \quad m \geq 1. \end{aligned}$$

For $m \geq n \geq 2$, the multiplication of $*_q$ in the algebra $A_q$ gives

$$Y_m *_q Y_n = \frac{q}{q+1} Y_{m+n} + \frac{1}{(q+1)q^{n-1}} Y_{m-n} + \frac{q-1}{q+1} \sum_{j=1}^{n-1} \frac{Y_{m+n-2j}}{q^j}.$$

**Theorem 4.6.** *The Banach algebras $L^1(K\backslash G/K)$ and $A_q$ are isomorphic. The isomorphism sends $\delta_{\Omega_0}$ to $Y_0$ and $\delta_{\Omega_1}$ to $Y_1$.*

*Proof.* We note that normalized basis $\{Y_j\}$ for $A_q$ is a rescaling of the characteristic functions $\delta_{\Omega_j} \in L^1(K\backslash G/K)$ as

$$Y_j(j) = \frac{1}{|\Omega_j|} \int_{g \in G} \delta_{\Omega_j}(g) dg$$

so that all $Y_j$ have mass 1.

For all $\delta_{\Omega_j} \in L^1(K\backslash G/K)$ and $Y_j \in A_q$, define a linear map

$$\theta : L^1(K\backslash G/K) \longrightarrow A_q, \quad \theta\left(\frac{\delta_{\Omega_j}}{|\Omega_j|}\right) = Y_j.$$

We need to show that for all $m, n \in \mathbb{Z}_+$, $\theta\left(\frac{\delta_{\Omega_m}}{|\Omega_m|} * \frac{\delta_{\Omega_n}}{|\Omega_n|}\right) = \theta\left(\frac{\delta_{\Omega_m}}{|\Omega_m|}\right) *_q \theta\left(\frac{\delta_{\Omega_n}}{|\Omega_n|}\right) = Y_m *_q Y_n$.

When $n = 0$,

$$\begin{aligned} \frac{\delta_{\Omega_m} * \delta_{\Omega_0}(g)}{|\Omega_m||\Omega_0|} &= \frac{1}{|\Omega_m|} \int_{h \in G} \delta_{\Omega_m}(h) \delta_{\Omega_0}(h^{-1}g) dh \\ &= \frac{1}{|\Omega_m|} \int_{h \in G} \delta_{\Omega_m}(hg^{-1}) \delta_{\Omega_0}(h^{-1}) dh \\ &= \frac{1}{|\Omega_m|} \int_{h \in K} \delta_{\Omega_m}(hg^{-1}) dh \\ &= \frac{1}{|\Omega_m|} \delta_{\Omega_m}(g^{-1}) = \frac{1}{|\Omega_m|} \delta_{\Omega_m}(g). \end{aligned}$$

When $m \geq n$ and $n = 1$,

$$
\begin{aligned}
\frac{\delta_{\Omega_m} * \delta_{\Omega_1}(g)}{|\Omega_m||\Omega_1|} &= \frac{1}{|\Omega_m||\Omega_1|} \int_{h \in G} \delta_{\Omega_m}(h)\delta_{\Omega_1}(h^{-1}g)dh \\
&\text{(setting } h_1 = hg) \\
&= \frac{1}{|\Omega_m||\Omega_1|} \int_{h_1 \in G} \delta_{\Omega_m}(h_1 g^{-1})\delta_{\Omega_1}(h_1^{-1})dh_1 \\
&= \frac{1}{|\Omega_m||\Omega_1|} \int_{h_1 \in \Omega_1} \delta_{\Omega_m}(h_1 g^{-1})dh_1 \\
&= \frac{1}{|\Omega_m||\Omega_1|} \left(\delta_{\Omega_{m+1}} + q\delta_{\Omega_{m-1}}\right) \\
&= \frac{1}{(q+1)q^{m-1}(q+1)} \left((q+1)q^m Y'_{m+1} + (q+1)q^{m-1}Y'_{m-1}\right) \\
&= \left(\frac{q}{q+1}Y'_{m+1} + \frac{1}{q+1}Y'_{m-1}\right).
\end{aligned}
$$

When $m \geq n \geq 2$,

$$
\begin{aligned}
\frac{\delta_{\Omega_m} * \delta_{\Omega_n}(g)}{|\Omega_m||\Omega_n|} &= \frac{1}{|\Omega_m||\Omega_n|} \int_{h \in G} \delta_{\Omega_m}(h)\delta_{\Omega_n}(h^{-1}g)dh \\
&\text{(setting } h_1 = hg) \\
&= \frac{1}{|\Omega_m||\Omega_n|} \int_{h_1 \in G} \delta_{\Omega_m}(h_1 g^{-1})\delta_{\Omega_n}(h^{-1})dh_1 \\
&= \frac{1}{|\Omega_m||\Omega_n|} \int_{h_1 \in \Omega_n} \delta_{\Omega_m}(h_1 g^{-1})dh_1 \\
&\text{(enumerating } x \in \mathcal{V}_{m+n-2j} \text{ such that } h_1 g^{-1} \in \Omega_m \text{ and } d(x, g^{-1}x_0) = n) \\
&= \frac{1}{|\Omega_m||\Omega_n|} \left(\delta_{\Omega_{m+n}} + \sum_{j=1}^{n-1}(q-1)q^{j+1}\delta_{\Omega_{m+n-2j}} + q^n\delta_{\Omega_{m-n}}\right) \\
&= \left(\frac{q}{q+1}Y'_{m+n} + \frac{q-1}{q+1}\sum_{j=1}^{n-1}\frac{Y'_{m+n-2j}}{q^j} + \frac{1}{(q+1)q^{n-1}}Y'_{m-n}\right).
\end{aligned}
$$

Since the convolution multiplications in $L^1(K\backslash G/K)$ are commutative, we have shown that $\theta$ is an isomorphism between $L^1(K\backslash G/K)$ and $A_q$, which proves that these two algebras are isomorphic. $\square$

## 4.3 Random walk on $\mathbb{T}_q$, spherical functions and characters

To study the random walk on the vertices of the homogeneous tree $\mathbb{T}_q$, we define a linear operator, the *Laplace operator* on the vector space of bounded complex-valued function on $\mathcal{V}$. For a function $f_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathbb{C}$, we define the value of $(Lf_{\mathcal{V}})(x)$ to be the average of the values of all $f_{\mathcal{V}}(y)$, where $y$ is adjacent to $x$, expressed as

$$(Lf_{\mathcal{V}})(x) = \frac{1}{q+1} \sum_{d(x,y)=1} f_{\mathcal{V}}(y).$$

The Laplace operator can be used as the generator of the unweighted random walk on the vertices through adjacencies.

Under the bi-$K$-invariance condition; i.e., the function on $\mathcal{V}$ is radial, we can also define the corresponding Laplace operator to the radial function, $\tilde{L}\tilde{f} : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ where

$$\begin{aligned}
(\tilde{L}\tilde{f})(j) &= \frac{1}{q+1}\tilde{f}(j-1) + \frac{q}{q+1}\tilde{f}(j+1), \text{ for } j \geq 1 \\
(\tilde{L}\tilde{f})(0) &= \tilde{f}(1),
\end{aligned}$$

which can be seen as a weighted random walk between adjacent orbits of vertices.

A bounded radial function $\tilde{\psi} \in \ell^\infty(\mathbb{Z}_+)$ is an eigenfunction for the Laplace operator $\tilde{L}$ if it satisfies that for all $j \in \mathbb{Z}_+$, there exists $\lambda \in \mathbb{C}$ such that $\left(\tilde{L}\tilde{\psi}\right)(j) = \lambda\tilde{\psi}(j)$, where $\lambda$ is the eigenvalue. In this example, we can work out the values of $\tilde{\psi}$ from the adjacency relations

$$\tilde{\psi}(1) = \lambda\tilde{\psi}(0), \tag{4.1}$$

$$\tilde{\psi}(j+2) = \frac{q+1}{q}\tilde{\psi}(j+1) - \frac{1}{q}\tilde{\psi}(j) \text{ for } j \geq 0, \tag{4.2}$$

where $\tilde{\psi}(0) \neq 0$ and an extra condition on $\lambda$ will be needed for the eigenfunction to be bounded. We will work out the condition of boundedness explicitly by from the growth rate later in this section. From the identity above we can show that $\tilde{\psi}(m) = \sum_{j=0}^{m} a_{m,j}\lambda^j$ by induction. Note that we always have $a_{m,j} = 0$ if $m+j$ is odd. For $j \neq 0$, the coefficients $a_{m,j}$ are given by

$$a_{m,j} = (-1)^{\frac{m-j}{2}} q^{-\frac{m+j}{2}} \left( \binom{\frac{m+j}{2}-1}{j-1} (q+1)^{j-1} q + \binom{\frac{m+j}{2}-1}{j} (q+1)^j \right) \tilde{\psi}(0). \tag{4.3}$$

For $j = 0$, we have

$$a_{m,j} = (-1)^{\frac{m-j}{2}} q^{-\frac{m+j}{2}} \binom{\frac{m+j}{2}-1}{j} (q+1)^j \tilde{\psi}(0). \tag{4.4}$$

In Lemma 4.7 and Lemma 4.9, we explain the conversion between the powers of $Y_1$ and natural basis of $A_q$, given by the set $\{Y_j\}_{j=0}^{\infty}$. The coefficients in the conversion between the two set of basis are mostly used in numerical computations.

**Lemma 4.7.** *For the algebra $A_q$, set $Y_1^0 = Y_0$ The natural basis element $Y_m$ of $A_q$ can be* *expressed as a linear sum of powers of the generator* $Y_1$, $Y_m = \sum_{j=0}^{m} a_{m,j} Y_1^j$. *For* $j \neq 0$, *the coefficients* $a_{m,j}$ *are given by*

$$a_{m,j} = (-1)^{\frac{m-j}{2}} q^{-\frac{m+j}{2}} \left( \binom{\frac{m+j}{2} - 1}{j-1} (q+1)^{j-1} q + \binom{\frac{m+j}{2} - 1}{j} (q+1)^j \right). \tag{4.5}$$

*For* $j = 0$, *we have*

$$a_{m,j} = (-1)^{\frac{m-j}{2}} q^{-\frac{m+j}{2}} \binom{\frac{m+j}{2} - 1}{j} (q+1)^j. \tag{4.6}$$

*Proof.* First we have $a_{0,0} = 1$ and $a_{1,1} = 1$. For $m \geq 2$, the adjacency relation $Y_m Y_1 = \frac{q}{q+1} Y_{m+1} + \frac{1}{q+1} Y_{m-1}$ gives

$$\begin{aligned} a_{m,j} &= \frac{q}{q+1} a_{m-1,j-1} + \frac{1}{q+1} a_{m-1,j+1} \text{ for } j \geq 2, \\ a_{m,0} &= \frac{1}{q+1} a_{m-1,1}, \\ a_{m,1} &= a_{m-1,0} + \frac{1}{q+1} a_{m-1,1}. \end{aligned}$$

The values of $a_{m,j}$ can then be verified by induction. □

**Remark 4.8.** *As for the uniqueness, we know that the leading term* $a_{m,m}$ *is the first non-zero term. Every step by canceling the next term we can determine the other coefficients downwards. Note that the odd terms from the top are all positive and the even terms are negative. Although the sum of these coefficients* $\sum_{j=1}^{m} a_{m,j}$ *is always* 1, *the value of* $\sum_{j=1}^{m} |a_{m,j}|$ *grows exponentially with growth rate* $\frac{q+1}{q}$, *which is not a good control to the norm.*

**Lemma 4.9.** *The power of the generator* $Y_1^n$ *can be expanded as a linear sum of elements in terms of the natural basis* $Y_k$, $Y_1^n = \sum_{k=0}^{\lfloor n/2 \rfloor} c_{n,k} Y_{n-2k}$, *where*

$$c_{n,k} = \sum_{p=0}^{k-1} \left( \binom{n-p-1}{k-p} - \binom{n-p-1}{k-p-1} \right) (q+1)^{p-n+1} q^{n-1-p-k} + (q+1)^{k-n+1} q^{n-2k-1}.$$

*Proof.* Consider the expansion of $Y_1^{n+1}$ into $Y_1^n *_q Y_1$. For $n \geq 2$, the coefficients $c_{n+1,k}$

are given by

$$\begin{aligned}
c_{n+1,0} &= \frac{q}{q+1}c_{n,0}, \\
c_{n+1,k} &= \frac{q}{q+1}c_{n,k} + \frac{1}{q+1}c_{n,k-1} \text{ for } 1 \leq k < \lfloor\frac{n}{2}\rfloor, \\
c_{n+1,n/2} &= c_{n,n/2} + \frac{1}{q+1}c_{n,n/2-1} \text{ for even } n, \\
c_{n+1,(n+1)/2} &= \frac{1}{q+1}c_{n,(n-1)/2} \text{ for odd } n.
\end{aligned}$$

The values of $c_{n,k}$ can then be verified by induction. □

**Remark 4.10.** *Since the coefficients of the expansion are all nonegative, the power of $Y_1$ in terms of the natural basis $Y_k$ also have mass 1. However the expansion of $Y_m$ into powers of $Y_1$ in the previous proposition does not have good control of the norm, despite the sum of the coefficients is always 1.*

**Proposition 4.11.** *Let $(G, K)$ be the Gelfand pair as above and $\psi$ be a bi-$K$-invariant function on $G$. Let $\psi_{\mathcal{V}}$ and $\tilde{\psi}$ be the corresponding functions on $\mathcal{V}$ and the radial function. Then $\psi$ is a spherical function if and only if the bounded radial function $\tilde{\psi}$ is an eigenfunction for the Laplace operator $\tilde{L}$ with $\tilde{\psi}(0) = 1$.*

*Proof.* If $\psi$ is a spherical function, from $\psi(e) = 1$ we know that the corresponding radial function satisfies $\tilde{\psi}(0) = 1$. Let $\tilde{\psi}(1) = \lambda$, i.e. for all $g \in G$ such that $d(x_0, gx_0) = 1$, $\psi(g) = \lambda$.

Given an edge $(x, y)$ such that $gx_0 = x$ and $hx_0 = y$, we have $d(gx_0, hx_0) = 1$. Since $G$ acts transitively on $\mathcal{V}$, we obtain further adjacency relations as

$$d(x_0, g^{-1}hx_0) = d(x_0, kg^{-1}hx_0) = d(x, gkg^{-1}hx_0) = 1,$$

where there exists $(q + 1)$ distinct $k_j \in K$ such that $gk_jg^{-1}hx_0 = y_j$, the $(q + 1)$ distinct vertices which are adjacent to $x$. Note that the $(q + 1)$ vertices $k_jg^{-1}hx_0$ are the $(q + 1)$ neighbours of $x_0$, therefore each one of the set $K_j = \{k_j \in K | gk_jg^{-1}hx_0 = y_j\}$ will have mass $\frac{1}{q+1}$ by the Haar measure 1 of $K$, as the $(q+1)$ disjoint sets of $K_j$ form $K$. Therefore

$$\begin{aligned}
\tilde{\psi}(1)\tilde{\psi}(d(x,x_0)) &= \psi(g^{-1}h)\psi(g) \\
&= \int_{k\in K} \psi\left(gkg^{-1}h\right) dk \\
&= \int_{k\in K} \tilde{\psi}\left(d\left(gkg^{-1}hx_0, x_0\right)\right) dk \\
&= \frac{1}{q+1} \sum_{d(x,y_j)=1} \psi_{\mathcal{V}}(y_j) \\
&= (L\psi_{\mathcal{V}})(x) \\
&= (\tilde{L}\tilde{\psi})(d(x,x_0)),
\end{aligned}$$

which shows that $\tilde{\psi}$ is an eigenfunction for the Laplace operator $\tilde{L}$ with eigenvalue $\tilde{\psi}(1) = \lambda$.

Conversely, we assume that the radial function $\tilde{\psi}$ is an eigenfunction for the Laplace operator $\tilde{L}$ with eigenvalue $\lambda$ and $\tilde{\psi}(0) = 1$. The value of $\tilde{\psi}(1) = \lambda$ is the eigenvalue. We need to show that for all $g, h \in G$, $\int_K \psi(gkh)dk = \psi(g)\psi(h)$, where $\psi(g) = \tilde{\psi}(d(gx_0, x_0))$ for all $g \in G$.

For $d(gx_0, x_0) = m$ and $d(hx_0, x_0) = n$, we have $\psi(g) = \tilde{\psi}(m)$ and $\psi(h) = \tilde{\psi}(n)$. When $g \in K$, it is trivial to show that $\psi(e) = \tilde{\psi}(0) = 1$ and $\int_K \psi(gkh)dk = \psi(h)$. When $d(gx_0, x_0) = 1$, from analysis of the disjoint set in the previous proof, we have

$$\begin{aligned}
\int_K \psi(gkh)dk &= \frac{1}{q+1} \sum_{y|d(y,hx_0)=1} \psi_{\mathcal{V}}(y) \\
&= (L\psi_{\mathcal{V}})(hx_0) = \lambda\psi(h) = \tilde{\psi}(1)\tilde{\psi}(n).
\end{aligned}$$

When $m \geq n \geq 2$, from the weight of the orbits in the multiplications, we expand $\int_K \psi(gkh)dk$ in terms of powers of $\tilde{\psi}(1)$ and prove that the expansion is indeed the product of $\tilde{\psi}(m)$ and $\tilde{\psi}(n)$.

$$\begin{aligned}
\int_K \psi(gkh)dk &= \frac{q}{q+1}\tilde{\psi}(m+n) + \frac{1}{(q+1)q^{n-1}}\tilde{\psi}(m-n) + \frac{q-1}{q+1}\sum_{j=1}^{n-1}\frac{\tilde{\psi}(m+n-2j)}{q^j} \\
&= \frac{q}{q+1}\left(\sum_{k=0}^{m+n} a_{m+n,k}\tilde{\psi}(1)^k\right) + \frac{q}{q+1}\left(\sum_{k=0}^{m-n} a_{m-n,k}\tilde{\psi}(1)^k\right) \\
&\quad + \frac{q-1}{q+1}\left(\sum_{j=1}^{n-1}\frac{1}{q^j}\left(\sum_{k=0}^{m+n-2j} a_{m+n-2j,k}\tilde{\psi}(1)^k\right)\right) \\
&= \left(\sum_{j=0}^{m} a_{m,j}\tilde{\psi}(1)^j\right)\left(\sum_{k=0}^{n} a_{n,k}\tilde{\psi}(1)^k\right) \\
&= \tilde{\psi}(m)\tilde{\psi}(n),
\end{aligned}$$

which shows that $\psi$ is indeed a spherical function. $\square$

By Lemma 4.2 and Lemma 4.3 of Chapter II in [22], we can also define the bounded characters of $A_q$ from the spherical functions in $L^\infty(K\backslash G/K)$. A character $\chi$ of the commutative algebra $A_q$ is a bounded linear function $\chi : A_q \longrightarrow \mathbb{C}$ such that $\chi(Y_m Y_n) = \chi(Y_m)\chi(Y_n)$ for all $Y_m, Y_n \in A_q$. For an eigenfunction $\tilde{\psi}$ where $\tilde{\psi}(0) = 1$ and $\tilde{\psi}(1) = \lambda$, let $\chi_\lambda : A_q \longrightarrow \mathbb{C}$ be a function which satisfies $\chi_\lambda(Y_m) = \tilde{\psi}(m)$ for all $m \in \mathbb{Z}_+$. Then we have $\chi_\lambda(Y_0) = 1$ and $\chi_\lambda(Y_m)\chi_\lambda(Y_n) = \tilde{\psi}(m)\tilde{\psi}(n) = \chi_\lambda(Y_m Y_n)$ for all $m, n \in \mathbb{Z}_+$ from the previous isomorphism $\theta$, which shows that $\chi_\lambda$ is a character for $A_q$.

**Proposition 4.12.** *There exists a decomposed form for the bounded characters of the algebra $A_q$.*

*Proof.* We will now compute the values of characters and spherical functions in the decomposed form from the above linear relations between adjacent terms, using a *shift matrix.* We denote a *basic edge* by a column vector $\mathcal{Y}_j = \left(Y_{j+1}, Y_j\right)^T \in A_q^2$, the linear operator $\mathcal{S}_2 \in M_2(A_q)$ shifts the $\mathcal{Y}_j$ vector by one step to $\mathcal{Y}_{j+1}$ as $\mathcal{S}_2\mathcal{Y}_j = \mathcal{Y}_{j+1}$, explicitly expressed as

$$\begin{pmatrix} \frac{q+1}{q}Y_1 & -\frac{1}{q} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} Y_{j+1} \\ Y_j \end{pmatrix} = \begin{pmatrix} Y_{j+2} \\ Y_{j+1} \end{pmatrix}, \tag{4.7}$$

Given $\chi(Y_1) = \lambda$, the eigenvalues $\mu_1, \mu_2$ of the matrix $\chi\left(\mathcal{S}_2\right)$ are the solutions to

$$\mu^2 - \frac{q+1}{q}\lambda\mu + \frac{1}{q} = 0.$$

The corresponding eigenvectors of the distinct eigenvalues $\mu_1$ and $\mu_2$ are $(\mu_1, 1)^T$ and

$(\mu_2, 1)^T$ respectively. We verify that the vector $\chi(Y_1, Y_0) = (\lambda, 1)$ is a linear combination of the two eigenvectors with nonzero coefficients. Therefore we express $\chi(Y_1, Y_0)$ as the decomposed form

$$\chi(Y_1, Y_0) = \alpha_1(\mu_1, 1) + \alpha_2(\mu_2, 1) \tag{4.8}$$

where $\alpha_1, \alpha_2$ are determined by the value of $\lambda$ and both nonzero. When $\mu_1 \neq \mu_2$, we have

$$\alpha_1 = \frac{\lambda - \mu_2}{\mu_1 - \mu_2}, \quad \alpha_2 = \frac{\lambda - \mu_1}{\mu_2 - \mu_1}. \tag{4.9}$$

Hence we work out the characters from the eigenvectors and eigenvalues of the shift matrix as

$$\chi\left(Y_{j+1}, Y_j\right)^T = \chi\left(\mathcal{S}_2^j \mathcal{Y}_0\right)^T = \left(\alpha_1\mu_1^j(\mu_1, 1) + \alpha_2\mu_2^j(\mu_2, 1)\right)^T. \tag{4.10}$$

We cannot have the case where $|\mu_1| = |\mu_2| > 1$ as the product of $\mu_1$ and $\mu_2$ is fixed to be $1/q$. The only case we have $\mu_1 = \mu_2$ is when $\mu_1 = \mu_2 = q^{-1/2}$ and $\lambda = \frac{2q^{1/2}}{q+1}$. In this case, we compute the values of the characters as $\chi(Y_m) = \frac{2q}{q+1}\mu_1^m$ for $m \geq 1$. □

As the value of the character grows exponentially in terms of $\mu_1$ and $\mu_2$ and to be bounded by arbitrary powers of $j$, we require the norms of both $\mu_1$ and $\mu_2$ to be bounded by 1, to obtain the bounded characters. Therefore, every unordered pair

$$\left\{\{\mu_1, \mu_2\} : \frac{1}{q} \leq |\mu_1| \leq 1, \frac{1}{q} \leq |\mu_2| \leq 1, \mu_1\mu_2 = \frac{1}{q}\right\} \tag{4.11}$$

gives a unique bounded character in $A_q$.

Conversely, every character $\chi_\lambda$ in $A_q$ also determines an undered pair $(\mu_1, \mu_2)$. Since every character in $A_q$ gives a unique value of $\lambda = \chi_\lambda(Y_1)$ and the value $\lambda$ is the parameter that affects the solutions to the equation for the growth rate $\mu_1$ and $\mu_2$, thus the unordered pair $(\mu_1, \mu_2)$ is determined by the character $\chi_\lambda$.

**Lemma 4.13.** *The character space of $A_q$ is parametrized by an ellipse $M_A$ centered at the origin on the complex plane.*

*Proof.* A character $\chi_\lambda$ in $A_q$ is determined by the value $\chi_\lambda(Y_1) = \lambda$. Recall the symmetrized bi-disc [2], given by a set of points on the complex plane $\Gamma = \{(\mu_1 + \mu_2, \mu_1\mu_2) : 0 \leq |\mu_1| \leq 1, 0 \leq |\mu_2| \leq 1\}$. Let $s = \mu_1 + \mu_2$ and $p = \mu_1\mu_2$. Agler and Young showed that $(s, p) \in \Gamma$ if and only if

$$|s| \leq 2, \quad \text{and} \quad |s - \bar{s}p| + |p|^2 \leq 1.$$

Since we have a fixed $p = \mu_1\mu_2 = \frac{1}{q}$, the above inequality implies that

$$|q\mu - \bar{\mu}| \leq q - 1.$$

Set $\mu = x + iy$ in the complex plane, the character space $M_A$ we have is

$$|(q-1)x - i(q+1)y| \leq q-1,$$

which is an ellipse centered at the origin. We can also check that $|\lambda| \leq 1$ for all $q \geq 1$, which agrees with the property of the random walk. $\square$

**Remark 4.14.** *The boundary of the character space $\partial M_A$ is achieved when one of $\mu_1$ and $\mu_2$ has modulus 1 and the other has modulus $\frac{1}{q}$.*

## 4.4 Point derivations and b.a.i

**Definition 4.15.** *Let $A$ be an algebra and $\chi : A \longrightarrow \mathbb{C}$ be a character. A point derivation $D$ on $A$ is a linear function $D : A \to \mathbb{C}$ such that*

$$D(f_1 f_2) = D(f_1)\chi(f_2) + \chi(f_1) D(f_2), \text{ for all } f_1, f_2 \in A.$$

The function $D(f)$ is a point bi-module to be evaluated by the multiplication with a character in the character space.

**Lemma 4.16.** *There exists an essentially unique point derivation $D : A_q \to \mathbb{C}$ for every character $\chi_\lambda$ for which $\lambda \in M_A \backslash \partial M_A$.*

*Proof.* For a character $\chi_\lambda$ where $\chi_\lambda(Y_1) = \lambda$ is not on the boundary of the ellipse, i.e., both $|\mu_1|$, $|\mu_2|$ are strictly less than 1, we have the unique expressions of the eigenfunction and character as a polynomial of degree $m$

$$\tilde{\psi}(m) = \chi_\lambda(Y_m) = \sum_{j=0}^{m} a_{m,j}\lambda^j = \mathcal{P}_m(\lambda).$$

We can define a point derivation on the basis term $Y_m \in A_q$ as $D_\lambda : A_q \longrightarrow \mathbb{C}_\lambda$, where

$$D_\lambda(Y_m) := \frac{d}{dz'}\left(\mathcal{P}_m(z')\right)\Big|_{z'=\lambda} D_\lambda.$$

Set $z' = \frac{q+1}{q}(\mu_1' + \mu_2')$ and $\mu_1'\mu_2' = \frac{1}{q}$, we have

$$\mathcal{P}_m(z') = \frac{z' - \mu_2'}{\mu_1' - \mu_2'}\mu_1'^m + \frac{z' - \mu_1'}{\mu_2' - \mu_1'}\mu_1'^m,$$

which shows that $D_\lambda(Y_m)$ is indeed bounded.

Since $A_q$ is singly generated by $Y_1$, every element $Y_j$ of the natural basis of $A_q$ is a unique finite sum of powers of $Y_1$, the set of powers of $Y_1$ $\{Y_1^k\}_{k\in\mathbb{Z}_+}$ also form a basis of

$A_q$. Every $Y_1^k$ is also a finite sum of powers of $(Y_1 - \lambda Y_0)$. Thus $\{(Y_1 - \lambda Y_0)^n\}_{n\in\mathbb{Z}_+}$ is also a basis for $A_q$.

To prove the uniqueness of $D_\lambda$, we can choose the basis $\{(Y_1 - \lambda Y_0)^n\}_{n\in\mathbb{Z}_+}$ and the identity $Y_0$ for $A_q$. Given a point derivation $D : A_q \longrightarrow \mathbb{C}_\lambda$, we have

$$D((Y_1 - \lambda Y_0)^n) = nD(Y_1 - \lambda Y_0)(z' - \lambda)^{n-1}\Big|_{z'=\lambda} = \begin{cases} D(Y_1) \text{ if } n = 1 \\ 0 \text{ if } n \neq 1 \end{cases},$$

which shows that the point derivation $D$ is determined by the value of $D(Y_1)$. Therefore any linear multiple of $D$ is also a point derivation of $A_q$. □

**Definition 4.17.** *A bounded left approximate identity for a normed algebra $A$ is a bounded net $\{e_\alpha\}_{\alpha\in I}$ with the property $\lim_\alpha e_\alpha a = a$ for $a \in A$. Bounded right and two-sided approximate identities are defined similarly. [10]*

For the algebra $A_q$, we seek a sequence $(e_N)_{N\geq 1}$ which satisfies the condition of being a b.a.i.. Lemma 4.18 and Lemma 4.19 are based on a suggestion of the external examiner.

**Lemma 4.18.** *Let $A$ be a commutative Banach algebra, $J$ a closed ideal, $(e_N)_{N\geq 1}$ a bounded sequence in $J$. Suppose there is $x \in J$ such that $\overline{Ax} = J$ and $\lim_N xe_N = x$. Then $(e_N)$ is a b.a.i. for $J$.*

*Proof.* Let $b \in J$ and let $\epsilon > 0$. By assumption there exists $a \in A$ with $\|b - ax\| < \epsilon$. Then

$$\begin{aligned} \|be_N - b\| &\leq \big\|(b - ax)e_N\big\| + \big\|a(xe_N - x)\big\| + \|ax - b\| \\ &\leq \epsilon\|e_N\| + \|a\|\|xe_N - x\| + \epsilon. \end{aligned}$$

As $N \to \infty$ the middle term $\to 0$. Therefore, for all sufficiently large $N$, $\|be_N - b\| \leq \big(\sup_N\|e_N\| + 2\big)\,\epsilon$. Since $\epsilon$ is arbitrarily small, $\|be_N - b\| \to 0$. □

**Lemma 4.19.** *Let $\chi_\lambda$ be any character on $A_q$. Then every $f \in \ker(\chi_\lambda)$ may be approximated by elements of the form $g(Y_1)(Y_1 - \lambda Y_0)$ where $g$ is a polynomial.*

*Proof.* We know from Lemma 4.7 that there is a sequence of polynomials $(h_n)$ such that $\big\|h_n(Y_1) - f\big\| \to 0$. Define the new polynomials $\big(h'_n\big)$ by $h'_n(X) = h_n(X) - h_n(\lambda)$. Then

$$\begin{aligned} \big\|h'_n(Y_1) - f\big\| &= \Big\|h_n(Y_1) - \chi_\lambda\big(h_n(Y_1)\big)Y_0 + \chi_\lambda(f)Y_0\Big\| \\ &\leq \|h_n - f\| + \big|\chi_\lambda(f_n - f)\big| \to 0 \end{aligned}$$

and the polynomial $h'_n(X)$ has a root at $X = \lambda$, hence it has a factor of $X - \lambda$, so we may define $g_n(X) = h'_n(X)/(X - \lambda)$. □

**Proposition 4.20.** *There exists a b.a.i. for the kernel of every character $\chi_\lambda$ on the boundary of the character space.*

*Proof.* By Lemma 4.18 and Lemma 4.19, it suffices to find a bounded sequence $(e_N)$ in $\ker \chi_\lambda$ which also satisfies $e_N *_q (Y_1 - \lambda Y_0) \to Y_1 - \lambda Y_0$. We seek a solution of the form $e_N = \sum_{j=1}^{N-1} \alpha_j Y_j$.

