Homological Algebra and Friezes

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Abstract

Over the last decade frieze patterns, as introduced by Conway and Coxeter in the 1970's, have been generalised in many ways. One such exciting development is a homological interpretation of frieze patterns, which we call friezes. A frieze in the modern sense is a map from a triangulated category C to some ring. A frieze X is characterised by the propety that if $\tau x \to y \to x$ is an Auslander-Reiten triangle in C, then $X(\tau x)X(x) - X(y) = 1$. A canonical example of a frieze is the Caldero-Chapoton map.

The more general notion of a generalised frieze was introduced by Holm and Jørgensen in [25] and [26]. A generalised frieze X' carries the more general property that $X'(\tau x)X'(x) - X'(y) \in \{0, 1\}$. In [25] and [26] Holm and Jørgensen also introduced a modified Caldero-Chapoton map, which satisfies the properties of a generalised frieze.

This thesis consists of six chapters. The first chapter provides a detailed outline of the thesis, whilst setting some of the main results in context and explaining their significance.

The second chapter provides a necessary background to the notions used throughout the remaining four chapters. We introduce triangulated categories, the derived category, quivers and path algebras, Auslander-Reiten theory and cluster categories, including the polygonal models associated to the cluster categories of Dynkin types A_n and D_n.

The third chapter is based around the proof of a multiplication formula for the modified Caldero-Chapoton map, which significantly simplifies its computation in practice. We define Condition F for two maps α and β, and show that when our category is 2-Calabi-Yau, Condition F implies that the modified Caldero-Chapoton map is a generalised frieze. We then use this to prove our multiplication formula.

The definition of the modified Caldero-Chapoton map requires a rigid subcategory R that sits inside a cluster tilting subcategory T. Chapter 4 proves several results showing that in the case of the cluster category of Dynkin type A_n, the modified Caldero-Chapoton map depends only on the rigid subcategory R. These results then allow us to prove a general formula for the group $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$, which is used in the definition of the modified Caldero-Chapoton map.

Chapter 5 provides a comprehensive list of exchange triangles in the cluster category of Dynkin type D_n.

Chapter 6 then proves several similar results to Chapter 4 in the case of the cluster category of Dynkin type D_n. We prove that the modified Caldero-Chapoton map depends only on the rigid subcategory R before again producing a general formula for $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$.

Contents

Chapter 1

Thesis Outline

In this chapter we will give a brief introduction to this thesis by outlining the layout to each chapter, detailing the main results and explaining their significance.

A large amount of the work in this thesis is an extension of the results in [26], in which the notion of a generalised frieze is introduced, as well as a modified version of the Caldero-Chapoton map. We manage to improve upon a result in that paper, as well as answer some important questions posed by its authors.

1.1 Chapter 2: Introduction

Chapter 2 provides all the key and necessary background material required to read and understand the remaining four chapters.

We begin in Section 2.1 by discussing triangulated categories, which were first introduced in [44] in the 1960's by J. L. Verdier. Triangulated categories are a somewhat weaker notion than that of an abelian category, however carry some relatable ideas. We introduce the notion of a (distinguished) triangle, which is often thought of as the triangulated analogue of a short exact sequence. In Definition 2.1.1 we define a triangulated category before giving some key properties in the propositions that follow.

Section 2.2 demonstrates how one constructs the derived category, a very well studied example of a triangulated category. We begin by defining a multiplicative system of morphisms S inside some triangulated category C, before detailing how to construct the category $S^{-1}\mathsf{C}$, the localisation of C by S. We give details of the make-up of the set of morphisms in $S^{-1}\mathsf{C}$, and then introduce the localisation functor $L : \mathsf{C} \to S^{-1}\mathsf{C}$. The remainder of Section 2.2 then shows how one can construct the derived category from the category of chain complexes $\mathsf{C}(\mathcal{A})$ over an additive category $\mathcal{A}$. We give a description of $\mathsf{C}(\mathcal{A})$ before passing to the homotopy category $\mathsf{K}(\mathcal{A})$, itself a triangulated category. We describe the triangles in $\mathsf{K}(\mathcal{A})$ and then show how to construct the derived category

$\mathsf{D}(\mathcal{A})$ by localising $\mathsf{K}(\mathcal{A})$ with respect to the set of quasi isomorphisms, which satisfy the properties of a multiplicative system. The final part of the section contains a discussion on how the derived category is triangulated.

Section 2.3 introduces quivers, path algebras and quiver representations. After defining a quiver and giving some examples, we show how to construct the path algebra $\mathbb{C}Q$ of a quiver Q. The final part of the section then introduces quiver respresentations and gives a proposition telling us that the category of quiver representations over $\mathbb{C}$ is equivalent to $\mathsf{Mod}\,\mathbb{C}Q$, the category of modules over $\mathbb{C}Q$.

Section 2.4 talks about some basic notions of Auslander-Reiten theory. Auslander-Reiten triangles are a much used object in this thesis and this section works towards their definition and key properties. We define left minimal almost split morphisms and right minimal almost split morphisms, and use these to define almost split sequences, more commonly known as Auslander-Reiten sequences. After giving some key properties, we define their triangulated analogue, namely Auslander-Reiten triangles. We provide a brief introduction to Serre functors and the existence of Auslander-Reiten triangles before ending the section with the defintion and examples of the Auslander-Reiten quiver. We provide some emphasis to the importance of the Auslander-Reiten quiver, and in our examples demonstrate how it allows us to read off Auslander-Reiten triangles.

Section 2.5 is the final section of the chapter and is based around the cluster category, a further example of a triangulated category and the one which we use most often in this thesis. The cluster category was introduced by Buan, Marsh, Reineke, Reiten and Todorov in [8], and is widely considered to be the categorification of the so-called cluster algebras introduced by Fomin and Zelevinsky in [17]. We give a basic definition of the cluster category $\mathsf{C}(Q)$, where Q is a suitable quiver, as well as some of its key properties. The later parts of the section are dedicated to describing the polygon model associated to $\mathsf{C}(A_n)$ and the once punctured polygon model associated to $\mathsf{C}(D_n)$. Here, $\mathsf{C}(A_n)$ denotes the cluster category of Dynkin type A_n and $\mathsf{C}(D_n)$ denotes the cluster category of Dynkin type D_n. We describe models in which indecomposables in our respective categories correspond to internal diagonals in the polygon, whilst rigid subcategories correspond to polygon dissections and cluster tilting subcategories correspond to polygon triangulations. In describing these models, we also draw the Auslander-Reiten quivers in each case, as well as demonstrating what the functor Σ and exchange triangles correspond to inside these models.

1.2 Chapter 3: A Multiplication Formula for the Modified Caldero-Chapoton Map

Chapter 3 focuses on the Modified Caldero-Chapoton map introduced by Holm and Jørgensen in [26], and the main results add to and improve upon the results given there.

The original Caldero-Chapoton map was introduced by Caldero and Chapoton in [10] and is a map $\gamma_T : \mathsf{C}(Q) \to A(Q)$, where $\mathsf{C}(Q)$ is a cluster category and $A(Q)$ is the corresponding cluster algebra. The map depends on a cluster tilting object T and it is known that the map γ_T sends so-called "reachable" indecomposable objects in $\mathsf{C}(Q)$ to cluster variables in $A(Q)$, a result that makes firm the idea that cluster categories are a categorification of cluster algebras.

It is also known that the Caldero-Chapoton map satisfies the properties of a frieze. A frieze is a map $X : \operatorname{obj} \mathsf{C} \to A$, where C is some triangulated category with Auslander-Reiten triangles and A is a ring, such that the following exponential conditions are satisfied:

$$X(0) = 1 \text{ and } X(a \oplus b) = X(a)X(b), \tag{1.1}$$

and if $\tau x \to y \to x$ is an Auslander-Reiten triangle, then

$$X(\tau x)X(x) - X(y) = 1.$$

These friezes are a seen as a modern day version of the combinatorial objects known as Conway-Coxeter friezes introduced in the 1970s in [13] and [14].

The modified Caldero-Chapoton map takes a more general viewpoint than γ_T and is a map $\rho_R : \operatorname{obj} \mathsf{C} \to A$ for some suitable triangulated category C and a commutative ring A. Here, as well as being defined on a more general category, the modified Caldero-Chapoton map relies on a rigid object R, a weaker notion than that of a cluster tilting object.

The modified Caldero-Chapoton map is defined as follows:

$$\rho(c) = \alpha(c) \sum_{e \in \mathrm{K}_0(\mathsf{fl}\,\mathsf{R})} \chi(\mathrm{Gr}_e(Gc))\beta(e), \tag{1.2}$$

where Gr_e is a certain Grassmannian, χ is the Euler characteristic and α and β are maps satisfying exponential properties similar to those in (1.1). It is proved in [26] that when α and β satisfy a technical "frieze-like" condition, the map ρ_R satisfies the properties of a generalised frieze. A generalised frieze X' is a similarly defined map to a frieze that again satisfies the exponential properties in (1.1), however allows the more general property that

$$X'(\tau x)X'(x) - X'(y) \in \{0, 1\}.$$

In Section 3.2 we define what it is for α and β to satisfy Condition F, a significantly less technical definition than frieze-like, before subsequently proving that when our category C is 2-Calabi-Yau, Condition F manages to replace the frieze-like definition from [26]. That is, we prove that when α and β satisfy Condition F, the map ρ_R is a generalised frieze.

Holm and Jørgensen also provide defintions of maps α and β which satisfy their frieze-like condition. We show in Section 3.3 that the same α and β also satisfy Condition F, which in turn recovers a main theorem from [26], stating that the definitions of α and β provided turn ρ_R into a generalised frieze.

In [25], Holm and Jørgensen provide a multiplication formula for ρ_R in a special case, namely the case when α and β are both set equal to one. This multiplication formula is an iterative formula and significantly simplifies the computation of ρ_R in practice, allowing a calculation without needing to compute the Grassmannian given in the definition. In Section 3.4, we outline the proof to the multiplication formula in [25], before adapting this proof in Section 3.5 to prove a multiplication formula for the definition of ρ_R from (1.2). We give a description of this formula here.

Let $R \in \operatorname{obj} \mathsf{C}$ be our rigid object and consider the subcategory $\mathsf{R} = \operatorname{add} R$, which is rigid in the sense that $\operatorname{Hom}_{\mathsf{C}}(\mathsf{R}, \Sigma\mathsf{R}) = 0$. We denote by $\operatorname{indec} \mathsf{C}$ the set of indecomposables in C and by $\operatorname{indec} \mathsf{R}$ the set of indecomposables in R. Now, let $m \in \operatorname{indec} \mathsf{C}$ and $r \in \operatorname{indec} \mathsf{R}$ be such that $\dim_{\mathbb{C}}\operatorname{Ext}^1_{\mathsf{C}}(m, r) = \dim_{\mathbb{C}}\operatorname{Ext}^1_{\mathsf{C}}(r, m) = 1$. Then, there are exchange triangles

$$m \to a \to r \to \Sigma m \quad \text{and} \quad r \to b \to m \to \Sigma r.$$

Our multiplication formula tells us that

$$\rho(r)\rho(m) = \rho(a) + \rho(b).$$

Following the proof of the formula, we explain how this formula can be applied iteratively to compute $\rho(m)$, reducing our computation to merely computing ρ for indecomposables x for which $\operatorname{Gr}(Gx)$ is a single point.

The final section in Chapter 3 provides two examples demonstrating the computation of ρ_R using the multiplication formula. In the first example, we compute the same example given in [26] to demonstrate that the multiplication formula does indeed make the computations simpler. Our second example is then in the case of $\mathsf{C}(A_9)$, a larger example which again shows the simplicity of the multiplication formula.

1.3 Chapter 4: Properties of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ in $\mathsf{C}(A_n)$.

In Chapter 4 we again work with ρ_R along with the definitions of α and β that satisfy Condition F. An essential construction in these definitions of α and β is the group we call

$\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$. We choose a rigid subcategory R and a cluster tilting subcategory T such that $\mathsf{R} \subseteq \mathsf{T}$. This means that there is some subcategory S such that

$$\operatorname{indec}\mathsf{T} = \operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S}.$$

The group $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, known as the split Grothendieck group of T, is the free abelian group generated by $[t]$ for $t \in \operatorname{indec}\mathsf{T}$. For each $t \in \operatorname{indec}\mathsf{T}$, there is an indecomposable $t^* \in \operatorname{indec}\mathsf{C}$ (known as the mutation of t) such that replacing t with t^* creates a new cluster tilting subcategory. The subgroup N is then generated by the difference in middle terms between exchange triangles

$$s \to A \to s^* \quad \text{and} \quad s^* \to A' \to s$$

for indecomposables $s \in \operatorname{indec}\mathsf{S}$. Additionally note that the definitions of α and β depend on an exponential map $\varepsilon : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to A$.

All the results in Chapter 4 are in the situation where $\mathsf{C} = \mathsf{C}(A_n)$. In the first result, which is in Section 4.2, we work towards is a proof that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ depends only on the choice of subcategory R, that is, it is independent of the choice of S. We recall the polygonal model that is associated to $\mathsf{C}(A_n)$ and consider two different cluster tilting subcategories T and T^*, where $\operatorname{indec}\mathsf{T}^*$ is the cluster tilting subcategory obtained by replacing a diagonal $s \in \operatorname{indec}\mathsf{S}$ with its mutation. This leads to two quotient groups $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ and $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*$. To prove that our quotient is independent of the choice of S, we construct an isomorphism between $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ and $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*$. The construction of this isomorphism relies heavily on the polygonal model associated to $\mathsf{C}(A_n)$.

The final part of Section 4.2 uses our result about the quotient to prove Theorem 4.2.5, which says that for two choices of cluster tilting subcategory T and T', we can construct a map $\varepsilon' : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N' \to A$ from $\varepsilon : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to A$ such that $\rho_R = \rho'_R$. Here, ρ'_R is the modified Caldero-Chapoton map defined with respect to T'. This theorem further verifies that ρ_R is indeed dependent only on a rigid object and not a cluster tilting object.

Since we know that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is independent of the choice of S, we are able in Section 4.3 to prove a general formula for $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ that depends on the number of indeccomposables in R. This formula is proved by induction. Our proof takes advantage of the fact that $\operatorname{indec}\mathsf{R}$ can be viewed as splitting our polygon P into smaller polygons (or cells), whilst $\operatorname{indec}\mathsf{S}$ then provides a triangulation of each of these cells. In our induction, the base case considers a polygon triangulation in which there are no diagonals in R. To induct, we introduce a gluing prodecure in which we glue cells to this polygon and identify the glued edges with indecomposables in R, thus creating a larger polygon and a larger cluster tilting subcategory. We are able to relate the quotient $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ before and after the gluing of a cell, and using a snake lemma argument produce a general formula given by the rank of the quotient.

1.4 Chapter 5: Exchange triangles in $\mathsf{C}(D_n)$

Chapter 5 is a precursor to Chapter 6, in which we prove similar results to Chapter 4, regarding the independence of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ under of choice of subcategory S in $\mathsf{C}(D_n)$. To prove that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is independent of the choice of S, we again require knowledge of exchange triangles. Chapter 5 provides a full classification of exchange triangles in $\mathsf{C}(D_n)$ as their construction using the polygon is somewhat less obvious than those in $\mathsf{C}(A_n)$. The comprehensive list of exchange triangles is given in Theorem 5.1.3, and then Section 5.3 provides a full proof to the theorem. Before proving the theorem, we provide several lemmas giving properties of triangles in $\mathsf{C}(D_n)$ which are heavily relied upon in each case of the proof. These lemmas can be found in Sections 5.1 and 5.2.

To know what the exchange triangles look like, we must know what their middle terms are. More formally, let a and a^* be two indecomposables in $\mathsf{C}(D_n)$ such that $\dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathsf{C}}(a, a^*) = \dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathsf{C}}(a^*, a) = 1$. Then, we know that there are exchange triangles

$$a \to e \to a^* \quad \text{and} \quad a^* \to e' \to a.$$

Theorem 5.1.3 tells us how to find e and e' using the polygon model associated to $\mathsf{C}(D_n)$. The proof of the theorem uses the lemmas proved earlier to narrow down the possibilities for indecomposables summands of e and e'. Once narrowed down, it verifies that each of the possibilites remaining must in fact be a summand of the corresponding middle term.

1.5 Chapter 6: Properties of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ in $\mathsf{C}(D_n)$

Chapter 6 largely follows the same pattern as Chapter 4, but in the case when $\mathsf{C} = \mathsf{C}(D_n)$. In order to prove the results in this chapter, in Section 6.1 we first give detailed descriptions of the possible types of triangulations of the once punctured polygon. These important descriptions allow us to discuss the models in greater depth and provide a basis upon which we can formulate arguments to our various results.

For the following discussions, we note that in the once punctured polygon model associated to $\mathsf{C}(D_n)$, there are two types of internal diagonal. Diagonals whose endpoints both meet the exterior edge of the polygon are known as arcs, whilst diagonals that have an endpoint meeting the puncture are known as spokes.

Our first result proves again that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is independent of our choice of subcategory S. We again consider two cluster tilting subcategories T and T^* differing by one indecomposable (not belonging to R), and construct an isomorphism between the corresponding quotient groups $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ and $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*$. Since the polygon model associated to $\mathsf{C}(D_n)$ is significantly more complicated that the one associated to $\mathsf{C}(A_n)$, our proof requires seven cases, each dependent on the type of triangulation we have and where inside the triangulation the replaced indecomposable sits. For completeness, we also provide a verification that all cases are covered.

Section 6.2 then provides a general formula for $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ dependent on the number of indecomposables inside R. This proof is again by induction, and again based around a gluing procedure allowing us to increase the number of indecomposables inside R. Again, since the polygon model is somewhat complicated, we require verification that any triangulation of a once punctured polygon can indeed be constructed through the gluing procedure. We verify this by introducing an ungluing procedure in which we repeatedly remove "cells" from a triangulation, creating what we call the central region of our triangulation. This is a triangulation of a once punctured polygon in which any indecomposables in R correspond to spokes.

The so-called central regions give the base case to our induction. We provide results giving formulae for $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ for central regions, and then these results are the base cases of the final theorem. We describe the gluing procedure, which glues non-punctured polygons on to the triangulation, and there are three cases to consider, depending on where on a polygon we glue the cell and what the central region looks like.

The final formula is given in Theorem 6.2.15. Using the gluing induction, the proof again uses a snake lemma argument to relate $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ before and after the gluing, and produce a formula given by the rank of the quotient.

Chapter 2

Introduction

2.1 Triangulated Categories

Triangulated categories were first introduced by J.L. Verdier in his thesis in the 1960's, which has been reprinted in [44]. These categories have been studied in great depth since their introduction, and many categories with a triangulated structure, including cluster categories, are widely studied in the modern day.

A triangulated category consists of an additive category $\mathcal{A}$, a suspension functor $\Sigma : \mathcal{A} \to \mathcal{A}$, and a certain collection of diagrams of the form

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X,$$

which are known as (distinguished) triangles. Here, X, Y, Z are objects in $\mathcal{A}$, whilst α, β, γ are morphisms. Additionally, the suspension functor $\Sigma : \mathcal{A} \to \mathcal{A}$, also known as the shift functor, is an automorphism. In particular, Σ^{-1} exists.

A morphism of triangles is a triple (f, g, h) of morphisms in $\mathcal{A}$ that make the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "f"] & Y \arrow[r, "\beta"] \arrow[d, "g"] & Z \arrow[r, "\gamma"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z' \arrow[r, "\gamma'"] & \Sigma X'.
\end{tikzcd}$$

Naturally, the morphism of triangles is an isomorphism of triangles if each of f, g, h are isomorphisms in $\mathcal{A}$.

It is widely accepted that in a triangulated category the triangles play a role similar to that of short exact sequences in an abelian category. The formal definition of a triangulated category follows.

Definition 2.1.1. A triangulated category is a triple $(\mathcal{A}, \Sigma, \Delta)$, where $\mathcal{A}$ is an additive category with $\Sigma : \mathcal{A} \to \mathcal{A}$ an additive automorphism and Δ is a collection of (distinguished) triangles that satisfy the following list of axioms:

- **(TR1):**
  - Δ is closed under isomorphisms.
  - For each $X \in \operatorname{obj} \mathcal{A}$, the following is a triangle: $X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow \Sigma X$.
  - For each morphism $\alpha : X \to Y$ in $\mathcal{A}$, there is a triangle $X \xrightarrow{\alpha} Y \longrightarrow Z \longrightarrow \Sigma X$.
- **(TR2):** $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ is a triangle if and only if $Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X \xrightarrow{-\Sigma\alpha} \Sigma Y$ is a triangle.
- **(TR3):** Given two distinguished triangles $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ and $X' \xrightarrow{\alpha'} Y' \xrightarrow{\beta'} Z' \xrightarrow{\gamma'} \Sigma X'$, then for each commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "f"] & Y \arrow[r, "\beta"] \arrow[d, "g"] & Z \arrow[r, "\gamma"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z' \arrow[r, "\gamma'"] & \Sigma X',
\end{tikzcd}$$

  there is a morphism $h : Z \to Z'$ in $\mathcal{A}$ completing the diagram to a morphism of triangles.
- **(TR4): (Octahedral Axiom)** Given triangles $X \xrightarrow{\alpha} Y \longrightarrow Z' \longrightarrow \Sigma X$, $Y \xrightarrow{\beta} Z \longrightarrow X' \longrightarrow \Sigma Y$ and $X \xrightarrow{\beta\alpha} Z \longrightarrow Y' \longrightarrow \Sigma X$, there is a triangle $Z' \longrightarrow Y' \longrightarrow X' \longrightarrow \Sigma Z'$ that makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "\beta"] & Z' \arrow[r] \arrow[d] & \Sigma X' \arrow[d, equal] \\
X \arrow[r, "\beta\alpha"] \arrow[d] & Z \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & \Sigma X' \arrow[d] \\
0 \arrow[r] \arrow[d] & X' \arrow[r, equal] \arrow[d] & X' \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma X \arrow[r, "-\Sigma\alpha"] & \Sigma Y \arrow[r] & \Sigma Z' \arrow[r] & \Sigma^2 X'.
\end{tikzcd} \tag{2.1}$$

  Here, every row and every column of the diagram is a triangle.

Remark 2.1.2. We refer to (TR2) as the "rolling" axiom as it allows us to "roll" a triangle to produce more triangles. Note that we may carry out this rolling in either direction. This means that if $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ is a triangle, then so are $Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X \xrightarrow{-\Sigma\alpha} \Sigma Y$ and $\Sigma^{-1} Z \xrightarrow{-\Sigma^{-1}\gamma} X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$. Obviously, we can roll each triangle as many times as we want.

Remark 2.1.3. With the exception of the octahedral axiom, the axioms used in the definition are clearly motivated. We attempt here to add some context and motivation to the octahedral axiom. Given a morphism $X \xrightarrow{\beta\alpha} Z$ in $\mathcal{A}$ that factorises through some object Y, we have a commutative square

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, equal] & Y \arrow[d, "\beta"] \\ X \arrow[r, "\beta\alpha"] & Z. \end{tikzcd}$$

Now, recalling from (TR1) that each of the morphisms α, β and $\beta\alpha$ can be completed to triangles $X \xrightarrow{\alpha} Y \longrightarrow Z' \longrightarrow \Sigma X$, $Y \xrightarrow{\beta} Z \longrightarrow X' \longrightarrow \Sigma Y$ and $X \xrightarrow{\beta\alpha} Z \longrightarrow Y' \longrightarrow \Sigma X$, the octahedral axiom tells us that these triangles can be used to create a fourth triangle $Z' \longrightarrow Y' \longrightarrow X' \longrightarrow \Sigma Z'$, and hence the commutative diagram in (2.1).

There are several versions of the octahedral axiom in the literature, which are all equivalent to (TR4). The version given in (TR4) is from the book by Neeman [38], which contains several other versions as well. See also [32] and [39] for further versions.

Remark 2.1.4. We give one further version of the octahedral axiom here. We will call it (TR4') and note that it is an equivalent condition to (TR4).

- **(TR4'):** Given triangles $X' \longrightarrow Y \xrightarrow{\beta} Z \longrightarrow \Sigma X'$, $Z' \longrightarrow X \xrightarrow{\alpha} Y \longrightarrow \Sigma Z'$ and $Y' \longrightarrow X \xrightarrow{\beta\alpha} Z \longrightarrow \Sigma Y'$, there is a triangle $Z' \longrightarrow Y' \longrightarrow X' \longrightarrow \Sigma X'$ that makes the following diagram commute,

$$\begin{tikzcd} \Sigma^{-2}Z \arrow[r] \arrow[d] & \Sigma^{-1}X' \arrow[r] \arrow[d] & \Sigma^{-1}Y \arrow[r, "-\Sigma^{-1}\beta"] \arrow[d] & \Sigma^{-1}Z \arrow[d] \\ 0 \arrow[r] \arrow[d] & Z' \arrow[r, equal] \arrow[d] & Z' \arrow[r] \arrow[d] & 0 \arrow[d] \\ \Sigma^{-1}Z \arrow[r] \arrow[d, equal] & Y' \arrow[r] \arrow[d] & X \arrow[r, "\beta\alpha"] \arrow[d, "\alpha"] & Z \arrow[d, equal] \\ \Sigma^{-1}Z \arrow[r] & X' \arrow[r] & Y \arrow[r, "\beta"] & Z. \end{tikzcd}$$

Here, as with (TR4), every row and every column is a triangle. Note that we have applied (TR2) to each of our triangles before constructing the commutative diagram.

The following propositions highlight some important and well known properties of triangles and triangulated categories.

Proposition 2.1.5. *(Composition of morphisms) Let $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ be a triangle. Then, $\beta \circ \alpha = \gamma \circ \beta = 0$, that is, the composition of any two morphisms in a triangle is zero.*

Proof. Note first that as a consequence of (TR2) we are only required to show one of the equalities. We will show that $\beta \circ \alpha = 0$.

By (TR2), rolling our triangle shows that $Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X \xrightarrow{-\Sigma\alpha} \Sigma Y$ is a triangle, and by (TR1) $Z \xrightarrow{1_Z} Z \longrightarrow 0 \longrightarrow \Sigma Z$ is also a triangle. Now, (TR3) allows us to complete the following commutative diagram to a morphism of triangles using the zero morphism labelled with the dashed arrow,

$$\begin{tikzcd}
Y \arrow[r, "\beta"] \arrow[d, "\beta"'] & Z \arrow[r, "\gamma"] \arrow[d, "1_Z"] & \Sigma X \arrow[r, "-\Sigma\alpha"] \arrow[d, dashed, "0"] & \Sigma Y \arrow[d, "\Sigma\beta"] \\
Z \arrow[r, "1_Z"] & Z \arrow[r] & 0 \arrow[r] & \Sigma Z.
\end{tikzcd}$$

The commutativity of the right hand square in the diagram tells us that $\Sigma\beta \circ (-\Sigma\alpha) = 0$. Thus, $-\Sigma(\beta \circ \alpha) = 0$, and since Σ is an automorphism, we can conclude that $\beta \circ \alpha = 0$, as required. $\square$

Remark 2.1.6. We use this remark to set up notation for the following proposition. We denote by F_A the covariant functor $\mathrm{Hom}_{\mathcal{A}}(A, -)$ where A is some object in $\mathcal{A}$. Note that $F_A : \mathcal{A} \to \mathsf{Ab}$, where Ab denotes the category of abelian groups. Also, for some morphism $f : V \to W$ in $\mathcal{A}$, we denote by f_* the morphism $\mathrm{Hom}_{\mathcal{A}}(A, f) : \mathrm{Hom}_{\mathcal{A}}(A, V) \to \mathrm{Hom}_{\mathcal{A}}(A, W)$ induced by f.

Proposition 2.1.7. *(Long exact sequences) Let $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ be a triangle. Then, for any $A \in \mathrm{obj}\,\mathcal{A}$ the following sequence is long exact,*

$$\cdots \longrightarrow F_A(\Sigma^i X) \xrightarrow{\Sigma^i\alpha_*} F_A(\Sigma^i Y) \xrightarrow{\Sigma^i\beta_*} F_A(\Sigma^i Z) \xrightarrow{\Sigma^i\gamma_*} F_A(\Sigma^{i+1} X) \longrightarrow \cdots . \tag{2.2}$$

Proof. Due to the rolling axiom in (TR2), it suffices to show that the sequence in (2.2) is exact at $F_A(\Sigma^i Y)$, that is, $\mathrm{Ker}\,\Sigma^i\beta_* = \mathrm{Im}\,\Sigma^i\alpha_*$.

Firstly, since we know by Proposition 2.1.5 that the composition of two consecutive morphisms in a triangle is zero, we know that $\Sigma^i\beta \circ \Sigma^i\alpha = 0$, and it follows that $\Sigma^i\beta_* \circ \Sigma^i\alpha_* = 0$. Thus, $\mathrm{Im}\,\Sigma^i\alpha_* \subseteq \mathrm{Ker}\,\Sigma^i\beta_*$.

It now remains to show the converse. Let $f \in \mathrm{Ker}\,\Sigma^i\beta_*$ be given, that is $f : A \to \Sigma^i Y$ in $\mathcal{A}$ such that $\Sigma^i\beta_*(f) = 0$. Now, using (TR1) and (TR2), there are triangles $A \longrightarrow 0 \longrightarrow \Sigma A \xrightarrow{1_{\Sigma A}} \Sigma A$ and $\Sigma^i Y \xrightarrow{(-1)^i\Sigma^i\beta} \Sigma^i Z \xrightarrow{(-1)^i\Sigma^i\gamma} \Sigma^{i+1} X \xrightarrow{(-1)^{i+1}\Sigma^{i+1}\alpha} \Sigma^{i+1} Y$, which fit into the following commutative diagram that can be completed with the dashed arrow h using (TR3),

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "1_{\Sigma A}"] \arrow[d, dashed, "h"] & \Sigma A \arrow[d, "\Sigma f"] \\
\Sigma^i Y \arrow[r, "(-1)^i\Sigma^i\beta"'] & \Sigma^i Z \arrow[r, "(-1)^i\Sigma^i\gamma"'] & \Sigma^{i+1} X \arrow[r, "(-1)^{i+1}\Sigma^{i+1}\alpha"'] & \Sigma^{i+1} Y.
\end{tikzcd}$$

Note that the left hand square commutes since we have assumed that $f \in \operatorname{Ker} \Sigma^i \beta_*$. The commutativity of the right hand square gives $(-1)^{i+1}\Sigma^{i+1}\alpha \circ h = \Sigma f$, and so $f = (-1)^{i+1}\Sigma^i \alpha \circ \Sigma^{-1} h$. Noticing that $\Sigma^{-1}h \in F_A(\Sigma^i X)$, it follows immediately that $f \in \operatorname{Im} \Sigma^i \alpha_*$, as required. ☐

Remark 2.1.8. It can be proved in a similar way to Proposition 2.1.7 that for each object $A \in \mathcal{A}$, the contravariant functor $G_A = \operatorname{Hom}_{\mathcal{A}}(-, A)$ satisfies the property that for a triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$, the following sequence is long exact:

$$\cdots \longrightarrow G_A(\Sigma^{i+1} X) \xrightarrow{\Sigma^i \gamma^*} G_A(\Sigma^i Z) \xrightarrow{\Sigma^i \beta^*} G_A(\Sigma^i Y) \xrightarrow{\Sigma^i \alpha^*} G_A(\Sigma^i X) \longrightarrow \cdots .$$

Here, for a morphism $g : V \to W$ in $\mathcal{A}$, we denote by g^* the morphism $\operatorname{Hom}_{\mathcal{A}}(g, A) : \operatorname{Hom}_{\mathcal{A}}(W, A) \to \operatorname{Hom}_{\mathcal{A}}(V, A)$ induced by g.

The following definition is from [38, chap. 1.1].

Definition 2.1.9. Let $\mathcal{A}$ be a triangulated category and let C be an abelian category. We say that a covariant functor $F : \mathcal{A} \to \mathsf{C}$ is a homological functor if for any distinguished triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow \Sigma X$ the following sequence in C is exact,

$$F(X) \xrightarrow{F(\alpha)} F(Y) \xrightarrow{F(\beta)} F(Z).$$

Similarly, we say that a contravariant functor $G : \mathcal{A} \to \mathsf{C}$ is a cohomological functor if for every triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow \Sigma X$ the following sequence in C is exact,

$$G(Z) \xrightarrow{G(\beta)} G(Y) \xrightarrow{G(\alpha)} G(X).$$

Remark 2.1.10. Notice that the functor F_A from Proposition 2.1.7 is a homological functor, whilst the functor G_A from Remark 2.1.8 is a cohomological functor.

Proposition 2.1.11. *(Triangulated 5-lemma) Consider the following morphism of triangles,*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'. \end{tikzcd}$$

If f and g are isomorphisms, then so is h.

Proof. Recall the definition of the cohomological functor $G_A = \operatorname{Hom}_{\mathcal{A}}(-, A)$ for some object $A \in \operatorname{obj} \mathcal{A}$. Consider G_Z and recall that applying it to each term of a triangle will produce a long exact sequence. Thus, there is the following commutative diagram where each row is an exact sequence,

$$\begin{tikzcd}
G_Z(\Sigma Y') \arrow[r] \arrow[d, "\Sigma g^*"] & G_Z(\Sigma X') \arrow[r] \arrow[d, "\Sigma f^*"] & G_Z(Z') \arrow[r] \arrow[d, "h^*"] & G_Z(Y') \arrow[r] \arrow[d, "g^*"] & G_Z(X') \arrow[d, "f^*"] \\
G_Z(\Sigma Y) \arrow[r] & G_Z(\Sigma X) \arrow[r] & G_Z(Z) \arrow[r] & G_Z(Y) \arrow[r] & G_Z(X).
\end{tikzcd}$$

Now, since f and g are known to be isomorphisms, it is clear that Σf and Σg are also isomorphisms. Thus, each of $f^*, g^*, \Sigma f^*$ and Σg^* are also isomorphisms. Applying the usual 5-lemma tells us that h^* is also an isomorphism. Thus, there exists $p \in G_Z(Z') = \mathrm{Hom}_{\mathcal{A}}(Z', Z)$ such that $p \circ h = 1_Z$, showing that h has a left inverse.

A verification that h has a right inverse can be found in the proof of [27, prop. 4.3]. □

With the aim of defining split triangles, we now make the definition of split monomorphisms and split epimorphisms, noting that these definitions make sense in a general, not necessarily triangulated, category.

Definition 2.1.12. Let C be a category and let $\alpha : X \to Y$ be a morphism in C, then

1. We say that α is a split monomorphism if there exists a morphism $\gamma : Y \to X$ in C such that $\gamma \circ \alpha = 1_X$. Split monomorphisms are also known as sections.

2. We say that α is a split epimorphism if there exists a morphism $\gamma : Y \to X$ in C such that $\alpha \circ \gamma = 1_Y$. Split epimorphisms are also known as retractions.

This allows us to define split triangles.

Definition 2.1.13. We say that a triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ is a split triangle if α is a split monomorphism and β is a split epimorphism.

Proposition 2.1.14. *(Split triangles) Let C be a triangulated category and let $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ be a triangle in C. The following statements are equivalent:*

1. $\gamma = 0$.

2. *α is a split monomorphism.*

3. *β is a split epimorphism.*

Proof. We prove here that (1) and (3) are equivalent. A proof that (1) and (2) are equivalent is similar and can be found in the proof of [22, chap. I lem. 1.3].

Assume that $\gamma = 0$ and consider the triangle $0 \longrightarrow Z \xrightarrow{1_Z} Z \longrightarrow 0$, which is a triangle by (TR1). Using (TR2) and (TR3) the following commutative diagram can be completed to an isomorphism of triangles using β',

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & Z \arrow[r, "1_Z"] \arrow[d, dashed, "\beta'"] & Z \arrow[r] \arrow[d, "1_Z"] & 0 \arrow[d] \\
X \arrow[r, "\alpha"'] & Y \arrow[r, "\beta"'] & Z \arrow[r, "0"'] & \Sigma X.
\end{tikzcd}$$

The commutativity of the centre square tells us that $\beta \circ \beta' = 1_Z$, and so β is a split epimorphism.

Conversely, if β is a split epimorphism, then there exists a morphism $\beta' : Z \to Y$ in C such that $\beta \circ \beta' = 1_Z$. Using (TR2) and (TR3), the following commutative diagram can be completed by the dashed arrow to a morphism of triangles:

$$\begin{tikzcd}
Z \arrow[r, "1_Z"] \arrow[d, "\beta'"'] & Z \arrow[r] \arrow[d, "1_Z"] & 0 \arrow[r] \arrow[d, dashed] & \Sigma Z \arrow[d, "\Sigma\beta'"] \\
Y \arrow[r, "\beta"'] & Z \arrow[r, "\gamma"'] & \Sigma X \arrow[r, "-\Sigma\alpha"'] & \Sigma Y.
\end{tikzcd}$$

The commutativity of the centre square clearly tells us that $\gamma = 0$, as required. $\square$

In order to define exchange triangles, we first define a $\mathbb{C}$-linear category and the functor $\mathrm{Ext}^1_{\mathsf{C}}$.

Definition 2.1.15. A category C is called $\mathbb{C}$-linear if the following two conditions are satisfied:

1. For each pair of objects $X, Y \in \operatorname{obj} \mathsf{C}$ we have that $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ carries a $\mathbb{C}$-vector space structure.

2. Composition of morphisms in C is a bilinear map.

Definition 2.1.16. Let C be a triangulated category. We define the functor $\mathrm{Ext}^1_{\mathsf{C}}$ by

$$\mathrm{Ext}^1_{\mathsf{C}}(X, Y) = \mathrm{Hom}_{\mathsf{C}}(X, \Sigma Y),$$

for objects $X, Y \in \mathsf{C}$.

Remark 2.1.17. The classical definition of $\mathrm{Ext}^1_{\mathsf{C}}$ is given in terms of short exact sequences. The more modern version given in Definition 2.1.16 is equivalent to the classical version, and is the one we use throughout this thesis.

Definition 2.1.18. Let C be a $\mathbb{C}$-linear, triangulated category and let $X, Y \in \operatorname{obj} \mathsf{C}$ such that

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathsf{C}}(X, Y) = \dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathsf{C}}(Y, X) = 1.$$

Then, there exists nonsplit triangles,

$$X \to A \to Y \to \Sigma X \quad \text{and} \quad Y \to B \to X \to \Sigma Y,$$

in C, called exchange triangles.

2.2 The Derived Category

A key example of a triangulated category is the derived category, which we construct in this section. In order to do so, we first give definitions that are required in the construction.

The following definition of a multiplicative system of morphisms follows that in [45, sec. 10.3].

Definition 2.2.1. Let C be a category and S a collection of morphisms in C. We say that S is a multiplicative system of morphisms in C if the following axioms hold,

- **(FR1):**
  - S is closed under composition: if $s : X \to Y$ and $t : Y \to Z$ are in S, then $t \circ s : X \to Z$ is also in S.
  - For each object X in C, we have the identity map $1_X : X \to X$ in S.
- **(FR2):** (*Ore condition*) If $s : Z \to Y$ is in S, then for every $f : X \to Y$ in C there is a commutative diagram
$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "t"'] & Z \arrow[d, "s"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd} \tag{2.3}$$
  where $t : W \to X$ is in S. Similarly, the dual must hold: if $t : W \to Z$ is in S, then for every $g : W \to X$ in C there is a commutative diagram,
$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "t"'] & X \arrow[d, "s"] \\ Z \arrow[r, "f"'] & Y, \end{tikzcd} \tag{2.4}$$
  where $s : X \to Y$ is in S.
- **(FR3):** (*Cancellation property*) If $f, g : X \to Y$ are morphisms in C, then the following two conditions are equivalent:
  - There exists $s : Y \to Z$ in S such that $s \circ f = s \circ g$.

– There exists $t : W \to X$ in S such that $f \circ t = g \circ t$.

Remark 2.2.2. The following slogan is often associated to Diagram (2.3) in the Ore condition: "$s^{-1} \circ f = g \circ t^{-1}$". The slogan for Diagram (2.4) is "$g \circ t^{-1} = s^{-1} \circ f$".

We now describe the localisation of a category C with respect to a multiplicative system of morphisms S. To do this, we follow the description in [19, sec. III.2]. We denote the localisation of C with respect to S by $S^{-1}\mathsf{C}$. We have

$$\operatorname{obj} S^{-1}\mathsf{C} = \operatorname{obj} \mathsf{C},$$

whilst morphisms in $S^{-1}\mathsf{C}$ can be represented by expressions of the form $f \circ s^{-1}$, where s is a morphism in S and f is a morphism in C. This expression can be represented by the diagram

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y, \end{tikzcd}$$

which we call a "roof", and denote by $(s, f) : X \to Y$. The morphisms in $S^{-1}\mathsf{C}$ are then equivalence classes of diagrams of this form. We should therefore describe what it means for two such roofs to be equivalent.

- (*Equivalence of roofs*) We say that two roofs $(s, f) : X \to Y$ and $(s', f') : X \to Y$, represented by the following diagrams,

  $$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & Z' \arrow[dl, "s'"'] \arrow[dr, "f'"] & \\ X & & Y, \end{tikzcd}$$

  are equivalent, and therefore represent the same morphism in $S^{-1}\mathsf{C}$, if there exists a roof $(t, g) : Z \to Z'$ making the following diagram commute,

  $$\begin{tikzcd} & & Z'' \arrow[dl, "t"'] \arrow[dr, "g"] & & \\ & Z \arrow[dl, "s"'] \arrow[drrr, "f"' near end] & & Z' \arrow[dr, "f'"] \arrow[dlll, "s'" near end] & \\ X & & & & Y. \end{tikzcd}$$

  A proof that this does indeed define an equivalence relation can be found in [19, lem. III.2.8].

- (*Composition of roofs*) We should also discuss how to compose two morphisms in

$S^{-1}\mathsf{C}$. Consider two roofs $(s,f) : X \to Y$ and $(t,g) : Y \to Z$ represented in the following diagram,

$$\begin{tikzcd} & V \arrow[dl, "s"'] \arrow[dr, "f"] & & W \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z. \end{tikzcd}$$

The Ore condition (FR2) allows us to turn this into the following commutative diagram:

$$\begin{tikzcd} & & U \arrow[dl, dashed, "u"'] \arrow[dr, dashed, "h"] & & \\ & V \arrow[dl, "s"'] \arrow[dr, "f"] & & W \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z. \end{tikzcd}$$

with $u \in S$. The composition is then represented by the roof $(s \circ u, g \circ h) : X \to Z$, given by the diagram

$$\begin{tikzcd} & U \arrow[dl, "s\circ u"'] \arrow[dr, "g\circ h"] & \\ X & & Z. \end{tikzcd}$$

- (*The identity roof*) Finally, we describe the identity morphism for an object X in $S^{-1}\mathsf{C}$. This is represented by the roof $(1_X, 1_X)$, given by

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[dr, "1_X"] & \\ X & & X. \end{tikzcd}$$

Definition 2.2.3. We define the localisation functor $L : \mathsf{C} \to S^{-1}\mathsf{C}$ as follows:

- L acts as the identity on objects in C, i.e. $L(X) = X$ for each $X \in \operatorname{obj} \mathsf{C}$.
- A morphism $f : X \to Y$ in C is sent to the equivalence class of the roof

$$\begin{tikzcd} & X \arrow[dl, equal] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

in $S^{-1}\mathsf{C}$.

Proposition 2.2.4. *The localisation functor $L : \mathsf{C} \to S^{-1}\mathsf{C}$ satisfies the following properties:*

1. *If $q \in S$, then $L(q)$ is an isomorphism in $S^{-1}\mathsf{C}$.*

2. *Let* D *be any category and let* $F : \mathsf{C} \to \mathsf{D}$ *be a functor that sends morphisms in* S *to isomorphisms in* D*. Then, there is a unique functor* $F' : S^{-1}\mathsf{C} \to \mathsf{D}$ *making the following diagram commute,*

$$\begin{tikzcd} \mathsf{C} \arrow[r, "L"] \arrow[d, "F"'] & S^{-1}\mathsf{C} \arrow[dl, dashed, "F'"] \\ \mathsf{D}. & \end{tikzcd} \tag{2.5}$$

Proof. We provide a formula for the functor F', whilst a full detailed proof of the proposition can be found in [27, thm. 7.10]. We require that $F = F' \circ L$. It is easy to see that the following definition of F' makes Diagram (2.5) commute.

- For objects $X \in \mathsf{C}$, we define $F'(X) := F(X)$.
- For morphisms $f \in \mathsf{C}$, define $F'(f) := F(f)$, and for $q \in S$, define $F'(q^{-1}) := F(q)^{-1}$. Note that this makes sense since $F(q)$ is assumed to be an isomorphism.

□

2.2.1 Constructing the Derived Category

We will now explain how to construct the derived category using the homotopy category of chain complexes. We should first construct the category of chain complexes over an abelian category $\mathcal{A}$. The following can be found in [24, sec. IV.1].

Definition 2.2.5. Let $\mathcal{A}$ be an abelian category. A chain complex $C = \{C_n, \partial_n^C\}_{n\in\mathbb{Z}}$ is a sequence of objects and connecting morphisms $\{\partial_n^C : C_n \to C_{n-1}\}$ in $\mathcal{A}$ with the property that $\partial_n^C \circ \partial_{n+1}^C = 0$.

Remark 2.2.6. A chain complex C is therefore a diagram of objects and morphisms in $\mathcal{A}$ of the form

$$C = \cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}^C} C_n \xrightarrow{\partial_n^C} C_{n-1} \xrightarrow{\partial_{n-1}^C} C_{n-2} \longrightarrow \cdots,$$

where the composition of consecutive ∂_i^Cs is zero. We refer to the connecting morphisms as differentials. Notice that the property $\partial_n^C \circ \partial_{n+1}^C = 0$ means that $\operatorname{Im} \partial_{n+1}^C \subseteq \operatorname{Ker} \partial_n^C$, but does not mean that the sequence is exact.

Definition 2.2.7. Let C and D be chain complexes, then a morphism of chain complexes (also known as a chain map) $\varphi : C \to D$ is a family of morphisms $\{\varphi_n : C_n \to D_n\}$ with the property that $\partial_n^D \varphi_n = \varphi_{n-1}\partial_n^C$. That is, the following diagram commutes,

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}^C"] \arrow[d, "\varphi_{n+1}"] & C_n \arrow[r, "\partial_n^C"] \arrow[d, "\varphi_n"] & C_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}^D"'] & D_n \arrow[r, "\partial_n^D"'] & D_{n-1} \arrow[r] & \cdots .
\end{tikzcd}$$

Definition 2.2.8. Let $\mathcal{A}$ be an abelian category. Define $\mathsf{C}(\mathcal{A})$ to be the category of chain complexes whose objects are chain complexes and whose morphisms are given by morphisms of chain complexes.

Remark 2.2.9. $\mathsf{C}(\mathcal{A})$ is an abelian category, see [45, thm. 1.2.3].

We now introduce the important notion of homology. For this, let R be a ring with identity, and let $\mathcal{A} = \mathsf{Mod}\, R$, the category of left R-modules.

Definition 2.2.10. Let $C = \{C_n, \partial_n^C\}_{n\in\mathbb{Z}}$ be a chain complex of R-modules. Define the n^{th} homology module $H_n(C)$ of C by

$$H_n(C) := \operatorname{Ker} \partial_n^C / \operatorname{Im} \partial_{n+1}^C.$$

Remark 2.2.11.

1. Notice that $H_n(C)$ is itself an R-module.
2. If $H_n(C) = 0$ for each $n \in \mathbb{Z}$, then C is an exact complex. That is, it is an exact sequence.
3. Let $\varphi : C \to D$ be a morphism of chain complexes. Then, there is an induced map on homology $H_n(\varphi) : H_n(C) \to H_n(D)$, given by

$$H_n(\varphi)(x + \operatorname{Im} \partial_{n+1}^C) = \varphi_n(x) + \operatorname{Im} \partial_{n+1}^D.$$

This is a well-defined map and allows us to define the homology functor $H_n(-) : \mathsf{C}(\mathcal{A}) \to \mathcal{A}$, which is a covariant functor.

We are now in a position to begin constructing the homotopy category of chain complexes. It makes sense to first explain what is meant by homotopy.

Definition 2.2.12. Let C and D be chain complexes and let $\varphi : C \to D$ be a chain map. We say that φ is null-homotopic if for each $n \in \mathbb{Z}$ there exists a morphism $\sigma_n : C_n \to D_{n+1}$ such that

$$\varphi_n = \partial_{n+1}^D \circ \sigma_n + \sigma_{n-1} \circ \partial_n^C. \tag{2.6}$$

Remark 2.2.13. We can view the above definition with the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[d, "\varphi_{n+1}"'] & C_n \arrow[r, "\partial^C_n"] \arrow[d, "\varphi_n"] \arrow[dl, dashed, "\sigma_n"'] & C_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"] \arrow[dl, dashed, "\sigma_{n-1}"'] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"'] & D_n \arrow[r, "\partial^D_n"'] & D_{n-1} \arrow[r] & \cdots .
\end{tikzcd}$$

Equation (2.6) can then be viewed as the parallelogram of morphisms enclosing the vertical morphism φ_n.

Definition 2.2.14. Let $\varphi, \gamma : C \to D$ be chain maps. We say that φ and γ are homotopic if $\varphi - \gamma$ is null-homotopic. If φ and γ are homotopic, we denote this by $\varphi \sim \gamma$.

Proposition 2.2.15. *The homotopy relation $\sim$ is an equivalence relation.*

Proof. Symmetry and reflexivity are clear. We show transitivity here.

Let $\varphi, \gamma, \phi : C \to D$ be chain maps and let $\varphi \sim \gamma$ and $\gamma \sim \phi$. We need to show $\varphi \sim \phi$. Since $\varphi \sim \gamma$, we know that there exist morphisms $\sigma_n : C_n \to D_{n+1}$ such that

$$\varphi_n - \gamma_n = \partial^D_{n+1} \circ \sigma_n + \sigma_{n-1} \circ \partial^C_n.$$

Rearranging, we get

$$\varphi_n = \gamma_n + \partial^D_{n+1} \circ \sigma_n + \sigma_{n-1} \circ \partial^C_n. \tag{2.7}$$

Since $\gamma \sim \phi$, we know that there exist morphisms $\tau_n : C_n \to D_{n+1}$ such that

$$\gamma_n - \phi_n = \partial^D_{n+1} \circ \tau_n + \tau_{n-1} \circ \partial^C_n.$$

Rearranging, we get

$$-\phi_n = -\gamma_n + \partial^D_{n+1} \circ \tau_n + \tau_{n-1} \circ \partial^C_n. \tag{2.8}$$

From (2.7) and (2.8), we see that

$$\begin{aligned}
\varphi_n - \phi_n &= \gamma_n + \partial^D_{n+1} \circ \sigma_n + \sigma_{n-1} \circ \partial^C_n - \gamma_n + \partial^D_{n+1} \circ \tau_n + \tau_{n-1} \circ \partial^C_n \\
&= \partial^D_{n+1} \circ (\sigma_n + \tau_n) + (\sigma_{n-1} + \tau_{n-1}) \circ \partial^C_n.
\end{aligned}$$

Thus, the morphisms $(\sigma + \tau)_n := (\sigma_n + \tau_n) : C_n \to D_{n+1}$ are sufficient to show that $\varphi - \phi$ is null-homotopic, and so $\varphi \sim \phi$. $\square$

Proposition 2.2.16. *Let $\varphi, \gamma : C \to D$ be two chain maps and let $\varphi \sim \gamma$. Then, $H_n(\varphi) = H_n(\gamma)$.*

Proof. Consider the following diagram,

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[d, "\varphi_{n+1}-\gamma_{n+1}"'] & C_n \arrow[r, "\partial^C_n"] \arrow[d, "\varphi_n-\gamma_n"'] \arrow[dl, dashed, "\sigma_n"] & C_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}-\gamma_{n-1}"] \arrow[dl, dashed, "\sigma_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"'] & D_n \arrow[r, "\partial^D_n"'] & D_{n-1} \arrow[r] & \cdots .
\end{tikzcd}$$

Since $\varphi \sim \gamma$, we can assume that for each $n \in \mathbb{Z}$, we have

$$\varphi_n - \gamma_n = \partial^D_{n+1} \circ \sigma_n + \sigma_{n-1} \circ \partial^C_n.$$

Rearranging gives

$$\varphi_n = \gamma_n + \partial^D_{n+1} \circ \sigma_n + \sigma_{n-1} \circ \partial^C_n. \tag{2.9}$$

Recall the definition of $H_n(\varphi)$ from Item 3 of Remark 2.2.11, and see that for $x \in \operatorname{Ker} \partial^C_n$,

$$\begin{aligned} H_n(\varphi)(x + \operatorname{Im} \partial^C_{n+1}) &= \varphi_n(x) + \operatorname{Im} \partial^D_{n+1} \\ &= \gamma_n(x) + \partial^D_{n+1}\sigma_n(x) + \sigma_{n-1}\partial^C_n(x) + \operatorname{Im} \partial^D_{n+1}. \end{aligned} \tag{2.10}$$

Here, the first $=$ is by definition and the second $=$ is by (2.9). Now, since $x \in \operatorname{Ker} \partial^C_n$, we see that $\sigma_{n-1}\partial^C_n(x) = 0$. Additionally, $\partial^D_{n+1}\sigma_n(x) \in \operatorname{Im} \partial^D_{n+1}$, and so (2.10) becomes

$$\begin{aligned} \varphi_n(x) + \operatorname{Im} \partial^D_{n+1} &= \gamma_n(x) + \operatorname{Im} \partial^D_{n+1} \\ &= H_n(\gamma)(x + \operatorname{Im} \partial^C_{n+1}), \end{aligned}$$

as required. □

Definition 2.2.17. Let $\mathcal{A}$ be an abelian category. We define the homotopy category of chain complexes in the following way:

- $\operatorname{obj} \mathsf{K}(\mathcal{A}) = \operatorname{obj} \mathsf{C}(\mathcal{A})$.
- $\operatorname{Hom}_{\mathsf{K}(\mathcal{A})}(C, D) = \operatorname{Hom}_{\mathsf{C}(\mathcal{A})}(C, D)/\sim$, i.e. morphisms in $\mathsf{K}(\mathcal{A})$ are equivalence classes of morphisms in $\mathsf{C}(\mathcal{A})$ modulo homotopy.

Remark 2.2.18. We verify here that composition of morphisms in $\mathsf{K}(\mathcal{A})$ is well-defined. Let $\varphi, \gamma : C \to D$ be morphisms in $\mathsf{K}(\mathcal{A})$ such that $\varphi \sim \gamma$. Also, let $\alpha : D \to E$ be a morphism in $\mathsf{K}(\mathcal{A})$. We need to verify that $\alpha\varphi \sim \alpha\gamma$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[d, "\varphi_{n+1}-\gamma_{n+1}"'] & C_n \arrow[r, "\partial^C_n"] \arrow[d, "\varphi_n-\gamma_n"] \arrow[dl, dashed, "\sigma_n"'] & C_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}-\gamma_{n-1}"] \arrow[dl, dashed, "\sigma_{n-1}"'] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"] \arrow[d, "\alpha_{n+1}"'] & D_n \arrow[r, "\partial^D_n"] \arrow[d, "\alpha_n"] & D_{n-1} \arrow[r] \arrow[d, "\alpha_{n-1}"] & \cdots \\
\cdots \arrow[r] & E_{n+1} \arrow[r, "\partial^E_{n+1}"'] & E_n \arrow[r, "\partial^E_n"'] & E_{n-1} \arrow[r] & \cdots
\end{tikzcd} \tag{2.11}$$

We know that for each $n \in \mathbb{Z}$ there is a morphism $\sigma_n : C_n \to D_{n+1}$ satisfying

$$\varphi_n - \gamma_n = \partial^D_{n+1} \circ \sigma_n + \sigma_{n-1} \circ \partial^C_n.$$

With the help of Diagram (2.11), notice then that

$$\begin{aligned}
\alpha_n \circ (\varphi_n - \gamma_n) &= \alpha_n \circ (\partial^D_{n+1} \circ \sigma_n + \sigma_{n-1} \circ \partial^C_n) \\
&= \alpha_n \circ \partial^D_{n+1} \circ \sigma_n + \alpha_n \circ \sigma_{n-1} \circ \partial^C_n \\
&= \partial^E_{n+1} \circ (\alpha_{n+1} \circ \sigma_n) + (\alpha_n \circ \sigma_{n-1}) \circ \partial^C_n.
\end{aligned}$$

Thus, the family $(\alpha_{n+1} \circ \sigma_n)_{n\in\mathbb{Z}}$ of morphisms $\alpha_{n+1} \circ \sigma_n : C_n \to E_{n+1}$ show that $\alpha\circ\varphi \sim \alpha\circ\gamma$. A similar argument shows that if $\varphi, \gamma : C \to D$ are homotopic and $\beta : B \to C$ is another morphism in $\mathsf{K}(\mathcal{A})$, then $\varphi \circ \beta \sim \gamma \circ \beta$. Thus, composition of morphisms in $\mathsf{K}(\mathcal{A})$ is well-defined.

Remark 2.2.19. We note here that $\mathsf{K}(\mathcal{A})$ is an additive category. The addition of morphisms is defined by addition of representatives of morphisms, and it is easy to see that this is well-defined. The hom spaces in $\mathsf{K}(\mathcal{A})$ then inherit the structure of an abelian group from the category $\mathsf{C}(\mathcal{A})$. Composition of morphisms in $\mathsf{K}(\mathcal{A})$ is bilinear, and this property is also inherited from $\mathsf{C}(\mathcal{A})$. The zero object is also the same as in $\mathsf{C}(\mathcal{A})$, which is just the chain complex with the zero object from $\mathcal{A}$ in each degree. Finally, the universal property of a coproduct also holds in $\mathsf{K}(\mathcal{A})$, and an easy verification of this can be found in [27, prop. 1.7].

We should describe the triangulated structure of $\mathsf{K}(\mathcal{A})$, starting with the suspension functor Σ.

Consider a chain complex C represented by the diagram,

$$C = \cdots \longrightarrow C_{n+2} \xrightarrow{\partial^C_{n+2}} C_{n+1} \xrightarrow{\partial^C_{n+1}} \overbrace{C_n}^{\text{n}^{\text{th}} \text{ degree}} \xrightarrow{\partial^C_n} C_{n-1} \xrightarrow{\partial^C_{n-1}} C_{n-2} \longrightarrow \cdots.$$

Then, ΣC is obtained by "shifting" each object one degree to the left and flipping the sign of ∂:

$$\Sigma C = \cdots \longrightarrow C_{n+1} \xrightarrow{-\partial^C_{n+1}} C_n \xrightarrow{-\partial^C_n} \overbrace{C_{n-1}}^{\text{n}^{\text{th}} \text{ degree}} \xrightarrow{-\partial^C_{n-1}} C_{n-2} \xrightarrow{-\partial^C_{n-2}} C_{n-3} \longrightarrow \cdots .$$

i.e. $(\Sigma C)_n = C_{n-1}$.

Mapping cones are described in the following way: Consider a chain map $\varphi : C \to D$. The mapping cone of φ, denoted $M(\varphi)$, is a chain complex where

$$M(\varphi)_n = C_{n-1} \oplus D_n$$

is the object in the n^{th} degree of $M(\varphi)$. The differentials $\partial_n^{M(\varphi)} : M(\varphi)_n \to M(\varphi)_{n-1}$ are given by

$$\partial_n^{M(\varphi)} := \begin{pmatrix} -\partial^C_{n-1} & 0 \\ \varphi_{n-1} & \partial^D_n \end{pmatrix}.$$

The canonical inclusion $i : D \hookrightarrow M(\varphi)$, given by

$$i_n := \begin{pmatrix} 0 \\ 1_{D_n} \end{pmatrix},$$

and the canonical projection $p : M(\varphi) \twoheadrightarrow \Sigma C$, given by

$$p_n := \begin{pmatrix} 1_{C_{n-1}} & 0 \end{pmatrix},$$

are both chain maps. We now have the necessary constructions in order to define triangles in $\mathsf{K}(\mathcal{A})$.

Definition 2.2.20. A sequence of objects and morphisms in $\mathsf{K}(\mathcal{A})$ of the form

$$C \xrightarrow{\varphi} D \xrightarrow{i} M(\varphi) \xrightarrow{p} \Sigma C$$

is called a standard triangle in $\mathsf{K}(\mathcal{A})$. In particular, it is a triangle. Additionally, any other sequence of objects and morphisms in $\mathsf{K}(\mathcal{A})$ that is isomorphic (in $\mathsf{K}(\mathcal{A})$) to a standard triangle is also a triangle.

Remark 2.2.21. For clarity, we provide in Figure 2.1 a detailed diagram of a standard triangle in $\mathsf{K}(\mathcal{A})$.

We therefore have the following proposition.

Proposition 2.2.22. $\mathsf{K}(\mathcal{A})$ *with the functor* Σ *and the triangles from above is a triangulated category.*

$$
\begin{tikzcd}
C = \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[d, "\varphi_{n+1}"] & C_n \arrow[r, "\partial^C_n"] \arrow[d, "\varphi_n"] & C_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"] & \cdots \\
D = \cdots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] & D_n \arrow[r, "\partial^D_n"] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] & D_{n-1} \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] & \cdots \\
M(\varphi) = \cdots \arrow[r] & C_n \oplus D_{n+1} \arrow[r, "\left(\begin{smallmatrix} -\partial^C_n & 0 \\ \varphi_n & \partial^D_{n+1} \end{smallmatrix}\right)"] \arrow[d, "(1\ 0)"] & C_{n-1} \oplus D_n \arrow[r, "\left(\begin{smallmatrix} -\partial^C_{n-1} & 0 \\ \varphi_{n-1} & \partial^D_n \end{smallmatrix}\right)"] \arrow[d, "(1\ 0)"] & C_{n-2} \oplus D_{n-1} \arrow[r] \arrow[d, "(1\ 0)"] & \cdots \\
\Sigma C = \cdots \arrow[r] & C_n \arrow[r, "-\partial^C_n"] & C_{n-1} \arrow[r, "-\partial^C_{n-1}"] & C_{n-2} \arrow[r] & \cdots
\end{tikzcd}
$$

Figure 2.1: A standard triangle in $\mathsf{K}(\mathcal{A})$.

Proof. See [27, sec. 6]. $\square$

We are now in a position to explain how the constructions in the homotopy category lead to the derived category. We first introduce the notion of a quasi-isomorphism.