For $f = Y_1 - \lambda Y_0$,

$$\begin{aligned} e_N f = & \left(-\lambda\alpha_0 + \frac{1}{q+1}\alpha_1\right) Y_0 + \left(\alpha_0 + \frac{1}{q+1}\alpha_2 - \lambda\alpha_1\right) Y_1 \\ & + \sum_{j=2}^{N-2} \left(-\lambda\alpha_j + \frac{q}{q+1}\alpha_{j-1} + \frac{1}{q+1}\alpha_{j+1}\right) Y_j \\ & + \left(\frac{q}{q+1}\alpha_{N-2} - \lambda\alpha_{N-1}\right) Y_{N-1} + \frac{q}{q+1}\alpha_{N-1} Y_N. \end{aligned}$$

Set $\beta_k = \alpha_{N-k}$. We choose $\alpha_j$ such that all middle terms vanish i.e. $-\lambda\alpha_j + \frac{q}{q+1}\alpha_{j-1} + \frac{1}{q+1}\alpha_{j+1} = 0$, then

$$\lambda\beta_k = \frac{q}{q+1}\beta_{k+1} + \frac{1}{q+1}\beta_{k-1} \text{ for } k = 2, \dots, N-2,$$

which has the same form as for the difference equation to the random walk with the shift matrix in Equation (4.7), and the terms in the eigenfunction corresponding to eigenvalue $\lambda$. By Remark 4.14, we consider the characters at the boundary of the character space; i.e., one of $\mu_1$ and $\mu_2$ has modulus 1 and the other has modulus $\frac{1}{q}$. Set $|\mu_1| = 1$ and $|\mu_2| = \frac{1}{q}$. Any middle terms $\alpha_j$ of the form $\alpha_j = A\mu_1^{-j} + B\mu_2^{-j}$ vanish for $Y_j$ when $1 \le j \le N-1$. Choose $\alpha_0 = 1$ and $B = 0$.

For the condition $\chi_\lambda(e_N) = \sum_{j=0}^{N-1} \alpha_j \chi_\lambda(Y_j) = 0$, the equation

$$\begin{aligned} \sum_{j=0}^{N-1} \alpha_j \chi_\lambda(Y_j) &= 1 + \sum_{k=1}^{N-1} A\mu_1^{-k} \left(\mu_1^k \left(\frac{\lambda-\mu_2}{\mu_1-\mu_2}\right) + \mu_2^k \left(\frac{\lambda-\mu_1}{\mu_2-\mu_1}\right)\right) \\ &= 1 + A\left((N-1)\left(\frac{\lambda-\mu_2}{\mu_1-\mu_2}\right) + \left(\frac{\lambda-\mu_1}{\mu_2-\mu_1}\right)\sum_{k=1}^{N-1}\left(\frac{\mu_2}{\mu_1}\right)^k\right) \\ &= 0 \end{aligned}$$

holds when

$$A = -\left((N-1)\left(\frac{\lambda-\mu_2}{\mu_1-\mu_2}\right) + \left(\frac{\lambda-\mu_1}{\mu_2-\mu_1}\right)\sum_{k=1}^{N-1}\left(\frac{\mu_2}{\mu_1}\right)^k\right)^{-1}.$$

Since $|\frac{\mu_2}{\mu_1}| = \frac{1}{q} < 1$ and $\mu_1 \neq \mu_2 \neq 0$, we can choose large $N$ such that

$$(N-1)\left(\frac{\lambda-\mu_2}{\mu_1-\mu_2}\right) + \left(\frac{\lambda-\mu_1}{\mu_2-\mu_1}\right)\sum_{k=1}^{N-1}\left(\frac{\mu_2}{\mu_1}\right)^k \neq 0.$$

Note that $\lambda \neq \mu_2$, as shown in Equation (4.9) as the nonzero coefficients in the decomposed form of the characters. Since $|A|^{-1}$ grows like $O(N)$, we have $\sup_N \|e_N\| < \infty$ and $\|f *_q e_N - f\| \to 0$.

We introduce another variable $Z_k = (q+1)q^{k/2-1}Y_k$, which has a simpler form in the relations of the multiplication:

$$\begin{aligned}
Z_j * Z_1 &= Z_{j+1} + Z_{j-1} \text{ for } j \geq 1,\\
Z_j * Z_0 &= \frac{q}{q+1} Z_j\\
Z_m * Z_n &= Z_{m+n} + Z_{m-n} + \frac{q-1}{q}\sum_{j=1}^{n-1} Z_{m+n-2j} \text{ for } m > n \geq 2.
\end{aligned}$$

The power of $Z_1$ and $Y_1$ can be computed as

$$\begin{aligned}
Y_1^n &= \left(\frac{\sqrt{q}}{q+1}\right)^n Z_1^n\\
&= (q+1)^{-n} q^{\frac{n}{2}} \left(\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\left(\sum_{m=0}^{k} C(n-1-k, k-m)\left(\frac{q}{q+1}\right)^m\right) Z_{n-2k}\right)\\
&= (q+1)^{1-n} q^{n} \left(\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\left(\sum_{m=0}^{k} C(n-1-k, k-m)\left(\frac{q}{q+1}\right)^m\right) q^{-1-k} Y_{n-2k}\right),
\end{aligned}$$

where $C(a,b)$ is the *generalized Catalan number* $\binom{a+b}{b} - \binom{a+b}{b-1}$. Conversely, by induction we can also rewrite $Y_n$ as a polynomial of $Y_1$ and $Y_0$, where $Y_0$ acts as the constant term in the polynomial. For an element $f = \sum_{j=0}^{n} \alpha_j Y_j = p_n(Y_1)$ in the algebra, the character $\chi_\lambda$ can be calculated by $\chi_\lambda(f) = p_n(\lambda)$, which is a polynomial in $\lambda$. If $\chi_\lambda(f) = 0$, then $p_n(Y_1)$ can be factorized and one of the factors is $(Y_1 - \lambda Y_0)$.

We have shown that $e_N(Y_1 - \lambda Y_0) \to (Y_1 - \lambda Y_0)$. Since $f$ can be factorized as $(Y_1 - \lambda Y_0)g$ when $\chi_\lambda(f) = 0$, we have $e_N f = e_N(Y_1 - \lambda Y_0)g = (Y_1 - \lambda Y_0)g$, which proves that $e_N$ is a b.a.i. for the character $\chi_\lambda$. □

**Remark 4.21.** *If there exists a bounded approximate identity which is in the kernel of a character $\chi_\lambda$, there cannot exist a nontrivial point derivation. Given a b.a.i. $e_N \in \ker \chi_\lambda$ such that $e_N f \to f$ and $\chi_\lambda(e_N) = \chi_\lambda(f) = 0$, for all $f \in \ker \chi_\lambda$, a point derivation $D_\lambda$*

*satisfies*

$$D_\lambda(f) = D_\lambda(e_N f) = e_N D_\lambda(f) + D_\lambda(e_N) f = 0$$

*For* $f \notin \ker \chi_\lambda$*, we can write* $f$ *as* $f = e_N f' + cY_0$*, where* $D_\lambda(cY_0) = 0$ *as* $Y_0$ *is the identity of* $A_q$*.*

## 4.5 The isomorphism to a weighted subalebra of $\ell^1(\mathbb{Z})$

As the multiplications in $A_q$ are not balanced, we seek an isomorphism between $A_q$ and another commutative algebra on $\mathbb{Z}_+$ with a weighted $\ell^1$-norm. First we define the $\omega_R$-weighted $\ell^1$-norm on $\mathbb{Z}_+$.

**Definition 4.22.** *Let* $Z_j$ *be the characteristic function for* $j \in \mathbb{Z}_+$*. The* $\omega_R$*-norm of* $Z_j$ *is given by*

$$\left\|Z_j\right\|_{\omega_R} = R^j.$$

Consider the normalized symmetric Laurent polynomials on with one variable $z$, $Z_a = \frac{1}{2}\left(z^a + z^{-a}\right)$ for $a \in \mathbb{Z}_+$. The multiplications satisfy $Z_a * Z_b = \frac{1}{2}\left(Z_{a+b} + Z_{|a-b|}\right)$.

We define a commutative algebra on $\mathbb{Z}_+$ with the $\omega_R$ norm and the multiplications between the characteristic functions $\left\{Z_j\right\}_{j\in\mathbb{Z}_+}$ satisfy the rules from the symmetric Laurent polynomials on $\mathbb{Z}$.

**Definition 4.23.** *Let* $*$ *be the multiplication rule given by the normalized symmetric Laurent polynomials with one variable and* $\left\{Z_j\right\}_{j\in\mathbb{Z}_+}$ *be the characteristic functions on* $\mathbb{Z}_+$*. The algebra* $\mathcal{A}_{2,\omega_R}$ *is defined to be*

$$\mathcal{A}_{2,\omega_R} := \ell^1\left(\mathbb{Z}_+, *, \omega_R\right) = \left\{ f = \sum_{j=0}^{\infty} c_j Z_j \,\middle|\, \|f\|_{\omega_R} = \sum_{j=0}^{\infty} |c_j| R^j < \infty \right\}.$$

**Remark 4.24.** *We can extend the definition of* $\omega_R$*-norm to* $\mathbb{Z}$*; i.e.,* $\left\|Z_j\right\|_{\omega_R} = R^{|j|}$ *for* $j \in \mathbb{Z}$*. Then the algebra* $\mathcal{A}_{2,\omega_R}$ *is the* $S_2$*-invariant subalgebra of* $\ell^1\left(\mathbb{Z}, *, \omega_R\right)$*, which is given by* $\mathcal{A}_{2,\omega_R} = \ell^1\left(\mathbb{Z}_+, \omega_R\right)^{S_2}$*.*

**Theorem 4.25.** *There exists an isomorphism between the Hecke algebra* $A_q$ *and* $\mathcal{A}_{2,\omega_R}$ *when* $R = q^{1/2}$*.*

*Proof.* The algebra $A_q$ has generator $Y_1$ and every element in the natural basis can be written as

$$Y_m = \sum_{j=0}^{m} a_{m,j} Y_1^j.$$

The algebra $\mathcal{A}_{2,\omega_R}$ has generator $Z_1 = \frac{1}{2}\left(z^{-1}+z\right)$ and every basis element can be written as

$$Z_m = \sum_{j=0}^{m} a'_{m,j} Z_1^j.$$

For $\check{f} = \sum_{j=0}^{\infty} \beta_j Z_j \in \mathcal{A}_{2,\omega_R}$, the $\omega_R$-weighted $\ell^1$ norm is given by

$$\left\|\check{f}\right\|_{\omega_R} = \sum_{j=0}^{\infty} \left(\left|\beta_j\right| R^j\right).$$

We scale the generator $Y_1$ of $A_q$ to

$$Y_1' = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2} Y_1.$$

Define $Y_0' = Y_0$ and $Y'_{m+1} = 2Y'_m Y'_1 - Y'_{m-1}$ for $m \geq 1$. We have the following identities

$$\begin{aligned}
Y'_m *_q Y'_0 &= Y'_m, \\
Y'_m *_q Y'_1 &= \frac{1}{2}\left(Y'_{m+1} + Y'_{m-1}\right), \text{ when } m \geq 1, \\
Y'_m *_q Y'_n &= \frac{1}{2}\left(Y'_{m+n} + Y'_{m-n}\right), \text{ when } m \geq n \geq 0,
\end{aligned}$$

which satisfy the same multiplication rules for the basis elements $\left\{Z_j\right\}_{j=0}^{\infty}$ of $\mathcal{A}_{2,\omega_R}$. For $m \geq 2$, we compute $Y'_m = \sum_{j=0}^{m} \beta_{m,j} Y_j$ explicitly. The nonzero coefficients $\beta_{m,j}$ are given by

$$\begin{aligned}
\beta_{m,m} &= \left(\frac{q+1}{2q}\right) q^{\frac{m}{2}} \text{ for all } m \geq 2, \\
\beta_{m,0} &= -\left(\frac{q-1}{2}\right) q^{-\frac{m}{2}} \text{ when } m \text{ is even}, \\
\beta_{m,m-2k} &= -q^{-2k}\left(q-1\right)\beta_{m,m} = -\left(\frac{q^2-1}{2q}\right) q^{\frac{m}{2}-2k} \text{ for all } 1 \leq k < \frac{m}{2}.
\end{aligned}$$

We verify that

$$\sup_m q^{-m/2}\left\|Y'_m\right\| < \infty.$$

Define a linear map $\theta_{R,q} : \mathcal{A}_{2,\omega_R} \longrightarrow A_q$ by

$$\theta_{R,q}\left(\sum_{j=0}^{\infty} \alpha_j Z_j\right) = \sum_{j=0}^{\infty} \alpha_j Y'_j.$$

From the multiplication rules above, we apply the identities to show that $\theta_{R,q}$ is a homomorphism as

$$\theta_{R,q}\left(Z_m Z_n\right) = Y'_m *_q Y'_n, \text{ for all } m,n \in \mathbb{Z}_+.$$

When $R = q^{\frac{1}{2}}$, we have

$$\sup_m \frac{\left\|Y'_m\right\|}{\left\|Z_m\right\|} < \infty.$$

Therefore the map $\theta_{R,q}$ is well-defined, linear and bounded from $\mathcal{A}_{2,\omega_R}$ to $A_q$.

Conversely, we can scale the generator $Z_1$ for $\mathcal{A}_{2,\omega_R}$ to be

$$Z'_1 = \frac{2}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} Z_1.$$

For $m \geq 2$, define the elements $Z'_m \in \mathcal{A}_{2,\omega_R}$ by induction

$$Z'_m = \frac{q+1}{q} Z'_{m-1} Z'_1 - \frac{1}{q} Z'_{m-2}.$$

We have the following identities

$$\begin{aligned}
Z'_m Z'_0 &= Z'_m, \\
Z'_m Z'_1 &= \frac{q}{q+1} Z'_{m+1} + \frac{1}{q+1} Z'_{m-1} \text{ when } m \geq 1.
\end{aligned}$$

For $m \geq 2$, we compute $Z'_m = \sum_{j=0}^{m} \gamma_{m,j} Z_j$ explicitly. The nonzero coefficients $\gamma_{m,j}$ are given by

$$\begin{aligned}
\gamma_{m,m} &= \frac{2q^{-m+1}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}, \\
\gamma_{m,0} &= \frac{q-1}{2} q^{-2m}, \\
\gamma_{m,m-2k} &= (q-1) q^{-m-2k} \text{ for all } 1 \leq k < \frac{m}{2}.
\end{aligned}$$

When $R = q^{\frac{1}{2}}$, we verify that

$$\sup_m \left\|Z'_m\right\|_{\omega_R} < \infty.$$

Define a linear map $\theta_{q,R} : A_q \longrightarrow \mathcal{A}_{2,\omega_R}$ by

$$\theta_{q,R}\left(\sum_{j=0}^{\infty} \alpha_j Y_j\right) = \sum_{j=0}^{\infty} \alpha_j Z'_j.$$

From the multiplication rules above, we apply the identities to show that $\theta_{q,R}$ is a homomorphism as

$$\theta_{q,R}\left(Y_m *_q Y_n\right) = Z'_m Z'_n, \text{ for all } m,n \in \mathbb{Z}_+.$$

When $R = q^{\frac{1}{2}}$, we have
$$\sup_m \frac{\|Z'_m\|}{\|Y_m\|} < \infty.$$
Therefore the map $\theta_{q,R}$ is well-defined, linear and bounded from $A_q$ to $\mathcal{A}_{2,\omega_R}$. $\square$

# Chapter 5

# Algebras of automorphism groups of buildings and the $\omega_R$-norm

For the projective general linear group over the $p$-adic numbers and $p$-adic integers, we show that the quotient space $PGL_n\left(\mathbb{Q}_p\right)/PGL_n\left(\mathbb{Z}_p\right)$ has the structure of a Bruhat-Tits building [1]. The building is homogeneous where the degree of every vertex is equal and determined by the prime number $p$. The group $PGL_n(\mathbb{Q}_p)$ acts on the equivalence classes of $n$-dimensional $p$-adic integer lattices which correspond to the vertices of the building.

We define the algebra of the bi-invariant summable functions on the vertices of the building given by the Gelfand pair $(PGL_n(\mathbb{Q}_p), PGL_n(\mathbb{Z}_p))$ [47]. There will be a set of $(n-1)$ distinct and independent Laplace operators generated by different types of adjacency relations on the building. The Hecke algebra $A_{n,p}$ on $\mathbb{Z}_+^n$ with $\ell^1$ norm, which is isomorphic to the discretized bi-invariant subalgebra of the group algebra, will also have $(n-1)$ Hecke operators as the generators. There is a correspondence between eigenfunctions, spherical functions and characters. The shift matrices for each algebra are more complicated to compute. For $n = 2$ and $n = 3$ we conjecture an explicit description of the character space and the existence of bounded approximate identities for the ideals given by certain points in the space. In particular, we will study the Gelfand pair $\Big(PGL_3\left(\mathbb{Q}_p\right), PGL_3\left(\mathbb{Z}_p\right)\Big)$ and the related $\tilde{A}_2$ building and lattices [46].

By rescaling the normalized orbit, we obtain the multiplication rules of $A_{n,p}$ with the same coefficients as the algebra of summable functions on the weighted $(n-1)$-dimensional permutohedral lattice, $\ell^1(\Lambda_n, \omega_R)$ where every vertex corresponds to a monomial. These two algebras have isomorphic character spaces determined by the prime number $p$ and the weight conditions $\omega_R$ for the rescaling. The isomorphism between the character spaces suggest that these two algebras are isomorphic to each other. We finish with a conjecture that there exists an isomorphism between $A_{n,p}$ and the weighted subalgebra. The computation of derivations and bounded approximate identities will be easier and enable us to analyze the cohomology groups in later problems.

## 5.1 Lattices of $p$-adic numbers and the buildings

Throughout this section we denote by $\mathbb{Q}_p$ and $\mathbb{Z}_p$ the $p$-adic rational numbers and the $p$-adic integers. Let $G = PGL_n(\mathbb{Q}_p)$ be the projective general linear group over $p$-adic rational numbers, which is locally compact. Let the subgroup $K = PGL_n(\mathbb{Z}_p)$ be the projective general linear group over $p$-adic integers. We recall some definitions of lattices on $p$-adic numbers and show that the quotient space $G/K$ is isomorphic to the Bruhat-Tits building.

**Definition 5.1.** *Given a set of $n$ linearly independent vectors $\{\underline{v_i} | i = 1, \ldots, n, \underline{v_i} \in \mathbb{Q}_p^n\}$, an $n$-dimensional lattice of $\mathbb{Q}_p$ is a set of vectors $\{\underline{v} = \sum_{i=1}^n \alpha_i \underline{v_i} | \alpha_i \in \mathbb{Z}_p\}$.*

If we write $\{\alpha_i \underline{v_i}\}$ as column vectors, then the $n \times n$ matrix $g = (\alpha_1 \underline{v_1} | \alpha_2 \underline{v_2} | \ldots | \alpha_n \underline{v_n})$ is an element in the general linear group $GL_n(\mathbb{Q}_p)$. Let $P_n(V)$ be the set of diagonal matrices which are powers of the diagonal matrix $\operatorname{diag}(p, p, \ldots, p)$. The projective general linear group $G = PGL_n(\mathbb{Q}_p)$ can therefore be expressed explicitly as the quotient group $G = GL_n(\mathbb{Q}_p)/P_n(V)$.

We know that every $g \in PGL_n(\mathbb{Q}_p)$ can be written as a product of two matrices $g = (\underline{u_1} | \underline{u_2} | \ldots | \underline{u_n})k = Mk$, where $M = (\underline{u_1} | \underline{u_2} | \ldots | \underline{u_n}) \in PGL_n(\mathbb{Q}_p)$ is an upper triangular matrix and $k \in PGL_n(\mathbb{Z}_p)$. We can assume without loss of generality that all column vectors $\{\underline{u}_i\}_{i=1}^n$ form upper triangular matrix $(\underline{u_1} | \underline{u_2} | \ldots | \underline{u_n})$ such that the diagonal entries have value $p^{a_i}$ all $i = 1, \ldots, n$. This shows that $\{\underline{u_i} | i = 1, \ldots, n, \underline{u_i} \in \mathbb{Q}_p^n\}$ is also a basis of the lattice.

Let $\mathcal{L}$ be the set of all equivalence classes of lattices in $\mathbb{Q}_p$ and $L_M$ be a lattice generated by $M \in PGL_n(\mathbb{Q}_p)$. Two lattices $L_{(\underline{u_1}|\underline{u_2}|\ldots|\underline{u_n})}$ and $L_{(\underline{v_1}|\underline{v_2}|\ldots|\underline{v_n})}$ are physically the same if and only if for all $(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_p^n$ there exists a unique $(\beta_1, \ldots, \beta_n) \in \mathbb{Z}_p^n$ such that $\sum_{i=1}^n \alpha_i \underline{u_i} = \sum_{i=1}^n \beta_i \underline{v_i}$, and for all $(\beta_1, \ldots, \beta_n) \in \mathbb{Z}_p^n$, there exists a unique $(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_p^n$ such that $\sum_{i=1}^n \alpha_i \underline{u_i} = \sum_{i=1}^n \beta_i \underline{v_i}$.

**Remark 5.2.** *We denote $L_M$ the lattice with basis consisting of the column vectors in $M$. The (projective) equivalence class $[L_{(\underline{u_1}|\underline{u_2}|\ldots|\underline{u_n})}]$ [45] consists of lattices of the form $L_{(\alpha\underline{u_1}|\alpha\underline{u_2}|\ldots|\alpha\underline{u_n})}$ and $L_{(p^l\underline{u_1}|p^l\underline{u_2}|\ldots|p^l\underline{u_n})}$, where $\alpha$ is invertible in $\mathbb{Q}_p$ and $l$ is an arbitrary integer. Therefore the quotient $G/K$ is isomorphic to the set of equivalence classes of $n$-dimensional lattices of $p$-adic numbers.*

**Definition 5.3.** *The lattice $L_{(\underline{u_1}|\ldots|\underline{u_n})}$ is said to be a* sublattice *of the lattice $L_{(\underline{v_1}|\ldots|\underline{v_n})}$ if we have*

$$L_{(\underline{u_1}|\ldots|\underline{u_n})} \subsetneq L_{(\underline{v_1}|\ldots|\underline{v_n})}.$$

The lattice $L_{(\underline{pu_1}|\ldots|\underline{pu_n})}$ is clearly a sublattice of $L_{(\underline{u_1}|\ldots|\underline{u_n})}$.

**Definition 5.4.** *Two lattices $L_{(\underline{u_1}|\ldots|\underline{u_n})}$ and $L_{(\underline{v_1}|\ldots|\underline{v_n})}$ are* adjacent *if we have*

$$L_{(\underline{pu_1}|\ldots|\underline{pu_n})} \subsetneq L_{(\underline{v_1}|\ldots|\underline{v_n})} \subsetneq L_{(\underline{u_1}|\ldots|\underline{u_n})}.$$

There are at most $n$ lattices $M_0, M_1, \ldots, M_{n-1}$ which are pairwise adjacent.

**Example 5.5.** *Let $M_0 = I$ such that $L_{M_0}$ is the lattice generated by the identity matrix and $M_j =$diag$(p, p, \ldots, p, 1, 1, \ldots, 1)$ for $j = 1, \ldots n-1$, the diagonal matrix with the first $j$ diagonal entries to be $p$ and the last $(n-j)$ diagonal entries to be $1$. Then we have $L_{diag(p,p,\ldots,p)M_0} \subsetneq L_{M_{n-1}} \subsetneq L_{M_{n-2}} \subsetneq \ldots \subsetneq M_1 \subsetneq L_{M_0}$.*

We explicitly list all the sets $\mathcal{M}_l$ of different families of upper triangular matrices $M$ such that $L_{\mathrm{diag}(p,p,\ldots,p)M_0} \subsetneq L_M \subsetneq L_{M_0} = L_I$, for different families $l = 1, \ldots, n-1$ as

$$\mathcal{M}_l = \left\{ M \in PGL_n(\mathbb{Q}_p) \;\middle|\; \begin{array}{l} M_{ij} = 0 \text{ when } i > j, \\ M_{ii} \in \{1, p\}, \\ \det M = p^l, \\ M_{ij} \in \{0, 1, \ldots, n-1\}, \text{ when } i < j, M_{ii} = p \text{ and } M_{jj} = 1, \\ 0 \text{ elsewhere} \end{array} \right\}.$$

Note that the disjoint union of the right cosets $\mathcal{M}_l K = \{MK | M \in \mathcal{M}_l\}$ is a double coset for $G = PGL_n(\mathbb{Q}_p)$ and $K = PGL_n(\mathbb{Z}_p) \subset G$.

**Definition 5.6.** *A lattice $y$ generated by $M_y$ is said to be a type-$i$ neighbour of another lattice $x$ generated by $M_x$ if there exists an element $M' \in \mathcal{M}_i$ such that $M_y = M'M_x$. We denote this adjacency relation by $y \sim_i x$.*

**Remark 5.7.** *Note that $y \sim_i x$ if and only if $x \sim_{n-i} y$.*

The size of every family $\mathcal{M}_l$ can be computed by enumerating the number of matrices in the above set. The number of lattices in $\mathcal{M}_l$ is given by

$$|\mathcal{M}_l| = \sigma_l\left(1, p, \ldots, p^{n-1}\right) = \sum_{0 \le k_1 < k_2 < \ldots < k_l \le n-1} p^{\sum_{i=1}^{l} k_i}, \tag{5.1}$$

the $l$-th elementary symmetric polynomial of $\left(1, p, \ldots, p^{n-1}\right)$.

Define a graph $\mathcal{G}_n$ whose vertices $\mathcal{V}$ correspond to the set of equivalence classes of lattices above. Two vertices are adjacent and connected by an edge if their corresponding lattices are connected by the description of sublattice relations above. The graph $\mathcal{G}_n$ have both the combinatorial and geometric structure of a building [30]. The building is locally finite and homogeneous and the degree of every vertex is determined by the prime number $p$. Note that the infinite homogeneous tree $\mathbb{T}_q$ also has these properties.

The group $G = PGL_n(\mathbb{Q}_p)$ is an automorphism group acting on the equivalence classes of lattices and $K$ is the stabilizer of the standard lattice generated by the identity matrix. The isomorphism between the vertices $\mathcal{V}$ of the building and the equivalence classes of lattices shows that $G$ acts transitively on the $\mathcal{V}$ and $K$ stabilizes a fixed vertex $x_0$ which corresponds to the identity matrix [13]. Thus we have $\mathcal{L} \simeq G/K \simeq \mathcal{V}$.

By Proposition 4.4.3 of [12], the double coset $\Omega_{\underline{\tilde{a}}} = Kg_{\underline{\tilde{a}}}K$ consists of all elements which can be written as the form $k_1\mathrm{diag}(p^{a_1}, \ldots, p^{a_n})k_2$ as the *Cartan decomposition*, where $k_1, k_2 \in K$. We will clarify the notations which relate to the powers of the diagonal entries.

Pick a representative vector $\underline{a} = (a_1, \ldots, a_n)$ in the equivalence class of $(\mathbb{Z}^n)^{S_n}/(1, 1, \ldots, 1)$. Another vector $\underline{b} = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ is said to be in the same equivalence class with $\underline{a}$ if they differ by a multiple of the vector $(1, 1, \ldots, 1)$ or by a permutation of the entries. There exists a unique $\mathbf{m}$-vector in $\mathbb{Z}_+^{n-1}$ from the equivalence class of $\underline{\tilde{a}}$-vectors in $\mathbb{Z}^n/(1, 1, \ldots, 1)$ with monotonic descending order given by

$$\mathbf{m} = (m_1, m_2, \ldots, m_{n-1}) = (a_1 - a_2, a_2 - a_3, \ldots, a_{n-1} - a_n),$$

where $\underline{\tilde{a}} = a_1 \geq \ldots \geq a_n$. Thus we can alternatively denote the orbit $\Omega_{\underline{\tilde{a}}}$ by $\Omega_{\mathbf{m}}$. Let $\{\underline{e_i}\}_{i=1}^n$ be the set of canonical basis for $\mathbb{Z}^n/(1, 1, \ldots, 1)$ and $\{\mathbf{e_j}\}_{j=1}^{n-1}$ be the set of canonical basis for $\mathbb{Z}_+^{n-1}$. Note that $\sum_{i=1}^n \underline{e_i} = \underline{0}$. We can see that the space $\mathbb{Z}^n/(1, 1, \ldots, 1)$ is isomorphic to the $(n-1)$-dimensional permutohedral lattice, denoted by $\Lambda_{n-1}$. The $\mathbf{m}$-vectors in $\mathbb{Z}_+^{n-1}$ are in a *Weyl chamber* of the lattice.

## 5.2 The Gelfand pair $(PGL_n(\mathbb{Q}_p), PGL_n(\mathbb{Z}_p))$

We will show that the algebra of the integrable bi-$K$-invariant functions on $G$, $L^1(K\backslash G/K)$ is commutative and isomorphic to a Banach algebra $\ell^1(\mathbb{Z}_+^{n-1}, *_p)$. We will use the following facts to show that $(G, K)$ is a Gelfand pair .

- $G$ is locally compact and unimodular by Lemma 8.1.5 in [65];
- $K$ is a compact subgroup of $G$ by Proposition 8.1 in [22];
- the transpose $g^T$ is also in the double coset $KgK$ as $g = k_1\mathrm{diag}(p^{a_1}, \ldots, p^{a_n})k_2$ implies that $g^T = k_2^T\mathrm{diag}(p^{a_1}, \ldots, p^{a_n})k_1^T$;
- the transpose operation $g \to g^T$ preserves the Haar measure as $G$ is unimodular by Lemma 8.1.5 of [65].

**Theorem 5.8.** *Let $G = PGL_n(\mathbb{Q}_p)$ and $K = PGL_n(\mathbb{Z}_p)$. The pair $(G, K)$ is a Gelfand pair; i.e., the bi-$K$-invariant functions on $G$ are commutative under convolution.*

*Proof.* For all $f_1, f_2 \in L^1(K\backslash G/K)$ and $g \in G$, we have the convolution between $f_1$ and $f_2$ as

$$\begin{aligned}
f_1 * f_2(g) &= \int_{h\in G} f_1(h)\, f_2\left(h^{-1}g\right) dh \\
\left(\text{setting } h_1 = h^{-1}g\right) & \\
&= \int_{h_1\in G} f_1\left(gh_1^{-1}\right) f_2(h_1)\, dh_1 \\
&= \int_{h_1\in G} f_1\left(\left(gh_1^{-1}\right)^T\right) f_2\left(h_1^T\right) dh_1 \\
&= \int_{h_1\in G} f_1\left(\left(h_1^{-1}\right)^T g^T\right) f_2\left(h_1^T\right) dh_1 \\
\left(\text{setting } h_2 = h_1^T\right) & \\
&= \int_{h_2\in G} f_2(h_2)\, f_1\left(h_2^{-1}g^T\right) dh_2 \\
&= f_2 * f_1\left(g^T\right).
\end{aligned}$$

Since $g^T \in KgK$, then by Lemma 4.2, we have $f_2 * f_1(g^T) = f_2 * f_1(g)$. Thus $f_1 * f_2 = f_2 * f_1$ for all $f_1, f_2 \in L^1(K\backslash G/K)$, which shows that the pair $(G, K)$ is a Gelfand pair. $\square$

Note that for $n \geq 3$, $(G, K)$ is not a symmetric Gelfand pair as $Kg^{-1}K \neq KgK$ for most $g \in G$.

We will now define a basis for the integrable bi-$K$-invariant functions $L^1(K\backslash G/K)$, the functions on the equivalence classes of lattices and the functions on $\mathbb{Z}_+^{n-1}$. We fix a Haar measure such that the subgroup $K$ has normalized mass 1.

We define a basis for $L^1(K\backslash G/K)$ by the scaled characteristic function as

$$Y'_{\underline{\tilde{a}}}(g) = \frac{1}{\left|\Omega_{\underline{\tilde{a}}}\right|}\delta_{\Omega_{\underline{\tilde{a}}}}(g) = \begin{cases} \frac{1}{\left|\Omega_{\underline{\tilde{a}}}\right|} \text{ if } g \in \Omega_j \\ 0 \text{ elsewhere} \end{cases}.$$

We also define the summable functions on the equivalence classes of lattices which are isomorphic to the vertices $\mathcal{V}$ of the building, $f_{\mathcal{L}} \in \ell^1(\mathcal{L})$ as

$$f_{\mathcal{L}} : G/K \longrightarrow \mathbb{C}, \quad f_{\mathcal{L}}\left(L_g\right) = \int_{h\in gK} f(h)dh.$$

The basis for $\ell^1(\mathcal{L})$ is given by the characteristic function

$$\delta_{L_g}(L_{g'}) = \begin{cases} 1 \text{ if } L_{g'} = L_g \\ 0 \text{ elsewhere} \end{cases}.$$

We now define the summable functions on $\mathbb{Z}_+^{n-1}$, which are isomorphic to the vertices of a Weyl chamber of the $A_{n-1}$ lattice $\Lambda_{n-1}$, including the vertices on the boundary of the chamber. The function $\tilde{f}$ which corresponds to $f_{\mathcal{L}} \in \ell^1(\mathcal{L})$ and $f \in L^1(K\backslash G/K)$ are defined as

$$\tilde{f} : \mathbb{Z}_+^{n-1} \longrightarrow \mathbb{C}, \tilde{f}(\mathbf{m}) = |\Omega_{\mathbf{m}}|\, f_{\mathcal{L}}\left(L_g\right) \text{ for } g \in \Omega_{\mathbf{m}}.$$

The basis are given by the characteristic functions $Y_{\mathbf{m}}$ on the $\mathbf{m}$-vectors in $\mathbb{Z}_+^{n-1}$ as

$$Y_{\mathbf{m}}(\mathbf{x}) = \begin{cases} 1 \text{ if } \mathbf{x} = \mathbf{m} \\ 0 \text{ elsewhere} \end{cases}.$$

**Remark 5.9.** *Alternatively, for $\underline{a} \in \mathbb{Z}^n$, we are allowed to express $Y_{\mathbf{m}} = Y_{\underline{a}}$ if there exists $\underline{b} = (b_1, \ldots, b_n)$ such that $\mathbf{m} = (b_1 - b_2, \ldots, b_{n-1} - b_n)$, which is in the same equivalence class of $(\mathbb{Z}^n)^{S_n}/(1, 1, \ldots, 1)$ with the $\underline{a}$-vector.*

We will show that there exists an isomorphism between $L^1(K\backslash G/K)$ and a Banach algebra on $\mathbb{Z}_+^{n-1}$ with specific multiplication rules $*_p$ given by the properties of $\mathcal{G}_n$.

**Definition 5.10.** *We define the commutative algebra $A_{n,p} = \ell^1(\mathbb{Z}_+^{n-1}, *_p)$, where*

$$\begin{aligned} Y_{\tilde{\underline{a}}} *_p Y_0 =& Y_{\underline{a}}, \\ Y_{\tilde{\underline{a}}} *_p Y_{\mathbf{e_j}} =& \frac{1}{|\mathcal{M}_j|} \sum_{1 \le k_1 < k_2 < \ldots < k_j \le n} \left( p^{\sum_{i=1}^{j}(n-k_i)} Y_{\tilde{\underline{a}} + \sum_{i=1}^{j} \underline{e_{k_i}}} \right), \\ & \text{for all } j \in \{1, 2, \ldots, n-1\}. \end{aligned}$$

*Alternatively, we may write the above equations in the form of $Y_{\mathbf{m}} *_p Y_0$ and $Y_{\mathbf{m}} *_p Y_{\mathbf{e_j}}$ by the conversion stated in Remark 5.9.*

**Theorem 5.11.** *Recall the equivalence class of the $\underline{a}$-vectors in $\mathbb{Z}^n/(1, 1, \ldots, 1)$ with canonical basis $\{\underline{e_j}\}_{j=1}^n$ and the $\mathbf{m}$-vectors in $\mathbb{Z}_+^{n-1}$ with canonical basis $\{\mathbf{e_i}\}_{i=1}^{n-1}$. The algebra of bi-$K$-invariant functions $L^1(K\backslash G/K)$ is isomorphic to the commutative algebra $A_{n,p}$. The isomorphism sends the characteristic functions $Y'_{\tilde{\underline{a}}}$ of $L^1(K\backslash G/K)$ to the corresponding characteristic functions $Y_{\tilde{\underline{a}}}$ of $A_{n,p}$.*

*Proof.* For $n \geq 2$ the commutative algebra $A_{n,p}$ has got $(n-1)$ generators $\{Y_{\mathbf{e_j}}\}_{j=1}^{n-1}$; i.e., every $Y_{\tilde{\underline{a}}} \in A_{n,p}$ can be uniquely written as a linear sum of products of the generators.

For all normalized basis elements $Y'_{\underline{\tilde{a}}} \in L^1(K\backslash G/K)$ and $Y_{\underline{\tilde{a}}} \in A_{n,p}$, define a linear map

$$\theta : L^1(K\backslash G/K) \longrightarrow A_{n,p},\ \theta\left(Y'_{\underline{\tilde{a}}}\right) = Y_{\underline{\tilde{a}}}.$$

Denote $\mathbf{m} = (m_1, \ldots, m_{n-1}) \in \mathbb{Z}_+^{n-1}$ and a representative vector in the equivalence classes of $\mathbb{Z}^n/(1,\ldots,1)$, $\underline{\tilde{a}} = (a_1, \ldots, a_n)$, such that $m_k = a_k - a_{k-1}$ for all $k \in \{1, 2, \ldots, n-1\}$. We need to show that for all equivalence classes of the $\underline{\tilde{a}}$-vectors in $\mathbb{Z}^n/(1,\ldots,1)$ and every $j \in \{1, 2, \ldots, n-1\}$, $\theta(Y'_{\underline{\tilde{a}}} * Y'_{\mathbf{e_j}}) = \theta(Y'_{\underline{\tilde{a}}}) *_p \theta(Y'_{\mathbf{e_j}}) = Y_{\underline{\tilde{a}}} *_p Y_{\mathbf{e_j}}$. We will show the proof for the multiplication with $Y_{\mathbf{e_1}}$ and the multiplications with the rest of generators just follow from the usual steps.

$$\begin{aligned} Y'_{\underline{\tilde{a}}} * Y'_{\mathbf{e_1}}(g) &= \frac{1}{\left|\Omega_{\underline{\tilde{a}}}\right| \left|\Omega_{\mathbf{e_1}}\right|} \int_{h\in G} \delta_{\Omega_{\underline{\tilde{a}}}}(h)\, \delta_{\Omega_{\mathbf{e_1}}}\left(h^{-1}g\right) dh \\ &= \frac{1}{\left|\Omega_{\underline{\tilde{a}}}\right| \left|\Omega_{\mathbf{e_1}}\right|} \int_{h\in \Omega_{\underline{\tilde{a}}}} \delta_{\Omega_{\mathbf{e_1}}}\left(h^{-1}g\right) dh. \end{aligned}$$

For every $h^{-1} = k_1^{-1}\mathrm{diag}(p^{-a_1}, p^{-a_2}, \ldots, p^{-a_n})k_2^{-1}$ where $k_1, k_2 \in K$ and $h^{-1}g \in \Omega_{\mathbf{e_1}}$, we require $g = k_2\mathrm{diag}(p^{a_1}, p^{a_2}, \ldots, p^{a_n})Mk_3$, where $M \in \mathcal{M}_1$. For every upper triangular matrix $M \in \mathcal{M}_1$, if $M$ has value $p$ at its $i$-th diagonal entry, then $\mathrm{diag}(p^{a_1}, p^{a_2}, \ldots, p^{a_n})M$ is in the double coset $K\mathrm{diag}(p^{a_1}, \ldots, p^{a_i+1}, \ldots, p^{a_n})K$. Since there are precisely $p^{n-i}$ elements in $\mathcal{M}_1$ with value $p$ at its $i$-th entry, we can work out the integral of the characteristic function as

$$\begin{aligned} Y'_{\underline{\tilde{a}}} * Y'_{\mathbf{e_1}}(g) &= \frac{1}{\left|\Omega_{\mathbf{e_1}}\right|} \sum_{j=1}^{n} \left(p^{n-i} Y'_{\underline{\tilde{a}}+\underline{e_j}}\right) \\ &= \frac{1}{|\mathcal{M}_1|} \sum_{j=1}^{n} \left(p^{n-i} Y'_{\underline{\tilde{a}}+\underline{e_j}}\right), \end{aligned}$$

which shows that $\theta\left(Y'_{\underline{\tilde{a}}} * Y'_{\mathbf{e_1}}\right) = \theta\left(Y'_{\underline{\tilde{a}}}\right) *_p \theta\left(Y'_{\mathbf{e_1}}\right)$. Similarly, we can show this equality holds for the multiplications with all generators $\{Y_{\mathbf{e_j}}\}_{j=1}^{n-1}$, which shows that the two Banach algebras are isomorphic to each other. □

**Remark 5.12.** *For the algebra $A_q$ (Definition 4.5) where $q = p$ is a prime number, the two algebras $A_q$ and $A_{2,p}$ are the same. By Equation (5.1), we have $|\mathcal{M}_l| = p+1$. The multiplication rules $*_p$ and $*_q$ are of the same form with the same coefficients.*

$$Y_{\underline{\tilde{a}}} *_p Y_{\mathbf{e_1}} = \frac{1}{p+1} \left( \sum_{1\le k\le 2} p^{2-k} Y_{\underline{\tilde{a}}+\underline{e_k}} \right) = \frac{p}{p+1} Y_{\underline{\tilde{a}}+\underline{e_1}} + \frac{1}{p+1} Y_{\underline{\tilde{a}}+\underline{e_2}}$$

*When $n = 2$, the $\mathbf{m}$-vectors are just non-negative integers. The canonical basis of the $\underline{a}$-*

*vectors, $\underline{e_1}$ and $\underline{e_2}$, correspond to $+1$ and $-1$ in the $\mathbf{m}$-vectors. Then $(\underline{\tilde{a}} + \underline{e_1})$ corresponds to $m+1$ and $(\underline{\tilde{a}} + \underline{e_2})$ corresponds to $m-1$ in the $\mathbf{m}$-vectors.*

## 5.3 Typed Laplace operators and random walks

To study the random walk on the vertices of the building $\mathcal{G}_n$, we define a set of linear operators $\{L_i\}_{i=1}^{n-1}$, where $L_i$ is the type-$i$ Laplace operator on the vector space of bounded complex-valued functions on the set of equivalence classes of lattices $\mathcal{L}$ of the $p$-adic numbers.

For a function $f_{\mathcal{L}} : \mathcal{L} \longrightarrow \mathbb{C}$, we define the value of $(L_i f_{\mathcal{L}})(x)$ to be the average of the values of all $f_{\mathcal{L}}(y)$, where $y$ is a type-$i$ neighbour of $x$, expressed as

$$(L_i f_{\mathcal{L}})(x) = \frac{1}{\sigma_i\left(1, p, \ldots, p^{n-1}\right)} \sum_{y \sim_i x} f_{\mathcal{L}}(y).$$

Under the bi-$K$-invariance condition; i.e., the function on $\mathcal{L}$ is determined by its corresponding position on a fixed Weyl chamber of the $A_{n-1}$ integer lattice $\Lambda_{n-1}^{S_n}$, we can also define the typed Laplace operators on to the functions on the integer lattice $\Lambda_{n-1}^{S_n}$, $\tilde{L}_j \tilde{f} : \mathbb{Z}_+^{n-1} \longrightarrow \mathbb{C}$, as

$$\left(\tilde{L}_j \tilde{f}\right)(\mathbf{m}) = \frac{1}{|\mathcal{M}_j|} \sum_{1 \le k_1 < k_2 < \ldots < k_j \le n} \left( p^{\sum_{i=1}^{j}(n-k_i)} \tilde{f}\left(\mathbf{m} + \sum_{i=1}^{j}\left(\sum_{l=1}^{k_i}(\mathbf{e}_{n-l})\right)\right)\right),$$

for all $j \in \{1, 2, \ldots, n\}$. By definition, we can see that the typed Laplace operators are commutative, i.e.

$$\left(\tilde{L}_k\left(\tilde{L}_j \tilde{f}\right)\right)(\mathbf{m}) = \left(\tilde{L}_j\left(\tilde{L}_k \tilde{f}\right)\right)(\mathbf{m}).$$

The type-$j$ Laplace operator parametrizes a weighted random walk to $\binom{n}{j}$ directions on the integer lattice $\Lambda_{n-1}$. We will compute the eigenfunctions to the random walk, starting with $n = 3$, and generalize the result to all $n \in \mathbb{Z}_+$ in the next section.

**Proposition 5.13.** *Let $(G, K)$ be the Gelfand pair $(PGL_n(\mathbb{Q}_p), PGL_n(\mathbb{Z}_p))$ and $\psi \in L^\infty\left(K \backslash G / K\right)$ be a bounded bi-$K$-invariant function on $G$. Let $\psi_{\mathcal{V}}$ and $\tilde{\psi}$ be the corresponding functions on the equivalence classes of lattices $\mathcal{L}$ and the function on the integer lattice $\Lambda_{n-1}$. Then $\psi$ is a spherical function if and only if the bounded radial function $\tilde{\psi}$ is an eigenfunction for the Laplace operators $\{\tilde{L}_i\}_{i=1}^{n-1}$ with $\tilde{\psi}(\mathbf{0}) = 1$.*

*Proof.* If $\psi$ is a spherical function, from $\psi(e) = 1$ we know that the corresponding function on the $(n-1)$-dimensional permutohedral lattice $\Lambda_{n-1}$ has value $\tilde{\psi}(\mathbf{0}) = 1$ at $\mathbf{0}$. For the $\mathbf{m}$-vectors in $\mathbb{Z}_+^{n-1} \simeq \Lambda_{n-1}$, let $\tilde{\psi}\left(\mathbf{e_j}\right) = \lambda_j$ for all $j \in \{1, 2, \ldots, n-1\}$.

First we show that $\tilde{\psi}$ is an eigenfunction for the type-1 Laplace operator $\tilde{L}_1$. Let $x_0$ be the lattice generated by the identity matrix $I$. Given two lattices $x$ generated by $M_x \in PGL_n(\mathbb{Q}_p)$ and $y \in PGL_n(\mathbb{Q}_p)$ generated by $M_y$ such that $y \sim_1 x$, we know that $y$ is a type-1 neighbour of $x$. Let $M_x$ be in the double coset $\mathit{\Omega}_{\mathbf{m}}$. Since $G$ acts transitively on $\mathcal{L}$, we obtain more type-1 adjacency relations as

$$\begin{aligned}
&L_{M_y} \sim_1 L_{M_x} \\
\Longleftrightarrow &L_{M_x^{-1}M_y} \sim_1 L_I \\
\Longleftrightarrow &L_{kM_x^{-1}M_y} \sim_1 L_I \\
\Longleftrightarrow &L_{M_xkM_x^{-1}M_y} \sim_1 L_{M_x},
\end{aligned}$$

where the $(1+p+\ldots+p^{n-1})$ distinct $k_j \in \mathcal{M}_1$ such that every $M_xk_jM_x^{-1}M_y$ generates a a lattice $y_j$ which is a type-1 neighbour of $x$ and these corresponding $(1+p+\ldots+p^{n-1})$ right cosets are all disjoint. Not that each one of the set $K_j = \{k \in K | L_{M_xkM_x^{-1}M_y} = y_j\}$ will have mass $\frac{1}{(1+p+\ldots+p^{n-1})}$ by the Haar measure 1 of $K$, as the union of the disjoint sets $K_j$ is $K$. Therefore, from the properties of the spherical function, we have

$$\begin{aligned}
\tilde{\psi}(\mathbf{e_1})\,\tilde{\psi}(\mathbf{m}) &= \psi\left(M_x^{-1}M_y\right)\psi(M_x) \\
&= \int_{k\in K} \psi\left(M_xkM_x^{-1}M_y\right)dk \\
&= \frac{1}{q+1}\sum_{y_j\sim_1 x} \psi_{\mathcal{L}}(y_j) \\
&= (L_1\psi_{\mathcal{L}})(x) \\
&= \left(\tilde{L}_1\tilde{\psi}\right)(\mathbf{m}),
\end{aligned}$$

which shows that $\tilde{\psi}$ is an eigenfunction for the type-1 Laplace operator $\tilde{L}_1$. Similarly, we can show that $\tilde{\psi}$ is an eigenfunction for all $(n-1)$ typed Laplace operators with eigenvalues $\{\lambda_i\}_{i=1}^{n-1}$.

Conversely, we assume that the function on the $(n-1)$-dimensional permutohedral lattice $\tilde{\psi}$ is an eigenfunction for all of the typed Laplace operators $\{\tilde{L}_i\}_{i=1}^{n-1}$. The eigenvalues for the $(n-1)$ typed Laplace operators are $\tilde{\psi}(\mathbf{e_i}) = \lambda_i$ for all $i \in \{1,\ldots,n-1\}$. We need to show that for all $g,h \in G$, we have $\int_K \psi(gkh)dk = f(g)f(h)$. The steps of the proof is the same for all dimensions. Since we have proved the $n=2$ example, where the Gelfand pair $(PGL_2(\mathbb{Q}_p), PGL_2(\mathbb{Z}_p))$ is isomorphic to the automorphism group and a selected subgroup of $\mathbb{T}_q$ in Section 4.3, it is a repetition to show for general $n$-dimensional cases. □

## 5.4 The $\tilde{A}_2$ lattice $\Lambda_2$ and the algebra $A_{3,p}$

In this section, we consider the Hecke algebra of the Gelfand pair $(PGL_n(\mathbb{Q}_p), PGL_n(\mathbb{Z}_p))$ with $L^1$ norm and the algebra $A_{n,p}$ which are functions on the $\tilde{A}_{n-1}$ lattice. In Section 4.3, as shown in Equation (4.11), we have found the character space for the example of $n = 2$ which is the only example of a symmetric Gelfand pair. We now start with $n = 3$ and generalize the result to $n$-dimensional example based on our analysis for the $n = 2$ case. We compute the bounded characters of the algebra from a shift matrix and describe the character space $\mathbf{\Lambda} = \{\lambda_j\}_{j=1}^{n-1}$. When $n \geq 3$, the characters of the algebra $A_{n,p}$ are parametrized by more than three variables. We finish with two conjectures for the existence of point derivations and bounded approximate identities as the calculations are much more complicated than the $n = 2$ case.

In Section 5.1, we have shown that the algebra of integrable bi-$PGL_n(\mathbb{Z}_p)$-invariant functions on $PGL_n(\mathbb{Q}_p)$ is commutative and isomorphic to the Hecke algebra $A_{n,p}$ by Theorem 5.11. When $n = 3$, we can view $A_{3,p}$ as living on a Weyl chamber of the $\tilde{A}_2$ lattice $\Lambda_2$, which is a triangular grid. The lattice $\Lambda_2$ consist of three types of vertices which are determined by the determinant of the representative diagonal matrix in the double coset $K\mathrm{diag}(p^{a_1}, p^{a_2}, p^{a_3})K$. The type-$j$ vertices of $\Lambda_2$ correspond to the double cosets with the determinant of representative diagonal matrix equal to $p^j$. The types for the vertices can be easily computed by $(a_1 + a_2 + a_3)$ modulo 3. Every vertex on the $\tilde{A}_2$ lattice $\Lambda_2$ is adjacent to six vertices which form a hexagon. For a vertex of type $j$, three of its neighbours are of type $k$ and the other three are of type $l$, where $\{j, k, l\} = \{1, 2, 3\}$.

In this computation, we use the $\mathbf{m}$-vectors in $\mathbb{Z}_+^2$ with natural basis $\{\mathbf{e_1}, \mathbf{e_2}\}$. In the commutative algebra $A_{3,p}$, the multiplications with generators $Y_{\mathbf{e_1}}$ and $Y_{\mathbf{e_2}}$ are expressed in different forms. By definition 5.10, for $\mathbf{m} = (m_1, m_2)$, when both $m_1, m_2 > 0$, we have

$$\begin{aligned}
Y_{\mathbf{e_1}} Y_{(m_1,m_2)} &= \frac{1}{p^2+p+1}\left(p^2 Y_{(m_1+1,m_2)} + pY_{(m_1-1,m_2+1)} + 1Y_{(m_1,m_2-1)}\right), \\
Y_{\mathbf{e_2}} Y_{(m_1,m_2)} &= \frac{1}{p^2+p+1}\left(p^2 Y_{(m_1,m_2+1)} + pY_{(m_1+1,m_2-1)} + 1Y_{(m_1-1,m_2)}\right).
\end{aligned}$$

For $m_1 = 0$ or $m_2 = 0$, we have

$$\begin{aligned}
Y_{\mathbf{e_1}} Y_{\mathbf{0}} &= Y_{\mathbf{e_1}}, \\
Y_{\mathbf{e_2}} Y_{\mathbf{0}} &= Y_{\mathbf{e_2}}, \\
Y_{\mathbf{e_1}} Y_{(m_1,0)} &= \frac{1}{p^2+p+1}\left(p^2 Y_{(m_1+1,0)} + (p+1)\, Y_{(m_1-1,1)}\right), \\
Y_{\mathbf{e_2}} Y_{(m_1,0)} &= \frac{1}{p^2+p+1}\left(\left(p^2+p\right) Y_{(m_1,1)} + 1 Y_{(m_1-1,0)}\right), \\
Y_{\mathbf{e_1}} Y_{(0,m_2)} &= \frac{1}{p^2+p+1}\left(\left(p^2+p\right) Y_{(1,m_2)} + 1 Y_{(0,m_2-1)}\right), \\
Y_{\mathbf{e_2}} Y_{(0,m_2)} &= \frac{1}{p^2+p+1}\left(p^2 Y_{(0,m_2+1)} + (p+1)\, Y_{(1,m_2-1)}\right),
\end{aligned}$$

where the coefficients are slightly different to the general case due to the reflections on the boundaries of the Weyl chamber.

By proposition 5.13, the values for the spherical functions in the algebra of bi-$PGL_3(\mathbb{Z}_p)$-invariant functions on $PGL_3(\mathbb{Q}_p)$, the eigenfunctions for the Laplace operators and the characters for the algebra are given by the same form

$$\psi_{\lambda_1,\lambda_2}\left(\operatorname{diag}\left(p^{m_1+m_2}, p^{m_2}, 1\right)\right) = \tilde{\psi}_{\lambda_1,\lambda_2}(m_1, m_2) = \chi_{\lambda_1,\lambda_2}(Y_{\mathbf{m}}),$$

where

$$\begin{aligned}
\chi_{\lambda_1,\lambda_2}(Y_{\mathbf{0}}) &= 1, \\
\chi_{\lambda_1,\lambda_2}\left(Y_{\mathbf{e_1}}\right) &= \lambda_1, \\
\chi_{\lambda_1,\lambda_2}\left(Y_{\mathbf{e_2}}\right) &= \lambda_2.
\end{aligned}$$

Similar to Proposition 4.12, we seek the shift matrices for the algebra $A_{3,p}$ to obtain the decomposed form of the bounded characters.

**Proposition 5.14.** *There exists a decomposed form for the bounded characters of the algebra $A_{3,p}$.*

*Proof.* We define a column vector $\mathcal{Y}_{\mathbf{m}} = \left(Y_{\mathbf{m}}, Y_{\mathbf{m}+\mathbf{e_1}}, Y_{\mathbf{m}+\mathbf{e_2}}\right)^T \in A_{3,p}^3$, which can be seen as a vector with entries which are the basis of three vertices which form a triangle on the lattice. Define the linear operators $\mathcal{S}_{3,1}, \mathcal{S}_{3,2} \in M_3\left(A_{3,p}\right)$ such that $\mathcal{S}_{3,1}\mathcal{Y}_{\mathbf{m}} = \mathcal{Y}_{\mathbf{m}+\mathbf{e_1}}$ and $\mathcal{S}_{3,2}\mathcal{Y}_{\mathbf{m}} = \mathcal{Y}_{\mathbf{m}+\mathbf{e_2}}$. The linear operators $\mathcal{S}_{3,1}$ and $\mathcal{S}_{3,2}$ shift the column vector $\mathcal{Y}_{\mathbf{m}}$ along the $\mathbf{e_1}$ and $\mathbf{e_2}$ directions by one step, respectively. For $\mathbf{m}' = \left(m_1', m_2'\right)$ and $\mathbf{m} = (m_1, m_2)$ where $m_1' > m_1$ and $m_1' > m_1$, we have $\mathcal{Y}_{\mathbf{m}'} = \mathcal{S}_{3,1}^{m_1'-m_1}\mathcal{S}_{3,2}^{m_2'-m_2}\mathcal{Y}_{\mathbf{m}}$. If we choose $m_1$ or

$m_2$ to be 0, then the shift relations can be explicitly expressed as

$$\mathcal{S}_{3,1}\begin{pmatrix} Y_{(m_1,0)} \\ Y_{(m_1+1,0)} \\ Y_{(m_1,1)} \end{pmatrix} = \begin{pmatrix} Y_{(m_1+1,0)} \\ Y_{(m_1+2,0)} \\ Y_{(m_1+1,1)} \end{pmatrix},$$

where

$$\mathcal{S}_{3,1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{(p^2+p+1)}{p^2+p}Y_{\mathbf{e_1}} & -\frac{p+1}{p^2} \\ -\frac{1}{p^2+p} & \frac{(p^2+p+1)}{p^2+p}Y_{\mathbf{e_2}} & 0 \end{pmatrix},$$

and

$$\mathcal{S}_{3,2}\begin{pmatrix} Y_{(0,m_2)} \\ Y_{(1,m_2)} \\ Y_{(0,m_2+1)} \end{pmatrix} = \begin{pmatrix} Y_{(0,m_2+1)} \\ Y_{(1,m_2+1)} \\ Y_{(0,m_2+2)} \end{pmatrix},$$

where

$$\mathcal{S}_{3,2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{(p^2+p+1)}{p^2+p}Y_{\mathbf{e_2}} & -\frac{p+1}{p^2} \\ -\frac{1}{p^2+p} & \frac{(p^2+p+1)}{p^2+p}Y_{\mathbf{e_1}} & 0 \end{pmatrix}.$$

Define a function $\check{\chi}_{\lambda_1,\lambda_2} : M_3(A_{3,p}) \longrightarrow M_3(\mathbb{C})$ by

$$\check{\chi}_{\lambda_1,\lambda_2}\left(\mathcal{S}_{ij}\right) = \chi_{\lambda_1,\lambda_2}\left(\mathcal{S}_{ij}\right) M_{ij},$$

where $\mathcal{S}_{ij}$ is the matrix entry in $M_3(A_{3,p})$ and $M_{ij}$ is a matrix unit in $M_3(\mathbb{C})$.

Similar to the method of computing the characters of $A_q$ in Section 4.2, we use the shift matrices $\mathcal{S}_{3,1}$ and $\mathcal{S}_{3,2}$ to obtain

$$\check{\chi}_{\lambda_1,\lambda_2}\left(\mathcal{Y}_{\mathbf{m}}\right) = \check{\chi}_{\lambda_1,\lambda_2}\left(\mathcal{S}_{3,1}^{m_1}\mathcal{S}_{3,2}^{m_2}\mathcal{Y}_{\mathbf{0}}\right) = \check{\chi}_{\lambda_1,\lambda_2}\left(\mathcal{S}_{3,1}^{m_1}\right)\check{\chi}_{\lambda_1,\lambda_2}\left(\mathcal{S}_{3,2}^{m_2}\right)\check{\chi}_{\lambda_1,\lambda_2}\left(\mathcal{Y}_{\mathbf{0}}\right). \tag{5.2}$$

We verify that the column vector $\chi\left(\mathcal{Y}_{\mathbf{0}}\right) = (1,\lambda_1,\lambda_2)^T$ is not an eigenvector of $\check{\chi}_{\lambda_1,\lambda_2}(\mathcal{S}_{3,1})$ or $\check{\chi}_{\lambda_1,\lambda_2}(\mathcal{S}_{3,2})$. Therefore all eigenvalues of $\check{\chi}_{\lambda_1,\lambda_2}(\mathcal{S}_{3,1})$ or $\check{\chi}_{\lambda_1,\lambda_2}(\mathcal{S}_{3,2})$ contribute to the growth rate in the decomposed form of characters.

By Equation (5.2), the values of $\check{\chi}_{\lambda_1,\lambda_2}\left(\mathcal{Y}_{\mathbf{m}}\right)$ are obtained by computing the powers of the two shift matrices. Therefore, for the character $\chi_{\lambda_1,\lambda_2} : A_{3,p} \longrightarrow \mathbb{C}$ to be bounded on all $Y_{\mathbf{m}}$, we require both matrices $\check{\chi}_{\lambda_1,\lambda_2}(\mathcal{S}_{3,1})$ and $\check{\chi}_{\lambda_1,\lambda_2}(\mathcal{S}_{3,2})$ have eigenvalues bounded by modulus 1, which indicate the growth rate of $\chi_{\lambda_1,\lambda_2}(Y_{\mathbf{m}})$ in terms of $m_1$ and $m_2$.

The eigenvalues of $\check{\chi}_{\lambda_1,\lambda_2}(\mathcal{S}_{3,1})$ and $\check{\chi}_{\lambda_1,\lambda_2}(\mathcal{S}_{3,2})$ are the solutions of the equations

$$\mu_1^3 - \lambda_1\left(1+\frac{1}{p}+\frac{1}{p^2}\right)\mu_1^2 + \lambda_2\left(\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}\right)\mu_1 - \frac{1}{p^3} = 0,$$
$$\mu_2^3 - \lambda_2\left(1+\frac{1}{p}+\frac{1}{p^2}\right)\mu_2^2 + \lambda_1\left(\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}\right)\mu_2 - \frac{1}{p^3} = 0,$$

which are obtained from the characteristic polynomials of $\check{\chi}_{\lambda_1,\lambda_2}(\mathcal{S}_{3,1})$ and $\check{\chi}_{\lambda_1,\lambda_2}(\mathcal{S}_{3,2})$.