Definition 2.2.23. Let $\varphi : C \to D$ be a morphism in $\mathsf{C}(\mathcal{A})$. Then, we say that φ is a quasi-isomorphism if $H_n(\varphi) : H_n(C) \to H_n(D)$ is an isomorphism for each $n \in \mathbb{Z}$, i.e. φ induces isomorphisms in homology.

Proposition 2.2.24. *The set Q of quasi-isomorphisms in $\mathsf{K}(\mathcal{A})$ is a multiplicative system of morphisms, i.e. it satisfies (FR1) to (FR3) from Definition 2.2.1.*

Proof. See [27, lem. 7.11]. $\square$

Remark 2.2.25. The set of quasi-isomorphisms in $\mathsf{C}(\mathcal{A})$ is not a multiplicative system of morphisms. The proof of Proposition 2.2.24 relies heavily upon the triangulated structure of $\mathsf{K}(\mathcal{A})$.

Definition 2.2.26. Let $\mathcal{A}$ be an abelian category and denote by Q the set of quasi-isomorphisms in the homotopy category $\mathsf{K}(\mathcal{A})$. Then, we define the derived category $\mathsf{D}(\mathcal{A})$ as follows,

$$\mathsf{D}(\mathcal{A}) = Q^{-1}\mathsf{K}(\mathcal{A}),$$

that is, the derived category is the localisation of the homotopy category with respect to Q.

Remark 2.2.27. Objects and morphisms in $\mathsf{D}(\mathcal{A})$ are given as follows.

$$\operatorname{obj}\mathsf{D}(\mathcal{A}) = \operatorname{obj}\mathsf{K}(\mathcal{A}),$$

whilst morphisms can be represented by roofs of the form

$$\begin{tikzcd} & Z \arrow[dl, "q"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

where $q \in Q$ and $f \in \operatorname{Hom}_{\mathsf{K}(\mathcal{A})}(Z, Y)$.

It will often be useful to consider the bounded derived category, which we define here.

Definition 2.2.28. The bounded derived category, denoted by $\mathsf{D}^b(\mathcal{A})$, is the full subcategory of $\mathsf{D}(\mathcal{A})$ whose objects are bounded chain complexes.

2.2.2 The Derived Category is Triangulated

We explain briefly in this section how the derived category becomes a triangulated category.

Recall the definition of the localisation functor. By Proposition 2.2.4, in this case, the localisation functor $L : \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ satisfies the following properties:

1. If $q \in Q$ is a quasi-isomorphism in $\mathsf{K}(\mathcal{A})$, then $L(q)$ is an isomorphism in $\mathsf{D}(\mathcal{A})$.
2. Let C be any category and let $F : \mathsf{K}(\mathcal{A}) \to \mathsf{C}$ be a functor that sends quasi-isomorphisms in $\mathsf{K}(\mathcal{A})$ to isomorphisms in C. Then, there is a unique functor $F' : \mathsf{D}(\mathcal{A}) \to \mathsf{C}$ making the following diagram commute,

$$\begin{tikzcd} \mathsf{K}(\mathcal{A}) \arrow[r, "L"] \arrow[d, "F"'] & \mathsf{D}(\mathcal{A}) \arrow[dl, dashed, "F'"] \\ \mathsf{C}. & \end{tikzcd} \tag{2.12}$$

Remark 2.2.29. Item 2 from above is the universal property of the derived category $\mathsf{D}(\mathcal{A})$ and implies that $\mathsf{D}(\mathcal{A})$ is unique up to equivalence of categories.

Proposition 2.2.30. $\mathsf{D}(\mathcal{A})$ *is a triangulated category.*

Proof. See [23, prop. I.3.2]. □

Remark 2.2.31. Although we don't provide a proof to Proposition 2.2.30, we do describe here some key points of the triangulated structure of $\mathsf{D}(\mathcal{A})$:

- The suspension functor Σ acts the same on objects in $\mathsf{D}(\mathcal{A})$ as in $\mathsf{K}(\mathcal{A})$, whilst for a morphism f in $\mathsf{D}(\mathcal{A})$ represented by the roof

$$\begin{tikzcd} & Z \arrow[dl, "q"'] \arrow[dr, "f"] & \\ X & & Y, \end{tikzcd}$$

the morphism Σf is represented by the roof

$$\begin{tikzcd} & \Sigma Z \arrow[dl, "\Sigma q"'] \arrow[dr, "\Sigma f"] & \\ \Sigma X & & \Sigma Y. \end{tikzcd}$$

Here, Σf and Σq are the shifts of f and q in $\mathsf{K}(\mathcal{A})$.

- A triangle in $\mathsf{D}(\mathcal{A})$ is any diagram isomorphic to the image of a triangle in $\mathsf{K}(\mathcal{A})$ under the localisation functor L.

Remark 2.2.32. Since L sends quasi-isomorphisms in $\mathsf{K}(\mathcal{A})$ to isomorphisms in $\mathsf{D}(\mathcal{A})$, the derived category $\mathsf{D}(\mathcal{A})$ must contain more isomorphisms. It therefore also makes sense that $\mathsf{D}(\mathcal{A})$ contains more triangles than $\mathsf{K}(\mathcal{A})$.

Remark 2.2.33. It is easy to see that $\mathsf{D}^b(\mathcal{A})$ is also a triangulated category.

2.3 Quivers, Path Algebras and Quiver Representations

We use this short section to introduce some basic notions associated to quivers and quiver representations. The constructions in this section, which can be found in [2], are crucial to the work in the remainder of this thesis.

Definition 2.3.1. A quiver $Q = (Q_0, Q_1, s, t)$ is a quadruple, where

- Q_0 is a set of vertices.
- Q_1 is a set of arrows between vertices.
- $s : Q_1 \to Q_0$ is a map associating to an arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$.
- $t : Q_1 \to Q_0$ is a map associating to an arrow $\alpha \in Q_1$ its target $t(\alpha) \in Q_0$.

Example 2.3.2. Below are some examples of quivers:

- A quiver of Dynkin type A_n:

$$\overset{1}{\circ} \xrightarrow[\alpha_1]{} \overset{2}{\circ} \xrightarrow[\alpha_2]{} \overset{3}{\circ} \longrightarrow \cdots \longrightarrow \overset{n-1}{\circ} \xrightarrow[\alpha_{n-1}]{} \overset{n}{\circ}$$

- A quiver of Dynkin type D_n:

$$
\begin{tikzcd}
\overset{1}{\circ} \arrow[dr, "\alpha_1"] & & & & & \\
& \overset{3}{\circ} \arrow[r, "\alpha_3"'] & \overset{4}{\circ} \arrow[r] & \cdots \arrow[r] & \overset{n-1}{\circ} \arrow[r, "\alpha_{n-1}"'] & \overset{n}{\circ} \\
\overset{2}{\circ} \arrow[ur, "\alpha_2"'] & & & & &
\end{tikzcd}
$$

- The Kronecker quiver:

$$
\begin{tikzcd}
\overset{1}{\circ} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \overset{2}{\circ}
\end{tikzcd}
$$

Definition 2.3.3. Let $Q = (Q_0, Q_1, s, t)$ be a quiver.

1. A path of length n in Q is a sequence of arrows,

$$\alpha_n \alpha_{n-1} \cdots \alpha_2 \alpha_1,$$

where $s(\alpha_i) = t(\alpha_{i-1})$. We also associate to each $a \in Q_0$ a trivial path e_a.

2. Let $p = \alpha_n \alpha_{n-1} \cdots \alpha_1$ and $q = \beta_m \beta_{m-1} \cdots \beta_1$ be paths with $t(\beta_m) = s(\alpha_1)$. Then, the concatenation of p and q, denoted pq, is the path

$$\alpha_n \alpha_{n-1} \cdots \alpha_1 \beta_m \beta_{m-1} \cdots \beta_1.$$

We now define some commonly used terminology associated to quivers.

Definition 2.3.4. Let Q be a quiver.

1. A cycle in Q is a path of length $n \geq 1$ whose source and target coincide.
2. A cycle of length n is called an n-cycle.
3. A cycle of length one is called a loop.
4. If Q contains no cycles, then we say that Q is acyclic.
5. Q is finite if Q_0 and Q_1 are both finite sets.
6. Q is connected if its underlying graph $\overline{Q}$, obtained by replacing all arrows of Q with undirected edges, is connected.

Definition 2.3.5. Let Q be a quiver. The path algebra of Q, denoted $\mathbb{C}Q$, is the $\mathbb{C}$-algebra whose underlying $\mathbb{C}$-vector space has as its basis the set of all paths in Q. The product $p \cdot q$ of two paths $p = \alpha_n \alpha_{n-1} \cdots \alpha_2 \alpha_1$ and $q = \beta_m \beta_{m-1} \cdots \beta_2 \beta_1$ is given by

$$p \cdot q = pq = \alpha_n \alpha_{n-1} \cdots \alpha_1 \beta_m \beta_{m-1} \cdots \beta_1,$$

if $t(\beta_m) = s(\alpha_1)$. If $t(\beta_m) \neq s(\alpha_1)$, then their product is given by

$$p \cdot q = 0.$$

We now introduce representations of quivers and the category of representations. The following can all be found in [2, chap. III].

Definition 2.3.6. Let Q be a finite quiver. A representation $M = (M_a, \varphi_\alpha)_{a \in Q_0, \alpha \in Q_1}$ of Q is defined as follows:

1. To each vertex $a \in Q_0$, we associate a $\mathbb{C}$-vector space M_a.
2. To each arrow $a \xrightarrow{\alpha} b$ in Q_1, we associate a linear map $\varphi_\alpha : M_a \to M_b$.

Example 2.3.7. Below are some examples of quiver representations:

- A quiver representation of A_4:

$$\mathbb{C} \xrightarrow{1_\mathbb{C}} \mathbb{C} \xrightarrow{1_\mathbb{C}} \mathbb{C} \xrightarrow{1_\mathbb{C}} \mathbb{C}$$

- A quiver representation of D_5:

$$\begin{tikzcd} \mathbb{C} \arrow[dr, "1_\mathbb{C}"] & & & \\ & \mathbb{C} \arrow[r] & 0 \arrow[r] & 0 \\ \mathbb{C} \arrow[ur, "1_\mathbb{C}"'] & & & \end{tikzcd}$$

- A quiver representation of the Kronecker quiver:

$$\begin{tikzcd} \mathbb{C} \arrow[r, shift left, "\left(\begin{smallmatrix} 1_\mathbb{C} \\ 0 \end{smallmatrix}\right)"] \arrow[r, shift right, "\left(\begin{smallmatrix} 0 \\ 1_\mathbb{C} \end{smallmatrix}\right)"'] & \mathbb{C}^2 \end{tikzcd}$$

Morphisms between quiver representations can be defined in the following way.

Definition 2.3.8. Let Q be a finite quiver and let $M = (M_a, \varphi_\alpha)$, $N = (N_a, \phi_\alpha)$ and $P = (P_a, \gamma_\alpha)$ be representations of Q.

1. A morphism $f : M \to N$ is a family of $\mathbb{C}$-linear maps $(f_a : M_a \to N_a)_{a \in Q_0}$, where for each arrow $a \xrightarrow{\alpha} b$ in Q_1, the following diagram commutes,

$$\begin{tikzcd} M_a \arrow[r, "\varphi_\alpha"] \arrow[d, "f_a"'] & M_b \arrow[d, "f_b"] \\ N_a \arrow[r, "\phi_\alpha"'] & N_b. \end{tikzcd}$$

2. The composition of two morphisms $f : M \to N$ and $g : N \to P$ is defined in the obvious way. That is, if $f = (f_a)_{a \in Q_0}$ and $g = (g_a)_{a \in Q_0}$, then $g \circ f := (g_a \circ f_a)_{a \in Q_0}$.

Definition 2.3.9. For a finite quiver Q we can define the category $\mathrm{Rep}_{\mathbb{C}}Q$ of quiver representations in the following way:

- Objects are quiver representations as described in Definition 2.3.6.
- Morphisms are morphisms of quiver representations as described in Definition 2.3.8.

Additionally, we denote by $\mathrm{rep}_{\mathbb{C}}Q$ the full subcategory of $\mathrm{Rep}_{\mathbb{C}}Q$ whose objects are finite dimensional representations of Q.

The following key propositions are both from [2, chap. III].

Proposition 2.3.10. *Let Q be a finite quiver. Then, $\mathrm{Rep}_{\mathbb{C}}Q$ and $\mathrm{rep}_{\mathbb{C}}Q$ are both abelian categories.*

Proof. See [2, III lem. 1.3] □

Proposition 2.3.11. *Let Q be a finite, connected, acyclic quiver and denote by $\mathbb{C}Q$ the path algebra of Q. The, there is an equivalence of categories,*

$$\mathrm{Rep}_{\mathbb{C}}Q \cong \mathsf{Mod}\mathbb{C}Q,$$

which restricts to,

$$\mathrm{rep}_{\mathbb{C}}Q \cong \mathsf{mod}\mathbb{C}Q.$$

Proof. See [2, III thm. 1.6]. □

2.4 Auslander-Reiten Theory

In this section we introduce some basic yet essential notions of Auslander-Reiten theory. Our main objects of study will be the triangulated analogues of Auslander-Reiten sequences, known as Auslander-Reiten triangles, and the Auslander-Reiten quiver of a category. Throughout the section we will give important definitions, results and examples that are crucial to this thesis. An extensive introduction to Auslander-Reiten theory can be found mainly in [5], but also in [2], whilst an introduction to Auslander-Reiten theory from a triangulated viewpoint can be found in [22] and [36].

We begin this section with some key definitions. Here, we let Λ be a finite dimensional $\mathbb{C}$-algebra and consider the category $\mathsf{mod}\,\Lambda$ of finitely generated left-Λ-modules. The following definitions can all be found in [2].

Definition 2.4.1. Let $P \in \mathsf{mod}\,\Lambda$. We say that P is projective if for any epimorphism $f : X \twoheadrightarrow Y$ and any morphism $g : P \to Y$ there is a morphism $h : P \to X$ that makes the following diagram commute:

$$\begin{tikzcd}
& P \arrow[dl, dashed, "h"'] \arrow[d, "g"] & \\
X \arrow[r, two heads, "f"'] & Y \arrow[r] & 0.
\end{tikzcd}$$

Remark 2.4.2. It is proved in [2, lem. I.5.3] that for each $X \in \mathsf{mod}\,\Lambda$ there is an exact sequence

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \twoheadrightarrow X \to 0$$

in $\mathsf{mod}\,\Lambda$, where each $P_i \in \mathsf{mod}\,\Lambda$ is a projective module.

Definition 2.4.3. Let $X \in \mathsf{mod}\,\Lambda$. A projective resolution of X is an exact sequence

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \twoheadrightarrow X \to 0$$

in $\mathsf{mod}\,\Lambda$, where each P_i is a projective Λ-module.

Definition 2.4.4. Let $X \in \mathsf{mod}\,\Lambda$. The projective dimension of X, denoted $\operatorname{pd} X$ is the smallest nonnegative integer n such that X has a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \twoheadrightarrow X \to 0,$$

of length n. If no such n exists, we say that X has infinite projective dimension.

Definition 2.4.5. Let Λ be a finite dimensional $\mathbb{C}$-algebra. We define the global dimension of Λ by

$$\operatorname{gl.dim} \Lambda = \max\{\operatorname{pd} X \ : \ X \in \mathsf{mod}\,\Lambda\}.$$

We now give the definition of an indecomposable object.

Definition 2.4.6. Let C be an additive category and let X be an object in C. We say that X is indecomposable if X is nonzero and $X = X_1 \oplus X_2$ implies that either $X_1 = 0$ or $X_2 = 0$.

Recall from Definition 2.1.12 the notion of a split monomorphism and the notion of a split epimorphism. We use these to define what it means for a morphism to be left minimal almost split and right minimal almost split. Our definition follows that in [2, IV def. 1.1], however we work in a general category C.

Definition 2.4.7. Let C be a category and let X, Y, Z be objects in C. Then,

1. A morphism $f : X \to Y$ in C is called left minimal if every $h \in \mathrm{End}_{\mathsf{C}}(Y)$ such that $h \circ f = f$ is an automorphism.

2. Dually, a morphism $g : Y \to Z$ in C is called right minimal if every $h \in \mathrm{End}_{\mathsf{C}}(Y)$ such that $g \circ h = g$ is an automorphism.

3. A morphism $f : X \to Y$ in C is called left almost split if the following conditions are satisfied:

   (a) f is not a split monomorphism.

   (b) For each $a : X \to A$ in C that is not a split monomorphism, there exists $a' : Y \to A$ in C making the following diagram commute,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[dl, dashed, "a'"] \\ A. & \end{tikzcd}$$

4. Dually, a morphism $g : Y \to Z$ is called right almost split if the following conditions are satisfied:

   (a) g is not a split epimorphism.

   (b) For every morphism $b : B \to Z$ in C that is not a split epimorphism, there exists a morphism $b' : B \to Y$ in C making the following diagram commute,

$$\begin{tikzcd} & B \arrow[dl, dashed, "b'"'] \arrow[d, "b"] \\ Y \arrow[r, "g"] & Z. \end{tikzcd}$$

5. A morphism $f : X \to Y$ in C is called left minimal almost split if it is both left minimal and left almost split.

6. Dually, a morphism $g : Y \to Z$ in C is called right minimal almost split if it is both right minimal and right almost split.

Almost split morphisms are very closely related to indecomposable objects, as the following proposition from [2, IV.1] shows.

Proposition 2.4.8. *Let* C *be an additive category.*

1. *If* $f : X \to Y$ *in* C *is a left almost split morphism, then* X *is indecomposable.*

2. *If* $g : Y \to Z$ *in* C *is a right almost split morphism, then* Z *is indecomposable.*

Proof. Here we prove (2). The proof of (1) is similar and can be found in the proof of [2, IV lem. 1.3].

Assume Z is not indecomposable, that is, $Z = Z_1 \oplus Z_2$ where Z_1 and Z_2 are both nonzero. Denote by $i_1 : Z_1 \hookrightarrow Z$ and $i_2 : Z_2 \hookrightarrow Z$ inclusions and note that clearly i_1 and i_2 are not split epimorphisms. Thus, by $4(b)$ of Definition 2.4.7 there must be morphisms $v_1 : Z_1 \to Y$ and $v_2 : Z_2 \to Y$ making the following diagrams commute,

$$\begin{tikzcd} & Z_1 \arrow[dl, dashed, "v_1"'] \arrow[d, hook, "i_1"] \\ Y \arrow[r, "g"'] & Z_1 \oplus Z_2, \end{tikzcd} \qquad \begin{tikzcd} & Z_2 \arrow[dl, dashed, "v_2"'] \arrow[d, hook, "i_2"] \\ Y \arrow[r, "g"'] & Z_1 \oplus Z_2. \end{tikzcd}$$

That is, $g \circ v_1 = i_1$ and $g \circ v_2 = i_2$. Consider the morphism $v = (v_1\ v_2) : Z \to Y$. Clearly, v satisfies that $g \circ v = 1_Z$, meaning that g is a split epimorphism. This is a contradiction to the assumption that g is right almost split, and so Z must in fact be indecomposable. $\square$

The following proposition, also taken from [2], shows that left minimal almost split morphisms are uniquely determined by their sources, whilst right minimal almost split morphisms are uniquely determined by their targets.

Proposition 2.4.9. 1. *Let* $f : X \to Y$ *and* $f' : X \to Y'$ *be left almost split morphisms in* C*. Then, there exists an isomorphism* $a : Y \to Y'$ *making the following diagram commute,*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f'"'] & Y \arrow[dl, dashed, "a"] \\ Y'. & \end{tikzcd}$$

2. *Let* $g : Y \to Z$ *and* $g' : Y' \to Z$ *be right minimal almost split morphisms in* C*. Then, there exists an isomorphism* $b : Y \to Y'$ *making the following diagram commute,*

$$\begin{tikzcd} & Y \arrow[dl, dashed, "b"'] \arrow[d, "g"] \\ Y' \arrow[r, "g'"'] & Z. \end{tikzcd}$$

Proof. We prove (2) here. The proof of (1) is similar and is given in the proof of [2, IV prop. 1.2].

We need to find an isomorphism $b : Y \to Y'$ in C such that $g' \circ b = g$. Well, since g' is a right almost split morphism, there exists $b : Y \to Y'$ in C such that

$$g' \circ b = g. \tag{2.13}$$

We are done if we can show that b is an isomorphism. Well, since g is right almost split, there is also a morphism $b' : Y' \to Y$ in C such that

$$g \circ b' = g'. \tag{2.14}$$

Combining (2.13) and (2.14) gives $g = g \circ b' \circ b$ and $g' = g' \circ b \circ b'$. Since g and g' are both right minimal, it follows that $b \circ b'$ and $b' \circ b$ are automorphisms, and so b (and b') is an isomorphism, as required. $\square$

We now switch to the setting of an abelian category in order to define Auslander-Reiten sequences, also known as almost split sequences.

Definition 2.4.10. Let C be an abelian category. We say that a short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in C is an Auslander-Reiten (or almost split) sequence if f is left minimal almost split and g is right minimal almost split.

Remark 2.4.11.
1. Existence of such Auslander-Reiten sequences is not obvious, nor true in general, however an existence theorem is given in [5, V thm. 1.15].
2. It follows immediately from Proposition 2.4.8 that if
$$0 \to X \to Y \to Z \to 0$$
is an Auslander-Reiten sequence, then the end terms X and Z must be indecomposable.
3. It follows from Proposition 2.4.9 that an Auslander-Reiten sequence is uniquely determined by each of its end terms; that is, if $0 \to X \to Y \to Z \to 0$ and $0 \to X' \to Y' \to Z' \to 0$ are two Auslander-Reiten sequences, then the following statements are equivalent:

(a) The two sequences are isomorphic in that there is a commutative diagram,

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & 0, \end{tikzcd}$$

where every vertical arrow is an isomorphism.

(b) There is an isomorphism $h : X \to X'$.

(c) There is an isomorphism $k : Z \to Z'$.

Before moving on to the situation of a triangulated category, we make the following definition.

Definition 2.4.12. Let $\mathcal{A}$ be an additive category. We say that $\mathcal{A}$ is Krull-Schmidt if any object in $\mathcal{A}$ is isomorphic to a finite direct sum of indecomposable objects with local endomorphism rings in $\mathcal{A}$.

We now move to the setting of a triangulated category with the aim of defining an analogue of Auslander-Reiten sequences, namely Auslander-Reiten triangles. All the triangulated categories that we work in are Krull-Schmidt categories, and so we will make the definition in the case of a Krull-Schmidt category. We first give the following result that can be found in [22, chap. I.1].

Proposition 2.4.13. *Let C be a triangulated category and let $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ be a triangle in C. The following statements are equivalent:*

1. *If $f : W \to Z$ in C is not a split epimorphism, then there is a morphism $f' : W \to Y$ in C making the following diagram commute,*

$$\begin{tikzcd} & W \arrow[dl, dashed, "f'"'] \arrow[d, "f"] \\ Y \arrow[r, "\beta"'] & Z. \end{tikzcd}$$

2. *If $f : W \to Z$ is not a split epimorphism, then $\gamma \circ f = 0$.*

Proof. Consider $F_W(-) = \mathrm{Hom}_{\mathsf{C}}(W, -)$ and recall from Propositon 2.1.7 that applying $F_W(-)$ to our triangle produces the following exact sequence,

$$F_W(X) \xrightarrow{\alpha_*} F_W(Y) \xrightarrow{\beta_*} F_W(Z) \xrightarrow{\gamma_*} F_W(\Sigma X). \tag{2.15}$$

Assume first that $f \in F_W(Z)$ is not a split epimorphism and that there exists $f' \in F_W(Y)$ such that $\beta \circ f' = f$. Thus, $f \in \mathrm{Im}\, \beta_*$. Now, by the exactness of (2.15), we know that $f \in \mathrm{Ker}\, \gamma_*$, and so $\gamma \circ f = 0$.

Conversely, assume that $f \in F_W(Z)$ is not a split epimorphism and that $\gamma \circ f = 0$. Thus, $f \in \operatorname{Ker} \gamma_*$. By the exactness of (2.15) we therefore have $f \in \operatorname{Im} \beta_*$ and so there exists $f' \in F_W(Y)$ such that $\beta \circ f' = f$, as required. □

We now define Auslander-Reiten triangles. The following definition is taken from [22, chap. 4].

Definition 2.4.14. Let C be a triangulated Krull-Schmidt category. A triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ in C is called as Auslander-Reiten triangle if the following conditions are satisfied:

- **(AR1):** X and Z are indecomposable objects.
- **(AR2):** $\gamma \neq 0$.
- **(AR3):** If $f : W \to Z$ is not a split epimorphism, then there exists $f' : W \to Y$ making the following diagram commute,

$$\begin{tikzcd} & W \arrow[dl, dashed, "f'"'] \arrow[d, "f"] \\ Y \arrow[r, "\beta"'] & Z. \end{tikzcd}$$

Remark 2.4.15. Notice that Propositions 2.1.14 and 2.4.13 provide equivalent conditions to (AR2) and (AR3).

The following self-duality proposition can also be found in [22, chap. 4].

Proposition 2.4.16. *Let $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ be an Auslander-Reiten triangle. Then, if $f : X \to W$ is not a split monomorphism, there exists $f' : Y \to W$ making the following diagram commute,*

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & Y \arrow[dl, dashed, "f'"] \\ W. & \end{tikzcd}$$

Proof. See [22, chap. 4]. □

Remark 2.4.17. Notice that if $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ is an Auslander-Reiten triangle, then α is left almost split and β is right almost split. Indeed, the relevant properties for α follow from Proposition 2.4.16. The right almost split conditions for β are satisfied due to (AR3).

Definition 2.4.18. Let C be a triangulated category. Given an object Z in C, assume that $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ is an Auslander-Reiten triangle. Then, we denote the object X by τZ, where τ is called the Auslander-Reiten translation of C.

Note in the above definition that τZ is only defined up to isomorphism.

Remark 2.4.19. Due to [36, lem. 2.3], if $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ is an Auslander-Reiten triangle, then both X and Z have local endomorphism rings.

The above remark leads to the following definition about the existence of Auslander-Reiten triangles.

Definition 2.4.20. Let C be a triangulated category.

1. Suppose that for each object X in C with local endomorphism ring, there exists an Auslander-Reiten triangle,
$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X. \tag{2.16}$$
Then, we say that C has left Auslander-Reiten triangles.

2. Suppose that for each object Z in C with local endomorphism ring, there exists an Auslander-Reiten triangle as in (2.16). Then, we say that C has right Auslander-Reiten triangles.

3. We say that C has Auslander-Reiten triangles if it has both left and right Auslander-Reiten triangles.

The authors in [42] provide a criterion for the existence of Auslander-Reiten triangles. In order to describe this criterion, we now define the notion of Serre functor. From here on, we assume that C is a Hom-finite, $\mathbb{C}$-linear, Krull-Schmidt, triangulated category.

Definition 2.4.21. A Serre functor is an additive automorphism $S : \mathsf{C} \to \mathsf{C}$ such that there are isomorphisms
$$\mathrm{Hom}_{\mathsf{C}}(X, Y) = D\mathrm{Hom}_{\mathsf{C}}(Y, SX),$$
which are natural in $X, Y \in \operatorname{obj} \mathsf{C}$. Here, $D = \mathrm{Hom}_{\mathbb{C}}(-, \mathbb{C})$ denotes the duality functor.

Proposition 2.4.22. *The following statements are equivalent:*

1. C *has a Serre functor.*

2. C *has Auslander-Reiten triangles.*

Proof. [42, thm. I.2.4]. □

Remark 2.4.23. Note that when the conditions from Proposition 2.4.22 are satisfied we have $\tau = \Sigma^{-1}S$.

Let Λ be a finite dimensional $\mathbb{C}$-algebra and consider the abelian category $\mathsf{mod}\,\Lambda$ of finitely generated Λ-modules. We denote by $\mathsf{D}^b(\Lambda) = \mathsf{D}^b(\mathsf{mod}\,\Lambda)$ the bounded derived category of the abelian category $\mathsf{mod}\,\Lambda$. This category will be used in the following section to construct the cluster category. The following proposition, proved in [21], tells us when such a category has Auslander-Reiten triangles.

Proposition 2.4.24. *Let Λ be a finite dimensional $\mathbb{C}$-algebra. Then, $\mathsf{D}^b(\Lambda)$ has Auslander-Reiten triangles if and only if Λ has finite global dimension.*

Proof. See [20, cor. 1.5]. $\square$

Remark 2.4.25. If Q is a finite quiver with no loops or cycles, then the path algebra $\mathbb{C}Q$ has finite global dimension. Thus, the bounded derived category $\mathsf{D}^b(\mathsf{mod}\,\mathbb{C}Q)$ has Auslander-Reiten triangles by Proposition 2.4.24, and a Serre functor by Proposition 2.4.22.

We now move towards defining the Auslander-Reiten quiver of a category. We will give a definition, some key examples and explain some important properties behind its construction. We first require the notion of an irreducible morphism. This definition is taken from [2, IV def. 1.4].

Definition 2.4.26. Let C be a category. A morphism $f : X \to Y$ in C is called irreducible if the following conditions hold:

1. f is neither a split monomorphism nor a split epimorphism.
2. If $f = f_1 \circ f_2$ then either f_1 is a split epimorphism or f_2 is a split monomorphism.

We now introduce the radical of a category. The following definiton and remark can be found in [2, IV.4].

Definition 2.4.27. Let $\mathcal{A}$ be an additive category and let $X, Y \in \operatorname{indec}\mathcal{A}$.

1. We define $\operatorname{rad}_{\mathcal{A}}(X, Y)$ to be to be the vector space of all noninvertible morphisms from X to Y.
2. We define $\operatorname{rad}^2_{\mathcal{A}}(X, Y)$ to consist of all morphisms of the form $g \circ f$, where $f \in \operatorname{rad}_{\mathcal{A}}(X, Z)$ and $g \in \operatorname{rad}_{\mathcal{A}}(Z, Y)$ for some (not necessarily indecomposable) object $Z \in \mathcal{A}$.

Remark 2.4.28. It is proved in [2, IV.1, 1.6] that a morphism $f : X \to Y$ is irreducible if and only if $f \in \operatorname{rad}_{\mathcal{A}}(X, Y) \setminus \operatorname{rad}^2_{\mathcal{A}}(X, Y)$. Thus, the quotient

$$\operatorname{Irr}(X, Y) = \operatorname{rad}_{\mathcal{A}}(X, Y)/\operatorname{rad}^2_{\mathcal{A}}(X, Y)$$

known as the irreducible space of morphisms, measure the number of irreducible morphisms from X to Y.

We are now able to define the Auslander-Reiten quiver. The following definiton can be found in [2, IV.4. 4.6].

Definition 2.4.29. Let $\mathcal{A}$ be an additive category. The Auslander-Reiten quiver $\Gamma = \Gamma(\mathcal{A})$ of $\mathcal{A}$ is defined as follows:

1. The vertex set Γ_0 of Γ is given by the set of isomorphism classes $[X]$ of indecomposable objects X in $\mathcal{A}$.

2. Let $[X]$ and $[Y]$ be vertices of Γ. The arrows $[X] \to [Y]$ are in bijective correspondence with the vectors of a basis of the vector space $\operatorname{Irr}(X, Y)$.

Remark 2.4.30. In practice, if $[X]$ and $[Y]$ are two vertices of the Auslander-Reiten quiver, then we draw an arrow $[X] \to [Y]$ if there exists an irreducible morphism from X to Y. Moreover, we put an integer valuation on each arrow in the Auslander-Reiten quiver, which corresponds to the number of irreducible morphisms between the two objects, that is the number of elements in a basis of $\operatorname{Irr}(X, Y)$. In every case that we consider, the valuation on every arrow if just one, so for simplicity we will drop the valuation and merely draw a single arrow.

Example 2.4.31. Consider the quiver of Dynkin type A_3:

$$1 \longrightarrow 2 \longrightarrow 3$$

We draw below the Auslander-Reiten quiver for $\mathsf{mod}\,\mathbb{C}A_3$. We denote by P_i the indecomposable projectives, by I_i the indecomposable injectives, and by S_i the simple modules.

$$\begin{tikzcd}
 & & P_1 = I_3 \arrow[dr] & & \\
 & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[dr] & \\
P_3 = S_3 \arrow[ur] & & S_2 \arrow[ur] & & S_1 = I_1.
\end{tikzcd} \tag{2.17}$$

Given by their quiver representations, these modules are as in in Figure 2.2. In general, the Auslander-Reiten quiver for $\mathsf{mod}\,\mathbb{C}A_n$ is a triangle as in (2.17), but with n rows.

Example 2.4.32. Using the Auslander-Reiten quiver for $\mathsf{mod}\,\mathbb{C}A_3$ from the example

$$
\begin{array}{ll}
S_1 = I_1 = \mathbb{C} \longrightarrow 0 \longrightarrow 0 & S_2 = 0 \longrightarrow \mathbb{C} \longrightarrow 0 \\
S_3 = P_3 = 0 \longrightarrow 0 \longrightarrow \mathbb{C} & I_2 = \mathbb{C} \xrightarrow{1_{\mathbb{C}}} \mathbb{C} \longrightarrow 0 \\
P_1 = I_3 = \mathbb{C} \xrightarrow{1_{\mathbb{C}}} \mathbb{C} \xrightarrow{1_{\mathbb{C}}} \mathbb{C} & P_2 = 0 \longrightarrow \mathbb{C} \xrightarrow{1_{\mathbb{C}}} \mathbb{C}.
\end{array}
$$

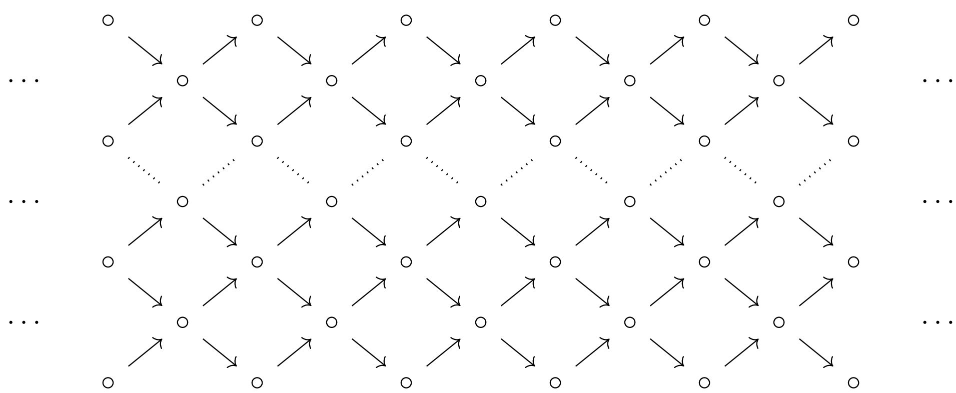
Figure 2.2: The projective, injective and simple modules in $\mathsf{C}(A_3)$, given by their quiver respresentations.

above, we draw below the Auslander-Reiten quiver for $\mathsf{D}^b(\operatorname{\mathsf{mod}} \mathbb{C}A_3)$, see [21, cor. 4.5(i)].

$$
\begin{tikzcd}
 & \Sigma^{-1}S_1 \arrow[dr] & & P_1 \arrow[dr] & & \Sigma P_3 \arrow[dr] & & \Sigma S_2 \arrow[dr] & & \Sigma S_1 & \\
\cdots & & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & \Sigma P_2 \arrow[ur] \arrow[dr] & & \Sigma I_2 \arrow[ur] \arrow[dr] & & \cdots \\
 & P_3 \arrow[ur] & & S_2 \arrow[ur] & & S_1 \arrow[ur] & & \Sigma P_1 \arrow[ur] & & \Sigma^2 P_3 &
\end{tikzcd}
$$

Notice that the part drawn in blue is the Auslander-Reiten quiver for $\operatorname{\mathsf{mod}} \mathbb{C}A_3$. There are then irreducible morphisms from I_2 to ΣP_3 and from S_1 to ΣP_2, represented by the red arrows. This then allows us to repeat the Auslander-Reiten quiver of $\operatorname{\mathsf{mod}} \mathbb{C}A_3$, having applied Σ and having flipped the Auslander-Reiten quiver of $\operatorname{\mathsf{mod}} \mathbb{C}A_3$. This can be seen in the above diagram. Repeating this process using both Σ and Σ^{-1} produces the full Auslander-Reiten quiver.

Example 2.4.33. We draw below the general case of Example 2.4.32, that is, the Auslander-Reiten quiver for $\mathsf{D}^b(\operatorname{\mathsf{mod}} \mathbb{C}A_n)$. Again, see [21, cor. 4.5(i)]. Notice that there are n rows.



The following proposition is given in [29, lem. 2.2], and is a direct consequence of [21, prop. 3.5].

Proposition 2.4.34. *Let $X \to Y \to Z \to \Sigma X$ be an Auslander-Reiten triangle, and let $Y = \bigoplus_i Y_i$ be a decomposition of Y into its indecomposable summands. Also, let Y' be some indecomposable. The following statements are equivalent:*

1. *There exists an irreducible morphism $f : X \to Y'$.*
2. *There exists an irreducible morphism $g : Y' \to Z$.*
3. *Y' is isomorphic to an indecomposable summand Y_i of Y.*

Remark 2.4.35. As a consequence of Proposition 2.4.34, the Auslander-Reiten quiver allows us to read off Auslander-Reiten triangles. Looking at the Auslander-Reiten quiver of $\mathsf{D}^b(\mathsf{mod}\,\mathbb{C}A_n)$ in Example 2.4.33, if

$$\begin{tikzcd} & Y_1 \arrow[dr] & \\ X \arrow[ur] \arrow[dr] & & Z \\ & Y_2 \arrow[ur] & \end{tikzcd}$$

is a diamond in the quiver, then

$$X \to Y_1 \oplus Y_2 \to Z \to \Sigma X,$$

is an Auslander-Reiten triangle. Note that if X and Z are on the upper boundary of the Auslander-Reiten quiver, then Y_1 should be taken as zero, whilst if X and Z are on the lower boundary, then Y_2 is taken to be zero. Recall from Definition 2.4.18 that $X = \tau Z$, and so we know that applying τ to an indecomposable corresponds to shifting one vertex to the left in the Auslander-Reiten quiver, i.e. any diamond in the Auslander-Reiten quiver is of the form

$$\begin{tikzcd} & Y_1 \arrow[dr] & \\ \tau Z \arrow[ur] \arrow[dr] & & Z. \\ & Y_2 \arrow[ur] & \end{tikzcd}$$

2.5 Cluster Categories

In this section we construct a further example of a triangulated category, namely the cluster category. It will be one of the main objects of study throughout this thesis, and was first introduced by Buan, Marsh, Reineke, Reiten and Todorov in [8] as a means of understanding the 'decorated quiver representations' introduced by Reineke, Marsh and Zelevinsky in [37]. The cluster category is widely acknowledged as the categorification of cluster algebras, introduced by Fomin and Zelevinsky in [17]. We use the following remark to set up some notation before defining the cluster category.

Remark 2.5.1. Let Q be a quiver with no loops or 2-cycles, and consider the bounded derived category $\mathsf{D}^b(\mathsf{mod}\,\mathbb{C}Q)$. Recall that $\mathsf{D}^b(\mathsf{mod}\,\mathbb{C}Q)$ has a suspension functor Σ and a Serre functor S. The Auslander-Reiten translation is given by $\tau = S\Sigma^{-1}$.

Definition 2.5.2. Let Q be a finite quiver with no loops or 2-cycles. The cluster category of type Q, denoted by $\mathsf{C}(Q)$, is defined as the following orbit category,

$$\mathsf{C}(Q) = \mathsf{D}^b(\mathsf{mod}\,\mathbb{C}Q)/(\Sigma \circ \tau^{-1}).$$

Definition 2.5.3. $\mathsf{C}(Q)$ is the orbit category of $\mathsf{D}(Q) = \mathsf{D}^b(\mathsf{mod}\,\mathbb{C}Q)$ with respect to the functor $\Sigma \circ \tau^{-1}$. Objects and morphisms in $\mathsf{C}(Q)$ are given as follows:

- $\operatorname{obj} \mathsf{C}(Q) = \operatorname{obj} \mathsf{D}(Q)$.
- For $X, Y \in \operatorname{obj} \mathsf{C}(Q)$, we have

$$\operatorname{Hom}_{\mathsf{C}(Q)}(X, Y) = \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_{\mathsf{D}(Q)}(X, (\Sigma\tau^{-1})^n Y).$$

We will give a list of important properties of $\mathsf{C}(Q)$, one of which is 2-Calabi-Yau, which we define below.

Definition 2.5.4. Let C be a triangulated category with a Serre functor S. We say that C is a 2-Calabi-Yau category if S is naturally equivalent to Σ^2.

Remark 2.5.5. The cluster category $\mathsf{C}(Q)$ has the following properties, each due to [8]. It is triangulated, $\mathbb{C}$-linear, Hom-finite, Krull-Schmidt and 2-Calabi-Yau, that is $S = \Sigma^2$. It is also essentially small and has split idempotents.

Proposition 2.5.6. *Let $X, Y \in \operatorname{obj} \mathsf{C}(Q)$. Then,*

$$\operatorname{Ext}^1_{\mathsf{C}(Q)}(X, Y) \cong \operatorname{Ext}^1_{\mathsf{C}(Q)}(Y, X).$$

Proof. See [8, prop. 1.7(b)]. □

2.5.1 The Cluster Category of Dynkin Type A_n

Independently to the work in [8], Caldero, Chapoton and Schiffler defined in [11] an equivalent category to $\mathsf{C}(Q)$ for Dynkin quivers of type A_n, where $n \geq 3$ is an integer. Their construction uses a nice combinatorial model, and this is what we introduce in this section. We first draw the Auslander-Reiten quiver for $\mathsf{C}(A_n)$ using the following example.

Example 2.5.7. Let $Q = A_5$, the Dynkin quiver of type A_5. The Auslander-Reiten quiver for $\mathsf{C}(A_5)$ is given as follows,

$$\begin{tikzcd}[column sep=tiny, row sep=small]
& \circ \arrow[dr] & & \circ \arrow[dr] & & e \arrow[dr] & & \Sigma a \arrow[dr] & & a \arrow[dr] & & \circ \arrow[dr] & & \circ & \\
\cdots & & \circ \arrow[ur]\arrow[dr] & & d \arrow[ur]\arrow[dr] & & \circ \arrow[ur]\arrow[dr] & & \Sigma b \arrow[ur]\arrow[dr] & & b \arrow[ur]\arrow[dr] & & \circ \arrow[ur]\arrow[dr] & & \cdots \\
& \circ \arrow[ur]\arrow[dr] & & c \arrow[ur]\arrow[dr] & & \circ \arrow[ur]\arrow[dr] & & \circ \arrow[ur]\arrow[dr] & & \Sigma c \arrow[ur]\arrow[dr] & & c \arrow[ur]\arrow[dr] & & \circ & \\
\cdots & & b \arrow[ur]\arrow[dr] & & \circ \arrow[ur]\arrow[dr] & & \circ \arrow[ur]\arrow[dr] & & \circ \arrow[ur]\arrow[dr] & & \Sigma d \arrow[ur]\arrow[dr] & & d \arrow[ur]\arrow[dr] & & \cdots \\
& a \arrow[ur] & & \circ \arrow[ur] & & \circ \arrow[ur] & & \circ \arrow[ur] & & \circ \arrow[ur] & & \Sigma e \arrow[ur] & & e. &
\end{tikzcd}$$

Notice that there is a certain identification in the Auslander-Reiten quiver, due to the functor $\Sigma \circ \tau^{-1}$. The "first slice" of indecomposables, labelled with a, b, c, d, e in the diagram, is flipped and repeated, as in the diagram above. Clearly, this means that the Auslander-Reiten quiver for $\mathsf{C}(A_5)$ is finite. Note that in general, the Auslander-Reiten quiver for $\mathsf{C}(A_n)$ has n rows. A coordinate system can be put on the Auslander-Reiten quiver, as follows:

$$\begin{tikzcd}[column sep=tiny, row sep=small]
\{4,6\} \arrow[dr] \arrow[dddd, dash, dashed] & & \{3,5\} \arrow[dr] & & \{0,6\} \arrow[dr] & & \{1,7\} \arrow[dr] & & \{0,2\} \arrow[dddd, dash, dashed] \\
& \{3,6\} \arrow[ur]\arrow[dr] & & \{0,5\} \arrow[ur]\arrow[dr] & & \{1,6\} \arrow[ur]\arrow[dr] & & \{2,7\} \arrow[ur]\arrow[dr] & \\
\{3,7\} \arrow[ur]\arrow[dr] & & \{0,4\} \arrow[ur]\arrow[dr] & & \{1,5\} \arrow[ur]\arrow[dr] & & \{2,6\} \arrow[ur]\arrow[dr] & & \{3,7\} \\
& \{0,3\} \arrow[ur]\arrow[dr] & & \{1,4\} \arrow[ur]\arrow[dr] & & \{2,5\} \arrow[ur]\arrow[dr] & & \{3,6\} \arrow[ur]\arrow[dr] & \\
\{0,2\} \arrow[ur] & & \{1,3\} \arrow[ur] & & \{2,4\} \arrow[ur] & & \{3,5\} \arrow[ur] & & \{4,6\}.
\end{tikzcd}$$

Note that the order of the coordinates does not matter, that is, $\{i, j\} = \{j, i\}$, and each coordinate is taken modulo $n + 3$. We should think of each coordinate pair $\{i, j\}$ as an interior diagonal connecting vertices i and j of a regular $(n + 3)$-gon with vertex set $\{0, 1, \ldots, n + 2\}$. The dashed lines are identified, forming a Möbius strip.

Remark 2.5.8. As before, if

$$\begin{tikzcd}
& Y_1 \arrow[dr] & \\
X \arrow[ur]\arrow[dr] & & Z \\
& Y_2 \arrow[ur] &
\end{tikzcd}$$

is a diamond in the Auslander-Reiten quiver of $\mathsf{C}(A_n)$, then

$$X \to Y_1 \oplus Y_2 \to Z \to \Sigma X$$

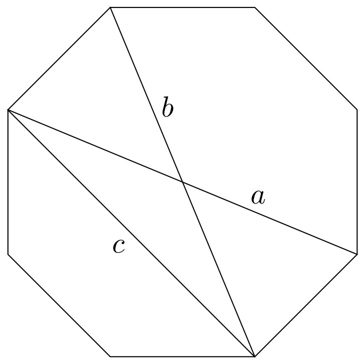


Figure 2.3: The polygon model associated to $\mathsf{C}(A_5)$.

is an Auslander-Reiten triangle, and in particular, $X = \tau Z$. Again, if X and Z sit on the upper boundary of the Auslander-Reiten quiver, then Y_1 is taken to be zero, whilst if X and Z sit on the lower boundary, then Y_2 is zero. From the first diagram in Example 2.5.7, we can see that each diamond is of the form,

$$\begin{tikzcd} & Y_1 \arrow[dr] & \\ \Sigma Z \arrow[ur] \arrow[dr] & & Z. \\ & Y_2 \arrow[ur] & \end{tikzcd}$$

$\mathsf{C}(A_n)$ is 2-Calabi-Yau so $S = \Sigma^2$ and $\tau = S\Sigma^{-1} = \Sigma$.

We now introduce the combinatorial model associated to $\mathsf{C}(A_n)$. Let $n \geq 3$ be an integer and let P be a regular $(n+3)$-gon. We say that a diagonal $\{i, j\}$ of P is a line in the interior of P connecting two non-neighbouring vertices i and j. We then consider the set of diagonals in P. See a, b and c in Figure 2.3 for examples of diagonals in P.

We make the following remark on notation that we will use regularly throughout this thesis.

Remark 2.5.9. For a category C, we denote by $\operatorname{indec} \mathsf{C}$ the set of indecomposables in C.

There is a bijection

$$\operatorname{indec} \mathsf{C}(A_n) \leftrightarrow \{\text{diagonals of } P\},$$

and so we can identify the set of diagonals of P with the set of indecomposables in C. We also identify any exterior edge of P with the zero object in $\mathsf{C}(A_n)$.

Let $\{0, 1, \ldots, n+2\}$ be the vertex set of P. Since a diagonal connects two vertices of P, we can think of a set $\{i, j\}$, for i and j vertices of P, as a diagonal in P. This coordinate system is the coordinate system used in drawing the Auslander-Reiten quiver in Example 2.5.7.

Definition 2.5.10. Let P be a regular $(n+3)$-gon with vertices labelled in a positive orientation by the set $\{0, 1, \ldots, n+2\}$. Then, let $\{i, j\}$ be a diagonal in P associated to an indecomposable in $\mathsf{C}(A_n)$. The object $\Sigma\{i, j\} \in \operatorname{indec} \mathsf{C}(A_n)$ corresponds to the diagonal $\{i-1, j-1\}$ obtained by "shifting" each endpoint of $\{i, j\}$ one vertex clockwise in P.

Remark 2.5.11. In Figure 2.3, we have $\Sigma a = b$.

Definition 2.5.12. We say that two diagonals in an $(n+3)$-gon P cross if they intersect each other in the interior of P.

Remark 2.5.13. In Figure 2.3, a and b cross, whilst neither a nor b crosses c. Note also that by definition, a diagonal does not cross itself.

We have the following highly convenient formula that tells us the dimension of $\operatorname{Ext}^1_{\mathsf{C}(A_n)}$ spaces. Let $a, b \in \operatorname{indec} \mathsf{C}(A_n)$, then

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathsf{C}(A_n)}(a, b) = \begin{cases} 1 & \text{if } a \text{ and } b \text{ cross,} \\ 0 & \text{if } a \text{ and } b \text{ do not cross.} \end{cases} \tag{2.18}$$

Remark 2.5.14. In Figure 2.3, we have

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathsf{C}(A_n)}(a, b) = 1,$$

whilst

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathsf{C}(A_n)}(a, c) = \dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathsf{C}(A_n)}(b, c) = 0.$$

Remark 2.5.15. Recalling that $\operatorname{Hom}(a, \Sigma b) = \operatorname{Ext}^1(a, b)$, we can compute dimensions of hom spaces in $\mathsf{C}(A_n)$ using (2.18). Indeed, for $a, b \in \operatorname{indec} \mathsf{C}(A_n)$, we have

$$\operatorname{Hom}_{\mathsf{C}(A_n)}(a, b) = \operatorname{Hom}_{\mathsf{C}(A_n)}(a, \Sigma(\Sigma^{-1} b)) = \operatorname{Ext}^1_{\mathsf{C}(A_n)}(a, \Sigma^{-1} b).$$

It follows from (2.18) that for all $a, b \in \operatorname{indec} \mathsf{C}(A_n)$,

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathsf{C}(A_n)}(a, b) = 0 \text{ or } 1.$$

$\mathsf{C}(A_n)$ is therefore what is known as a Schurian category.

Recall from Definition 2.1.18 the definition of exchange triangles. Our model allows us to compute the middle terms of such triangles.

Let $x, y \in \operatorname{indec} \mathsf{C}(A_n)$ with,

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathsf{C}(A_n)}(x, y) = \dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathsf{C}(A_n)}(y, x) = 1. \tag{2.19}$$

Then, let

$$x \longrightarrow A \longrightarrow y \xrightarrow{\delta} \Sigma x \quad \text{and} \quad y \longrightarrow B \longrightarrow x \xrightarrow{\gamma} \Sigma y,$$

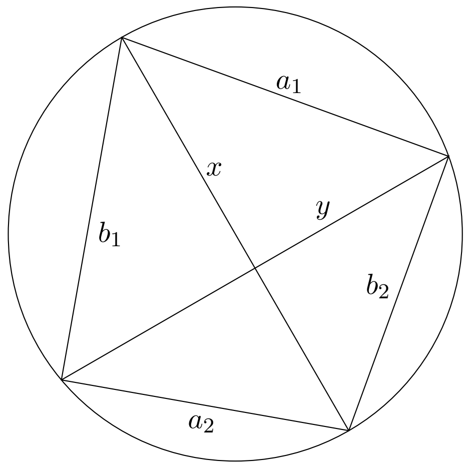


Figure 2.4: There are exchange triangles $x \to a_1 \oplus a_2 \to y$ and $y \to b_1 \oplus b_2 \to x$.

with $\delta, \gamma \neq 0$, be the ensuing exchange triangles. Clearly, due to (2.19), we know that the diagonals x and y cross, as in Figure 2.4. Connecting the endpoints of x and y, as in the figure, allows us to compute the middle terms of the exchange triangles, which are given as follows:

$$A = a_1 \oplus a_2 \quad \text{and} \quad B = b_1 \oplus b_2.$$

If any a_i or b_i are exterior edges of P, then the corresponding summands are zero.

Definition 2.5.16. Let C be a triangulated category and let $R \in \operatorname{obj} \mathsf{C}$. We say that R is a rigid object if

$$\operatorname{Hom}_{\mathsf{C}}(R, \Sigma R) = 0.$$

Remark 2.5.17. Recalling that $\operatorname{Hom}_{\mathsf{C}}(a, \Sigma b) = \operatorname{Ext}^1_{\mathsf{C}}(a, b)$, we see from (2.18) that every indecomposable object in $\mathsf{C}(A_n)$ is rigid.

Definition 2.5.18. Let C be a triangulated Krull-Schmidt category and let $R \in \operatorname{obj} \mathsf{C}$ be a rigid object. We define by $\mathsf{R} = \operatorname{add} R$ the rigid subcategory of C whose objects are finite direct sums of the summands of R. Notice that R is rigid in the sense that,

$$\operatorname{Hom}_{\mathsf{C}}(\mathsf{R}, \Sigma \mathsf{R}) = 0.$$

Definition 2.5.19. Let C be a triangulated category and let $T \in \operatorname{obj} \mathsf{C}$. We say that T is a cluster tilting object if,

$$\operatorname{Hom}_{\mathsf{C}}(T, \Sigma t) = 0 \Leftrightarrow t \in \operatorname{add} T \Leftrightarrow \operatorname{Hom}_{\mathsf{C}}(t, \Sigma T) = 0. \tag{2.20}$$

Remark 2.5.20. Cluster tilting objects have become a widely studied notion in triangulated categories, with a large amount of literature dedicated to their study, see for example [35].

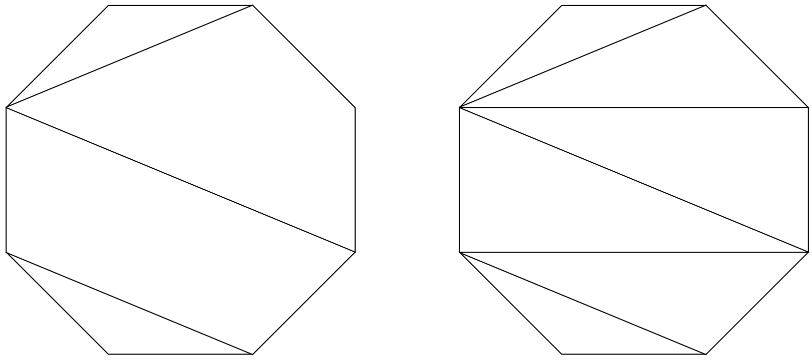


Figure 2.5: Indecomposables of a rigid subcatergory (left) and a cluster tilting subcategory (right) of $\mathsf{C}(A_5)$.

We give perhaps the most interesting property of cluster tilting objects in this remark. Indeed, let C be a triangulated category and let T be a cluster tilting object in C. Consider then the ideal $[\operatorname{add} T]$ generated by morphisms in C which factor through an object in $\operatorname{add} T$. By [35, sec.2.1, prop.], we then have the following equivalence of categories

$$\mathsf{C}/[\operatorname{add} T] \cong \mathsf{mod}\operatorname{End} T.$$

Here, notice that we have started with a triangulated category C, taken a quotient by an ideal constructed from the cluster tilting object T and obtained an abelian category.

Definition 2.5.21. The subcategory $\mathsf{T} = \operatorname{add} T$ from Definition 2.5.19, whose objects are finite direct sums of the summands of T, is a cluster tilting subcategory.

We are able to represent rigid and cluster tilting subcategories of $\mathsf{C}(A_n)$ using the polygon model:

- Let $\mathsf{R} \subseteq \mathsf{C}(A_n)$ be a rigid subcategory. Then, $\operatorname{indec} \mathsf{R}$ corresponds to a polygon dissection of the $(n+3)$-gon P, made up of non-crossing diagonals, as in the left hand side of Figure 2.5. This is easy to see since indecomposables correspond to diagonals in P, and since R is rigid, it follows from (2.18) that none of these diagonals cross each other, thus creating a polygon dissection.

- Let $\mathsf{T} \subseteq \mathsf{C}(A_n)$ be a cluster tilting subcategory. Then, $\operatorname{indec} \mathsf{T}$ corresponds to a polygon triangulation of P, again made up of non-crossing diagonals, as in the right hand side of Figure 2.5. This again follows from (2.18). We get a full triangulation in this case as a consequence of (2.20). Indeed, if $\operatorname{indec} \mathsf{T}$ did not correspond to a full triangulation, then there would be a diagonal not in T, crossing none of the diagonals in T, corresponding to a $t \notin \operatorname{add} T$ such that $\operatorname{Hom}_{\mathsf{C}}(T, \Sigma t) = 0$, a contradiction.

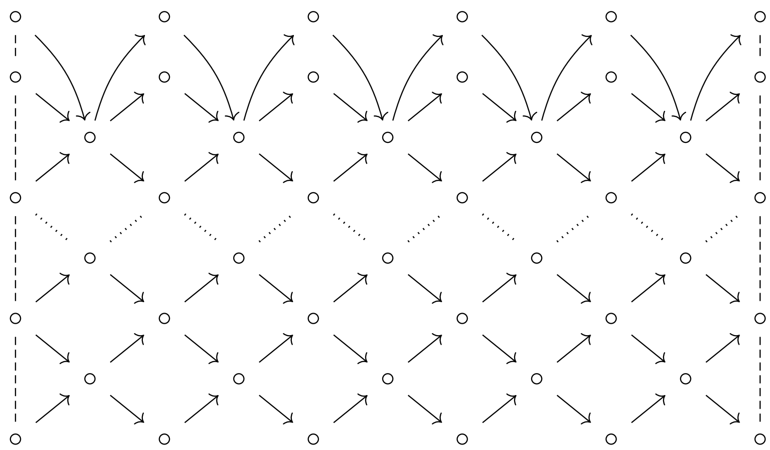


Figure 2.6: The Auslander-Reiten quiver for $\mathsf{C}(D_n)$.

Remark 2.5.22. Notice that in $\mathsf{C}(A_n)$, any maximal rigid subcategory is a cluster tilting subcategory. Note here that we say a rigid subcategory R is maximal rigid if $\mathrm{Ext}^1_{\mathsf{C}}(\mathsf{R} \cup \operatorname{add} M, \mathsf{R} \cup \operatorname{add} M) = 0$ implies that $M \in \mathsf{R}$.

2.5.2 The Cluster Category of Dynkin Type D_n.

In this subsection we describe the cluster category of Dynkin type D_n, which we denote by $\mathsf{C}(D_n)$. Firstly, the Auslander-Reiten quiver of $\mathsf{C}(D_n)$ is given in Figure 2.6. We can again read off Auslander-Reiten triangles from the quiver as follows. If

$$\begin{tikzcd} & Y_1 \arrow[dr] & \\ X \arrow[ur] \arrow[dr] & & Z \\ & Y_2 \arrow[ur] & \end{tikzcd}$$

is a diamond in the Auslander-Reiten quiver of $\mathsf{C}(D_n)$, then

$$X \to Y_1 \oplus Y_2 \to Z$$

is an Auslander-Reiten triangle in $\mathsf{C}(D_n)$.

For the top row of the quiver, if

$$\begin{tikzcd}
 & Y_1 \arrow[ddr, bend left] & \\
 & Y_2 \arrow[dr] & \\
X \arrow[uur, bend left] \arrow[ur] \arrow[dr] & & Z \\
 & Y_3 \arrow[ur] &
\end{tikzcd}$$

is a "mesh" in the Auslander-Reiten quiver, then

$$X \to Y_1 \oplus Y_2 \oplus Y_3 \to Z$$

is an Auslander-Reiten triangle.

Finally, looking at the top row again, if

$$\begin{tikzcd}
 & X_1 \arrow[ddr, bend left] & & Z_1 \arrow[ddr, bend left] & \\
 & X_2 \arrow[dr] & & Z_2 \arrow[dr] & \\
\circ \arrow[uur, bend left] \arrow[ur] & & Y \arrow[uur, bend left] \arrow[ur] & & \circ
\end{tikzcd}$$

sits in the Auslander-Reiten quiver, then

$$X_1 \to Y \to Z_1 \quad \text{and} \quad X_2 \to Y \to Z_2$$

are both Auslander-Reiten triangles.

Similarly to the case of $\mathsf{C}(A_n)$, there is also a nice combinatorial model associated to $\mathsf{C}(D_n)$. This was introduced by Schiffler in [43]. We introduce some of the key constructions of this combinatorial model here.

Let P be a regular n-gon, with a puncture at its centre. We may associate the set of indecomposables of $\mathsf{C}(D_n)$ with a certain set of diagonals of P. There are two types of diagonals, namely "spokes" and "arcs". Denote by a and b two vertices of P with $a \neq b$. Then, we denote by $M_{a,b}$ the diagonal in P that runs anticlockwise from a to b. This diagonal is called an arc, and we note that due to the puncture, the arcs $M_{a,b}$ and $M_{b,a}$ are distinct. Note also that if b is the anticlockwise neighbouring vertex of a, then $M_{a,b}$ lies on the edge of P. In this case, we identify $M_{a,b}$ with the zero object inside $\mathsf{C}(D_n)$. See Figure 2.7 for examples of arcs.