Let $\{\mu_{11},\mu_{12},\mu_{13}\}$ be the set of solutions to the first equation above with $|\mu_{11}| \geq |\mu_{12}| \geq |\mu_{13}|$ and $\{\mu_{21},\mu_{22},\mu_{23}\}$ be the set of solutions to the first equation above with $|\mu_{21}| \leq |\mu_{22}| \leq |\mu_{23}|$. We obtain the following identities from the solutions:

$$\begin{aligned}
&\sigma_3\left(\mu_{11},\mu_{12},\mu_{13}\right) = \sigma_3\left(\mu_{21},\mu_{22},\mu_{23}\right) = \sigma_3\left(1,\frac{1}{p},\frac{1}{p^2}\right)\\
&\lambda_1 = \frac{\sigma_1\left(\mu_{11},\mu_{12},\mu_{13}\right)}{\sigma_1\left(1,\frac{1}{p},\frac{1}{p^2}\right)} = \frac{\sigma_2\left(\mu_{21},\mu_{22},\mu_{23}\right)}{\sigma_2\left(1,\frac{1}{p},\frac{1}{p^2}\right)},\\
&\lambda_2 = \frac{\sigma_2\left(\mu_{11},\mu_{12},\mu_{13}\right)}{\sigma_2\left(1,\frac{1}{p},\frac{1}{p^2}\right)} = \frac{\sigma_1\left(\mu_{21},\mu_{22},\mu_{23}\right)}{\sigma_1\left(1,\frac{1}{p},\frac{1}{p^2}\right)},
\end{aligned}$$

where $\sigma_1,\sigma_2,\sigma_3$ denote the elementary symmetric polynomials. We can see that $\mu_{21} = \mu_{12}\mu_{13}/p^2$, $\mu_{22} = \mu_{11}\mu_{13}/p^2$ and $\mu_{23} = \mu_{11}\mu_{12}/p^2$. Therefore the solutions to the second equation is completely dependent on the first one. When the values of $\mu_{11},\mu_{12},\mu_{13}$ are distinct, the character $\chi_{\lambda_1,\lambda_2}$ can be explicitly written as the decomposed form

$$\begin{aligned}
\chi_{\lambda_1,\lambda_2}\left(Y_{(m_1,m_2)}\right) &= \sum_{0\leq j_1\leq m_1, 0\leq j_2\leq m_2} a_{\mathbf{m},j_1,j_2}\lambda_1^{j_1}\lambda_2^{j_2}\\
&= C_{123}\mu_{11}^{m_1}\mu_{12}^{m_2} + C_{132}\mu_{11}^{m_1}\mu_{13}^{m_2}\\
&+ C_{213}\mu_{12}^{m_1}\mu_{11}^{m_2} + C_{231}\mu_{12}^{m_1}\mu_{13}^{m_2}\\
&+ C_{312}\mu_{13}^{m_1}\mu_{11}^{m_2} + C_{321}\mu_{13}^{m_1}\mu_{12}^{m_2},
\end{aligned}$$

where $C_{ijk}$ are the *chamber coefficients* for the six Weyl chambers of the hexagonal lattice on the plane. $\square$

As we require the modulus of all solutions to be bounded by 1, the modulus all six values will also have a lower bound of $\frac{1}{p^2}$. Therefore the character $\chi_{\lambda_1,\lambda_2}$ is parametrized by an unordered triple $(\mu_{11},\mu_{12},\mu_{13})$ with fixed product $\frac{1}{p^3}$, which are three points on an annulus with interior radius $\frac{1}{p^2}$ and exterior radius 1 on the complex plane.

The above computation shows that given a bounded character $\chi_{\lambda_1,\lambda_2}$, we can find out the bounded growth rate which is a triple $(\mu_{11},\mu_{12},\mu_{13})$. And the elementary symmetric polynomials of the triple are equal to the values determined by $\lambda_1$ and $\lambda_2$.

**Conjecture 5.15.** *Conversely, if we have an unordered triple* $(\mu_{11}, \mu_{12}, \mu_{13})$ *with fixed product* $\frac{1}{p^3}$ *and every* $\mu_{1j}$ *satisfies* $\frac{1}{p^2} \leq |\mu_{1j}| \leq 1$, *we can find a bounded character* $\chi_{\lambda_1,\lambda_2}$ *with* $\lambda_1 = \frac{\sigma_1(\mu_{11},\mu_{12},\mu_{13})}{\sigma_1\left(1,\frac{1}{p},\frac{1}{p^2}\right)}$ *and* $\lambda_2 = \frac{\sigma_2(\mu_{11},\mu_{12},\mu_{13})}{\sigma_2\left(1,\frac{1}{p},\frac{1}{p^2}\right)}$. *This construction gives* $\chi_{\lambda_1,\lambda_2}(Y_{\mathbf{0}}) = 1$, $\chi_{\lambda_1,\lambda_2}(Y_{\mathbf{e_1}}) = \lambda_1$ *and* $\chi_{\lambda_1,\lambda_2}(Y_{\mathbf{e_2}}) = \lambda_2$.

We assume that $|\mu_{11}| \geq |\mu_{12}| \geq |\mu_{13}|$. Let $M_{A_{3,p}}$ be the space of $(\mu_{11}, \mu_{12}, \mu_{13})$ that parametrize the characters. Then we have

$$M_{A_{3,p}} = \left\{ (\mu_{11}, \mu_{12}, \mu_{13}) \middle| 1 \geq |\mu_{11}| \geq |\mu_{12}| \geq |\mu_{13}| \geq \frac{1}{p^2}, \mu_{11}\mu_{12}\mu_{13} = \frac{1}{p^3} \right\}. \tag{5.3}$$

**Remark 5.16.** *We observe that the triple* $(\mu_{11}, \mu_{12}, \mu_{13})$ *with a fixed product for a character of* $A_{3,p}$ *is only determined by two values on an annulus on the complex plane. Similar to the definition of the boundary of a symmetrized bi-disc [3], we clarify different types of boundary points of* $M_{A_{3,p}}$ *in Definition 5.17.*

**Definition 5.17.** *We say that the* topological boundary $M_{A_{3,p}}$ *is obtained when either* $|\mu_{11}| = 1$ *or* $|\mu_{13}| = \frac{1}{p^2}$ *is satisfied, and the* distinguished boundary *of* $M_{A_{3,p}}$ *is achieved when both* $|\mu_{11}| = 1$ *and* $|\mu_{13}| = \frac{1}{p^2}$ *are satisfied.*

We finish this section with the following conjectures which can probably be extended from Lemma 4.16 and Proposition 4.20. These two statements can probably be generalized to the cases for $n \geq 3$.

**Conjecture 5.18.** *If* $(\mu_{11}, \mu_{12}, \mu_{13})$ *does not belong to the distinguished boundary, then there is a non-trivial point derivation on* $A_{3,p}$.

**Conjecture 5.19.** *There exists a bounded approximate identity for every character* $\chi_{\lambda_1,\lambda_2}$ *where the corresponding triple* $(\mu_{11}, \mu_{12}, \mu_{13})$ *for* $(\lambda_1, \lambda_2)$ *is on the distinguished boundary of* $M_{A_{3,p}}$.

## 5.5 The weight condition on $\tilde{A}$ lattices

In this section, we aim to show that the Hecke algebra $A_{n,p}$ is isomorphic to another commutative Banach algebra $\mathcal{A}_{n,\omega_R} = \ell^1(\mathbb{Z}_+^{n-1}, \omega_R)$. The algebra $\mathcal{A}_{n,\omega_R}$ consist of all *weighted* bounded functions on the $\tilde{A}_{n-1}$ lattice which are invariant under $S_n$ action on the Weyl chambers. The weight condition $\omega_R$ of the lattice is parametrized by the prime number $p$ in the Hecke algebra $A_{n,p}$.

Consider the $\tilde{A}_{n-1}$ lattice $\Lambda_{n-1}$ where $n \geq 2$. Recall the equivalence classes of $\underline{a}$-vectors in $\mathbb{Z}^n/(1, 1, \ldots, 1)$ with the canonical basis $\{\underline{e_i}\}$, and the $\mathbf{m}$-vectors with basis $\{\mathbf{e_j}\}_{j=1}^{n-1}$ of $\mathbb{Z}^{n-1}$. Every vertex on $\Lambda_{n-1}$ can be uniquely expressed as a representative of an equivalence class of the $\underline{a}$-vectors or an $\mathbf{m}$-vector. Note that the entries of the $\underline{a}$-vectors do not have to be of descending order and the entries of the $\mathbf{m}$-vectors do not have to be positive for the vertices on all Weyl chambers of $\Lambda_{n-1}$.

To each $\underline{a}$-vector $\underline{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ we assign a monomial $(\underline{z})^{\underline{a}} = z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n}$, with the condition $\prod_{i=1}^n z_i = 1$ imposed to satisfy the quotient relation. The algebra of summable functions on the lattice $\ell^1(\Lambda_{n-1})$

$$\check{f} : \Lambda_{n-1} \longrightarrow \mathbb{C}$$

has the set of characteristic functions $\{\tilde{\chi}_{\underline{a}}\}_{\underline{a} \in \Lambda_{n-1}}$ as basis. The multiplication between two characteristic functions on this lattice is clearly commutative. The multiplication is explicitly given by the vector addition, generated by the multiplications of monomials which are defined above. Therefore the algebra $\ell^1(\Lambda_{n-1})$ is isomorphic to the set of infinite Laurent polynomials where the $\ell^1$ norm is bounded.

The lattice $\Lambda_{n-1}$ consist of $n!$ Weyl chambers, which are given by the elements in the symmetry group $\pi \in S_n$ to indicate the descending order $a_{\pi(1)} \geq a_{\pi(2)} \geq \cdots \geq a_{\pi(n)}$. Fix a value $R > 1$. We define the weight condition on the lattice to be

$$R\left(\underline{a}\right) = \left\| (\underline{z})^{\underline{a}} \right\|_{\Lambda_{n-1}} = R^{\sum_{j=1}^n (j - \frac{n+1}{2}) a_j},$$

where $a_1 \geq a_2 \geq \ldots \geq a_n$.

We define an algebra of bounded functions on the weighted lattice $\Lambda_{n-1}$ to be

$$\ell^1(\Lambda_{n-1}, \omega_R) = \left\{ \sum_{\underline{a} \in \Lambda_{n-1}} \left( \check{f}(\underline{a}) \prod_{i=1}^n z_i^{a_i} \right) \middle| \prod_{i=1}^n z_i = 1, \left\| \check{f} \right\|_{\omega_R} < \infty \right\}, \tag{5.4}$$

where $\omega_R$ is the weight condition that defines the weight $R\left(\underline{a}\right)$ on the corresponding lattice points and the weighted one norm of $\check{f}$ is bounded. The weight condition $\omega_R$ is defined to be invariant on the vertices which are invariant under the $S_n$ action on the $\Lambda_{n-1}$ lattice.

The algebra of the functions on the lattice has an $S_n$-invariant subalgebra generated by $Z_{\mathbf{m}}$. For $\mathbf{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{Z}_+^{n-1}$ in terms of the $\mathbf{m}$-vectors, the corresponding truncated $\underline{a}$-vector where $a_n = 0$ is given by $\check{\underline{a}} = (a_1, \ldots, a_{n-1}) \in \mathbb{Z}_+^{n-1}$, where the entries are in monotonic descending order. Therefore we have the basis of the $S_n$-invariant subalgebra as

$$Z_{\mathbf{m}} = Z_{\check{\underline{a}}} = \frac{1}{n!} \sum_{\pi \in S_n} \left( \prod_{j=1}^{n-1} z_{\pi(j)}^{a_j} \right), \tag{5.5}$$

for all $\check{\underline{a}} = (a_1, \ldots, a_{n-1}) \in \mathbb{Z}_+^{n-1}$ where the entries are in monotonic descending order. Note that the every basis element $Z_{\check{\underline{a}}}$ is given by the symmetric polynomials of $n$ variables with certain powers of every monomial.

**Definition 5.20.** *The $S_n$-invariant subalgebra $\mathcal{A}_{n,\omega_R} = \ell^1(\mathbb{Z}_+^{n-1}, \omega_R)$ is given by*

$$\mathcal{A}_{n,\omega_R} = \ell^1(\Lambda_{n-1}, \omega_R)^{S_n} = \left\{ \sum_{\mathbf{m} \in \mathbb{Z}_+^{n-1}} \left( f(\mathbf{m}) Z_{\mathbf{m}} \right) \middle| \, \|f\|_{\omega_R} < \infty \right\}.$$

Note that the weight condition is multiplicative in a fixed Weyl chamber $\Lambda_{n-1}^{S_n}$ as

$$R\left(\underline{\check{a}} + \underline{\check{b}}\right) = \left\| Z_{\underline{\check{a}}+\underline{\check{b}}} \right\|_{\Lambda_{n-1}} = \|Z_{\underline{\check{a}}}\|_{\Lambda_{n-1}} \left\| Z_{\underline{\check{b}}} \right\|_{\Lambda_{n-1}}.$$

Every lattice point has $n-1$ types of adjacent points, and the types are given by the corresponding typed vertices in the Hecke algebra. There are $n-1$ generators for the $S_n$-invariant subalgebra, namely $\{Z_{\mathbf{e_k}}\}_{k=1}^{n-1}$. The multiplication with a generator in the algebra $\mathcal{A}_{n,\omega_R}$ are given by the unbiased random walk on the lattice

$$Z_{\mathbf{m}} Z_{\mathbf{e_k}} = \frac{1}{\binom{n}{k}} \sum_{\mathbf{m}' \in \mathbb{Z}^{n-1}, \mathbf{m}' \sim_k \mathbf{m}} Z_{\mathbf{m}'}, \tag{5.6}$$

where $\sim_k$ denotes the relation of *type*-k adjacency on the vertices of $\Lambda_{n-1}$.

Recall the multiplication $Y_{\underline{\tilde{a}}} *_p Y_{\mathbf{e_j}}$ and $Y_{\mathbf{m}} *_p Y_{\mathbf{e_k}}$ in the algebra $A_{n,p}$ in Definition 5.10. By the conversion Equation (5.5), we seek a scaling $Z'_{\underline{a}} = C_{\underline{a}} Z_{\underline{a}}$ such that the multiplication between the scaled variables $Z'_{\underline{a}}$ have similar coefficients as the multiplications $Y_{\mathbf{m}} *_p Y_{\mathbf{e_k}}$ in $A_{n,p}$. The only possible scaling that gives the same magnitude as the coefficients in the multiplications between the generators of the $p$-adic algebras $A_{n,p}$ will be

$$C_{\underline{a}} = A_m p^{\sum_{j=1}^{n} (j - \frac{n+1}{2}) a_j},$$

for $a_1 \geq a_2 \geq \cdots \geq a_{n-1} > a_n = 0$ in the integer vector $\mathbf{a}$, where $A_m$ are the constants in the scaling functions and $m$ is the number of actual equalities in the array. We can check that the centre of the lattice which corresponds to the constant vector in the quotient always has weight 1 so that the scaling function is well defined on $\mathbb{Z}^n$ and $\mathbb{Z}^n/(1,1,\ldots,1)$.

Based on the isomorphism between the Hecke algebras $A_{2,p}$ and $\mathcal{A}_{2,\omega_R}$, we aim to reach the statement of a conjecture of the existence of an isomorphism between $A_{n,p}$ and $\mathcal{A}_{n,\omega_R}$ . Both algebras $A_{n,p}$ and $\mathcal{A}_{n,\omega_R}$ consist of bounded functions on a Weyl chamber of the $\tilde{A}_{n-1}$ lattice $\Lambda_{n-1}^{S_n}$. Theorem 5.11 showed that $A_{n,p}$ with $\ell^1$ norm is isomorphic to the Hecke algebra of the Gelfand pair $(PGL_n(\mathbb{Q}_p), PGL_n(\mathbb{Z}_p))$ with $L^1$ norm for all prime numbers $p$. The prime number $p$ parametrizes the condition of boundedness and the character space. The characters of $A_{n,p}$ are parametrized by $n$ points on an annulus with a fixed product.

When $p = 1$, the $\ell^1$ norm for the algebra $A_{n,1}$ is unweighted and $A_{n,1}$ is isomorphic to the bounded functions on a Weyl chamber of the $\tilde{A}_{n-1}$ lattice $\Lambda_{n-1}^{S_n}$. The characters of this algebra are parametrized by $n$ points on a circle with the fixed product to be 1, which is not homeomorphic to the annulus.

The following statement is generalized from Theorem 4.25 but requires further explicit computation to complete the proof.

**Conjecture 5.21.** *There exists an isomorphism between the Hecke algebra $A_{n,p}$ and $\mathcal{A}_{n,\omega_R}$.*

The aim is to find a bounded linear map $\theta_{p,R}$ from $A_{n,p}$ to $\mathcal{A}_{n,\omega_R}$ which is an homomorphism and to find the inverse map $\theta_{R,p}$. The steps follow from the proof of Theorem 4.25.

We define the elements $Y'_{\mathbf{m}} \in A_{n,p}$ for all $\mathbf{m} \in \mathbb{Z}_+^{n-1}$ where the 1-norms are uniformly bounded. We will construct these elements from the scaled generators of this algebra.

Both Banach algebras $A_{n,p}$ and $\mathcal{A}_{n,\omega_R}$ have $n-1$ generators, namely $\{Y_{\mathbf{e_k}}\}_{k=1}^{n-1}$ for $A_{n,p}$ and $\{Z_{\mathbf{e_k}}\}_{k=1}^{n-1}$ for $\mathcal{A}_{n,\omega_R}$. We keep the identity term $Y'_{\mathbf{0}} = Y_{\mathbf{0}}$ and scale the generators of $A_{n,p}$ to

$$Y'_{\mathbf{e_k}} = \frac{\sigma_j\left(1, \frac{1}{p}, \frac{1}{p^2}, \dots, \frac{1}{p^{n-1}}\right)}{\binom{n}{k}} p^{\frac{n-1}{2}k} Y_{\mathbf{e_k}}, \text{ for all } k = 1, \dots, n-1.$$

In the algebra $\mathcal{A}_{n,\omega_R}$, recall that every basis element $Z_{\mathbf{m}} \in \mathcal{A}_{n,\omega_R}$ can be constructed by induction $Z_{\mathbf{m}+\mathbf{e_k}} = \binom{n}{k} Z_{\mathbf{m}} Z_{\mathbf{e_k}} -$ lower terms, which can be easily obtained from the type-$k$ adjacency relation $Z_{\mathbf{m}} Z_{\mathbf{e_k}} = \frac{1}{\binom{n}{k}} \sum_{\mathbf{m}' \sim_k \mathbf{m}} Z_{\mathbf{m}'}$. Thus every $Z_{\mathbf{m}} \in \mathcal{A}_{n,\omega_R}$ can be written as a linear combinations of powers of the generators uniquely as

$$Z_{\mathbf{m}} = \sum_{\mathbf{l}=(l_1,\dots,l_{n-1}) \in \mathbb{Z}_+^{n-1}} \left( a'_{\mathbf{m},\mathbf{l}} \prod_{k=1}^{n-1} Z_{\mathbf{e_k}}^{l_k} \right). \tag{5.7}$$

Similarly, we define the type-$k$ adjacency relations between elements in the algebra $A_{n,p}$ as

$$Y'_{\mathbf{m}} Y'_{\mathbf{e_k}} = \frac{1}{\binom{n}{k}} \sum_{\mathbf{m}' \in \mathbb{Z}^{n-1}, \mathbf{m}' \sim_k \mathbf{m}} Y'_{\mathbf{m}'}, \text{ for all } k = 1, \dots, n-1. \tag{5.8}$$

These adjacency relations allow us to express the terms $Y'_{\mathbf{m}} \in A_{n,p}$ explicitly by induction for all $\mathbf{m} \in \mathbb{Z}_+^{n-1}$. We have all $Y'_{\mathbf{m}} \in A_{n,p}$ in terms of the linear combinations of powers of the scaled generators $Y'_{\mathbf{e_k}}$. By expanding the powers of the generators, we will have $Y'_{\mathbf{m}}$ in terms of a unique linear combination of the natural basis elements $Y_{\mathbf{j}} \in A_{n,p}$ as

$$\begin{aligned} Y'_{\mathbf{m}} &= \sum_{\mathbf{l}=(l_1,\ldots,l_{n-1})\in\mathbb{Z}_+^{n-1}} \left( a'_{\mathbf{m},\mathbf{l}} \prod_{k=1}^{n-1} Y'^{l_k}_{\mathbf{e_k}} \right) \\ &= \sum_{\mathbf{j}=(j_1,\ldots,j_{n-1})\in\mathbb{Z}_+^{n-1}} \left( \beta_{\mathbf{m},\mathbf{j}} Y_{\mathbf{j}} \right). \end{aligned}$$

We need to show that the 1-norm of $Y'_{\mathbf{m}}$, $\left\|Y'_{\mathbf{m}}\right\|_1 = \sum_{\mathbf{j}\in\mathbb{Z}_+^{n-1}} \left|\beta_{\mathbf{m},\mathbf{j}}\right|$ is uniformly bounded by some conditions on powers of $p$.

There exist linear relations between the values of $\beta_{\mathbf{m},\mathbf{j}}$. Such relations are obtained from the adjacency equations of $Y'_{\mathbf{m}} Y'_{\mathbf{e_k}} = \frac{1}{\binom{n}{k}} \sum_{\mathbf{m}'\sim_k \mathbf{m}} Y'_{\mathbf{m}'}$, where the coefficients for the expansions into the linear combinations of $Y_{\mathbf{j}}$ need to be equal for all $\mathbf{j} \in \mathbb{Z}_+^{n-1}$. Note that the value of $\beta_{\mathbf{m},\mathbf{j}}$ is always zero if $\mathbf{j}$ is not of the same type with $\mathbf{m}$.

Define the $\Lambda_{n,p}$-weight function on $\mathbb{Z}_+^{n-1}$ by

$$\begin{aligned} \|\mathbf{m}\|_{\Lambda_{n,p}} &= p^{\sum_{j=1}^{n-1}\left(\sum_{k=j}^{n-1}(j-\frac{n+1}{2})m_k\right)} \\ &= \|Z_{\mathbf{m}}\|_{\omega_R} \end{aligned}$$

when $R = p^{\frac{n-1}{2}}$.

By solving the linear equations of the adjacency relations, we can obtain all values of $\beta_{\mathbf{m},\mathbf{j}}$ explicitly as

$$\beta_{\mathbf{m},\mathbf{j}} = \check{\alpha}_{\mathbf{m},\mathbf{j}} \frac{\|\mathbf{m}\|_{\Lambda_{n,p}}}{\|\mathbf{m}-\mathbf{j}\|^2_{\Lambda_{n,p}}} = \check{\alpha}_{\mathbf{m},\mathbf{j}} \frac{\|Z_{\mathbf{m}}\|_{\omega_R}}{\left\|Z_{\mathbf{m}-\mathbf{j}}\right\|^2_{\omega_R}},$$

where $\check{\alpha}_{\mathbf{m},\mathbf{j}}$ is a constant which is determined only by whether $\mathbf{m}$ is on any boundary of the Weyl chamber and whether there are any zero entries in the vector $\mathbf{m}-\mathbf{j}$. Note that all $\check{\alpha}_{\mathbf{m},\mathbf{j}}$ are bounded by $\frac{\beta_{(1,1,\ldots,1),(1,1,\ldots,1)}}{\left\|(1,1,\ldots,1)\right\|_{\Lambda_{n,p}}}$. We note that $\mathbf{j} \in \Delta\mathbf{m}$ if $\beta_{\mathbf{m},\mathbf{j}} \neq 0$.

For $\mathbf{m} = \sum_{k=1}^{n-1} m_k \mathbf{e}_k$, the 1-norm of $Y'_{\mathbf{m}}$ can be computed by

$$\begin{aligned}
\left\|Y'_{\mathbf{m}}\right\|_1 &= \sum_{\mathbf{j}\in\mathbb{Z}_+^{n-1}} \left|\beta_{\mathbf{m},\mathbf{j}}\right| \\
&\leq \frac{\beta_{(1,1,\ldots,1),(1,1,\ldots,1)}\|\mathbf{m}\|_{\Lambda_{n,p}}}{\left\|(1,1,\ldots,1)\right\|_{\Lambda_{n,p}}} \sum_{\mathbf{j}\in\Delta\mathbf{m}} \frac{1}{\|\mathbf{m}-\mathbf{j}\|^2_{\Lambda_{n,p}}} \\
&\leq \frac{\beta_{(1,1,\ldots,1),(1,1,\ldots,1)}\|\mathbf{m}\|_{\Lambda_{n,p}}}{\left\|(1,1,\ldots,1)\right\|_{\Lambda_{n,p}}} \prod_{k=1}^{n-1} \left(\sum_{i_k=0}^{m_k} \frac{1}{\|i_k\mathbf{e}_k\|^2_{\Lambda_{n,p}}}\right) \\
&\leq \frac{\beta_{(1,1,\ldots,1),(1,1,\ldots,1)}\|\mathbf{m}\|_{\Lambda_{n,p}}}{\left\|(1,1,\ldots,1)\right\|_{\Lambda_{n,p}}} \prod_{k=1}^{n-1} \left(\sum_{i_k=0}^{\infty} \frac{1}{\|i_k\mathbf{e}_k\|^2_{\Lambda_{n,p}}}\right),
\end{aligned}$$

which is the product of $n-1$ sums of bounded geometric sequences. Thus, when $R = p^{(n-1)/2}$, there is a constant $K_{n,p}$ such that $\left\|Y'_{\mathbf{m}}\right\| \leq K_{n,p}\|\mathbf{m}\|_{\Lambda_{n,p}} = K_{n,p}\|Z_{\mathbf{m}}\|_{\omega_R}$ for all $\mathbf{m} \in \mathbb{Z}_+^{n-1}$.

Since the construction of $Y'_{\mathbf{m}}$ is unique for all $\mathbf{m} \in \mathbb{Z}_+^{n-1}$, we can alternatively obtain another basis of the algebra $A_{n,p}$, given by the set of elements $\left\{\frac{Y'_{\mathbf{m}}}{\|\mathbf{m}\|_{\Lambda_{n,p}}}\right\}$ where $\mathbf{m} \in \mathbb{Z}_+^{n-1}$.

Define a linear map $\theta_{p,R} : A_{n,p} \longrightarrow \mathcal{A}_{n,\omega_R}$ by

$$\theta_{p,R}\left(Y'_{\mathbf{m}}\right) = Z_{\mathbf{m}}, \text{ for all } \mathbf{m} \in \mathbb{Z}_+^{n-1}.$$

Since both $Y'_{\mathbf{m}}$ and $Z_{\mathbf{m}}$ are constructed by induction from the generators, they satisfy the same adjacency relations which are given by the type-$k$ adjacency equation in the two algebras. Thus we can check that the linear map $\theta_{p,R}$ is a homomorphism. Conjecture 5.21 would now follow if we knew that $\theta_{p,R}$ is surjective. This would require an upper bound on $\|Z_{\mathbf{m}}\|_{\omega_R}$ by some constant multiple of $\left\|Y'_{\mathbf{m}}\right\|$.

# Chapter 6

# Algebras with weighted norms and cohomology

In this chapter, we study the algebras of summable functions on weighted type $\tilde{A}$ and type $\tilde{B}$ lattices where the characteristic functions on the lattice points are generated by the corresponding Laurent polynomials. The weight conditions on the lattices are related to the size of different orbits on the $p$-adic buildings in Chapter 5. We also study the invariant subalgebras under the corresponding Weyl group actions on the components of the coordinates on the lattices. We clarify the character space for both cases, with or without the invariance conditions, and study the existence of point derivations.

For the higher cohomology groups of the algebras of functions on weighted lattices, we start with the algebras with single generator; e.g., summable functions on $\mathbb{Z}_+$ and $\mathbb{Z}$, with or without invariance conditions, with weighted or unweighted $\ell^1$ norm. We look at some classical methods to explicitly find the second cohomology groups and derive a inductive process on an example with the invariance condition under the weighted $\ell^1$ norm. Finally we generalize the method for higher simplicial and cyclic cohomology groups on the invariant subalgebras of functions on the weighted lattices with multi-generators.

## 6.1 Algebras of functions on weighted type $\tilde{A}$ lattices

In this section, we consider $\ell^1(\Lambda_{n-1}, \omega_R)$, the algebra of summable functions on the weighted $\tilde{A}_{n-1}$ lattice $\Lambda_{n-1}$ and the $S_n$-invariant subalgebra $\mathcal{A}_{n,\omega_R} = \ell^1(\Lambda_{n-1}, \omega_R)^{S_n}$. We will discuss the space of characters of the two algebras and the existence of point derivations. In addition, we find the space of simplicial derivations for $\mathcal{A}_{2,\omega_R}$, which is isomorphic to the algebra $A_q$ for the infinite homogeneous tree $\mathbb{T}_q$ in Chapter 4.

We recall the definitions of the algebra $\ell^1(\Lambda_{n-1}, \omega_R)$ and the invariant subalgebra

$\mathcal{A}_{n,\omega_R}$ from Section 5.5:

$$\ell^1(\Lambda_{n-1},\omega_R)=\left\{\sum_{\underline{a}\in\Lambda_{n-1}}\left(\check{f}(\underline{a})\prod_{i=1}^{n}z_i^{a_i}\right)\middle|\prod_{i=1}^{n}z_i=1,\left\|\check{f}\right\|_{\omega_R}<\infty\right\},$$

$$\mathcal{A}_{n,\omega_R}=\ell^1(\Lambda_{n-1},\omega_R)^{S_n}=\left\{\sum_{\mathbf{m}\in\mathbb{Z}_+^{n-1}}\left(f(\mathbf{m})Z_{\mathbf{m}}\right)\middle|\,\|f\|_{\omega_R}<\infty\right\}.$$

Given $n$ variables with fixed product 1, the basis of the algebra without the invariance condition $\ell^1(\Lambda_{n-1},\omega_R)$, is given by the Laurent monomials of the $n$ variables, $\underline{z}^{\underline{a}}=\prod_{j=1}^{n}z_j^{a_j}$ with $\prod_{j=1}^{n}z_j=1$ and $\underline{a}\in\Lambda_{n-1}$. The invariant subalgebra $\mathcal{A}_{n,\omega_R}$ is isomorphic to the Hecke algebra $A_{n,p}$ which corresponds to the building from the $p$-adic general linear groups, and thus is generated by the $(n-1)$ elementary symmetric polynomials of the $n$ variables. We obtain the space of characters of $\mathcal{A}_{n,\omega_R}$ as

$$M_{\mathcal{A}_{n,\omega_R}}=\left\{\left\{\mu_j\right\}_{j=1}^{n}\middle|\prod_{j=1}^{n}\mu_j=1,R^{-1}\le\left|\mu_j\right|\le R\text{ for }j=1,\dots,n\right\}$$

The characters on the generators $Z_{\mathbf{e}_i}$ are given by elementary symmetric polynomials of the set of $n$ numbers $\left\{\mu_j\right\}_{j=1}^{n}$ as

$$\chi_{\underline{\lambda}}\left(Z_{\mathbf{e}_i}\right)=\lambda_i=\sigma_i\left(\mu_1,\dots,\mu_n\right)\text{ for }i=1,\dots n-1.$$

The characters of the algebra $\ell^1(\Lambda_{n-1},\omega_R)$ are parametrized by $n$ ordered points on an annulus with fixed product 1. The space of characters of $\ell^1(\Lambda_{n-1},\omega_R)$ is given by

$$M_{\ell^1(\Lambda_{n-1},\omega_R)}=\left\{(\mu_1,\dots,\mu_n)\middle|\prod_{j=1}^{n}\mu_j=1,R^{-1}\le\left|\mu_j\right|\le R\text{ for }j=1,\dots,n\right\}.$$

The character $\tilde{\chi}_\mu$ on each variable $z_j$ is given by $\tilde{\chi}_\mu\left(z_j\right)=\mu_j$.

It can be shown that a character $\chi_\mu$ is determined by the sequence $\left\{\chi_\mu\left(z_j\right)\right\}_{j=1}^{n}$ with a condition imposed for the product of the $n$ values to be fixed. The set of analytic functions which determine a character indicate that every character is mapped to a set of $n$ points on an annulus on the complex plane with the fixed product.

The interior of the character space of the algebra $\ell^1\left(\Lambda_{n-1},\omega_R\right)$ is obtained when all $n$ points are not on the internal or external circle of the annulus. The topological boundary is obtained when at least one point is on the internal or external circle whereas the distinguished boundary is achieved when there are precisely $\lfloor\frac{n}{2}\rfloor$ points on the internal circle and $\lfloor\frac{n}{2}\rfloor$ points on the external circle of the annulus.

**Lemma 6.1.** *Let $\chi_{\underline{\mu}}$ be a character on $\ell^1(\Lambda_{n-1}, \omega_R)$ where $\chi(z_j) = \mu_j$ for all $j = 1, \ldots, n$. If $\chi_{\underline{\mu}}$ is not on the distinguished boundary of $M_{\ell^1(\Lambda_{n-1}, \omega_R)}$, the set of point derivations $\mathcal{H}^1\left(\ell^1(\Lambda_{n-1}, \omega_R), \mathbb{C}_{\underline{\mu}}\right)$ is in an $(n-1)$-dimensional subspace of the linear span of $\langle D_{\mu_1}, D_{\mu_2}, \ldots, D_{\mu_n}\rangle$ with $\sum_{j=1}^n \mu_j^{-1} D_{\mu_j} = 0$ and $D_{\mu_j} = D(z_j)$ for $j = 1, \ldots, n$.*

*Proof.* The functions in the algebra $\ell^1(\Lambda_{n-1}, \omega_R)$ are given by the linear sums of monomials $\underline{z}^{\underline{a}}$ where $\underline{a}$ is in the equivalence class of $\mathbb{Z}^n/(1, \ldots, 1)$. Note that the product of the $n$ variables is 1; i.e., $\prod_{j=1}^n z_j = 1$. A point bi-module $\mathbb{C}_\mu$ of the commutative algebra $\ell^1(\Lambda_{n-1}, \omega_R)$ is evaluated by

$$\underline{z}^{\underline{a}} D\left(\underline{z}^{\underline{b}}\right) \underline{z}^{\underline{c}} = \chi_\mu\left(\underline{z}^{\underline{a}}\right) \chi_\mu\left(\underline{z}^{\underline{c}}\right) D\left(\underline{z}^{\underline{b}}\right).$$

A point derivation is a bounded linear function $D : \ell^1(\Lambda_{n-1}, \omega_R) \longrightarrow \mathbb{C}_{\underline{\mu}}$ satisfying $D\left(\underline{z}^{\underline{a}}\underline{z}^{\underline{b}}\right) = \underline{z}^{\underline{a}} D\left(\underline{z}^{\underline{b}}\right) + D\left(\underline{z}^{\underline{a}}\right)\underline{z}^{\underline{b}}$ with $D(1) = 0$. We expand $D(1)$ as

$$\begin{aligned} D(z_1 z_2 \ldots z_n) &= z_2 z_3 \ldots z_n . D(z_1) + z_1 z_3 \ldots z_n . D(z_2) + \ldots + z_1 \ldots z_{n-1} D(z_n) \,\big|_{\mathbb{C}_{\underline{\mu}}} \\ &= \sum_{j=1}^n z_j^{-1} D(z_j) \,\big|_{\mathbb{C}_{\underline{\mu}}} \\ &= \sum_{j=1}^n \mu_j^{-1} D(z_j), \end{aligned}$$

which shows that the $n$ values of $\big(D(z_1), \ldots, D(z_n)\big)$ are actually in an $(n-1)$-dimensional subspace of $\mathbb{C}^n$.

We can always add any integer multiples of $(1, \ldots, 1)$ to a vector $\underline{a}$ such that the monomial $Z_{\underline{a}}$ have positive powers on all $n$ variables. Assume that all entries of $\underline{a}$ are positive, then the linear function $D$ can be expanded as

$$D\left(Z_{\underline{a}}\right) = \chi_{\underline{\mu}}\left(Z_{\underline{a}}\right) \sum_{j=1}^n a_j \mu_j^{-1} D(z_j), \tag{6.1}$$

which shows that the point derivations is determined by the $n$ values of $D_{\mu_j}$.

We will consider the special case where the set of $n$ points of the annulus for a character $\chi_\mu$ is on the topological boundary; i.e., $|\mu_j| = R$ or $R^{-1}$. When $|\mu_j| = R$, the function $D\left(z_j^{a_j}\right) = a_j \mu_j^{a_j - 1} D_{\mu_j}$ is not uniformly bounded by $\left\|z_j^{a_j}\right\|_{\omega_R}$ if $D_{\mu_j}$ is not zero. Since $D\left(z_j^{a_j}\right) = a_j \mu_j^{a_j-1} D_{\mu_j}$, when $|\mu_j| = R$ we have

$$\frac{\left|D\left(z_j^{a_j}\right)\right|}{\left\|z_j^{a_j}\right\|_{\omega_R}} = \frac{|a_j|\left|D_{\mu_j}\right|}{R}.$$

Therefore $\left|D_{\mu_j}\right| \leq R|a_j|^{-1}\|D\|$, for arbitrarily large $a_j$, hence $\left|D_{\mu_j}\right| = 0$. When $|\mu_j| = R^{-1}$,

$$\frac{\left|D\left(z_j^{-a_j}\right)\right|}{\left\|z_j^{-a_j}\right\|_{\omega_R}} = \frac{|a_j|\left|D_{\mu_j}\right|}{R},$$

and so by similar reasoning $\left|D_{\mu_j}\right| = 0$.

Therefore the value of $D_{\mu_j}$ must be zero if $\mu_j$ is on the internal or external circle of the annulus for the character $\chi_{\underline{\mu}}$. When there are maximum number of points on the boundary of the annulus, i.e. when we have a character on the distinguished boundary, all values $D_{\mu_j}$ must be zero. □

For $n = 2$, we consider the invariant subalgebra $\mathcal{A}_{2,\omega_R}$. By Theorem 4.25, when $R = q^{1/2}$, the algebra $\mathcal{A}_{2,\omega_R}$ is isomorphic to the Hecke algebra $A_q$ of the infinite homogeneous tree $\mathbb{T}_q$. The multiplications and change of variables are easier to compute if we consider $\mathcal{A}_{2,\omega_R}$ where the multiplications with the generator corresponds to a balanced random walk on the $\mathbb{Z}_+$ lattice. Explicitly, we have

$$\mathcal{A}_{2,\omega_R} = \ell^1\left(\Lambda_{n-1}, \omega_R\right)^{S_2} = \left\{ f = \sum_{j=0}^{\infty} c_j y_j \,\middle|\, \|f\|_{\omega_R} = \sum_{j=1}^{\infty} |c_j| R^j < \infty \right\}, \tag{6.2}$$

where $y_j = \frac{1}{2}\left(z^j + z^{-j}\right)$ with $y_j = y_{-j}$ and $y_j y_k = \frac{1}{2}\left(y_{j+k} + y_{j-k}\right)$.

**Definition 6.2.** *The simplicial derivations on an algebra $A$ is the set of linear functions $D : A \longrightarrow A^*$ such that*

$$D\left(f_1 f_2\right)\left(f_3\right) = D\left(f_1\right)\left(f_2 f_3\right) + D\left(f_2\right)\left(f_3 f_1\right), \text{ for all } f_1, f_2, f_3 \in A.$$

Since $A_q$ is singly generated by $Y_1$, we can derive that $D(1)(Y_m) = 0$ for all $m \in \mathbb{Z}_+$ and

$$D\left(Y_1^m\right)\left(Y_1^n\right) = \frac{m}{m+n} D\left(Y_1^{m+n}\right)\left(Y_0\right),$$

which shows that $D$ is determined by the sequence $\left\{D\left(Y_1^m\right)(1)\right\}_{m=1}^{\infty}$. It is not straightforward to compute the simplicial derivations of $A_q$ in terms of its natural basis. Since this commutative algebra has one single generator $Y_1$, it is possible to convert all $Y_m$ into a linear sum of powers of $Y_1$. The values of $a_{m,j}$ from Lemma 4.7 and $c_{n,k}$ from Lemma 4.9 will be used in the numerical computation.

Let $\mathcal{HH}^1(A_q)$ be the set of bounded simplicial derivations of $A_q$. The map from $\mathcal{HH}^1(A_q)$ to $A_q^*$ gives a correspondence from the simplicial derivation $D : A_q \longrightarrow A_q^*$ to a sequence $\tau$. Set $\tau_n = D(Y_n)(1)$ We aim to prove that there exists a simplicial derivation

$D : A_q \longrightarrow A_q^*$ if and only if the sequence $\tau$ is bounded. Given a bounded sequence on $\mathbb{Z}_+$ which determines the values of $\{D(Y_n)(1)\}_{n=0}^{\infty}$, we seek the general form of $D(Y_m)(Y_n)$ by converting the terms between the two different methods of expansions as

$$\begin{aligned}
D(Y_m)(Y_n) &= D\left(\sum_{j=0}^{m} a_{m,j}Y_1^j\right)\left(\sum_{k=0}^{n} a_{n,k}Y_1^k\right) \\
&= \sum_{j=0}^{m}\sum_{k=0}^{n}\left(a_{m,j}a_{n,k}\frac{j}{j+k}D(Y_1^{j+k})\,(Y_0)\right) \\
&= \sum_{j=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{\lfloor\frac{j+k}{2}\rfloor}\left(c_{j+k,l}a_{m,j}a_{n,k}\frac{j}{j+k}D(Y_{j+k-2l})\,(Y_0)\right) \\
&= \sum_{i=0}^{m+n} C_{m,n,i}D(Y_i)(Y_0) \\
&= \sum_{i=0}^{m+n} C_{m,n,i}\tau_i,
\end{aligned}$$

where the numerical computation suggests that $\sum_{i=1}^{m+n}|C_{m,n,i}| < 1$.

**Lemma 6.3.** *The simplicial derivation $D$ on the algebra $\mathcal{A}_{2,\omega_R}$ is determined by*

$$\begin{aligned}
D\left(y_a\right)\left(y_b\right) &= \frac{1}{2}\left(\frac{a}{a+b}D\left(y_{a+b}\right)(1) + \frac{a}{a-b}D\left(y_{a-b}\right)(1)\right) \quad \textit{for } a \neq \pm b, \\
D\left(y_n\right)\left(y_n\right) &= \frac{1}{4}D\left(y_{2n}\right)(1).
\end{aligned}$$

*Moreover, if we define $\tau_n = D(y_n)(1)$, then $\|D\| \leq \|\tau\|_\infty$.*

*Proof.* After defining the simplicial derivation in Definition 6.2, we showed that every simplicial derivation $D(y_a)(y_b)$ can be written as a linear sum of $\sum_{l=0}^{a+b} C_{a,b,l}D(y_l)(1)$. However we did not compute the values of the coefficients $C_{a,b,l}$ for the algebra $A_q$ of the infinite homogeneous tree. The boundedness also needs to be clarified in both $A_q$ the isomorphic algebra $\mathcal{A}_{2,\omega_R}$.

Let $\tau$ be a sequence such that $\tau_n = D(y_n)(1)$ for all $n \in \mathbb{Z}_+$. We need to show that for all pairs $(a,b) \in \mathbb{Z}_+^2$, the derivation $D\left(y_a\right)\left(y_b\right)$ is bounded by $R^{a+b}\|\tau\|$.

For $b = 0$, we have $D\left(y_a\right)\left(y_0\right) = D\left(y_a\right)\left(y_1\right)$. For $a = b = n$, it is clear that

$$D\left(y_n\right)\left(y_n\right) + D\left(y_n\right)\left(y_n\right) = D\left(y_ny_n\right)(1) = \frac{1}{2}D\left(y_{2n}\right)(1),$$

which implies that $D\left(y_n\right)\left(y_n\right) = \frac{1}{4}D\left(y_{2n}\right)(1)$.

From $D\left(y_ay_b\right)(1) = D\left(y_a\right)\left(y_b\right) + D\left(y_b\right)\left(y_a\right)$, we will know the value of $D\left(y_b\right)\left(y_a\right)$ once we know the value of $D\left(y_a\right)\left(y_b\right)$. Therefore we only need to find values of $D\left(y_a\right)\left(y_b\right)$ for all positive integer pairs where $b > a \geq 1$.

We then consider the derivation $D(y_m)(y_1)$. For $m \geq 2$, using the coefficients of $a_{m,j}$ from Lemma 4.7 with $q = 1$, the expansion gives

$$\begin{aligned}
&D(y_m)(y_1)\\
&= D\left(\sum_{j=0}^{m} a_{m,j} y_1^j\right)(y_1)\\
&= D\left(\sum_{j=0}^{m} \frac{j}{j+1} a_{m,j} y_1^{j+1}\right)(1)\\
&= D\left(\sum_{j=0}^{m} \frac{1}{2}\left(\frac{m}{m+1} a_{m+1,j+1} + \frac{m}{m-1} a_{m-1,j+1}\right) y_1^{j+1}\right)(1)\\
&= \frac{1}{2}\left(\frac{m}{m+1} D(y_{m+1})(1) + \frac{m}{m-1} D(y_{m-1})(1)\right).
\end{aligned}$$

For $N = a + b = 3$, the only case is $a = 2$ and $b = 1$, which is included in the previous calculation with $m = 2$.

We assume that $D(y_a)(y_b) = \frac{1}{2}\left(\frac{a}{a+b} D(y_{a+b})(1) + \frac{a}{a-b} D(y_{a-b})(1)\right)$ is true for all $b > a \geq 1$ and $a + b \leq N$. We have shown that it is true for all $a \geq 2$ with $b = 1$ and we can obtain the values of all $D(y_1)(y_a)$ from the values of $D(y_a)(y_1)$. For $a + b = N + 1$, we assume that it is true for all $1 \leq b \leq n$. When $m - n \neq 2$, for $a = m - 1$ and $b = n + 1$, we have

$$
\begin{aligned}
& D\left(y_{m-1}\right)\left(y_{n+1}\right) \\
= & D\left(y_{m-1}\right)\left(2 y_n y_1-y_{n-1}\right) \\
= & D\left(y_{m-1}\right)\left(2 y_n y_1\right)-D\left(y_{m-1}\right)\left(y_{n-1}\right) \\
= & D\left(2 y_n y_{m-1}\right)\left(y_1\right)-D\left(y_n\right)\left(2 y_1 y_{m-1}\right)-D\left(y_{m-1}\right)\left(y_{n-1}\right) \\
= & D\left(y_{m+n-1}\right)\left(y_1\right)+D\left(y_{m-n-1}\right)\left(y_1\right)-D\left(y_n\right)\left(y_m\right)-D\left(y_n\right)\left(y_{m-2}\right)-D\left(y_{m-1}\right)\left(y_{n-1}\right) \\
= & \frac{1}{2}\left(\frac{m+n-1}{m+n} D\left(y_{m+n-2}\right)(1)+\frac{m+n-1}{m+n-2} D\left(y_{m+n-2}\right)(1)\right) \\
& +\frac{1}{2}\left(\frac{m-n-1}{m-n} D\left(y_{m-n}\right)(1)+\frac{m-n-1}{m-n-2} D\left(y_{m-n-2}\right)(1)\right) \\
& -\frac{1}{2}\left(\frac{n}{m+n} D\left(y_{m+n}\right)(1)+\frac{n}{n-m} D\left(y_{m-n}\right)(1)\right) \\
& -\frac{1}{2}\left(\frac{n}{m+n} D\left(y_{m+n}\right)(1)+\frac{n}{n-m} D\left(y_{m-n}\right)(1)\right) \\
& -\frac{1}{2}\left(\frac{n}{m+n-2} D\left(y_{m+n-2}\right)(1)+\frac{n}{n-m+2} D\left(y_{n-m+2}\right)(1)\right) \\
& -\frac{1}{2}\left(\frac{m-1}{m+n-2} D\left(y_{m+n-2}\right)(1)+\frac{m-1}{m-n} D\left(y_{m-n}\right)(1)\right) \\
= & \frac{1}{2}\left(\frac{m-1}{m+n} D\left(y_{m+n}\right)(1)+\frac{m-1}{m-n-2} D\left(y_{m-n-2}\right)(1)\right) .
\end{aligned}
$$

The first term $\frac{m-1}{m+n} D\left(y_{m+n}\right)(1)$ is clearly bounded by $\|\tau\| R^{m+n}$. The maximum value of $\frac{m-1}{m-n-2}\left\|y_{m-n-2}\right\|$ is achieved when $m-n-2=\log R$. Hence the norm of the second term is uniformly bounded by $(m-1) e\|\tau\|$. Therefore we have

$$
\left|\frac{\frac{m-1}{m-n-2} D\left(y_{m-n-2}\right)(1)}{\left\|y_{m+1}\right\|\left\|y_{n-1}\right\|}\right| \leq e\|\tau\| m R^{-m} \leq \frac{e R^{-\frac{1}{\log R}}}{\log R}\|\tau\|=C\|\tau\|,
$$

which indicates that $D\left(y_{m-1}\right)\left(y_{n+1}\right)$ is bounded by $\frac{1}{2}(1+C)\|\tau\| R^{m+n}$.

Hence for all positive integer pairs $(a, b)$, the simplicial derivation is bounded if and only if the sequence $\tau$ is bounded. □

**Remark 6.4.** *The isomorphism between $A_q$ and $\mathcal{A}_{2, \omega_R}$ shows that there indeed exists a simplicial derivation for $A_q$, as well as the Hecke algebra $A_{2,p}$ with $\ell^1$ norm of the Gelfand pair $\left(PGL_2\left(\mathbb{Q}_p\right), PGL_2\left(\mathbb{Z}_p\right)\right)$.*

## 6.2 Algebras of functions on weighted type $\tilde{B}$ lattices

In this section, we consider $\ell^1(\mathbb{Z}^n, \omega_R)$, the algebra of functions on the $\mathbb{Z}^n$ integer lattice with respect to the $\omega_R$-weighted $\ell^1$ norm. We assume that the characteristic functions multiply as the Laurent polynomials of $n$ independent variables. In particular, we compute the invariant subalgebra $\mathcal{B}_{n,\omega_R} = \ell^1(\mathbb{Z}^n, \omega_R)^{B_n}$ where the functions on the lattice are invariant under the action of the Coxeter group $B_n$ which acts on the $n$-hypercube. We start from $n = 1$ and $n = 2$ and then generalize to higher dimensions by finding the space of characters and study the existence of the point derivations of the two algebras.

Given a point $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ on the integer grid, we express the characteristic function on at the point $\underline{a}$ as the Laurent monomial $\underline{z}^{\underline{a}} = \prod_{j=1}^n z_j^{a_j}$.

**Definition 6.5.** *The $\omega_R$ norm of a characteristic function $\underline{z}^{\underline{a}}$ on the $\mathbb{Z}^n$ integer lattice is given by a weighted $\ell^1$ norm as*

$$\|\underline{z}^{\underline{a}}\|_{\omega_R} = R^{\sum_{i=1}^n |a_i|}.$$

The Weyl group $B_n$ acts on the $\mathbb{Z}^n$ integer lattice by alternating the signs and permuting the $n$ entries of the coordinates. We have $|B_n| = 2^n n!$. The $\omega_R$ norm of a characteristic function $\|\underline{z}^{\underline{a}}\|_{\omega_R}$ is fixed under the $B_n$ action on the coordinates.

**Definition 6.6.** *The algebra of summable functions on the weighted $\mathbb{Z}^n$ integer lattice is given by*

$$\tilde{\mathcal{B}}_{n,\omega_R} = \ell^1(\mathbb{Z}^n, \omega_R) = \left\{ \tilde{f} = \sum_{\underline{a} \in \mathbb{Z}^n} c_{\underline{a}} \underline{z}^{\underline{a}} \;\middle|\; \left\|\tilde{f}\right\|_{\omega_R} < \infty \right\}.$$

For all $i = 1, \ldots, n$, a character $\chi_{\underline{\mu}}$ of the algebra $\ell^1(\mathbb{Z}^n, \omega_R)$ is given by

$$\chi_{\underline{\mu}}(z_i) = \mu_i,$$
$$\chi_{\underline{\mu}}\left(z_i^{-1}\right) = \mu_i^{-1},$$
$$\chi_{\underline{\mu}}(\underline{z}^{\underline{a}}) = \prod_{i=1}^n \mu_i^{a_i}.$$

For the characters to be bounded, we require $R^{-1} \leq |\mu_i| \leq R$ for all $i = 1, \ldots, n$. Thus the character space is parametrized by $n$ ordered points on an annulus where the radii are $R^{-1}$ and $R$ for the internal and external circle on the complex plane as

$$M_{\ell^1(\mathbb{Z}^n,\omega_R)} = \left\{ (\mu_1, \ldots, \mu_n) \middle| R^{-1} \leq |\mu_i| \leq R \text{ for all } j = 1, \ldots, n \right\}.$$

We consider the $B_n$-invariant subalebgra $\mathcal{B}_{n,\omega_R} = \ell^1(\mathbb{Z}^n, \omega_R)^{B_n}$. The values of the functions on the vertices of $\mathbb{Z}^n$ are invariant if and only if the coordinates of these points can be obtained from each other by alternating the signs and permuting the entries. There are $2^n n!$ Weyl chambers on the $\mathbb{Z}^n$ integer lattice. We pick a specific chamber which consists of the vertices $\underline{a} \in \mathbb{Z}^n$ such that $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$. Every vertex in this chamber corresponds to a vector in $\mathbb{Z}_+^n$ where the entries are in monotonic descending order. We use another vector $\mathbf{m} \in \mathbb{Z}_+^n$ which corresponds to a unique element in this chamber, given by $\mathbf{m} = (a_1 - a_2, a_2 - a_3, \ldots, a_{n-1} - a_n, a_n)$. The basis of $\mathcal{B}_{n,\omega_R}$ is given by the set of $B_n$-symmetric Laurent polynomials, denoted by $Z_{\mathbf{m}}$ or $Z_{\underline{a}}$ as

$$Z_{\mathbf{m}} = Z_{\underline{a}} = \frac{1}{|B_n|} \sum_{g \in B_n} \underline{z}^{\underline{a}^g}.$$

with $\left\|Z_{\underline{a}}\right\| = \left\|\underline{z}^{\underline{a}^g}\right\|$ for all $g \in B_n$. In particular, we have $\mathcal{B}_{1,\omega_R} = \mathcal{A}_{2,\omega_R}$.

Explicitly, the algebra $\mathcal{B}_{n,\omega_R}$ can be considered as summable functions on the lattice points of a Weyl chamber of the $\mathbb{Z}^n$ lattice. We denote the characteristic functions on the lattice points of the specific Weyl chamber by the $B_n$-symmetric Laurent polynomials from the algebra $\ell^1(\mathbb{Z}^n, \omega_R)$. We have

$$\mathcal{B}_{n,\omega_R} = \left\{ f = \sum_{\mathbf{m} \in \mathbb{Z}_+^n} c_{\mathbf{m}} Z_{\mathbf{m}} \;\middle|\; \|f\|_{\omega_R} < \infty \right\}.$$

**Proposition 6.7.** *Every $B_n$-symmetric Laurent polynomial $Z_{\mathbf{m}} \in \mathcal{B}_{n,\omega_R}$ is generated by the elementary symmetric polynomials of the set of polynomials*

$$\left\{ \left(z_1 + z_1^{-1}\right), \ldots, \left(z_n + z_n^{-1}\right) \right\},$$

*namely*

$$\sigma_j^{\pm} = \sigma_j \left( \left(z_1 + z_1^{-1}\right), \left(z_2 + z_2^{-1}\right), \ldots, \left(z_n + z_n^{-1}\right) \right), \quad 1 \leq j \leq n;$$

*i.e., every symmetric polynomial $Z_{\mathbf{m}}$ can be uniquely written as a linear sum of powers of the elementary symmetric polynomials as*

$$Z_{\mathbf{m}} = \mathcal{P}_{\mathbf{m}} \left(\sigma_1^{\pm}, \ldots, \sigma_n^{\pm}\right) = \sum_{\mathbf{j} \in \mathbb{Z}_+^n} \left( a_{\mathbf{m},\mathbf{j}} \prod_{k=1}^{n} \left(\sigma_k^{\pm}\right)^{j_k} \right).$$

*Proof.* The elementary symmetric polynomials $\sigma_j^{\pm}$ are in fact the element $Z_{\mathbf{e_j}} \in \mathcal{B}_{n,\omega_R}$, where $\{\mathbf{e_j}\}_{j=1}^n$ are the canonical basis of the $\mathbf{m}$ vectors in $\mathbb{Z}_+^n$. Let $Z_{\underline{a}} = Z_{\mathbf{m}}$ where $\mathbf{m} = (a_1 - a_2, a_2 - a_3, \ldots, a_{n-1} - a_n, a_n)$. Given any $\underline{a} \in \mathbb{Z}_+^n$ and the corresponding $\mathbf{m} \in \mathbb{Z}_+^n$, we expand the product $\prod_{k=1}^n \left(\sigma_k^{\pm}\right)^{m_k}$ and observe that the term with the highest power is indeed $Z_{\mathbf{m}}$ and the coefficient of this leading term is fixed to be 1. For $\underline{a} = (a_1, \ldots, a_n)$ and $\underline{b} = (b_1, \ldots, b_n)$, the expansion is given by

$$\begin{aligned}\prod_{k=1}^{n} \left(\sigma_k^{\pm}\right)^{m_k} &= Z_{\mathbf{m}} + \text{lower terms}\\ &= Z_{\underline{a}} + \sum_{\underline{b} \in \mathbb{Z}_+^n} \gamma_{\underline{a},\underline{b}} Z_{\underline{b}},\end{aligned}$$

where the coefficient $\gamma_{\underline{a},\underline{b}}$ is nonzero only when $b_i \leq a_i$ for all entries of $\underline{a}$ and $\underline{b}$. This step can be repeated until there are no other terms apart from the elementary symmetric polynomials. Then we can recover the symmetric polynomial $Z_{\mathbf{m}}$ in terms of a linear sum of powers of the elementary symmetric polynomials. The coefficients in this construction are fixed as we have no choice for the leading term thus have no alternative route for the repeated steps of the induction. □

We apply some concepts of the *symmetrized bi-disc* from [2] The symmetrized bi-disc is a set of points in $\mathbb{C}^2$, given by

$$\Gamma = \left\{(z+w, zw) \;\middle|\; |z| \leq 1, |w| \leq 1\right\} \subset \mathbb{C}^2.$$

The *topological boundary* of $\Gamma$ is achieved when either $z$ or $w$ has modulus 1. The *distinguished boundary* of $\Gamma$ is achieved when both $z$ and $w$ have modulus 1. The *royal variety* of $\Gamma$ is achieved when $z = w$.

**Lemma 6.8.** *The character space of $\mathcal{B}_{2,\omega_R}$, $M_{\mathcal{B}_{2,\omega_R}}$ is parametrized by the symmetrized bi-disc.*

*Proof.* The two elementary symmetric polynomials that generate this subalgebra are

$$\begin{aligned}\sigma_1^{\pm} &= Z_{1,0} = \left(z_1 + z_1^{-1}\right) + \left(z_2 + z_2^{-1}\right),\\ \sigma_2^{\pm} &= Z_{1,1} = z_1 z_2 + z_1^{-1} z_2 + z_1 z_2^{-1} + z_1^{-1} z_2^{-1} = \left(z_1 + z_1^{-1}\right)\left(z_2 + z_2^{-1}\right).\end{aligned}$$

Every character $\chi_{\underline{\lambda}}$ is determined by $\lambda_1 = \chi_{\underline{\lambda}}\left(\sigma_1^{\pm}\right)$ and $\lambda_2 = \chi_{\underline{\lambda}}\left(\sigma_2^{\pm}\right)$, which can be considered as the sum and the product of the pair $\left\{\left(z_1 + z_1^{-1}\right), \left(z_2 + z_2^{-1}\right)\right\}$.

As described in the examples of $A_q$ and $\mathcal{A}_{2,\omega_R}$, the character space of $A_q$ and $\mathcal{A}_{2,\omega_R}$, are both parametrized by an ellipse on the complex plane, which is homeomorphic to a unit disc. The boundary of the ellipse is achieved when $|z|$ or $|z^{-1}|$ is on the boundary of the annulus.

Therefore the character space of $\mathcal{B}_{2,\omega_R}$ is parametrized by the symmetrized bi-ellipse. Both ellipses are homeomorphic to the unit disc which implies that this character space is also parametrized by the symmetrized bi-disc. □

**Remark 6.9.** *In general, it can be proved that the character space of* $\mathcal{B}_{n,\omega_R}$*, is parametrized by the symmetrized* $n$*-disc*

$$\Gamma_n = \left\{ \left(\sigma_1\left(z_1,\ldots,z_n\right),\ldots,\sigma_n\left(z_1,\ldots,z_n\right)\right) \;\middle|\; |z_j| \leq 1 \text{ for all } j=1,\ldots,n \right\} \subset \mathbb{C}^n,$$

*which is homeomorphic to a set of* $n$ *unordered ellipses thus corresponds to the set of* $n$ *unordered points on an annulus of the complex plane.*

Let $\chi_{\underline{\lambda}}$ be a character on $\mathcal{B}_{n,\omega_R}$, the character on the elementary symmetric Laurent polynomials are evaluated as

$$\chi_{\underline{\lambda}}\left(\sigma_j^{\pm}\right) = \lambda_j = \sigma_j\left(\left(\mu_1+\mu_1^{-1}\right),\ldots,\left(\mu_n+\mu_n^{-1}\right)\right)$$

Similar to the point derivations of the algebra of functions on the type $\tilde{A}$ lattice, a point derivation $D \in \mathcal{H}^1\left(\ell^1\left(\mathbb{Z}^n,\omega_R\right),\mathbb{C}_{\underline{\lambda}}\right)$ for a character $\chi_{\underline{\lambda}}$, is determined by the $n$ values $\left(D(z_1),\ldots,D(z_n)\right)$. If the modulus of $\mu_j$ is equal to $R$ or $R^{-1}$, then we have $D(z_j)=0$.

We will now calculate the point derivations on the $B_n$-invariant subalgebra $\mathcal{B}_{n,\omega_R}$ by computing the derivative of the polynomial of the generators by the chain rule.

**Lemma 6.10.** *Given a character* $\chi_{\underline{\lambda}}$ *not on the distinguished boundary of the character space, the set of point derivations* $\mathcal{H}^1\left(\mathcal{B}_{n,\omega_R},\mathbb{C}_{\underline{\lambda}}\right)$ *is given by the linear span of* $\langle D_{\lambda_1}, D_{\lambda_2},\ldots,D_{\lambda_n}\rangle$ *with* $D\left(Z_{\mathbf{e_j}}\right) = D_{\lambda_j}$ *for all* $j=1,\ldots,n$*.*

*Proof.* For a character $\chi_{\underline{\lambda}}$ in the interior of the character space, i.e. $R > |\mu_i| > R^{-1}$ for all $j=1,2,\ldots,n$, we consider the unique expressions of the character as a polynomial of $\lambda_1,\lambda_2,\ldots,\lambda_n$, $\chi_{\underline{\lambda}}(Z_{\mathbf{m}}) = \mathcal{P}_{\mathbf{m}}\left(\lambda_1,\lambda_2,\ldots,\lambda_n\right)$.