For $\epsilon \in \{\pm 1\}$, we denote by $M^{\epsilon}_{a,a}$ a spoke at the vertex a. This is a diagonal from a to the puncture. Each vertex a permits two spokes, $M^{1}_{a,a}$ and $M^{-1}_{a,a}$, and we use a tag ϵ to distinguish between these. Again, see Figure 2.7 for examples of spokes. We say that

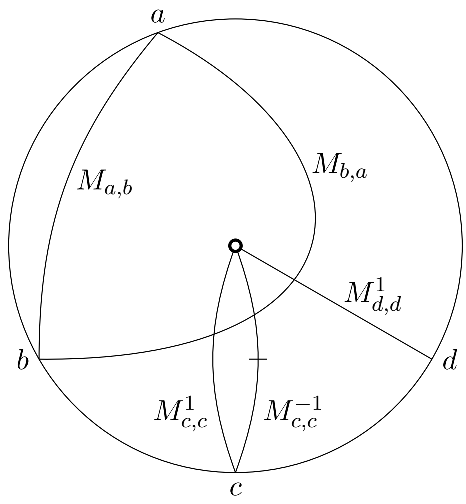


Figure 2.7: Examples of spokes and arcs inside the polygon. $M_{a,b}$ and $M_{b,a}$ are both arcs, whilst $M^1_{c,c}$, $M^{-1}_{c,c}$ and $M^1_{d,d}$ are spokes.

the set of diagonals of P is the set of spokes and arcs in P, and then there is the following bijection

$$\operatorname{indec} \mathsf{C}(D_n) \leftrightarrow \{\text{diagonals of } P\}.$$

We then identify the set of diagonals of P with the set of indecomposables in $\mathsf{C}(D_n)$.

Remark 2.5.23. Formally, we should view the diagonals defined above as isotopy classes of paths inside P. Choose two vertices a and b, and a path α from a to b that lies within the interior of P and does not pass through the same point twice, noting that the puncture is not a point in the interior and so the path cannot pass through the puncture. Then, $M_{a,b}$ is the isotopy class of the path α. In the case when $a = b$, there are two paths from a to a. One path that runs around the puncture, and one that doesn't. We represent the isotopy classes of these paths using straight lines from the vertex a to the puncture. We use the tag $\epsilon \in \{-1, 1\}$ to distinguish between the two isotopy classes $M^1_{a,a}$ and $M^{-1}_{a,a}$. This viewpoint allows the proceeding important discussion about crossings of diagonals inside P.

Crossings of diagonals inside P are defined as follows:

1. Consider two arcs $M_{a.b}$ and $M_{c,d}$ with $a \neq b$ and $c \neq d$. Let α be a representative of the isotopy class $M_{a,b}$ and β a representative of the isotopy class $M_{c,d}$. Then, we say that $M_{a,b}$ and $M_{c,d}$ cross if the minimum number of intersections between α and β in the interior of P is greater than zero. If the number of intersections is zero, then we say that they do not cross. In Figure 2.7, the arcs $M_{a,b}$ and $M_{b,a}$ do not cross.

2. Consider an arc $M_{a,b}$ with $a \neq b$ and a spoke $M^\epsilon_{c,c}$. Let α be a representative of $M_{a,b}$ and β a straight line from c to the puncture. Then, we say that $M_{a.b}$ and $M^\epsilon_{c,c}$ cross

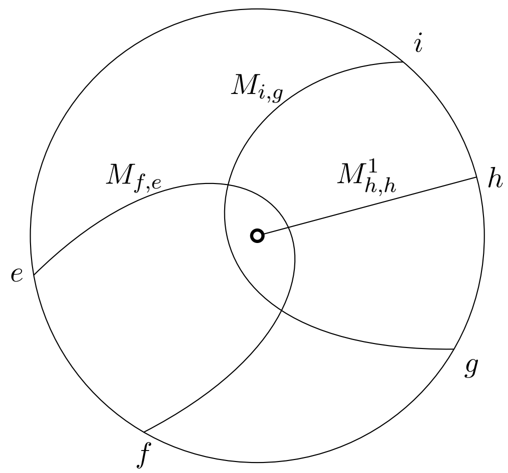


Figure 2.8: Crossings between diagonals in P.

if again, the minimum number of intersections between α and β in the interior of P in greater than zero. In Figure 2.7, the arc $M_{b,a}$ crosses all three spokes, whereas $M_{a,b}$ crosses none of the spokes.

3. Finally, we say that two spokes $M^{\epsilon}_{a,a}$ and $M^{\epsilon'}_{b,b}$ cross if $a \neq b$ and $\epsilon \neq \epsilon'$. Otherwise, we say that the two spokes do not cross. In Figure 2.7, $M^{-1}_{c,c}$ crosses $M^1_{d,d}$, however, neither spoke crosses $M^1_{c,c}$.

From the definitions of crossing above, we can now define the crossing number as in the following definition.

Definition 2.5.24. For two diagonals M and N, we define the crossing number of M and N, denoted by $e(M, N)$, as follows:

1. If $M = M_{a,b}$ with $a \neq b$ and $N = N_{c,d}$ with $c \neq d$, then $e(M, N)$ is the minimum number of intersections of representatives of isotopy classes of M and N.

2. If $M = M_{a,b}$ with $a \neq b$ and $N = N^{\epsilon}_{c,c}$, then

$$e(M, N) = \begin{cases} 1 & \text{if } M \text{ and } N \text{ cross,} \\ 0 & \text{if } M \text{ and } N \text{ do not cross.} \end{cases}$$

3. If $M = M^{\epsilon}_{a,a}$ and $N = N^{\epsilon'}_{b,b}$, then

$$e(M, N) = \begin{cases} 1 & \text{if } M \text{ and } N \text{ cross,} \\ 0 & \text{if } M \text{ and } N \text{ do not cross.} \end{cases}$$

Remark 2.5.25. For two arcs, the crossing number can be a maximum of two, whereas, for two spokes, or one arc and one spoke, the crossing number can clearly be at most one.

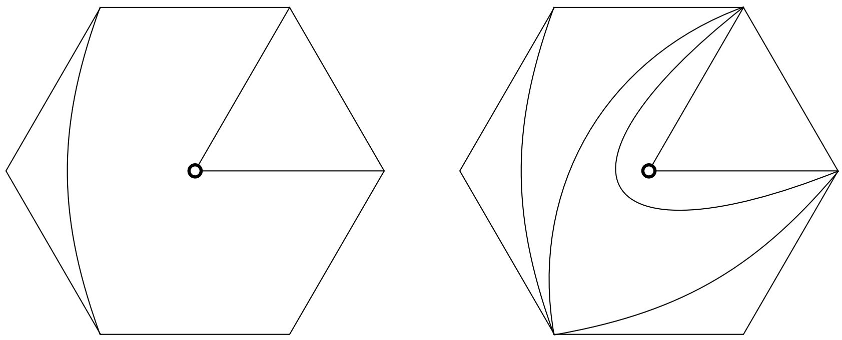


Figure 2.9: Indecomposables of a rigid subcategory (left) and a cluster tilting subcategory (right) of $\mathsf{C}(D_6)$

Remark 2.5.26. In Figure 2.8, we have the following crossing numbers:

$$e(M_{f,e}, M_{i,g}) = 2 \ , \ e(M_{f,e}, M^1_{h,h}) = 1 \ , \ e(M_{i,g}, M^1_{h,h}) = 0.$$

Now, due to [43, thm. 5.3], we have the following highly convenient property for two indecomposables M, N in $\mathsf{C}(D_n)$:

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathsf{C}}(M, N) = e(M, N). \tag{2.21}$$

All $\operatorname{Ext}^1_{\mathsf{C}}$ spaces between indecomposables in $\mathsf{C}(D_n)$ therefore have dimension zero, one or two.

Since $\mathsf{C}(D_n)$ is triangulated, it is equipped with a suspension functor Σ. This can be described combinatorially as follows:

1. If $M_{a,b}$, with $a \neq b$, is an arc, then

$$\Sigma M_{a,b} = M_{a',b'},$$

   where a' is the clockwise neighbouring vertex of a, and b' is the clockwise neighbouring vertex of b. Informally, applying Σ to an arc shifts each endpoint of the arc one vertex clockwise.

2. If $M^{\epsilon}_{a,a}$ is a spoke, then

$$\Sigma M^{\epsilon}_{a,a} = M^{\epsilon'}_{a',a'},$$

   where a' is again the clockwise neighbour of a, and $\epsilon' \neq \epsilon$. Informally, applying Σ to a spoke shifts the endpoint on the edge of P one vertex clockwise, whilst also changing the tagging of the spoke.

As with $\mathsf{C}(A_n)$, we are able to represent rigid and cluster tilting subcategories of $\mathsf{C}(A_n)$ using the polygon model:

- Let $\mathsf{R} \subseteq \mathsf{C}(D_n)$ be a rigid subcategory. Then, $\operatorname{indec} \mathsf{R}$ corresponds to a polygon dissection of the n-gon P, made up of non-crossing diagonals, as in the left hand side of Figure 2.9. This is easy to see since we know that indecomposables correspond to diagonals in P. Then, since R is rigid, it follows from (2.21) that these diagonals do not cross each other, thus creating a polygon dissection.

- Let $\mathsf{T} \subseteq \mathsf{C}(D_n)$ be a cluster tilting subcategory. Then, $\operatorname{indec} \mathsf{T}$ corresponds to a polygon triangulation of P, again made up of non-crossing diagonals, as in the right hand side of Figure 2.9. This again follows from (2.21). We get a full triangulation in this case as a direct consequence of (2.20). Indeed, if $\operatorname{indec} \mathsf{T}$ did not correspond to a full triangulation of P, then there would be a diagonal not in T, crossing none of the diagonals in T, corresponding to a $t \notin \operatorname{add} T$ such that $\operatorname{Hom}_{\mathsf{C}(D_n)}(T, \Sigma t) = 0$, a contradiction.

Remark 2.5.27. Notice that in $\mathsf{C}(D_n)$, any maximal rigid subcategory is a cluster tilting subcategory.

Chapter 3

A Multiplication Formula for the Modified Caldero-Chapoton Map

3.1 Introduction

3.1.1 Summary

This chapter focuses around two main topics: generalised friezes with integer values (see [25]) and generalised friezes taking values inside a Laurent polynomial ring (see [26]).

A frieze is a map $X : \operatorname{obj} \mathsf{C} \to A$, where C is some triangulated category with Auslander-Reiten (AR) triangles and A is a ring, such that the following exponential conditions are satisfied:

$$X(0) = 1 \text{ and } X(a \oplus b) = X(a)X(b), \tag{3.1}$$

and if $\tau x \to y \to x$ is an Auslander-Reiten triangle, then

$$X(\tau x)X(x) - X(y) = 1. \tag{3.2}$$

The canonical example of a frieze is the Caldero-Chapoton map, which is defined in Equation (3.11).

Generalised friezes are similarly defined maps $X' : \operatorname{obj} \mathsf{C} \to A$, also satisfying the exponential conditions in (3.1), however we permit the more general property that

$$X'(\tau x)X'(x) - X'(y) \in \{0, 1\}. \tag{3.3}$$

The canonical example of a generalised frieze is the modified Caldero-Chapoton map, which we introduce in Section 3.1.4. The arithmetic version π, with integer values, is defined in Equation (3.18), whilst the more general version ρ, taking values inside a Laurent polynomial ring, is defined in Equation (3.23).

The modified Caldero-Chapoton map was introduced in [26], and we improve and add to the results of that paper. When working with a 2-Calabi-Yau category, we manage to replace the technical "frieze-like" condition (see [26, def. 1.4]) for the maps α and β in the generalised Caldero-Chapoton map (Equation (3.23)), by our so-called Condition F (see Definition 3.2.1). This condition significantly simplifies the frieze-like condition and demonstrates the roles of α and β. We will see that α plays the role of a "generalised index", whilst β provides a correction term to α being exponential over a distinguished triangle.

We use this to establish a multiplication formula for the modified Caldero-Chapoton map ρ (see Theorem 3.5.3), allowing its computation in practice. In [26], the computation of ρ is not addressed. However, our multiplication formula does address the computation, and does so in a simpler manner than merely applying the definition. In particular, the formula allows us to compute values of ρ without calculating Euler characteristics of submodule Grassmannians which are otherwise part of the definition of ρ.

3.1.2 The Original Caldero-Chapoton Map

The original Caldero-Chapoton map was first introduced by Caldero and Chapoton in [10], and is widely acknowledged to make firm the idea that cluster categories are a categorification of so-called cluster algebras introduced by Fomin and Zelevinsky in [17]. We begin this section by giving a extremely brief idea behind the theory of cluster algebras. A much more comprehensive introduction can be found in [33].

Cluster Algebras Cluster algebras were first introduced around the year 2000. They are commutative algebras whose generators and relations are constructed recursively. Since their introduction, they have had a significant impact in many different areas of mathematics, including Poisson geometry, discrete dynamical systems, commutative and noncommutative algebraic geometry and in the representation theory of quivers and finite dimensional algebras. The following nice description of cluster algebras is from [33, sec. 2.1]:

A cluster algebra is a commutative $\mathbb{Q}$-algebra endowed with a set of distinguished generators (the cluster variables) grouped into overlapping subsets (the clusters) of constant cardinality (the rank) which are constructed recursively via mutation from an initial cluster. The set of cluster variables can be finite of infinite.

Let $\bar{Q}$ be a quiver and consider the field $\mathbb{Q}(x_1, \ldots, x_n)$. To the quiver $\bar{Q}$ there is a notion of quiver mutation (at a chosen vertex). A description of this can be found in [33, sec. 3.1]. Now, a seed is a pair (Q, u), where Q is a quiver and u is a sequence $u_1, \ldots, u_n$ of elements of $\mathbb{Q}(x_1, \ldots, x_n)$ which form a transcendence basis of $\mathbb{Q}(x_1, \ldots, x_n)$ over $\mathbb{Q}$. Again, for a vertex k of Q, there is notion of seed mutation. This produces a new seed (Q', u'), where Q' is obtained from the quiver mutation mentioned before, and u' is obtained by replacing u_k with a new element u'_k, which is defined by a so-called "exchange relation", see [33, sec. 3.2].

Now, for a fixed quiver Q, the initial seed of Q is $(Q, \{x_1, \dots, x_n\})$. A cluster associated to Q is a sequence u which appears in a seed (Q'', u) that can be obtained from the inital seed through a series of mutations. The elements of these clusters are called cluster variables. The cluster algebra $A(Q)$ is the $\mathbb{Q}$-subalgebra of $\mathbb{Q}(x_1, \dots, x_n)$ generated by the cluster variables.

This brings us back to our introduction of the original Caldero-Chapoton map. Let Q be a finite quiver with no loops or 2-cycles. Then, denote by $\mathsf{C}(Q)$ the associated cluster category and by $A(Q)$ the associated cluster algebra. Let $T \in \operatorname{obj} \mathsf{C}(Q)$ be a cluster tilting object. The original Caldero-Chapoton map γ_T, which depends on T, is a map,

$$\gamma_T : \operatorname{obj} \mathsf{C}(Q) \to A(Q),$$

which carries the salient property that it sends so-called "reachable" indecomposable objects in $\mathsf{C}(Q)$ to cluster variables in $A(Q)$.

Note that from now, when there is no risk of confusion, we will drop T from the subscript of γ_T and denote the Caldero-Chapoton map by γ. We now describe some necessary constructions in order to define γ. We will make the definition of γ on a general 2-Calabi-Yau category with a cluster tilting object T, and the definition that we give, along with the necessary constructions, can all be found in [30, sec. 1].

The Category $\mathsf{Mod}\,\mathsf{T}$

Let C be a $\mathbb{C}$-linear, Hom-finite, Krull-Schmidt, 2-Calabi-Yau, triangulated category with suspension functor Σ. Note that the Serre functor for C is $S = \Sigma^2$. Recall our discussion on cluster tilting objects from Remark 2.5.20, with the definition given in Definition 2.5.19. Let $T \in \operatorname{obj} \mathsf{C}$ be a cluster tilting object in the sense of Definiton 2.5.19, and denote by $\mathsf{T} = \operatorname{add} T$ the corresponding cluster tilting subcategory of C.

We denote by $\mathsf{Mod}\,\mathsf{T}$ the abelian category whose objects are $\mathbb{C}$-linear contravariant functors $\mathsf{T} \to \mathsf{Vect}\,\mathbb{C}$, where $\mathsf{Vect}\,\mathbb{C}$ denotes the category of vector spaces over $\mathbb{C}$. The morphisms in $\mathsf{Mod}\,\mathsf{T}$ are natural transformations between the contravariant functors. $\mathsf{Mod}\,\mathsf{T}$ is an abelian category in the following sense. Let $F, G, H \in \operatorname{obj} \mathsf{Mod}\,\mathsf{T}$. Then, the sequence

$$F \to G \to H$$

in $\mathsf{Mod}\,\mathsf{T}$ is an exact seqence if and only if the following sequence is exact for each $t \in \operatorname{obj} \mathsf{T}$:

$$F(t) \to G(t) \to H(t).$$

See [3, sec. 2].

We then define the functor

$$\begin{aligned} \overline{G} : \mathsf{C} &\to \mathsf{Mod}\,\mathsf{T} \\ c &\mapsto \mathrm{Hom}_{\mathsf{C}}(-, \Sigma c)|_{\mathsf{T}}. \end{aligned} \tag{3.4}$$

K-theory

For each object $t \in \mathrm{obj}\,\mathsf{T}$ there is a projective object

$$\overline{P}_t = \mathrm{Hom}_{\mathsf{C}}(-, t)|_{\mathsf{T}},$$

in $\mathsf{Mod}\,\mathsf{T}$. Denote by $\mathrm{rad}\,\overline{P}_t$ the radical of $\overline{P}_t$ and then define the top of $\overline{P}_t$ by

$$\mathrm{top}\,\overline{P}_t := \overline{P}_t / \mathrm{rad}\,\overline{P}_t.$$

By [4, prop. 2.3(b)], the simple objects in $\mathsf{Mod}\,\mathsf{T}$ are then given by

$$\overline{S}_t = \mathrm{top}\,\overline{P}_t, \tag{3.5}$$

for t indecomposable in T. Now, denote by $\mathsf{fl}\,\mathsf{T} \subseteq \mathsf{Mod}\,\mathsf{T}$ the full subcategory of $\mathsf{Mod}\,\mathsf{T}$ consisting of all finite length objects. An object $F \in \mathrm{obj}\,\mathsf{Mod}\,\mathsf{T}$ has finite length if it has a composition series of finite length. Note also that $\mathsf{fl}\,\mathsf{T}$ is itself an abelian category.

The Grothendieck group of the abelian category $\mathrm{fl}\,\mathsf{T}$ is denoted $\mathrm{K}_0(\mathsf{fl}\,\mathsf{T})$ and is the free abelian group with $[\overline{S}_t]$, for $t \in \mathrm{indec}\,\mathsf{T}$, as generators. Here, we use $[F]$ to denote the K_0-class of an object $F \in \mathrm{obj}\,\mathsf{fl}\,\mathsf{T}$. Since each object $F \in \mathrm{obj}\,\mathsf{fl}\,\mathsf{T}$ has finite length, the K_0-class $[F]$ is given by the finite sum

$$[F] = \Sigma[\overline{S}_i],$$

where $\overline{S}_i$ are the simple quotients from a composition series of F.

We define the split Grothendieck group, denoted by $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, of the cluster tilting subcategory T to be the free abelian group on the generators $[t]$ for $t \in \mathrm{indec}\,\mathsf{T}$. Here, we again use the notation $[a]$ to denote the $\mathrm{K}_0^{\mathrm{split}}$-class of an object $a \in \mathrm{obj}\,\mathsf{T}$. We should note that for two objects $a, b \in \mathrm{obj}\,\mathsf{T}$, we have

$$[a \oplus b] = [a] + [b].$$

Indices and Coindices

For an object $c \in \mathrm{obj}\,\mathsf{C}$, we may define the index and coindex of c with respect to the cluster tilting subcategory T. These notions were first introduced by Palu in [40].

Definition 3.1.1. Let $c \in \operatorname{obj} \mathsf{C}$. Then, by [34], there exists a triangle

$$t_1 \to t_0 \to c,$$

in C with $t_1, t_0 \in \mathsf{T}$. The index of c with respect to T is the following well-defined element of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$:

$$\operatorname{ind}_{\mathsf{T}}(c) = [t_0] - [t_1].$$

Definition 3.1.2. Let $c \in \operatorname{obj} \mathsf{C}$. Then, there exists a triangle

$$c \to \Sigma^2 t^0 \to \Sigma^2 t^1,$$

in C with $t^0, t^1 \in \mathsf{T}$. The coindex of c with respect to T is the following well-defined element of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$:

$$\operatorname{coind}_{\mathsf{T}}(c) = [t^0] - [t^1].$$

Remark 3.1.3. A verification that the index and coindex are well-defined elements of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$ can be found in [40, lem. 2.1].

In the cluster category of Dynkin type A_n, there is a combinatorial formula to compute the index of an object $c \in \operatorname{obj} \mathsf{C}(A_n)$. This formula is due to Jørgensen and Yakimov in [31], and we will introduce it later in this chapter when computing some examples in $\mathsf{C}(A_n)$.

Grassmannians

The Caldero-Chapoton map involves computing the Euler characteristic of a Grassmannian, and so we introduce some key notation and properties here.

For $F \in \operatorname{\mathsf{Mod}} \mathsf{T}$, we denote by $\operatorname{Gr}(F)$ the algebraic variety of submodules of F with finite length in $\operatorname{\mathsf{Mod}} \mathsf{T}$. Fix $e \in \mathrm{K}_0(\operatorname{\mathsf{fl}} \mathsf{T})$, then we denote by $\operatorname{Gr}_e(F)$ the Grassmannian of submodules consisting of those points corresponding to submodules $F' \subseteq F$ with finite length that satisfy $[F'] = e$. The nature of all the Grassmannians as complex algebraic varieties means that Euler characteristic makes sense. If, as in our Dynkin type A_n examples, $\operatorname{Gr}_e(F)$ has only finitely many points, then its Euler characteristic is simply the number of those points.

Let $\eta : F \to G$ be a morphism in $\operatorname{\mathsf{fl}} \mathsf{T}$. Then, there are constructible maps of Grassmannians,

$$\begin{aligned} \operatorname{Gr}(F) &\to \operatorname{Gr}(G) \\ F' &\mapsto \eta F' \end{aligned}$$

and

$$\begin{aligned} \mathrm{Gr}(G) &\to \mathrm{Gr}(F) \\ G' &\mapsto \eta^{-1}G'. \end{aligned}$$

Note also that the image and inverse image of a constructible set under a constructible map are also constructible sets. The definitions of constructible sets and constructible maps can be found in [41, sec. 2.1].

Now, let $x, z \in \operatorname{obj} \mathsf{C}$ be such that $\overline{G}x, \overline{G}z \in \mathsf{fl}\,\mathsf{T}$. Assume also that there are the following triangles in C,

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma x \quad \text{and} \quad z \xrightarrow{f'} y' \xrightarrow{g'} x \xrightarrow{h'} \Sigma z. \tag{3.6}$$

Since $\overline{G}$ is a homological functor, there are exact sequences

$$\overline{G}x \xrightarrow{\overline{G}f} \overline{G}y \xrightarrow{\overline{G}g} \overline{G}z \quad \text{and} \quad \overline{G}z \xrightarrow{\overline{G}f'} \overline{G}y' \xrightarrow{\overline{G}g'} \overline{G}x.$$

Note that since $\overline{G}x$ and $\overline{G}z$ have finite length in $\mathsf{Mod}\,\mathsf{T}$, then so do $\overline{G}y$ and $\overline{G}y'$.

For $e_1, e_2 \in \mathrm{K}_0(\mathsf{fl}\,\mathsf{T})$, define the constructible subset $Y_{e_1,e_2} \subseteq \mathrm{Gr}(\overline{G}y)$ by

$$Y_{e_1,e_2} = \{Y \subseteq \overline{G}y \mid [(\overline{G}f)^{-1}Y] = e_1, [(\overline{G}g)Y] = e_2\}.$$

There is then a morphism π_{e_1,e_2} of algebraic varieties,

$$\begin{aligned} Y_{e_1,e_2} &\xrightarrow{\pi_{e_1,e_2}} \mathrm{Gr}_{e_1}(\overline{G}x) \times \mathrm{Gr}_{e_2}(\overline{G}z) \\ Y &\longmapsto ((\overline{G}f)^{-1}Y, (\overline{G}g)Y). \end{aligned}$$

Again, for $e_1, e_2 \in \mathrm{K}_0(\mathsf{fl}\,\mathsf{T})$, define a constructible subset $Y'_{e_1,e_2} \subseteq \mathrm{Gr}(\overline{G}y')$ by

$$Y'_{e_1,e_2} = \{Y' \subseteq \overline{G}y' \mid [(\overline{G}f')^{-1}Y'] = e_2, [(\overline{G}g')Y'] = e_1\}.$$

There is then another morphism π'_{e_1,e_2} of algebraic varieties,

$$\begin{aligned} Y'_{e_1,e_2} &\xrightarrow{\pi'_{e_1,e_2}} \mathrm{Gr}_{e_1}(\overline{G}x) \times \mathrm{Gr}_{e_2}(\overline{G}z) \\ Y' &\longmapsto ((\overline{G}g')Y', (\overline{G}f')^{-1}Y'). \end{aligned}$$

We then have the following results, where more detailed arguments can be found in [30, sec. 1.6].

Assume that the triangles in (3.6) are split, i.e. $h, h' = 0$. Then, from the first triangle, we know that

$$\Sigma^{-1}z \xrightarrow{-\Sigma^{-1}h} x \xrightarrow{f} y \xrightarrow{g} z$$

is also a triangle, and since $h = 0$, it follows that $-\Sigma^{-1}h = 0$ also. Applying $\overline{G}$ gives the following exact sequence,

$$\overline{G}\Sigma^{-1}z \xrightarrow{0} \overline{G}x \xrightarrow{\overline{G}f} \overline{G}y \xrightarrow{\overline{G}g} \overline{G}z \xrightarrow{0} \overline{G}\Sigma x.$$

By the exactness of the sequence, we know $\overline{G}f$ must be injective and $\overline{G}g$ must be surjective. Thus, there is a short exact sequence,

$$0 \longrightarrow \overline{G}x \xrightarrow{\overline{G}f} \overline{G}y \xrightarrow{\overline{G}g} \overline{G}z \longrightarrow 0.$$

We note that this short exact sequence is also split. It follows that for $e \in \mathrm{K}_0(\mathsf{fl}\,\mathsf{T})$, we have

$$\mathrm{Gr}_e(\overline{G}y) = \bigsqcup_{e_1+e_2=e} Y_{e_1,e_2}. \tag{3.7}$$

Here $\bigsqcup$ denotes the disjoint union.

Now, denoting by χ the usual Euler characteristic, we also have that π_{e_1,e_2} is a surjection and has fibres equal to affine spaces, see [18, p.93 ex.], so we have

$$\chi(\mathrm{Gr}_{e_1}(\overline{G}x) \times \mathrm{Gr}_{e_2}(\overline{G}z)) = \chi(Y_{e_1,e_2}). \tag{3.8}$$

An easy calculation using (3.7) and (3.8) then gives

$$\sum_{e_1+e_2=e} \chi\big(\mathrm{Gr}_{e_1}(\overline{G}x) \times \mathrm{Gr}_{e_2}(\overline{G}z)\big) = \chi(\mathrm{Gr}_e(\overline{G}y)).$$

Assume now that $\dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathsf{C}}(x,z) = \dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathsf{C}}(z,x) = 1$ and that in the triangles (3.6) we have $h' \neq 0$. The $h \neq 0$ and morphisms π_{e_1,e_2} and π'_{e_1,e_2} then induce a surjection

$$\left(\pi_{e_1,e_2} \ \pi'_{e_1,e_2} \right) : Y_{e_1,e_2} \bigsqcup Y'_{e_1,e_2} \twoheadrightarrow \mathrm{Gr}_{e_1}(\overline{G}x) \times \mathrm{Gr}_{e_2}(\overline{G}z),$$

with the property that for each $M \in \mathrm{Gr}_{e_1}(\overline{G}x) \times \mathrm{Gr}_{e_2}(\overline{G}z)$ we have either $M \in \mathrm{Im}\,\pi_{e_1,e_2}$ or $M \in \mathrm{Im}\,\pi'_{e_1,e_2}$, but not both, see [40, prop. 4.3]. Again, since the fibres are equal to affine spaces, we have

$$\chi(\mathrm{Gr}_{e_1}(\overline{G}x) \times \mathrm{Gr}_{e_2}(\overline{G}z)) = \chi(Y_{e_1,e_2}) + \chi(Y'_{e_1,e_2}).$$

Defining the Caldero-Chapoton Map

We now approach the definition of the Caldero-Chapoton map γ. We first introduce the notion of mutation in T using the following remark.

Remark 3.1.4. For each $t \in \operatorname{indec} \mathsf{T}$, there is a unique $t^* \in \operatorname{indec} \mathsf{C}$ such that the subcategory $\mathsf{T}^* \subseteq \mathsf{C}$, defined by

$$\operatorname{indec} \mathsf{T}^* = \{\operatorname{indec} \mathsf{T} \setminus \{t\}\} \cup \{t^*\},$$

is cluster tilting, see [28, thm. 5.3]. The indecomposable t^* is called the mutation of t. There are then exchange triangles

$$t \to A \to t^* \quad \text{and} \quad t^* \to A' \to t, \tag{3.9}$$

where $A, A' \in \operatorname{add}(\operatorname{indec} \mathsf{T} \setminus \{t\})$.

There is a homomorphism $\bar{\theta} : \mathrm{K}_0(\mathsf{fl}\,\mathsf{T}) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, defined by

$$\bar{\theta}([\overline{S}_t]) = [A'] - [A]. \tag{3.10}$$

Here, A and A' are the middle terms of the exchange triangles for t, as in (3.9).

The Caldero-Chapoton map takes values inside the Laurent polynomial ring $\mathbb{Z}[x_t^{\pm 1}]_{t \in \operatorname{indec} \mathsf{T}}$, where for each $t \in \operatorname{indec} \mathsf{T}$, we create a variable x_t. Now, for $\sigma = \sum_t \sigma_t [t] \in \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, we define the element $x^\sigma \in \mathbb{Z}[x_t^{\pm 1}]_{t \in \mathsf{T}}$ by

$$x^\sigma = \prod_t x_t^{\sigma_t}.$$

This allows us to define the Caldero-Chapoton map. Indeed, $\gamma_T : \operatorname{obj} \mathsf{C} \to \mathbb{Z}[x_t^{\pm 1}]_{t \in \mathsf{T}}$ is defined as

$$\gamma_T(c) = x^{-\operatorname{coind}_{\mathsf{T}}(\Sigma c)} \sum_{e \in \mathrm{K}_0(\mathsf{fl}\,\mathsf{T})} \chi(\mathrm{Gr}_e(\overline{G}c)) x^{\bar{\theta}(e)}. \tag{3.11}$$

We note that this formula only makes sense for $c \in \operatorname{obj} \mathsf{C}$ such that $\overline{G}c$ has finite length.

Remark 3.1.5. It is a well known property of γ that it is a frieze, that is, it satisfies the property in (3.2), as well as the exponential conditions in (3.1). See [1, def. 1.1], [10, prop. 3.10] and [16, thm.].

3.1.3 Frieze Patterns

Frieze patterns were first introduced by Conway and Coxeter in [13] and [14]. An example of a frieze pattern, known as a Conway-Coxeter frieze, is given in Figure 3.1. This frieze pattern is obtained from the original Caldero-Chapoton map γ, defined in (3.11), by omitting the arrows from the Auslander-Reiten quiver of $\mathsf{C}(A_7)$, setting each x_t equal to 1 and replacing each vertex by the value of γ applied to that indecomposable.

$$
\begin{array}{ccccccccccc}
4 & & 4 & & 1 & & 2 & & 2 & & 4 \\
& 15 & & 3 & & 1 & & 3 & & 7 & \\
11 & & 11 & & 2 & & 1 & & 10 & & 5 \\
& 8 & & 7 & & 1 & & 3 & & 7 & \\
5 & & 5 & & 3 & & 2 & & 2 & & 11 \\
& 3 & & 2 & & 5 & & 1 & & 3 & \\
4 & & 1 & & 3 & & 2 & & 1 & & 4
\end{array}
$$

Figure 3.1: A frieze on the cluster category of Dynkin type A_7. The dotted lines are identified with opposite orientations.

In formal terms, for some positive integer n, a frieze pattern is an array of n offset rows of positive integers. Each diamond

$$
\begin{array}{ccc}
& \alpha & \\
\beta & & \eta \\
& \delta &
\end{array}
$$

in a frieze pattern satisfies a so-called determinant property in that

$$\beta\eta - \alpha\delta = 1. \tag{3.12}$$

If β and η are on the top row of the frieze, then the determinant property becomes

$$\beta\eta - \delta = 1,$$

and if β and η are on the bottom row, the determinant property becomes

$$\beta\eta - \alpha = 1.$$

Remark 3.1.6. Recall that if

$$
\begin{tikzcd}
& b \arrow[dr] & \\
a \arrow[ur] \arrow[dr] & & d \\
& c \arrow[ur] &
\end{tikzcd}
$$

is a diamond in the Auslander-Reiten quiver of $\mathsf{C}(A_n)$, then

$$a \to b \oplus c \to d$$

is an Auslander-Reiten triangle. If a and d sit on the upper boundary of the Auslander-

Reiten quiver, then b should be taken as zero, whereas if a and d sit on the lower boundary, then c is taken to be zero. Note then that a frieze X on $\mathsf{C}(A_n)$, as defined in (3.2), satisfies

$$X(a)X(d) - X(b)X(c) = 1. \tag{3.13}$$

Recall from Remark 3.1.5 that the original Caldero-Chapoton map γ is a frieze. By virtue of Equation (3.13), the map γ satisfies the determinant property in (3.12).

Frieze patterns are known to be invariant under a glide reflection. A region of the frieze pattern, known as a fundamental domain, is enough to produce the whole frieze pattern by repeatedly performing a glide reflection.

Long before the introduction of cluster categories, Broline, Crowe and Isaacs in [6] gave a purely combinatorial algorithm for determining the n rows of a frieze pattern, given a triangulation of a regular $(n+3)$-gon. We will describe this algorithm here.

Choose a triangulation of a regular $(n+3)$-gon P. We define nonnegative integers $f_{i,j}$, for vertices i and j, using the following inductive procedure. Define first that $f_{i,i} = 0$. Also note that it was proved in [6] that $f_{i,j} = f_{j,i}$.

Fix a vertex i of P. Firstly, notice that i is a vertex of at least one triangle in the triangulation of P. Choose one such triangle τ. Then, if j is another vertex of τ, we set $f_{i,j} = 1$. Therefore, for every triangle τ that contains i as a vertex, we have a method to compute $f_{i,j}$ whenever j is another vertex of τ. Now, if i is not a vertex of every triangle in the triangulation, then there must be a triangle τ' sharing an edge (k,l) with a triangle τ, for which τ is a triangle with i as a vertex. Notice that we know the values $f_{i,k}$ and $f_{i,l}$ since k and l are vertices of τ. Assume now that j is the third vertex of τ'. We have the following formula to compute $f_{i,j}$:

$$f_{i,j} = f_{i,k} + f_{i,l}. \tag{3.14}$$

We may then assume that any other triangle τ'' in the triangulation shares an edge (k,l) with a triangle τ''' for which $f_{i,k}$ and $f_{i,l}$ are known. The formula in (3.14) can then be used to compute $f_{i,j}$.

3.1.4 A Modified Caldero-Chapoton Map

Setup. We assume in the rest of this chapter that C is an essentially small, $\mathbb{C}$-linear, Hom-finite, 2-Calabi-Yau, triangulated category, which is Krull-Schmidt and has Auslander-Reiten triangles.

In [25], Holm and Jørgensen introduce a modified version of the Caldero-Chapoton map, which we denote by π, that relies on a rigid object $R \in \operatorname{obj}\mathsf{C}$, a much weaker condition than that of being a cluster tilting object. Recall that an object R is rigid if

$$\mathrm{Hom}_{\mathsf{C}}(R, \Sigma R) = 0.$$

We also note that the definition of π in [25] does not require that the category is 2-Calabi-Yau, allowing a category C that is more general than a cluster category. However, the results that we provide in this chapter do require that C is 2-Calabi-Yau, and so we will make the relevant definitions with this added assumption.

Let $R \in \operatorname{obj} \mathsf{C}$ be a rigid object, and consider the functorially finite subcategory $\mathsf{R} \subseteq \mathsf{C}$, given by $\mathsf{R} = \operatorname{add}(R)$, which is closed under direct sums and summands, and is rigid in the sense that $\mathrm{Hom}_{\mathsf{C}}(\mathsf{R}, \Sigma \mathsf{R}) = 0$. We assume also that C has a cluster tilting subcategory T. We additionally require that $\mathsf{R} \subseteq \mathsf{T}$. Note finally that the Auslander-Reiten translation for C is

$$\tau = \Sigma,$$

and its Serre functor is $S = \Sigma^2$.

Denote by $\mathsf{Mod}\,\mathsf{R}$ the abelian category whose objects are contravariant functors $\mathsf{R} \to \mathsf{Vect}\,\mathbb{C}$. The morphisms in $\mathsf{Mod}\,\mathsf{R}$ are natural transformations between contravariant functors $\mathsf{R} \to \mathsf{Vect}\,\mathbb{C}$. $\mathsf{Mod}\,\mathsf{R}$ is then abelian in the following sense. Let $F, G, H \in \operatorname{obj} \mathsf{Mod}\,\mathsf{R}$. Then, the sequence

$$F \to G \to H,$$

in $\mathsf{Mod}\,\mathsf{R}$ is an exact sequence if and only if the following sequence is exact for each $r \in \operatorname{obj} \mathsf{R}$:

$$F(r) \to G(r) \to H(r).$$

See [3, sec. 2].

As with $\mathsf{Mod}\,\mathsf{T}$ from earlier, for each $r \in \operatorname{obj} \mathsf{R}$ there is a projective object

$$P_r = \mathrm{Hom}_{\mathsf{C}}(-, r)|_{\mathsf{R}} \tag{3.15}$$

in $\mathsf{Mod}\,\mathsf{R}$. By [4, prop. 2.3(b)], the simple objects in $\mathsf{Mod}\,\mathsf{R}$ are given by

$$S_r = \operatorname{top} P_r, \tag{3.16}$$

where $\operatorname{top} S_r$ is defined by

$$\operatorname{top} S_r := P_r / \operatorname{rad} P_r.$$

There is a functor

$$\begin{aligned} G : \mathsf{C} &\to \mathsf{Mod}\,\mathsf{R} \\ c &\mapsto \mathrm{Hom}_{\mathsf{C}}(-, \Sigma c)|_{\mathsf{R}}. \end{aligned} \tag{3.17}$$

For some object $c \in \operatorname{obj} \mathsf{C}$, the modified Caldero-Chapoton map π from [25] is then defined by the formula:

$$\pi(c) = \chi(\mathrm{Gr}(Gc)), \tag{3.18}$$

where Gr denotes the Grassmannian of submodules and χ is the Euler characteristic defined by cohomology with compact support (see [18, p. 93]).

It is proved in [25] that π is a generalised frieze; that is, as well as the exponential properties given in Equation (3.1) it satisfies the property given in Equation (3.3).

A multiplication formula for computing π is also proved in [25]. Let $m \in \operatorname{indec} \mathsf{C}$ and $r \in \operatorname{indec} \mathsf{R}$ satisfy that $\mathrm{Ext}^1_{\mathsf{C}}(m, r)$ and $\mathrm{Ext}^1_{\mathsf{C}}(r, m)$ both have dimension one over $\mathbb{C}$. Then, there are nonsplit triangles

$$m \xrightarrow{\mu} a \xrightarrow{\gamma} r \xrightarrow{\delta} \Sigma m \ , \ r \xrightarrow{\sigma} b \xrightarrow{\eta} m \xrightarrow{\zeta} \Sigma r, \tag{3.19}$$

that are unique up to isomorphism. It is proved in [25] that

$$\pi(m) = \pi(a) + \pi(b). \tag{3.20}$$

This formula can be applied iteratively to compute values of π.

In [26] Holm and Jørgensen redefine the modified Caldero-Chapoton map in a more general manner (the work in [25] is a special case of that in [26]). This is the map that we work with throughout this chaper. Let A be some commutative ring, and let the maps

$$\alpha : \operatorname{obj} \mathsf{C} \to A \quad \text{and} \quad \beta : \mathrm{K}_0(\mathsf{fl}\, \mathsf{R}) \to A \tag{3.21}$$

be exponential maps in the sense that

$$\begin{aligned} \alpha(0) = 1 \ , \ \alpha(x \oplus y) = \alpha(x)\alpha(y), \\ \beta(0) = 1 \ , \ \beta(e + f) = \beta(e)\beta(f). \end{aligned} \tag{3.22}$$

Then, they define $\rho : \operatorname{obj} \mathsf{C} \to A$ by

$$\rho(c) = \alpha(c) \sum_{e \in \mathrm{K}_0(\mathsf{fl}\, \mathsf{R})} \chi(\mathrm{Gr}_e(Gc))\beta(e). \tag{3.23}$$

Here, the sum is taken over $e \in \mathrm{K}_0(\mathsf{fl}\, \mathsf{R})$, where $\mathsf{fl}\, \mathsf{R}$ denotes the full subcategory of $\mathsf{Mod}\, \mathsf{R}$, consisting of the finite length objects, and $\mathrm{K}_0(\mathsf{fl}\, \mathsf{R})$ denotes the Grothendieck group of this abelian category. As with the original Caldero-Chapton map, $\mathrm{Gr}_e(Gc)$ denotes the Grassmannian of subobjects $M \subseteq Gc$ with K_0-class satisfying $[M] = e$.

For two objects $a, b \in \mathsf{C}$ such that Ga and Gb have finite length, it is known by [26, prop. 1.3] that ρ is also exponential; that is,

$$\rho(0) = 1 \ , \ \rho(a \oplus b) = \rho(a)\rho(b).$$

It therefore suffices to calculate ρ for each indecomposable object in C. Note that the formula for ρ only makes sense when Gc has finite length in $\mathsf{Mod}\,\mathsf{R}$. That is, we require that $Gc \in \mathsf{fl}\,\mathsf{R}$.

It is proved in [26, thm. 1.6] that when the maps α and β satisfy a technical "frieze-like" condition, given in [26, def. 1.4], the map ρ becomes a generalised frieze, as in Equation (3.3).

3.2 Condition F on the maps α and β

We continue under the setup of Section 3.1.4. Consider the exponential maps α and β introduced earlier in Equation (3.21). The following definition describes a condition on α and β, which will later be used in Section 3.5 to prove a multiplication formula for ρ.

Definition 3.2.1. (Condition F). We say that the maps α and β satisfy Condition F if, for each triangle

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma x \tag{3.24}$$

in C, such that Gx, Gy and Gz have finite length in $\mathsf{Mod}\,\mathsf{R}$, the following property holds:

$$\alpha(y) = \alpha(x \oplus z)\beta([\mathrm{Ker}\, Gf]).$$

We now prove a theorem showing that in the case when C is 2-Calabi-Yau, Condition F can replace the frieze-like condition defined in [26, def. 1.4].

Theorem 3.2.2. *Assume that the exponential maps $\alpha : \mathrm{obj}\,\mathsf{C} \to A$ and $\beta : K_0(\mathsf{fl}\,\mathsf{R}) \to A$ from Equation (3.21) satisfy Condition F from Definition 3.2.1. Then, the modified Caldero-Chapoton map ρ, defined in Equation (3.23), is a generalised frieze in the sense that, where defined, it satisfies Equation (3.3), as well as the exponential conditions in (3.1).*

Proof. Note that by [26, thm. 1.6], if

$$\Delta = \big(\Sigma c \xrightarrow{\xi} b \longrightarrow c\big)$$

is an Auslander-Reiten triangle in C such that Gc and $G\Sigma c$ have finite length in $\mathsf{Mod}\,\mathsf{R}$, and α and β are frieze-like maps for Δ in the sense of [26, def. 1.4], then $\rho(\Sigma c)\rho(c) - \rho(b) \in \{0, 1\}$. Hence, it suffices to show that α and β are frieze-like for each Δ. We first note that if $c \notin \mathsf{R} \cup \Sigma^{-1}\mathsf{R}$, then $G(\Delta)$ is a short exact sequence, see [25, lem. 1.12(iii)]. We now check the three cases of [26, def. 1.4].

Case (i): Assume $c \notin \mathsf{R} \cup \Sigma^{-1}\mathsf{R}$ and $G(\Delta)$ is a split short exact sequence. That is,

$$0 \longrightarrow G(\Sigma c) \xrightarrow{G\xi} Gb \longrightarrow Gc \longrightarrow 0$$

is split short exact. It follows immediately that $G\xi$ has trivial kernel. Applying Condition F to Δ, we obtain

$$\begin{aligned}\alpha(b) &= \alpha(c \oplus \Sigma c)\beta([\operatorname{Ker} G\xi]) \\ &= \alpha(c \oplus \Sigma c)\beta(0) \\ &= \alpha(c \oplus \Sigma c),\end{aligned}$$

where the final $=$ is due to β being exponential.

Case (ii), first part: Assume $c \notin \mathsf{R} \cup \Sigma^{-1}\mathsf{R}$ and $G(\Delta)$ is a nonsplit short exact sequence. Then, by the same working as in Case (i), we see that

$$\alpha(b) = \alpha(c \oplus \Sigma c).$$

Now, consider the following triangle in C

$$c \xrightarrow{\nu} 0 \longrightarrow \Sigma c \xrightarrow{1_{\Sigma c}} \Sigma c.$$

Applying Condition F and using the fact that α is exponential, we obtain

$$\begin{aligned}1 &= \alpha(0) \\ &= \alpha(c \oplus \Sigma c)\beta([\operatorname{Ker} G\nu]) \\ &= \alpha(c \oplus \Sigma c)\beta([Gc]).\end{aligned}$$

We note that this manipulation works for any $c \in \operatorname{obj} \mathsf{C}$.

Case (ii), second part: Let $c = \Sigma^{-1}r \in \Sigma^{-1}\mathsf{R}$. We showed in the first part of Case (ii) that

$$\alpha(c \oplus \Sigma c)\beta([Gc]) = 1.$$

Now, by [25, lem. 1.12(i)], $G(\Delta)$ becomes

$$G(\Delta) = 0 \to \operatorname{rad} P_r \to P_r.$$

Recall here from (3.16) that P_r is the indecomposable projective $\operatorname{Hom}_{\mathsf{R}}(-, r)$ in $\mathsf{Mod}\,\mathsf{R}$, described in [25, sec. 1.5]. Therefore, $G\xi$ has zero kernel, and applying Condition F shows

$$\begin{aligned}\alpha(b) &= \alpha(c \oplus \Sigma c)\beta([\operatorname{Ker} G\xi]) \\ &= \alpha(c \oplus \Sigma c)\beta(0) \\ &= \alpha(c \oplus \Sigma c).\end{aligned}$$

Case (iii): Let $c = r \in \mathsf{R}$ and again consider the triangle

$$c \xrightarrow{\nu} 0 \longrightarrow \Sigma c \xrightarrow{1_{\Sigma c}} \Sigma c,$$

in C. Applying Condition F, we see

$$\begin{aligned}1 &= \alpha(0) \\ &= \alpha(c \oplus \Sigma c)\beta([\operatorname{Ker} G\nu]) \\ &= \alpha(c \oplus \Sigma c)\beta(0) \\ &= \alpha(c \oplus \Sigma c),\end{aligned}$$

where the third $=$ is since $G(c) = G(r) = 0$ and hence $\operatorname{Ker} G\nu = 0$.
Now, by [25, lem. 1.12(ii)], applying G to Δ gives the exact sequence

$$I_r \xrightarrow{G\xi} \operatorname{corad} I_r \longrightarrow 0,$$

where I_r is the indecomposable injective $\operatorname{Hom}_{\mathsf{R}}(-, \Sigma^2 r)$ in $\mathsf{Mod}\,\mathsf{R}$, described in [25, sec. 1.10] and $\operatorname{corad} I_r$ is its coradical. Additionally, by [25, sec. 1.10], we know that there is the following short exact sequence:

$$0 \to S_r \to I_r \to \operatorname{corad} I_r \to 0.$$

Here, recall from (3.16) that S_r in $\mathsf{Mod}\,\mathsf{R}$ is the simple object supported at r. It follows that $\operatorname{corad} I_r \cong I_r/S_r$ and we see that $\operatorname{Ker} G\xi = S_r$. So, by applying Condition F once again, we see that

$$\begin{aligned}\alpha(b) &= \alpha(c \oplus \Sigma c)\beta([\operatorname{Ker} G\xi]) \\ &= \beta([\operatorname{Ker} G\xi]) \\ &= \beta([S_r]),\end{aligned}$$

where the second $=$ is since $\alpha(c \oplus \Sigma c) = 1$. □

3.3 Constructing Maps that Satisfy Condition F

We again continue under the setup of Section 3.1.4. We will show that there exist maps α and β satisfying Condition F, namely those given in [26, def. 2.8]. Let us first look at the necessary constructions behind the definitions of α and β in [26, def. 2.8].

Recall that T is some cluster tilting subcategory of C with $\mathsf{R} \subseteq \mathsf{T}$. Define S to be the full subcategory of C which is closed under direct sums and summands and has

$$\operatorname{indec} \mathsf{S} = \operatorname{indec} \mathsf{T} \setminus \operatorname{indec} \mathsf{R}. \tag{3.25}$$

Then, consider the subgroup

$$N = \left\langle [a] - [a'] \;\middle|\; \begin{array}{l} s^* \to a \to s \ , \ s \to a' \to s^* \text{ are exchange} \\ \text{triangles with } s \in \operatorname{indec} \mathsf{S} \end{array} \right\rangle \tag{3.26}$$

of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, defined in [26, def. 2.4]. Here, recall the exchange triangles from Remark 3.1.4, and the definition of the split Grothendieck group $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$ of the additive category T from Section 3.1.2. We then denote by Q the canonical surjection

$$Q \,:\, \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \twoheadrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \ , \ Q([t]) = [t] + N.$$

Before giving the definition of α and β from [26, def. 2.8], it remains to recall how to construct the homomorphism θ from [26, sec. 2.6]. We do this by following the constructions in [26, sec. 2.5]. Since $\mathsf{R} \subseteq \mathsf{T}$, the inclusion functor $i \,:\, \mathsf{R} \hookrightarrow \mathsf{T}$ induces the exact functor

$$i^* :\, \operatorname{Mod} \mathsf{T} \to \operatorname{Mod} \mathsf{R}, \ i^*(F) = F|_{\mathsf{R}}.$$

Recall from (3.5) that for each $t \in \operatorname{indec} \mathsf{T}$, there is a simple object $\overline{S}_t \in \operatorname{Mod} \mathsf{T}$ supported at t. One sees that

$$i^*\overline{S}_t = \begin{cases} S_t & \text{if } t \in \operatorname{indec} \mathsf{R}, \\ 0 & \text{if } t \in \operatorname{indec} \mathsf{S}, \end{cases}$$

where S_t denotes the simple object in $\operatorname{Mod} \mathsf{R}$ supported at t. Due to i^* being exact, we can restrict it to the subcategories $\operatorname{fl} \mathsf{T}$ and $\operatorname{fl} \mathsf{R}$, made up of the finite length objects in $\operatorname{Mod} \mathsf{T}$ and $\operatorname{Mod} \mathsf{R}$, respectively. Then, there is an induced (surjective) group homomorphism

$$\kappa :\, \mathrm{K}_0(\operatorname{fl} \mathsf{T}) \twoheadrightarrow \mathrm{K}_0(\operatorname{fl} \mathsf{R}),$$

with the obvious property that

$$\kappa([\overline{S}_t]) = \begin{cases} [S_t] & \text{if } t \in \operatorname{indec} \mathsf{R}, \\ 0 & \text{if } t \in \operatorname{indec} \mathsf{S}. \end{cases}$$

It is also not hard to see that for the functor $\overline{G} : \mathsf{C} \to \mathsf{Mod}\,\mathsf{T}$, defined in (3.4), we have the property that $i^*\overline{G} = G$.

We define θ to be the group homomorphism making the following diagram commute:

$$\begin{tikzcd} \mathrm{K}_0(\mathsf{fl}\,\mathsf{T}) \arrow[r, "\bar{\theta}"] \arrow[d, "\kappa"] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \arrow[d, "Q"] \\ \mathrm{K}_0(\mathsf{fl}\,\mathsf{R}) \arrow[r, "\theta"'] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N. \end{tikzcd} \tag{3.27}$$

Here, recall the definition of $\bar{\theta}$ from (3.10).

Now, from [26, def. 2.8] the maps $\alpha\colon \mathrm{obj}\,\mathsf{C} \to A$ and $\beta : \mathrm{K}_0(\mathsf{fl}\,\mathsf{R}) \to A$ to a suitable ring A can be defined by

$$\alpha(c) = \varepsilon Q(\mathrm{ind}_{\mathsf{T}}(c)) \quad \text{and} \quad \beta(e) = \varepsilon\theta(e), \tag{3.28}$$

where $\varepsilon : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to A$ is a suitably chosen exponential map, meaning that

$$\varepsilon(0) = 1\ ,\ \varepsilon(a + b) = \varepsilon(a)\varepsilon(b). \tag{3.29}$$

Here, a and b denote two elements of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$.

Lemma 3.3.1. *The maps α and β from (3.28) satisfy Condition F.*

Proof. Consider the following triangle

$$x \xrightarrow{\varphi} y \xrightarrow{\omega} z \xrightarrow{\psi} \Sigma x \tag{3.30}$$

from (3.24). Then, we have

$$\begin{aligned} \alpha(y) &= \varepsilon Q(\mathrm{ind}_{\mathsf{T}}(y)) \\ &= \varepsilon Q(\mathrm{ind}_{\mathsf{T}}(x) + \mathrm{ind}_{\mathsf{T}}(z) - \mathrm{ind}_{\mathsf{T}}(C) - \mathrm{ind}_{\mathsf{T}}(\Sigma^{-1}C)) \\ &= \varepsilon(\mathrm{ind}_{\mathsf{T}}(x) + \mathrm{ind}_{\mathsf{T}}(z) - \mathrm{ind}_{\mathsf{T}}(C) - \mathrm{ind}_{\mathsf{T}}(\Sigma^{-1}C) + N) \\ &= \varepsilon(\mathrm{ind}_{\mathsf{T}}(x) + N)\varepsilon(\mathrm{ind}_{\mathsf{T}}(z) + N)\varepsilon(-\mathrm{ind}_{\mathsf{T}}(C) - \mathrm{ind}_{\mathsf{T}}(\Sigma^{-1}C) + N) \\ &= (*), \end{aligned}$$

where C in C is some lifting of $\mathrm{Coker}\,\overline{G}(\Sigma^{-1}\omega)$ in the sense that $\overline{G}\Sigma^{-1}C = \mathrm{Coker}\,\overline{G}(\Sigma^{-1}\omega)$. In the above manipulation, the second $=$ is due to [41, prop. 2.2] and the penultimate $=$ occurs since ε is exponential, see Equation (3.29).

In addition,

$$\begin{aligned}\alpha(x \oplus z)\beta([\operatorname{Ker} G\varphi]) &= \alpha(x)\alpha(z)\beta([\operatorname{Ker} G\varphi]) \\ &= \varepsilon Q(\operatorname{ind}_{\mathsf{T}}(x))\varepsilon Q(\operatorname{ind}_{\mathsf{T}}(z))\beta([\operatorname{Ker} G\varphi]) \\ &= \varepsilon(\operatorname{ind}_{\mathsf{T}}(x)+N)\varepsilon(\operatorname{ind}_{\mathsf{T}}(z)+N)\varepsilon\theta([\operatorname{Ker} G\varphi]), \\ &= (**),\end{aligned}$$

where the first $=$ is due to α being exponential and the penultimate $=$ is just by the definition of β.

Now, using the property that $i^*\overline{G} = G$, it follows that

$$[\operatorname{Ker} G\varphi] = [\operatorname{Ker} i^*\overline{G}\varphi] \overset{(1)}{=} [i^*\operatorname{Ker} \overline{G}\varphi] \overset{(2)}{=} \kappa[\operatorname{Ker} \overline{G}\varphi],$$

where (1) follows from i^* being an exact functor, and (2) from the definition of κ. We can now manipulate the expression $(**)$ further:

$$\begin{aligned}(**) &= \varepsilon(\operatorname{ind}_{\mathsf{T}}(x)+N)\varepsilon(\operatorname{ind}_{\mathsf{T}}(z)+N)\varepsilon\theta\big(\kappa\big([\operatorname{Ker}\overline{G}\varphi]\big)\big) \\ &= \varepsilon(\operatorname{ind}_{\mathsf{T}}(x)+N)\varepsilon(\operatorname{ind}_{\mathsf{T}}(z)+N)\varepsilon Q\big(\bar{\theta}([\operatorname{Ker}\overline{G}\varphi])\big) \\ &= \varepsilon(\operatorname{ind}_{\mathsf{T}}(x)+N)\varepsilon(\operatorname{ind}_{\mathsf{T}}(z)+N)\varepsilon\big(\bar{\theta}([\operatorname{Ker}\overline{G}\varphi])+N\big) \\ &= (***),\end{aligned}$$

where the second equality is due to the commutativity of Diagram (3.27).

Comparing $(*)$ to $(***)$, we see that the required equality for Condition F is satisfied if

$$\bar{\theta}([\operatorname{Ker}\overline{G}\varphi]) = -(\operatorname{ind}_{\mathsf{T}}(C) + \operatorname{ind}_{\mathsf{T}}(\Sigma^{-1}C)). \tag{3.31}$$

Making use of the "rolling" property on our triangle in (3.30), we obtain the following sequence:

$$\Sigma^{-1}y \xrightarrow{-\Sigma^{-1}\omega} \Sigma^{-1}z \xrightarrow{-\Sigma^{-1}\psi} x \xrightarrow{\varphi} y \xrightarrow{\omega} z,$$

where any four consecutive terms form a triangle. Furthermore, since $\overline{G}$ is a homological functor, we may apply it to this sequence and produce the following long exact sequence in $\operatorname{fl}\mathsf{T}$:

$$\overline{G}\Sigma^{-1}y \xrightarrow{-\overline{G}\Sigma^{-1}\omega} \overline{G}\Sigma^{-1}z \xrightarrow{-\overline{G}\Sigma^{-1}\psi} \overline{G}x \xrightarrow{\overline{G}\varphi} \overline{G}y \xrightarrow{\overline{G}\omega} \overline{G}z. \tag{3.32}$$

This shows $\operatorname{Coker}\overline{G}\Sigma^{-1}\omega = \operatorname{Ker}\overline{G}\varphi$. Moreover, C is chosen such that $\overline{G}\Sigma^{-1}C = \operatorname{Coker}\overline{G}\Sigma^{-1}\omega$, and hence $\operatorname{Ker} \overline{G}\varphi = \overline{G}\Sigma^{-1}C$. We can hence compute as follows:

$$\begin{aligned}\bar{\theta}([\operatorname{Ker} \overline{G}\varphi]) &= \bar{\theta}([\overline{G}\Sigma^{-1}C]) \\ &= -(\operatorname{ind}_{\mathsf{T}}(\Sigma^{-1}C) + \operatorname{ind}_{\mathsf{T}}(\Sigma(\Sigma^{-1}C))) \\ &= -(\operatorname{ind}_{\mathsf{T}}(C) + \operatorname{ind}_{\mathsf{T}}(\Sigma^{-1}C)),\end{aligned}$$

where the second = is due to [26, lem. 2.10]. We can now see that Equation (3.31) holds, and hence the lemma is proved. □

Remark 3.3.2. Through Lemma 3.3.1 and Theorem 3.2.2, we have managed to recover [26, thm. 2.11]. Indeed, [26, thm. 2.11] states that when

$$\Delta = \Sigma c \to b \to c$$

is an Auslander-Reiten triangle in C such that $\overline{G}c$ and $\overline{G}(\Sigma c)$ have finite length in $\operatorname{\mathsf{Mod}} \mathsf{T}$, the maps α and β from Equation (3.28) satisfy the frieze-like condition given in [26, def. 1.4]. By Lemma 3.3.1, we know that α and β as defined in Equation (3.28) satisfy Condition F. Theorem 3.2.2 proves that any α and β satisfying Condition F also satisfy the frieze-like condition for Δ (recovering [26, thm. 2.11]). Hence by [26, thm. 1.6], these α and β turn ρ into a generalised frieze.

3.4 The Multiplication Formula from [25]

In this section we demonstrate some of the technicalities behind the proof of the multiplication formula for π from Equation (3.20), proved in [25, prop. 4.4]. This is done with a view of proving a similar formula for ρ in Section 3.5. Following the setup of [25, sec. 4], for this section we do not require a cluster tilting subcategory T, as the theory in [25] uses only the rigid subcategory R.

Let $m \in \operatorname{indec} \mathsf{C}$ and $r \in \operatorname{indec} \mathsf{R}$ be indecomposable objects such that $\operatorname{Ext}^1_{\mathsf{C}}(r, m)$ and $\operatorname{Ext}^1_{\mathsf{C}}(m, r)$ both have dimension one over $\mathbb{C}$. As in [25, rem. 4.2], this allows us to construct the following nonsplit triangles in C:

$$m \xrightarrow{\mu} a \xrightarrow{\gamma} r \xrightarrow{\delta} \Sigma m \tag{3.33}$$

and

$$r \xrightarrow{\sigma} b \xrightarrow{\eta} m \xrightarrow{\zeta} \Sigma r, \tag{3.34}$$

with δ and ζ nonzero. Note that "rolling" the first triangle gives

$$\Sigma^{-1} r \xrightarrow{-\Sigma^{-1}\delta} m \xrightarrow{\mu} a \xrightarrow{\gamma} r,$$

which is also a triangle in C. Applying the functor G to both the "rolled" triangle and the triangle in (3.34) gives the following exact sequences in $\mathsf{Mod}\,\mathsf{R}$, obtained in [25]:

$$G(\Sigma^{-1}r) \xrightarrow{-G(\Sigma^{-1}\delta)} Gm \xrightarrow{G\mu} Ga \longrightarrow 0 \tag{3.35}$$

and

$$0 \longrightarrow Gb \xrightarrow{G\eta} Gm \xrightarrow{G\zeta} G(\Sigma r). \tag{3.36}$$

Remark 3.4.1. 1. The zeros arise in each exact sequence due to $G(r) = \mathrm{Hom}(-, \Sigma r)|_{\mathsf{R}}$ being the zero functor. Indeed, since R is rigid, evaluating $G(r)$ at any x in R will make the corresponding Hom-space zero.

2. The exact sequences are in $\mathsf{Mod}\,\mathsf{R}$; that is, each term is a $\mathbb{C}$-linear contravariant functor $\mathsf{R} \to \mathsf{Vect}\,\mathbb{C}$.