We define a point derivation $D_{\lambda_j} : \mathcal{B}_{n,\omega_R} \longrightarrow \mathbb{C}_{\underline{\lambda}}$ by

$$D_{\lambda_j}\left(Z_{\mathbf{m}}\right) := \frac{d}{dy_j}\left(\mathcal{P}_{\mathbf{m}}\left(y_1,y_2,\ldots,y_n\right)\right)\Big|_{y_1=\lambda_1,\ldots,y_n=\lambda_n} D_{\lambda_j},$$

for all $j=1,2,\ldots,n$. This shows that every point derivation $D$ is determined by the set of values $\left\{D_{\lambda_j}\right\}_{j=1}^{n}$. We need to find the condition for the boundedness for all characters $\chi_{\underline{\lambda}}$ in the character space.

We consider the symmetric polynomial which corresponds to $Z_{\underline{a}}$ as a homogeneous polynomial with positive powers of $2n$ variables. namely $f_{\underline{a}}\left(z_1,\ldots,z_n,z_1^{-1},\ldots,z_n^{-1}\right)$. For

$n = 2$,

$$\begin{aligned} Z_{\underline{a}} =& f_{\underline{a}}\left(z_1, z_2, z_1^{-1}, z_2^{-1}\right) \\ =& \quad z_1^{a_1} z_2^{a_2} + \left(z_1^{-1}\right)^{a_1} z_2^{a_2} + z_1^{a_1}\left(z_2^{-1}\right)^{a_2} + \left(z_1^{-1}\right)^{a_1}\left(z_2^{-1}\right)^{a_2} \\ &+ z_1^{a_2} z_2^{a_1} + \left(z_1^{-1}\right)^{a_2} z_2^{a_1} + z_1^{a_2}\left(z_2^{-1}\right)^{a_1} + \left(z_1^{-1}\right)^{a_2}\left(z_2^{-1}\right)^{a_1} \end{aligned}$$

We set $y_j = \sigma_j\left(\left(x_1 + x_{-1}\right), \ldots, \left(x_n + x_{-n}\right)\right)$, the $j$-th elementary symmetric polynomial of the $n$ sums of pairs $\left(x_1 + x_{-1}\right), \ldots, \left(x_n + x_{-n}\right)$. For a continuous function $f_{\underline{a}}$ on the $2n$ variables $x_1, \ldots, x_n, x_{-1}, \ldots, x_{-n}$, we have

$$f_{\underline{a}}\left(\underline{x}_{\pm}\right) = f_{\underline{a}}\left(x_1, \ldots, x_n, x_{-1}, \ldots, x_{-n}\right) = \mathcal{P}_{\mathbf{m}}\left(y_1, y_2, \ldots, y_n\right).$$

The derivative can also be evaluated as

$$\begin{aligned} & \frac{d}{dy_j}\left(\mathcal{P}_{\mathbf{m}}\left(y_1, y_2, \ldots, y_n\right)\right)\Big|_{y_1=\lambda_1,\ldots,y_n=\lambda_n} \\ =& \quad \frac{d}{dy_j} f_{\underline{a}}\left(\underline{x}_{\pm}\right)\Big|_{x_1=\mu_1,\ldots,x_n=\mu_n,x_{-1}=\mu^{-1},\ldots,x_{-n}=\mu_n^{-1}} \\ =& \sum_{k=1}^{n} \frac{dx_k}{dy_j}\frac{\partial}{\partial x_k} f_{\underline{a}}\left(\underline{x}_{\pm}\right)\Big|_{x_1=\mu_1,\ldots,x_n=\mu_n,x_{-1}=\mu^{-1},\ldots,x_{-n}=\mu_n^{-1}} \\ &+ \sum_{k=1}^{n} \frac{dx_{-k}}{dy_j}\frac{\partial}{\partial x_{-k}} f_{\underline{a}}\left(\underline{x}_{\pm}\right)\Big|_{x_1=\mu_1,\ldots,x_n=\mu_n,x_{-1}=\mu^{-1},\ldots,x_{-n}=\mu_n^{-1}} \\ =& \sum_{g\in B_n} \frac{d\left(x_{g(1)}\ldots x_{g(j)}\right)}{dy_j}\frac{\partial}{\partial\left(x_{g(1)}\ldots x_{g(j)}\right)} f_{\mathbf{m}}\left(\underline{x}_{\pm}\right)\Big|_{x_1=\mu_1,\ldots,x_n=\mu_n,x_{-1}=\mu^{-1},\ldots,x_{-n}=\mu_n^{-1}}. \end{aligned}$$

Note that none of the values of $\frac{dx_k}{dy_j}$ and $\frac{dx_{-k}}{dy_j}$ depend on the choice of $\mathbf{m}$ or $\underline{a}$. When $R^{-1} < |\mu_k| < R$ for all $k = 1, \ldots, n$, all partial derivatives $\frac{\partial}{\partial x_k} f_{\underline{a}}\left(\underline{x}_{\pm}\right)$ and $\frac{\partial}{\partial x_{-k}} f_{\underline{a}}\left(\underline{x}_{\pm}\right)$ are uniformly bounded by $\left\|Z_{\underline{a}}\right\|_{\omega_R}$.

We consider the case for the character $\chi_{\underline{\lambda}}$ on the topological boundary of the character space. When the corresponding unordered set of points on the annulus, $\{\mu_1, \ldots, \mu_n\}$ has precisely $k$ values with modulus equal to $R$ or $R^{-1}$, there exists an element $g$ in the Weyl group $B_n$ such that the partial derivative $\frac{\partial}{\partial\left(x_{g(1)}\ldots x_{g(j)}\right)} f_{\mathbf{m}}\left(\underline{x}_{\pm}\right)$ is unbounded if $j \le k$. In this case we require $D_{\lambda_j}$ to be zero for the point derivations to be bounded for all $1 \le j \le k$.

Furthermore, when a character is on the distinguished boundary of the character space, i.e. all $n$ values of $\{\mu_1, \ldots, \mu_n\}$ are equal to $R$ or $R^{-1}$, all $D_{\lambda_j}$ need to be zero, i.e. there does not exist any nontrivial point derivations. □

## 6.3 Higher cohomology of algebras with single generator

In this section, we consider the examples of higher cohomology groups of algebras of functions on $\mathbb{Z}_+$ with weighted and unweighted $\ell^1$ norm. We also consider the higher cohomology groups of functions on $\mathbb{Z}$, with or without the invariance condition of $S_2$ for the weighted case.

### 6.3.1 Second cohomology of algebras on $\mathbb{Z}$ and $\mathbb{Z}_+$

**Definition 6.11.** *A bounded* 2*-linear map $\phi$ from $\mathcal{A}^2$ to the bimodule $Y$ is a* 2-cocycle *if for all $a, b, c \in \mathcal{A}$, we have*

$$\delta\phi(a, b, c) = a\phi(b, c) - \phi(ab, c) + \phi(a, bc) - \phi(a, b)c = 0.$$

**Definition 6.12.** *A bounded* 2*-linear map $\delta\psi$ from $\mathcal{A}$ to the bimodule $Y$ is a* 2-coboundary *if*

$$\delta\psi(a, b) = \psi(a)b - \psi(ab) + a\psi(b)$$

*for some bounded linear map $\psi$ from $\mathcal{A}$ to $Y$.*

We denote the linear space of the 2-cocycles by $\mathcal{Z}^2(\mathcal{A}, Y)$ and the linear space of the 2-coboundaries by $\mathcal{B}^2(\mathcal{A}, Y)$. The second cohomology group $\mathcal{H}^2(\mathcal{A}, Y)$ is defined by the quotient

$$\mathcal{H}^2(\mathcal{A}, Y) = \frac{\mathcal{Z}^2(\mathcal{A}, Y)}{\mathcal{B}^2(\mathcal{A}, Y)}.$$

The proofs for Lemma 6.13 and Lemma 6.14 are well known. We will apply the ideas from the two lemmas to the algebras with the weighted $\ell^1$ norm.

**Lemma 6.13.** $\mathcal{H}^2\left(\ell^1\left(\mathbb{Z}_+\right), \mathbb{C}_0\right) = 0.$

*Proof.* A bounded 2-linear map $\phi$ from $\ell^1\left(\mathbb{Z}_+\right)$ to the point bimodule $\mathbb{C}_0$ is a 2-cocycle if it satisfies

$$\begin{aligned}\delta\phi\left(z^{a_1}, z^{a_2}, z^{a_3}\right) = z^{a_1}\phi\left(z^{a_2}, z^{a_3}\right) - \phi\left(z^{a_1+a_2}, z^{a_3}\right) \\ + \phi\left(z^{a_1}, z^{a_2+a_3}\right) - \phi\left(z^{a_1}, z^{a_2}\right)z^{a_3} = 0.\end{aligned}$$

When $n = a_1 + a_2 + a_3$ and $a_1, a_3 \neq 0$, the 2-cocycle equation shows that $\phi\left(z^{a_1+a_2}, z^{a_3}\right) = \phi\left(z^{a_1}, z^{a_2+a_3}\right) = f\left(z^n\right)$ for some bounded function $f$. When $a_1 \neq 0$ and $a_3 = 0$, we have $\delta\phi\left(z^{a_1}, z^{a_2}, z^0\right) = -\phi\left(z^{a_1+a_2}, 1\right) = 0.$

Define $\psi\left(z^{n+1}\right) := -\phi\left(z^{n}, z\right)$ for $n \geq 0$ and $\psi(1) := \phi(1,1)$. For $a, b \neq 0$,

$$\begin{aligned} \delta\psi(z^{a}, z^{b}) &= \psi(z^{a})z^{b} - \psi(z^{a+b}) + z^{a}\psi(z^{b}) \\ &= -\psi(z^{a+b}) = \phi(z^{a+b-1}, z). \end{aligned}$$

We also have $\delta\psi(z^{a}, 1) = 0$ when $a \geq 1$, $\delta\psi(1, z^{b}) = 0$ when $b \geq 1$ and $\delta\psi(1,1) = \psi(1) = \phi(1,1)$.

Let $\phi' = \phi - \delta\psi$. For $a, b \neq 0$, we have $\phi'(z^{a}, z^{b}) = \phi(z^{a}, z^{b}) - \delta\psi(z^{a}, z^{b}) = f(z^{a+b}) - f(z^{a+b-1+1}) = 0$. With $\phi'(1,1) = 0, \phi'(z^{a}, 1) = \phi'(1, z^{b}) = 0$, we obtain that the function $\psi$ cobounds $\phi$.

□

**Lemma 6.14.** $\mathcal{H}^{2}\left(\ell^{1}\left(\mathbb{Z}_{+}\right), \mathbb{C}_{\frac{1}{2}}\right) = 0.$

*Proof.* We consider the algebra of summable functions on $\mathbb{Z}_{+}$ with the unweighted $\ell^{1}$ norm. We write the characteristic functions as the monomials $\left\{z^{j}\right\}_{j=0}^{\infty}$.

The character space of the algebra $\ell^{1}\left(\mathbb{Z}_{+}\right)$ is given by a unit disc centered at the origin on the complex plane. We consider the point module $\mathbb{C}_{\frac{1}{2}}$.

Set $w = z - \frac{1}{2}$ and we have $\|w\|^{n} = \left(\frac{3}{2}\right)^{n}$. The 2-cocycles equation can be written in the form of powers of $w$. The set of powers of $w$ span the set of powers of $z$ and $w^{n} = \left(z - \frac{1}{2}\right)^{n} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \left(\frac{1}{2}\right)^{n-j} z^{j}$. We will define a 2-coboundary function $\psi$ that cobounds $\phi$ and then show that $\psi$ is bounded.

Define $\psi\left(w^{n+1}\right) := -\phi\left(w^{n}, w\right)$ for $n \geq 0$ and $\psi(1) := \phi(1,1)$. We apply the proof in Lemma 6.13. For $a, b \neq 0$, we have

$$\begin{aligned} \delta\psi\left(w^{a}, w^{b}\right) &= \psi\left(w^{a}\right) w^{b} - \psi\left(w^{a+b}\right) + w^{a}\psi\left(w^{b}\right) \\ &= -\psi\left(w^{a+b}\right) = \phi\left(w^{a+b-1}, w\right). \end{aligned}$$

We also have $\delta\psi(w^{a}, 1) = 0$ when $a \geq 1$, $\delta\psi(1, w^{b}) = 0$ when $b \geq 1$ and $\delta\psi(1,1) = \psi(1) = \phi(1,1)$.

Let $\phi' = \phi - \delta\psi$. We can show that $\phi'(w^{a}, w^{b}) = 0$ for all $a, b \in \mathbb{Z}_{+}$ therefore $\phi'(z^{a}, z^{b}) = 0$ as the powers of $z$ are spanned by the powers of $w$.

We will show that $\psi$ is bounded by induction. First we verify that

$$\psi\left(z\right) = \psi\left(w + \frac{1}{2}\right) = -\phi\left(1, w\right) + \frac{1}{2}\phi(1,1) = -\phi\left(1, z\right) + \phi(1,1),$$

which indicates $\left|\psi\left(z\right)\right| \leq 2\|\phi\|$.

We assume that $\left|\psi\left(z^{m}\right)\right| \leq 2\|\phi\| + 2\left|\phi(1,z)\right|$ for all $m \leq n$.

Consider $\delta\psi\left(z^n, z\right)$, as $\psi$ and $\phi$ satisfy the inequality

$$\left|\phi(z^n, z)\right| = \left|\delta\psi(z^n, z)\right| = \left|\frac{1}{2^n}\psi(z) - \psi(z^{n+1}) + \psi(z^n)\frac{1}{2}\right| \leq \|\phi\|\,,$$

we have

$$\begin{aligned}
\left|\psi(z^{n+1})\right| &\leq \|\phi\| + \frac{1}{2^n}\left|\psi(z)\right| + \frac{1}{2}\left|\psi(z^n)\right| \\
&\leq \|\phi\| + \frac{1}{2^n}\left|\psi(z)\right| + \frac{1}{2}\|\phi\| + \frac{1}{2^{n-1}}\left|\psi(z)\right| + \left|\psi(z^{n-1})\right| \\
&\dots \\
&\leq 2\|\phi\| + 2\left|\psi(z)\right| \\
&\leq 2\|\phi\| + 2\left|\phi(1, z)\right|,
\end{aligned}$$

which proves that $\left|\psi(z^{n+1})\right|$ is indeed bounded by $2\|\phi\| + 2\left|\phi(1, z)\right|$. Hence $\psi$ is well defined. $\square$

The proof for the following statement applies a similar method from Lemma 6.14. In general, we assume the weight condition $\omega_R$ to have $R > 1$.

**Lemma 6.15.** $\mathcal{H}^2\left(\ell^1\left(\mathbb{Z}_+, \omega_R\right), \mathbb{C}_1\right) = 0$.

*Proof.* We consider the algebra of summable functions on $\mathbb{Z}_+$ with the $\omega_R$ weighted $\ell^1$ norm. We write the characteristic functions as the monomials $\left\{z^j\right\}_{j=0}^{\infty}$. With the weight condition $\omega_R$, a monomial $z^n$ has weight $R^n$.

The character space of the algebra $l^1\left(\mathbb{Z}_+, \omega_R\right)$ is given by a disc with radius $R$ centered at the origin on the complex plane. Consider the point module $\mathbb{C}_1$ where the point 1 is in the interior of the character space.

The method is similar to the proof of Lemma 6.14. Set $w_+ = z - 1$. Then $w_+^n$ has weight $\left\|w_+^n\right\| = (R+1)^n$. Define $\psi(w_+^{n+1}) := -\phi(w_+^n, w_+)$ for $n \geq 0$ and $\psi(1) := \phi(1, 1)$. We can show that $\phi' = \phi - \delta\psi$ on $l^1(\mathbb{Z}_+, \omega_R)^2$ is always zero; i.e., $\phi'(w_+^a, w_+^b) = 0$ and $\phi'(z^a, z^b) = 0$ for all $a, b \in \mathbb{Z}_+$. To prove that $\psi$ is bounded, we have the inequality

$$\left|\phi(z^n, z)\right| = \left|\delta\psi(z^n, z)\right| = \left|\psi(z) - \psi(z^{n+1}) + \psi(z^n)\right| \leq R^{n+1}\|\phi\|\,,$$

to give

$$\begin{aligned}
\left|\psi(z^{n+1})\right| &\leq R^{n+1}\|\phi\| + \left|\psi(z)\right| + \left|\psi(z^n)\right| \\
&\leq R^{n+1}\|\phi\| + \left|\psi(z)\right| + R^n\|\phi\| + \left|\psi(z)\right| + \left|\psi(z^{n-1})\right| \\
&\dots \\
&\leq \frac{R}{R-1}R^{n+1}\|\phi\| + (n+1)\left|\psi(z)\right| \\
&\leq R^{n+1}\left(\frac{R}{R-1} + \frac{R}{e\log R}\right)\|\phi\| .
\end{aligned}$$

which proves that $\psi$ is bounded and $\|\psi\| \leq \left(\frac{R}{R-1} + \frac{R}{e\log R}\right)\|\phi\|$. □

The below statement is a new result.

**Proposition 6.16.** $\mathcal{H}^2\left(\ell^1\left(\mathbb{Z},\omega_R\right),\mathbb{C}_1\right) = 0$.

*Proof.* We define $\mathcal{A}_+ = \ell^1(\mathbb{Z}_+,\omega_R)$ and $\mathcal{A}_+ = \ell^1(\mathbb{Z}_-,\omega_R)$, therefore $\mathcal{A} = \ell^1(\mathbb{Z},\omega_R) = \mathcal{A}_+ \oplus \mathcal{A}_-$. We use the Laurent monomials as the basis and use the weight condition $\|z^m\| = R^{|m|}$ on the characteristic functions.

Define $w_+ = z-1$ and $w_- = z^{-1}-1$. We have $-w_+w_- = w_+ + w_-$. Then the positive and negative powers of $z$ can be expressed in terms of $w_+$ and $w_-$ separately. We have shown that $\mathcal{H}^2(\mathcal{A}_+,\mathbb{C}_1) = 0$ in Lemma 6.15 and we can show that $\mathcal{H}^2(\mathcal{A}_-,\mathbb{C}_1) = 0$ with the same method.

Define $\psi(1) := \phi(1,1)$ which agrees on both $\mathcal{A}_+$ and $\mathcal{A}_-$. Define $\psi(w_+^{n+1}) := -\phi(w_+^n, w_+)$ and $\psi(w_-^{n+1}) := -\phi(w_-^n, w_-)$. By the inequalities shown at the end of Lemma 6.15, $\psi$ is bounded on both $\mathbb{Z}_+$ and $\mathbb{Z}_-$ therefore bounded on all $\mathbb{Z}$. Set $\phi' = \phi - \delta\psi$. We have $\phi'(z^a,z^b) = 0$, $\phi'(z^{-a},z^{-b}) = 0$, $\phi'(w_+^a,w_+^b) = 0$ and $\phi'(w_-^a,w_-^b) = 0$ for all $a,b \in \mathbb{Z}_+$.

We will now need to show what conditions $\phi'(z^a,z^b)$ satisfy when $a > 0, b < 0$ or $a < 0, b > 0$. Consider the 2-cocycles equation, for all $a_1,a_2,a_3 \in \mathbb{Z}$,

$$\begin{aligned}
&\delta\phi'(z^{a_1},z^{a_2})(z^{a_3}) \\
&= z^{a_1}\phi'(z^{a_2},z^{a_3}) - \phi'(z^{a_1+a_2},z^{a_3}) + \phi'(z^{a_1},z^{a_2+a_3}) - \phi'(z^{a_1},z^{a_2})z^{a_3} = 0. \qquad (\star)
\end{aligned}$$

By taking $a_1 = 1$, $a_2 = -1$ and $a_3 = 1$, we have

$$\phi'(z^{-1},z) - \phi'(1,z) + \phi'(z,1) - \phi'(z,z^{-1}) = 0.$$

The middle two terms vanish, hence $\phi'(z^{-1},z) = \phi'(z,z^{-1})$.

Suppose $a \geq 1$. Taking $a_1 = -a$, $a_2 = 1$ and $a_3 = -1$ in Equation ($\star$), we have

$$\phi'(z^{-a},z^{-1}) - \phi'(z^{-a+1},z^{-1}) + \phi'(z^{-a},1) - \phi'(z^{-a},z^1) = 0.$$

The middle two terms vanish, hence $\phi(z^{-a}, z) = \phi'(z, z^{-1})$.

If $b \geq j \geq 1$, then taking $a_1 = b$, $a_2 = -j$ and $a_3 = 1$ in Equation $(\star)$, we have

$$\phi'(z^{-j}, z) - \phi'(z^{b-j}, z) + \phi'(z^b, z^{-j+1}) - \phi'(z^b, z^{-j}) = 0.$$

The second term vanishes, hence

$$\phi'(z^b, z^{-j}) - \phi'(z^b, z^{-j+1}) = \phi'(z^{-j}, z) = \phi'(z, z^{-1}) \qquad (\star\star)$$

For $b \geq a \geq 1$, a summation of Equation $(\star\star)$ gives

$$\sum_{j=1}^{a} \left(\phi'(z^b, z^{-j}) - \phi'(z^b, z^{-j+1})\right) = \phi'(z^b, z^{-a}) - \phi'(z^b, 1) = \phi'(z^b, z^{-a}) = a\phi'(z, z^{-1}).$$

Similarly, by taking $a_1 = 1$, $a_2 = -j$ and $a_3 = b$ and apply the summation as above, we obtain that for all $b \geq a \geq 1$,

$$\phi'(z^{-a}, z^b) = a\phi'(z^{-1}, z) = \phi'(z^b, z^{-a}).$$

We then consider the cases where the signs are alternated. The analysis is by swapping $z$ and $z^{-1}$ in the previous arguments. For all $b \leq a \leq -1$, we have

$$\phi'(z^b, z^{-a}) = \phi'(z^{-a}, z^b) = a\phi'(z, z^{-1}).$$

Therefore, for any $m, n \geq 1$, we have shown that

$$\phi'\left(z^{\pm m}, z^{\mp n}\right) = \min(m, n)\phi'\left(z, z^{-1}\right).$$

Now define $\psi'(z^a) := \frac{1}{2}|a|\phi'(z, z^{-1})$, which shows that $\psi'$ is uniformly bounded by $\|\phi\|'$, hence uniformly bounded by $\|\phi\|$. For $a, b$ both non-negative or non-positive, we have $\delta\psi'(z^a, z^b) = 0$ and $\delta\psi'(z^a, z^{-b}) = \min\{|a|, |b|\}\phi'(z, z^{-1})$, which proves that $\phi'' = \phi' - \delta\psi' = \phi - \delta(\psi + \psi') = 0$, where $\psi'' = \psi + \psi'$ is uniformly bounded by $\|\phi\|$. $\square$

**Theorem 6.17.** $\mathcal{H}^2\left(\mathcal{A}_{2,\omega_R}, \mathbb{C}_1\right) = 0$.

*Proof.* We consider the $S_2$-invariant subalgebra of $\ell^1(\mathbb{Z}, \omega_R)$. The algebra $\mathcal{A}_{2,\omega_R}$ is generated by a single generator $y_1 = \frac{z+z^{-1}}{2}$ and all variables of the form $y_m = \frac{z^m + z^{-m}}{2}$ can be written as $y_m = \sum_{j=0}^{m} a_{m,j} y_1^j$. Similarly, we set $w = y_1 - 1$ and define $\psi\left(w^{n+1}\right) := \phi\left(w^n, w\right)$ for $n \geq 0$ and $\psi(1) = \phi(1,1)$. Now we have three sets of elements: the powers of $\frac{w}{R+1}$, the powers of $\frac{y_1}{R}$ and $\left\{\frac{y_j}{R^j}\right\}_{j=1}^{\infty}$. It is obvious that these three sets have the same span; i.e., each set of elements can be expressed as a linear sum of other set of elements. From Lemma 6.15 we know that $\phi - \delta\psi$ vanishes on the powers of $w$. Therefore we need to check that $\psi$ is uniformly bounded by $\|\phi\|$ on this set $\left\{\frac{y_j}{R^j}\right\}_{j=1}^{\infty}$ against their weight. Recall the coefficients of $a_{m,j}$ from Lemma 4.7 and $c_{n,k}$ from Lemma 4.9 with $q = 1$. We compute $\psi(y_m)$ explicitly from the relations between the variables of $y_m$, $y_1$ and $w$ as

$$\begin{aligned}
\psi(y_m) &= \psi\left(\sum_{j=0}^{m} a_{m,j} y_1^j\right) \\
&= \psi\left(\sum_{j=0}^{m} a_{m,j} (w+1)^j\right) \\
&= \psi\left(\sum_{j=0}^{m} a_{m,j} \left(\sum_{k=0}^{j} \binom{j}{k} w^k\right)\right) \\
&= \psi\left(\sum_{j=1}^{m} a_{m,j} \left(\sum_{k=1}^{j} \binom{j}{k} w^k\right)\right) + \phi(1,1) \\
&= -\sum_{j=1}^{m} a_{m,j} \left(\sum_{k=1}^{j} \binom{j}{k} \phi\left(w^{k-1}, w\right)\right) + \phi(1,1) \\
&= -\sum_{j=1}^{m} a_{m,j} \left(\sum_{k=1}^{j} \binom{j}{k} \phi\left((y_1-1)^{k-1}, w\right)\right) + \phi(1,1) \\
&= -\sum_{j=1}^{m} a_{m,j} \left(\sum_{k=1}^{j} \binom{j}{k} \phi\left(\sum_{l=1}^{k-1} \binom{k-1}{l} y_1^l (-1)^{k-1-l}, w\right)\right) + \phi(1,1) \\
&= -\sum_{j=1}^{m} a_{m,j} \left(\sum_{k=1}^{j} \binom{j}{k} \phi\left(\sum_{l=1}^{k-1} \left(\binom{k-1}{l} (-1)^{k-1-l} \sum_{i=0}^{\lfloor l/2 \rfloor} c_{l,i} y_i\right), w\right)\right) + \phi(1,1).
\end{aligned}$$

The numerical computation suggests

$$\psi(y_m) = -\phi\left(2\sum_{p=1}^{m-1} p y_{m-p}, y_1 - 1\right) + \left(2m^2 - m + 1\right)\phi(1,1). \tag{6.3}$$

This can be proved by induction from the relation

$$\psi(y_{n+1}) = 2\psi(y_n) + 2\psi(y_1) - \psi(y_{n-1}) - 2\phi(y_n, y_1).$$

By Equation (6.3), we calculate the size of $\left|\psi\left(y_m\right)\right|$ by

$$\left\|\psi\left(y_m\right)\right\| \leq \left(\sum_{p=1}^{m-1} 2(R+1)pR^{m-p} + \left(2m^2 - m + 1\right)\right)\|\phi\|.$$

The first term is bounded by $\frac{2R(R+1)}{(R-1)^2}\|y_m\|$ and the second term is bounded by $\frac{8R^{-\frac{2}{\log R}}}{(\log R)^2}\|y_m\|$. Hence for all weight conditions $\omega_R$ where $R > 1$, the $\psi$ function is well defined. $\square$

**Definition 6.18.** *An $n$-cochain $\phi \in \mathcal{C}^n(\mathcal{A})$ is* cyclic *if*

$$\phi\left(a_0, a_1, \ldots, a_{n-1}\right) = (-1)^n\left(a_1, \ldots, a_{n-1}, a_0\right)$$

*for all $a_0, a_1, \ldots, a_{n-1} \in \mathcal{A}$.*

A bounded 2-linear map $\phi$ from $\mathcal{A}^2$ to the dual module $\mathcal{A}'$ is cyclic if for all $a_1, a_2, a_0 \in \mathcal{A}$, we have

$$\phi\left(a_1, a_2\right)\left(a_0\right) = \phi\left(a_0, a_1\right)\left(a_2\right) = \phi\left(a_2, a_0\right)\left(a_1\right).$$

The cyclic map $\phi$ is a cyclic 2-cocycle if $\delta\phi = 0$.

A bounded linear map $\psi$ from $\mathcal{A}$ to the dual module $\mathcal{A}'$ is cyclic or antisymmetric if for all $a_1, a_0 \in \mathcal{A}$, we have

$$\psi\left(a_1\right)\left(a_0\right) = -\psi\left(a_0\right)\left(a_1\right).$$

The map $\delta\psi$ is a cyclic 2-coboundary if $\delta\psi\left(a_1, a_2\right)\left(a_0\right) = \psi\left(a_1\right)\left(a_2 a_0\right) - \psi\left(a_1 a_2\right)\left(a_0\right) + \psi\left(a_2\right)\left(a_0 a_1\right)$. We denote the linear space of the cyclic 2-cocycles by $\mathcal{ZC}^2(\mathcal{A})$ and the linear space of the 2-coboundaries by $\mathcal{BC}^2(\mathcal{A})$. The cyclic cohomology group $\mathcal{HC}^2(\mathcal{A})$ is defined by the quotient

$$\mathcal{HC}^2(\mathcal{A}) = \frac{\mathcal{ZC}^2(\mathcal{A})}{\mathcal{BC}^2(\mathcal{A})}.$$

**Lemma 6.19.** *$\mathcal{HC}^2\left(\ell^1\left(\mathbb{Z}_+, \omega_R\right)\right) \simeq \mathbb{C}^1$.*

*Proof.* Consider the algebra $\mathcal{A} = \ell^1\left(\mathbb{Z}_+, \omega_R\right)$. The dual module is defined to be $z^{a_1}\phi(z^{a_2}, z^{a_3}) = \phi(z^{a_2}, z^{a_3})(z^{a_0} z^{a_1})$. The 2-cocycles equation is given by

$$\begin{aligned}
\delta\phi(z^{a_1}, z^{a_2}, z^{a_3})(z^{a_0}) &= \phi(z^{a_2}, z^{a_3})(z^{a_0+a_1}) - \phi(z^{a_1+a_2}, z^{a_3})(z^{a_0}) \\
&\quad + \phi(z^{a_1}, z^{a_2+a_3})(z^{a_0}) - \phi(z^{a_1}, z^{a_2})(z^{a_3+a_0}) \\
&= 0.
\end{aligned}$$

For $a+b>0$, we define

$$\psi(z^a)(z^b) = \frac{1}{a+b}\left(\sum_{i=1}^{a}\phi(z^i, z^{a-i})(z^b) - \sum_{j=1}^{b}\phi(z^a, z^j)(z^{b-j})\right),$$

where a summation term is zero if $a=0$ or $b=0$. Given a bounded $\phi$, we have

$$\left|\psi(z^a)(z^b)\right| \leq \frac{1}{a+b}\left(\sum_{i=1}^{a}\|\phi\|\, R^{i+a-i+b} + \sum_{j=1}^{b}\|\phi\|\, R^{a+j+b-j}\right) = \|\phi\|\, R^{a+b}, \tag{6.4}$$

which shows that $\left|\psi(z^a)(z^b)\right|$ is bounded by $\|\phi\|$ against the weight of $(z^a)(z^b)$. We expand the 2-coboundary equation $\delta\psi(z^{a_1}, z^{a_2})(z^{a_3})$ from the $\psi$ defined above and obtain that $\delta\psi(z^{a_1}, z^{a_2})(z^{a_3}) = -\phi(z^{a_1}, z^{a_2})(z^{a_3})$. The only function which cannot be cobounded is the function $\phi\,(1,1)\,(1)$ supported by the triple $\left(z^0, z^0, z^0\right)$ as $\delta\,(1,1)\,(1) = \psi(1)(1) = 0$. Therefore we have $\mathcal{HC}^2(l^1(\mathbb{Z}_+, \omega_R)) \simeq \mathbb{C}^1$. $\square$

We will now move to the invariant subalgebra and show that the second cyclic cohomology group is also one-dimensional.