Letting Gr denote the Grassmannian of submodules, we have injective morphisms of algebraic varieties,

$$\mathrm{Gr}(Ga) \overset{\xi}{\hookrightarrow} \mathrm{Gr}(Gm) \overset{\nu}{\hookleftarrow} \mathrm{Gr}(Gb),$$

$$P \longmapsto (G\mu)^{-1}(P),$$

$$(G\eta)(N) \longleftarrow N.$$

It was proved in [25, Lemma 4.3] that if M in $\mathsf{Mod}\,\mathsf{R}$ is some submodule of Gm, then either $M \subseteq \mathrm{Im}\, G\eta$ or $\mathrm{Ker}\, G\mu \subseteq M$, but not both. This means that for $M \subseteq Gm$ we can find either a submodule $N \subseteq Gb$ such that $(G\eta)(N) = M$ or we can find $P \subseteq Ga$ such that $(G\mu)^{-1}(P) = M$. Hence, the submodule M is either of the form $\nu(N) = (G\eta)(N)$ or $\xi(P) = (G\mu)^{-1}(P)$, but not both.

It is therefore clear that $\mathrm{Gr}(Gm)$ is equal to the disjoint union of the images of ξ and ν. That is,

$$\mathrm{Gr}(Gm) \cong \mathrm{Gr}(Gb) \bigsqcup \mathrm{Gr}(Ga).$$

Remark 3.4.2. 1. We should note that for M in $\mathsf{Mod}\,\mathsf{R}$, the Grassmannian $\mathrm{Gr}(M)$ is an algebraic variety. Therefore, it makes sense to calculate the Euler characteristic of $\mathrm{Gr}(Gm)$, $\mathrm{Gr}(Ga)$ and $\mathrm{Gr}(Gb)$.

2. In addition, we note that ξ and ν are both constructible maps, hence the images of ξ and ν form constructible subsets in $\mathrm{Gr}(Gm)$. See [41, Section 2.1] for the definitions of a constructible map and a constructible set.

The following statement then follows in [25]:

$$\chi(\mathrm{Gr}(Gm)) = \chi(\mathrm{Gr}(Gb)) + \chi(\mathrm{Gr}(Ga)), \tag{3.37}$$

where χ again denotes the Euler characteristic defined by cohomology with compact support (see [18, p. 93]). Using Remark 3.4.2, since the images of ξ and ν are constructible sets inside $\mathrm{Gr}(Gm)$, we know that χ is additive (see [18, p. 92, item (3)]), which gives the above equality in (3.37). That is

$$\pi(m) = \pi(a) + \pi(b).$$

3.5 Adaptation of the Multiplication Formula to [26]

This section builds on the material covered in the previous section and makes necessary adjustments and additions in order to obtain the multiplication formula for ρ, given in Theorem 3.5.3. Clearly, now that we are back working with ρ, we again require the setup of Section 3.1.4; that is, we need a cluster tilting subcategory T, with $\mathsf{R} \subseteq \mathsf{T}$.

As with π, we look to understand how to evaluate ρ for some $m \in \operatorname{indec} \mathsf{C}$. In the definition of ρ, we take a sum over $e \in \mathrm{K}_0(\mathsf{fl}\,\mathsf{R})$. In order to do this, we will require knowledge of the Grothendieck group $\mathrm{K}_0(\mathsf{fl}\,\mathsf{R})$ and the K_0-classes of some of its key elements.

Firstly, we know

$$[\nu N] = [N].$$

Indeed, by definition, $\nu N = (G\eta)(N)$, and since $G\eta$ is injective, $(G\eta)(N)$ and N have the same composition series. Hence, the above equality is true.

To find $[\xi P]$, we first note that by definition, $\xi P = (G\mu)^{-1}P$, and therefore, $[\xi P] = [(G\mu)^{-1}P]$. A consequence of the Second Isomorphism Theorem is that a composition series of $(G\mu)^{-1}P$ can be obtained by concatenating composition series of P and of $\operatorname{Ker} G\mu$. That is, $[(G\mu)^{-1}(P)] = [P] + [\operatorname{Ker} G\mu]$. So, the K_0-classes are

$$[\nu N] = [N] \text{ and } [\xi P] = [P] + [\operatorname{Ker} G\mu].$$

Now that we have some useful information about the K_0-classes, we can take a more in depth look at ρ. First consider the following lemma.

Lemma 3.5.1. *Let* $m \in \operatorname{indec} \mathsf{C}$ *such that* $\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathsf{C}}(m, \mathsf{R}) = \dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathsf{C}}(\mathsf{R}, m) = 0$. *Then,* $\rho(m) = \alpha(m)$.

Proof. Let m be as in the lemma. We compute $\rho(m)$:

$$\begin{aligned}
\rho(m) &= \alpha(m)\sum_e \chi\Big(\mathrm{Gr}_e(G(m))\Big)\beta(e) \\
&= \alpha(m)\sum_e \chi(\mathrm{Gr}_e(0))\beta(e) \\
&= \alpha(m)\beta(0) \\
&= \alpha(m).
\end{aligned}$$

In the above calculation, the third $=$ is due to $\chi(\mathrm{Gr}_e(0))$ being zero for all nonzero $e \in \mathrm{K}_0(\mathsf{fl}\,\mathsf{R})$ and one when $e = 0$. The last $=$ is since β is exponential. $\square$

In particular, notice from this lemma that $\rho(r) = \alpha(r)$ for each $r \in \mathsf{R}$. Now, consider $\rho(r)\rho(m)$ for $m \in \mathrm{indec}\,\mathsf{C}$:

$$\begin{aligned}
\rho(r)\rho(m) &= \alpha(r)\alpha(m)\sum_e \chi(\mathrm{Gr}_e(Gm))\beta(e) \\
&= \alpha(r)\alpha(m)\Bigg(\sum_e \chi(\mathrm{Im}\,\xi \cap \mathrm{Gr}_e(Gm)) + \chi(\mathrm{Im}\,\nu \cap \mathrm{Gr}_e(Gm))\Bigg)\beta(e),
\end{aligned}$$

where the second equality arises from $\mathrm{Gr}(Gm)$ being the disjoint union of the images of ξ and ν. We now make an important remark about the two intersections in the second equality above.

Remark 3.5.2.

1. The first intersection is given by:

$$\begin{aligned}
\mathrm{Im}\,\xi \cap \mathrm{Gr}_e(Gm) &= \{\xi P \mid [\xi P] = e\} \\
&= \{\xi P \mid [P] = e - [\mathrm{Ker}\,G\mu]\} \\
&= \xi(\mathrm{Gr}_{e-[\mathrm{Ker}G\mu]}(Ga)).
\end{aligned}$$

   Here, we used the fact that $[\xi P] = [P] + [\mathrm{Ker}G\mu]$.

2. The second intersection can be obtained in a similar way:

$$\begin{aligned}
\mathrm{Im}\,\nu \cap \mathrm{Gr}_e(Gm) &= \{\nu N \mid [\nu N] = e\} \\
&= \{\nu N \mid [N] = e\} \\
&= \nu(\mathrm{Gr}_e(Gb)).
\end{aligned}$$

Using this remark, we can continue to calculate $\rho(r)\rho(m)$, obtaining

$$\begin{aligned}\rho(r)\rho(m) &= \alpha(r)\alpha(m)\sum_e \Big(\chi(\xi(\mathrm{Gr}_{e-[\mathrm{Ker}\,G\mu]}(Ga))) + \chi(\nu(\mathrm{Gr}_e(Gb)))\Big)\beta(e) \\ &= \alpha(r)\alpha(m)\sum_e \Big(\chi(\mathrm{Gr}_{e-[\mathrm{Ker}\,G\mu]}(Ga)) + \chi(\mathrm{Gr}_e(Gb))\Big)\beta(e). \qquad (3.38)\end{aligned}$$

We can discard ξ and ν in the final expression since they are both embeddings.

Theorem 3.5.3. *Let $m \in \operatorname{indec}\mathsf{C}$ and $r \in \operatorname{indec}\mathsf{R}$ such that $\mathrm{Ext}^1_{\mathsf{C}}(r,m)$ and $\mathrm{Ext}^1_{\mathsf{C}}(m,r)$ both have dimension one over $\mathbb{C}$. Then, there are nonsplit triangles*

$$m \xrightarrow{\mu} a \xrightarrow{\gamma} r \xrightarrow{\delta} \Sigma m \qquad \text{and} \qquad r \xrightarrow{\sigma} b \xrightarrow{\eta} m \xrightarrow{\zeta} \Sigma r,$$

with δ and ζ nonzero. Let Gm have finite length in $\mathsf{Mod}\,\mathsf{R}$, then

$$\rho(r)\rho(m) = \rho(a) + \rho(b).$$

Proof. We first note that since Gm has finite length, then so do Ga and Gb. This follows immediately from the exact sequences in (3.35) and (3.36).

Now, by making the substitution $f = e - [\mathrm{Ker}\,G\mu]$ in Equation (3.38), observe that

$$\begin{aligned}\rho(r)\rho(m) &= \alpha(r)\alpha(m)\sum_f \chi\big(\mathrm{Gr}_f(Ga)\big)\beta(f + [\mathrm{Ker}\,G\mu]) \\ &\quad + \alpha(r)\alpha(m)\sum_e \chi\big(\mathrm{Gr}_e(Gb)\big)\beta(e) \\ &\overset{(a)}{=} \alpha(r)\alpha(m)\sum_f \chi\big(\mathrm{Gr}_f(Ga)\big)\beta(f)\beta([\mathrm{Ker}\,G\mu]) \\ &\quad + \alpha(r)\alpha(m)\sum_e \chi\big(\mathrm{Gr}_e(Gb)\big)\beta(e) \\ &\overset{(b)}{=} \alpha(r)\alpha(m)\beta([\mathrm{Ker}\,G\mu])\sum_f \chi\big(\mathrm{Gr}_f(Ga)\big)\beta(f) \\ &\quad + \alpha(r)\alpha(m)\sum_e \chi\big(\mathrm{Gr}_e(Gb)\big)\beta(e). \qquad (3.39)\end{aligned}$$

Here, (a) is due to β being exponential and (b) is due to $\beta([\mathrm{Ker}\,G\mu])$ being a constant.

Now, consider $\mathrm{Ker}\,G\sigma$. Since $G(r) = 0$, then $\mathrm{Ker}\,G\sigma = 0$, and clearly $[\mathrm{Ker}\,G\sigma] = 0$. The map β is exponential, and therefore $\beta([\mathrm{Ker}\,G\sigma]) = \beta(0) = 1$. We can insert this into

Equation (3.39) and see that

$$\begin{aligned}\rho(r)\rho(m) &= \alpha(r)\alpha(m)\beta([\operatorname{Ker} G\mu]) \sum_f \chi\big(\operatorname{Gr}_f(Ga)\big)\beta(f) \\ &\quad + \alpha(r)\alpha(m)\beta([\operatorname{Ker} G\sigma]) \sum_e \chi\big(\operatorname{Gr}_e(Gb)\big)\beta(e).\end{aligned} \tag{3.40}$$

Applying Condition F to our two triangles in the theorem, whilst remembering that α is exponential, gives

$$\alpha(a) = \alpha(r)\alpha(m)\beta([\operatorname{Ker} G\mu])$$
$$\alpha(b) = \alpha(r)\alpha(m)\beta([\operatorname{Ker} G\sigma]).$$

Returning these equalities into Equation (3.40), the expression for $\rho(r)\rho(m)$ becomes

$$\begin{aligned}\rho(r)\rho(m) &= \alpha(a) \sum_f \chi\big(\operatorname{Gr}_f(Ga)\big)\beta(f) + \alpha(b) \sum_e \big(\operatorname{Gr}_e(Gb)\big)\beta(e) \\ &= \rho(a) + \rho(b).\end{aligned}$$

□

3.6 Examples

In this section, we will use two examples to demonstrate the multiplication formula for ρ in Theorem 3.5.3. These examples will be in $\mathsf{C}(A_n)$, so it is important to recall the polygonal model associated to $\mathsf{C}(A_n)$ introduced in Chapter 2. In the first example we will recompute the same example as in [26, sec. 3], using the multiplication formula to significantly simplify the computation. Our second example will then be a somewhat larger example in $\mathsf{C}(A_9)$.

3.6.1 Example for $\mathsf{C}(A_5)$

To compute our first example, we refer to the setup of [26, sec. 3]; that is, we set $\mathsf{C} = \mathsf{C}(A_5)$, the cluster category of Dynkin type A_5. Thus, the indecomposables of C can be identified with the diagonals on a regular 8-gon. As in [26] we will denote by $\{a, b\}$ the indecomposable corresponding to the diagonal connecting the vertices a and b. We use the same polygon triangulation as in [26, sec. 3]; that is, indec R corresponds to the red diagonals in Figure 3.4 and indec S corresponds to the blue diagonals. Hence, T contains the following indecomposable objects

$$\{1,7\}\ ,\ \{2,4\}\ ,\ \{2,5\}\ ,\ \{2,7\}\ ,\ \{5,7\},$$

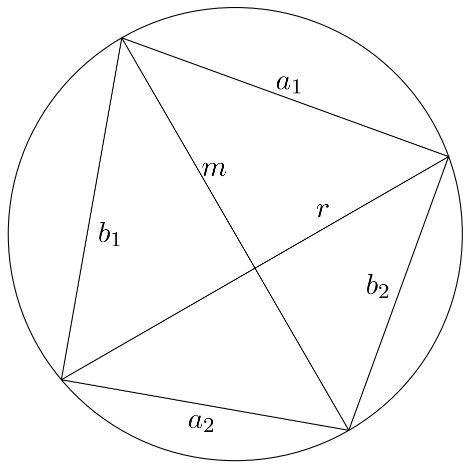


Figure 3.2: There are triangles $m \to a_1 \oplus a_2 \to r$ and $r \to b_1 \oplus b_2 \to m$. We therefore have $\rho(r)\rho(m) = \rho(a_1)\rho(a_2) + \rho(b_1)\rho(b_2)$.

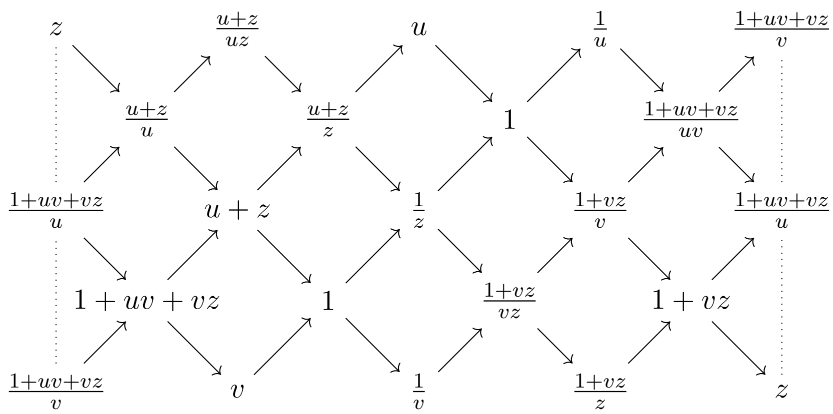


Figure 3.3: The Auslander-Reiten quiver of the cluster category of Dynkin type A_5. The vertices have been replaced with values of the modified Caldero-Chapoton map ρ. Again, the dotted lines are identified with oppostite orientations.

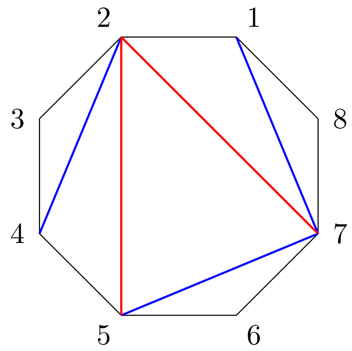


Figure 3.4: Red diagonals correspond to indecomposables in indec R and blue diagonals correspond to indecomposables in indec S.

whilst the indecomposables in S are:

$$\{1,7\}\ ,\ \{2,4\}\ ,\ \{5,7\}.$$

Recall from our description of $\mathsf{C}(A_n)$ in Section 2.5.1 that there are exchange triangles as described in Figure 3.2. The indecomposables in S therefore fit in the following exchange triangles:

$$\begin{array}{ll}
\{1,7\} \longrightarrow \{2,7\} \longrightarrow \{2,8\} & \qquad \{2,8\} \longrightarrow 0 \longrightarrow \{1,7\} \\
\{2,4\} \longrightarrow \{2,5\} \longrightarrow \{3,5\} & \qquad \{3,5\} \longrightarrow 0 \longrightarrow \{2,4\} \\
\{5,7\} \longrightarrow \{2,5\} \longrightarrow \{2,6\} & \qquad \{2,6\} \longrightarrow \{2,7\} \longrightarrow \{5,7\}.
\end{array}$$

Then, applying the definition of N from (3.26), it is easily seen that

$$N = \big\langle [2,5],[2,7] \big\rangle.$$

Here, we denote by $[a,b]$ the $\mathrm{K}_0^{\mathrm{split}}$-class of the indecomposable $\{a,b\}$. We also have that

$$\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N = \Big\langle [1,7]+N,\ [2,4]+N,\ [5,7]+N \Big\rangle$$

is a free abelian group on three generators. Let the exponential map $\varepsilon : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to \mathbb{Z}[u^{\pm 1}, v^{\pm 1}, z^{\pm 1}]$ be given by

$$\varepsilon([1,7]+N) = u\ ,\ \varepsilon([2,4]+N) = v\ ,\ \varepsilon([5,7]+N) = z. \tag{3.41}$$

We will now demonstrate how to calculate $\rho(\{4,6\})$ using an alternative method to that in [26, ex. 3.5]. We will compute it by applying the multiplication formula for ρ in Theorem 3.5.3. Since $\dim_{\mathbb{C}}\mathrm{Ext}^1(\{4,6\},\{2,5\}) = \dim_{\mathbb{C}}\mathrm{Ext}^1(\{2,5\},\{4,6\}) = 1$, we may set $r = \{2,5\}$, and using Figure 3.2, we know that $\{4,6\}$ sits in the following nonsplit triangles:

$$\{4,6\} \to \{2,4\} \to \{2,5\}\ ,\ \{2,5\} \to \{2,6\} \to \{4,6\}.$$

Applying Theorem 3.5.3, we get the following equality:

$$\rho(\{2,5\})\rho(\{4,6\}) = \rho(\{2,4\}) + \rho(\{2,6\}). \tag{3.42}$$

Due to the fact that the diagonals corresponding to $\{2,5\}, \{2,4\}$ and $\{2,6\}$ do not cross any diagonals in $\operatorname{indec} \mathsf{R}$, it is immediate from Lemma 3.5.1 that Equation (3.42) becomes

$$\alpha(\{2,5\})\rho(\{4,6\}) = \alpha(\{2,4\}) + \alpha(\{2,6\}). \tag{3.43}$$

In order to calculate α of each of the indecomposables $\{2,5\}, \{2,4\}$ and $\{2,6\}$, we first calculate their indices with respect to T.

$\{2,5\}$ sits in the following triangle:

$$0 \to \{2,5\} \to \{2,5\},$$

and since $\{2,5\} \in \operatorname{indec} \mathsf{T}$, we see that

$$\operatorname{ind}_{\mathsf{T}}(\{2,5\}) = [2,5]. \tag{3.44}$$

By the same logic,

$$\operatorname{ind}_{\mathsf{T}}(\{2,4\}) = [2,4]. \tag{3.45}$$

We note that one of the exchange triangles for $\{2,6\}$ is

$$\{5,7\} \to \{2,5\} \to \{2,6\},$$

and hence

$$\operatorname{ind}_{\mathsf{T}}(\{2,6\}) = [2,5] - [5,7]. \tag{3.46}$$

Since $[2,5] \in N$, using the definition of α from (3.28), we see that

$$\begin{aligned}
\alpha(\{2,5\}) &= \varepsilon Q(\operatorname{ind}_{\mathsf{T}}(\{2,5\}) \\
&= \varepsilon Q([2,5]) \\
&= \varepsilon([2,5] + N) \\
&= \varepsilon(0) \\
&= 1,
\end{aligned}$$

and hence, the left hand side of Equation (3.43) becomes $\alpha(\{2,5\})\rho(\{4,6\}) = \rho(\{4,6\})$. Substituting back into Equation (3.43), we obtain

$$\begin{aligned}\rho(\{4,6\}) &= \alpha(\{2,4\}) + \alpha(\{2,6\}) \\ &= \varepsilon Q(\mathrm{ind}_{\mathsf{T}}(\{2,4\})) + \varepsilon Q(\mathrm{ind}_{\mathsf{T}}(\{2,6\})) \\ &\overset{(1)}{=} \varepsilon Q([2,4]) + \varepsilon Q([2,5] - [5,7]) \\ &= \varepsilon([2,4] + N) + \varepsilon(-[5,7] + N) \\ &\overset{(2)}{=} v + z^{-1} \\ &= \frac{1 + vz}{z},\end{aligned}$$

where (1) is by substituting the values from Equations (3.45) and (3.46), and (2) is due to the definition of ε from Equation (3.41). We notice here that this is indeed the same result for $\rho(\{4,6\})$ obtained in [26, ex. 3.5].

3.6.2 Example for $\mathsf{C}(A_9)$

In this section, we demonstrate the use of the multiplication formula with a larger example. We compute values of ρ for all indecomposables in $\mathsf{C}(A_9)$ for the polygon triangulation given in Figure 3.6. These values are given in the Auslander-Reiten quiver in Figure 3.7. In this example, we will compute the index of a given indecomposable using a formula given by Jørgensen and Yakimov in [31]. We introduce this formula first.

An Index Formula

Consider the cluster category $\mathsf{C}(A_n)$ and denote by T a cluster tilting subcategory of $\mathsf{C}(A_n)$. Recall that $\mathrm{indec}\,\mathsf{T}$ corresponds to a triangulation of a regular $(n+3)$-gon P. Consider $c \in \mathrm{indec}\,\mathsf{C}(A_n)$, and denote by e and f the endpoints of c, as in the left hand side of Figure 3.5. In this section, we will provide a formula for $\mathrm{ind}_{\mathsf{T}}(c)$. The following remark introduces some key notation.

Remark 3.6.1.

1. Let a be a vertex of P, we denote by a^+ the anticlockwise neighbouring vertex of a and by a^- the clockwise neighbouring vertex of a.
2. For two vertices a and b of P with $a \neq b$, we denote by $[a,b]$ the set of vertices of P between a and b that are anticlockwise of a. Note that a and b are both included in the interval $[a,b]$.

We now give the following remark which provides us with a sequence of vertices of P from which we can define a formula for $\mathrm{ind}_{\mathsf{T}}(c)$. A proof of the claims in this remark is given in [31, prop. 6.2].

Remark 3.6.2. Let $c \in \operatorname{indec} \mathsf{C}(A_n)$ be as in the left hand side of Figure 3.5. There exists an integer $n \geq 0$ and a sequence of vertices $e_0, e_1, e_2, \ldots, e_{2n+1}$ of P, as in the right hand side of Figure 3.5, such that the following statements are satisfied:

1. $e < e_1 < e_3 < \cdots < f < \cdots < e_6 < e_4 < e_2$.
2. $e_0 = e$ and $e_{2n+1} = f$.
3. $\{e_i, e_{i+1}\} \in \operatorname{indec} \mathsf{T}$.
4. e_1 is the final vertex in $[e_0^+, f]$ that is connected to e_0 by a diagonal in T, i.e. $\{e_0, e_1\}$ is the longest diagonal in T from e_0 to the right hand portion of the polygon in Figure 3.5. Note that if there are no diagonals from e_0 to $[e_0^+, f]$, then we set $e_1 = e_0^+$.
5. For $i \geq 1$, we have
   (a) e_{2i} is the first vertex in $[f^+, e_{2i-2}^-]$ which is connected to e_{2i-1} by a diagonal in T, i.e. $\{e_{2i-1}, e_{2i}\}$ is the longest diagonal in T from e_{2i-1} to $[f^+, e_{2i-2}^-]$.
   (b) e_{2i+1} is the last vertex in $[e_{2i-1}^+, f]$ which is connected to e_{2i} by a diagonal in T, i.e. $\{e_{2i}, e_{2i+1}\}$ is the longest diagonal in T from e_{2i} to $[e_{2i-1}^+, f]$.

   Note here that 'first' and 'last' are used with respect to the cyclic ordering o the vertices of P.

The formula for the $\operatorname{ind}_{\mathsf{T}}(c)$ is then given by

$$\operatorname{ind}_{\mathsf{T}}(c) = \sum_{i=0}^{2i} (-1)^i [\{e_i, e_{i+1}\}]. \tag{3.47}$$

Computing an Example in $\mathsf{C}(A_9)$

Now that we have a formula for computing the index, we move on to our example in $\mathsf{C}(A_9)$. Consider the triangulation of the 12-gon given in Figure 3.6. This triangulation corresponds to the set of indecomposables from a cluster tilting subcategory $\mathsf{T} \subseteq \mathsf{C}(A_9)$, where the red diagonals correspond to indecomposables in the rigid subcategory R and the blue diagonals to indecomposables in the subcategory S. Hence, T contains the following indecomposables,

$$\{2,4\}\ ,\ \{2,5\}\ ,\ \{2,9\}\ ,\ \{2,12\}\ ,\ \{5,8\}\ ,\ \{5,9\}\ ,\ \{6,8\}\ ,\ \{9,12\}\ ,\ \{10,12\},$$

whilst the indecomposable objects in S are:

$$\{2,5\}\ ,\ \{5,8\}\ ,\ \{6,8\}\ ,\ \{9,12\}\ ,\ \{10,12\}.$$

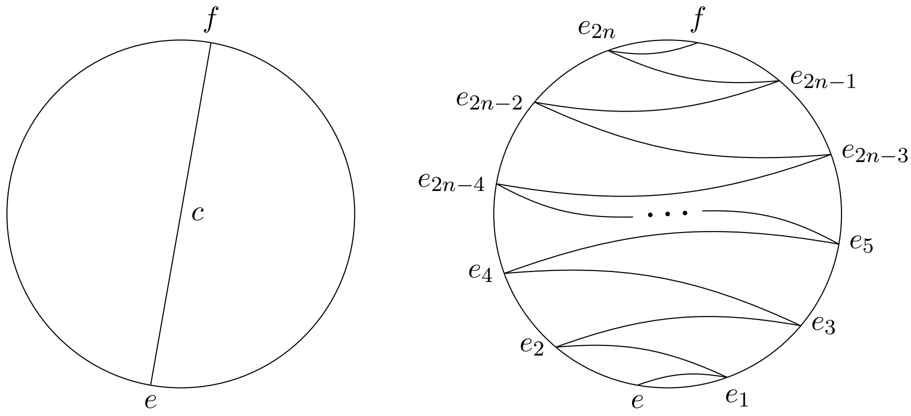


Figure 3.5: The sequence of vertices $e_0, e_1, \ldots, e_{2n+1}$ from Remark 3.6.2.

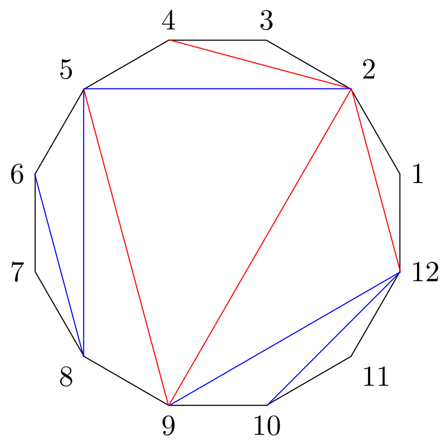


Figure 3.6: The indecomposables of a cluster tilting subcategory of $\mathsf{C}(A_9)$. The red diagonals correspond to indecomposables in R, whilst the blue diagonals correspond to indecomposables in S.

Recalling from Figure 3.2 how to construct exchange triangles in $\mathsf{C}(A_n)$, we know that the indecomposables in S fit into the following exchange triangles:

$$\begin{array}{ll}
\{2,5\} \longrightarrow \{2,9\} \longrightarrow \{4,9\} & \{4,9\} \to \{2,4\} \oplus \{5,9\} \to \{2,5\} \\
\{5,8\} \to \{5,9\} \oplus \{6,8\} \to \{6,9\} & \{6,9\} \longrightarrow 0 \longrightarrow \{5,8\} \\
\{6,8\} \longrightarrow 0 \longrightarrow \{5,7\} & \{5,7\} \longrightarrow \{5,8\} \longrightarrow \{6,8\} \\
\{9,12\} \to \{2,9\} \oplus \{10,12\} \to \{2,10\} & \{2,10\} \longrightarrow \{2,12\} \longrightarrow \{9,12\} \\
\{10,12\} \longrightarrow 0 \longrightarrow \{9,11\} & \{9,11\} \longrightarrow \{9,12\} \longrightarrow \{10,12\}.
\end{array}$$

The middle terms of the exchange triangles can then be used to compute N, which has the following presentation,

$$N = \Big\langle [2,4]+[5,9]-[2,9],\ [5,9]+[6,8],\ [5,8],\ [2,12]-[10,12]-[2,9],\ [9,12] \Big\rangle. \quad (3.48)$$

Here, recall that we denote by $[a,b]$ the $\mathrm{K}_0^{\mathrm{split}}$-class of the indecomposable $\{a,b\}$. Notice from this that in the quotient $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$, we have

$$[5,8]+N = [9,12]+N = 0,$$

whilst

$$\begin{aligned} [5,9]+N &= [2,9]-[2,4]+N, \\ [6,8]+N &= [2,4]-[2,9]+N, \\ [10,12]+N &= [2,12]-[2,9]+N. \end{aligned}$$

This shows that

$$\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N = \Big\langle [2,4]+N,\ [2,9]+N,\ [2,12]+N,\ [2,5]+N \Big\rangle$$

is a free abelian group with four generators. Then let the exponential map $\varepsilon : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to \mathbb{Z}[v^{\pm}, w^{\pm}, y^{\pm}, z^{\pm}]$ be given by

$$\varepsilon([2,4]+N) = v, \quad \varepsilon([2,9]+N) = w, \quad \varepsilon([2,12]+N) = y, \quad \varepsilon([2,5]+N) = z. \quad (3.49)$$

We will now demonstrate how to compute the value at one of the vertices in the Auslander-Reiten quiver in Figure 3.7 using the multiplication formula. We will compute $\rho(\{7,11\})$. Set $r_1 = \{2,9\} \in \operatorname{indec} \mathsf{R}$ and notice that

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathsf{C}(A_9)}(\{7,11\},\{2,9\}) = \dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathsf{C}(A_9)}(\{2,9\},\{7,11\}) = 1.$$

Thus, there are exchange triangles

$$\{7,11\} \twoheadrightarrow \{2,7\} \oplus \{9,11\} \twoheadrightarrow \{2,9\} \qquad \{2,9\} \twoheadrightarrow \{7,9\} \oplus \{2,11\} \twoheadrightarrow \{7,11\},$$

and the multiplication formula tells us that

$$\rho(\{2,9\})\rho(\{7,11\}) = \rho(\{2,7\})\rho(\{9,11\}) + \rho(\{7,9\})\rho(\{2,11\}). \quad (3.50)$$

We first compute $\rho(\{2,7\})$ by setting $r_2 = \{5,9\}$ and noticing that

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathsf{C}(A_9)}(\{2,7\},\{5,9\}) = \dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathsf{C}(A_9)}(\{5,9\},\{2,7\}) = 1.$$

Thus, there are exchange triangles

$$\{2,7\} \to \{5,7\} \oplus \{2,9\} \to \{5,9\} \quad , \quad \{5,9\} \to \{2,5\} \oplus \{7,9\} \to \{2,7\},$$

and the multiplication formula tells us that

$$\rho(\{5,9\})\rho(\{2,7\}) = \rho(\{5,7\})\rho(\{2,9\}) + \rho(\{2,5\})\rho(\{7,9\}). \tag{3.51}$$

We make the following computations. Recall from Lemma 3.5.1 that if a diagonal m crosses no diagonals in R then $\rho(m) = \alpha(m)$. Recall also the definition of α from (3.28) and that if $t \in \operatorname{indec} \mathsf{T}$, then $\operatorname{ind}_{\mathsf{T}}(t) = [t]$. We have

$$\begin{aligned}
\rho(\{5,9\}) &= \alpha(\{5,9\}) \\
&= \varepsilon Q(\operatorname{ind}_{\mathsf{T}}(\{5,9\})) \\
&= \varepsilon Q([5,9]) \\
&= \varepsilon([2,9] - [2,4] + N) \\
&= \varepsilon([2,9] + N)\varepsilon(-[2,4] + N) \\
&= \frac{w}{v}.
\end{aligned}$$

Here, the fourth $=$ uses the presentation of N in (3.48). For $\{5,7\}$, we have

$$\begin{aligned}
\rho(\{5,7\}) &= \alpha(\{5,7\}) \\
&= \varepsilon Q(\operatorname{ind}_{\mathsf{T}}(\{5,7\})) \\
&= \varepsilon Q(-[6,8]) \\
&= \varepsilon([5,9] + N) \\
&= \frac{w}{v}.
\end{aligned}$$

Here, starting at vertex 7, the zigzag for $\operatorname{ind}_{\mathsf{T}}(\{5,7\})$ is $\{7,8\}, \{6,8\}, \{5,6\}$, and so applying the formula from (3.47) gives $\operatorname{ind}_{\mathsf{T}}(\{5,7\}) = -[6,8]$, which is used in the third $=$. The final $=$ is by the computations for $\rho(\{5,9\})$. For $\{2,9\}$, we have

$$\begin{aligned}
\rho(\{2,9\}) &= \alpha(\{2,9\}) \\
&= \varepsilon Q(\operatorname{ind}_{\mathsf{T}}(\{2,9\})) \\
&= \varepsilon Q([2,9]) \\
&= \varepsilon([2,9] + N) \\
&= w.
\end{aligned}$$

For $\{2,5\}$ we have:

$$\begin{aligned}\rho(\{2,5\}) &= \alpha(\{2,5\}) \\ &= \varepsilon Q(\mathrm{ind}_{\mathsf{T}}(\{2,5\})) \\ &= \varepsilon Q([2,5]) \\ &= \varepsilon([2,5]+N) \\ &= z.\end{aligned}$$

For $\{7,9\}$ we have:

$$\begin{aligned}\rho(\{7,9\}) &= \alpha(\{7,9\}) \\ &= \varepsilon Q(\mathrm{ind}_{\mathsf{T}}(\{7,9\})) \\ &= \varepsilon Q([5,9]-[5,8]) \\ &= \varepsilon([5,9]-[5,8]+N) \\ &= \varepsilon([5,9]+N) \\ &= \frac{w}{v}.\end{aligned}$$

Here, starting at vertex 9, the zigzag for $\mathrm{ind}_{\mathsf{T}}(\{7,9\})$ is $\{5,9\}, \{5,8\}, \{7,8\}$, and so applying the formula from (3.47) gives $\mathrm{ind}_{\mathsf{T}}(\{7,9\}) = [5,9]-[5,8]$, which is used in the third $=$. The penultimate $=$ uses the presentation of N in (3.48) and the final $=$ is by the computations for $\rho(\{5,9\})$.

Now, we can substitute the above computations back into Equation (3.51) and see that

$$wv^{-1}\rho(\{2,7\}) = wv^{-1}w + zwv^{-1}.$$

Cancelling wv^{-1} from each term shows that

$$\rho(\{2,7\}) = w + z.$$

Now, we need to compute the missing values from Equation (3.50). For $\{9,11\}$ we have

$$\begin{aligned}\rho(\{9,11\}) &= \alpha(\{9,11\}) \\ &= \varepsilon Q(\mathrm{ind}_{\mathsf{T}}(\{9,11\})) \\ &= \varepsilon Q(-[10,12]) \\ &= \varepsilon([2,9]-[2,12]+N) \\ &= \varepsilon([2,9]+N)\varepsilon(-[2,12]+N) \\ &= \frac{w}{y}.\end{aligned}$$

Here, starting at vertex 11, the zigzag for $\mathrm{ind}_{\mathsf{T}}(\{9,11\})$ is $\{11,12\},\{10,12\},\{9,10\}$, and so applying the formula from (3.47) gives us $\mathrm{ind}_{\mathsf{T}}(\{9,11\}) = -[10,12]$, which is used in the third $=$. The fourth $=$ uses the presentation of N in (3.48). For $\{2,11\}$ we have:

$$\begin{aligned}\rho(\{2,11\}) &= \alpha(\{2,11\}) \\ &= \varepsilon Q(\mathrm{ind}_{\mathsf{T}}(\{2,11\})) \\ &= \varepsilon Q([2,9]-[9,12]) \\ &= \varepsilon([2,9]-[9,12]+N) \\ &= \varepsilon([2,9]+N)\varepsilon(0) \\ &= w.\end{aligned}$$

Here, starting at vertex 11, the zigzag for $\mathrm{ind}_{\mathsf{T}}(\{2,11\})$ is $\{11,12\},\{9,12\},\{2,9\}$, and so applying the formula from (3.47) gives us $\mathrm{ind}_{\mathsf{T}}(\{2,11\}) = [2,9]-[9,12]$, which is used in the third $=$.

Now, we can substitute all our values into Equation (3.50) to compute $\rho(\{7,11\})$. We have,

$$w\rho(\{7,11\}) = (w+z)wy^{-1} + wv^{-1}w.$$

Cancelling w from each term gives

$$\rho(\{7,11\}) = \frac{w+z}{y} + \frac{w}{v},$$

and rearranging gives

$$\rho(\{7,11\}) = \frac{vw+vz+wy}{vy}.$$

Remark 3.6.3. In general, the formula from Theorem 3.5.3 can be applied iteratively to calculate $\rho(m)$. Indeed,

$$\rho(r)\rho(m) = \rho(a) + \rho(b)$$

is an iterative formula on m, and hence, calculating ρ of each indecomposable in C can

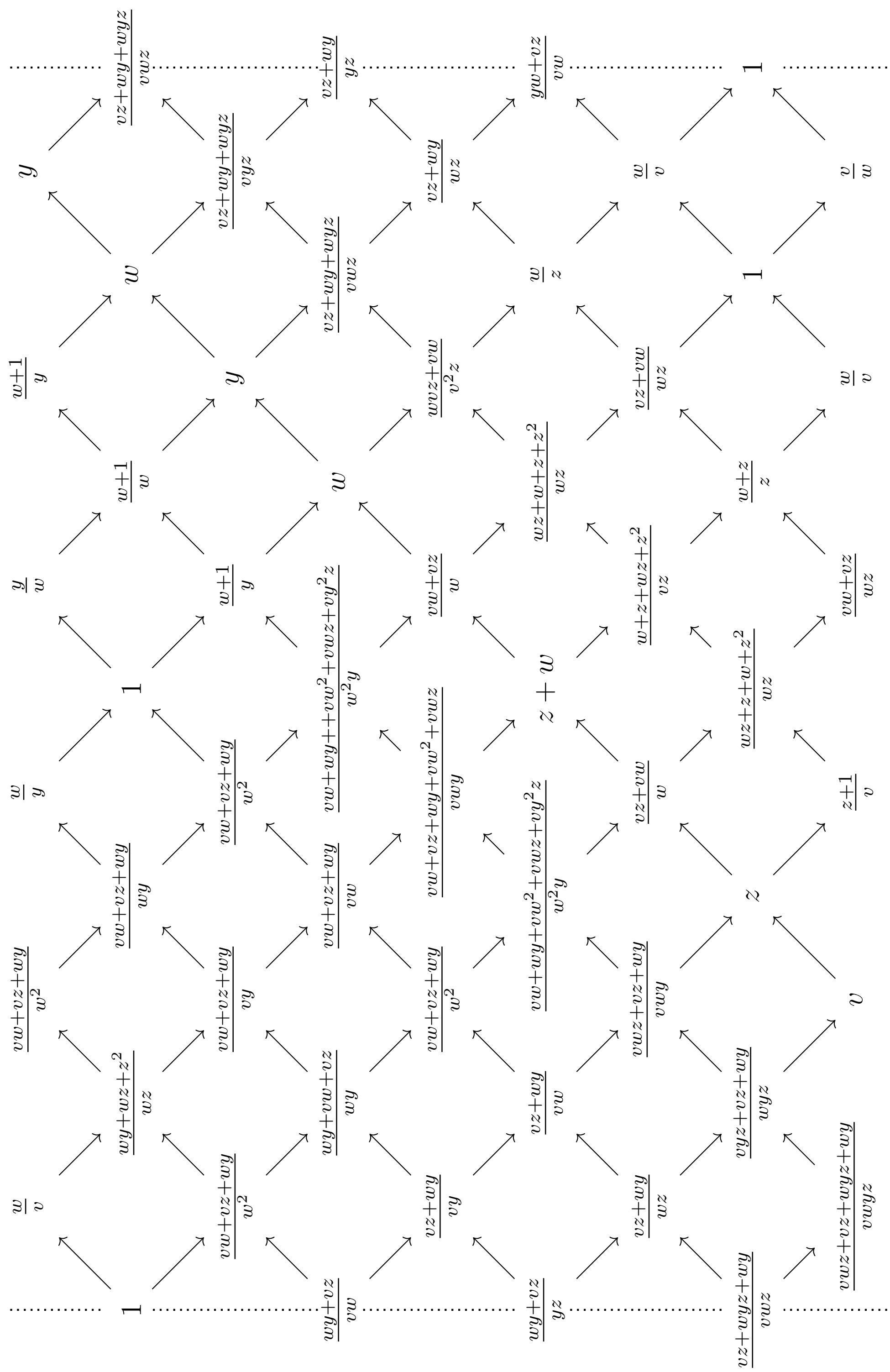


Figure 3.7: The Auslander-Reiten quiver of $\mathsf{C}(A_9)$. The vertices have been replaced with the values of ρ. The dotted lines are identified with opposite orientations.

be reduced to calculating ρ of the indecomposables in C whose corresponding diagonals in the $(n+3)$-gon do not cross any of the diagonals in R. Namely, it is clear from Figure 3.2 that each of a_1, a_2, b_1 and b_2 sit inside "smaller" polygons than m. Here, the smaller polygons are those obtained from r dissecting the (n+3)-gon. Since R consists of only noncrossing diagonals, the remaining diagonals in R sit inside these smaller polygons. Reapplying Theorem 3.5.3 to each of a_1, a_2, b_1 and b_2 will again create a series of even smaller polygons, containing a new a or b. After repeated iterations, this process will eventually terminate at the stage when the new a or b does not cross any of the diagonals in R.

Now, in the case when a diagonal, say m', does not cross a diagonal in R, we know by Lemma 3.5.1 that $\rho(m') = \alpha(m')$. Computing $\alpha(m')$ is done by calculating the index of m', and then applying the maps Q and ε. Hence, finding $\rho(m)$ for each $m \in \operatorname{indec} \mathsf{C}$ can be reduced by Theorem 3.5.3 to computing the index of each of the indecomposables in C whose corresponding diagonals do not cross any diagonals in R.

Chapter 4

Properties of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ in $\mathsf{C}(A_n)$

In this chapter we prove some important properties of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ in $\mathsf{C}(A_n)$, where $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$ is the split Grothendieck group of the cluster tilting subcategory T, and N is the subgroup of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$ defined in (3.26).

Recall from Chapter 3 that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is an essential object in our study of generalised friezes. Indeed, in Section 3.3, we provide exponential maps α and β, involving $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$, which satisfy Condition F from Definition 3.2.1. Their definitions depend on an exponential map $\varepsilon : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to A$, where A is a ring. The group $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ determines which maps ε can choose. In turn, by Theorem 3.2.2, α and β turn the modified Caldero-Chapoton map ρ, defined in (3.23), into a generalised frieze. The map ρ depends only on the rigid subcategory $\mathsf{R} \subseteq \mathsf{T}$.

This chapter is laid out as follows. Section 4.1 recalls some of the important constructions from Chapters 2 and 3. Then, Section 4.2 shows in Proposition 4.2.3 that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is dependent only on the choice of rigid subcategory R, resulting in Theorem 4.2.5. In Section 4.3, we are then able to prove Theorem 4.3.2, which provides us with a general formula for the quotient $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$.

4.1 Introduction

In this section we recall some of the important constructions from the first two chapters that will be required throughout this chapter. We recall these notions in brief, and note that more details can be found in the first two chapters.

For some integer $n \geq 2$, consider the cluster category of Dynkin type A_n (see [8], [11]), which we denote by $\mathsf{C} = \mathsf{C}(A_n)$. Note that C is triangulated, essentially small, Hom-finite, $\mathbb{C}$-linear, Krull-Schmidt and 2-Calabi-Yau. Recall from Section 2.5.1 that the set of indecomposables of $\mathsf{C}(A_n)$, denoted $\operatorname{indec} \mathsf{C}(A_n)$, is in bijection with the set of diagonals of a regular $(n+3)$-gon P. Any edge of P is associated with the zero object in $\mathsf{C}(A_n)$, and the suspension functor Σ applied to an indecomposable corresponds to shifting the endpoints of the corresponding diagonal one vertex clockwise. That is, if the vertices of P are labelled with the set $\{0, 1, \ldots, n+2\}$, with a positive orientation, then for some indecomposable $\{i, j\}$, where i, j are vertices of P, we have

$$\Sigma\{i, j\} = \{i-1, j-1\}.$$

Here, these coordinates should be taken modulo $n+3$.

Recall that an object R is rigid if $\mathrm{Hom}_{\mathsf{C}}(R, \Sigma R) = 0$. We consider a rigid object R, and set $\mathsf{R} = \operatorname{add} R$, the rigid subcategory consisting of all finite direct sums of the summands of R. This subcategory is rigid in the sense that

$$\mathrm{Hom}_{\mathsf{C}}(\mathsf{R}, \Sigma\mathsf{R}) = 0.$$

The set of indecomposables of R, denoted $\operatorname{indec}\mathsf{R}$, correspond to a polygon dissection of P, consisting of noncrossing diagonals. We may then choose a cluster-tilting subcategory T such that $\mathsf{R} \subseteq \mathsf{T}$, whose indecomposables correspond to a triangulation of P, again consisting of noncrossing diagonals. Hence, there is a subcategory S such that

$$\operatorname{indec}\mathsf{T} = \operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S},$$

whose indecomposables turn the polygon dissection from R in to a triangluation.

Recall from Section 3.1.4 the modified Caldero-Chapoton map $\rho : \operatorname{obj}\mathsf{C} \to A$, defined in [26] by

$$\rho(c) = \alpha(c) \sum_e \chi(\mathrm{Gr}_e(Gc))\beta(e),$$

where A is some commutative ring and $c \in \operatorname{obj}\mathsf{C}$. Here, χ denotes the Euler characteristic, Gr_e the Grassmannian of submodules $M \subseteq Gc$ with K_0-class $[M] = e$, where $e \in \mathrm{K}_0(\mathsf{fl}\,\mathsf{R})$, the Grothendieck group of the abelian category $\mathsf{fl}\,\mathsf{R}$. Note that $\mathsf{fl}\,\mathsf{R}$ is the full subcategory of $\mathsf{Mod}\,\mathsf{R}$, consisting of all finite length modules in the category of $\mathbb{C}$-linear contravariant functors $\mathsf{R} \to \mathrm{Vect}\,\mathbb{C}$. For each $r \in \operatorname{indec}\mathsf{R}$, there is a simple object S_r, see [25, sec. 1.8]. Note then that $\mathrm{K}_0(\mathsf{fl}\,\mathsf{R})$ is the free group on generators $[S_r]$ for $r \in \operatorname{indec}\mathsf{R}$. The functor $G : \mathsf{C} \to \mathsf{Mod}\,\mathsf{R}$ is defined by

$$G(c) = \mathrm{Hom}_{\mathsf{C}}(-, \Sigma c)|_{\mathsf{R}},$$

and the maps $\alpha : \operatorname{obj}\mathsf{C} \to A$ and $\beta : \mathrm{K}_0(\mathsf{fl}\,\mathsf{R}) \to A$ are exponential maps in the sense that

$$\alpha(0) = 1 \ , \ \alpha(x \oplus y) = \alpha(x)\alpha(y),$$

$$\beta(0) = 1 \ , \ \beta(e+f) = \beta(e)\beta(f),$$

which also satisfy the "frieze-like" condition given in [26, def. 1.4].

Recall from Section 3.3 the definitions of α and β, given in [26, sec. 2], where

$$\alpha(c) = \varepsilon Q(\mathrm{ind}_{\mathsf{T}}(c)) \text{ and } \beta(e) = \varepsilon\theta(e).$$

Full definitions of each of these components, along with the commutative square for θ, can be found in Section 3.3. We note here that $\mathrm{ind}_{\mathsf{T}}(-)$ denotes the index, calculated with respect to T, and $Q : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \twoheadrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the canonical surjection, where $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$ is the split Grothendieck group of the additive category T. That is, $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$ is the finitely generated free abelian group with $\mathrm{indec}\,\mathsf{T}$ as its basis set.

There is a functor $\overline{G} : \mathsf{C} \to \mathsf{Mod}\,\mathsf{T}$, defined by

$$\overline{G}(c) = \mathrm{Hom}_{\mathsf{C}}(-, \Sigma c)|_{\mathsf{T}}.$$

It is clear that $i^*\overline{G} = G$, where $i^* : \mathrm{Mod}\,\mathsf{T} \to \mathrm{Mod}\,\mathsf{R}$ is the exact restriction functor induced by the inclusion $i : \mathsf{R} \to \mathsf{T}$.

Recall from Remark 3.1.4 that for each $t \in \mathrm{indec}\,\mathsf{T}$, there is a unique $t^* \in \mathrm{indec}\,\mathsf{C}$, called the mutation of t, such that replacing t with t^* in $\mathrm{indec}\,\mathsf{T}$ forms a new cluster tilting subcategory T^*. There are then exchange triangles

$$t \to B \to t^* \ , \ t^* \to A \to t,$$

with $A, B \in \mathrm{add}\,(\,\mathrm{indec}\,\mathsf{T} \setminus t)$, and we then define the subgroup N by

$$N = \left\langle [A] - [B] \;\middle|\; \begin{array}{l} s^* \to A \to s \ , \ s \to B \to s^* \text{ are exchange} \\ \text{triangles with } s \in \mathrm{indec}\,\mathsf{S} \end{array} \right\rangle, \tag{4.1}$$

see [26, def. 2.4]. For each $s \in \mathrm{indec}\,\mathsf{S}$, we will denote by $n(s)$ the generator of N associated to s; that is, the generator $[A]-[B]$ where $[A]$ and $[B]$ are the middle terms of the exchange triangles for s, as in the definition of N in (3.26).

4.2 Independence of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ Under the Choice of Subcategory S.

In this section, we show that in the case when $\mathsf{C} = \mathsf{C}(A_n)$, the results in [26] are independent of the choice of the subcategory S; that is, they are only dependent on the rigid subcategory R. If R is fixed, we can make two different choices, say S and S', obtaining

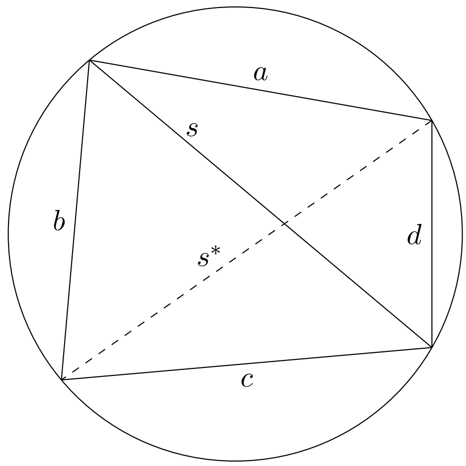


Figure 4.1: Selected diagonals from a triangulation of P. Here, s^* is the mutation of s and is the unique diagonal that recompletes the triangulation after the removal of s. Note that the regions between each of a, b, c, d and the boundary of P are filled with triangulations.

two different cluster tilting subcategories T and T', defined by

$$\operatorname{indec}\mathsf{T} = \operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S},$$

$$\operatorname{indec}\mathsf{T}' = \operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S}'.$$

We will show that when we calculate $\rho(c)$ for some object c in C, the outcome is independent of whether we calculate $\rho(c)$ with respect to T or T'.

We start by showing that

$$\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \cong \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'.$$

We first discuss some finer details of the make up of T and T' in terms of triangulations of P. In each case, R splits P into a set of polygons with fewer vertices. Hence, $\operatorname{indec}\mathsf{S}$ identifies with a triangulation of each of these "smaller" polygons. It makes sense to consider such triangulations of P that differ by only one diagonal (see Remark 4.2.1). That is, there are two subcategories corresponding to these triangulations, say S and S^*, such that $\operatorname{indec}\mathsf{S}$ and $\operatorname{indec}\mathsf{S}^*$ differ by one indecomposable. The different indecomposable s^* in S^* is obtained by removing a single indecomposable s in S and replacing it with its mutation, see [25, Section 2.3] and Figure 4.1. Then $\operatorname{indec}\mathsf{S}^*$ is given by

$$\operatorname{indec}\mathsf{S}^* = \{s^*\} \cup \{\operatorname{indec}\mathsf{S} \setminus \{s\}\}. \tag{4.2}$$

The cluster tilting subcategories $\operatorname{indec}\mathsf{T}$ and $\operatorname{indec}\mathsf{T}^*$ also differ by one indecomposable and their corresponding triangulations of P will differ by one diagonal. We have:

$$\operatorname{indec}\mathsf{T} = \operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S}, \tag{4.3}$$

$$\operatorname{indec}\mathsf{T}^* = \operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S}^*. \tag{4.4}$$

Remark 4.2.1. It is known that from one triangulation of P, one may obtain any other triangulation of P through some sequence of mutations of its diagonals. It follows that from one choice of $\operatorname{indec}\mathsf{S}$, one may retrieve any other choice of $\operatorname{indec}\mathsf{S}$ through a sequence of mutations of indecomposables in S. As a consequence of this, it makes sense to consider the two subcategories S and S^*, defined in Equation (4.2), differing by just one indecomposable. Indeed, any result proved using S and S^* can be generalised for any two subcategories S and S', not necessarily differing by just one indecomposable.

We now show how to construct an isomorphism

$$\varphi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \xrightarrow{\sim} \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*.$$

We will obtain φ from an isomorphism $\xi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)$ sending elements in N to elements in N^*. The generators for the subgroup N are given by differences of middle terms of the exchange triangles, see (4.1). The only diagonals whose exchange triangles are affected by the mutation of the indecomposable s are those that form one of the edges of the enclosing quadrilateral of s and s^*, see Figure 4.2 where the diagonals whose associated exchange triangles are affected by mutating s are i, j, k, l.

Excluding s, there are clearly at most four diagonals whose exchange triangles are affected by the mutation, and we note here that it is also possible for less than four to be affected. This is since any edge of the enclosing quadrilateral of s and s^* could sit on the boundary of P, in which case it is associated to the zero object in C, or any edge could be associated to an element of $\operatorname{indec}\mathsf{R}$. In the latter case, there is no generator of N associated to the indecomposable in R since the generators for N are constructed using the middle terms of mutation triangles coming from elements in $\operatorname{indec}\mathsf{S}$, see (4.1). Since there are at most four diagonals whose exchange triangles are affected, there are at most four generators that are affected by the mutation. We will therefore consider the change to four generators in N, and verify later in Remark 4.2.4 why this suffices for the case when less than four generators are affected.

Consider the two diagrams in Figure 4.2 and assume that each is a collection of diagonals sat in P, identified with indecomposables in $\operatorname{indec}\mathsf{S}$ and $\operatorname{indec}\mathsf{S}^*$, respectively.

We observe that s is the diagonal that we remove and replace by its mutation, and i, j, k, l is its enclosing quadrilateral. By definiton of a triangulation, for a diagonal of the quadrilateral to be nonzero (i.e. not an edge of P), there must be at least one triangle on

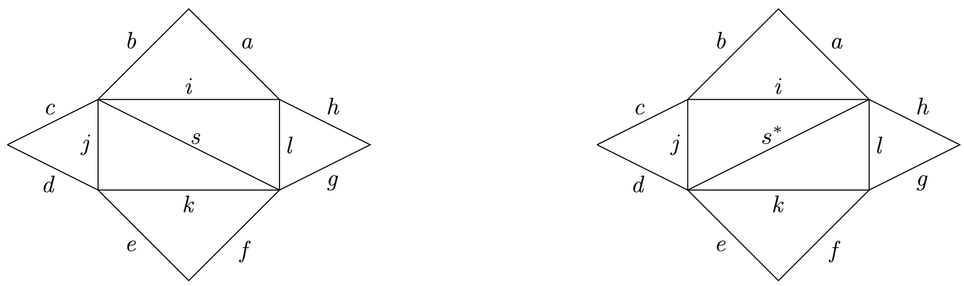


Figure 4.2: Collection of diagonals corresponding to indecomposables in indec S and indec S^*.

its exterior side. These triangles are those added to the diagonals constituting the quadrilateral i, j, k, l. We calculate some exchange triangles and generators for the subgroups N and N^*.

Exchange triangles in T:

$$\begin{array}{cc}
\vdots & \vdots \\
i \longrightarrow b \oplus l \longrightarrow i^* & i^* \longrightarrow a \oplus s \longrightarrow i \\
j \longrightarrow s \oplus d \longrightarrow j^* & j^* \longrightarrow c \oplus k \longrightarrow j \\
k \longrightarrow j \oplus f \longrightarrow k^* & k^* \longrightarrow s \oplus e \longrightarrow k \\
l \longrightarrow h \oplus s \longrightarrow l^* & l^* \longrightarrow i \oplus g \longrightarrow l \\
s \longrightarrow i \oplus k \longrightarrow s^* & s^* \longrightarrow j \oplus l \longrightarrow s.
\end{array}$$

Now, using the definition of the subgroup N of $K_0^{\mathrm{split}}(\mathsf{T})$ from Equation (4.1), we have the following generators in N, where $[a]$ denotes the K_0^{split}-class of a. Note that we also use the relation $[a \oplus b] = [a] + [b]$ for objects a, b.

$$\begin{aligned}
n(i) &= [a] + [s] - [b] - [l], & n(j) &= [c] + [k] - [s] - [d], \\
n(k) &= [s] + [e] - [j] - [f], & n(l) &= [i] + [g] - [h] - [s] \\
n(s) &= [j] + [l] - [i] - [k].
\end{aligned} \tag{4.5}$$

We have omitted the other exchange triangles and subgroup generators as they are identical for T and T^*.

T^* gives the following exchange triangles. Here we denote the mutation of i as $\bar{i^*}$ (and similarly for the other diagonals of the enclosing quadrilateral for s), as they differ from those in the exchange triangles for T.

$$\begin{array}{ccc}
\vdots & & \vdots \\
i \longrightarrow b \oplus s^* \longrightarrow \bar{i^*} & & \bar{i^*} \longrightarrow j \oplus a \longrightarrow i \\
j \longrightarrow i \oplus d \longrightarrow \bar{j^*} & & \bar{j^*} \longrightarrow c \oplus s^* \longrightarrow j \\
k \longrightarrow f \oplus s^* \longrightarrow \bar{k^*} & & \bar{k^*} \longrightarrow e \oplus l \longrightarrow k \\
l \longrightarrow k \oplus h \longrightarrow \bar{l^*} & & \bar{l^*} \longrightarrow g \oplus s^* \longrightarrow l \\
s^* \longrightarrow j \oplus l \longrightarrow s & & s \longrightarrow i \oplus k \longrightarrow s^*.
\end{array}$$

Hence, N^* has the following generators:

$$\begin{aligned}
n^*(i) &= [j] + [a] - [b] - [s^*], & n^*(j) &= [c] + [s^*] - [i] - [d], \\
n^*(k) &= [e] + [l] - [f] - [s^*], & n^*(l) &= [g] + [s^*] - [k] - [h] \\
n^*(s^*) &= [i] + [k] - [j] - [l].
\end{aligned} \tag{4.6}$$

Let us now denote the exchange triangles for s as follows:

$$s \to A' \to s^* \to \Sigma s \ , \ s^* \to A \to s \to \Sigma s^*. \tag{4.7}$$

Then, from [15, sec. 3], there are two homomorphisms $\phi_+ : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)$ and $\phi_- : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)$ defined by

$$\phi_+([s]) = [A] - [s^*] \ , \ \phi_-([s]) = [A'] - [s^*]$$

and fixing all other basis elements of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$. We may then define the map $\phi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)$ by

$$\phi([x]) = \begin{cases} \phi_+([x]) & \text{if } [x:s] \geq 0 \\ \phi_-([x]) & \text{if } [x:s] \leq 0, \end{cases} \tag{4.8}$$

where $[x : s]$ is the coefficient of $[s]$ in the expression for $[x]$. By [15, sec. 3, thm.], ϕ intertwines the indices with respect to T and T^*, that is,

$$\phi(\mathrm{ind}_{\mathsf{T}}(x)) = \mathrm{ind}_{\mathsf{T}^*}(x). \tag{4.9}$$

Now, using ϕ, we may construct the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \arrow[r, "Q"] \arrow[d, "\phi"] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \arrow[d, "\varphi"] \\
\operatorname{indec}\mathsf{C} \arrow[ur, "\mathrm{ind}_{\mathsf{T}}"] \arrow[dr, "\mathrm{ind}_{\mathsf{T}^*}"'] & & \\
 & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*) \arrow[r, "Q^*"'] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*.
\end{tikzcd} \tag{4.10}$$

Here, $\varphi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*$ is the homomorphism between the quotient groups, induced by ϕ_+. It is defined by

$$\varphi([x]+N) = \begin{cases} [A]-[s^*]+N^* & \text{if } x = s \\ [x]+N^* & \text{if } x \neq s. \end{cases}$$

We show in Proposition 4.2.2 that φ is well defined. By this construction, the square in Diagram (4.10) immediately commutes. Indeed,

$$\begin{aligned} Q^*\phi_+([s]) &= Q^*([A]-[s^*]) \\ &= [A]-[s^*]+N^* \\ &= \varphi Q([s]). \end{aligned}$$

Note that ϕ_+ and φ could have been replaced by ϕ_- and

$$\begin{aligned} \bar{\varphi} : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N &\longrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^* \\ [s]+N &\longmapsto [A']-[s^*]+N^*. \end{aligned}$$

We note later in Remark 4.2.4, Item 2 that in fact $\varphi = \bar{\varphi}$. The triangle in (4.10) commutes by Equation (4.9).

Proposition 4.2.2. *Assume that* $s \in \operatorname{indec}\mathsf{T}$ *sits in the exchange triangles in (4.7). Then, the linear maps* $\varphi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*$*, defined by*

$$\varphi([x]+N) = \begin{cases} [A]-[s^*]+N^* & \text{if } x = s \\ [x] & \text{if } x \neq s, \end{cases}$$

and $\varphi' : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^* \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$*, defined by*

$$\varphi'([x]+N^*) = \begin{cases} [A]-[s]+N & \text{if } x = s^* \\ [x] & \text{if } x \neq s^*, \end{cases}$$

are well defined homomorphisms.

Proof. In order to prove the proposition, we continue working with the diagonals in Figure 4.2, and note that $[A] = [j \oplus l] = [j] + [l]$ and $[A'] = [i \oplus k] = [i] + [k]$.

We define homomorphisms

$$\begin{aligned} \xi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) &\longrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*) \\ [s] &\longmapsto [A] - [s^*] = [j] + [l] - [s^*] \end{aligned}$$

and

$$\begin{aligned} \xi' : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*) &\longrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \\ [s^*] &\longmapsto [A] - [s] = [j] + [l] - [s], \end{aligned}$$

where all other basis elements are fixed by both homomorphisms. Note that the map ξ is just the map ϕ_+, defined earlier, and ξ' is the same definition, but in reverse. The following calculations would also work should we define the maps ξ and ξ' based on the map ϕ_-. We now check that ξ sends each generator of N in (4.5) to a linear combination of generators of N^* in (4.6). Then ξ induces the (well-defined) homomorphism φ.
For $n(i)$:

$$\begin{aligned} \xi\big([a] + [s] - [b] - [l]\big) &= [a] - [b] - [l] + [j] + [l] - [s^*] \\ &= [a] + [j] - [b] - [s^*] \\ &= n^*(i). \end{aligned}$$

For $n(j)$:

$$\begin{aligned} \xi\big([c] + [k] - [s] - [d]\big) &= [c] + [k] - [d] - [j] - [l] + [s^*] \\ &= \big([c] + [s^*] - [i] - [d]\big) + \big([i] + [k] - [j] - [l]\big) \\ &= n^*(j) + n^*(s^*). \end{aligned}$$

For $n(k)$:

$$\begin{aligned} \xi\big([s] + [e] - [j] - [f]\big) &= [j] + [l] - [s^*] + [e] - [j] - [f] \\ &= [l] + [e] - [f] - [s^*] \\ &= n^*(k). \end{aligned}$$

For $n(l)$:

$$\begin{aligned}\xi\big([i]+[g]-[h]-[s]\big) &= [i]+[g]-[h]-[j]-[l]+[s^*]\\ &= \big([g]+[s^*]-[k]-[h]\big)+\big([i]+[k]-[j]-[l]\big)\\ &= n^*(l)+n^*(s^*).\end{aligned}$$

There is nothing to check for generator $n(s)$ since ξ fixes all the basis elements in this generator, and so $\xi(n(s)) = -n^*(s^*)$. It can be checked in a similar way that ξ' is also a homomorphism sending elements of N^* to elements of N. Then ξ' induces the (well-defined) homomorphism φ'.

For $n^*(i)$:

$$\begin{aligned}\xi'([a]+[j]-[b]-[s^*]) &= [a]+[j]-[b]-[j]-[l]+[s]\\ &= [a]+[s]-[b]-[l]\\ &= n(i).\end{aligned}$$

For $n^*(j)$:

$$\begin{aligned}\xi'([c]+[s^*]-[i]-[d]) &= [c]-[i]-[d]+[j]+[l]-[s]\\ &= ([c]+[k]-[s]-[d])+([j]+[l]-[i]-[k])\\ &= n(j)+n(s).\end{aligned}$$

For $n^*(k)$:

$$\begin{aligned}\xi'([l]+[e]-[f]-[s^*]) &= [l]+[e]-[f]-[j]-[l]+[s]\\ &= [e]+[s]-[f]-[j]\\ &= n(k).\end{aligned}$$

For $n^*(l)$:

$$\begin{aligned}\xi'([g]+[s^*]-[k]-[h]) &= [g]-[k]-[h]+[j]+[l]-[s]\\ &= ([i]+[g]-[h]-[s])+([j]+[l]-[i]-[k])\\ &= n(l)+n(s).\end{aligned}$$

Again, there is nothing to check for $n^*(s^*)$ as ξ' fixes all basis elements in this generator and so $\xi'(n^*(s^*)) = -n(s)$.

□

Proposition 4.2.3. *The quotient group*

$$\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$$

depends only on the subcategory R*.*

Proof. Note that by Remark 4.2.1, it suffices to prove that the quotient group is invariant under the mutation of one indecomposable in the subcategory S. Therefore, we again consider the subcategories S and S^*, where $\operatorname{indec}\mathsf{S}^*$ is formed from $\operatorname{indec}\mathsf{S}$, as in Equation (4.2). This then gives two cluster tilting subcategories T and T^*, described in Equations (4.3) and (4.4), differing by one indecomposable. In order to show that the quotient group is independent of the choice of S, we will show that the homomorphism φ, defined in Propositon 4.2.2, is in fact an isomorphism, with φ' as its inverse.

This clearly only needs to be verified for $[s]+N$ and $[s^*]+N^*$. Indeed,

$$\begin{aligned}\varphi'\varphi([s]+N) &= \varphi'([j]+[l]-[s^*]+N^*)\\ &= [j]+[l]-[j]-[l]+[s]+N\\ &= [s]+N,\end{aligned}$$

$$\begin{aligned}\varphi\varphi'([s^*]+N^*) &= \varphi([j]+[l]-[s]+N)\\ &= [j]+[l]-[j]-[l]+[s^*]+N^*\\ &= [s^*]+N^*,\end{aligned}$$

as required.

□

Remark 4.2.4.

1. Notice that the proofs of Propositions 4.2.2 and 4.2.3 still apply if one or more of the diagonals in Figure 4.2 are either identified with the zero object in C or with an indecomposable in $\operatorname{indec}\mathsf{R}$. For $x \in \operatorname{indec}\mathsf{S}$ notice that either $\xi(n(x)) = n^*(x)$ or $\xi(n(x)) = n^*(x)+n^*(s^*)$, both of which are linear combinations of generators of N^*. Now, if $x \in \operatorname{indec}\mathsf{R}$ or if $x=0$, then the would-be generator $n^*(x)$ is missing from N^*, since a generator $n^*(x')$ of N^* is only defined when $x' \in \operatorname{indec}\mathsf{S}$. However, this is not an issue since the would-be generator $n(x)$ of N is also missing for the same reason. A similar statement also works in the case of ξ'. Hence, if any of the diagonals Figure 4.2 were indeed identified with either the zero object or an indecomposable in R, the the maps φ and φ' would still be well defined isomorphisms.

2. Additionally notice that φ can be redefined in the following way

$$\begin{aligned}\bar{\varphi} : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N &\longrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*\\ [s]+N &\longmapsto [A']-[s^*]+N^*,\end{aligned}$$

with obvious inverse. With a small calculation, and noting that $[A'] - [A] \in N^*$, one sees that in fact $\varphi = \bar{\varphi}$.

Theorem 4.2.5. *Let* S *and* S' *be two subcategories such that the subcategories* T *and* T' *defined by*

$$\operatorname{indec}\mathsf{T} = \operatorname{indec}\mathsf{S} \cup \operatorname{indec}\mathsf{R} \quad \text{and} \quad \operatorname{indec}\mathsf{T}' = \operatorname{indec}\mathsf{S}' \cup \operatorname{indec}\mathsf{R}$$

are cluster tilting. Then, from the map $\varepsilon : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to A$*, one may construct* $\varepsilon' : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N' \to A$ *such that*

$$\rho(c) = \rho'(c),$$

for each c *in* C*. Here,* ρ *(respectively* ρ'*) denotes the modified Caldero-Chapoton map calculated with respect to* R, S *and* ε *(respectively* R, S' *and* ε'*), see Section 4.1.*

Proof. Again, noting Remark 4.2.1, it suffices to show the above holds true for the subcategories S and S^*, differing by one indecomposable. The result in the theorem then follows. We remark also that the only parts of the formula for ρ that rely on the cluster tilting subcategory are α and β.

For an object c in C, recall that the definiton of α is given by

$$\alpha(c) = \varepsilon Q(\operatorname{ind}_{\mathsf{T}}(c)). \tag{4.11}$$

Again, consider the cluster tilting subcategories T and T^* and denote by $\operatorname{ind}_{\mathsf{T}}(c)$ and $\operatorname{ind}_{\mathsf{T}^*}(c)$ the index of c with respect to T and T^*, respectively. Denote by α (respectively α^*) calculating (4.11) with respect to ε, R and S (respectively $\varepsilon^*, \mathsf{R}$ and S^*), that is $\alpha^*(c) = \varepsilon^* Q^*(\operatorname{ind}_{\mathsf{T}^*}(c))$. Then, computing $\alpha(c)$ and $\alpha^*(c)$ can be represented by the diagram below:

$$\begin{tikzcd}
 & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \arrow[r, "Q"] \arrow[d, "\phi"'] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \arrow[d, "\varphi"] \arrow[dr, "\varepsilon"] & \\
\operatorname{indec}\mathsf{C} \arrow[ur, "\operatorname{ind}_{\mathsf{T}}"] \arrow[dr, "\operatorname{ind}_{\mathsf{T}^*}"'] & & & A \\
 & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*) \arrow[r, "Q^*"'] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^* \arrow[ur, "\varepsilon^*"'] &
\end{tikzcd} \tag{4.12}$$

Here, ϕ is the map defined in (4.8), and we should note that since it is not a linear map and $\operatorname{indec}\mathsf{C}$ is just a set, one should view the diagram as merely a diagram of sets. It needs to be shown that the diagram commutes.

It is known from Diagram (4.10) that the left triangle and centre square both commute.

Now, since φ is a bijection, we may define

$$\varepsilon^* = \varepsilon \circ \varphi^{-1},$$

making the right triangle commute. Thus, the whole diagram is commutative and $\alpha = \alpha^*$.

It now remains to show the same for β. Recall that for $e \in \mathrm{K}_0(\mathsf{fl}\,\mathsf{R})$, the map $\beta : \mathrm{K}_0(\mathsf{fl}\,\mathsf{R}) \to A$ is defined by

$$\beta(e) = \varepsilon\theta(e), \tag{4.13}$$

where θ is defined as the homomorphism making the square in (3.27) commute. Denote by β^* the computation of β with respect to $\varepsilon^*, \mathsf{R}$ and S^*, that is $\beta^*(e) = \varepsilon^*\theta^*(e)$. Then, computing β and β^* can be represented by the bottom face of Diagram (4.14). With the exception of the bottom face, the diagram is known to commute by (3.27) and (4.12). In order to show that $\beta = \beta^*$, we therefore need to show that this bottom face commutes; that is, we should show that $\theta\varphi = \theta^*$. It is enough to show this on each $[S_r] \in \mathrm{K}_0(\mathsf{fl}\,\mathsf{R})$, where $r \in \operatorname{indec}\mathsf{R}$.

$$\begin{tikzcd}
 & \mathrm{K}_0(\mathsf{fl}\,\mathsf{T}^*) \arrow[rr, "\bar\theta^*"] \arrow[dd, two heads, "\kappa^*"] & & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*) \arrow[dd, two heads, "Q^*"] & \\
\mathrm{K}_0(\mathsf{fl}\,\mathsf{T}) \arrow[rr, "\bar\theta"] \arrow[ddd, two heads, "\kappa"] & & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \arrow[ur, "\phi"] \arrow[ddd, two heads, "Q"] & & \\
 & \mathrm{K}_0(\mathsf{fl}\,\mathsf{R}) \arrow[rr, "\theta^*"] \arrow[ddl, equal] & & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^* \arrow[dr, "\varepsilon^*"] & \\
 & & & & A \\
\mathrm{K}_0(\mathsf{fl}\,\mathsf{R}) \arrow[rr, "\theta"] & & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \arrow[uur, "\varphi"] \arrow[urr, "\varepsilon"'] & &
\end{tikzcd} \tag{4.14}$$

For a given $c \in \operatorname{indec}\mathsf{C}$, consider $[\overline{G}c] \in \mathrm{K}_0(\mathsf{fl}\,\mathsf{T})$. Then,

$$\begin{aligned}
\varphi\theta\kappa([\overline{G}c]) &= \varphi Q\bar\theta([\overline{G}c]) \\
&\overset{(1)}{=} \varphi Q(-\mathrm{ind}_{\mathsf{T}}(c) - \mathrm{ind}_{\mathsf{T}}(\Sigma c)) \\
&= Q^*\phi(-\mathrm{ind}_{\mathsf{T}}(c) - \mathrm{ind}_{\mathsf{T}}(\Sigma c)) \\
&\overset{(2)}{=} Q^*(-\mathrm{ind}_{\mathsf{T}^*}(c) - \mathrm{ind}_{\mathsf{T}^*}(\Sigma c)) \\
&\overset{(3)}{=} Q^*\bar\theta^*([\overline{G}^*c]) \\
&= \theta^*\kappa^*([\overline{G}^*c]).
\end{aligned} \tag{4.15}$$

Here, (1) and (3) follow from [26, lem. 2.10(i)], whilst (2) holds by (4.9). All the other equalities in the manipulation follow from the commutativity of the front, back and side faces of Diagram (4.14). Notationally, note that any map above with a "$*$" is the obvious map taken with respect to R, S^* and T^*.

Now, we also know that

$$\kappa([\overline{G}c]) = [Gc] = \kappa^*[\overline{G}^*c],$$

hence, as a consequence of the manipulation in (4.15), we have

$$\varphi\theta([Gc]) = \theta^*([Gc]). \tag{4.16}$$

Now, let c be a diagonal in $\operatorname{indec}\mathsf{C}$ that crosses some $r \in \operatorname{indec}\mathsf{R}$, but does not cross any other indecomposables in R. Then, $Gc = S_r$ and Equation (4.16) becomes

$$\varphi\theta([S_r]) = \theta^*([S_r]).$$

This can be done for each $r \in \operatorname{indec}\mathsf{R}$, thus $\varphi\theta = \theta^*$, and so the bottom face of Diagram (4.14) commutes. Therefore,

$$\beta = \beta^*,$$

as required. □

4.3 Computation of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ in $\mathsf{C}(A_n)$.

In this section, we will prove a general formula, dependent on R, for the group $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$. We will do this by inducting on the number of indecomposables in R. We note that since R corresponds to a polygon dissection of P, it is actually splitting P into a series of "cells", where each cell is a subpolygon of P. The subcategory S then corresponds to a triangulation of each of these cells, providing a full triangulation of P corresponding to the cluster tilting subcategory T. For the remainder of this section, we will say that a cell is 'even', if it has an even number of vertices, and 'odd' if it has an odd number of vertices.

In order to talk more in depth about producing a general formula for the quotient group, we should first introduce the notion of "gluing" cells. If we have a polygon P equipped with a dissection R, we can say that increasing the number of indecomposables in R (which we denote by $|\mathsf{R}|$), is equivalent to gluing on a new cell to the existing polygon. We do this by identifying the glued edge with a new rigid indecomposable, hence forming a larger rigid subcategory, that we will denote by R'. More formally, from the cluster-tilting subcategory T, given by

$$\operatorname{indec}\mathsf{T} = \operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S}, \tag{4.17}$$

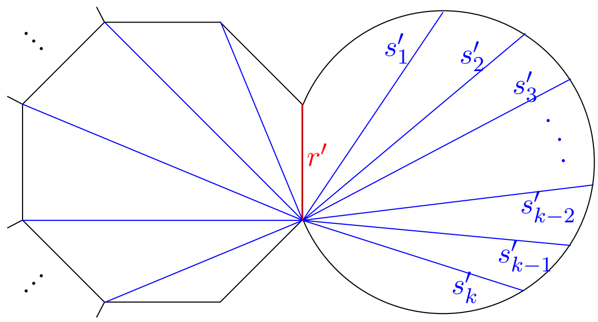


Figure 4.3: Gluing on a new cell and identifying the glued edge with a new rigid indecomposable r'. Proposition 4.2.3 allows us to arrange the internal diagonals of each cell in a fan.

of the cluster category C, we construct a larger cluster tilting subcategory

$$\operatorname{indec}\mathsf{T}' = \operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S} \cup \{r'\} \cup \operatorname{indec}\mathsf{S}', \tag{4.18}$$

of a larger cluster category C'. Here,

$$\operatorname{indec}\mathsf{R}' = \operatorname{indec}\mathsf{R} \cup \{r'\}, \tag{4.19}$$

is the new, larger rigid subcategory of C'. Also, $\operatorname{indec}\mathsf{S}'$ corresponds to those indecomposables associated with the diagonals in the triangulation of the new cell. From (4.17), we obtain $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, N and hence $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$, whilst from (4.18), we get the larger groups $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$, N' and $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$. For the free groups, the ranks can easily be calculated by counting the generators. We have

$$\begin{aligned} \operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})) &= |\mathsf{R}| + |\mathsf{S}|, \\ \operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')) &= |\mathsf{R}| + |\mathsf{S}| + |\mathsf{S}'| + 1. \end{aligned} \tag{4.20}$$

Here, we use the notation $|\ |$ to denote the number of isomorphism classes of indecomposables in that particular subcategory. For example, $|\mathsf{R}|$ denotes the number of indecomposables in R.

The process of gluing on a new cell is pictured in Figure 4.3. Here, the 8-gon on the left corresponds to the original cluster category, of which T is a cluster-tilting subcategory, and the polygon on the right is the glued cell. In the figure, blue diagonals correspond to indecomposables in S, and the red to an indecomposable in R. Note here that we have arranged the indecomposables in S in a fan, which allows us simpler computations later. Proposition 4.2.3 allows us to arrange them in this way.

There are two injections $\varphi : N \hookrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$ and $\varphi' : N' \hookrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$, as well as a canonical surjection $\kappa' : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}') \twoheadrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, defined by

$$\kappa'([t]) = \begin{cases} [t], & \text{if } t \in \operatorname{indec} \mathsf{T} \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that κ' induces a well-defined map

$$\nu' : N' \twoheadrightarrow N,$$

satisfying $\varphi \circ \nu' = \kappa' \circ \varphi'$. We should verify that $\kappa'\varphi'(N') \subseteq N$. Recall that for $s \in \operatorname{indec} \mathsf{S}$, we denote by $n(s)$ the generator of N associated to s. Also, for $s' \in \operatorname{indec} \mathsf{S}'$, denote by $n'(s')$ the generator of N' associated to s'. Notice that if $s \in \operatorname{indec} \mathsf{S}$, then either $n'(s)$ consists entirely of indecomposables in T, or $n'(s) = n(s) + r'$. In either case, since κ' sends the $\mathrm{K}_0^{\mathrm{split}}$-class of r' to zero and acts as the identity on the $\mathrm{K}_0^{\mathrm{split}}$-classes of all indecomposables in T, it is easy to see that $\kappa'\varphi'(n'(s)) = n(s)$.

On the other hand, if $s' \in \operatorname{indec} \mathsf{S}'$, then $n'(s')$ consists entirely of indecomposables outside T, and so $\kappa'\varphi'(n'(s')) = 0$. Thus, $\kappa'\varphi'(N') \subseteq N$. In fact this verification shows that,

$$\nu'(n'(s')) = \begin{cases} n(s') & \text{if } s' \in \operatorname{indec}\mathsf{S}, \\ 0 & \text{if } s' \in \operatorname{indec} \mathsf{S}', \end{cases} \tag{4.21}$$

and in particular, ν' is surjective. We then get a commutative square

$$\begin{tikzcd} N' \arrow[r, hook, "\varphi'"'] \arrow[d, two heads, "\nu'"'] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}') \arrow[d, two heads, "\kappa'"] \\ N \arrow[r, hook, "\varphi"'] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}). \end{tikzcd}$$

Taking the cokernels of the injections, we obtain a diagram of short exact sequences, completed by λ' using the short-5-lemma,

$$\begin{tikzcd} 0 \arrow[r] & N' \arrow[r, hook, "\varphi'"'] \arrow[d, two heads, "\nu'"] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}') \arrow[r, two heads] \arrow[d, two heads, "\kappa'"] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N' \arrow[r] \arrow[d, "\lambda'"] & 0 \\ 0 \arrow[r] & N \arrow[r, hook, "\varphi"'] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \arrow[r, two heads] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \arrow[r] & 0. \end{tikzcd}$$

Applying the Snake Lemma gives the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}\nu' \arrow[r] \arrow[d, hook] & F \arrow[r] \arrow[d, hook, "\mu"] & \operatorname{Ker}\lambda' \arrow[d, hook] \arrow[dddll, rounded corners] & \\
0 \arrow[r] & N' \arrow[r, hook, "\varphi'"] \arrow[d, two heads, "\nu'"] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}') \arrow[r, two heads] \arrow[d, two heads, "\kappa'"] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N' \arrow[r] \arrow[d, two heads, "\lambda'"] & 0 \\
0 \arrow[r] & N \arrow[r, hook, "\varphi"] \arrow[d] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \arrow[r, two heads] \arrow[d] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd} \tag{4.22}$$

As $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$ is the free group on $\operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S}$, and $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$ is the free group on $\operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S} \cup \{r'\} \cup \operatorname{indec}\mathsf{S}'$, while κ' kills the generators corresponding to $\{r'\} \cup \operatorname{indec}\mathsf{S}'$, we know that $\operatorname{Ker}\kappa'$ must be free on $\{r'\} \cup \operatorname{indec}\mathsf{S}'$, and so we denote $\operatorname{Ker}\kappa'$ by F in the above diagram. It is clear that

$$\operatorname{rank}(F) = |\mathsf{S}'| + 1. \tag{4.23}$$

Additionally, due to ν' and κ' being known surjections, their cokernels are zero. Adding to this that the Snake Lemma produces a long exact sequence yields that $\operatorname{Coker}\lambda' = 0$, and hence λ' is also a surjection. Note finally that the vertical sequences in Diagram (4.22) are also short exact.

Lemma 4.3.1. *$Ker\,\lambda'$ is cyclic.*

Proof. Since the top row of Diagram 4.22 forms a short exact sequence, we know that

$$\operatorname{Ker}\lambda' \cong F/\operatorname{Ker}\nu'.$$

We also know that $\operatorname{Ker}\nu'$ contains $n'(s')$ for each $s' \in \operatorname{indec}\mathsf{S}'$, see Equation (4.21). We show that $\operatorname{Ker}\lambda'$ is generated by one element in each of two cases; the case when the glued cell in Figure 4.3 is even, and the case when it is odd.

In the even case, k in Figure 4.3 must be odd, and so $k-1$ even. Computing $n'(s')$ for each $s' \in \operatorname{indec}\mathsf{S}'$ gives the elements

$$s_2' - r', s_3' - s_1', s_4' - s_2', \ldots, s_k' - s_{k-2}', s_{k-1}'.$$

It is easy to see from these relations that in $F/\operatorname{Ker}\nu'$,

$$r' + \operatorname{Ker}\nu' = s_2' + \operatorname{Ker}\nu' = s_4' + \operatorname{Ker}\nu' = \cdots = s_{k-1}' + \operatorname{Ker}\nu' = 0,$$

and

$$s'_i + \operatorname{Ker}\nu' = s'_j + \operatorname{Ker}\nu',$$

for any odd i, j. Thus, r' and all s'_i for i even become zero in the quotient, and all s'_i for i odd become equal. Thus, in the even case, $\operatorname{Ker}\lambda'$ has one generator, and is therefore cyclic.

In the odd case, k in Figure 4.3 must be even, and $k-1$ odd. The relations

$$s'_2 - r', s'_3 - s'_1, s'_4 - s'_2, \ldots, s'_k - s'_{k-2}, s'_{k-1},$$

show that in $\operatorname{Ker}\lambda'$,

$$s'_i + \operatorname{Ker}\nu' = 0,$$

for all i odd, and

$$r' + \operatorname{Ker}\nu' = s'_2 + \operatorname{Ker}\nu' = s'_4 + \operatorname{Ker}\nu' = \cdots = s'_k + \operatorname{Ker}\nu'.$$

Thus, in $\operatorname{Ker}\lambda'$, each s'_i for i odd becomes zero, and r' becomes equal to every s'_i for i even. Again, there is only one generator, and so $\operatorname{Ker}\lambda'$ is cyclic. □

Theorem 4.3.2. $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ *is the finitely generated free abelian group on n generators, where*

$$n = \begin{cases} |\mathsf{R}|+1, & \text{if each cell is even} \\ |\mathsf{R}|, & \text{otherwise.} \end{cases}$$

Before proving the theorem, we make the following important remark.

Remark 4.3.3. In the proof of the theorem, we use the fact that for a short exact sequence of abelian groups

$$0 \to A \to B \to C \to 0,$$

we have the property that

$$\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C).$$

This property follows from the well known facts that for a finitely generated abelian group X, we have $\operatorname{rank}(X) = \dim_{\mathbb{Q}}(\mathbb{Q}\otimes_{\mathbb{Z}} X)$, and that the functor $\mathbb{Q}\otimes_{\mathbb{Z}} -$ preserves the exactness of a short exact sequence. This reduces the problem to computing $\dim_{\mathbb{Q}}$ of vector spaces in a short exact sequence, for which we know the required equality holds.

Proof. We will set up the following notation that we will use for this proof. For a polygon P, we consider a triangulation made up of diagonals corresponding to indecomposables in the rigid subcategory R and the subcategory S of the cluster category C. We denote by $|\mathsf{R}|$ and $|\mathsf{S}|$ the number of indecomposables in these respective subcategories. Note also

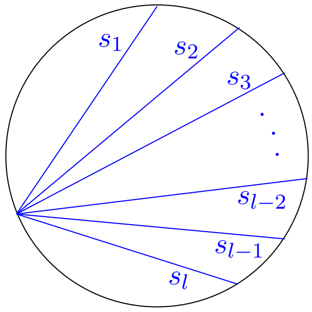


Figure 4.4: When $|\mathsf{R}| = 0$, the polygon P is the only cell. The blue diagonals again correspond to indecomposable objects in S.

that the additive closure of R and S is the cluster tilting subcategory $\mathsf{T} \subseteq \mathsf{C}$, as defined in (4.17). We then glue on a new cell to the polygon P, creating a larger polygon P'. The triangulation of P' is made up of diagonals associated to indecomposables in the subcategories R', S and S' of a larger cluster category C'. The additive closure of these subcategories is the cluster tilting subcategory $\mathsf{T}' \subseteq \mathsf{C}'$, as defined in (4.18). We denote by $|\mathsf{S}'|$ the number of indecomposables in S', whilst it is known that $|\mathsf{R}'| = |\mathsf{R}| + 1$, see (4.19).

Now, we induct on $|\mathsf{R}|$ in two cases, starting first with the case when not all cells have an even number of edges.

Case i. (Not all cells are even) Base: Let $|\mathsf{R}| = 0$, then the polygon P is the only cell, and we can assume that it has an odd number of vertices. Then, for some even l, the polygon is as in Figure 4.4. Calculating N produces:

$$N = \langle s_2, s_3 - s_1, s_4 - s_2, \dots, s_{l-1} - s_{l-3}, s_l - s_{l-2}, s_{l-1} \rangle.$$

We note that each generator (except s_2 and s_{l-1}) is of the form $s_i - s_j$, where $i = j + 2$. In particular, other than s_1 and s_l, each s_i appears in precisely two generators, and even s_i's appear with even s_i's, whilst odd s_i's appear with odd s_i's. Notice that s_2 can be deleted from the generator $s_4 - s_2$, leaving s_4 as a generator. Subsequently, s_4 can be deleted from the generator $s_6 - s_4$, leaving s_6 as a generator. Repeating this process leaves each s_i for i even as a generator for N. Now, since l is even, $l - 1$ must be odd, and repeating the same process starting by deleting s_{l-1} from the generator $s_{l-1} - s_{l-3}$ will show that

$$N = \langle s_1, s_2, \dots, s_{l-1}, s_l \rangle = \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}).$$

It then immediately follows that

$$\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N = 0,$$

as required.

Induction: Assume that $|\mathsf{R}| \geq 0$ and that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on $|\mathsf{R}|$ generators.

Step: Now assume that we glue on a new cell to the existing polygon, obtaining the new, larger subcategory indec T' from Equation (4.18). We aim to show that

$$\mathrm{rank}\big(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'\big) = |\mathsf{R}| + 1 = \mathrm{rank}\big(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N\big) + 1.$$

We first compute the rank of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ by repeatedly using the property from Remark 4.3.3 that if

$$0 \to A \to B \to C \to 0,$$

is a short exact sequence, then

$$\mathrm{rank}(B) = \mathrm{rank}(A) + \mathrm{rank}(C).$$

Firstly, we know by assumption that

$$\mathrm{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N) = |\mathsf{R}|.$$

Also, recall the ranks of the free groups $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$ and F from (4.20) and (4.23). Now, from the bottom short exact sequence in Diagram (4.22), we get that

$$\mathrm{rank}(N) = |\mathsf{S}|.$$

Now, since when we glue on a cell, we add at most $|\mathsf{S}'|$ generators to N in order to form N', we see that

$$\mathrm{rank}(N') \leq |\mathsf{S}| + |\mathsf{S}'|,$$

meaning that computing ranks in the left vertical short exact sequence, we obtain

$$\mathrm{rank}(\mathrm{Ker}\,\nu') \leq |\mathsf{S}'|.$$

Penultimately, the top short exact sequence reveals that $\mathrm{rank}(\mathrm{Ker}\,\lambda') \geq 1$, but since $\mathrm{Ker}\,\lambda'$ is cyclic by Lemma 4.3.1, we must have

$$\mathrm{rank}(\mathrm{Ker}\,\lambda') = 1.$$

Finally, calculating ranks in the right hand short exact sequence shows that

$$\mathrm{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N') = |\mathsf{R}| + 1,$$

as required.

It now remains to show that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ is free. Well, in the right hand short exact sequence, as $\operatorname{Ker}\lambda'$ is cyclic of rank one, it must be free. In addition, since we have assumed that the final object $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is also free, the sequence must be split exact. Thus, $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N' = \operatorname{Ker}\lambda' \oplus \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$, which are both free, and so $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ is also free.

Case ii. (All cells are even). *Base*: Assume that $|\mathsf{R}| = 0$, and that the only cell has an even number of edges. Then, for some odd l, we obtain the situation in Figure 4.4. Computing N produces

$$N = \big\langle s_2, s_3 - s_1, s_4 - s_2, \ldots, s_{l-1} - s_{l-3}, s_l - s_{l-2}, s_{l-1}\big\rangle.$$

Starting the "deleting" process by deleting s_2 from the generator $s_4 - s_2$, and continuing as in Case i, we realise that the process will terminate by deleting s_{l-3} from the generator $s_{l-1} - s_{l-3}$. This will leave two copies of the generator s_{l-1} in the presentation of N, meaning that one copy can be deleted. N then becomes:

$$N = \big\langle s_2, s_4, \ldots, s_{l-1}, s_3 - s_1, s_5 - s_3, \ldots, s_l - s_{l-2}\big\rangle.$$

Computing the quotient, it is easy to see that all generators s_i of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, for i even, will become zero in the quotient group $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$, whilst all generators s_j for j odd will be set equal. It follows that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on $|\mathsf{R}| + 1 = 1$ generators.

Induction: Assume that $|\mathsf{R}| \geq 0$ and that each of the $|\mathsf{R}| + 1$ cells has an even number of edges. Assume that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on $|\mathsf{R}| + 1$ generators.

Step: Now assume that we glue on an extra cell with an even number of edges and obtain the new, larger subcategory in Equation (4.18). We aim to show that gluing on this new even cell has increased the rank of the quotient group by one; that is

$$\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N') = |\mathsf{R}| + 2 = \operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N) + 1.$$

We show first that $\operatorname{rank}(N') \leq |\mathsf{S}| + |\mathsf{S}'| - 1$: We know by assumption that

$$\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N) = |\mathsf{R}| + 1,$$

and by construction that

$$\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})) = |\mathsf{R}| + |\mathsf{S}|.$$

Thus, since the bottom row of Diagram (4.22) is a short exact sequence, we see that

$$\operatorname{rank}(N) = |\mathsf{S}| - 1.$$

As $\operatorname{rank}(N)$ is strictly less than the number $|\mathsf{S}|$ of generators $n(s)$ of N, there must be a nontrivial linear relation between the n(s). Denote this relation by

$$\sum_{s \in \operatorname{indec} \mathsf{S}} \alpha(s) n(s) = 0, \tag{4.24}$$

where each $\alpha(s) \in \mathbb{Z}$ and not all $\alpha(s)$ are zero. To show that $\operatorname{rank}(N') \leq |\mathsf{S}| + |\mathsf{S}'| - 1$, we will show that there is also a nontrivial linear relation between the generators $n'(s')$ of N'.

Consider the generators $n'(s')$ of N' for each $s' \in \operatorname{indec} \mathsf{S}'$. These are the generators coming from the glued cell, as in Figure 4.3. They are given by the set

$$\{n'(s') | s' \in \operatorname{indec}\mathsf{S}'\} = \{s'_2 - r', s'_3 - s'_1, s'_4 - s'_2, \ldots, s'_{k-1} - s'_{k-3}, s'_k - s'_{k-2}, s'_{k-1}\},$$

where k is odd, and so $k-1$ even. Consider the following linear combination of these generators:

$$s'_{k-1} - (s'_{k-1} - s'_{k-3}) - \cdots - (s'_4 - s'_2) - (s'_2 - r') = r',$$

which is

$$n'(s'_k) - n'(s'_{k-2}) - \cdots - n'(s'_3) - n'(s'_1) = r'. \tag{4.25}$$

This shows that $r' \in N'$.

Now consider the following linear combination of generators of N' coming from the indecomposables in S:

$$x = \sum_{s \in \operatorname{indec} \mathsf{S}} \alpha(s) n'(s) \in N'. \tag{4.26}$$

Then, using (4.21) and Diagram (4.22), we know

$$\begin{aligned} \kappa'\varphi'(x) &= \varphi\nu'(x) \\ &= \sum_{s \in \operatorname{indec} \mathsf{S}} \alpha(s) \nu'(n'(s)) \\ &= \sum_{s \in \operatorname{indec} \mathsf{S}} \alpha(s) n(s) \\ &= 0, \end{aligned}$$

where the third = is by (4.21) and the final = is due to the relation in (4.24). This shows that $\varphi'(x) \in \operatorname{Ker} \kappa'$, and so there must be some $y \in F$ such that $\mu(y) = \varphi'(x)$. Now, φ' is an inclusion, and so $\varphi'(x) = x \in \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$. Consider $x \in \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$ in terms of the generators of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$; that is, in terms of the elements of $\operatorname{indec} \mathsf{R} \cup \operatorname{indec} \mathsf{S} \cup \{r'\} \cup \operatorname{indec} \mathsf{S}'$. Well, by construction, the coefficients of the generators coming from $\operatorname{indec}\mathsf{S}'$ must be zero as the summation in (4.26) for x is taken over $s \in \operatorname{indec}\mathsf{S}$. Thus, the unique linear combination of generators from $\operatorname{indec}\mathsf{S}' \cup \{r'\}$ that make up y must have zero coefficients for each $s' \in \operatorname{indec}\mathsf{S}'$. Therefore, for some $\alpha \in \mathbb{Z}$

$$y = \alpha r'.$$

Therefore,

$$\varphi'(x) = \mu(\alpha r'). \tag{4.27}$$

Now, since $x \in N'$ and $r' \in N'$, we know $x - \alpha r' \in N'$. Computing $\varphi'(x - \alpha r')$, we obtain

$$\begin{aligned}\varphi'(x - \alpha r') &= \varphi'(x) - \alpha\varphi'(r')\\ &= \alpha r' - \alpha r'\\ &= 0,\end{aligned}$$

where the second $=$ uses the equality in (4.27), as well as the fact that $\mu(\alpha r') = \alpha r'$ as μ is just an inclusion. As $\varphi'(x - \alpha r') = 0$, it follows that $x - \alpha r' = 0$ since φ' is an inclusion, and so we have found a linear relation of generators of N'. It now remains to show that this relation is nontrivial; that is, not all coefficients are zero. Well,

$$x - \alpha r' = 0,$$

and substituting using the equalities in (4.26) and (4.25) gives:

$$\sum_{s \in \operatorname{indec}\mathsf{S}} \alpha(s) n'(s) - \alpha\Big(n'(s'_k) - n'(s'_{k-2}) - \cdots - n'(s'_3) - n'(s'_1)\Big) = 0.$$

This is clearly a nontrivial relation as we know by assumption that at least one $\alpha(s)$ is nonzero. Since N' is generated by $|\mathsf{S}| + |\mathsf{S}'|$ generators, and we have found a relation between these generators, we can conclude that

$$\operatorname{rank}(N') \leq |\mathsf{S}| + |\mathsf{S}'| - 1.$$

We now compute $\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N')$ using the short exact sequences in Diagram (4.22) as in Case i. Note that we already know the following ranks:

$$\begin{aligned}
\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N) &= |\mathsf{R}| + 1, \\
\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})) &= |\mathsf{R}| + |\mathsf{S}|, \\
\operatorname{rank}(N) &= |\mathsf{S}| - 1, \\
\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')) &= |\mathsf{R}| + |\mathsf{S}| + |\mathsf{S}'| + 1, \\
\operatorname{rank}(N') &\leq |\mathsf{S}| + |\mathsf{S}'| - 1, \\
\operatorname{rank}(F) &= |\mathsf{S}'| + 1.
\end{aligned}$$

Now, using the left vertical short exact sequence, we know that

$$\operatorname{rank}(\operatorname{Ker}\nu') \leq |\mathsf{S}'|.$$

Then, using the top short exact sequence, we compute $\operatorname{rank}(\operatorname{Ker}\lambda') \geq 1$, but as we know $\operatorname{Ker}\lambda'$ to be cyclic by Lemma 4.3.1, we see that

$$\operatorname{rank}(\operatorname{Ker}\lambda') = 1.$$

Finally, using the right vertical short exact sequence we conclude that

$$\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N') = |\mathsf{R}| + 2,$$

as required.

It now remains to show that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ is free. Well, in the right hand short exact sequence, as $\operatorname{Ker}\lambda'$ is cyclic of rank one, it must be free. In addition, since we have assumed that the final object $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is also free, the sequence must be split exact. Thus, $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N' \cong \operatorname{Ker}\lambda' \oplus \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$, which are both free, and so $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ is also free. □

Chapter 5

Exchange Triangles in $\mathsf{C}(D_n)$

5.1 Properties of Triangles in $\mathsf{C}(D_n)$

Consider the category $\mathsf{C} = \mathsf{C}(D_n)$, the cluster category of Dynkin type D_n, where $n < \infty$. Recall from Section 2.5.2 the punctured polygon model associated to $\mathsf{C}(D_n)$. In this chapter, we will give a comprehensive list of exchange triangles in $\mathsf{C}(D_n)$, which can be seen in Theorem 5.1.3.

The following lemma will be used later in Section 5.3 in the proof of Theorem 5.1.3.

Lemma 5.1.1. *For $a, a^* \in \operatorname{indec} \mathsf{C}$, let*

$$a \to e \to a^*, \tag{5.1}$$

be a triangle in C, and let t be a diagonal in P such that t crosses a direct summand of e. Then, t crosses at least one of a or a^*.

Proof. Applying $\operatorname{Ext}^1_{\mathsf{C}}(t, -)$ to the triangle in (5.1) produces the following exact sequence:

$$\operatorname{Ext}^1_{\mathsf{C}}(t, a) \to \operatorname{Ext}^1_{\mathsf{C}}(t, e) \to \operatorname{Ext}^1_{\mathsf{C}}(t, a^*). \tag{5.2}$$

Since each $\operatorname{Ext}^1_{\mathsf{C}}$ space is nonzero precisely when the two terms cross, we know that if t crosses a direct summand of e, then $\operatorname{Ext}^1_{\mathsf{C}}(t, e) \neq 0$. By the exactness of the sequence in (5.2), it follows that at least one of $\operatorname{Ext}^1_{\mathsf{C}}(t, a)$ or $\operatorname{Ext}^1_{\mathsf{C}}(t, a^*)$ must also be nonzero, and so t must cross at least one of a or a^*. $\square$

Lemma 5.1.2. *Let $a, a^* \in \operatorname{indec} \mathsf{C}$ satisfy that $\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathsf{C}}(a, a^*) = 1$. Let $a \to e \to a^* \xrightarrow{\delta} \Sigma a$ be the ensuing nonsplit triangle. Then, $\operatorname{Ext}^1_{\mathsf{C}}(a, e) = \operatorname{Ext}^1_{\mathsf{C}}(a^*, e) = 0$.*

Proof. Rolling the nonsplit triangles produces the triangle,

$$a^* \xrightarrow{\delta} \Sigma a \longrightarrow \Sigma e \longrightarrow \Sigma a^*,$$

and then applying the functor $\mathrm{Hom}_{\mathsf{C}}(a^*, -)$ produces the following long exact sequence:

$$\mathrm{Hom}_{\mathsf{C}}(a^*, a^*) \xrightarrow{\delta_*} \mathrm{Hom}_{\mathsf{C}}(a^*, \Sigma a) \xrightarrow{\alpha} \mathrm{Hom}_{\mathsf{C}}(a^*, \Sigma e) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(a^*, \Sigma a^*). \tag{5.3}$$

Since a^* does not cross itself, it is known immediately that $\mathrm{Hom}_{\mathsf{C}}(a^*, \Sigma a^*) = 0$. Now, since $\delta \neq 0$, the induced map δ_* must also be nonzero. Adding to this that $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathsf{C}}(a^*, \Sigma a) = 1$, the map δ_* must be surjective. Due to the exactness of the sequence in (5.3), this tells us that $\alpha = 0$, and so $\mathrm{Hom}_{\mathsf{C}}(a^*, \Sigma e) = 0$, as required.

We may prove the second equality in a similar fashion. Rolling the nonsplit triangle produces the following triangle:

$$\Sigma^{-1}a \longrightarrow \Sigma^{-1}e \longrightarrow \Sigma^{-1}a^* \xrightarrow{-\Sigma^{-1}\delta} a.$$

Then, applying the contravariant functor $\mathrm{Hom}_{\mathsf{C}}(-, a)$ produces the following long exact sequence:

$$\mathrm{Hom}_{\mathsf{C}}(a, a) \xrightarrow{-\Sigma^{-1}\delta^*} \mathrm{Hom}_{\mathsf{C}}(\Sigma^{-1}a^*, a) \xrightarrow{\beta} \mathrm{Hom}_{\mathsf{C}}(\Sigma^{-1}e, a) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(\Sigma^{-1}a, a). \tag{5.4}$$

Noting that $\mathrm{Hom}_{\mathsf{C}}(a, \Sigma a) = 0$, we know that $\mathrm{Hom}_{\mathsf{C}}(\Sigma^{-1}a, a) = 0$. Now, as $\delta \neq 0$, we must have $-\Sigma^{-1}\delta \neq 0$ and so the induced map $-\Sigma^{-1}\delta^*$ must also be nonzero. Since $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathsf{C}}(a^*, \Sigma a) = 1$, we know that $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathsf{C}}(\Sigma^{-1}a^*, a) = 1$, and so $-\Sigma^{-1}\delta^*$ must be surjective. Due to the exactness of the sequence in (5.4), this tells us that $\beta = 0$ and so $\mathrm{Hom}_{\mathsf{C}}(\Sigma^{-1}e, a) = 0$. It then follows immediately that $\mathrm{Hom}_{\mathsf{C}}(e, \Sigma a) = 0$, and since $\mathsf{C}(D_n)$ is 2-Calabi-Yau, we know that $\mathrm{Hom}_{\mathsf{C}}(a, \Sigma e) = 0$, as required. □

The following theorem gives a full construction of all exchange triangles in $\mathsf{C}(D_n)$. These exchange triangles are nonsplit extensions between indecomposables that have a one dimensional $\mathrm{Ext}^1_{\mathsf{C}}$ space between them. Section 5.3 provides a full proof to the theorem.

Theorem 5.1.3. *A full list of exchange triangles in $\mathsf{C}(D_n)$ is given as follows:*

1. *Figure 5.1,*
$$a \to A_2 \oplus A_3 \to a^* \ , \ a^* \to A_1 \oplus A_4 \oplus A_5 \to a.$$
2. *Figure 5.2,*
$$b \to A_1 \to b^* \ , \ b^* \to A_2 \to b.$$
3. *Figure 5.3,*
$$f \to A_2 \oplus A_4 \to f^* \ , \ f^* \to A_1 \oplus A_3 \to f.$$
4. *Figure 5.4,*
$$h \to A_1 \oplus A_3 \to h^* \ , \ h^* \to A_2 \oplus A_4 \to h.$$

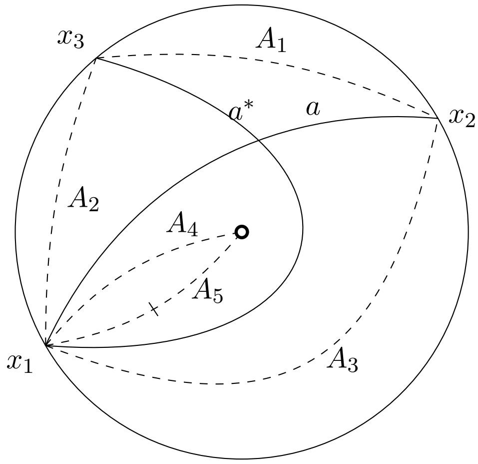


Figure 5.1: Theorem 5.1.3, Case 1. The exchange triangles for a and a^* are: $a \to A_2 \oplus A_3 \to a^*$ and $a^* \to A_1 \oplus A_4 \oplus A_5 \to a$.

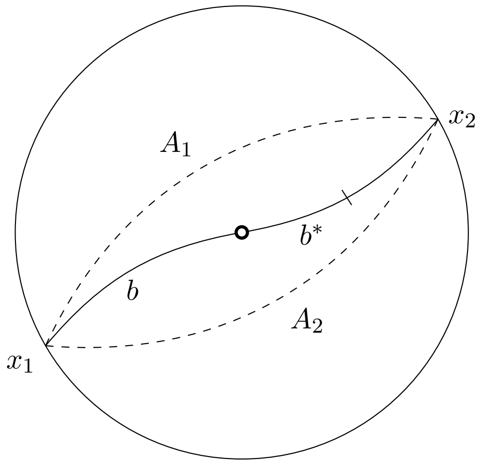


Figure 5.2: Theorem 5.1.3, Case 2. The exchange triangles for b and b^* are: $b \to A_1 \to b^*$ and $b^* \to A_2 \to b$.

5.2 Nonsplit triangles in C

In this section, we will prove the following useful lemmas, which will be heavily relied upon in order to compute exchange triangles in $\mathsf{C}(D_n)$, and hence prove Theorem 5.1.3, in the following section. We will set up this section using a more general category C, before reverting back to $\mathsf{C}(D_n)$ in the following section. Here, assume that C is $\mathbb{C}$-linear, Hom-finite and Krull-Schmidt.

In these lemmas, note that if a particular map is given by an $n \times m$ matrix, then we refer to each entry in the matrix as a component of that map.

Lemma 5.2.1. *Let a^* be an indecomposable in C, and assume that for some object $a \in$*

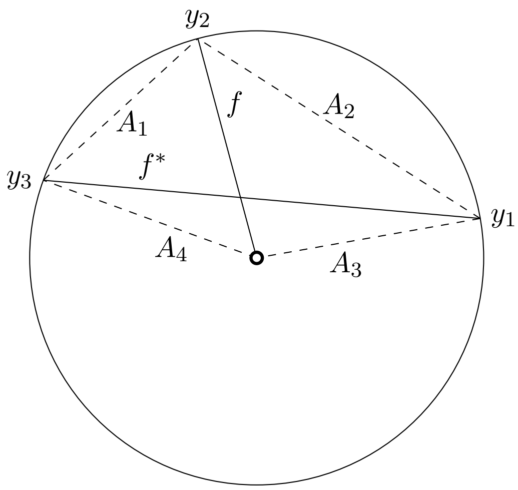


Figure 5.3: Theorem 5.1.3, Case 3. The exchange triangles for f and f^* are: $f \to A_2 \oplus A_4 \to f^*$ and $f^* \to A_1 \oplus A_3 \to f$.

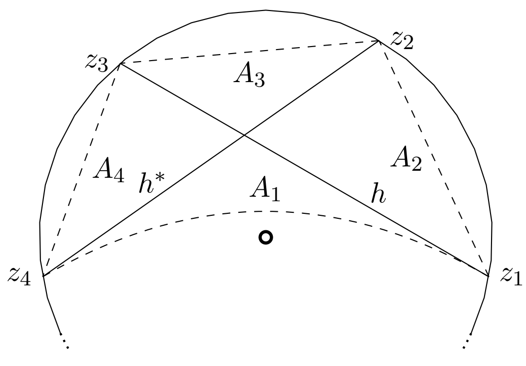


Figure 5.4: Theorem 5.1.3, Case 4. The exchange triangles for h and h^* are: $h \to A_1 \oplus A_3 \to h^*$ and $h^* \to A_2 \oplus A_4 \to h$.

obj C, *we have the following nonsplit triangle:*

$$a \xrightarrow{\alpha} e \xrightarrow{\varepsilon} a^* \xrightarrow{\alpha^*} \Sigma a. \tag{5.5}$$

That is, $\alpha^* \neq 0$. *Then, each row of the matrix* α *has a nonzero entry.*

Proof. The claim is empty if $e = 0$ and easy if e is indecomposable, so assume that for some indecomposable e_1 and some nonzero object e_2, that $e = e_1 \oplus e_2$. Then, we can write the triangle in (5.5) as

$$a \xrightarrow{\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)} e_1 \oplus e_2 \xrightarrow{(\,\varepsilon_1\ \varepsilon_2\,)} a^* \xrightarrow{\alpha^*} \Sigma a. \tag{5.6}$$

Assume $\alpha_1 = 0$, that is, $\alpha = \left(\begin{smallmatrix}0\\ \alpha_2\end{smallmatrix}\right)$. Then, we have the following commutative square:

$$\begin{tikzcd}
a \arrow[r, "\alpha_2"] \arrow[d, equal] & e_2 \arrow[d, "\left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right)"] \\
a \arrow[r, "\left(\begin{smallmatrix}0\\ \alpha_2\end{smallmatrix}\right)"'] & e_1 \oplus e_2.
\end{tikzcd}$$

By [38, prop. 1.4.6], we may apply the octahedral axiom using triangle (5.6), as well as the following triangles:

$$e_2 \xrightarrow{\left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right)} e_1 \oplus e_2 \xrightarrow{(\,1\ 0\,)} e_1 \xrightarrow{0} \Sigma e_2, \tag{5.7}$$

$$a \xrightarrow{\alpha_2} e_2 \longrightarrow b \longrightarrow \Sigma a. \tag{5.8}$$

Here, b is the mapping cone of α_2. The commutative square then gives rise to the following commutative diagram, where each row and each column is a triangle:

$$\begin{tikzcd}
a \arrow[r, "\alpha_2"] \arrow[d, equal, "1"'] & e_2 \arrow[r] \arrow[d, "\left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right)"] & b \arrow[r] \arrow[d] & \Sigma a \arrow[d, equal] \\
a \arrow[r, "\left(\begin{smallmatrix}0\\ \alpha_2\end{smallmatrix}\right)"] \arrow[d] & e_1 \oplus e_2 \arrow[r, "(\,\varepsilon_1\ \varepsilon_2\,)"] \arrow[d, "(\,1\ 0\,)"] & a^* \arrow[r, "\alpha^*"] \arrow[d] & \Sigma a \arrow[d] \\
0 \arrow[r] \arrow[d] & e_1 \arrow[r, equal, "1"] \arrow[d, "0"] & e_1 \arrow[r] \arrow[d, "\beta"] & 0 \arrow[d] \\
\Sigma a \arrow[r] & \Sigma e_2 \arrow[r] & \Sigma b \arrow[r] & \Sigma^2 a.
\end{tikzcd} \tag{5.9}$$

Now, due to the triangle in (5.7) being split, the anticlockwise composition in the bottom centre square of Diagram (5.9) is zero. Due to the commutativity of the square, the clockwise composition is also zero, and we conclude that $\beta = 0$, as the other morphism in the composition is the identity. Now, since $\beta = 0$, the triangle

$$b \longrightarrow a^* \longrightarrow e_1 \xrightarrow{\beta} \Sigma b$$

is split, and hence $a^* \cong b \oplus e_1$. Since a^* is an indecomposable, we see that $b = 0$, and therefore the triangle in (5.8) becomes

$$a \xrightarrow{\alpha_2} e_2 \longrightarrow 0 \longrightarrow \Sigma a.$$

As the mapping cone of α_2 is zero, α_2 must be an isomorphism. Using the rolling axiom on the triangle in (5.6), we obtain the triangle

$$\Sigma^{-1}a^* \xrightarrow{-\Sigma^{-1}a^*} a \xrightarrow{\left(\begin{smallmatrix} 0 \\ \alpha_2 \end{smallmatrix}\right)} e_1 \oplus e_2 \xrightarrow{(\,\varepsilon_1\ \varepsilon_2\,)} a^*,$$

and since the composition of two consecutive morphisms in a triangle is zero, we see that

$$\left(\begin{smallmatrix} 0 \\ \alpha_2 \end{smallmatrix}\right) \circ \Sigma^{-1}\alpha^* = 0,$$

and hence

$$\alpha_2 \circ \Sigma^{-1}\alpha^* = 0.$$

Since α_2 is an isomorphism, this must imply that

$$\Sigma^{-1}\alpha^* = 0,$$

and hence

$$\alpha^* = 0,$$

which is a contradiction to the triangle in (5.5) being nonsplit.

□

Lemma 5.2.2. *Let a be an indecomposable in C, and assume that for some object $a^* \in$* $\operatorname{obj}\mathsf{C}$*, we have the following nonsplit triangle:*

$$a \xrightarrow{\alpha} e \xrightarrow{\varepsilon} a^* \xrightarrow{\alpha^*} \Sigma a. \tag{5.10}$$

That is, $\alpha^* \neq 0$. Then, each column of the matrix ε has a nonzero entry.

Proof. The claim is empty if $e = 0$ and easy if e is indecomposable, so assume $e = e_1 \oplus e_2$, where $e_1 \in \operatorname{indec}\mathsf{C}$ and e_2 is a nonzero object of C. Then, the triangle in (5.10) can be rewritten as

$$a \xrightarrow{\left(\begin{smallmatrix} \alpha_1 \\ \alpha_2 \end{smallmatrix}\right)} e_1 \oplus e_2 \xrightarrow{(\,\varepsilon_1\ \varepsilon_2\,)} a^* \xrightarrow{\alpha^*} \Sigma a. \tag{5.11}$$

Assume that $\varepsilon_1 = 0$. Then, we can factorise $(0\ \varepsilon_2)$ using the following commutative square:

$$\begin{tikzcd}
e_1 \oplus e_2 \arrow[r, "(0\ \varepsilon_2)"] \arrow[d, "(0\ 1)"'] & a^* \arrow[d, equal] \\
e_2 \arrow[r, "\varepsilon_2"'] & a^*.
\end{tikzcd}$$

Now, consider the triangle in (5.11), as well as the following two triangles:

$$e_1 \xrightarrow{\binom{1}{0}} e_1 \oplus e_2 \xrightarrow{(0\ 1)} e_2 \xrightarrow{0} \Sigma e_1 \tag{5.12}$$

and

$$c \longrightarrow e_2 \xrightarrow{\varepsilon_2} a^* \longrightarrow \Sigma c, \tag{5.13}$$

where c is some unknown object of C. By [38, prop. 1.4.6], we may apply the octahedral axiom to these three triangles. In the following diagram, we apply the rolling axiom to each triangle, before applying the octahedral axiom.

$$\begin{tikzcd}
\Sigma^{-2}a^* \arrow[r] \arrow[d] & \Sigma^{-1}c \arrow[r] \arrow[d, "\gamma"] & \Sigma^{-1}e_1 \arrow[r] \arrow[d, "0"] & \Sigma^{-1}a^* \arrow[d] \\
0 \arrow[r] \arrow[d] & e_1 \arrow[r, equal] \arrow[d] & e_1 \arrow[r] \arrow[d, "\binom{1}{0}"] & 0 \arrow[d] \\
\Sigma^{-1}a^* \arrow[r] \arrow[d, equal] & a \arrow[r, "\binom{\alpha_1}{\alpha_2}"] \arrow[d] & e_1 \oplus e_2 \arrow[r, "(0\ \varepsilon_2)"] \arrow[d, "(0\ 1)"] & a^* \arrow[d, equal] \\
\Sigma^{-1}a^* \arrow[r] & c \arrow[r] & e_2 \arrow[r] & a^*.
\end{tikzcd} \tag{5.14}$$

Now, since the third morphism in the triangle in (5.12) is zero, it follows that the clockwise composition in the top centre square of Diagram (5.14) is zero. Due to the commutativity of the square, the anticlockwise composition is also zero, and since one of these composing morphisms is the identity, it is immediate that $\gamma = 0$. Hence,

$$\Sigma^{-1}c \xrightarrow{0} e_1 \longrightarrow a \longrightarrow c$$

is split, and rolling this triangle, we see that

$$e_1 \longrightarrow a \longrightarrow c \xrightarrow{0} \Sigma^{-1}c$$

is also split. It follows that $a \cong e_1 \oplus c$, and since $a \in \operatorname{indec} \mathsf{C}$, we can conclude that $c = 0$.

Hence, the triangle in (5.13) becomes

$$0 \longrightarrow e_2 \xrightarrow{\varepsilon_2} a^* \longrightarrow 0,$$

and thus ε_2 is an isomorphism. Since composing two morphisms in a triangle is zero, we see from the triangle in (5.11) that

$$\alpha^* \circ (0\ \varepsilon_2) = 0,$$

and hence

$$\alpha^* \circ \varepsilon_2 = 0.$$

Since ε_2 is an isomorphism, we see that $\alpha^* = 0$, a contradiction since the triangle in (5.11) is nonsplit.

$\square$

Lemma 5.2.3. *For $a, a^* \in \operatorname{indec} \mathsf{C}$, let*

$$a \xrightarrow{\alpha} e \longrightarrow a^* \tag{5.15}$$

be a nonsplit triangle in C. Also, assume that if e_i is an indecomposable summand of e, then $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathsf{C}}(a, e_i) = 1$. Then, e has no repeated indecomposable summands.

Proof. Assume $e = e_1 \oplus e_1 \oplus e_2$, where $e_1 \in \operatorname{indec} \mathsf{C}$. Then, the triangle in (5.15) can be written as

$$a \xrightarrow{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}} e \longrightarrow a^*.$$

By Lemma 5.2.1, we know that α_1 and α_2 are both nonzero. Additionally, since $\mathrm{Hom}_{\mathsf{C}}(a, e_i)$ is one dimensional for $i \in \{1, 2\}$, we know that $\alpha_1 = s \cdot \alpha_2$ for some $s \in \mathbb{C}$.

Now, consider the following square:

$$\begin{tikzcd} a \arrow[r, "\begin{pmatrix} \alpha_1 \\ 0 \\ \alpha_3 \end{pmatrix}"] \arrow[d, equal] & e_1 \oplus e_1 \oplus e_2 \arrow[d, "\delta"] \\ a \arrow[r, "\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}"'] & e_1 \oplus e_1 \oplus e_2, \end{tikzcd} \tag{5.16}$$

where $\delta = \left(\begin{smallmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Note that it commutes since

$$\begin{pmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \circ \begin{pmatrix} \alpha_1 \\ 0 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ s \cdot \alpha_1 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

Observe also that δ is an isomorphism and its inverse is $\left(\begin{smallmatrix} 1 & 0 & 0 \\ -s & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. By the axioms of a triangulated category, we may complete Diagram (5.16) to the following morphism of triangles:

$$\begin{tikzcd} a \arrow[r, "\left(\begin{smallmatrix} \alpha_1 \\ 0 \\ \alpha_3 \end{smallmatrix}\right)"] \arrow[d, equal] & e_1 \oplus e_1 \oplus e_2 \arrow[r] \arrow[d, "\delta"] & a^* \arrow[d, "\varphi"] \\ a \arrow[r, "\left(\begin{smallmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{smallmatrix}\right)"'] & e_1 \oplus e_1 \oplus e_3 \arrow[r] & a^*. \end{tikzcd} \tag{5.17}$$

Then, by the triangulated-5-lemma (see [23, prop. 1.1 (c)]), since δ and the identity are isomorphisms, we know that φ must be an isomorphism as well. Hence, Diagram (5.17) is an isomorphism of triangles, which leads to a contradiction to Lemma 5.2.1 since the top triangle has a zero component in the first map.

□

Lemma 5.2.4. *Let $a, a^* \in \operatorname{indec} \mathsf{C}$ and let*

$$a \xrightarrow{\alpha} e \xrightarrow{\varepsilon} a^* \longrightarrow \Sigma a$$

be a nonsplit triangle in C. Then,

$$\operatorname{Ext}^1_{\mathsf{C}}(\Sigma a, e_i) \neq 0 \quad \textit{and} \quad \operatorname{Ext}^1_{\mathsf{C}}(e_i, \Sigma^{-1} a^*) \neq 0,$$

for each indecomposable summand e_i of e.

Proof. Since each component of α and ε are nonzero by Lemmas 5.2.1 and 5.2.2, we know that if e_i is an indecomposable summand of e, then

$$\operatorname{Hom}(a, e_i) \neq 0 \quad \text{and} \quad \operatorname{Hom}(e_i, a^*) \neq 0.$$

This is true if and only if

$$\operatorname{Hom}\big(\Sigma^{-1}(\Sigma a), e_i\big) \neq 0 \quad \text{and} \quad \operatorname{Hom}\big(e_i, \Sigma(\Sigma^{-1} a^*)\big) \neq 0,$$

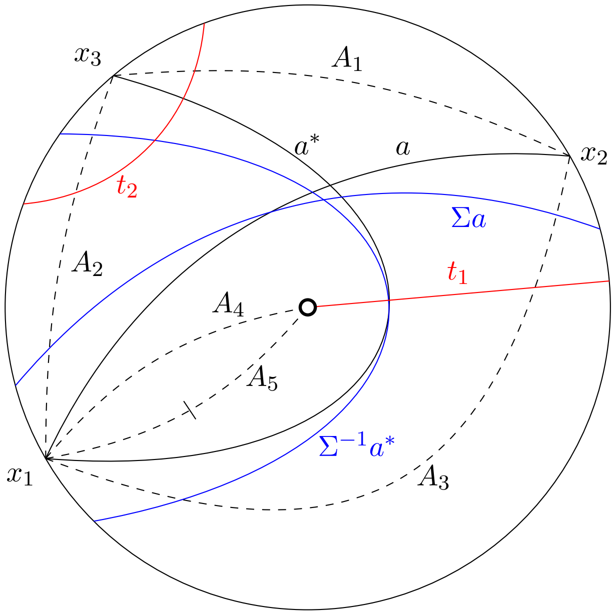


Figure 5.5: Calculating the exchange triangle $a \to e \to a^*$.

which holds if and only if

$$\mathrm{Ext}^1(\Sigma a, e_i) \neq 0 \quad \text{and} \quad \mathrm{Ext}^1(e_i, \Sigma^{-1}a^*) \neq 0,$$

as required. $\square$

5.3 Exchange Triangles in Type D_n

We consider all possible pairs of diagonals in P with crossing number one, and then compute their exchange triangles. The cases follow the statements of Theorem 5.1.3, of which this section provides a full proof.