**Theorem 6.20.** *For $R > \sqrt{2}$, $\mathcal{HC}^2(\mathcal{A}_{2,\omega_R}) \simeq \mathbb{C}^1$.*

*Proof.* The only term we cannot cobound is $\phi\,(1,1)\,(1)$. Given $\phi \in \mathcal{ZC}^2(\mathcal{A}_{2,\omega_R})$, we will find bounded $\psi\left(y_{m_1}\right)\left(y_{m_2}\right)$ for all $a, b \in \mathbb{Z}_+$ such that $\delta\psi = \phi$.

As the algebra $\mathcal{A}_{2,\omega_R}$ is generated by a single generator $y_1$, it is possible to apply the similar construction of $\psi$ in the proof of $\mathcal{HC}^2(l^1(\mathbb{Z}_+, \omega_R))$ in Lemma 6.19 to define $\psi\left(y_1^a\right)\left(y_1^b\right)$ as

$$\psi\left(y_1^a\right)\left(y_1^b\right) = -\left(\sum_{i=1}^{a}\phi\left(y_1^i, y_1^{a-i}\right)\left(y_1^b\right) - \sum_{j=1}^{b}\phi\left(y_1^a, y_1^j\right)\left(y_1^{b-j}\right)\right).$$

Therefore $\psi\left(y_{m_1}\right)\left(y_{m_2}\right)$ can be expressed as a linear combination as

$$\psi\left(y_{m_1}\right)\left(y_{m_2}\right) = \sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2} a_{m_1,j_1} a_{m_2,j_2}\psi\left(y_1^{j_1}\right)\left(y_1^{j_2}\right).$$

This construction makes $\psi$ antisymmetric and $\delta\psi = \phi$. We will now show that $\psi$ is bounded.

First, we consider $\delta\psi\,(y_0, y_0)\,(y_n)$. This gives us $\psi\,(y_0)\,(y_n) = \phi\,(y_0, y_0)\,(y_n)$. The norm estimate is straightforward to show that $\left|\psi\,(y_0)\,(y_n)\right| = \left|\phi\,(y_0, y_0)\,(y_n)\right| \leq R^n\|\phi\|$. By the antisymmetry property, we have $\psi\,(y_a)\,(y_a) = 0$.

Then we consider the first example that needs conversion into powers of $y_1$, $\psi(y_1)(y_2)$. The expansion gives

$$\begin{aligned}\psi(y_1)(y_2) &= \psi(y_1)\left(2y_1^2 - y_0\right)\\ &= 2\psi(y_1)\left(y_1^2\right) - \psi(y_1)(y_0)\\ &= -\frac{2}{3}\left(\phi(y_1, y_0)\left(y_1^2\right) - \phi(y_1, y_1)(y_1) - \phi\left(y_1, y_1^2\right)(y_0)\right) + \phi(y_0, y_0)(y_1)\\ &= -\frac{1}{3}\phi(y_1, y_0)(y_2) + \frac{1}{3}\phi(y_1, y_2)(y_0) + \frac{2}{3}\phi(y_1, y_1)(y_1) + \phi(y_1, y_1)(y_1),\end{aligned}$$

which shows that $\left|\psi(y_1)(y_2)\right| \leq 2R^3\|\phi\|$.

We will prove by induction that all $\left|\psi\left(y_{m_1}\right)\left(y_{m_2}\right)\right|$ are bounded by $R^{m_1+m_2}\|\phi\|\,\frac{2R^2+1}{R^2-2}$.

Let $K_N = \max\{|\psi(y_{m_1})(y_{m_2}) : m_1 + m_2 \leq N\}$. By our previous remarks, $K_2 \leq R^2\|\phi\|$ and $K_3 \leq 2R^3\|\phi\|$.

We have verified the cases of $m_1 + m_2 \leq 3$, $m_1$ or $m_2 = 0$, and $m_1 = m_2$. By the antisymmetry property, we are now left to check the cases where $1 \leq m_1 < \frac{m_1+m_2}{2}$.

Let $N = m_1 + m_2$, and choose $a$ such that

$$\left|\psi(y_a)(y_{N-a})\right| := \max\left\{\left|\psi(y_i)(y_{N-i})\right| : 1 \leq i \leq \frac{N-1}{2}\right\}.$$

This set is nonempty as $\left|\psi(y_1)(y_{N-1})\right|$ is in the set. We have

$$\delta\psi(y_a, y_a)(y_{N-2a}) = 2\psi(y_a)(y_a y_{N-2a}) - \psi(y_a y_a)(y_{N-2a}) = \phi(y_a, y_a)(y_{N-2a}),$$

which simplifies to

$$2\psi(y_a)(y_{N-a}) - \psi(y_{2a})(y_{N-2a}) = 2\psi(y_a)\left(y_{|N-3a|}\right) - \psi(y_0)(y_{N-2a}) + 2\phi(y_a, y_a)(y_{N-2a}).$$

As $\psi(y_a)(y_{N-a})$ is chosen to be the function which has the largest norm of all $\psi\left(y_{m_1}\right)\left(y_{m_2}\right)$ such that $m_1 + m_2 = N$, we have $\left|\psi(y_{2a})(y_{N-2a})\right| \leq \left|\psi(y_a)(y_{N-a})\right|$. Hence,

$$\begin{aligned}\left|\psi(y_a)(y_{N-a})\right| &\leq \left|2\psi(y_a)(y_{N-a}) - \psi(y_a)(y_{N-a})\right|\\ &\leq \left|2\psi(y_a)(y_{N-a}) - \psi(y_{2a})(y_{N-2a})\right|.\end{aligned}$$

And we use the setting that $a$ is a positive integer and strictly less than $\frac{N}{2}$ to have $a \leq \frac{N-1}{2}$ and $a + |N - 3a| \leq N - 2$, which shows that $\left|\psi(y_a)\left(y_{|N-3a|}\right)\right| \leq K_{N-2}$.

The inequalities give

$$\begin{aligned}
\left|\psi\left(y_a\right)\left(y_{N-a}\right)\right| &\leq \left|2\psi\left(y_a\right)\left(y_{N-a}\right)-\psi\left(y_{2a}\right)\left(y_{N-2a}\right)\right| \\
&\leq 2\left|\psi\left(y_a\right)\left(y_{N-3a}\right)\right|+\left|\psi\left(y_0\right)\left(y_{N-2a}\right)\right|+2\left|\phi\left(y_a,y_a\right)\left(y_{N-2a}\right)\right| \\
&\leq 2\left|\psi\left(y_a\right)\left(y_{N-3a}\right)\right|+\left|\phi\left(y_0,y_0\right)\left(y_{N-2a}\right)\right|+2\left|\phi\left(y_a,y_a\right)\left(y_{N-2a}\right)\right| \\
&\leq 2K_{N-2}+R^{N-2}\|\phi\|+2R^{N}\|\phi\|\,.
\end{aligned}$$

Using the inductive hypothesis for $K_{N-2}$, we obtain that

$$K_N \leq R^{N-2}\|\phi\|\left(\frac{2(1+2R^2)}{R^2-2}+1+2R^2\right)\leq R^N\|\phi\|\,\frac{2R^2+1}{R^2-2},$$

which shows that $\psi$ is indeed bounded for $R>\sqrt{2}$. □

### 6.3.2 Simplicial and cyclic cohomology of algebras on $\mathbb{Z}_+$

In this section, we study higher simplicial and cyclic cohomology groups of some singly generated algebras. We first consider the algebra $\ell^1\left(\mathbb{Z}_+\right)$ and the higher simplicial and cyclic cohomology groups $\mathcal{HH}^n\left(\ell^1\left(\mathbb{Z}_+\right)\right)$ and $\mathcal{HC}^n\left(\ell^1\left(\mathbb{Z}_+\right)\right)$. We present the method introduced in [20] with an explicit construction of cobounding in the algebra $\ell^1\left(\mathbb{Z}_+\right)$. The aim is to apply a similar method to the algebra $\mathcal{A}_{2,\omega_R}$ with the weighted $\omega_R$ norm. We use a finite induction method to find an explicit construction of the coboundaries for the higher simplicial and cyclic cohomology groups. We use a setting of similar notations in [44], [33], [34] and [32] to define the cohomology groups.

Let $\mathcal{A}$ be a Banach algebra and the dual $\mathcal{A}'$ be an $\mathcal{A}$-bimodule. Let the $n$-cochain $\phi$ be a bounded $n$-linear map from $\mathcal{A}^n$ to $\mathcal{A}'$, denoted by $\phi\in C^n\left(\mathcal{A},\mathcal{A}'\right)$.

**Definition 6.21.** *The $n$-cochain $\phi$ is* cyclic *if we have*

$$\phi\left(w_1,w_2,\ldots,w_n\right)\left(w_0\right)=(-1)^n\phi\left(w_0,w_1,\ldots,w_{n-1}\right)\left(w_n\right).$$

**Definition 6.22.** *We define a map $\delta: C^n\left(\mathcal{A},\mathcal{A}'\right)\longrightarrow C^{n+1}\left(\mathcal{A},\mathcal{A}'\right)$ as*

$$\begin{aligned}
\delta^n\phi\left(w_1,w_2,\ldots,w_{n+1}\right)\left(w_0\right)= \quad & \phi\left(w_2,\ldots,w_{n+1}\right)\left(w_0w_1\right) \\
&+\sum_{j=1}^{n}(-1)^j\phi\left(w_1,w_2,\ldots,w_jw_{j+1},w_{n+1}\right)\left(w_0\right) \\
&+(-1)^{n+1}\phi\left(w_1,\ldots,w_n\right)\left(w_{n+1}w_0\right).
\end{aligned}$$

**Definition 6.23.** *The $n$-cochain $\phi$ is an $n$-*cocycle *if $\delta\phi=0$.*

**Definition 6.24.** *The $n$-cochain $\phi$ is an $n$-*coboundary *if there exists $\psi \in C^{n-1}\left(\mathcal{A}, \mathcal{A}'\right)$ such that $\phi = \delta^{n-1}\psi$.*

**Definition 6.25.** *The $n$-cochain $\phi$ is a cyclic $n$-*coboundary *if there exists cyclic $\psi \in C^{n-1}\left(\mathcal{A}, \mathcal{A}'\right)$ such that $\phi = \delta^{n-1}\psi$.*

We denote the linear space of all $n$-cocycles by $\mathcal{Z}^n\left(\mathcal{A}, \mathcal{A}'\right)$ and the linear space of all $n$-coboundaries by $\mathcal{B}^n\left(\mathcal{A}, \mathcal{A}'\right)$. We use the fact that $\mathcal{B}^n\left(\mathcal{A}, \mathcal{A}'\right)$ is a subset of $\mathcal{Z}^n\left(\mathcal{A}, \mathcal{A}'\right)$. The $n$-th simplicial cohomology group is defined by the quotient

$$\mathcal{H}\mathcal{H}^n\left(\mathcal{A}\right) = \mathcal{H}^n\left(\mathcal{A}, \mathcal{A}'\right) = \frac{\mathcal{Z}^n\left(\mathcal{A}, \mathcal{A}'\right)}{\mathcal{B}^n\left(\mathcal{A}, \mathcal{A}'\right)}.$$

We denote the linear space of all cyclic $n$-cocycles by $\mathcal{ZC}^n\left(\mathcal{A}, \mathcal{A}'\right)$ and the linear space of all cyclic $n$-coboundaries by $\mathcal{BC}^n\left(\mathcal{A}, \mathcal{A}'\right)$. The $n$-th cyclic cohomology group is defined by the quotient

$$\mathcal{HC}^n\left(\mathcal{A}\right) = \frac{\mathcal{ZC}^n\left(\mathcal{A}, \mathcal{A}'\right)}{\mathcal{BC}^n\left(\mathcal{A}, \mathcal{A}'\right)}.$$

For $w = w_1 \otimes \cdots \otimes w_n \otimes w_0 \in \bigotimes^{n+1}\mathcal{A}$, we write $\phi(w)$ as the $n$-cochain $\phi\left(w_1, w_2, \ldots, w_n\right)\left(w_0\right)$.

We start with a discussion on the pre-dual of the algebra $A_+$. We consider the algebra $A_+ = \ell^1\left(\mathbb{Z}_+\right)$ where the characteristic functions at all nonnegative integers $m \in \mathbb{Z}_+$ are given by the monomials $z^m$. The algebra $A_+$ is singly generated by $z$. First we consider the subalgebra of polynomials with finite sums of powers $\tilde{A}_+$ and then extend to the Banach algebra $A_+$.

**Definition 6.26.** *Define single-variable-splitting map $\tilde{s} : \tilde{A}_+ \longrightarrow \tilde{A}_+ \otimes \tilde{A}_+$ as $\tilde{s}\left(z^n\right) = \sum_{i=1}^{n} z^{a-i} \otimes z^i$ for $n > 0$ and $\tilde{s}\left(1\right) = 0$.*

**Definition 6.27.** *Define the product map $\pi : \tilde{A}_+ \otimes \tilde{A}_+ \longrightarrow \tilde{A}_+$ as $\pi\left(z^a \otimes z^b\right) = z^a z^b = z^{a+b}$.*

Note that $\frac{1}{n}\left(\pi\tilde{s}\right)\left(z^n\right) = z^n$, which is the identity map of the monomial $z^n$ for $n > 0$.

We consider an elementary tensor product $w = z^{a_1} \otimes \cdots \otimes z^{a_j} \otimes z^{a_{j+1}} \otimes \cdots \otimes z^{a_{n+1}} \otimes z^{a_0}$ for the algebra $\tilde{A}_+$ . We always assume that $n \geq 1$. Let $N = \sum_{k=0}^{n+1} a_k$ be the total degree of $w$.

**Definition 6.28.** *Define the* face *maps $d_j^n : \bigotimes^{n+2}\tilde{A}_+ \longrightarrow \bigotimes^{n+1}\tilde{A}_+$ as*

$$d_j^n\left(w\right) = (-1)^j z^{a_1} \otimes \cdots \otimes z^{a_j} z^{a_{j+1}} \otimes \cdots \otimes z^{a_{n+1}} \otimes z^{a_0} \text{ for } 1 \leq j \leq n+1.$$

**Definition 6.29.** *Define the* wrap around pinching *map $d_0^n : \bigotimes^{n+2}\tilde{A}_+ \longrightarrow \bigotimes^{n+1}\tilde{A}_+$ as*

$$d_0^n\left(w\right) = z^{a_2} \otimes \cdots \otimes z^{a_{n+1}} \otimes z^{a_0} z^{a_1}.$$

**Definition 6.30.** *Define the* splitting *map* $s_k^n := \bigotimes^{n+1} \tilde{A}_+ \longrightarrow \bigotimes^{n+2} \tilde{A}_+$ *on the $k$-th variable of an elementary tensor product $w$ as*

$$s_k^n(w) = \frac{(-1)^j}{N} z^{a_1} \otimes \cdots \otimes \tilde{s}(z^{a_k}) \otimes \cdots \otimes z^{a_{n+1}} \otimes z^{a_0} \text{ for } 1 \leq k \leq n+1.$$

Let $w = w_1 \otimes \cdots \otimes w_n \in \bigotimes^{n+1} \mathcal{A}$ be a tensor product of the algebra $\mathcal{A}$. Let the cyclic equivalence relation be generated by the map $t : \bigotimes^{n+1} \mathcal{A} \longrightarrow \bigotimes^{n+1} \mathcal{A}$ be defined as

$$t(w) = (-1)^n w_2 \otimes \cdots \otimes w_{n+1} \otimes w_1.$$

**Definition 6.31.** *The* cyclic equivalence class $\mathcal{CC}_n(\mathcal{A})$ *is a subspace of* $\bigotimes^{n+1} \mathcal{A}$ *generated by the cyclic equivalence relation as* $\mathcal{CC}_n(\mathcal{A}) := \bigotimes^{n+1} \mathcal{A}/\langle w - t(w) \rangle \subset \bigotimes^{n+1} \mathcal{A}$.

Set $d^n = \sum_{j=0}^{n+1} d_j^n$ and $s^n = \sum_{k=1}^{n+1} s_k^n$. Sometimes we write $d$ and $s$ as the abbreviation of $d^n$ and $s^n$.

**Lemma 6.32.** *The proof is stated in [20]. For* $w \in \mathcal{CC}_n\left(\tilde{A}_+\right)$, *we have*

$$\left(s^{n-1} d^{n-1} + d^n s^n\right)(w) = w.$$

*Proof.* We expand $\left(s^{n-1} d^{n-1} + s^n d^n\right)(w)$ and obtain the cancellations as the following terms:

$$\begin{aligned}
&\sum_{k=1}^{n+1} d_k^n s_k^n(w) = w, \\
&s_k^{n-1} d_j^{n-1} + d_j^n s_{k+1}^n = 0, \text{ for } 1 \leq j < k \leq n, \\
&s_k^{n-1} d_j^{n-1} + d_{j+1}^n s_k^n = 0, \text{ for } 1 \leq k < j \leq n, \\
&s_k^{n-1} d_0^{n-1} + d_0^n s_{k+1}^n = 0, \text{ for } 1 \leq k \leq n-1, \\
&s_k^{n-1} d_k^{n-1} + d_{k+1}^n s_k^n + d_k^n s_{k+1}^n = 0, \text{ for } 1 \leq k \leq n \\
&s_n^{n-1} d_0^{n-1} + d_0^n s_{n+1}^n + d_0^n s_1^n \equiv 0,
\end{aligned}$$

which shows that

$$\left(s^{n-1} d^{n-1} + d^n s^n\right)(w) = \sum_{k=1}^{n+1} d_k^n s_k^n(w) = w.$$

∎

**Definition 6.33.** *Let* $w = z^{a_1} \otimes \cdots \otimes z^{a_{n+1}}$ *be an elementary product in* $\bigotimes^{n+1} \tilde{A}_+$. *We define another split map* $s_0^n := \bigotimes^{n+1} \tilde{A}_+ \longrightarrow \bigotimes^{n+2} \tilde{A}_+$ *on an elementary tensor product*

*$w$ as*

$$s_0^n(w) = \frac{(-1)^{n+1}}{N} \sum_{i=1}^{a_1} \left( z^{a_1 - i} \otimes \cdots \otimes z^{a_{n+1}+i} \otimes z^0 \right)$$

*for $a_1 \neq 0$. And $s_0^n(w) = 0$ if $a_1 = 0$.*

Set $s'^n = s^n + s_0^n$.

**Lemma 6.34.** *For $w \in \bigotimes^{n+1} \tilde{A}_+$, we have*

$$\left( s'^{n-1} d^{n-1} + d^n s'^n \right)(w) = w.$$

*Proof.* We consider the additional terms apart from the terms in Lemma 6.32. The additional terms cancel as follows:

$$\begin{aligned}
&s_0 d_j + d_j s_0 = 0, \text{ for } 2 \leq j \leq n-1, \\
&d_0 s_1 + d_{n+1} s_0 = 0, \\
&s_n d_0 + d_0 s_{n+1} + d_0 s_0 = 0, \\
&s_0 d_0 + s_0 d_1 + d_1 s_0 = 0.
\end{aligned}$$

Hence we have $\left( s'^{n-1} d^{n-1} + d^n s'^n \right)(w) = w$. □

**Remark 6.35.** *Note that the maps $d_j$, $s_k$, $\sum_{j=0}^{n} d_j$ and $\sum_{k=1}^{n+1} s_k$ are all continuous on the tensor product of algebra of finite sums of powers of $z$ and the space is dense. The splitting maps are all bounded because of the averaging factor $\frac{1}{N}$ in the coefficient. Therefore we can apply the same setting to the Banach algebra of infinite sums of powers $A_+$.*

The two theorems below are presented in [20].

**Theorem 6.36.** *Let $n$ be a positive integer greater than 1. We have*

1. *$\mathcal{HC}^n(A_+) \simeq \mathbb{C}^1$ if $n$ is even.*
2. *$\mathcal{HC}^n(A_+) \simeq 0$ if $n$ is odd.*

*Proof.* Let $\phi(w)$ be a cyclic $n$-cycle. We consider the function $\phi$ where $\phi\left(z^0, \ldots, z^0\right)\left(z^0\right) = 1$ and 0 elsewhere. If $n$ is even and the function $\psi$ cobounds $\phi$, then we have $\phi = \delta\psi$. As $\psi\left(z^0, \ldots, z^0\right)\left(z^0\right) = 0$ and $\delta\psi\left(z^0, \ldots, z^0\right)\left(z^0\right) = 0$, the function $\phi$ cannot be cobounded. If $n$ is odd, we have $\phi\left(z^0, \ldots, z^0\right)\left(z^0\right) = 0$.

Recall the operator $s$ and $d$ on the predual of the tensor products. Set $\psi(w) = \phi\left(s(w)\right)$. The dual of $s$ on the function $\psi$ is given by $s^*\psi(w) = \phi(w)$.

We apply the identity map $(sd + ds)$ on $w$ in Lemma 6.32 and obtain that

$$\phi\left((sd + ds)\, w\right) = \delta s^* \phi(w) + s^* \delta\phi(w) = \delta\psi(w).$$

□

**Theorem 6.37.** *Let $n$ be a positive integer greater than 1. We have $\mathcal{HH}^n(A_+) \simeq 0$.*

*Proof.* Let $\phi(w)$ be an $n$-cocycle and $\psi$ be an $(n-1)$-cochain. If $n$ is odd, $\delta\phi(1,\ldots,1)(1) = \phi(1,\ldots,1)(1) = 0$. If $n$ is even, set $\psi(1,\ldots,1)(1) = \phi(1,\ldots,1)(1)$. In general, we set $\psi = s'^*\phi$. We apply the identity map $\left(s'd + ds'\right)$ on $w$ in Lemma 6.34 and obtain that

$$\phi\left(\left(s'd + ds'\right)w\right) = \delta s'^*\phi(w) + s'^*\delta\phi(w) = \delta\psi(w).$$

□

We now move to the invariant subalgebra $\mathcal{A}_{2,\omega_R}$ with a single generator $y_1$ and the characteristic functions on $\mathbb{Z}_+$ multiply as $y_a y_b = \frac{1}{2}\left(y_{a+b} + y_{|a-b|}\right)$. We consider the algebra of weighted polynomials $\tilde{\mathcal{A}}_{2,\omega_R}$ given by the finite powers of $y_1$ with the same multiplication rule to $\mathcal{A}_{2,\omega_R}$.

**Definition 6.38.** *For $w = \sum_{j=0}^{N} \alpha_j y_j \in \tilde{\mathcal{A}}_{2,\omega_R}$, the* degree *of $w$, $\deg(w)$ is defined to be the largest $j$ where $\alpha_j \neq 0$.*

The algebra $\mathcal{A}_{2,\omega_R}$ can be extended from $\tilde{\mathcal{A}}_{2,\omega_R}$ given by the infinite sums of powers of $y_1$. Define the map $\tilde{s} : \tilde{\mathcal{A}}_{2,\omega_R} \longrightarrow \tilde{\mathcal{A}}_{2,\omega_R} \otimes \tilde{\mathcal{A}}_{2,\omega_R}$ as

$$\tilde{s}\left(y_n\right) = 2\sum_{i=1}^{n-1} y_{n-i} \otimes y_i + y_0 \otimes y_n \tag{6.5}$$

for $n \geq 2$. We also define $\tilde{s}\left(y_1\right) = y_0 \otimes y_1$ and $\tilde{s}\left(y_0\right) = 0$. We define the same product map $\pi$ as in the algebra $A_+$ where $\pi\left(y_a \otimes y_b\right) = y_a y_b$. Note that the map $\pi\tilde{s}$ on the algebra $\tilde{\mathcal{A}}_{2,\omega_R}$ has got a similar form to the map on $A_+$ as

$$\pi\tilde{s}\left(y_n\right) = ny_n + \left(\text{terms with degree strictly less than } n\right).$$

We can check that the lower terms in this expansion are uniformly bounded against the weight of $y_n$.

Given an elementary tensor product $w = y_{a_1} \otimes \cdots \otimes y_{a_j} \otimes y_{a_{j+1}} \otimes \cdots \otimes y_{a_{n+1}} \otimes y_{a_0}$, we define the pinching and splitting maps similarly to the maps on the tensor product of $A_+$.

**Definition 6.39.** *Define the* face *maps $d_j^n : \bigotimes^{n+2} \tilde{\mathcal{A}}_{2,\omega_R} \longrightarrow \bigotimes^{n+1} \tilde{\mathcal{A}}_{2,\omega_R}$ as*

$$d_j^n\left(w\right) = (-1)^j y_{a_1} \otimes \cdots \otimes y_{a_j} y_{a_{j+1}} \otimes \cdots \otimes y_{a_{n+1}} \otimes y_{a_0} \text{ for } 1 \leq j \leq n+1.$$

**Definition 6.40.** *Define the* wrap around pinching *map $d_0^n : \bigotimes^{n+2} \tilde{\mathcal{A}}_{2,\omega_R} \longrightarrow \bigotimes^{n+1} \tilde{\mathcal{A}}_{2,\omega_R}$ as*

$$d_0^n\left(w\right) = y_{a_2} \otimes \cdots \otimes y_{a_{n+1}} \otimes y_{a_0} y_{a_1}.$$

**Definition 6.41.** *Define the* splitting *map* $s_k^n := \bigotimes^{n+1} \tilde{\mathcal{A}}_{2,\omega_R} \longrightarrow \bigotimes^{n+2} \tilde{\mathcal{A}}_{2,\omega_R}$ *on the* $k$*-th variable of a non-trivial elementary tensor product* $w$ *as*

$$s_k^n\left(w\right) = \frac{(-1)^j}{N} y_{a_1} \otimes \cdots \otimes \tilde{s}\left(y_{a_k}\right) \otimes \cdots \otimes y_{a_{n+1}} \otimes y_{a_0} \text{ for } 1 \leq j \leq n+1,$$

*where* $N = \sum_{k=1}^{n} a_k > 0$ *is the called the degree of* $w$*.*

For $n \geq 2$, set $d^n = \sum_{j=0}^{n+1} d_j^n$ and $s^n = \sum_{k=1}^{n+1} s_k^n$. We will show that the map $\left(s^{n-1}d^{n-1} + s^n d^n\right)$, abbreviated as $sd + ds$, is an approximation of the identity map; i.e., an identity map plus an error term with small norm.

Consider a tensor product $W = W_1 \otimes \cdots \otimes W_{n+1} \in \bigotimes^{n+1} \tilde{\mathcal{A}}_{2,\omega_R}$ where $W_j \in \tilde{\mathcal{A}}_{2,\omega_R}$ for $j = 1, \ldots n+1$. Define the map $t : \bigotimes^{n+1} \tilde{\mathcal{A}}_{2,\omega_R} \longrightarrow \bigotimes^{n+1} \tilde{\mathcal{A}}_{2,\omega_R}$ to be

$$t\left(W\right) = (-1)^n W_2 \otimes \cdots \otimes W_{n+1} \otimes W_1.$$

Define the cyclic equivalence class $\mathcal{CC}_n\left(\tilde{\mathcal{A}}_{2,\omega_R}\right) = \bigotimes^{n+1} \tilde{\mathcal{A}}_{2,\omega_R} / \langle W - t(W) \rangle \subseteq \bigotimes^{n+1} \tilde{\mathcal{A}}_{2,\omega_R}$ to be the subspace which is generated by the cyclic equivalence relation.

**Lemma 6.42.** *Let* $w = y_{a_1} \otimes \cdots \otimes y_{a_j} \otimes y_{a_{j+1}} \otimes \cdots \otimes y_{a_{n+1}}$ *be an elementary tensor product in the cyclic equivalence class* $\mathcal{CC}_n\left(\tilde{\mathcal{A}}_{2,\omega_R}\right)$ *with degree* $N \geq 2$*. Then* $\left(sd + ds - 1\right)\left(w\right)$ *has degree at most* $N - 2$*.*

*Proof.* We expand $\left(s^{n-1}d^{n-1} + d^n s^n\right)$ as

$$\sum_{k=1}^{n} \sum_{j=0}^{n} s_k^{n-1} d_j^{n-1} + \sum_{k=0}^{n+1} \sum_{k=1}^{n+1} d_j^n s_k^n$$

and consider the following different types of terms.

$$\begin{aligned}
&\text{Type 1} : d_k^n s_k^n, \text{ for } 1 \leq k \leq n+1, \\
&\text{Type 2} : s_k^{n-1} d_j^{n-1} + d_j^n s_{k+1}^n, \text{ for } 1 \leq j < k \leq n, \\
&\text{Type 3} : s_k^{n-1} d_j^{n-1} + d_{j+1}^n s_k^n, \text{ for } 1 \leq k < j \leq n, \\
&\text{Type 4} : s_k^{n-1} d_0^{n-1} + d_0^n s_{k+1}^n, \text{ for } 1 \leq k \leq n-1, \\
&\text{Type 5} : s_k^{n-1} d_k^{n-1} + d_{k+1}^n s_k^n + d_k^n s_{k+1}^n, \text{ for } 1 \leq k \leq n \\
&\text{Type 6} : s_n^{n-1} d_0^{n-1} + d_0^n s_{n+1}^n + d_0^n s_1^n.
\end{aligned}$$

For the elementary tensor product $w = y_{a_1} \otimes \cdots \otimes y_{a_j} \otimes y_{a_{j+1}} \otimes \cdots \otimes y_{a_{n+1}}$, we have

$$\begin{aligned}
&\text{Type 1: for } 1 \le k \le n+1,\ d_k^n s_k^n (w) \\
&= \frac{1}{N} y_{a_1} \otimes \cdots \otimes \left( a_k y_{a_k} + \sum_{i=1}^{a_k - 1} y_{|a_k - 2i|} \right) \otimes \cdots \otimes y_{a_{n+1}} \\
&= \frac{a_k}{N} w + \text{terms with degree at most } N-2;
\end{aligned}$$

$$\begin{aligned}
&\text{Type 2: for } 1 \le j < k \le n,\ \left( s_k^{n-1} d_j^{n-1} + d_j^n s_{k+1}^n \right)(w) = \\
&(-1)^{j+k} \left( \frac{1}{N} - \frac{1}{N - 2\min\{a_j, a_{j+1}\}} \right) y_{a_1} \otimes \cdots \otimes y_{|a_j - a_{j+1}|} \cdots \otimes \tilde{s}\left(y_{a_k}\right) \otimes \cdots \otimes y_{a_{n+1}};
\end{aligned}$$

$$\begin{aligned}
&\text{Type 3: for } 1 \le j < k \le n,\ \left( s_k^{n-1} d_j^{n-1} + d_j^n s_{k+1}^n \right)(w) = \\
&(-1)^{j+k} \left( \frac{1}{N} - \frac{1}{N - 2\min\{a_j, a_{j+1}\}} \right) y_{a_1} \otimes \cdots \otimes \tilde{s}\left(y_{a_k}\right) \cdots \otimes y_{|a_j - a_{j+1}|} \otimes \cdots \otimes y_{a_{n+1}};
\end{aligned}$$

$$\begin{aligned}
&\text{Type 4: for } 1 \le k \le n-1,\ \left( s_k^{n-1} d_0^{n-1} + d_0^n s_{k+1}^n \right)(w) = \\
&(-1)^{k} \left( \frac{1}{N} - \frac{1}{N - 2\min\{a_0, a_{n+1}\}} \right) y_{a_2} \otimes \cdots \otimes \tilde{s}\left(y_{a_k}\right) \cdots \otimes y_{a_n} \otimes y_{|a_0 - a_{n+1}|};
\end{aligned}$$

$$\begin{aligned}
&\text{Type 5: for } 1 \le k \le n,\ \left( s_k^{n-1} d_k^{n-1} + d_{k+1}^n s_k^n + d_k^n s_{k+1}^n \right)(w) = \\
&(-1)^{k} \left( \frac{1}{N} - \frac{1}{N - 2\min\{a_k, a_{k+1}\}} \right) y_{a_1} \otimes \cdots \otimes \tilde{s}\left( y_{|a_k - a_{k-1}|} \right) \otimes \cdots \otimes y_{a_{n+1}};
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&\text{Type 6: } \left( s_n^{n-1} d_0^{n-1} + d_0^n s_{n+1}^n + d_0^n s_1^n \right)(w) \\
&\equiv \left( \frac{1}{N} - \frac{1}{N - 2\min\{a_1, a_{n+1}\}} \right) y_{a_2} \otimes \cdots \otimes y_{a_n} \otimes \tilde{s}\left( y_{|a_1 - a_{n-1}|} \right),
\end{aligned}$$

which consists of terms of degree at most $N-2$ under the cyclic equivalence relation.