Case 1. (Figure 5.1, exchange triangles for a and a^*.)

In this case, we prove Statement 1 of Theorem 5.1.3. To compute the exchange triangles for a, we will now refer to Figures 5.5 and 5.6, which are more detailed versions of Figure 5.1 from the formulation of the theorem.

Since a and a^* intersect each other once in the interior of P, their crossing number is one, and so $\dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathsf{C}}(a, a^*) = \dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathsf{C}}(a^*, a) = 1$. Therefore, a fits in an exchange triangle of the form

$$a \xrightarrow{\xi} e \xrightarrow{\zeta} a^*, \tag{5.18}$$

where e is the object that we wish to compute. Notice first that the crossing between a and a^* dissects P in to four "subcells"; three peripheral cells enclosed by a, a^* and the edge of P, and one central subcell containing the puncture. By Lemma 5.1.2, we know that $\mathrm{Ext}^1_{\mathsf{C}}(a, e) = \mathrm{Ext}^1_{\mathsf{C}}(a^*, e) = 0$, and so each indecomposable summand e_i of e must be contained fully within one of the subcells. This leaves the diagonals $A_1, \ldots, A_5$ from Figure 5.5 as the only candidates for summands of e_i. This is true since any other diagonal contained fully within one of the subcells intersects a diagonal t that intersects neither a nor a^*, contradicting Lemma 5.1.1. Note that one such diagonal t can easily be drawn by using an A_i as an endpoint.

Now, due to Lemma 5.2.1 and Lemma 5.2.2 all the components of ξ and ζ in the triangle from (5.18) must be nonzero. Then, by Lemma 5.2.4, if e_i is a summand of e, we see that

$$\mathrm{Ext}^1(\Sigma a, e_i) \neq 0 \quad \text{and} \quad \mathrm{Ext}^1(e_i, \Sigma^{-1} a^*) \neq 0.$$

The diagonals Σa and $\Sigma^{-1} a^*$ are given by the blue diagonals in Figure 5.5, and it is clear that A_2 and A_3 are the only candidates that cross both, so are the only remaining possibilities for summands of e.

We note here that when x_3 is the clockwise neighbour of x_1, the arc A_2 becomes zero as it then lies on the edge of P. In addition, if x_1 is the clockwise neighbour of x_2, then A_3 becomes zero. We now show that $e = A_2 \oplus A_3$.

Assume $e = 0$, that is, neither A_2 nor A_3 are summands of e. Then, the triangle in (5.18) becomes

$$a \to 0 \to a^*,$$

meaning that $a^* = \Sigma a$, which is a clear contradiction as both a and a^* share a common endpoint.

Assume $e = A_2$, that is, A_3 is not a summand. Rolling the triangle from (5.18), using the rolling axiom, gives the triangle

$$e \to a^* \to \Sigma a,$$

and reapplying $\mathrm{Ext}^1(t, -)$ to this gives the following exact sequence:

$$\mathrm{Ext}^1(t, e) \to \mathrm{Ext}^1(t, a^*) \to \mathrm{Ext}(t, \Sigma a). \tag{5.19}$$

By setting $t = t_1$ from Figure 5.5, we see that both $\mathrm{Ext}^1_{\mathsf{C}}(t, e) = 0$ and $\mathrm{Ext}^1_{\mathsf{C}}(t, \Sigma a) = 0$, whilst $\mathrm{Ext}^1_{\mathsf{C}}(t, a^*) \neq 0$. This is a contradiction to the exactness of the sequence in (5.19), and hence, A_3 must be a summand of e. Notice here that such a t_1 cannot be drawn in the case when $x_1^+ = x_2^-$. However, in this exceptional case, $A_3 = 0$ as it is an exterior edge of P, and so A_2 remains the only candidate for a summand of e.

For a vertex x of P, denote by x^+ the anticlockwise neighbouring vertex of x, and by x^- the clockwise neighbouring vertex. Also, for two vertices x and y, denote by $[x, y]$ the

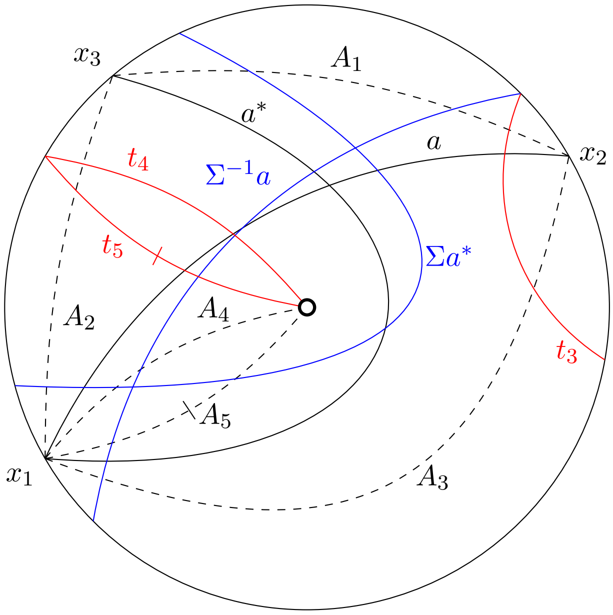


Figure 5.6: Calculating the exchange triangle $a^* \to e' \to a$.

set of vertices of P between x and y that is anticlockwise of x. Both x and y are in the interval $[x, y]$. Note that the peripheral endpoint of t_1 from Figure 5.5 should be in the interval $[x_1^+, x_2^-]$.

Assume $e = A_3$, that is, A_2 is not a summand of e. Then, set $t = t_2$ in Figure 5.5 and see that both $\mathrm{Ext}^1_{\mathsf{C}}(t, e) = 0$ and $\mathrm{Ext}^1_{\mathsf{C}}(t, \Sigma a) = 0$, whilst $\mathrm{Ext}^1_{\mathsf{C}}(t, a^*) \neq 0$, which is again a contradiction to the exactness of the sequence in (5.19). Note that one endpoint of t_2 should be in the interval $[x_3^+, x_1^-]$ and the other endpoint should be in the interval $[x_2, x_3^-]$. Hence A_2 is a summand of e. Notice here that when $x_3 = x_1^-$ such a t_2 cannot be drawn. However, in this case, $A_2 = 0$ and so A_3 is the only remaining candidate for a summand of e.

For each indecomposable direct summand e_i of e, we have $\dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathsf{C}}(\Sigma a, e_i) = 1$, and since $\mathrm{Ext}^1_{\mathsf{C}}(\Sigma a, e_i) = \mathrm{Hom}_{\mathsf{C}}\big(\Sigma^{-1}(\Sigma a), e_i\big) = \mathrm{Hom}_{\mathsf{C}}(a, e_i)$, we also know that $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathsf{C}}(a, e_i) = 1$. Then, by Lemma 5.2.3 we know that e has no repeat summands, that is, it contains only one copy of each of A_2 and A_3. We therefore see that

$$e = A_2 \oplus A_3.$$

We now compute the other exchange triangle, which takes the form

$$a^* \xrightarrow{\xi'} e' \xrightarrow{\zeta'} a. \tag{5.20}$$

Using Lemma 5.1.1 and the same logic as the previous part, it is clear that $A_1, \ldots, A_5$

are the only candidates for summands of e'. Now, by Lemma 5.2.1 and Lemma 5.2.2 we require that all the components of ξ' and ζ' are nonzero. Then, by Lemma 5.2.4, if e'_i is a summand of e', we need

$$\mathrm{Ext}^1(\Sigma a^*, e'_i) \neq 0 \quad \text{and} \quad \mathrm{Ext}^1(e'_i, \Sigma^{-1} a) \neq 0.$$

The diagonals Σa^* and $\Sigma^{-1}a$ are given by the blue diagonals in Figure 5.6. Since A_1, A_4 and A_5 are the only candidates to cross both blue lines, they remain the only possibilities for summands of e'.

We note here that when x_2 is the clockwise neighbour of x_3, the arc A_1 lies on the edge of P and is hence zero. We now show that $e' = A_1 \oplus A_4 \oplus A_5$.

Since $a \neq \Sigma a^*$, we know that $e' \neq 0$, and so at least one of A_1, A_4 or A_5 must be a summand of e'.

Assume $e' = A_4 \oplus A_5$, that is, A_1 is not a summand of e'. Rolling the triangle in (5.20) produces the following triangle:

$$e' \to a \to \Sigma a^*,$$

and applying $\mathrm{Ext}^1_{\mathsf{C}}(t, -)$ produces the exact sequence:

$$\mathrm{Ext}^1_{\mathsf{C}}(t, e') \to \mathrm{Ext}^1_{\mathsf{C}}(t, a) \to \mathrm{Ext}^1_{\mathsf{C}}(t, \Sigma a^*). \tag{5.21}$$

Let $t = t_3$ from Figure 5.6, then we see that both $\mathrm{Ext}^1_{\mathsf{C}}(t, e') = 0$ and $\mathrm{Ext}^1_{\mathsf{C}}(t, \Sigma a^*) = 0$, whilst $\mathrm{Ext}^1_{\mathsf{C}}(t, a) \neq 0$. This produces a contradiction to the exactness of the sequence in (5.21), and hence A_1 must be a summand of e'. Note that one endpoint of t_3 should be in the interval $[x_1^+, x_2]$, and the other endpoint should be in the interval $[x_2^+, x_3^-]$. Notice here that if $x_2^+ = x_3$, then such a t_3 cannot be drawn. However, in this exceptional case, $A_1 = 0$ and so A_4 and A_5 are the only remaining candidates for summands of e'.

Assume $e' = A_1 \oplus A_4$, that is, A_5 is not a summand of e'. Set $t = t_4$ in Figure 5.6, and see that both $\mathrm{Ext}^1_{\mathsf{C}}(t, e') = 0$ and $\mathrm{Ext}^1_{\mathsf{C}}(t, \Sigma a^*) = 0$, whilst $\mathrm{Ext}^1_{\mathsf{C}}(t, a) \neq 0$. This again contradicts the exactness of the sequence in (5.21). Hence, A_5 is also a summand of e'. Here, the peripheral endpoint of t_4 should be in the interval $[x_3, x_1^-]$.

Assume now that $e' = A_1 \oplus A_5$, that is, A_4 is not a summand. Then, by setting $t = t_5$ in Figure 5.6, we obtain the same contradiction. Here, the peripheral endpoint of t_5 should be in the interval $[x_3, x_1^-]$. A_4 is therefore a summand of e'.

For each summand e'_i of e', we have that $\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathsf{C}}(\Sigma a^*, e'_i) = 1$ and since $\mathrm{Ext}^1_{\mathsf{C}}(\Sigma a^*, e'_i) = \mathrm{Hom}_{\mathsf{C}}\big(\Sigma^{-1}(\Sigma a^*), e'_i\big) = \mathrm{Hom}_{\mathsf{C}}(a^*, e'_i)$, we know $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathsf{C}}(a^*, e'_i) = 1$. Then, by Lemma 5.2.3 we know that e' has no repeated summands, that is, it contains only one copy of each of A_1, A_4 and A_5. We therefore conclude that $e' = A_1 \oplus A_4 \oplus A_5$, and see that the two

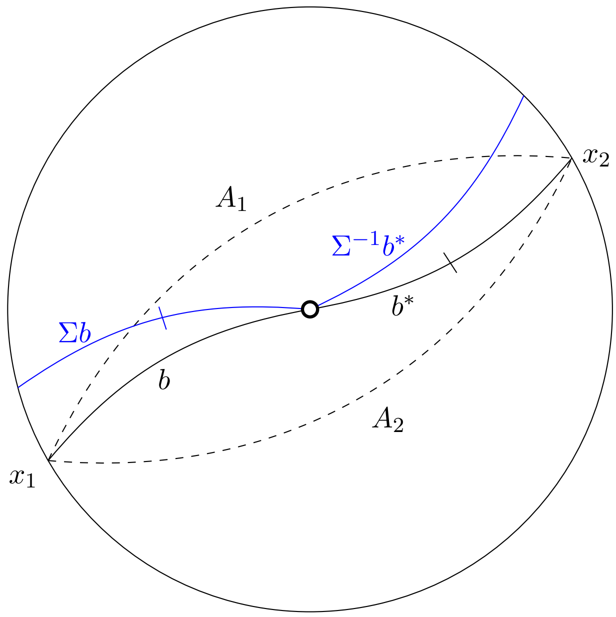


Figure 5.7: Calculating the exchange triangle $b \to e \to b^*$.

exchange triangles for a are

$$a \to A_2 \oplus A_3 \to a^* \ , \ a^* \to A_1 \oplus A_4 \oplus A_5 \to a.$$

Case 2. (Figure 5.2, exchange triangles for b and b^*.)

In this case, we prove Statement 2 of Theorem 5.1.3. We will refer to Figure 5.7 in order to compute the exchange triangles for b, as this is a more detailed version of Figure 5.2 from the formulation of the theorem.

Since b and b^* are both spokes which cross, their crossing number is one, and so $\dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathsf{C}}(b, b^*) = \dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathsf{C}}(b^*, b) = 1$ Therefore, b sits in an exchange triangle of the form

$$b \xrightarrow{\xi} e \xrightarrow{\zeta} b^*. \tag{5.22}$$

Notice first that the crossing between b and b^* splits P into two peripheral subcells enclosed by b, b^* and the edge of P. By Lemma 5.1.2, we know that $\mathrm{Ext}^1_{\mathsf{C}}(b, e) = \mathrm{Ext}^1_{\mathsf{C}}(b^*, e) = 0$, and so each indecomposable summand e_i of e must be fully contained within one of the peripheral subcells. This leaves the arcs A_1 and A_2 from Figure 5.7 as the only candidates for summands of e, since any other arc contained within one of these subcells intersects an arc t that itself intersects neither b nor b^*, a contradiction to Lemma 5.1.1. Such a diagonal t can easily be found by using an A_i as an endpoint. Note also that no spoke can be a candidate for a summand of e as it would cross one of b or b^*, contradicting Lemma 5.1.2.

By Lemma 5.2.1 and Lemma 5.2.2, we require that all components of ξ and ζ are nonzero. By Lemma 5.2.4, if e_i is a summand of e, then we require that

$$\mathrm{Ext}^1_{\mathsf{C}}(\Sigma b, e_i) \neq 0 \quad \text{and} \quad \mathrm{Ext}^1_{\mathsf{C}}(e_i, \Sigma^{-1} b^*) \neq 0.$$

The indecomposables Σb and $\Sigma^{-1} b^*$ are given by the blue diagonals in Figure 5.7. Since A_1 is the only candidate crossing both of these arcs, it remains the only possibility for a summand of e.

We now note a special case of Figure 5.7, namely when $b^* = \Sigma b$. In this particular case, A_1 is on the edge of P, and is hence zero. It then becomes clear that $e = 0$.

Now, assume that $b^* \neq \Sigma b$. Then, since $e = 0$ leads to a clear contradiction, we may conclude that A_1 is a summand of e.

For each summand e_i of e, we have that $\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathsf{C}}(\Sigma b, e_i) = 1$, and since $\mathrm{Ext}^1_{\mathsf{C}}(\Sigma b, e_i) = \mathrm{Hom}_{\mathsf{C}}\big(\Sigma^{-1}(\Sigma b), e_i\big) = \mathrm{Hom}_{\mathsf{C}}(b, e_i)$, we know $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathsf{C}}(b, e_i) = 1$. Then, by Lemma 5.2.3 we know that e has no repeat summands, that is, it contains only one copy of A_1. We have therefore shown that

$$e = A_1.$$

The other exchange triangle is of the form

$$b^* \xrightarrow{\xi'} e' \xrightarrow{\zeta'} b. \tag{5.23}$$

It is clear that this case is completely symmetric to the previous triangle, and we maybe therefore conclude immediately that

$$e' = A_2.$$

So, the exchange triangles for b are

$$b \to A_1 \to b^* \ , \ b^* \to A_2 \to b.$$

Case 3. (Figure 5.3, exchange triangles for f and f^*.)

In this case, we prove Statement 3 of Theorem 5.1.3. For this, we will work with Figures 5.8 and 5.9, which are more detailed versions of Figure 5.3 from the formulation of the theorem.

Since f and f^* intersect each other once in the interior of P, their crossing number is one and so $\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathsf{C}}(f, f^*) = \dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathsf{C}}(f^*, f) = 1$. Thus, f sits in an exchange triangle of the form

$$f \xrightarrow{\xi} e \xrightarrow{\zeta} f^*. \tag{5.24}$$

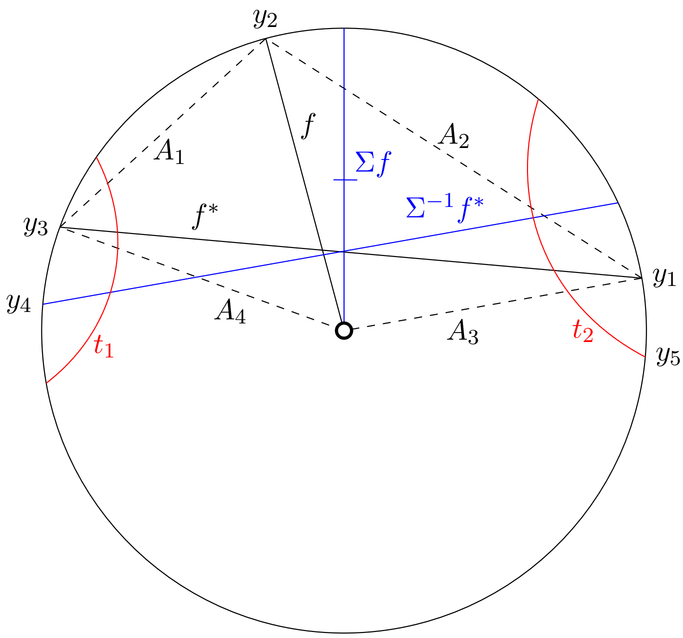


Figure 5.8: Calculating the exchange triangle $f \to e \to f^*$.

Notice first that the crossing between f and f^* splits P into three subcells, each enclosed by f, f^* and the edge of P. By Lemma 5.1.2 we know that $\mathrm{Ext}^1_{\mathsf{C}}(f, e) = \mathrm{Ext}^1_{\mathsf{C}}(f^*, e) = 0$, and so each indecomposable summand e_i of e must be fully contained within one of the three subcells. This leaves the diagonals $A_1, \ldots, A_4$ from Figure 5.8 as the only candidates for summands of e. Indeed, any other diagonal contained within one of the subcells intersects a diagonal t that intersects neither f nor f^*, which is a contradiction to Lemma 5.1.1. Such a diagonal t can easily be found by using an A_i as an endpoint. Note also that the spokes A_3 and A_4 both necessarily have the same tagging as the spoke f, otherwise a contradiction to Lemma 5.1.2 would occur.

By Lemma 5.2.1 and Lemma 5.2.2, we require that each component of ξ and ζ is nonzero. So, by Lemma 5.2.4, we know that if e_i is a summand of e, then

$$\mathrm{Ext}^1_{\mathsf{C}}(\Sigma f, e_i) \neq 0 \quad \text{and} \quad \mathrm{Ext}^1_{\mathsf{C}}(e_i, \Sigma^{-1} f^*) \neq 0.$$

The blue diagonals in Figure 5.8 correspond to Σf and $\Sigma^{-1} f^*$. Since A_2 and A_4 are the only two diagonals among the A_i to cross both Σf and $\Sigma^{-1} f^*$, they remain the only possibilities for summands of e. Note that Σf does indeed cross A_4 as they are both spokes with opposite taggings.

We now show that $e = A_2 \oplus A_4$. Assume $e = 0$, that is, neither A_2 nor A_4 is a summand of e. Then, the triangle in (5.24) becomes

$$f \to 0 \to f^*,$$

meaning that $f^* = \Sigma f$. This is a clear contradiction as f^* is an arc, whilst Σf is a spoke.

Assume $e = A_2$, that is, A_4 is not a summand. Then, rolling the triangle in (5.24), using the rolling axiom, gives the triangle

$$e \to f^* \to \Sigma f,$$

and applying $\mathrm{Ext}^1_{\mathsf{C}}(t, -)$ produces the following exact sequence:

$$\mathrm{Ext}^1_{\mathsf{C}}(t, e) \to \mathrm{Ext}^1_{\mathsf{C}}(t, f^*) \to \mathrm{Ext}^1_{\mathsf{C}}(t, \Sigma f). \tag{5.25}$$

Now, if we set $t = t_1$ from Figure 5.8, we see that both $\mathrm{Ext}^1_{\mathsf{C}}(t, e) = 0$ and $\mathrm{Ext}^1_{\mathsf{C}}(t, \Sigma f) = 0$. However, we also see that $\mathrm{Ext}^1_{\mathsf{C}}(t, f^*) \neq 0$, which produces a contradiction to the exactness of the sequence in (5.25). Note that one endpoint of t_1 should be in the interval $[y_2, y_3^-]$, and the other endpoint should be in the interval $[y_3^+, y_1^-]$. Hence, A_4 must be a summand of e.

Assume $e = A_4$, that is, A_2 is not a summand of e. Then, if we set $t = t_2$ from Figure 5.8, we see that $\mathrm{Ext}^1_{\mathsf{C}}(t, e) = 0$ and $\mathrm{Ext}^1_{\mathsf{C}}(t, \Sigma f) = 0$. However, we again see that $\mathrm{Ext}^1_{\mathsf{C}}(t, f^*) \neq 0$, producing another contradiction to the exactness of the sequence in (5.25). One endpoint of t_2 should be in the interval $[y_3^+, y_1^-]$, and the other endpoint should be in the interval $[y_1^+, y_2^-]$. Hence, A_2 is also a summand of e. Notice that in the case when $y_1^+ = y_2$ such a t_2 cannot be drawn. However, in this exceptional case, $A_2 = 0$ and so A_4 is the only candidate for a summand of e.

For each summand e_i of e, we have that $\dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathsf{C}}(\Sigma f, e_i) = 1$, and since $\mathrm{Ext}^1_{\mathsf{C}}(\Sigma f, e_i) = \mathrm{Hom}_{\mathsf{C}}\big(\Sigma^{-1}(\Sigma f), e_i\big) = \mathrm{Hom}_{\mathsf{C}}(f, e_i)$, we know $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathsf{C}}(f, e_i) = 1$. Then, by Lemma 5.2.3 we know that e has no repeat summands, that is, it contains only one copy of each of A_2 and A_4. We have therefore shown that

$$e = A_2 \oplus A_4.$$

The other exchange triangle takes the form

$$f^* \xrightarrow{\xi'} e' \xrightarrow{\zeta'} f. \tag{5.26}$$

Again, using Lemma 5.1.1 and the same logic as with the previous triangle, the only candidates for summands of e' are the diagonals A_1, A_2, A_3, A_4. By Lemma 5.2.1 and Lemma 5.2.2, we know that each component of ξ' and ζ' is nonzero. By Lemma 5.2.4, we therefore know that if e'_i is a summand of e', then

$$\mathrm{Ext}^1_{\mathsf{C}}(\Sigma f^*, e'_i) \neq 0 \quad \text{and} \quad \mathrm{Ext}^1_{\mathsf{C}}(e'_i, \Sigma^{-1} f) \neq 0.$$

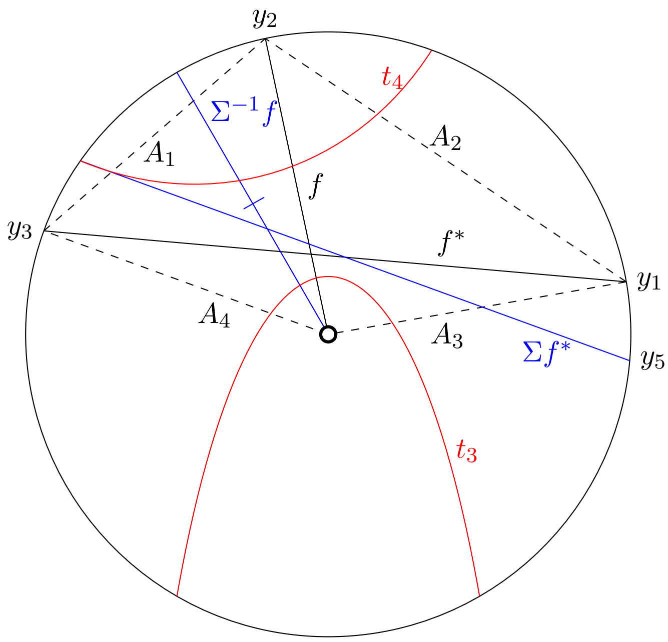


Figure 5.9: Calculating the exchange triangle $f^* \to e' \to f$.

Now, Σf^* and $\Sigma^{-1}f$ correspond to the blue diagonals in Figure 5.9, and since A_1 and A_3 are the only candidates that cross them both, they remain the only possibilities for summands of e'.

We now show that $e' = A_1 \oplus A_3$. Since $f \neq \Sigma^{-1}f^*$, at least one of A_1 or A_3 must be a summand of e'.

Assume $e' = A_1$, that is A_3 is not a summand of e. Then, rolling the triangle in (5.26), using the rolling axiom, produces the following triangle:

$$e' \to f \to \Sigma f^*,$$

and applying $\mathrm{Ext}^1_{\mathsf{C}}(t, -)$ to this sequence creates the following long exact sequence:

$$\mathrm{Ext}^1_{\mathsf{C}}(t, e') \to \mathrm{Ext}^1_{\mathsf{C}}(t, f) \to \mathrm{Ext}^1_{\mathsf{C}}(t, \Sigma f^*). \tag{5.27}$$

Now, if we set $t = t_3$ from Figure 5.9, then we see that $\mathrm{Ext}^1_{\mathsf{C}}(t, e') = 0$ and $\mathrm{Ext}^1_{\mathsf{C}}(t, \Sigma f^*) = 0$, whilst $\mathrm{Ext}^1_{\mathsf{C}}(t, f) \neq 0$. This produces a contradiction to the exactness of the sequence in (5.27). Note that both endpoints of t_3 should be in the interval $[y_3^+, y_1^-]$, and the arc should wrap around the puncture as in the figure. We then see that A_3 must be a summand of e'.

Assume $e' = A_3$, that is A_1 is not a summand of e'. Then, if we set $t = t_4$ from Figure 5.9, we see that both $\mathrm{Ext}^1_{\mathsf{C}}(t, e') = 0$ and $\mathrm{Ext}^1_{\mathsf{C}}(t, \Sigma f^*) = 0$, whilst $\mathrm{Ext}^1_{\mathsf{C}}(t, f) \neq 0$. This is again a contradiction to the exactness of the sequence in (5.27), and hence A_1 must

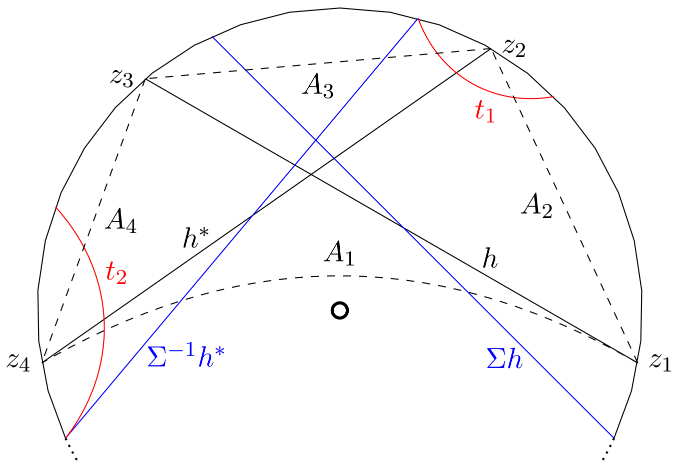


Figure 5.10: Calculating the exchange triangle $h \to e \to h^*$.

also be a summand of e'. Here, one endpoint of t_4 should be in the interval $[y_1, y_2^-]$ and the other endpoint should be in the interval $[y_2^+, y_3^-]$. Notice here that in the case when $y_2^+ = y_3$ such a t_4 cannot be drawn. However, in this case, $A_1 = 0$ and so A_3 is the only candidate for a summand of e'.

For each summand e_i' of e', we have that $\dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathsf{C}}(\Sigma f^*, e_i') = 1$, and since $\mathrm{Ext}^1_{\mathsf{C}}(\Sigma f^*, e_i') = \mathrm{Hom}_{\mathsf{C}}\big(\Sigma^{-1}(\Sigma f^*), e_i'\big) = \mathrm{Hom}_{\mathsf{C}}(f^*, e_i')$, we know $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathsf{C}}(f^*, e_i') = 1$. Then, by Lemma 5.2.3 we know that e' has no repeat summands, that is, it contains only one copy of each of A_1 and A_3. We therefore see that

$$e' = A_1 \oplus A_3,$$

and the exchange triangles for f are:

$$f \to A_2 \oplus A_4 \to f^* \;,\; f^* \to A_1 \oplus A_3 \to f.$$

Case 4. (Figure 5.4, exchange triangles for h and h^*.) In this case, we prove Statement 4 of Theorem 5.1.3. For this, we will refer to Figure 5.10, which is a more detailed version of Figure 5.4 from the formulation of the theorem.

Since h and h^* intersect once in the interior of P, their crossing number is one and so $\dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathsf{C}}(h, h^*) = \dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathsf{C}}(h^*, h) = 1$. Therefore, h sits in an exchange triangle of the form

$$h \xrightarrow{\xi} e \xrightarrow{\zeta} h^*. \tag{5.28}$$

Notice first that the crossing between h and h^* splits P into four subcells, each enclosed by h, h^* and the edge of P. We know by Lemma 5.1.2 that $\mathrm{Ext}^1_{\mathsf{C}}(h, e) = \mathrm{Ext}^1_{\mathsf{C}}(h^*, e) = 0$, and so each indecomposable summand e_i of e must be fully contained within one of these subcells. This leaves the arcs $A_1, \ldots, A_4$ as the only candidates for summands of e. Indeed, any other diagonal contained within one of the three subcells intersects a diagonal t that itself intersects neither h nor h^*, which is a contradiction to Lemma 5.1.1. Such a t can easily be found by using an A_i as an endpoint, or if the proposed diagonal is the arc from z_1 to z_4 that wraps the opposite way around the puncture to A_1, then t can be a spoke.

By Lemma 5.2.1 and Lemma 5.2.2, we require that all components of ξ and ζ are nonzero. By Lemma 5.2.4, we know that if e_i is a summand of e, then

$$\mathrm{Ext}^1_{\mathsf{C}}(\Sigma h, e_i) \neq 0 \quad \text{and} \quad \mathrm{Ext}^1_{\mathsf{C}}(e_i, \Sigma^{-1} h^*) \neq 0.$$

The blue diagonals in Figure 5.10 correspond to the indecomposables Σh and $\Sigma^{-1}h^*$, and since A_1 and A_3 are the only candidates to cross both of these diagonals, they remain the only possibilities for summands of e.

We now show that $e = A_1 \oplus A_3$. First note the special case when $h^* = \Sigma h$. In this case, both A_1 and A_3 are on the edge of P and are hence zero. This forces $e = 0$.

Now, assume that $h^* \neq \Sigma h$. If $e = 0$, then the triangle in (5.28) will read

$$h \to 0 \to h^*,$$

meaning that $h^* = \Sigma h$, a contradiction to our assumption. Hence, at least one of A_1 or A_3 must be a summand of e.

Assume $e = A_1$, that is, A_3 is not a summand. Then, rolling the triangle in (5.28) using the rolling axiom produces the following triangle:

$$e \to h^* \to \Sigma h,$$

and applying $\mathrm{Ext}^1_{\mathsf{C}}(t, -)$ will create a long exact sequence:

$$\mathrm{Ext}^1_{\mathsf{C}}(t, e) \to \mathrm{Ext}^1_{\mathsf{C}}(t, h^*) \to \mathrm{Ext}^1_{\mathsf{C}}(t, \Sigma h). \tag{5.29}$$

Now, if we set $t = t_1$ from Figure 5.10, we see that both $\mathrm{Ext}^1_{\mathsf{C}}(t, e) = 0$ and $\mathrm{Ext}^1_{\mathsf{C}}(t, \Sigma h) = 0$, whilst $\mathrm{Ext}^1_{\mathsf{C}}(t, h^*) \neq 0$. This is a contradiction to the exactness of the sequence in (5.29), and thus A_3 must be a summand of e. Here, one endpoint of t_1 should be in the interval $[z_1, z_2^-]$, whilst the other endpoint should be in the interval $[z_2^+, z_3^-]$. Notice that in the case when $z_2^+ = z_3$ such a t_1 cannot be drawn. However, in such a case, $A_3 = 0$ and so A_1 is the only candidate for a summand of e.

Assume $e = A_3$, that is, A_1 is not a summand of e. Then, by setting $t = t_2$ in Figure 5.10, we see that $\mathrm{Ext}^1_{\mathsf{C}}(t, e) = 0$ and $\mathrm{Ext}^1_{\mathsf{C}}(t, \Sigma h) = 0$. However, since $\mathrm{Ext}^1_{\mathsf{C}}(t, h^*) \neq 0$, we obtain another contradiction to the exactness of the sequence in (5.29). Here, one endpoint of t_2 should be in the interval $[z_3+, z_4^-]$, whilst the other endpoint should be in the interval $[z_4^+, z_1^-]$. Hence, A_1 is also a summand.

For each summand e_i of e, we have that $\dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathsf{C}}(\Sigma h, e_i) = 1$, and since $\mathrm{Ext}^1_{\mathsf{C}}(\Sigma h, e_i) = \mathrm{Hom}_{\mathsf{C}}\big(\Sigma^{-1}(\Sigma h), e_i\big) = \mathrm{Hom}_{\mathsf{C}}(h, e_i)$, we know $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathsf{C}}(h, e_i) = 1$. Then, by Lemma 5.2.3 we know that e has no repeat summands, that is, it contains only one copy of each of A_1 and A_3. We therefore see that:

$$e = A_1 \oplus A_3.$$

The other exchange triangle takes the form

$$h^* \xrightarrow{\xi'} e' \xrightarrow{\zeta'} h. \tag{5.30}$$

It is clear that this case is completely symmetric to the previous exchange triangle, and thus, we may immediately conclude that

$$e' = A_2 \oplus A_4,$$

and so the exchange triangles for h are:

$$h \to A_1 \oplus A_3 \to h^* \ , \ h^* \to A_2 \oplus A_4 \to h.$$

We note that this case also demonstrates how to compute exchange triangles in $\mathsf{C}(A_n)$, the cluster category of Dynkin type A_n.

The above four cases cover all possibilities for exchange triangles in $\mathsf{C}(D_n)$. Theorem 5.1.3 has therefore been proved.

Case 4 from the above proof also demonstrates how to compute exchange triangles in $\mathsf{C}(A_n)$, the cluster category of Dynkin type A_n, and so we also have the following theorem.

Theorem 5.3.1. *Looking at Figure 5.11, exchange triangles in $\mathsf{C}(A_n)$ are as follows*

$$h \to A_1 \oplus A_3 \to h^* \ , \ h^* \to A_2 \oplus A_4 \to h.$$

Proof. Case 4 of the proof of Theorem 5.1.3 suffices to prove this. □

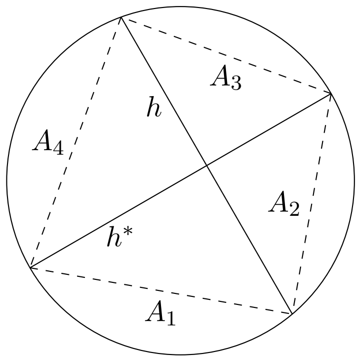


Figure 5.11: Theorem 5.3.1. The exchange triangles for h and h^* are: $h \to A_1 \oplus A_3 \to h^*$ and $h^* \to A_2 \oplus A_4 \to h$.

Chapter 6

Properties of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ in $\mathsf{C}(D_n)$

This chapter follows a similar pattern to that in Chapter 4. Here, in the case of $\mathsf{C}(D_n)$, we again prove properties of the quotient group $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$.

Recall from Chapter 3 that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is a vital object in our study of generalised friezes. Indeed, in Section 3.3, we provide exponential maps α and β, involving $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$, which satisfy Condition F from Definition 3.2.1. Their definitions depend on an exponential map $\varepsilon : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to A$, where A is a ring. The group $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ determines which maps can be chosen for ε. In turn, by Theorem 3.2.2, α and β turn the modified Caldero-Chapoton map ρ, defined in (3.23), into a generalised frieze. The map ρ depends only on the rigid subcategory R.

This chapter is split into two sections. Section 6.1 works towards proving Theorem 6.1.7 which shows that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ depends only on our choice of rigid subcategory $\mathsf{R} \subseteq \mathsf{T}$. Then, Section 6.2 is aimed at proving Theorem 6.2.15, which provides a general formula for $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$.

For the work in this chapter, we should recall the once punctured polygon model associated to $\mathsf{C}(D_n)$ that we introduced in Section 2.5.2.

6.1 Independent Choices for S in type D.

In this section we will show that in the case of $\mathsf{C} = \mathsf{C}(D_n)$, the quotient group $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is invariant under mutation of indecomposables in the subcategory S.

6.1.1 Background

As this section requires full triangulations of the once punctured polygon P, it makes sense first to document the possible types of triangulations in type D. In order to do this, we use the following remark to describe the notion of a cell, which we will use heavily throughout this chapter.

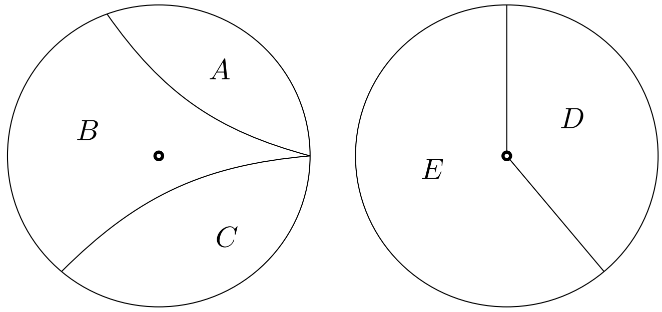


Figure 6.1: Any diagonals in R that dissect the polygon, split P into cells.

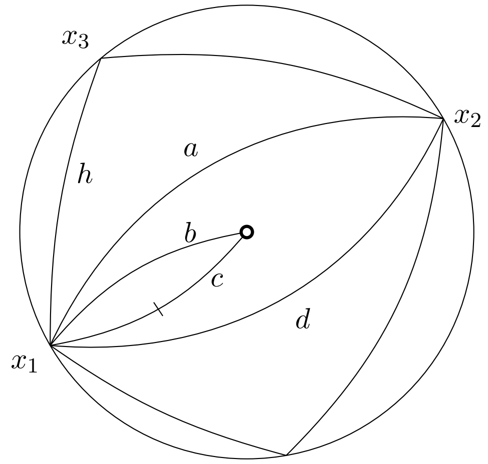


Figure 6.2: A selection of diagonals from a polygon triangulation corresponding to a cluster tilting object inside $\mathsf{C}(D_n)$.

Remark 6.1.1. When a diagonal in R, or series of diagonals in R, dissect the punctured polygon P, we refer to the resulting portions of P as "cells". The portions A, B, C, D, E in Figure 6.1 are examples of cells.

We categorise the triangulations of the punctured poylgon into the two following types:

1. Selected diagonals from the first type of triangulation are pictured in Figure 6.2. The triangulation contains both a single tagged spoke and a single untagged spoke from some vertex x_1 to the puncture, labelled in the figure as b and c. There are then two arcs connecting the vertex x_1 to another vertex x_2 of P, one either side of the puncture. These are the arcs labelled a and d in the figure. We should note that if x_2 is a neighbouring vertex of x_1, then either a or c will be on the edge of P, and hence will be zero. Each of the arcs a and d create a subpolygon between themself and the edge of P. Each of these subpolygons contains a triangulation made up of noncrossing arcs; the diagonal h in the figure is an example of one such arc. Together with the diagonals a, b, c and d, this gives a full triangulation of the polygon P.

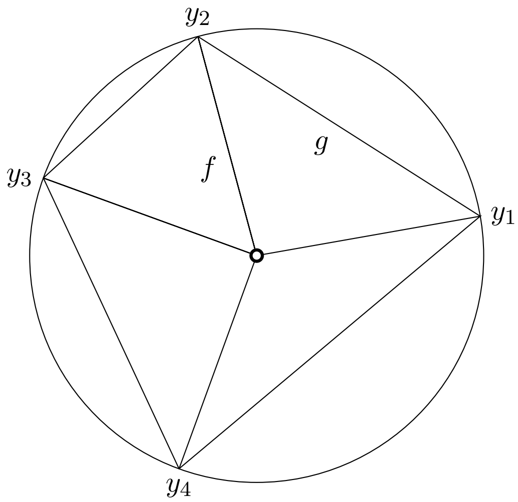


Figure 6.3: A selection of diagonals from a polygon triangulation corresponding to a cluster tilting object inside $\mathsf{C}(D_n)$.

2. Selected diagonals from the second type of triangulation are pictured in Figure 6.3. There are $n \geq 2$ spokes, all either tagged or untagged, from the puncture to different vertices on the edge of P. The spoke f in the figure is one such spoke. The endpoint at the edge of P of each spoke is then connected by arcs to the endpoints of its two neighbouring spokes. The diagonal g in the figure is one such arc. As with the first type of triangulation, the subpolygons between these arcs and the edge of P contain triangulations by noncrossing arcs. These then complete a full triangulation of P.

Remark 6.1.2. We remark that the above two cases are the only types of triangulation possible.

It is clear that for $n \geq 2$ spokes, the above two cases cover all possibilities. Indeed, if there are two spokes, then there are two cases: If the two spokes both share a common endpoint on the edge of P, then they must have opposite taggings. In this case, Figure 6.2 details the triangulation. If the two spokes do not share the same endpoint on the edge of P, then they must both share the same tagging. In which case, Figure 6.3 details the triangulation. If there are more than two spokes, then they must all share the same tagging. In this case, Figure 6.3 also covers the full triangulation.

We finally note that it is not possible to have fewer than two spokes. Choose any triangulation of P, which may contain both spokes and arcs. Choose an arc a and notice that it splits P into two subpolygons A and B, as in Figure 6.4. One subpolygon A contains the puncture, whilst the other B contains a triangulation made up solely of arcs. Removing B, whilst identifying the arc a with an exterior edge, leaves the smaller polygon A, see the left hand side of Figure 6.4. Now, the triangulation of A contains strictly fewer arcs than that of P. Thus, we may continue this process with a new arc. This process will

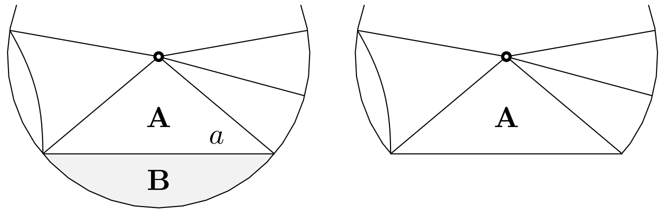


Figure 6.4: Removing exterior subpolygons of P leaves a cental subpolygon containing the puncture. A triangulation of this subpolygon, and thus a triangulation of P, must contain at least two spokes.

terminate at the stage when we are left with a central subpolygon whose triangulation contains no arcs. The central subpolygon, which contains the puncture, can have no fewer than two vertices. Assume, as a minimal case, that the central subpolygon does indeed have two vertices. We will discuss here how to construct a triangulation of this subpolygon, showing that there must be at least two spokes in any such triangulation. Clearly, we may insert a spoke from one vertex to the puncture. We may then insert either another spoke of the same tagging from the other vertex to the puncture, or another spoke of opposite tagging from the same vertex to the puncture. Neither of these spokes cross the first, and we now have a full triangulation. Note that no arc could have been inserted as it would be homotopic to the edge of P. Should the central subpolygon have more than two vertices, it is clear that a triangulation of this will also contain at least two spokes.

We now set up the necessary constructions that we will use in order to prove that the quotient $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is invariant under mutation of indecomposables in S. Consider some rigid subcategory R and two cluster tilting subcategories T and T' such that $\mathsf{R} \subseteq \mathsf{T}$ and $\mathsf{R} \subseteq \mathsf{T}'$. That is, there are two subcategories S and S' such that

$$\operatorname{indec}\mathsf{T} = \operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S},$$

and

$$\operatorname{indec}\mathsf{T}' = \operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S}'.$$

From these constructions, we can calculate two quotient groups $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ and $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$, where N is the subgroup calculated with respect to T and N' is the subgroup calculated with respect to T'. We will show that

$$\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \cong \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'.$$

Now, it is known that the indecomposables in R provide a polygon dissection of the punctured polygon P with n vertices. This splits P into cells, whilst $\operatorname{indec}\mathsf{S}$ then provides a full triangulation of each of these cells. Since any triangulation of P can be retrieved from any other triangulation of P by a series of mutations, it is enough to consider S and

S' that differ by a single mutation. Indeed, consider the cluster tilting subcategory T^* defined by

$$\operatorname{indec}\mathsf{T}^* = \operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S}^*,$$

where

$$\operatorname{indec}\mathsf{S}^* = \big\{\operatorname{indec}\mathsf{S} \setminus \{s\}\big\} \cup \{s^*\},$$

where s^* is the mutation of the chosen indecomposable $s \in \operatorname{indec}\mathsf{S}$. The triangulations coming from $\operatorname{indec}\mathsf{T}$ and $\operatorname{indec}\mathsf{T}^*$ clearly differ by one diagonal.

We will construct an isomorphism

$$\varphi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*.$$

Since generators of N come from the middle terms of the mutation triangles associated to each diagonal in $\operatorname{indec}\mathsf{S}$, we have several cases to consider. This is due to Theorem 5.1.3. As with $\mathsf{C}(A_n)$, the generators of N that are affected by the mutation are precisely those generators coming from those diagonals that make up the enclosing quadrilateral of s. Assume that s and s^* sit in the exchange triangles

$$s \to B \to s^* \ , \ s^* \to B' \to s. \tag{6.1}$$

The following lemma introduces well defined maps between the quotient groups.

Lemma 6.1.3. *The following maps are well defined, noting in each case that t in T is indecomposable,*

1. $\varphi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*$*, defined by*

$$\varphi([t]+N) = \begin{cases} [B]-[s^*]+N^* & \text{if } t = s, \\ [t]+N^* & \text{if } t \neq s. \end{cases}$$

2. $\varphi' : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*$*, defined by*

$$\varphi'([t]+N) = \begin{cases} [B']-[s^*]+N^* & \text{if } t = s, \\ [t]+N^* & \text{if } t \neq s. \end{cases}$$

3. $\eta : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^* \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$*, defined by*

$$\eta([t]+N^*) = \begin{cases} [B]-[s]+N & \text{if } t = s^*, \\ [t]+N & \text{if } t \neq s^*. \end{cases}$$

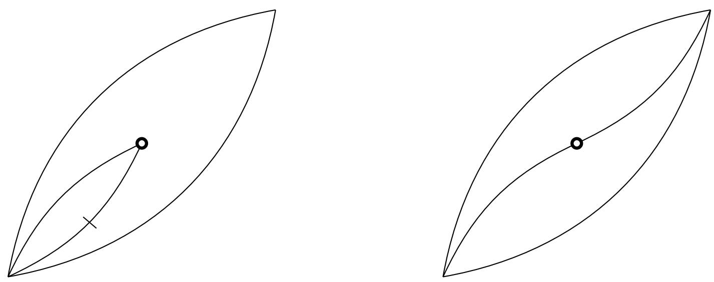


Figure 6.5: The two versions of the self-folded triangle.

4. $\eta' : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^* \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$*, defined by*

$$\eta([t] + N^*) = \begin{cases} [B'] - [s] + N & \text{if } t = s^*, \\ [t] + N & \text{if } t \neq s^*. \end{cases}$$

We also have that $\varphi = \varphi'$ *and* $\eta = \eta'$*.*

Section 6.1.2 provides a full proof to this lemma. This proof requires multiple cases, and to show that we have checked all cases, we will use positions of the so-called self-folded triangle in the triangulation. We describe what the self-folded triangle is in the following remark.

Remark 6.1.4. When a triangulation of P contains only two spokes, we call the region containing the puncture a "self-folded" triangle. This is something of the form in Figure 6.5. It is made up of the two spokes, as well as two arcs that share one or both endpoints. Each arc wraps in opposite directions around the puncture. Note that it is also possible for one of the arcs to be on the edge of P if the other arc wraps around the puncture to its neighbouring vertex.

There are two versions of the self-folded triangle, depending on where the two spokes meet the edge of P. The first version is when both spokes meet the edge of P at the same vertex, and hence one is tagged, whilst the other untagged. On the other hand, the spokes could meet the edge at different vertices, meaning that they must have the same tagging. This is the second version.

6.1.2 Well Defined Maps $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*$.

In this section, we provide a full proof to Lemma 6.1.3. The proof requires seven cases, and Remark 6.1.5 verifies that we have covered all necessary cases by considering each possible position in the triangulation that the self-folded triangle can sit.

For this proof, we will require the following notation: For each indecomposable $a \in \operatorname{indec}\mathsf{S}$, there is an associated generator for N. We will denote this generator by $n(a)$.

Similarly, for $a^* \in \operatorname{indec}\mathsf{S}^*$, denote the generator of N^* associated to a^* by $n^*(a^*)$.

Firstly, it is easy to see that $\varphi = \varphi'$ and $\eta = \eta'$. Indeed, φ and φ' are clearly equal on the $\mathrm{K}_0^{\mathrm{split}}$-classes of all diagonals other than s. Hence, we only need to see that $[B] - [s^*] + N^* = [B'] - [s^*] + N^*$. This is true since $[B] - [B'] = n^*(s^*)$ is a generator of N^*. Thus, $[B] + N^* = [B'] + N^*$, and therefore, $\varphi = \varphi'$. By the same reasoning, we also see that $\eta = \eta'$. We are therefore only required to check that φ and η are well defined.

To check that φ is well defined in each case, we define the map $\zeta : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)$ by

$$\zeta([t]) = \begin{cases} [B] - [s^*] & \text{if } t = s, \\ [t] & \text{if } t \neq s \end{cases}$$

for t indecomposable, and check that it sends generators of N to linear combinations of generators of N^*. To show that η is well defined, we define the map $\xi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$ by

$$\xi([t]) = \begin{cases} [B] - [s] & \text{if } t = s^* \\ [t] & \text{if } t \neq s^* \end{cases}$$

for t indecomposable, and check that it sends generators of N^* to linear combinations of generators of N. Before proving each of our cases, we give the following remark stating every possible case, and how the proof will cover each of them.

Remark 6.1.5. Noting that since there is only one puncture, any triangulation of P can contain at most one self-folded triangle, we consider all the possible positions in the triangulation that, if it exists, the self-folded triangle can sit. We also consider the two different forms of the self-folded triangle, and which diagonals we can mutate in each case. Recall that the diagonals whose associated generators are affected by the mutation form an "enclosing quadrilateral" Q of s and s^*. The diagonals forming the edges of the enclosing quadrilateral are precisely the diagonals forming the edges of the subpolygon inside which s and s^* both sit. The first two bullet points cover the situation when there is no self-folded triangle in the triangulation. The next three bullet points cover when the self-folded triangle sits inside the quadrilateral, whilst the sixth and seventh bullet points cover when it appears as an edge of the quadrilateral. The final bullet point covers the situation when the self-folded triangle is neither inside, nor on the edge of the enclosing quadrilateral.

- There is no self-folded triangle and a spoke from $\operatorname{indec}\mathsf{T}$ is mutated and replaced with an arc to form $\operatorname{indec}\mathsf{T}^*$. See Figure 6.6. Case 1 covers this.

- There is no self-folded triangle and an arc from $\operatorname{indec}\mathsf{T}$ is mutated and replaced with an arc to form $\operatorname{indec}\mathsf{T}^*$. See Figure 6.12. Case 7 covers this.

- The self-folded triangle is inside the quadrilateral Q, and a spoke in $\operatorname{indec}\mathsf{T}$ is mutated to form indec T^*. Note that this mutation changes the make-up of the self-folded triangle, so covers the case when both spokes meet the edge at the same vertex, as well as the case when they meet the edge at different vertices. See Figure 6.7. This is covered in Case 2.

- The self-folded triangle is inside the quadrilateral Q, and the two spokes meet the edge of P at the same vertex. We then choose to mutate an arc of the self-folded triangle. See Figure 6.8. This is covered in Case 3.
- The self-folded triangle is inside the quadrilateral Q again, and the two spokes meet the edge of P at different vertices. We again choose to mutate an arc of the self-folded triangle. See Figure 6.9. This is covered in Case 4.
- The self-folded triangle shares an edge with the quadrilateral Q, and both spokes meet the edge of P at the same vertex. See Figure 6.10. This is covered in Case 5.
- The self-folded triangle again shares an edge with the quadrilateral Q, and the two spokes meet the edge of P at different vertices. See Figure 6.11. This is covered in Case 6.
- The self-folded triangle is neither inside the quadrilateral Q, nor shares an edge with the quadrilateral. See Figure 6.12. This is covered in Case 7.

Remark 6.1.6. In each of the seven cases, we consider a collection of diagonals that form the edges of the enclosing quadrilateral Q and diagonals that form triangles on the exterior edges of the enclosing quadrilateral Q. These collections are precisely those diagonals pictured in the figures appearing with each case. We will assume that every diagonal in each collection is both nonzero and belongs to S. However, a problem may arise when we allow certain diagonals to be zero or belong to R. We verify here that no such problem occurs in the case of ζ, and therefore φ; however a similar verification holds for ξ, and therefore η. It is clear from the seven cases that for each $x \in \operatorname{indec} \mathsf{S}$,

$$\zeta(n(x)) = \begin{cases} n^*(x) \text{ or } n^*(x) + n^*(s^*), & \text{if } x \neq s \\ -n^*(s^*) & \text{if } x = s, \end{cases}$$

all of which are linear combinations of generators of N^*. Now, if $x \in \operatorname{indec} \mathsf{R}$ or if $x = 0$, then the would-be generator $n^*(x)$ is missing from N^*, since a generator $n^*(x')$ of N^* is only defined when $x' \in \operatorname{indec} \mathsf{S}$. However, this is not an issue since the would-be generator $n(x)$ of N is also missing for the same reason. As mentioned, a similar verification holds for ξ.

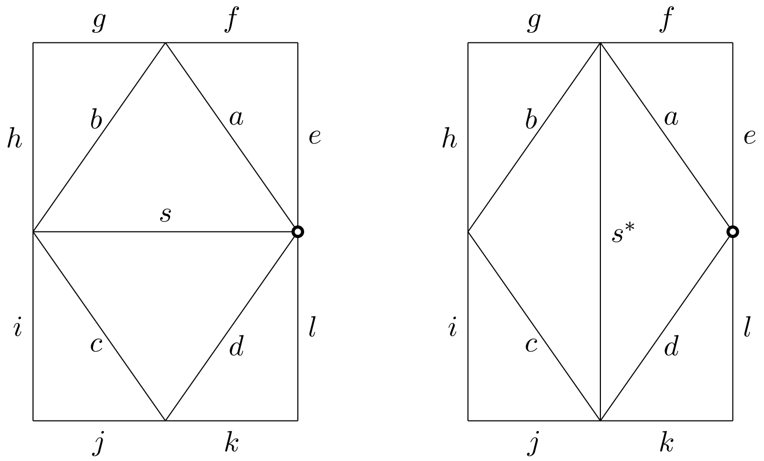


Figure 6.6: Case 1: There is no self-folded triangle in the triangulation of P.

Case 1. In this case the triangulation contains no self-folded triangle, and so there must be more than two spokes. We will demonstrate the case when a spoke in S is mutated and replaced by an arc to form S^*. The case when there is no self-folded triangle and an arc is replaced with another arc to form S^* is covered later in Case 7. The collection of diagonals in Figure 6.6 demonstrates this situation. Here, the diagonals in the left hand picture correspond to indecomposables in T, whilst the diagonals in the right hand picture correspond to indecomposables in T^*. We replace the diagonal s in $\operatorname{indec}\mathsf{T}$ with its mutation s^* in order to form $\operatorname{indec}\mathsf{T}^*$. Notice that the diagonals a, b, c, d are the edges of the enclosing quadrilateral of s and s^*. Hence, the generators $n(a), n(b), n(c), n(d)$ of N are the only ones containing $[s]$, and the generators $n^*(a), n^*(b), n^*(c), n^*(d)$ of N^* are the only ones containing $[s^*]$. Thus, to see that φ is well defined, we must check that ζ sends each of $n(a), n(b), n(c), n(d), n(s)$ to a linear combination of generators of N^*, and to check that η is well defined, we must show that ξ sends each of $n^*(a), n^*(b), n^*(c), n^*(d), n^*(s^*)$ to a linear combination of generators of N. Notice that the generator associated to every diagonal other than a, b, c, d, s, s^* appears as both a generator of N and a generator of N^*. There is therefore nothing to check for these generators.

We assume that a, b, c, d are nonzero and correspond to indecomposables in S. Since they are nonzero, there must be a triangle between themselves and the edge of P. These triangles are made up of the diagonals $e, f, \ldots, l$ in Figure 6.6. As discussed earlier in Remark 6.1.6 this suffices to prove that φ and η are well defined in a general case.

Using the results in Theorem 5.1.3, we can compute exchange triangles in T for the diagonals a, b, c, d, s. These exchange triangles are:

$$
\begin{array}{ll}
s \longrightarrow b \oplus d \longrightarrow s^*, & s^* \longrightarrow a \oplus c \longrightarrow s, \\
a \longrightarrow s \oplus f \longrightarrow a^*, & a^* \longrightarrow b \oplus e \longrightarrow a, \\
b \longrightarrow a \oplus h \longrightarrow b^*, & b^* \longrightarrow g \oplus s \longrightarrow b, \\
c \longrightarrow j \oplus s \longrightarrow c^*, & c^* \longrightarrow i \oplus d \longrightarrow c, \\
d \longrightarrow l \oplus c \longrightarrow d^*, & d^* \longrightarrow k \oplus s \longrightarrow d.
\end{array}
$$

From this we see that N has the following generators:

$$
\begin{array}{ll}
n(s) = [a] + [c] - [b] - [d], & n(a) = [b] + [e] - [s] - [f], \\
n(b) = [g] + [s] - [a] - [h], & n(c) = [i] + [d] - [j] - [s], \\
n(d) = [k] + [s] - [l] - [c]. &
\end{array}
\tag{6.2}
$$

The exchange triangles in T^* for s^*, a, b, c, d are:

$$
\begin{array}{ll}
s^* \longrightarrow a \oplus c \longrightarrow s, & s \longrightarrow b \oplus d \longrightarrow s^*, \\
a \longrightarrow d \oplus f \longrightarrow \bar{a}^*, & \bar{a}^* \longrightarrow e \oplus s^* \longrightarrow a, \\
b \longrightarrow h \oplus s^* \longrightarrow \bar{b}^*, & \bar{b}^* \longrightarrow c \oplus g \longrightarrow b, \\
c \longrightarrow b \oplus j \longrightarrow \bar{c}^*, & \bar{c}^* \longrightarrow i \oplus s^* \longrightarrow c, \\
d \longrightarrow l \oplus s^* \longrightarrow \bar{d}^*, & \bar{d}^* \longrightarrow a \oplus k \longrightarrow d,
\end{array}
$$

and thus N^* has the following generators:

$$
\begin{array}{ll}
n^*(s^*) = [b] + [d] - [a] - [c], & n^*(a) = [e] + [s^*] - [d] - [f], \\
n^*(b) = [c] + [g] - [h] - [s^*], & n^*(c) = [i] + [s^*] - [b] - [j], \\
n^*(d) = [a] + [k] - [l] - [s^*]. &
\end{array}
\tag{6.3}
$$

Note that in this case $[B] = [b] + [d]$, and so, $\varphi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*$ is defined by

$$
\varphi([t] + N) = \begin{cases} [b] + [d] - [s^*] + N^* & \text{if } t = s, \\ [t] + N^* & \text{if } t \neq s. \end{cases}
$$

In order to show that φ is well defined, we show that $\zeta : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)$, defined by

$$
\zeta([t]) = \begin{cases} [b] + [d] - [s^*] & \text{if } t = s, \\ [t] & \text{if } t \neq s, \end{cases}
$$

sends generators from N in (6.2) to linear combinations of generators of N^* in (6.3). Table 6.1 shows the generators of N in the left column and their images under ζ in the right hand column. Notice that for each $x \in \operatorname{indec}\mathsf{S}$, the map ζ has the property that either $\zeta(n(x)) = n^*(x)$ or $\zeta(n(x)) = n^*(x)+n^*(s^*)$. Therefore, since φ is the map between the quotient groups that is induced by ζ, it must be well defined.

$n(-) \in N$	$\zeta(n(-)) \in N^*$
$\boldsymbol{n(a)} = [b] + [e] - [s] - [f]$	$\boldsymbol{n^*(a)} = [e] + [s^*] - [d] - [f]$
$\boldsymbol{n(b)} = [g] + [s] - [a] - [h]$	$\boldsymbol{n^*(b) + n^*(s^*)} = [b] + [d] + [g] - [a] - [h] - [s^*]$
$\boldsymbol{n(c)} = [i] + [d] - [j] - [s]$	$\boldsymbol{n^*(c)} = [i] + [s^*] - [b] - [j]$
$\boldsymbol{n(d)} = [k] + [s] - [l] - [c]$	$\boldsymbol{n^*(d) + n^*(s^*)} = [b] + [d] + [k] - [c] - [l] - [s^*]$
$\boldsymbol{n(s)} = [a] + [c] - [b] - [d]$	$\boldsymbol{-n^*(s^*)} = [a] + [c] - [b] - [d]$

Table 6.1: ζ sends generators of N to linear combinations of generators of N^*.

Now, to see that η is well defined, we use the map $\xi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, defined by

$$\xi([t]) = \begin{cases} [b] + [d] - [s] & \text{if } t = s^* \\ [t] & \text{if } t \neq s^*, \end{cases}$$

and see that it sends the generators of N^* in (6.3) to linear combinations of generators of N in (6.2). Indeed, Table 6.2 shows the generators of N^* in the left hand column and their images under ξ in the right hand column. Notice that for each $x \in \operatorname{indec}\mathsf{S}^*$, the map ξ has the property that either $\xi(n^*(x)) = n(x)$ or $\xi(n^*(x)) = n(x) + n(s)$. Thus, since η is the map between the quotient groups induced by ξ, we see that it is well defined.

$n^*(-) \in N^*$	$\xi(n^*(-)) \in N$
$\boldsymbol{n^*(a)} = [e] + [s^*] - [d] - [f]$	$\boldsymbol{n(a)} = [b] + [e] - [s] - [f]$
$\boldsymbol{n^*(b)} = [c] + [g] - [h] - [s^*]$	$\boldsymbol{n(b) + n(s)} = [c] + [g] + [s] - [b] - [d] - [h]$
$\boldsymbol{n^*(c)} = [i] + [s^*] - [b] - [j]$	$\boldsymbol{n(c)} = [i] + [d] - [j] - [s]$
$\boldsymbol{n^*(d)} = [a] + [k] - [l] - [s^*]$	$\boldsymbol{n(d) + n(s)} = [a] + [k] + [s] - [b] - [d] - [l]$
$\boldsymbol{n^*(s^*)} = [b] + [d] - [a] - [c]$	$\boldsymbol{-n(s)} = [b] + [d] - [a] - [c]$

Table 6.2: ξ sends generators of N^* to linear combinations of generators of N.

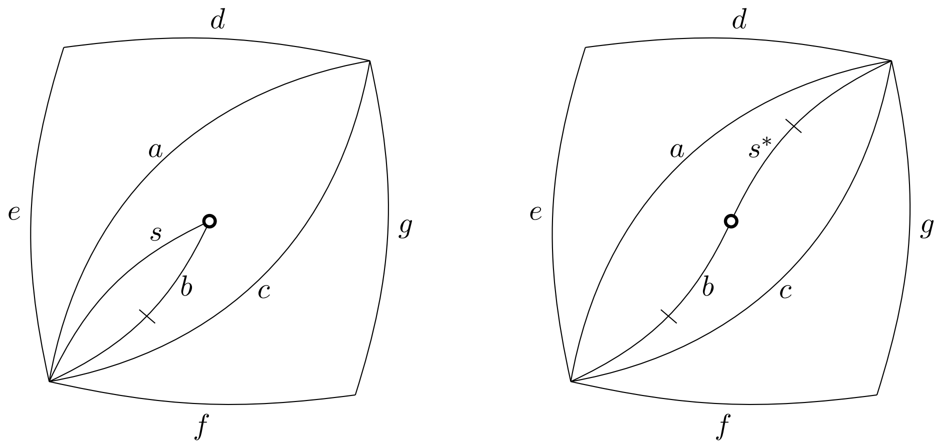


Figure 6.7: Case 2: We mutate and replace a spoke in the self-folded triangle in T, and form T^*.

Case 2. This is the case when a spoke inside the self-folded triangle is mutated to form T^*. We illustrate this here by the collections of diagonals in Figure 6.7, where the collection on the left corresponds to T, and the collection on the right corresponds to T^*. Again, we replace $s \in \operatorname{indec}\mathsf{T}$ with its mutation s^* in order to form $\operatorname{indec}\mathsf{T}^*$. Notice that the diagonals a, b, c are the diagonals forming the enclosing quadrilateral of s and s^*, and so it is the generators $n(a), n(b), n(c)$ of N that contain $[s]$ and $n^*(a), n^*(b), n^*(c)$ of N^* that contain $[s^*]$. Therefore, in order to see that φ is well defined we must check that ζ sends the generators $n(a), n(b), n(c), n(s)$ to linear combinatons of generators of N^*, and to see that η is well defined, we must check that ξ sends $n^*(a), n^*(b), n^*(c), n^*(s^*)$ to linear combinations of generators of N. Notice that the generators coming from all diagonals other than a, b, c, s, s^* appear as generators of both N and N^*, and so there is nothing to check for them.

We will assume again that each of a, b, c correspond to an indecomposable in S; that is, they are both nonzero and not in R. For them to be nonzero, there must be a triangle between each of them and the edge of P. These triangles correspond to the diagonals d, e, f, g in Figure 6.7.

Using the results in Theorem 5.1.3, we can compute the exchange triangles in T for the diagonals a, b, c, s. These exchange triangles are:

$$
\begin{aligned}
& s \longrightarrow a \longrightarrow s^*, && s^* \longrightarrow c \longrightarrow s, \\
& a \longrightarrow c \oplus e \longrightarrow a^*, && a^* \longrightarrow b \oplus d \oplus s \longrightarrow a, \\
& b \longrightarrow a \longrightarrow b^*, && b^* \longrightarrow c \longrightarrow b, \\
& c \longrightarrow g \oplus b \oplus s \longrightarrow c^*, && c^* \longrightarrow a \oplus f \longrightarrow c.
\end{aligned}
$$

From this we see that N has the following generators:

$$\begin{aligned} n(s) &= [c]-[a], & n(a) &= [b]+[d]+[s]-[c]-[e], \\ n(c) &= [a]+[f]-[b]-[g]-[s]. \end{aligned} \tag{6.4}$$

Note here that $n(s)$ and $n(b)$ are the same.

The exchange triangles in T^* for the diagonals s^*, a, b, c are:

$$\begin{array}{ll} s^* \longrightarrow c \longrightarrow s, & s \longrightarrow a \longrightarrow s^*, \\ a \longrightarrow e \oplus s^* \longrightarrow \bar{a^*}, & \bar{a^*} \longrightarrow b \oplus d \longrightarrow a, \\ b \longrightarrow a \longrightarrow b^*, & b^* \longrightarrow c \longrightarrow b, \\ c \longrightarrow b \oplus g \longrightarrow \bar{c^*}, & \bar{c^*} \longrightarrow s^* \oplus f \longrightarrow c. \end{array}$$

N^* has the following generators:

$$\begin{aligned} n^*(s^*) &= [a]-[c], & n^*(a) &= [b]+[d]-[e]-[s^*], \\ n^*(c) &= [s^*]+[f]-[b]-[g]. \end{aligned} \tag{6.5}$$

Notice that $n^*(b) = -n^*(s^*)$, and so $n^*(b)$ is not included in the list above. Now, note that in this case, $[B] = a$, and so $\varphi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*$ is defined by

$$\varphi([t]+N) = \begin{cases} [a]-[s^*]+N^* & \text{if } t = s, \\ [t]+N^* & \text{if } t \neq s. \end{cases}$$

In order to show that φ is well defined, we consider the map $\zeta : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)$, defined by

$$\zeta([t]) = \begin{cases} [a]-[s^*] & \text{if } t = s, \\ [t] & \text{if } t \neq s, \end{cases}$$

and show that it sends generators of N to linear combinations of generators of N^*.

$n(-) \in N$	$\zeta(n(-)) \in N^*$
$\boldsymbol{n(a)} = [b]+[d]+[s]-[e]-[c]$	$\boldsymbol{n^*(a)+n^*(s^*)} = [a]+[b]+[d]-[c]-[e]-[s^*]$
$\boldsymbol{n(c)} = [a]+[f]-[b]-[g]-[s]$	$\boldsymbol{n^*(c)} = [f]+[s^*]-[g]-[b]$
$\boldsymbol{n(s)} = [c]-[a]$	$\boldsymbol{-n^*(s^*)} = [c]-[a]$

Table 6.3: ζ sends generators of N to linear combinations of generators of N^*.

Table 6.3 shows the generators of N in the left hand column and their images under

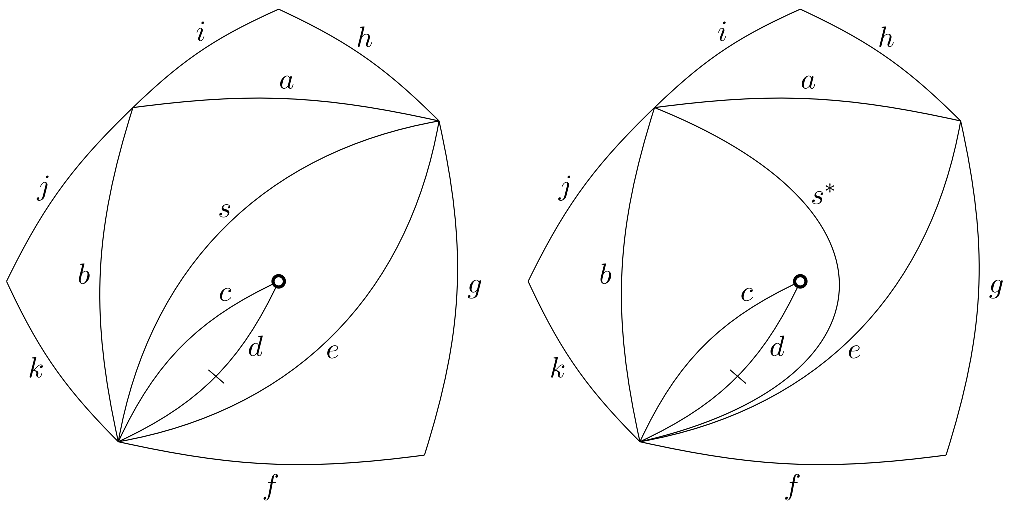


Figure 6.8: Case 3: We replace an arc of the self-folded triangle in T with its mutation, and form T^*. Here, both spokes of the self-folded triangle meet the edge of P at the same vertex.

ζ in the right hand column. This table shows that for each $x \in \operatorname{indec} \mathsf{S}$, the map ζ has the property that either $\zeta(n(x)) = n^*(x)$ or $\zeta(n(x)) = n^*(x) + n^*(s^*)$. Therefore, since φ is the map between $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ and $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*$ induced by ζ, we know that it is well defined.

In order to show that η is well defined, we check that $\xi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, defined by

$$\xi([t]) = \begin{cases} [a] - [s] & \text{if } t = s^*, \\ [t] & \text{if } t \neq s^*, \end{cases}$$

sends generators of N^* to generators of N.

$n^*(-) \in N^*$	$\xi(n^*(-)) \in N$
$\boldsymbol{n^*(a)} = [b] + [d] - [e] - [s^*]$	$\boldsymbol{n(a) + n(s)} = [b]+[d]+[s]-[a]-[e]$
$\boldsymbol{n^*(c)} = [f] + [s^*] - [b] - [g]$	$\boldsymbol{n(c)} = [f] - [b] - [g] + ([a] - [s]$
$\boldsymbol{n^*(s^*)} = [a] - [c]$	$\boldsymbol{-n(s)} = [a] - [c]$

Table 6.4: ξ sends generators of N^* to linear combinations of generators of N.

Table 6.4 gives the generators of N^* in the left hand column and their images under ξ in the right hand column. It shows that for each $x \in \operatorname{indec} \mathsf{S}^*$, we have either $\xi(n^*(x)) = n(x)$ or $\xi(n^*(x)) = n(x) + n(s)$, and since η is the map from $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*$ to $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ induced by ξ, it must be well defined.

Case 3. This is the case when an arc of the self-folded triangle is mutated to form T^* and both spokes inside the self-folded triangle meet the edge of P at the same vertex. Figure 6.8

illustrates this situation, where the collection of diagonals on the left corresponds to T, and the collection of diagonals on the right corresponds to T^*. Again, $s \in \operatorname{indec}\mathsf{T}$ is replaced by its mutation s^* in order to form $\operatorname{indec}\mathsf{T}^*$. Note that the generators for N that are affected by this mutation are $n(a), n(b), n(c), n(d), n(e)$, as the diagonals a, b, c, d, e are the diagonals forming the enclosing quadrilateral of s in Figure 6.8. These are the generators containing the $\mathrm{K}_0^{\mathrm{split}}$-class of s, so in order to check that φ is well defined, we must check that ζ sends each of these generators as well as $n(s)$ to linear combinations of generators of N^*. Similarly, a, b, c, d, e are the diagonals forming the enclosing quadrilateral of s^*, and so the generators $n^*(a), n^*(b), n^*(c), n^*(d), n^*(e)$ are the generators of N^* containing the $\mathrm{K}_0^{\mathrm{split}}$-class of s^*. So, in order to check that η is well defined, we should check that ξ sends each of these generators as well as $n^*(s^*)$ to linear combinations of generators of N. Notice that the generators of all diagonals other than a, b, c, d, e, s, s^* appear as both generators of N and N^*, and so there is nothing to check for them.

For now, we assume all of the diagonals in Figure 6.8 to be nonzero and not in R; that is, they all correspond to indecomposables in S. By definition of the triangulation of P, for each of a, b, c, d, e to be nonzero, that is, not on the edge of P, there must be a triangle between each of them and the edge of P. These triangles correspond to the remaining diagonals in Figure 6.8.

Since all diagonals in the figure are nonzero and not in R, each must produce an associated generator for the subgroup. To check that φ is well defined, we check that it maps generators of N to linear combinations of generators in N^*.

Using the results of Theorem 5.1.3, the exchange triangles in T for the diagonals a, b, c, d, e and s in are as follows:

$$
\begin{aligned}
&s \longrightarrow e \oplus b \longrightarrow s^*, && s^* \longrightarrow a \oplus c \oplus d \longrightarrow s,\\
&a \longrightarrow i \oplus s \longrightarrow a^*, && a^* \longrightarrow h \oplus b \longrightarrow a,\\
&b \longrightarrow a \oplus k \longrightarrow b^*, && b^* \longrightarrow s \oplus j \longrightarrow b,\\
&c \longrightarrow s \longrightarrow c^*, && c^* \longrightarrow e \longrightarrow c,\\
&d \longrightarrow s \longrightarrow d^*, && d^* \longrightarrow e \longrightarrow d,\\
&e \longrightarrow c \oplus d \oplus g \longrightarrow e^*, && e^* \longrightarrow s \oplus f \longrightarrow e.
\end{aligned}
$$

The subgroup N has the following generators:

$$
\begin{aligned}
n(s) &= [a] + [c] + [d] - [e] - [b], & n(a) &= [h] + [b] - [i] - [s],\\
n(b) &= [s] + [j] - [a] - [k], & n(c) &= [e] - [s],\\
n(e) &= [s] + [f] - [c] - [d] - [g].
\end{aligned}
\tag{6.6}
$$

Since $n(c) = n(d)$, we omit $n(d)$ from the list above.

The exchange triangles in T^* for the diagonals a, b, c, d, e and s^* are:

$$\begin{array}{ll}
s^* \longrightarrow a \oplus c \oplus d \longrightarrow s, & s \longrightarrow b \oplus e \longrightarrow s^*, \\
a \longrightarrow i \oplus e \longrightarrow \bar{a^*}, & \bar{a^*} \longrightarrow s^* \oplus h \longrightarrow a, \\
b \longrightarrow s^* \oplus k \longrightarrow \bar{b^*}, & \bar{b^*} \longrightarrow c \oplus d \oplus j \longrightarrow b, \\
c \longrightarrow b \longrightarrow \bar{c^*}, & \bar{c^*} \longrightarrow s^* \longrightarrow c, \\
d \longrightarrow b \longrightarrow \bar{d^*}, & \bar{d^*} \longrightarrow s^* \longrightarrow d, \\
e \longrightarrow s^* \oplus g \longrightarrow \bar{e^*}, & \bar{e^*} \longrightarrow a \oplus f \longrightarrow e.
\end{array}$$

The subgroup N^* has the following generators:

$$\begin{array}{cc}
n^*(s^*) = [e] + [b] - [a] - [c] - [d], & n^*(a) = [h] + [s^*] - [i] - [e], \\
n^*(b) = [c] + [d] + [j] - [s^*] - [k] & n^*(c) = [s^*] - [b], \\
n^*(e) = [a] + [f] - [s^*] - [g]. &
\end{array} \tag{6.7}$$

Again, since $n^*(c) = n^*(d)$, we omit one of these generators from the list.

Now, note that in this case, $B = [e] + [b]$, and so $\varphi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*$ is defined as follows

$$\varphi([t] + N) = \begin{cases} [b] + [e] - [s^*] + N^* & \text{if } t = s, \\ [t] + N^* & \text{if } t \neq s. \end{cases}$$

In order to show that φ is well defined, we consider the map $\zeta : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)$, defined by

$$\zeta([t]) = \begin{cases} [b] + [e] - [s^*] & \text{if } t = s, \\ [t] & \text{if } t \neq s, \end{cases}$$

and show that it send generators of N to generators of N^*.

An easy computation allows us to compute the values of ζ on the generators of N. Table 6.5 shows the generators of N in the left column and their images under ζ in the right column. This table shows that for any $x \in \operatorname{indec} \mathsf{S}$, we have either $\zeta(n(x)) = n^*(x)$ or $\zeta(n(x)) = n^*(x) + n^*(s^*)$, and since φ is the map from $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ to $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*$ induced by ζ, it must be well defined.

To check that η is well defined, we consider the map $\xi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, defined by

$$\xi([t]) = \begin{cases} [b] + [e] - [s] & \text{if } t = s^*, \\ [t] & \text{if } t \neq s^*, \end{cases}$$

$n(-) \in N$	$\zeta(n(-)) \in N^*$
$\boldsymbol{n(a)} = [h] + [b] - [i] - [s]$	$\boldsymbol{n^*(a)} = [h] + [s^*] - [i] - [e]$
$\boldsymbol{n(b)} = [s] + [j] - [a] - [k]$	$\boldsymbol{n^*(b) + n^*(s^*)} = [e] + [b] + [j] - [s^*] - [a] - [k]$
$\boldsymbol{n(c)} = [e] - [s]$	$\boldsymbol{n^*(c)} = [s^*] - [b]$
$\boldsymbol{n(e)} = [s] + [f] - [c] - [d] - [g]$	$\boldsymbol{n^*(e) + n^*(s^*)} = [e] + [b] + [f] - [c] - [d] - [g] - [s^*]$
$\boldsymbol{n(s)} = [a] + [c] + [d] - [e] - [b]$	$\boldsymbol{-n^*(s^*)} = [a] + [c] + [d] - [e] - [b]$

Table 6.5: ζ sends generators of N to linear combinations of generators of N^*

and show that is sends generators of N^* to linear combinations of generators of N.

Table 6.6 shows the generators of N^* in the left column and their images under ξ in the right hand column. The table shows that ξ has the property that for each $x \in \operatorname{indec} \mathsf{S}^*$, we have either $\xi(n^*(x)) = n(x)$ or $\xi(n^*(x)) = n(x) + n(s)$, and since η is the map between the quotient groups induced by ξ, we see that it is well defined.

$n^*(-) \in N^*$	$\xi(n^*(-)) \in N$
$\boldsymbol{n^*(a)} = [s^*] + [h] - [i] - [e]$	$\boldsymbol{n(a)} = [b] + [h] - [i] - [s]$
$\boldsymbol{n^*(b)} = [c] + [d] + [j] - [s^*] - [k]$	$\boldsymbol{n(b) + n(s)} = [c] + [d] + [j] + [s] - [k] - [e] - [b]$
$\boldsymbol{n^*(c)} = [s^*] - [b]$	$\boldsymbol{n(c)} = [e] - [s]$
$\boldsymbol{n^*(e)} = [a] + [f] - [s^*] - [g]$	$\boldsymbol{n(e) + n(s)} = [a] + [f] + [s] - [b] - [e] - [g]$
$\boldsymbol{n^*(s^*)} = [b] + [e] - [a] - [c] - [d]$	$\boldsymbol{-n(s)} = [b] + [e] - [a] - [c] - [d]$

Table 6.6: ξ sends generators of N^* to linear combinations of generators of N.

Case 4. This is the case when an arc of the self-folded triangle is mutated and replaced by a spoke to form a new cluster tilting subcategory. This case demonstrates a situation where one cluster tilting subcategory contains a self-folded triangle, whilst the other does not contain one. The situation described is pictured using the collection of diagonals in Figure 6.9, which we will refer to for the rest of this case. Here, s is the arc that we mutate and replace by s^* to form $\operatorname{indec} \mathsf{T}^*$. The diagonals a, b, c, d are those that make up the enclosing quadrilateral of s and s^*, and it is therefore the subgroup generators $n(a), n(b), n(c), n(d)$ that contain $[s]$ and $n^*(a), n^*(b), n^*(c), n^*(d)$ that contain $[s^*]$. As before, to check that φ is well defined, we must show that ζ sends each of $n(a), n(b), n(c), n(d), n(s)$ to a linear combination of generators of N^*, and to show that η is well defined, we check that ξ

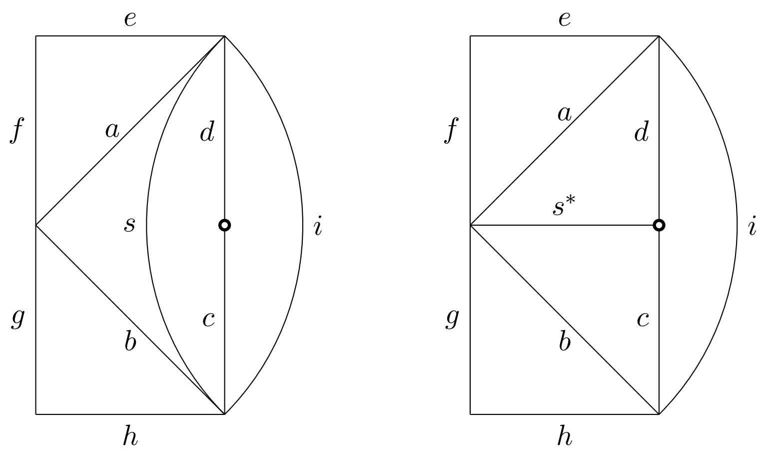


Figure 6.9: Case 4: We replace an arc of the self-folded triangle with its mutation and form a new cluster tilting subcategory. Here, both spokes of the self-folded triangle meet the edge of P at different vertices.

sends each of $n^*(a), n^*(b), n^*(c), n^*(d), n^*(s^*)$ to a linear combination of generators of N. Since the generators associated to all diagonals other than a, b, c, d, s, s^* appear as both generators of N and N^*, there is nothing to check for them.

We assume that each of the diagonals a, b, c, d are nonzero and not in R. Since they are nonzero, there must be a triangle between each of them and the edge of P. These triangles correspond to the diagonals $e, f, \ldots, i$ in the figure. Note that since we have specified that the triangulation contains only 3 spokes, we are forced to include the arc i on the exterior edges of c and d.

Using Theorem 5.1.3, we can compute the exchange triangles in T for s, a, b, c, d. These are as follows:

$$\begin{aligned}
&s \longrightarrow b \oplus d \longrightarrow s^*, \qquad && s^* \longrightarrow a \oplus c \longrightarrow s,\\
&a \longrightarrow f \oplus s \longrightarrow a^*, && a^* \longrightarrow b \oplus e \longrightarrow a,\\
&b \longrightarrow a \oplus h \longrightarrow b^*, && b^* \longrightarrow g \oplus s \longrightarrow b,\\
&c \longrightarrow s \longrightarrow c^*, && c^* \longrightarrow i \longrightarrow c,\\
&d \longrightarrow i \longrightarrow d^*, && d^* \longrightarrow s \longrightarrow d,
\end{aligned}$$

The middle terms of the exchange triangles form the generators of N as follows:

$$\begin{aligned}
n(s) &= [a] + [c] - [b] - [d], \qquad & n(a) &= [b] + [e] - [f] - [s],\\
n(b) &= [g] + [s] - [a] - [h], & n(c) &= [i] - [s].
\end{aligned} \tag{6.8}$$

Notice that $n(c) = -n(d)$, and so we omit the generator $n(d)$ from the above list of generators.

The exchange triangles in T^* for s^*, a, b, c, d are:

$$\begin{array}{ll}
s^* \longrightarrow a \oplus c \longrightarrow s, & s \longrightarrow b \oplus d \longrightarrow s^*, \\
a \longrightarrow d \oplus f \longrightarrow \bar{a}^*, & \bar{a}^* \longrightarrow e \oplus s^* \longrightarrow a, \\
b \longrightarrow h \oplus s^* \longrightarrow \bar{b}^*, & \bar{b}^* \longrightarrow c \oplus g \longrightarrow b, \\
c \longrightarrow b \oplus d \longrightarrow \bar{c}^*, & \bar{c}^* \longrightarrow i \oplus s^* \longrightarrow c, \\
d \longrightarrow i \oplus s^* \longrightarrow \bar{d}^*, & \bar{d}^* \longrightarrow a \oplus c \longrightarrow d.
\end{array}$$

and then N^* has the following generators:

$$\begin{array}{ll}
n^*(s^*) = [b] + [d] - [a] - [c], & n^*(a) = [e] + [s^*] - [d] - [f], \\
n^*(b) = [c] + [g] - [h] - [s^*], & n^*(c) = [i] + [s^*] - [b] - [d].
\end{array} \tag{6.9}$$

Notice here that since in the quotient $[a] + [c] + N^* = [b] + [d] + N^*$, the middle terms of the exchange triangles for c and d give equal generators in N^*. We therefore omit one of these generators from the above list.

In this case, $[B] = [b] + [d]$, and so $\varphi : K_0^{\text{split}}(\mathsf{T})/N \to K_0^{\text{split}}(\mathsf{T}^*)/N^*$ is defined by

$$\varphi([t] + N) = \begin{cases} [b] + [d] - [s^*] + N^* & \text{if } t = s, \\ [t] + N^* & \text{if } t \neq s. \end{cases}$$

Now, in order to check that φ is well defined, we use the map $\zeta : K_0^{\text{split}}(\mathsf{T}) \to K_0^{\text{split}}(\mathsf{T}^*)$, given by

$$\zeta([t]) = \begin{cases} [a] + [c] - [s^*] & \text{if } t = s, \\ [t] & \text{if } t \neq s. \end{cases}$$

and show that it sends generators of N to linear combinations of generators of N^*. Indeed, Table 6.7 shows the generators of N in the left column and their images under ζ in the right column. We see that for each $x \in \operatorname{indec} \mathsf{S}$, the map ζ carries the property that either $\zeta(n(x)) = n^*(x)$ or $\zeta(n(x)) = n^*(x) + n^*(s^*)$. Therefore, since ζ induces the map φ between the quotient groups, φ is well defined.

In order to see that η is well defined, we define the map $\xi : K_0^{\text{split}}(\mathsf{T}^*) \to K_0^{\text{split}}(\mathsf{T})$, by

$$\xi([t]) = \begin{cases} [b] + [d] - [s] & \text{if } t = s^*, \\ [t] & \text{if } t \neq s^*. \end{cases}$$

and show that it maps generators of N^* in (6.9) to linear combinations of generators of N in (6.8). Table 6.8 shows the generators of N^* in the left hand column and their images under ξ in the right hand column. For each $x \in \operatorname{indec} \mathsf{S}^*$, the map ξ clearly has

$n(-) \in N$	$\zeta(n(-)) \in N^*$
$\boldsymbol{n(a)} = [b] + [e] - [f] - [s]$	$\boldsymbol{n^*(a)} = [e] + [s^*] - [d] - [f]$
$\boldsymbol{n(b)} = [g] + [s] - [a] - [h]$	$\boldsymbol{n^*(b) + n^*(s^*)} = [b] + [d] + [g] - [a] - [h] - [s^*]$
$\boldsymbol{n(c)} = [i] - [s]$	$\boldsymbol{n^*(c)} = [i] + [s^*] - [b] - [d]$
$\boldsymbol{n(s)} = [a] + [c] - [b] - [d]$	$\boldsymbol{-n^*(s^*)} = [a] + [c] - [b] - [d]$

Table 6.7: ζ sends generators of N to linear combinations of generators of N^*.

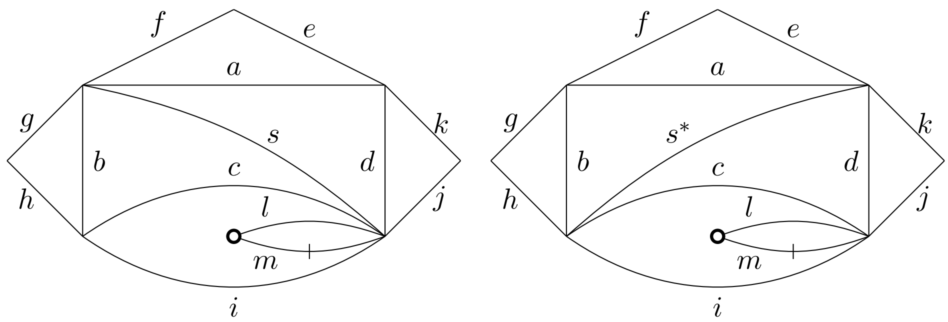


Figure 6.10: Case 5: Collections of diagonals in P corresponding to indecomposables in T and T^*.

the property that either $\xi(n^*(x)) = n(x)$ or $\xi(n^*(x)) = n(x) + n(s)$. We can therefore conclude the the map between the quotient groups induced by ξ, namely η, is well defined.

$n^*(-) \in N^*$	$\xi(n^*(-)) \in N$
$\boldsymbol{n^*(a)} = [e] + [s^*] - [d] - [f]$	$\boldsymbol{n(a)} = [b] + [e] - [f] - [s]$
$\boldsymbol{n^*(b)} = [c] + [g] - [h] - [s^*]$	$\boldsymbol{n(b) + n(s)} = [c] + [g] + [s] - [b] - [d] - [h]$
$\boldsymbol{n^*(c)} = [i] + [s^*] - [b] - [d]$	$\boldsymbol{n(c)} = [i] - [s]$
$\boldsymbol{n^*(s^*)} = [b] + [d] - [a] - [c]$	$\boldsymbol{-n(s)} = [b] + [d] - [a] - [c]$

Table 6.8: ξ sends generators of N^* to linear combinations of generators of N.

Case 5. This case and the following case demonstrate the situations when the self-folded triangle shares an edge with the enclosing quadrilateral of s and s^*. In Case 5, both spokes inside the self-folded triangle meet the edge of P at the same vertex. The collection of diagonals in Figure 6.10 illustrates the situation, where the collection on the left corresponds to T and the collection the right corresponds to T^*. Again, to form $\operatorname{indec}\mathsf{T}^*$, we remove $s \in \operatorname{indec}\mathsf{T}$ and replace it with its mutation s^*. Notice that a, b, c, d are the diagonals that form the enclosing quadrilateral of s and s^*, and thus the generators $n(a), n(b), n(c), n(d)$

of N are those containing $[s]$, and the generators $n^*(a), n^*(b), n^*(c), n^*(d)$ of N^* are those containing $[s^*]$. In order to see that φ is well defined we check that ζ sends each of $n(a), n(b), n(c), n(d), n(s)$ to a linear combination of generators of N^*, and in order to show that η is well defined we check that ξ sends each of $n^*(a), n^*(b), n^*(c), n^*(d), n^*(s^*)$ to a linear combination of generators of N. Since generators associated to all diagonals other than a, b, c, d, s, s^* appear as both generators of N and N^*, there is nothing to check for them.

We again assume that each of a, b, c, d are nonzero and not in R . They therefore have a triangle between themselves and the edge of P, and in this case we let one of these triangles be the self-folded triangle. These triangles correspond to the diagonals $e, f, \dots, m$ in Figure 6.10.

Since the diagonals a, b, d are contained within a region of P that contains a triangulation formed by noncrossing arcs, we know already from Section 4.2 that ζ maps $n(a), n(b), n(d)$ to linear combinations of generators of N^* and ξ maps $n^*(a), n^*(b), n^*(d)$ to linear combinations of generators of N. It is therefore appropriate to only check the corresponding statements for the generators associated to c and s.

Using Theorem 5.1.3, we can compute the exchange triangles in T for c and s. These are as follows:

$$s \longrightarrow a \oplus c \longrightarrow s^*, \qquad s^* \longrightarrow b \oplus d \longrightarrow s,$$
$$c \longrightarrow b \oplus l \oplus m \longrightarrow c^*, \qquad c^* \longrightarrow i \oplus s \longrightarrow c.$$

The middle terms of these then form generators for N as follows:

$$n(s) = [b] + [d] - [a] - [c], \qquad n(c) = [i] + [s] - [b] - [l] - [m]. \tag{6.10}$$

In T^* the exchange triangles for s^* and c are:

$$s^* \longrightarrow b \oplus d \longrightarrow s, \qquad s \longrightarrow a \oplus c \longrightarrow s^*,$$
$$c \longrightarrow s^* \oplus l \oplus m \longrightarrow \bar{c}^*, \qquad \bar{c}^* \longrightarrow d \oplus i \longrightarrow c,$$

and therefore N^* has the following generators:

$$n^*(s^*) = [a] + [c] - [b] - [d], \qquad n^*(c) = [d] + [i] - [s^*] - [l] - [m]. \tag{6.11}$$

Note now that in this case, $\varphi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*$ is defined by

$$\varphi([t] + N) = \begin{cases} [a] + [c] - [s^*] + N^* & \text{if } t = s, \\ [t] + N^* & \text{if } t \neq s. \end{cases}$$

and in order to show that it is well defined we use the map $\zeta : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)$, defined by

$$\zeta([t]) = \begin{cases} [a]+[c]-[s^*] & \text{if } t = s, \\ [t] & \text{if } t \neq s, \end{cases}$$

and show that it maps the generators of N to linear combinations of generators of N^*. Table 6.9 shows generators of N in the left column and their images under ζ in the right hand column. We can see from the table that either $\zeta(n(x)) = n^*(x)$ or $\zeta(n(x)) = n^*(x) + n^*(s^*)$, whenever $x \in \operatorname{indec} \mathsf{S}$. As φ is the map between the quotient groups that is induced by ζ, we can conclude that it is well defined.

$n(-) \in N$	$\zeta(n(-)) \in N^*$
$\boldsymbol{n(c)} = [i]+[s]-[b]-[l]-[m]$	$\boldsymbol{n^*(c)+n^*(s^*)} = [a]+[c]+[i]-[b]-[l]-[m]-[s^*]$
$\boldsymbol{n(s)} = [b]+[d]-[a]-[c]$	$\boldsymbol{-n^*(s^*)} = [b]+[d]-[a]-[c]$

Table 6.9: ζ sends generators of N to linear combinations of generators of N^*.

To show that η is well defined, we once again use the map $\xi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, given by

$$\xi([t]) = \begin{cases} [a]+[c]-[s] & \text{if } t = s^*, \\ [t] & \text{if } t \neq s^*. \end{cases}$$

and show that it sends the generators of N^* to linear combinations of generators of N. Table 6.12 shows the generators of N^* in the left hand column and their images under ξ in the right hand column. We see that either $\xi(n^*(x)) = n(x)$ or $\xi(n^*(x)) = n(x) + n(s)$ for each $x \in \operatorname{indec} \mathsf{S}^*$. Since η is the map between the quotient groups that is induced by ξ we know that it must also be well defined.

$n^*(-) \in N^*$	$\xi(n^*(-)) \in N$
$\boldsymbol{n^*(c)} = [d]+[i]-[s^*]-[l]-[m]$	$\boldsymbol{n(c)+n(s)} = [a]+[c]+[i]-[b]-[l]-[m]-[s^*]$
$\boldsymbol{n^*(s^*)} = [a]+[c]-[b]-[d]$	$\boldsymbol{-n(s)} = [a]+[c]-[b]-[d]$

Table 6.10: ξ sends generators of N^* to linear combinations of generators of N.

Case 6. This case is similar to the previous, but considers the situation when the two spokes inside the self-folded triangle meet the edge of P at different vertices. Other than this, our setup is identical to the previous case. This case is pictured using the collections of diagonals in Figure 6.11, and we again assume that every diagonal in this figure is both

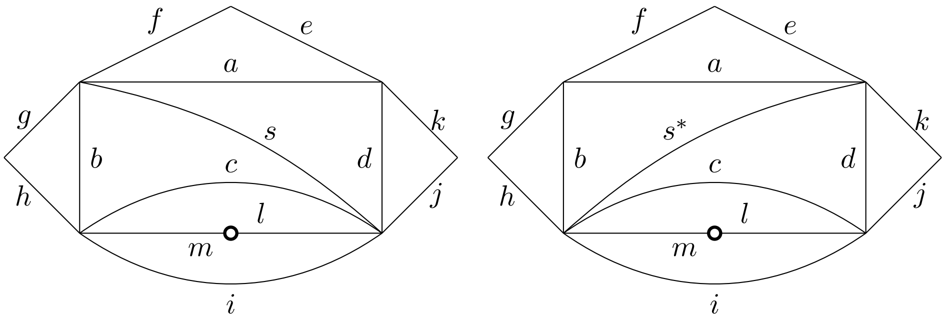


Figure 6.11: Case 6: Collections of diagonals in P corresponding to indecomposables in T and T^*.

nonzero and not in R. Due to the logic in the previous case, we know already that ζ maps $n(a), n(b), n(d)$ to N^* and ξ maps $n^*(a), n^*(b), n^*(d)$ to N. The generators associated to c and s or s^* in each subgroup are therefore the only ones to check.

Using Theorem 5.1.3, we compute the exchange triangles in T for s and c as follows:

$$s \longrightarrow a \oplus c \longrightarrow s^*, \qquad s^* \longrightarrow b \oplus d \longrightarrow s,$$
$$c \longrightarrow b \oplus l \longrightarrow c^*, \qquad c^* \longrightarrow m \oplus s \longrightarrow c.$$

The subgroup N has the following generators, among others:

$$n(s) = [b] + [d] - [a] - [c], \qquad n(c) = [m] + [s] - [b] - [l]. \tag{6.12}$$

The exchange triangles in T^* for s^* and c are

$$s^* \longrightarrow b \oplus d \longrightarrow s, \qquad s \longrightarrow a \oplus c \longrightarrow s^*,$$
$$c \longrightarrow l \oplus s^* \longrightarrow \bar{c}^*, \qquad \bar{c}^* \longrightarrow d \oplus m \longrightarrow c,$$

and therefore N^* has the following generators, among others:

$$n^*(s^*) = [a] + [c] - [b] - [d], \qquad n^*(c) = [d] + [m] - [l] - [s^*. \tag{6.13}$$

Note that here $[B] = [a] + [c]$, and so $\varphi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*$ is defined by

$$\varphi([t] + N) = \begin{cases} [a] + [c] - [s^*] + N^* & \text{if } t = s, \\ [t] + N^* & \text{if } t \neq s. \end{cases}$$

To check that φ is well defined, we use the map $\zeta : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)$, given by

$$\zeta([t]) = \begin{cases} [a] + [c] - [s^*] & \text{if } t = s, \\ [t] & \text{if } t \neq s, \end{cases}$$

and show that it ζ sends generators of N to linear combinations of generators of N^*. Indeed, Table 6.11 contains the generators of N in the left column and their images under ζ in the right hand column. It is clear that either $\zeta(n(x)) = n^*(x)$ or $\zeta(n(x)) = n^*(x)+n^*(s^*)$ for each $x \in \operatorname{indec} \mathsf{S}$. As φ is the map between the quotient groups that is induced by ζ, it must be well defined.

$n(-) \in N$	$\zeta(n(-)) \in N^*$
$\boldsymbol{n(c)} = [m] + [s] - [b] - [l]]$	$\boldsymbol{n^*(c) + n^*(s^*)} = [a] + [c] + [m] - [b] - [l] - [s^*]$
$\boldsymbol{n(s)} = [b] + [d] - [a] - [c]$	$\boldsymbol{-n^*(s^*)} = [b] + [d] - [a] - [c]$

Table 6.11: ζ sends generators of N to linear combinations of generators of N^*.

In order to show that η is well defined we use the map $\xi : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, defined by

$$\xi([t]) = \begin{cases} [a] + [c] - [s] & \text{if } t = s^*, \\ [t] & \text{if } t \neq s^*, \end{cases}$$

and show that it sends the generators of N^* to linear combinations of generators of N. Well, Table 6.12 shows the generators of N^* in the left column and their images under ξ in the right column. We see from this that either $\xi(n^*(x)) = n(x)$ or $\xi(n^*(x)) = n(x) + n(s)$ for each $x \in \operatorname{indec} \mathsf{S}^*$. As η is the map between the quotient groups induced by ξ we can conclude that it is well defined.

$n^*(-) \in N^*$	$\xi(n^*(-)) \in N$
$\boldsymbol{n^*(c)} = [d] + [m] - [l] - [s^*]$	$\boldsymbol{n(c) + n(s)} = [d]+[m]+[s]-[a]-[c]-[l]$
$\boldsymbol{n^*(s^*)} = [a] + [c] - [b] - [d]$	$\boldsymbol{-n(s)} = [a] + [c] - [b] - [d]$

Table 6.12: ξ sends generators of N^* to linear combinations of generators of N.

Case 7. This final case is similar to the previous two cases, however it covers the situation when the self-folded triangle does not appear in the enclosing quadrilateral of s and s^*. This case also suffices to cover the situation when there is no self-folded triangle and an arc from S is mutated and replaced by another arc to form S^*. Figure 6.12 illustrates the situation. Notice here that the diagonals a, b, c, d that form the enclosing quadrilateral of s and s^* sit in the interior of one of the subpolygons of P containing a triangulation made up of noncrossing arcs. This case is therefore equivalent to the situation in Section 4.2, and so it has already been proved that φ and η are well defined here.

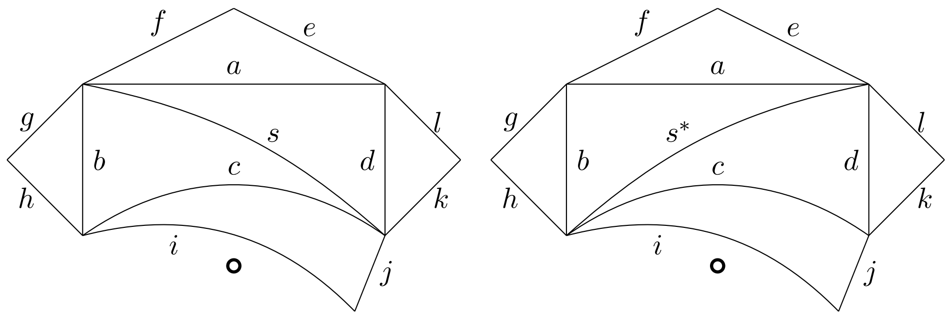


Figure 6.12: Case 7: Collections of diagonals in P corresponding to indecomposables in T and T^*.

Theorem 6.1.7. *The quotient group*

$$\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$$

depends only on the subcategory R*.*

Proof. Since any triangulation of P can be obtained from any other triangulation of P through a series of mutations of diagonals, it suffices to show that φ and η are inverse to each other. This shows that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ and $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}^*)/N^*$ are isomorphic.

Since $\varphi([x]+N) = [x]+N^*$ and $\eta([x]+N^*) = [x]+N$ for every $x \notin \{s, s^*\}$, it is clear that φ and η are inverse to each other for such x. We check s and s^*. Indeed,

$$\begin{aligned} \eta\varphi([s]+N) &= \eta([B]-[s^*]+N^*) \\ &= [B]-[B]+[s]+N \\ &= [s]+N, \end{aligned} \tag{6.14}$$

where for the second =, we recall that $[B]$ does not contain $[s^*]$. Also,

$$\begin{aligned} \varphi\eta([s^*]+N^*) &= \varphi([B]-[s]+N) \\ &= [B]-[B]+[s^*]+N^* \\ &= [s^*]+N^*, \end{aligned} \tag{6.15}$$

recalling for the second = that $[B]$ does not contain $[s]$. $\square$

6.2 Computing $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$

In this section we will compute a formula for $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ in the case when $\mathsf{C} = \mathsf{C}(D_n)$. This requires several stages which we outline here. The proof of our main formula will induct on the number of arcs in R in a triangulation of the punctured polygon P, made up of diagonals in R and S. It uses so-called "central regions" as a base case and then by "gluing" so-called "type A cells" to the central regions, we increase the number of arcs in R. Section 6.2.1 first defines the notions of a central region and of type A cells. It then goes on to describe an "ungluing" procedure, which allows us to construct any triangulation of a punctured polygon P, made up of diagonals in R and S, through the gluing procedure. In Section 6.2.2, we introduce the notion of "fitting", which provides us with a method of constructing any given central region, and then we describe the gluing procedure in detail in Section 6.2.3. In Section 6.2.4 we describe the three different cases of gluing and their effects on $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$. In each of the three cases, we construct a commutative square which we use in Section 6.2.5 to produce a Snake Lemma argument used to compute the rank of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ in the proof of our final formula. As mentioned, the proof of the main theorem is an inductive proof in which we use the central regions as a base case. There are two types of central region. Section 6.2.6 computes a formula for $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ for the first type of central region, and Section 6.2.7 computes a formula for $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ for the second type. Finally, in Section 6.2.8 we provide a general formula for $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ for a general punctured polygon P containing a triangulation made up of diagonals in R and S. After providing some examples, we prove the formula using the procedures and results from the previous sections.

6.2.1 Ungluing

We start here by describing an "ungluing" procedure that allows us to construct any given triangulation of a punctured polygon P, made up of diagonals in R and S, through a certain "gluing" procedure, which we describe in Section 6.2.2. Informally, given a punctured polygon P containing a triangulation made up of diagonals in R and S, the ungluing procedure describes a process for removing so-called "type A cells" from P in order to produce a "central region" of the triangulation. Before describing the ungluing procedure more formally, we first give definitions of the central region and of type A cells.

Definition 6.2.1. Let P be a punctured polygon containing a triangulation made up of diagonals in R and S. We say that P is a central region if every indecomposable in R corresponds to a spoke in the triangulation.

Definition 6.2.2. Let P be an unpunctured polygon. We say that P is a type A cell if it contains a triangulation made up entirely of diagonals in S.

Remark 6.2.3. (*The "ungluing" procedure*). Consider a triangulation of a punctured polygon P' made up of diagonals in R and S. The polygon dissection coming from R may contain both spokes and arcs. If it does contains arcs, then recall from Remark 6.1.1 that any arc r' splits P' into two cells; one cell A containing the puncture and another cell B containing a triangulation made up of noncrossing arcs in both R and S, as in Figure 6.13.

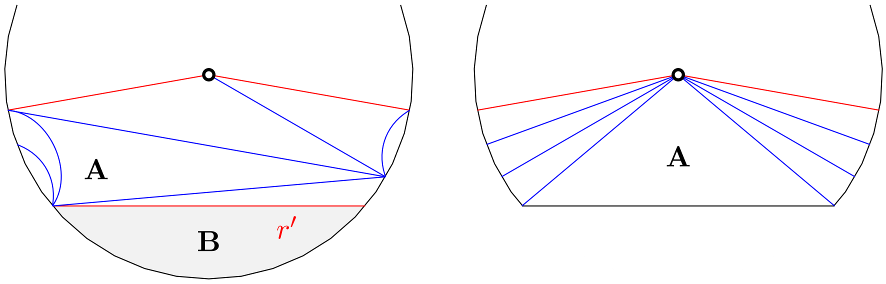


Figure 6.13: The ungluing procedure removes all exterior cells of the triangulation of P, leaving us with the central region. Here, red diagonals correspond to indecomposables in R and blue diagonals to indecomposables in S.

Choose an arc r' such that the cell B is a type A cell, that is, the triangulation of the cell B contains no diagonals in R. Remove the cell B from P', leaving the arc r' as an exterior edge of the remaining polygon A. Repeating this procedure with all remaining arcs in R will produce a situation where any remaining diagonals in R are spokes. The remaining cell contains the puncture, and is the central region C of the triangulation of P'. Recall that by Theorem 6.1.7, we may choose freely the arrangement of the diagonals in S in our triangulation. For convenience, in the case when every spoke in R has the same tagging, we specify that all diagonals in S in the central region are spokes, as in the figure. In the case when the triangulation contains spokes in R of opposite taggings, we specify that the diagonals in S in the central region are arranged in a fan, as is done later in Figure 6.25.

The following important remark gives detailed descriptions of the types of cells that we use throughout this chapter.

Remark 6.2.4. (*Types of cells*). The following five points each describe a different type of cell. Figure 6.14 pictures each type.

1. An unpunctured polygon triangulated by diagonals in S. These are precisely the type A cells from Definition 6.2.2. See Item 1 from Figure 6.14.

2. A polygon with the puncture as a vertex, triangulated by diagonals in S. We will refer to cells of this type as "wedge cells". See Item 2 of Figure 6.14.

3. A punctured polygon triangulated by diagonals in S. A cell of this type is in fact a central region that contains no diagonals in R. See Item 3 of Figure 6.14.

4. A punctured polygon triangulated by precisely one spoke in R and all other diagonals in S. A cell of this type is in fact a central region that contains one diagonal in R. See Item 4 of Figure 6.14.

5. A punctured polygon triangulated by two spokes in R which have opposite taggings,

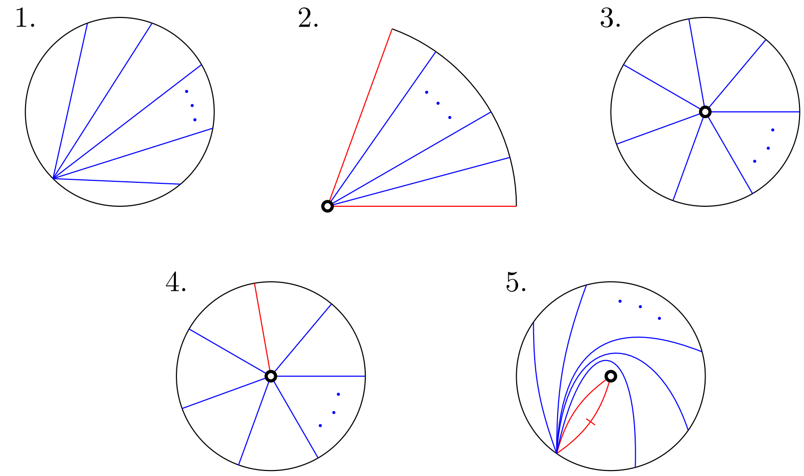


Figure 6.14: Different types of cells. Red diagonals correspond to indecomposables in R, whilst blue diagonals correspond to indecomposables in S.

and all other diagonals in S. A cell of this type is also a central region. See Item 5 of Figure 6.14.

We end the section with the following definition of the parity of each type of cell.

Definition 6.2.5. Let X be a cell. We define the parity of X as follows:

1. If X is a type A cell, i.e. a cell as described in Item 1 of Remark 6.2.4, then we say that X is even if it has an even number of vertices, and odd if it has an odd number of vertices.

2. If X is a cell as described in one of Items 2,3 or 5 in Remark 6.2.4, then we say that X is even if it has an even number of vertices, including the puncture, and we say that X is odd if it has an odd number of vertices, including the puncture.

3. If X is a cell as described in Item 4 of Remark 6.2.4, then we say that X is even if it has an even number of vertices, and odd if it has an odd number of vertices. In particular, notice in this case that we do not count the puncture as a vertex.

6.2.2 Fitting

In this section and the subsequent section, we demonstrate how to construct any given triangulation of a punctured polygon P made up of diagonals in R and S, using the cells described in Remark 6.2.4. The gluing procedure in the following section allows us to construct such a triangulation, starting with a central region and attaching type A cells. However, we first discuss in this section how to construct a given central region, using the cells described in Remark 6.2.4 and a fitting procedure allowing us to fit wedge cells to

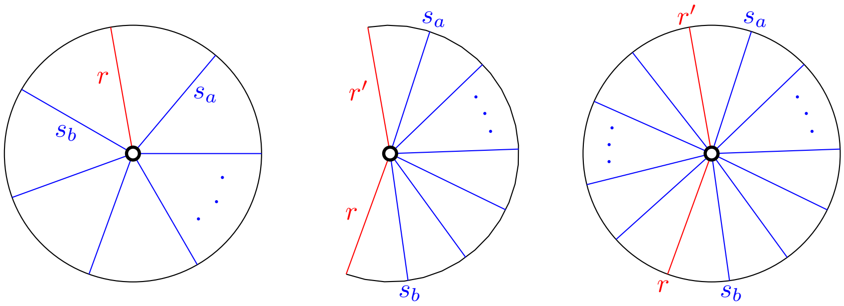


Figure 6.15: Fitting a new cell in order to create a central region. Red diagonals correspond to indecomposables in R, whilst blue diagonals correspond to indecomposables in S.

a central region. This section will be brief, however formal computations involving the fitting procedure will be given in Section 6.2.6 when computing $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ for central regions. The following remark describes the fitting procedure.

Remark 6.2.6. (*"Fitting"*). Choose a cell as described in Item 4 of Remark 6.2.4. We describe here how to fit a wedge cell to this construction, resulting in a central region. The following process is iterative, and after any number of iterations, the resulting polygon is a central region where all spokes in R have the same tagging. We note that we can clearly have zero iterations, as the cell described in Item 4 of Remark 6.2.4 is itself a central region.

Now, let C be a cell as described in Item 4 of Remark 6.2.4, see the left hand side of Figure 6.15. We fit a wedge cell in the following way. As in the centre of Figure 6.15, we separate the cell C at the spoke r in R, and insert a new spoke corresponding to a new rigid indecomosable r' in the original position of r, creating a wedge shaped cell. We then fit a new wedge cell to this construction by inserting spokes in S in order to complete to a triangulation of a larger punctured polygon P', as in the right hand side of Figure 6.15. The polygon P', which is a central region, can then be seen as the composite of two wedge cells. This fitting process can be carried out as many times as required, each time inserting a new spoke corresponding to a rigid indecomposable, and then fitting a wedge cell.

We give a categorical description of fitting in the more formal discussion in Section 6.2.6. The following remark notes each different way that we can obtain a central region.

Remark 6.2.7. (*Constructing central regions*). There are three possible ways in which we can construct a central region. They are as follows:

1. Choose a cell as described in Item 5 of Remark 6.2.4, see Item 5 of Figure 6.14. This is already a central region.

2. Choose a cell as described in Item 3 of Remark 6.2.4, see Item 3 of Figure 6.14. This is already a central region.

3. Choose a cell as described in Item 4 of Remark 6.2.4, see Item 4 of Figure 6.14. Then, fit as many wedge shaped cells as required using the fitting procedure described in Remark 6.2.6, see Figure 6.15. Note that the starting cell is itself a central region, and so it is possible to choose zero iterations of the fitting procedure.

We then separate the central regions constructed in the above list into the following two types. This separation will be useful for the gluing procedure described in the next section.

I Central regions produced in Item 1 of the above list will be known as central regions whose spokes in R have opposite taggings.

II Central regions produced in Items 2 or 3 of the above list will be known as central regions whose spokes all have the same tagging.

6.2.3 Gluing

It is not difficult to see that running the ungluing process in reverse allows us to construct any triangulation of P, made up of diagonals in R and S, starting with a central region C. The reverse of the ungluing process is called "gluing", and is described in the following remark.

Remark 6.2.8. (*"Gluing"*). Let P be a central region. Choose an exterior edge of P and identify it with a new rigid indecomposable r'. On to this new diagonal r', glue a type A cell P''. We now have a larger punctured polygon P'''. This process can be repeated by identifying any exterior edge of P''' with a new rigid indecomposable and once again gluing on a type A cell. This describes the gluing process, and it follows from the ungluing procedure that any triangulation of a punctured polygon P', made up of diagonals in R and S, can be constructed using the gluing process, starting with its central region C.

Categorically, the gluing process works as follows. From the cluster tilting subcategory $\mathsf{T} \subseteq \mathsf{C}$, defined by

$$\operatorname{indec}\mathsf{T} = \operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S},$$

we construct a new cluster tilting subcategory $\mathsf{T}' \subseteq \mathsf{C}'$ of a larger cluster category C'. The cluster tilting subcategory T' is defined as follows,

$$\operatorname{indec}\mathsf{T}' = \operatorname{indec}\mathsf{S} \cup \operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S}' \cup \{r'\}, \tag{6.16}$$

where r' is the new rigid indecomposable to which we glue a type A cell, and $\operatorname{indec}\mathsf{S}'$ corresponds to the set of diagonals triangulating the glued cell. The new rigid subcategory $\mathsf{R}' \subseteq \mathsf{T}'$ is given by,

$$\operatorname{indec}\mathsf{R}' = \operatorname{indec}\mathsf{R} \cup \{r'\}.$$

Now, from the cluster tilting subcategory T, we construct groups $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, N and the quotient $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$, whilst from T', we construct $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$, N', and thus $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$. Note that the ranks of the free groups $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$ and $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$ are easily computed by counting the number of generators, and so are given as follows:

$$\begin{aligned} \operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})) &= |\mathsf{R}| + |\mathsf{S}|, \\ \operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')) &= |\mathsf{R}| + |\mathsf{S}| + |\mathsf{S}'| + 1. \end{aligned} \tag{6.17}$$

Recall here that for a Krull-Schmidt category D, we denote by $|\mathsf{D}|$ the number of isomorphism classes of indecomposables in the category.

6.2.4 Relating $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ Before and After Gluing a Type A Cell: A Commutative Square

In this section we describe specific cases of gluing a type A cell. There are three cases to consider, and along with the work in the next section, we describe how the gluing affects the quotient group $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$.

In each of the below cases, we will describe formally the gluing of a cell to a polygon P containing a triangulation made up of diagonals in R and S. In particular, note that in every case, previous gluing is permitted. The cases differ by the make up of the central region and where on P we glue the new cell. The cases to consider are:

1. We glue a new cell directly to the central region C of a polygon P whose spokes all have the same tagging.

2. We glue a new cell directly to the central region C of a polygon P whose spokes in R have opposite taggings.

3. We glue a new cell to an already glued cell.

In each of the three cases, we will show that the surjection $\kappa' : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}') \twoheadrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, defined by

$$\kappa'([t]) = \begin{cases} [t], & \text{if } t \in \operatorname{indec} \mathsf{T}, \\ 0, & \text{otherwise}, \end{cases}$$

induces a map $\nu' : N' \to N$ that carries the property that,

$$\nu'(n'(s')) = \begin{cases} n(s') & \text{if } s' \in \operatorname{indec} \mathsf{S}, \\ 0 & \text{if } s' \in \operatorname{indec} \mathsf{S}'. \end{cases}$$

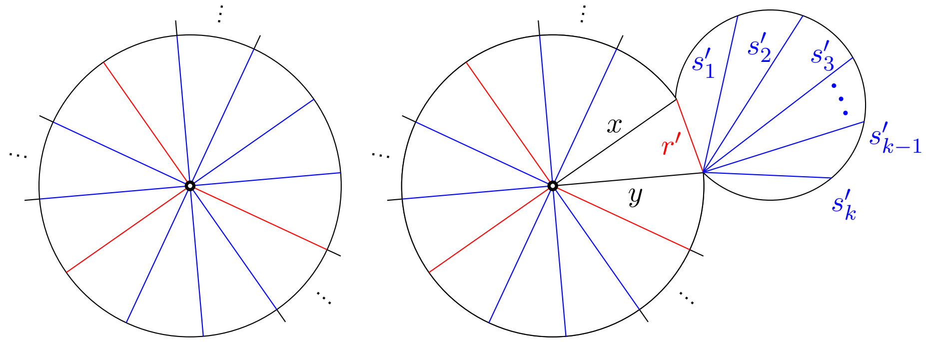


Figure 6.16: Gluing a cell to the central region of a polygon that has either no spokes in R, or whose spokes in R all have the same tagging. The red diagonals correspond to indecomposables in R, and the blue diagonals to indecomposables in S.

These maps then fit into a commutative square,

$$\begin{tikzcd} N' \arrow[r, hook, "\varphi'"] \arrow[d, "\nu'"'] & K_0^{\text{split}}(\mathsf{T}') \arrow[d, "\kappa'"] \\ N \arrow[r, hook, "\varphi"'] & K_0^{\text{split}}(\mathsf{T}). \end{tikzcd} \tag{6.18}$$

This commutative square will then be used in the following section to produce a Snake Lemma argument, upon which we will rely heavily for the proof of the main theorem. We start with the gluing of a new cell directly to a central region that either contains no spokes in R, or whose spokes in R all have the same tagging.

Case 1: Gluing directly to the Central Region of a Polygon Whose Spokes All Have the Same Tagging

Consider a triangulation, made up of diagonals in R and S, of a punctured polgyon P, and consider its central region C. Assume that every spoke in the triangulation of C has the same tagging. Now, mutate the diagonals in S such that every diagonal inside C is a spoke, as in the left hand side of Figure 6.16. Then, choose an exterior edge of P, which is also an exterior edge of C, and identify it with a new rigid indecomposable r' before gluing on a type A cell using the gluing procedure in the previous section. Recall that this procedure constructs the larger cluster tilting subcategory T', defined in (6.16). The right hand side of Figure 6.16 shows part of the new polygon P'. Here, the blue diagonals correspond to indecomposables in S or S' and the red diagonals to indecomposables in R. Note that we have drawn the diagonals x and y in black as they may belong to either R or S.

In order to compute $K_0^{\text{split}}(\mathsf{T})/N$ and $K_0^{\text{split}}(\mathsf{T}')/N'$, we should compute N and N'. By Theorem 6.1.7, we are able to choose freely the arrangement of the diagonals in S and

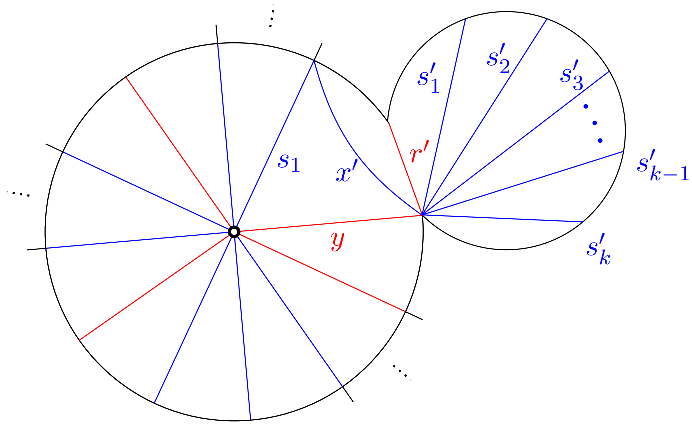


Figure 6.17: When $x \in \operatorname{indec} \mathsf{S}$ and $y \in \operatorname{indec} \mathsf{R}$, we arrange the diagonals in S and S' as in the figure. Here, x' is the mutation of x from Figure 6.16.