We rename the terms of degree less than $N$ as the *error terms.* The sum of the type $d_k s_k$ terms in the summation is precisely $w$ plus error terms which have degree at most $N-2$. All the other types of terms in the summation have degree at most $N-2$; i.e., these are all error terms. We will then estimate the size of the error terms. $\square$

Set $w_0 = w$ and $w_j = (sd + ds - 1)^j (w)$.

**Corollary 6.43.** *For all $j \geq \lfloor \frac{N}{2} \rfloor + 1$, we have $w_j = 0$.*

*Proof.* As $w_{j+1} = (sd + ds - 1)\left(w_j\right)$, we can see that the highest possible degree of $w_{j+1}$ is always 2 less than the highest possible degree of $w_j$, otherwise 0. Therefore $w_j = (sd + ds - 1)^j (w)$ has degree at most $N - 2j$ for $j \leq \lfloor \frac{N}{2} \rfloor$. After apply the $(sd + ds - 1)$ map $\lfloor \frac{N}{2} \rfloor$ times on the original $w$, the result will have degree 0. □

As the norm of $w_1$ may be greater than the norm of $w$, we seek an alternative way to estimate the error terms in the inductive process of the map $(sd + ds - 1)$.

**Lemma 6.44.** *Given the weight condition $\omega_R$, for all $R > 1$, there exists a positive integer $N_0$ and a constant $C_{N_0}$ such that $\left\|(sd + ds - 1)(w)\right\|$ is bounded by $C_{N_0}\|w\|$ if $N < N_0$ or bounded by $\frac{1}{2}\|w\|$ if $N \geq N_0$.*

*Proof.* For large $N$, consider different types of error terms in the expansion of $(sd + ds - 1)(y_N)$ from Lemma 6.42. The norm of Type 1 error terms are estimated by the summation $\left\|\sum_{i=1}^{a_k-1} y_{|a_k-2i|}\right\|$ with other fixed coefficients, which is a geometric progression. If $a_j = 0$, then we get value zero for the coefficient $\left(\frac{1}{N} - \frac{1}{N-2\min\{a_j,a_{j+1}\}}\right)$ in the Type 2, 3, 4 or 5 expansion. The norm of Type 2, 3, 4 and 5 error terms are estimated to $\frac{1}{N}R^{-2a_j}\|w_0\|$ or $\frac{1}{N}R^{-2a_{j+1}}\|w_0\|$, which is bounded by $\frac{4}{NR^2}$. We compute the size of the first error terms $w_1$ by adding up all types of error terms to obtain that

$$\|w_1\| = \left\|(sd + ds - 1)(w)\right\| \leq \left(\frac{4(n+1)^2}{N\left(R^2 - 1\right)} + \frac{4n^2}{NR^2}\right)\|w_0\|.$$

We will later pick a positive integer $N_0$ and for the degree of $w$ where $N \geq N_0$ and define

$$w_1 = w_{1,\text{high}} + w_{1,\text{low},0}.$$

For $N < N_0$, we write

$$w_1 = w_{1,\text{low},0},$$

where $w_{1,\text{high}}$ consists of terms with degree at least $N_0$ of $w_1$ and $w_{1,\text{low},0}$ consists of terms with degree less than $N_0$ of $w_1$ and $w_{1,\text{low},0}$.

When $N \geq N_0$, we can compute the size of the two sets of error terms as

$$\left\|w_{1,\text{high}}\right\| \leq \frac{4(n+1)^2 R^N}{N\left(R^2 - 1\right)} + \frac{4n^2 R^N}{NR^2},$$

$$\left\|w_{1,\text{low},0}\right\| \leq \frac{4(n+1)^2 R^{N_0}}{N\left(R^2 - 1\right)} + \frac{4n^2 R^{N_0}}{NR^2}.$$

Fix $N_0 = \frac{16(n+1)^2}{(R^2-1)}$. Then for all $N \geq N_0$, we have $\|w_1\| \leq \frac{1}{2}\|w_0\|$, which indicates both $\|w_{1,\text{high}}\|$ and $\|w_{1,\text{low},0}\|$ bounded by $\frac{1}{2}\|w_0\|$.

When $N < N_0$, we have

$$\|w_{1,\text{low},0}\| = \|w_1\| \leq \frac{8(n+1)^2}{R^2-1}\|w_0\| = C_{N_0}\|w_0\|.$$

□

For $N \geq N_0$, we write $w_0 = w_{0,\text{high}}$. Then $(sd + ds - 1)(w)$ may consist of terms of degree both greater or less than $N_0$, i.e. $w_1 = w_1 = w_{1,\text{high}} + w_{1,\text{low},0}$. We define

$$(sd + ds - 1)\left(w_{j,\text{high}}\right) = w_{j+1,\text{high}} + w_{j+1,\text{low},0}$$

and

$$(sd + ds - 1)\, w_{j,\text{low},k} = w_{j,\text{low},k+1},$$

where the high and low in the subscripts indicate the terms with large and small degree. Every specified error term is precisely obtained from a unique previous error term. Therefore there does not exist any interactions between the error terms with large and small degree during the computation of the powers of $(sd + ds - 1)$ and we use the diagram below to show the relation as

$$\begin{array}{lllll}
w_{0,\text{high}} & \longrightarrow w_{1,\text{low},1} & \longrightarrow \cdots & \longrightarrow w_{1,\text{low},\lfloor\frac{N_0}{2}\rfloor} & \longrightarrow 0 \\
\downarrow & & & & \\
w_{1,\text{high}} & \longrightarrow w_{2,\text{low},1} & \longrightarrow \cdots & \longrightarrow w_{2,\text{low},\lfloor\frac{N_0}{2}\rfloor} & \longrightarrow 0 \\
\downarrow & & & & \\
\vdots & & & & \\
\downarrow & & & & \\
w_{\lfloor\frac{N-N_0}{2}\rfloor,\text{high}} & \longrightarrow w_{\lfloor\frac{N-N_0}{2}\rfloor+1,\text{low},1} & \longrightarrow \cdots & \longrightarrow w_{\lfloor\frac{N-N_0}{2}\rfloor+1,\text{low},\lfloor\frac{N_0}{2}\rfloor} & \longrightarrow 0 \\
\downarrow & & & & \\
0 & & & &
\end{array}$$

where the vertical arrows indicate the the large error terms and the horizontal arrows indicate the small error terms when applying the $(sd + ds - 1)$ map. The vertical arrows only come out from the large error terms therefore the arrows don't cross. This setting implies that

$$w_j = w_{j,\text{high}} + \sum_{k=0}^{j} w_{j-k,\text{low},k}.$$

We have a norm control of the error terms as

$$\begin{aligned} \left\|w_{j+1,\text{low},0}\right\| &\leq \frac{1}{2}\left\|w_{j,\text{high}}\right\| \\ \left\|w_{j+1,\text{high}}\right\| &\leq \frac{1}{2}\left\|w_{j,\text{high}}\right\| \\ \left\|w_{j,\text{low},k+1}\right\| &\leq C_{N_0}\left\|w_{j,\text{low},k}\right\|, \end{aligned}$$

for all $j, k \geq 0$. From the degree-decreasing lemma above, we also have $\frac{1}{2}\left\|w_{j,\text{high}}\right\| = 0$ for all $j \geq \lfloor\frac{N-N_0}{2}\rfloor$ and $\left\|w_{j,\text{low},k}\right\| = 0$ for all $k \geq \lfloor\frac{N_0}{2}\rfloor$.

To compute the norm of all error terms precisely, we have

$$\sum_{j=0}^{\lfloor\frac{N-N_0}{2}\rfloor} \left\|w_{j,\text{high}}\right\| \leq \sum_{j=0}^{\lfloor\frac{N-N_0}{2}\rfloor} 2^{-j}\|w\| \leq \|w\|$$

and

$$\begin{aligned} \sum_{k}^{\lfloor\frac{N_0}{2}\rfloor} \left\|w_{j+1,\text{low},k+1}\right\| &\leq \sum_{k}^{\lfloor\frac{N_0}{2}\rfloor} C_{N_0}^{k}\left\|w_j\right\| \\ &\leq C_{\text{low}}\left\|w_j\right\|. \end{aligned}$$

For $N < N_0$, we have

$$\sum_{k=1}^{\lfloor\frac{N}{2}\rfloor}\left\|w_{1,\text{low},k}\right\| \leq C_{N_0}\|w\|.$$

As in Remark 6.35, we extend the setting of $s$ and $d$ maps to the Banach algebra $\mathcal{A}_{2,\omega_R}$ and compute the simplicial and cyclic cohomology groups with a verification of the boundedness to the coboundaries. We apply the same argument to the $n$-cocycle $\phi(1,\ldots,1)(1)$ in Theorem 6.36 and Theorem 6.37 to check whether the function supported by terms with zero degree can be cobounded.

**Theorem 6.45.** *Let $n$ be a positive integer greater than 1. We have*

1. $\mathcal{HC}^n\left(\mathcal{A}_{2,\omega_R}\right) \simeq \mathbb{C}^1$ *if $n$ is even.*

2. $\mathcal{HC}^n\left(\mathcal{A}_{2,\omega_R}\right) \simeq 0$ *if $n$ is odd.*

*Proof.* As $\phi$ is cyclic, we cannot cobound $\phi\left(y_0,\ldots,y_0\right)\left(y_0\right)$ and its scalar multiples when $n$ is even. When $n$ is odd, we have $\phi\left(y_0,\ldots,y_0\right)\left(y_0\right) = 0$. Given a bounded function in the cyclic $n$-cocylce $\phi \in \mathcal{ZC}^n\left(\mathcal{A}_{\omega_R}\right)$ where the total degree of the monomials in $\phi$ are not zero, we shall construct a bounded cyclic function $\psi$ such that $\delta\psi = \phi$.

We write $\phi(w) = \phi\left(y_{a_1},\dots,y_{a_n}\right)\left(y_{a_{n+1}}\right)$ for $w = y_{a_1}\otimes\cdots\otimes y_{a_{n+1}}$ and $\psi(w) = \phi\left(y_{a_1},\dots,y_{a_{n-1}}\right)\left(y_{a_n}\right)$ for $w = y_{a_1}\otimes\cdots\otimes y_{a_n}$. The $d$ operator in the pre-dual corresponds to the dual operator $\delta$ where $\delta\phi(w) = \phi(dw) = 0$.

Set $\phi_0 = \phi$. Define $\psi_0(w) := s^*\phi(w) = \phi\big(s(w)\big)$. We have

$$\phi\big((sd+ds)\,w\big) = \phi(w+w_1) = \delta s^*\phi(w) + s^*\delta\phi(w) = \delta\psi_0(w),$$

which implies

$$\delta\psi_0(w) = \phi(w) + \phi(w_1),$$

Now define $w_j = (sd+ds-1)^j\,w_0$ and $\psi_j(w) := s^*\phi\left(w_j\right)$. We have

$$\delta\psi_j(w) = \delta s^*\phi\left(w_j\right) + s^*\delta\phi\left(w_j\right) = \phi\left((sd+ds)\,w_j\right) = \phi\left(w_j\right) + \phi\left(w_{j+1}\right).$$

Let $\psi = \sum_{j=0}^{\lfloor\frac{N}{2}\rfloor}(-1)^j\psi_j$, we have

$$\delta\psi(w) = \sum_{j=0}^{\lfloor\frac{N}{2}\rfloor}(-1)^j\left(\phi\left(w_j\right) + \phi\left(w_{j+1}\right)\right) = \phi(w_0) + (-1)^{\lfloor\frac{N}{2}\rfloor}\phi\left(w_{\lfloor\frac{N}{2}\rfloor}\right) = \phi(w),$$

which shows that $\psi$ cobounds $\phi$ when the degree $N$ is nonzero.

We will now show that $\psi$ is indeed bounded. If $N < N_0$, we simply have $w_1 = w_{1,\text{low},0}$ and

$$\|w_1\| = \left\|w_{1,\text{low},0}\right\| \le C_{N_0}\|w\|.$$

In this case, we can compute $\|\phi\|$ straightforwardly as

$$\begin{aligned}
\left|\psi(w)\right| &\le \sum_{j=0}^{\lfloor\frac{N}{2}\rfloor}\psi_j(w)\\
&\le \sum_{j=0}^{\lfloor\frac{N}{2}\rfloor}\phi\left(s^n w_j\right)\\
&\le \sum_{j=0}^{\lfloor\frac{N}{2}\rfloor}\left\|w_j\right\|\|s^n\|\|\phi\|\\
&\le (2C_{\text{low}}+2)\|\phi\|\|w\|.
\end{aligned}$$

When $N \geq N_0$, we have

$$
\begin{aligned}
\left|\psi\left(w\right)\right| &\leq \sum_{j=0}^{\lfloor \frac{N}{2} \rfloor} \left|\psi_j\left(w\right)\right| \\
&\leq \sum_{j=0}^{\lfloor \frac{N}{2} \rfloor} \left|\phi\left(s^n w_j\right)\right| \\
&\leq \sum_{j=0}^{\lfloor \frac{N-N_0}{2} \rfloor} \|s^n\| \|\phi\| \left\|w_{j,\mathrm{high}}\right\| + \sum_{j=1}^{\lfloor \frac{N-N_0}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{N_0}{2} \rfloor} \|s^n\| \|\phi\| \left\|w_{j,\mathrm{low},k}\right\| \\
&\leq \|s^n\| \|\phi\| \|w\| + C_{\mathrm{low}} \|\phi\| \|w\| \\
&\leq (2C_{\mathrm{low}} + 4) \|\phi\| \|w\| ,
\end{aligned}
$$

which shows that $\psi$ is bounded by $(2C_{\mathrm{low}} + 4)$ for all $N \geq 1$.

□

We define another splitting map $s_0^n := \bigotimes^{n+1} \tilde{\mathcal{A}}_{2,\omega_R} \longrightarrow \bigotimes^{n+2} \tilde{\mathcal{A}}_{2,\omega_R}$ as

$$
\begin{aligned}
s_0^n\left(w\right) = & \frac{(-1)^{n+1}}{N} \sum_{i=1}^{a_1 - 1} 2 y_{a_1 - i} \otimes \cdots \otimes y_{a_n} \otimes y_{a_{n+1} + i} \otimes y_0 \\
& + y_0 \otimes \cdots \otimes y_{a_n} \otimes y_{a_{n+1} + a_1} \otimes y_0
\end{aligned}
$$

for $a_1 \neq 0$ and $s_0^n\left(w\right) = 0$ for $a_1 = 0$.

**Lemma 6.46.** *Set $s' = s + s_0$. Let $w \in \bigotimes^{n+1} \tilde{\mathcal{A}}_{2,\omega_R}$ with degree $N \geq 2$. Then $\left(s'd + ds' - 1\right)\left(w\right)$ has degree at most $N - 2$.*

*Proof.* We compare the new map to the previous $\left(s'd + ds' - 1\right)$ and notice that the following terms are additional to the terms computed in the cyclic case:

$$
\begin{aligned}
& s_0 d_j + d_j s_0 \text{ for } 2 \leq j \leq n - 1, \\
& d_0 s_1 + d_{n+1} s_0, \\
& s_n d_0 + d_0 s_{n+1} + d_0 s_0, \\
& s_0 d_0 + s_0 d_1 + d_1 s_0.
\end{aligned}
$$

The expansion of these terms give

$$\text{for } 2 \le j \le n-1,\ \left(s_0 d_j + d_j s_0\right)(w) =$$
$$(-1)^{j+n+1}\left(\frac{1}{N} - \frac{1}{N - 2\min\{a_j, a_{j+1}\}}\right)\sum_{i=1}^{a_1-1} y_{a_1-i}\otimes\cdots\otimes y_{|a_j-a_{j+1}|}\cdots\otimes y_{a_{n+1}+i}\otimes y_0$$
$$+(-1)^{j+n+1}\left(\frac{1}{N} - \frac{1}{N - 2\min\{a_j, a_{j+1}\}}\right) y_0\otimes\cdots\otimes y_{|a_j-a_{j+1}|}\cdots\otimes y_{a_{n+1}+a_1}\otimes y_0;$$
$$\left(s_n d_0 + d_0 s_{n+1} + d_0 s_0\right)(w) =$$
$$\left(\frac{1}{N} - \frac{1}{N - 2\min\{a_1, a_{n+1}\}}\right) y_{a_2}\otimes\cdots\otimes y_{a_n}\otimes\tilde{s}\left(y_{|a_1-a_{n-1}|}\right);$$
$$\left(s_0 d_0 + s_0 d_1 + d_1 s_0\right)(w) =$$
$$\left(\frac{1}{N} - \frac{1}{N - 2\min\{a_1, a_2\}}\right)\sum_{i=1}^{|a_1-a_2|} y_{|a_1-a_2|-i}\otimes\cdots\otimes y_{a_n}\otimes y_{a_{n+1}+i}\otimes y_0$$
$$+\left(\frac{1}{N} - \frac{1}{N - 2\min\{a_1, a_2\}}\right) y_0\otimes\cdots\otimes y_{a_n}\otimes y_{a_{n+1}+|a_1-a_2|}\otimes y_0$$
$$\text{for } a_1 \ge 2,\ \left(d_0 s_1 + d_{n+1} s_0\right)(w) = -\frac{1}{N}\sum_{i=1}^{a_1-1} y_{a_1-i}\otimes\cdots\otimes y_{a_n}\otimes y_{|a_{n+1}-i|},$$
$$\text{for } a_1 \le 2,\ \left(d_0 s_1 + d_{n+1} s_0\right)(w) = 0,$$

which shows that these terms consist of elementary tensor products of degree at most $N-2$. ☐

Now we estimate the norm of error terms after $j$ steps of inductive process; i.e., to compute the size of $\left(s'd + ds' - 1\right)^j(w)$.

**Lemma 6.47.** *Given the weight condition $\omega_R$, for all $R$ not close to 1, there exists $N_0'$ such that $\left\|\left(s'd + ds' - 1\right)(w)\right\|$ is bounded by $H_{N_0}\|w\|$ if $N < N_0'$ or bounded by $\frac{1}{2}\|w\|$ if $N \ge N_0'$.*

*Proof.* We have computed the extra error terms in $\left(s'd + ds' - 1\right)(w)$ which are not included in the calculation of $(sd + ds - 1)(w)$. From the expansions we observe that

$$\left\|\left(s'd + ds' - 1\right)(w) - (sd + ds - 1)(w)\right\| \le \left\|(sd + ds - 1)(w)\right\|,$$

which implies $\left\|\left(s'd + ds' - 1\right)(w)\right\| \le 2\left\|(sd + ds - 1)(w)\right\|$. Note that there is a factor of $\frac{1}{N}$ in both norms. Therefore we use the value of $N_0$ obtained from Lemma 6.44, and set $N_0' = 2N_0$. This completes the classification of the high and low error terms similar to the cyclic case. ☐

Set $w'_0 = w$ and $w'_j = \left(s'd + ds' - 1\right)^j(w)$. For $N \geq N'_0$, we write $w'_0 = w'_{0,\text{high}}$. Then $\left(s'd + ds' - 1\right)(w)$ may consist of terms both greater or less than $N'_0$, i.e. $w'_1 = w'_{1,\text{high}} + w'_{1,\text{low},0}$. We define

$$\left(s'd + ds' - 1\right)\left(w'_{j,\text{high}}\right) = w'_{j+1,\text{high}} + w'_{j+1,\text{low},0}$$

and

$$\left(s'd + ds' - 1\right) w'_{j,\text{low},k} = w'_{j,\text{low},k+1},$$

as we defined $w_j$ with high and low terms for $(sd + ds - 1)^j(w)$

**Theorem 6.48.** *Let $n$ be a positive integer greater than 1. We have $\mathcal{HH}\left(\mathcal{A}_{2,\omega_R}\right) \simeq 0$.*

*Proof.* The proof follows the same the steps as in Theorem 6.45. □

## 6.4 Future work

In this section, we consider the higher simplicial cohomology groups for the invariant subalgebras $\mathcal{A}_{k+1,\omega_R}$ and $\mathcal{B}_{k,\omega_R}$ on the weighted type $\tilde{A}$ and type $\tilde{B}$ lattices for $k \geq 2$. The computations for the cohomology groups and the verification for the boundedness of the coboundaries are yet to be done. Both $\mathcal{A}_{k+1,\omega_R}$ and $\mathcal{B}_{k,\omega_R}$ have $k$ generators hence it is natural to consider the analysis of the higher simplicial cohomology groups of the algebra $\ell^1\left(\mathbb{Z}_+^k\right)$ in [32].

The algebra $\ell^1\left(\mathbb{Z}_+^k\right)$ has $k$ generators, namely the set of variables $\left\{z_j\right\}_{j=1}^k$ for the polynomials. Set $\mathcal{A} = \ell^1(\mathbb{Z}_+)$ and $\mathcal{I} = \ell^1(\mathbb{N})$, a closed ideal of $\mathcal{A}$. Theorem 7.5 in [32] states that up to topological isomorphism,

1. $\mathcal{HH}^n\left(\ell^1\left(\mathbb{Z}_+^k\right)\right) \simeq 0$ if $n > k$.
2. $\mathcal{HH}^n\left(\ell^1\left(\mathbb{Z}_+^k\right)\right) = \bigoplus^{\binom{k}{n}}\left[\left(\mathcal{I}^{\hat{\otimes}^n}\hat{\bigotimes}\mathcal{A}^{\hat{\otimes}^{k-n}}\right)'\right]$ if $n \leq k$.

We consider the pre-dual $\hat{\bigotimes}^{n+1}\ell^1\left(\mathbb{Z}_+^k\right)$. For $n > k$, there exists a bounded contracting homotopy map $s^{n+1} : \hat{\bigotimes}^{n+1}\ell^1\left(\mathbb{Z}_+^k\right) \longrightarrow \hat{\bigotimes}^{n+2}\ell^1\left(\mathbb{Z}_+^k\right)$ and a bounded face map $d^{n+1} : \hat{\bigotimes}^{n+2}\ell^1\left(\mathbb{Z}_+^k\right) \longrightarrow \hat{\bigotimes}^{n+1}\ell^1\left(\mathbb{Z}_+^k\right)$, abbreviated as $s$ and $d$ respectively, such that the map $\left(s^n d^n + d^{n+1}s^{n+1}\right)$ is the identity map. This shows that given an elementary product $w \in \hat{\bigotimes}^{n+1}\ell^1\left(\mathbb{Z}_+^k\right)$ with positive degree, we have $\left(s^n d^n + d^{n+1}s^{n+1}\right)(w) = w$. For a bounded $n$-cocycle $\phi$, we take the dual map $s^*$ to define $\psi = s^*\phi$ that cobounds $\phi$. The face map $d$ is given by the dual of the $\delta$ map defined in Definition 6.22 hence the map $d$ preserves the degree of $w$. We will analyse the property of the map $s$ and show that it maps an elementary tensor product $w$ to homogeneous terms with the same degree.

**Lemma 6.49.** *Let the map $s$ be defined as above and $w \in \hat{\bigotimes}^{n+1} \ell^1 \left(\mathbb{Z}_+^k\right)$ be an elementary tensor product with degree $N > 0$. There exists a map $s : \hat{\bigotimes}^{n+1} \ell^1 \left(\mathbb{Z}_+^k\right) \longrightarrow \hat{\bigotimes}^{n+2} \ell^1 \left(\mathbb{Z}_+^k\right)$ such that $s(w) \in \hat{\bigotimes}^{n+2} \ell^1 \left(\mathbb{Z}_+^k\right)$ also has degree $N > 0$.*

*Proof.* Since $(sd + ds)$ is the identity map, $(sd + ds)(w) = w$ has degree $N$. Let $w_{\deg j}$ be the terms in $w$ with degree $j$. Then we have

$$w = \sum_{j=0}^{\infty} \left( \left(ds(w)\right)_{\deg j} + sd(w)_{\deg j} \right) = d \left( \sum_{j=0}^{\infty} \left(s(w)\right)_{\deg j} \right) + \sum_{j=0}^{\infty} \left(sd(w)\right)_{\deg j} = w_{\deg N}.$$

Without loss of generality, we can set $\left(sd(w)\right)_{\deg j}$ and $\left(s(w)\right)_{\deg j}$ to be zero for all $j \neq N$. Set $\check{s}(w) = s(w)_{\deg N}$. Then $\check{s}$ is a map such that $(\check{s}d + d\check{s})$ is the identity map. Hence we can use $\check{s}$ as the contracting homotopy map $s$ where $s(w)$ consists of homogeneous terms of degree $N$. □

We aim to compute the higher simplicial cohomology groups for the algebras on the weighted type $\tilde{A}$ and type $\tilde{B}$ lattices and their invariant subalgebras under the Weyl group actions. The statement for the conjecture of the higher simplicial cohomology groups on these algebras are similar. Let $A$ be one of the following algebras or invariant subalgebras: $\ell^1 \left(\Lambda_k, \omega_R\right)$, $\ell^1 \left(\mathbb{Z}^k, \omega_R\right)$, $\mathcal{A}_{k+1,\omega_R}$ and $\mathcal{B}_{k,\omega_R}$. Up to an topological isomorphism,

1. $\mathcal{HH}^n \left(A\right) \simeq 0$ if $n > k$.
2. $\mathcal{HH}^n \left(A\right)$ is $\binom{k}{n}$-dimensional if $n \leq k$.

For $n > k$, in each one of the cases of the above algebras and subalgebras, there are some computational difficulties to obtain the precise bounded $n$-coboundaries.

The first example is $\tilde{\mathcal{A}}_{k+1,\omega_R} = \ell^1 \left(\Lambda_k, \omega_R\right)$, the algebra of summable functions on the $\omega_R$-weighted $k$-dimensional $\Lambda_k$ lattice. As described in Section 6.1, the characteristic functions on the vertices of $\Lambda_k$ are isomorphic to the monomials of $k$ variables with non-negative powers. The $k$ variables in the monomials can be used as the $k$ generators of the algebra. As the weight condition $\omega_R$ is multiplicative on the lattice $\Lambda_k$, for $n > k$, we define a bounded contracting homotopy map $s_{\tilde{A}} : \hat{\bigotimes}^{n+1} \tilde{\mathcal{A}}_{k+1,\omega_R} \longrightarrow \hat{\bigotimes}^{n+2} \tilde{\mathcal{A}}_{k+1,\omega_R}$ such that for all $n$-cocycles $\phi \in \mathcal{ZC}^n \left(\tilde{\mathcal{A}}_{k+1,\omega_R}\right)$, the bounded cochain $\psi = s^*_{\tilde{A}}\phi$ cobounds $\phi$. As the contracting homotopy map $s$ on the tensor product of the algebra $\ell^1 \left(\mathbb{Z}_+^k\right)$ is homogeneous, the contracting homotopy map $s$ on the tensor product of the algebra $\tilde{\mathcal{A}}_{k+1,\omega_R}$ also preserves the powers of the variables, thus preserve the norm on the $\omega_R$-weighted $\Lambda_k$ lattice.

We then consider $\tilde{\mathcal{B}}_{k,\omega_R} = \ell^1 \left(\mathbb{Z}^k, \omega_R\right)$, the algebra of summable functions on $\mathbb{Z}^k$ with the weight condition $\omega_R$. The construction of the contracting homotopy map $s_{\tilde{B}}$ :

$\hat{\bigotimes}^{n+1}\tilde{\mathcal{B}}_{k,\omega_R} \longrightarrow \hat{\bigotimes}^{n+2}\tilde{\mathcal{B}}_{k,\omega_R}$ on an elementary tensor product $w \in \hat{\bigotimes}^{n+1}\tilde{\mathcal{B}}_{k,\omega_R}$ is the same as the contracting homotopy map $s$ for the algebra $\ell^1\left(\mathbb{Z}_+^k\right)$ if the powers on all variables in $w$ are all positive or all negative. When there exist both positive and negative powers on the variables of $w$, the contracting homotopy map $s_{\tilde{B}}$ might be different. The analysis will also base on the construction of the 2-coboundaries of $\mathcal{HC}^2\left(\ell^1\left(\mathbb{Z},\omega_R\right)\right)$, where we used different constructions of $\psi$ for vertices on different sections of $\mathbb{Z}^2$.

The invariant subalgebras under the corresponding Weyl group actions, $\mathcal{A}_{k+1,\omega_R}$ and $\mathcal{B}_{k,\omega_R}$, can both be considered as algebras on $\mathbb{Z}_+^k$ with weight conditions $\omega_R$ and multiplication rules given by the symmetric Laurent polynomials. The face map $d$, which is the dual map of $\delta$ defined in Definition 6.22, is not homogeneous on an elementary tensor product $w \in \hat{\bigotimes}^{n+1}\mathcal{A}_{k+1,\omega_R}$ or $w \in \hat{\bigotimes}^{n+1}\mathcal{B}_{k,\omega_R}$. If we take a similar homogeneous contracting homotopy map $s_A$ or $s_B$ on an elementary tensor product $w$ of $\mathcal{A}_{k+1,\omega_R}$ or $\mathcal{B}_{k,\omega_R}$ as the contracting map $s$ on an elementary tensor product of $\ell^1\left(\mathbb{Z}_+^k\right)$, there might not exist any terms with the same degree of $w$ in $\left(s_Ad + ds_A\right)(w)$ or $\left(s_Bd + ds_B\right)(w)$; i.e., the maps $\left(s_Ad + ds_A\right)$ and $\left(s_Bd + ds_B\right)$ might not be the approximations of the identity maps on the two subalgebras.

However it might be possible to construct a contracting homotopy map on an elementary tensor product of the $k$ generators in the invariant subalgebras. In this case, it is hard to estimate the norm of the precise contracting homotopy map obtained from the operations on the powers of generators. One possible route is to analyse the properties of the coefficients used in the transform between the linear sums of symmetric polynomials and the powers of the generators which are the elementary symmetric polynomials.

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