S', and in order to make our computations of N and N' easier, we will choose a different arrangement depending on whether the diagonals x and y from Figure 6.16 correspond to elements in S or R. Consider the following cases:

1. $x, y \in \operatorname{indec} \mathsf{R}$. In this case, we arrange the indecomposables of S and S' as in Figure 6.16. Note that every generator of N is also a generator of N'. In addition to the generators of N, the subgroup N' also contains the generators $n'(s'_i)$, for $s'_i \in \operatorname{indec} \mathsf{S}'$.

2. $x \in \operatorname{indec} \mathsf{S}$ and $y \in \operatorname{indec} \mathsf{R}$. In this case, we will arrange the diagonals in S and S' as in Figure 6.17. Note here that the diagonal $x' \in \operatorname{indec} \mathsf{S}$ is the mutation of x. In this case note that every generator of N, with the exception of $n(x') = s_1 - y$, is also a generator of N'. For N', in addition to the unchanged generators of N, we have the generators $n'(x') = r' + s_1 - y$ and $n'(s'_i)$ for each $s'_i \in \operatorname{indec} \mathsf{S}'$.

3. $x, y \in \operatorname{indec} \mathsf{S}$. In this case, we again arrange the diagonals in S and S' as in Figure 6.17, noting this time that $y \in \operatorname{indec} \mathsf{S}$, and so should appear as a blue diagonal. The generators for N and N' are the same as in the previous case, whilst we should additionally note that $n(y) = n'(y)$.

In particular, if the triangulation of the central region C contains no spokes in R, then the third case from above will always describe the gluing situation.

Notice that the important feature of the arrangement of the S diagonals in each of the above cases is that at most one generator from N is affected by the gluing of the new cell. That is, we arrange the diagonals in S in such a way that there is at most one generator for N that does not remain a generator for N'. Assume that $s \in \operatorname{indec} \mathsf{S}$ is the diagonal whose generator changes, then we also know that $n'(s) = n(s) + r'$.

There are two injections $\varphi : N \hookrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$ and $\varphi' : N' \hookrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$, as well as a canonical surjection $\kappa' : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}') \twoheadrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, defined by

$$\kappa'([t]) = \begin{cases} [t], & \text{if } t \in \operatorname{indec} \mathsf{T}, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that κ' induces a map

$$\nu' : N' \twoheadrightarrow N$$

that satisfies $\varphi \circ \nu' = \kappa' \circ \varphi'$. We should verify here that $\kappa'\varphi'(N') \subseteq N$. Recall that if $s \in \operatorname{indec} \mathsf{S}$, then either $n'(s)$ consists entirely of indecomposables in T, or $n'(s) = r' + s_1 - y$, where s_1 is from Figure 6.17. In the first case it is clear that

$$\kappa'\varphi'(n'(s)) = n(s).$$

In the second case,

$$\kappa'\varphi'(n'(s)) = \kappa'\varphi'(r' + s_1 - y) = s_1 - y = n(s).$$

Thus, for $s \in \operatorname{indec} \mathsf{S}$, we see that $\kappa'\varphi'(n'(s)) \in N$. For $s' \in \operatorname{indec} \mathsf{S}'$, recall that $n'(s')$ consists entirely of indecomposables not belonging to T, so

$$\kappa'\varphi'(n'(s')) = 0,$$

and so $\kappa'\varphi'(n'(s')) \in N$ for $s' \in \operatorname{indec} \mathsf{S}'$. The above verification in fact shows that

$$\nu'(n'(s')) = \kappa'\varphi'(n'(s')) = \begin{cases} n(s') & \text{if } s' \in \operatorname{indec} \mathsf{S}, \\ 0 & \text{if } s' \in \operatorname{indec} \mathsf{S}'. \end{cases}$$

In particular, we see that ν' is surjective and that κ' and φ' fit into the commutative square in (6.18).

Case 2: Gluing directly to the Central Region of a Polygon Whose Spokes in R Have Opposite Taggings

We describe here the process when we glue a new cell on to the central region of a polygon whose spokes in R have opposite taggings.

Consider a triangulation of a punctured polygon P and consider its central region C, whose triangulation has the property that R contains spokes of different taggings, as in the left hand side of Figure 6.18. Note that the triangulation of C cannot contain any other spokes. Now, choose an exterior edge of P, which is also an exterior edge of C,

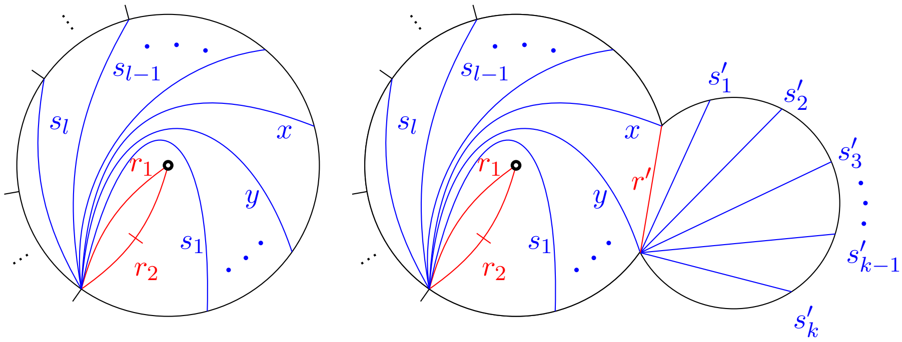


Figure 6.18: Gluing a cell to the central region of a polygon whose spokes in R have opposite taggings. Red diagonals correspond to indecomposables in R, whilst blue diagonals correspond to indecomposables in S or S'.

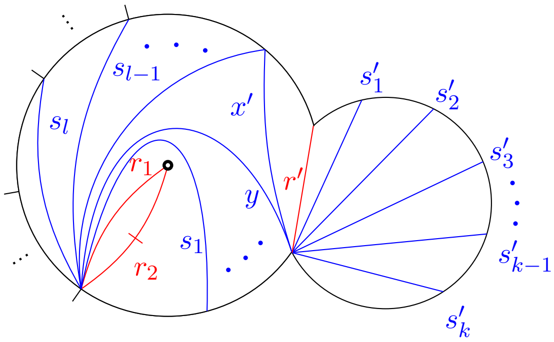


Figure 6.19: Replacing x from Figure 6.18 with x' allows an easier computation of the generators of N and N'.

and identify it with the new rigid indecomposable r', before gluing on a type A cell as described in the gluing process. This procedure produces a new larger polygon P', whose triangulation corresponds to T', defined in (6.16). Part of the new polygon P' is pictured in the right hand side of Figure 6.18. Here, $x, y \in \operatorname{indec} \mathsf{S}$ are the arcs that meet the vertices of the exterior edge r'.

In order to make the computation of the generators of N and N' easier, we rearrange the diagonals in S. Theorem 6.1.7 allows us to do this. Figure 6.19 shows the rearrangement. In this figure, x' is the mutation of x. Notice that if the type A cell is glued to the exterior edge sharing vertices with the endpoints of s_1, then the following computations still hold, replacing s_1 with its mutation.

Now, notice that every generator of N, with the exception of $n(x')$ is also a generator of N'. In the case of x', we can easily compute that $n'(x') = n(x') + r'$. The other generators of N' are given by $n'(s'_i)$, for each $s'_i \in \operatorname{indec} \mathsf{S}'$.

As with the gluing in the previous case, consider the maps $\varphi : N \hookrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$ and $\varphi' : N' \hookrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$, as well as the canonical surjection $\kappa' : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}') \twoheadrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, given by,

$$\kappa'([t]) = \begin{cases} [t], & \text{if } t \in \operatorname{indec} \mathsf{T}, \\ 0, & \text{otherwise.} \end{cases}$$

κ' induces a map $\nu' : N' \to N$ that satisfies $\varphi \circ \nu' = \kappa' \circ \varphi'$. We must check that the gluing we have described means that $\kappa'\varphi'(N') \subseteq N$. Indeed, if $s \in \operatorname{indec} \mathsf{S}$ and $s \neq x'$, then since $n'(s) = n(s)$ and $n(s)$ consists entirely of indecomposables in T, it is clear that

$$\kappa'\varphi'(n'(s)) = n(s).$$

If $s = x'$, then

$$\kappa'\varphi'(n(x') + r') = n(x'),$$

since $r' \notin \operatorname{indec} \mathsf{T}$ and $n(x')$ consists entirely of diagonals in $\operatorname{indec} \mathsf{T}$. Finally, if $s' \in \operatorname{indec} \mathsf{S}'$, then $n'(s')$ consists entirely of diagonals in $\operatorname{indec} \mathsf{T}' \setminus \operatorname{indec} \mathsf{T}$, and so

$$\kappa'\varphi'(n'(s')) = 0.$$

Thus, $\kappa'\varphi'(N') \subseteq N$. The above computations also shows that,

$$\nu'(n'(s')) = \begin{cases} n(s') & \text{if } s' \in \operatorname{indec}\mathsf{S}, \\ 0 & \text{if } s' \in \operatorname{indec} \mathsf{S}'. \end{cases}$$

Iin particular, ν' is surjective and κ' and ν' fit into the commutative square in (6.18).

Remark 6.2.9. Notice that in each of the two gluing procedures already described, the situation is less obvious should the central region have only two vertices. In the final section of this chapter, we demonstrate these small cases by computing the quotient in two examples. Example 6.2.16 considers the case when the central region has only two vertices and both spokes have the same tagging, whilst Example 6.2.17 computes the case when the central region has two vertices and the spokes have opposite taggings.

Case 3: Gluing on to a Glued Cell

We describe here the case when we glue a type A cell on to a cell that has already been glued on to a central region.

Notice that the cell upon which we glue the type A cell is by definition an unpunctured polygon containing a triangulation made up of noncrossing arcs in S. Our situation is therefore reminiscent of the gluing procedure used in the computation of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ when $\mathsf{C} = \mathsf{C}(A_n)$.

Choose a triangulation of the previously glued cell P. Recall from Theorem 6.1.7 that we may choose freely the arrangement of the diagonals in S, and so choose a fan, as in the left hand side of Figure 6.20. Now, choose an exterior edge of P and identify it

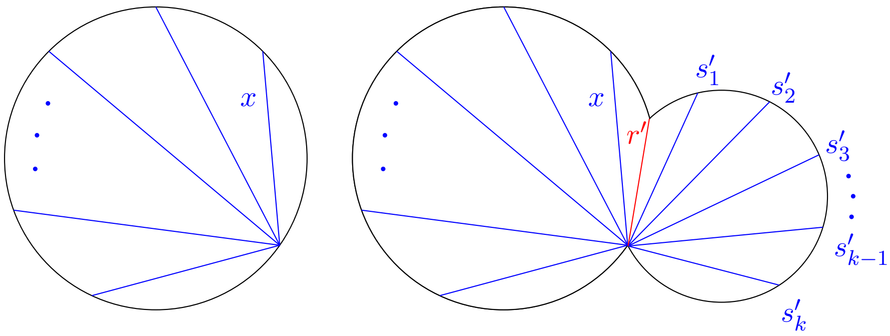


Figure 6.20: Gluing a new cell to a cell that has been previously glued on to a central region. Blue diagonals correspond to indecomposables in S or S'.

with the new rigid indecomposable r', before gluing the type A cell as described in the gluing procedure. This procedure produces a new larger polygon P', whose triangulation corresponds to T', defined in (6.16). Part of this new polygon P' is pictured in the right hand side of Figure 6.20.

Notice that every generator of N, with the exception of $n(x)$, is also a generator of N'. In the exceptional case, we have $n'(x) = n(x) + r'$. The other generators of N' are given by $n'(s'_i)$, for each $s'_i \in \operatorname{indec} \mathsf{S}'$.

As with the gluing in the previous cases, consider the maps $\varphi : N \hookrightarrow \mathrm{K}_0^{\text{split}}(\mathsf{T})$ and $\varphi' : N' \hookrightarrow \mathrm{K}_0^{\text{split}}(\mathsf{T}')$, as well as the canonical surjection $\kappa' : \mathrm{K}_0^{\text{split}}(\mathsf{T}') \twoheadrightarrow \mathrm{K}_0^{\text{split}}(\mathsf{T})$ given by,

$$\kappa'([t]) = \begin{cases} [t], & \text{if } t \in \operatorname{indec} \mathsf{T}, \\ 0, & \text{otherwise.} \end{cases}$$

κ' induces a map $\nu' : N' \to N$ that satisfies $\varphi \circ \nu' = \kappa' \circ \varphi'$. We will check that the gluing we have described means that $\kappa'\varphi'(N') \subseteq N$; that is, ν' sends generators of N' to generators of N. Indeed, if $s \in \operatorname{indec} \mathsf{S}$ and $s \neq x$, then since $n'(s) = n(s)$ and $n(s)$ consists entirely of indecomposables in T, it is clear that

$$\kappa'\varphi'(n'(s)) = n(s).$$

If $s = x$, then

$$\kappa'\varphi'(n(x) + r') = n(x),$$

since $r' \notin \operatorname{indec} \mathsf{T}$ and $n(x)$ consists entirely of diagonals in $\operatorname{indec} \mathsf{T}$. Finally, if $s' \in \operatorname{indec} \mathsf{S}'$, then $n'(s')$ consists entirely of diagonals in $\operatorname{indec} \mathsf{T}' \setminus \operatorname{indec} \mathsf{T}$, and so

$$\kappa'\varphi'(n'(s')) = 0.$$

Thus, $\kappa'\varphi'(N') \subseteq N$. The above computations also show that

$$\nu'(n'(s')) = \begin{cases} n(s') & \text{if } s' \in \operatorname{indec}\mathsf{S}, \\ 0 & \text{if } s' \in \operatorname{indec}\mathsf{S}'. \end{cases}$$

In particular, ν' is surjective, and κ' and ν' fit into the commutative square in (6.18).

6.2.5 Relating $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ Before and After Gluing a Type A Cell: Snake Lemma

In the three gluing descriptions from the previous section, we constructed the map $\nu' : N' \to N$, and showed that it is surjective with the property that

$$\nu'(n'(s')) = \begin{cases} n(s') & \text{if } s' \in \operatorname{indec}\mathsf{S}, \\ 0 & \text{if } s' \in \operatorname{indec}\mathsf{S}'. \end{cases} \tag{6.19}$$

We also know that ν' and κ' fit into a commutative square:

$$\begin{tikzcd} N' \arrow[r, hook, "\varphi'"] \arrow[d, "\nu'"'] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}') \arrow[d, "\kappa'"] \\ N \arrow[r, hook, "\varphi"'] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}). \end{tikzcd}$$

Taking cokernels of the two injections, we obtain a diagram of short exact sequences, which can be completed by λ', using the short-5-lemma,

$$\begin{tikzcd} 0 \arrow[r] & N' \arrow[r, hook, "\varphi'"'] \arrow[d, "\nu'"] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}') \arrow[r, two heads] \arrow[d, "\kappa'"] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N' \arrow[r] \arrow[d, "\lambda'"] & 0 \\ 0 \arrow[r] & N \arrow[r, hook, "\varphi"'] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \arrow[r, two heads] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \arrow[r] & 0. \end{tikzcd}$$

Applying the snake lemma gives the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}\nu' \arrow[r] \arrow[d, hook] & F \arrow[r] \arrow[d, hook, "\mu"] & \operatorname{Ker}\lambda' \arrow[d, hook] \arrow[dddll, rounded corners, to path={ -- ([xshift=4ex]\tikztostart.east) |- ([yshift=-1.5ex]\tikztostart.south |- 0,0) -| ([xshift=-12ex]\tikztotarget.west) |- (\tikztotarget)}] & \\
0 \arrow[r] & N' \arrow[r, hook, "\varphi'"] \arrow[d, two heads, "\nu'"] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}') \arrow[r, two heads] \arrow[d, two heads, "\kappa'"] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N' \arrow[r] \arrow[d, two heads, "\lambda'"] & 0 \\
0 \arrow[r] & N \arrow[r, hook, "\varphi"] \arrow[d] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \arrow[r, two heads] \arrow[d] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd} \tag{6.20}$$

Recall that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$ is the free group on $\operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S}$, and $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$ is the free group on $\operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S} \cup \{r'\} \cup \operatorname{indec}\mathsf{S}$. Now, since κ' kills the generators corresponding to $\{r'\} \cup \operatorname{indec}\mathsf{S}'$, we know that $\operatorname{Ker}\kappa'$ must be free on $\{r'\} \cup \operatorname{indec}\mathsf{S}'$, and so we denote $\operatorname{Ker}\kappa'$ by F in the above diagram. The rank of F is clearly

$$\operatorname{rank}(F) = |\mathsf{S}'| + 1. \tag{6.21}$$

Additionally, since ν' and κ' are known surjections, their cokernels are zero. Adding to this that the snake lemma produces a long exact sequence yields that $\operatorname{Coker}\lambda' = 0$, and so λ' is also a surjection. Finally note that each vertical sequence in Diagram (6.20) is also a short exact sequence.

Lemma 6.2.10. *Ker* λ' *is cyclic.*

Proof. Since the top row of Diagram (6.20) is a short exact sequence, we know that

$$\operatorname{Ker}\lambda' \cong F/\operatorname{Ker}\nu'.$$

We also know that for each $s' \in \operatorname{indec}\mathsf{S}'$, the generator $n'(s') \in \operatorname{Ker}\nu'$, see Equation (6.19). We show that $\operatorname{Ker}\lambda'$ is cyclic in each of two cases; the case when the glued cell P'' from Figures 6.17, 6.19 and 6.20 is even, and the case when it is odd.

Even case: Assume that the glued cell has l vertices, and that l is even. Then, since $k = l-3$, we know that k must be odd, and $k-1$ even. Using either Figure 6.17, 6.19 or 6.20, computing $n'(s')$ for each $s' \in \operatorname{indec}\mathsf{S}'$ produces the elements:

$$s_2' - r', s_3' - s_1', s_4' - s_2', \ldots, s_k' - s_{k-2}', s_{k-1}'. \tag{6.22}$$

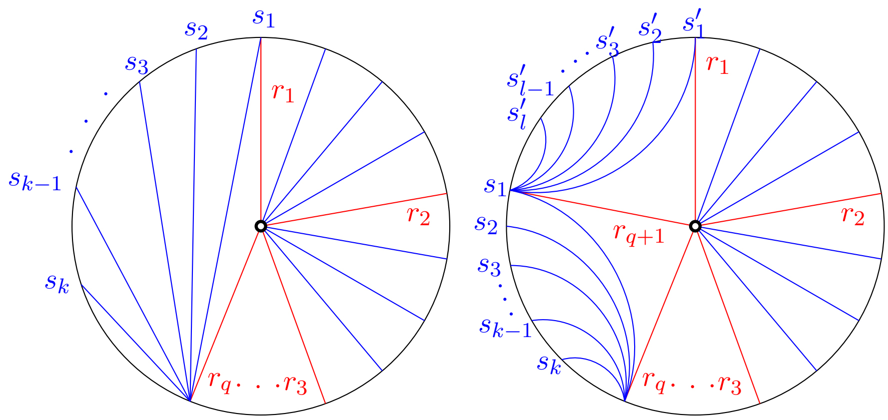


Figure 6.21: Fitting a new rigid indecomposable r_{q+1}, and creating a larger cluster tilting subcategory $\mathsf{T}' \subseteq \mathsf{C}'$. We may arrange the internal diagonals of each cell in this way by virtue of Theorem 6.1.7.

From these relations, we can deduce that in the quotient $F/\operatorname{Ker}\nu'$, we have

$$r' + \operatorname{Ker}\nu' = s'_i + \operatorname{Ker}\nu' = 0 + \operatorname{Ker}\nu',$$

for every even i, and

$$s'_i + \operatorname{Ker}\nu' = s'_j + \operatorname{Ker}\nu',$$

for any odd i, j. Thus, r' and all s'_i for i even become zero in the quotient, and all s'_i for i odd become equal. So, $\operatorname{Ker}\lambda' \cong F/\operatorname{Ker}\nu'$ has one generator and is therefore cyclic in the even case.

Odd case: If the number of vertices l of P'' is odd, then $k = l-3$ must be even and so $k-1$ odd. From the relations $n'(s')$ in (6.22), we see that in the quotient

$$r' + \operatorname{Ker}\nu' = s'_i + \operatorname{Ker}\nu',$$

for each even i, and

$$s'_i + \operatorname{Ker}\nu' = 0 + \operatorname{Ker}\nu',$$

for any odd i. Therefore, in the quotient r' is equal to every s'_i for i even, and $s'_i = 0$ for every i odd. Again, $\operatorname{Ker}\lambda'$ has one generator and so is cyclic. $\square$

6.2.6 Computation of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ for a Central Region: All Spokes Have the Same Tagging

As mentioned previously, we will use the central region as the base case for an inductive proof of our main theorem. In this section, and the subsequent section, we will compute a general formula for $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ for a central region. This current section deals with

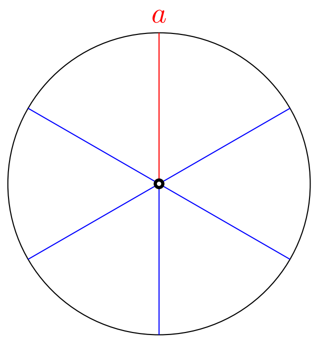


Figure 6.22: When $|\mathsf{R}| = 1$, the polygon P is the only cell.

the case when all spokes in the triangulation of the central region have the same tagging, whilst the next subsection deals with the case when there are spokes in R with opposite taggings. It will turn out that this is sufficient for the base case of our final induction.

Indeed, let P be a central region whose spokes all have the same tagging. Recall that R dissects P into wedge cells. The cluster tilting subcategory T then corresponds to a full triangulation of P, and so, in addition to the dissection provided by R, it provides a triangulation of each of the cells. The diagonals inside these cells correspond to indecomposables inside the subcategory S, where

$$\operatorname{indec} \mathsf{T} = \operatorname{indec} \mathsf{R} \cup \operatorname{indec} \mathsf{S}. \tag{6.23}$$

An example of a triangulation of this kind is given in Figure 6.21, where the red spokes correspond to indecomposables in R, and the blue diagonals to indecomposables in S.

In order to talk more in detail about producing a general formula for the quotient group $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$, we require the notion of fitting, which was introduced in Section 6.2.2. We describe here the fitting of a wedge cell to a central region that is made up of previously fitted wedge cells. Given a dissection R of P that consists only of spokes of the same tagging, we will produce a larger punctured polygon P' with dissection R' by fitting a new spoke r_{q+1} as in Figure 6.21. In this figure, the polygon on the left corresponds to the original triangulation of P, and the figure on the right demonstrates how to fit r_{q+1} and form a triangulation of P'. Formally, from the cluster tilting subcategory T of the cluster category C, defined in (6.23), we construct a larger cluster tilting subcategory

$$\operatorname{indec} \mathsf{T}' = \operatorname{indec} \mathsf{R} \cup \operatorname{indec} \mathsf{S} \cup \{r_{q+1}\} \cup \operatorname{indec} \mathsf{S}',$$

of a larger cluster category C'. Here,

$$\operatorname{indec} \mathsf{R}' = \operatorname{indec} \mathsf{R} \cup \{r_{q+1}\}$$

is the new, larger rigid subcategory of C'. Also, $\operatorname{indec}\mathsf{S}'$ corresponds to the diagonals in the triangulation of the new cell created by fitting r_{q+1}. From T we obtain the group $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, the subgroup N, and thus the quotient $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$, whilst from T' we obtain the larger group $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$, the subgroup N', and the quotient $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$. The ranks of the free groups $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$ and $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$ can be computed easily by counting generators, and so are given by

$$\begin{aligned} \operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})) &= |\mathsf{R}| + |\mathsf{S}|, \\ \operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')) &= |\mathsf{R}| + |\mathsf{S}| + |\mathsf{S}'| + 1. \end{aligned} \tag{6.24}$$

Using Figure 6.21, we now compute generators of the subgroups N and N', omitting those generators that come from the unlabelled blue diagonals as they are the same for each subgroup. We have:

$$N = \left\langle \; \ldots, r_q - r_1 - s_2,\, s_1 - s_3,\, s_2 - s_4, \ldots, s_{k-2} - s_k, s_{k-1} \; \right\rangle.$$

In particular, note that $n(s_1) = r_q - r_1 - s_2$. Computing N', we obtain:

$$N' = \left\langle \begin{array}{c} \ldots, r_q - r_{q+1} - s_2,\, s_1 - s_3,\, s_2 - s_4, \ldots, s_{k-2} - s_k, s_{k-1}, \\ \ldots, r_{q+1} - r_1 - s_2',\, s_1' - s_3',\, s_2' - s_4', \ldots, s_{l-2}' - s_l',\, s_{l-1}' \end{array} \right\rangle.$$

In particular note that $n'(s_1) = r_q - r_{q+1} - s_2$ and $n'(s_1') = r_{q+1} - r_1 - s_2'$.

Consider the two inclusions $\varphi' : N' \hookrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$ and $\varphi : N \hookrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, as well as the surjection $\kappa' : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}') \twoheadrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, defined by

$$\kappa'([t]) = \begin{cases} [t] & \text{if } t \in \operatorname{indec}\mathsf{T}, \\ [r_1] & \text{if } t = r_{q+1} \\ 0 & \text{otherwise.} \end{cases} \tag{6.25}$$

Note that $\operatorname{Ker}\kappa'$ is free on $\{s' | s' \in \operatorname{indec}\mathsf{S}'\} \cup \{r_{q+1} - r_1\}$. Indeed, consider the following element that is clearly in $\operatorname{Ker}\kappa'$:

$$x = \alpha_1 s_1' + \alpha_2 s_2' + \cdots + \alpha_l s_l' + \beta(r_{q+1} - r_1),$$

where $\alpha_i, \beta \in \mathbb{Z}$. Now, we can construct an obvious isomorphism between $\operatorname{Ker}\kappa'$ and $\mathbb{Z}^{l+1}$ by mapping x to the vector

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_l \\ \beta \end{pmatrix} \in \mathbb{Z}^{l+1},$$

giving us the required result.

Now, define $\nu' : N' \to N$ to be the homorphism that makes the following diagram commute:

$$\begin{tikzcd} N' \arrow[r, hook, "\varphi'"] \arrow[d, "\nu'"'] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}') \arrow[d, "\kappa'"] \\ N \arrow[r, hook, "\varphi"'] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}), \end{tikzcd} \tag{6.26}$$

that is, the homomorphism that satisfies $\varphi\circ\nu' = \kappa'\circ\varphi'$. We must check that $\kappa'\varphi'(N') \subseteq N$. There are four cases to consider:

1. Let $s \in \operatorname{indec}\mathsf{S}$ and $s \neq s_1$. Then, since the expression for $n'(s)$ contains elements only in $\operatorname{indec}\mathsf{T}$, it is clear that
$$\kappa'\varphi'(n'(s)) = n(s).$$

2. Let $s' \in \operatorname{indec}\mathsf{S}'$ and $s' \neq s'_1$. Then, since the expression for $n'(s')$ consists only of elements in $\operatorname{indec}\mathsf{T}'$ and does not include r_{q+1}, it is clear that
$$\kappa'\varphi'(n'(s')) = 0.$$

3. Consider s_1,
$$\begin{aligned} \kappa'\varphi'(n'(s_1)) &= \kappa'(r_q - r_{q+1} - s_2) \\ &= r_q - r_1 - s_2 \\ &= n(s_1) \end{aligned}$$

4. Consider s'_1,
$$\begin{aligned} \kappa'\varphi'(n'(s'_1)) &= \kappa'(r_1 - r_{q+1} - s'_2) \\ &= r_1 - r_1 \\ &= 0. \end{aligned}$$

The above four cases show that ν' has the property that

$$\nu'(n'(s')) = \begin{cases} n(s') & \text{if } s' \in \operatorname{indec}\mathsf{S}, \\ 0 & \text{if } s' \in \operatorname{indec}\mathsf{S}'. \end{cases} \tag{6.27}$$

In particular, notice that ν' is surjective.

Now, we may complete each row of the square in (6.26) to a short exact sequence. There is a unique homomorphism λ' which completes to the following commutative diagram of short exact sequences:

$$
\begin{tikzcd}
0 \arrow[r] & N' \arrow[r, hook, "\varphi'"'] \arrow[d, "\nu'"] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}') \arrow[r, two heads] \arrow[d, "\kappa'"] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N' \arrow[r] \arrow[d, "\lambda'"] & 0 \\
0 \arrow[r] & N \arrow[r, hook, "\varphi"'] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \arrow[r, two heads] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \arrow[r] & 0.
\end{tikzcd}
$$

Applying the Snake Lemma yields

$$
\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}\nu' \arrow[r] \arrow[d, hook] & F \arrow[r] \arrow[d, hook, "\mu"] & \operatorname{Ker}\lambda' \arrow[d, hook] \arrow[dddll, rounded corners] & \\
0 \arrow[r] & N' \arrow[r, hook, "\varphi'"] \arrow[d, two heads, "\nu'"] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}') \arrow[r, two heads] \arrow[d, two heads, "\kappa'"] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N' \arrow[r] \arrow[d, two heads, "\lambda'"] & 0 \\
0 \arrow[r] & N \arrow[r, hook, "\varphi"] \arrow[d] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \arrow[r, two heads] \arrow[d] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}
\tag{6.28}
$$

Here, as $\operatorname{Ker}\kappa'$ is free on $\{s'|s' \in \operatorname{indec}\mathsf{S}'\} \cup \{r_{q+1} - r_1\}$, we denote it by F. It is also clear that

$$
\operatorname{rank}(F) = |\mathsf{S}'| + 1. \tag{6.29}
$$

Since κ' and ν' are known surjections, they have trivial cokernels. It then follows from the long exact sequence produced by the snake lemma that $\operatorname{Coker}\lambda' = 0$, and so λ' must also be a surjection. Notice also that each row and column in Diagram (6.28) forms a short exact sequence.

Lemma 6.2.11. $\operatorname{Ker}\lambda'$ *is cyclic.*

Proof. Since the top row of Diagram (6.28) forms a short exact sequence, we know

$$
\operatorname{Ker}\lambda' \cong F/\operatorname{Ker}\nu'. \tag{6.30}
$$

Note that $\operatorname{Ker}\nu'$ contains $n'(s')$ for each $s' \in \operatorname{indec}\mathsf{S}'$ by (6.27), and recall that F is free on $\{s'|s' \in \operatorname{indec}\mathsf{S}'\} \cup \{r_{q+1} - r_1\}$.

We show $\operatorname{Ker}\lambda'$ is cyclic in two cases; the case when the fitted cell is even, and the case when it is odd.

Even case. Assume that the fitted cell is even, then l in Figure 6.21 is odd, and so $l-1$ must be even. Consider the relations $n'(s')$ in $\operatorname{Ker}\nu'$, given by

$$
r_{q+1} - r_1 - s'_2, s'_1 - s'_3, s'_2 - s'_4, \ldots, s'_{l-2} - s'_l, s'_{l-1}. \tag{6.31}
$$

It is easy to see from this that in the quotient in (6.30) we have

$$s'_i + \operatorname{Ker}\nu' = r_{q+1} - r_1 + \operatorname{Ker}\nu' = 0 + \operatorname{Ker}\nu',$$

for each i even, and

$$s'_i + \operatorname{Ker}\nu' = s'_j + \operatorname{Ker}\nu',$$

when i, j are both odd. Thus, the quotient has one generator, and in the even case, $\operatorname{Ker}\lambda'$ is cyclic.

Odd case. Assume that the fitted cell is odd, then l in Figure 6.21 is even, and so $l-1$ must be odd. From the relations in (6.31), we see that in the quotient in (6.30)

$$s'_i + \operatorname{Ker}\nu' = 0 + \operatorname{Ker}\nu',$$

for each i odd, and

$$r_{q+1} - r_1 + \operatorname{Ker}\nu' = s'_i + \operatorname{Ker}\nu',$$

for each i even. Again, there is just one generator, and so $\operatorname{Ker}\lambda'$ is cyclic. □

We now go ahead and prove the main result for this section. Recall our use of even and odd terminology in Definition 6.2.5 for each different type of cell described in Remark 6.2.4. Recall in particular that when $|\mathsf{R}| = 0$, the punctured polygon P is itself a cell.

Theorem 6.2.12. *1. If $|\mathsf{R}| = 0$, then $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on n generators, where*

$$n = \begin{cases} 1 & \text{if } P \text{ is even,} \\ 2 & \text{if } P \text{ is odd.} \end{cases}$$

2. If $|\mathsf{R}| > 0$, then $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on n generators, where

$$n = \begin{cases} |\mathsf{R}| + 1 & \text{if every cell is even,} \\ |\mathsf{R}| & \text{otherwise.} \end{cases}$$

Proof. We prove Item 1 of the theorem first of all. Consider the case when $|\mathsf{R}| = 0$, as pictured in Figure 6.23. Then, we compute the generators $n(s)$ as follows:

$$s_2 - s_k, s_3 - s_1, s_4 - s_2, \ldots, s_k - s_{k-2}, s_1 - s_{k-1}. \tag{6.32}$$

We prove the statement in the following two cases:

Case 1. (Even). Assume P is even; that is, k in the figure is odd. We see from the

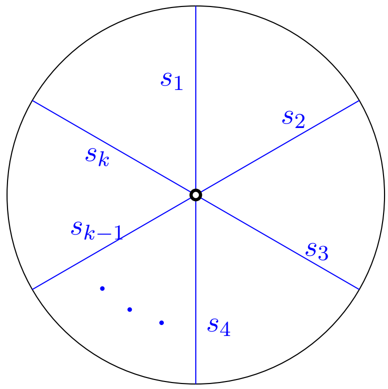


Figure 6.23: Theorem 6.2.12, Item 1: $|\mathsf{R}| = 0$. All diagonals correspond to indecomposables in S. Theorem 6.1.7 allows us to specify that they are all spokes.

generators $s_2 - s_k$ and $s_1 - s_{k-1}$ in (6.32) that in the quotient $K_0^{\mathrm{split}}(\mathsf{T})/N$ we have

$$s_i + N = s_j + N,$$

for all i, j. Thus, $K_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on one generator, as required.

Case 2. (Odd). Assume P is odd; that is, k is even. Then, from the relations $s_2 - s_k$ and $s_1 - s_{k-1}$ in (6.32), we see that in the quotient $K_0^{\mathrm{split}}(\mathsf{T})/N$ we have

$$s_i + N = s_j + N,$$

when i, j both have the same parity. Thus, $K_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on two generators, as required.

We now prove Item 2 of the theorem using induction. We will start with the case where not every cell is even.

Case 1. (Not all cells are even.) *Base*: Assume $|\mathsf{R}| = 1$, as pictured in Figure 6.24. Assume also that the unique cell is odd; that is k in the figure is even, and so $k - 1$ is odd.

Computing the generators $n(s)$ of N gives the relations:

$$s_2 - r, s_3 - s_1, s_4 - s_2, \ldots, s_k - s_{k-2}, r - s_{k-1}.$$

From these generators we can see that

$$r + N = s_i + N,$$

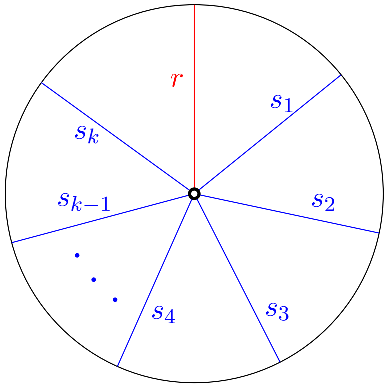


Figure 6.24: Theorem 6.2.12, Item 2: $|\mathsf{R}| = 1$. Blue diagonals correspond to indecomposables in S, and red to the sole indecomposable in R. Theorem 6.1.7 allows us to specify that all blue diagonals are spokes.

for each i. Thus, in the quotient, all generators become equal and so $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on $|\mathsf{R}| = 1$ generators, as required.

Induction: Assume for some $|\mathsf{R}| > 0$ that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on $|\mathsf{R}|$ generators.

Step: Assume now that we have fitted a new cell as described in Figure 6.21, and formed the cluster tilting subcategory $\mathsf{T}' \subseteq \mathsf{C}'$, defined by

$$\operatorname{indec}\mathsf{T}' = \operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S} \cup \{r_{q+1}\} \cup \operatorname{indec}\mathsf{S}'.$$

From this we construct the quotient group $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$, and we will show that

$$\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N') = |\mathsf{R}| + 1 = \operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N).$$

In order to prove this inequality, we will count ranks in Diagram (6.28), by repeatedly using the result from Remark 4.3.3 that for a short exact sequence

$$0 \to A \to B \to C \to 0,$$

we have

$$\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C).$$

Now, we know by assumption that

$$\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N) = |\mathsf{R}|,$$

and recall the ranks of the free groups $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$ and F from (6.24) and (6.29).

Now, from the bottom short exact sequence in Diagram (6.28), we know

$$\operatorname{rank}(N) = |\mathsf{S}|,$$

and since when fitting a cell we add $|\mathsf{S}'|$ generators to N in order to form N', we know that

$$\operatorname{rank}(N') \leq |\mathsf{S}| + |\mathsf{S}'|.$$

Thus, computing ranks in the left vertical short exact sequence in Diagram (6.28), we see that

$$\operatorname{rank}(\operatorname{Ker}\nu') \leq |\mathsf{S}'|.$$

The top short exact sequence then reveals that $\operatorname{rank}(\operatorname{Ker}\lambda') \geq 1$, but by Lemma 6.2.11, $\operatorname{Ker}\lambda'$ is cyclic and so

$$\operatorname{rank}(\operatorname{Ker}\lambda') = 1.$$

Finally, the vertical short exact sequence on the right shows that

$$\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N') = |\mathsf{R}| + 1,$$

as required.

It now remains to show that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ is free. Well, since $\operatorname{Ker}\lambda'$ is cyclic of rank one, it must be free, and since $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is free by assumption, we know that the vertical short exact sequence on the right is in fact split exact. This follows since the final object is free. Thus, $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N' \cong \operatorname{Ker}\lambda' \oplus \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$, which is the direct sum of two free abelian groups, and so it follows that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ must be free abelian itself.

Case 2. (All cells are even). *Base*: Assume that $|\mathsf{R}| = 1$, and that the only cell is even; that is, k in Figure 6.24 is odd, and thus $k-1$ even.

Computing the generators $n(s)$ of N produces the relations:

$$s_2 - r, s_3 - s_1, s_4 - s_2, \ldots, s_k - s_{k-2}, r - s_{k-1}.$$

We can see from these relations that in the quotient $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$

$$s_i + N = s_j + N,$$

when i, j are both odd, and from $s_2 - r$ and $r - s_{k-1}$ in particular, we can see that,

$$r + N = s_i + N,$$

for every even i. Thus, $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on $|\mathsf{R}|+1 = 2$

generators, as required.

Induction: Assume for $|\mathsf{R}| \geq 0$ that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on $|\mathsf{R}| + 1$ generators.

Step: Now assume we have fitted a new spoke as in Figure 6.21, and formed $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$. We show here that

$$\mathrm{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N') = |\mathsf{R}| + 2 = \mathrm{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N) + 1.$$

First, we show that $\mathrm{rank}(N') \leq |\mathsf{S}| + |\mathsf{S}'| - 1$: We know by assumption that

$$\mathrm{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N) = |\mathsf{R}| + 1,$$

and recall from (6.24) that

$$\mathrm{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})) = |\mathsf{R}| + |\mathsf{S}|.$$

Thus, from the bottom short exact sequence in Diagram (6.28), we know that

$$\mathrm{rank}(N) = |\mathsf{S}| - 1.$$

As the rank of this subgroup is strictly less than the number of generators of the form $n(s)$, there must be a nontrivial linear relation between these generators. Denote this relation by

$$\sum_{s \in \mathrm{indec}\,\mathsf{S}} \alpha(s) n(s) = 0, \tag{6.33}$$

where $\alpha(s) \in \mathbb{Z}$, and not all $\alpha(s)$ are zero. In order to show $\mathrm{rank}(N') \leq |\mathsf{S}| + |\mathsf{S}'| - 1$, we must find a nontrivial relation of the generators $n'(s'')$ of N', for $s'' \in \mathrm{indec}\,\mathsf{S} \cup \mathrm{indec}\,\mathsf{S}'$. Consider the set of relations $n'(s')$ for $s' \in \mathrm{indec}\,\mathsf{S}'$, given by

$$\{n'(s') | s' \in \mathrm{indec}\,\mathsf{S}'\} = \{r_{q+1} - r_1 - s'_2,\, s'_1 - s'_3,\, s'_2 - s'_4, \ldots, s'_{l-2} - s'_l,\, s'_{l-1}\},$$

where $l - 1$ is even because we have glued an even cell. Consider the following linear combination of these generators

$$(r_{q+1} - r_1 - s'_2) + (s'_2 - s'_4) + (s'_4 - s'_6) + \cdots + (s'_{l-3} - s'_{l-1}) + s'_{l-1} = r_{q+1} - r_1.$$

That is,

$$n'(s'_1) + n'(s'_3) + n'(s'_5) + \cdots + n'(s'_{l-2}) + n'(s'_l) = r_{q+1} - r_1, \tag{6.34}$$

and so $r_{q+1} - r_1 \in N'$. Now, let $x \in N'$ be the following linear combination

$$x = \sum_{s \in \operatorname{indec} \mathsf{S}} \alpha(s) n'(s) \in N', \tag{6.35}$$

where $\alpha(s)$ is as in (6.33). Then, using (6.27) and Diagram (6.28), we know that

$$\begin{aligned}
\kappa' \varphi'(x) &= \varphi \nu'(x) \\
&= \sum_{s \in \operatorname{indec} \mathsf{S}} \alpha(s) \nu'(n'(s)) \\
&= \sum_{s \in \operatorname{indec} \mathsf{S}} \alpha(s) n(s) \\
&= 0,
\end{aligned}$$

where the third $=$ is due to (6.27) and the final $=$ is due to the relation in (6.33). Thus, $\varphi'(x) \in \operatorname{Ker} \kappa'$, and so there exists $y \in F$ such that $\mu(y) = \varphi'(x)$. Now, φ' is an inclusion, and so $\varphi'(x) = x \in \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$. Consider $x \in \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$ in terms of the generators of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$; that is, in terms of elements in $\operatorname{indec} \mathsf{R} \cup \operatorname{indec} \mathsf{S} \cup \{r_{q+1}\} \cup \operatorname{indec} \mathsf{S}'$. Well, by construction, the coefficients of the generators coming from the set $\operatorname{indec} \mathsf{S}'$ must be zero, as the summation in (6.35) is taken over $s \in \operatorname{indec} \mathsf{S}$. But F is the group generated by the s_i' and $r_{q+1} - r_1$, so $x = \mu(y)$ implies that y has the form

$$y = \alpha(r_{q+1} - r_1)$$

for some $\alpha \in \mathbb{Z}$, and so

$$\varphi'(x) = \mu(\alpha(r_{q+1} - r_1)). \tag{6.36}$$

Then, since $x \in N'$ and $r_{q+1} - r_1 \in N'$, we know $x - \alpha(r_{q+1} - r_1) \in N'$. Computing $\varphi'(x - \alpha(r_{q+1} - r_1))$, we see

$$\begin{aligned}
\varphi'(x - \alpha(r_{q+1} - r_1)) &= \varphi'(x) - \alpha\varphi'(r_{q+1} - r_1) \\
&= \alpha(r_{q+1} - r_1) - \alpha(r_{q+1} - r_1) \\
&= 0,
\end{aligned}$$

where the second $=$ uses the equality in (6.36), as well as the fact that μ and φ' are just inclusions. Since $\varphi'(x - \alpha(r_{q+1} - r_1)) = 0$, we must have that $x - \alpha(r_{q+1} - r_1) = 0$, and so we have found a relation between the generators of N'. It now remains to show that this relation is nontrivial; that is, not all the coefficients are zero. Well, rewriting

$$x - \alpha(r_{q+1} - r_1) \tag{6.37}$$

using the equalities in (6.34) and (6.35) gives

$$\sum_{s \in \operatorname{indec} \mathsf{S}} \alpha(s) n'(s) - \alpha\big(n'(s_1) + n'(s_3) + \cdots + n'(s_l')\big).$$

We know by assumption that at least one $\alpha(s)$ is nonzero, so our relation in (6.37) is nonzero. Now, since N' has $|\mathsf{S}|+|\mathsf{S}'|$ generators of the form $n'(s'')$, for $s'' \in \operatorname{indec} \mathsf{S} \cup \operatorname{indec} \mathsf{S}'$, and we have found a nontrivial relation between them, we know that

$$\operatorname{rank}(N') \leq |\mathsf{S}| + |\mathsf{S}'| - 1.$$

Now, as in Case 1, we compute $\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N')$ using the short exact sequences in Diagram (6.28). Recall the following ranks:

$$\begin{aligned} \operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N) &= |\mathsf{R}| + 1, \\ \operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})) &= |\mathsf{R}| + |\mathsf{S}|, \\ \operatorname{rank}(N) &= |\mathsf{S}| - 1, \\ \operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')) &= |\mathsf{R}| + |\mathsf{S}| + |\mathsf{S}'| + 1, \\ \operatorname{rank}(N') &\leq |\mathsf{S}| + |\mathsf{S}'| - 1, \\ \operatorname{rank}(F) &= |\mathsf{S}'| + 1. \end{aligned}$$

Using the left vertical short exact sequence, we can deduce that

$$\operatorname{rank}(\operatorname{Ker} \nu') \leq |\mathsf{S}'|.$$

Thus, the top short exact sequence tells us that $\operatorname{rank}(\operatorname{Ker} \lambda') \geq 1$, however, since $\operatorname{Ker} \lambda'$ is cyclic by Lemma 6.2.11, we in fact know that

$$\operatorname{rank}(\operatorname{Ker} \lambda') = 1.$$

Finally, the right vertical short exact sequence now shows that

$$\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N') = |\mathsf{R}| + 2,$$

as required.

The final thing to show is that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')N'$ is free. Well, since $\operatorname{Ker} \lambda'$ is cyclic and of rank one, it must be free. Adding this to our assumption that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is also free, we know that the right vertical short exact sequence in Diagram (6.28) splits, and so $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N' \cong \operatorname{Ker} \lambda' \oplus \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$. As both of these summands are free abelian, it follows that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ is also free abelian.

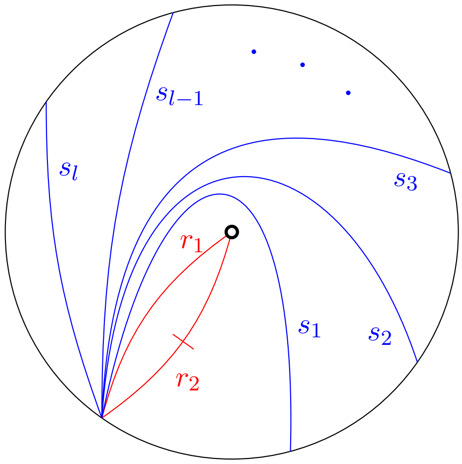


Figure 6.25: R contains spokes of both taggings. Theorem 6.1.7 allows us to arrange the arcs in S in a fan.

□

6.2.7 Computation of $\mathrm{K}_0^{\text{split}}(\mathsf{T})/N$ for a Central Region: Spokes in R Have Both Taggings

In this section, we will compute a general formula for $\mathrm{K}_0^{\text{split}}(\mathsf{T})/N$ for a central region, assuming that spokes in R have opposite taggings.

Recall that if a triangulation of the punctured polygon P contains spokes of both taggings, then the triangulation contains precisely two spokes which share both endpoints. We assume that both of these spokes belong to R. Now, as we have also assumed that every diagonal in R is a spoke, we know that every other diagonal in the triangulation of P, other than the two spokes, must correspond to an indecomposable in S. Additionally, every diagonal in S must be an arc. Figure 6.25 demonstrates this situtation, where the blue diagonals correspond to indecomposables in S and the red diagonals to indecomposables in R. As a consequence of Theorem 6.1.7, we are permitted to arrange the blue diagonals in a fan.

Theorem 6.2.13. *If* R *contains only spokes and* R *has spokes of different taggings, then* $\mathrm{K}_0^{\text{split}}(\mathsf{T})/N$ *is the finitely generated free abelian group on* $|\mathsf{R}| = 2$ *generators.*

Proof. Using Figure 6.25, we can compute the generators $n(s) \in N$ for $s \in \operatorname{indec} \mathsf{S}$. This produces the relations

$$r_1 + r_2 - s_2, s_1 - s_3, s_2 - s_4, \ldots, s_{l-2} - s_l, s_{l-1}. \tag{6.38}$$

There are two cases to consider; the case when l is odd, and the case when it is even.

Even case: If l is even, then we can see from the relations in (6.38) that in the quotient $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$, we have

$$r_1 + r_2 + N = s_i + N,$$

for each even i, whilst

$$s_i + N = 0 + N,$$

whenever i is odd. Here, $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ has two generators and is clearly free.

Odd case: If l is odd, we can see from the relations in (6.38) that in $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$, we have

$$r_1 + N = -r_2 + N,$$

as well as

$$s_i + N = s_j + N,$$

for i, j both odd, and

$$s_i + N = 0 + N,$$

whenever i is even. Again $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ has two generators and is clearly free. □

6.2.8 Computation of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ in General

As discussed previously, we will prove the upcoming theorem in this section using the results in Theorems 6.2.12 and 6.2.13. We consider a triangulation of the central region C and glue cells to this polygon using the methods discussed previously. We prove the theorem by inducting on the number of arcs corresponding to elements in R, and the following remark introduces important notation that allows us to do this.

Remark 6.2.14. Consider a rigid subcategory R whose indecomposables correspond to diagonals in a dissection of a punctured polygon P. We will denote the sets of indecomposables in R that correspond to spokes and arcs in the following way:

$$\operatorname{indec} \mathsf{R} = \operatorname{indec} \mathsf{R}_{\mathsf{s}} \cup \operatorname{indec} \mathsf{R}_{\mathsf{a}},$$

where $\operatorname{indec} \mathsf{R}_{\mathsf{s}}$ denotes those indecomposables that correspond to spokes, and $\operatorname{indec} \mathsf{R}_{\mathsf{a}}$ denotes those that correspond to arcs.

Recall our use of even and odd terminology in Definition 6.2.5 for the different types of cells introduced in Remark 6.2.4. Recall in particular that when $|\mathsf{R}_{\mathsf{s}}| \in \{0, 1\}$, the central region C of P is a cell.

Theorem 6.2.15. *1. If* $|\mathsf{R}_{\mathsf{s}}| > 0$*, then* $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ *is the finitely generated free abelian*

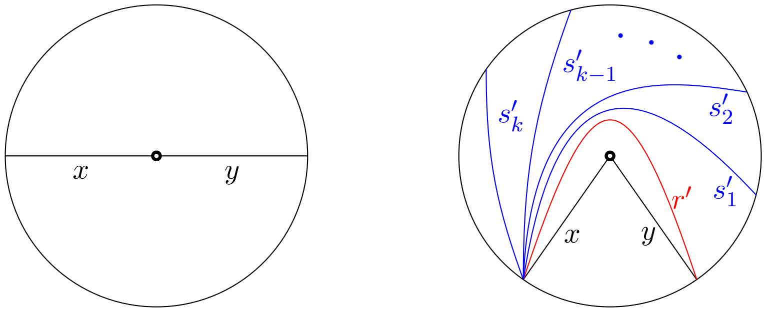


Figure 6.26: When the central region only has two vertices, a triangulation of it must consist of two spokes.

group on n generators, where

$$n = \begin{cases} |\mathsf{R}|+1 & \text{if all cells are even and all spokes in } \mathsf{R} \text{ have the same tagging,} \\ |\mathsf{R}| & \text{otherwise.} \end{cases}$$

2. *If $|\mathsf{R}_\mathsf{s}| = 0$, then $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on n generators, where*

$$n = \begin{cases} |\mathsf{R}|+2 & \text{if the central region } C \text{ of } P \text{ is odd and all other cells are even,} \\ |\mathsf{R}|+1 & \text{otherwise.} \end{cases}$$

Before proving the theorem, we discuss the following examples, which clarify some small and special cases of the theorem.

Example 6.2.16. We demonstrate in this example the case when the central region has two vertices. In this case, the central region will look as in the diagram on the left hand side of Figure 6.26. Here, the two spokes can belong to either S or R. It is clear that $N = 0$, and thus $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \cong \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, regardless of which subcategory x and y belong to.

Now, assume that we have glued on a new cell and formed the diagram on the right hand side of Figure 6.26, from which we can construct the groups $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$, N' and thus $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$. There are three cases to consider, depending on which subcategories x and y belong to. Each of these cases then splits into two, depending on whether the glued cell is odd or even.

1. ($x, y \in \mathsf{R}$). Note that in this case $|\mathsf{R}| = 3$, and $|\mathsf{R}_\mathsf{s}| = 2$. There are three cells; two wedge cells making up the central region, and a glued cell. Since each of the wedge cells are odd, in both of the following cases not all cells are even. Computing

generators of N' gives the relations:

$$r' - s'_2, s'_3 - s'_1, s'_4 - s'_2, \ldots, s'_k - s'_{k-2}, s'_{k-1}. \tag{6.39}$$

Even case: If the glued cell is even, then k must be odd. From the relations in (6.39), we see that in the quotient $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$, we have

$$r' + N' = s'_i + N' = 0 + N',$$

for each even i, whilst

$$s'_i + N' = s'_j + N',$$

whenever i and j are both odd. Note also that $x + N'$ and $y + N'$ are also elements in the quotient. Thus, the quotient is clearly the finitely generated free abelian group on $|\mathsf{R}| = 3$ generators.

Odd case: If the glued cell is odd, then k must be even. From the relations in (6.39), we see that in the quotient

$$s'_i + N' = 0 + N',$$

for every odd i, whilst

$$r' + N' = s'_i + N',$$

for every even i. Again, $x + N'$ and $y + N'$ are also elements of the quotient. $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ is again the finitely generated free abelian group on $|\mathsf{R}| = 3$ generators.

2. ($x \in \mathsf{R}$, $y \in \mathsf{S}$). Note in this case that $|\mathsf{R}| = 2$, and $|\mathsf{R}_{\mathsf{s}}| = 1$. Here, the central region is an even cell, thus when the glued cell is even, all cells in the construction are even, whereas this is not the case when the glued cell is odd. Computing generators of N' gives the relations

$$r', r' - s'_2, s'_3 - s'_1, s'_4 - s'_2, \ldots, s'_k - s'_{k-2}, s'_{k-1}. \tag{6.40}$$

Even case: If the glued cell is even, then k must be odd. From the relations in (6.40), we see that in $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$, we have

$$r' + N' = s'_i + N' = 0 + N',$$

for every even i, whereas

$$s'_i + N' = s'_j + N',$$

whenever i and j are both odd. Note also that $x + N'$ and $y + N'$ are both elements in the quotient, and so $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ is clearly the finitely generated free abelian

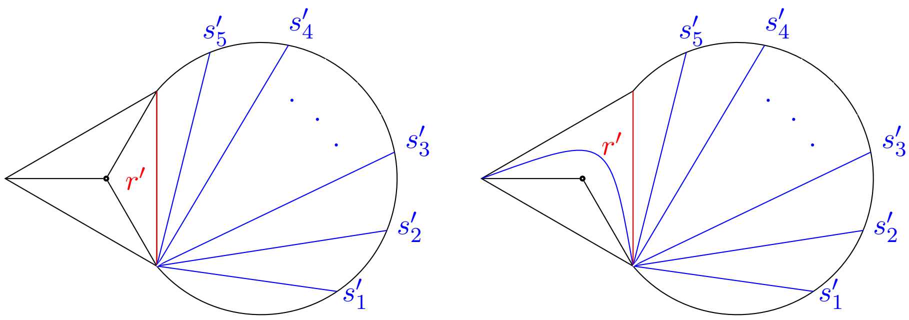


Figure 6.27: The gluing process when the cell containing the puncture has three vertices.

group on $|\mathsf{R}| + 1 = 3$ generators.

Odd case: If the glued cell is odd, then k is even. Here, the relations in (6.40) show that in the quotient

$$r' + N' = s'_i + N' = 0 + N',$$

for each i. Thus, as $x+N'$ and $y+N'$ are the only surviving generators, $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ is the finitely generated free abelian group on $|\mathsf{R}| = 2$ generators.

3. $(x, y \in \mathsf{S})$. Note in this case that $|\mathsf{R}| = 1$ and that $|\mathsf{R}_\mathsf{s}| = 0$, so we are working in the case of Statement 2 of Theorem 6.2.15. Note also that the original polygon P (on the left hand side of Figure 6.26) is odd. Computing generators of N' gives the same relations as the previous case; that is, those in (6.40). The odd and even cases are therefore the same as the previous case as well. Thus, when the glued cell is even, $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ is the finitely generated free abelian group on $|\mathsf{R}| + 2 = 3$ generators, and when the glued cell is odd, the quotient is the finitely generated free abelian group on $|\mathsf{R}| + 1 = 2$ generators.

The above three cases all concur with the statements in Theorem 6.2.15, and demonstrate how the gluing procedure works in the case when the cell containing the puncture only has two vertices. Notice now that when the cell containing the puncture has three or more vertices, we are able to work in the situations described in Figures 6.16 and 6.17. To clarify this, a diagram of the gluing in the case of three vertices is provided in Figure 6.27. The diagram on the left shows the gluing in the case from Figure 6.16, and the diagram on the right shows the gluing in the cases from Figure 6.17. Again, red diagonals correspond to indecomposables in R and blue diagonals to indecomposables in S. We permit any black spokes to be in either R or S.

Example 6.2.17. We demonstrate the special case when the central region has two vertices and R contains spokes of both taggings. Let $|\mathsf{R}| = 2$, and assume that the two spokes have opposite taggings. Then, the central region will look like the diagram on the left hand side of Figure 6.28. Since both $x, y \in \operatorname{indec} \mathsf{R}$, we know $N = 0$ and so $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \cong \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$.

Now, assume that we glue on a cell, forming the diagram on the right hand side of Figure 6.28, from which we can construct $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$, N', and so $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$. Computing the generators of N' gives the relations:

$$r' - s'_2, s'_3 - s'_1, s'_4 - s'_2, \ldots, s'_k - s'_{k-2}, s'_{k-1}. \tag{6.41}$$

There are two cases to consider; the case when the glued cell is even, and the case when it is odd.

Even case: If the glued cell is even, then k must be odd. The relations in (6.41) show that in the quotient

$$r' + N' = s'_i + N' = 0 + N',$$

for any even i, whereas

$$s'_i + N' = s'_j + N',$$

whenever i and j are both odd. Since x, y do not appear in any generators of N', we know $x + N'$ and $y + N'$ are both clearly generators of the quotient. Thus, $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ is the finitely generated free abelian group on $|\mathsf{R}| = 3$ generators, as stated in the theorem.

Odd case: When the glued cell is odd, k must be even. In this case, the relations in (6.41) show that in the quotient

$$r' + N' = s'_i + N',$$

for any even i, and

$$s'_i + N' = 0 + N',$$

for all odd i. Again, as x, y do not appear in any of the generators of N', they must be generators of the quotient. Hence, $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ is the finitely generated free abelian group on $|\mathsf{R}| = 3$ generators, as stated in the theorem.

Proof. (Theorem 6.2.15) Given any triangulation of the punctured polygon, apply the "ungluing" process described in Remark 6.2.3, leaving the central region C. We prove the results by inducting on $|\mathsf{R}_\mathsf{a}|$, and so in each of the statements of the theorem, the central region C will provide the base case for the induction. Theorems 6.2.12 and 6.2.13 provide a formula for $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ for any given triangulation of C made up of spokes in R and diagonals in S. We discuss here why this is the case. If the central region contains spokes all of the same tagging, then Theorem 6.2.12 gives a formula for the quotient. Should the central region contain spokes of both taggings, then there are two options; either both

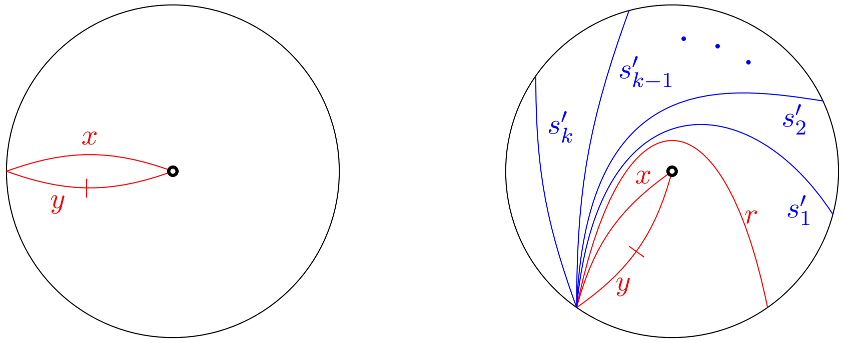


Figure 6.28: When R has spokes of both taggings, there must only be two spokes in the triangulation, and so the cell containing the puncture has two vertices.

spokes are in R, in which case Theorem 6.2.13 provides a formula for $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$, or at least one spoke is in S. In this case, choose one of the spokes in S and then mutate and replace it, leaving two spokes of the same tagging. Theorem 6.2.12 now provides a formula for the quotient of this central region. Note finally that if C contains no spokes in R, then Theorem 6.2.12 still provides a formula for the quotient.

We start by proving part 1 of Theorem 6.2.15, and do so by induction on $|\mathsf{R_a}|$. This will be done in two cases, starting with the "otherwise" case.

Case 1. (*Not all cells are even.*) We first work in Statement 1 of the theorem, and prove the "otherwise" case.

Base: Assume that $|\mathsf{R_s}| \geq 1$ and $|\mathsf{R_a}| = 0$. At least one of the following statements must be satisfied:

1. There is an odd cell.

2. R contains spokes of both taggings.

If the first statement is satisfied, the central region C must itself be odd if $|\mathsf{R_s}| = 1$, as in this case C is the only cell. However, if $|\mathsf{R_s}| > 1$ then C must contain an odd wedge cell. In either case, we know by Theorem 6.2.12 that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on $|\mathsf{R}|$ generators, as required.

If the second statement is satisfied, then we know by Theorem 6.2.13 that again $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the free abelian group on $|\mathsf{R}|$ generators. The base therefore holds.

Induction: Assume now that $|\mathsf{R_a}| > 0$, and that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on $|\mathsf{R}|$ generators.

Step: Assume now that we have implemented the gluing process described earlier, and formed the larger cluster tilting subcategory $\mathsf{T}' \subseteq \mathsf{C}'$ from Equation (6.16). We will show that

$$\mathrm{rank}\big(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'\big) = |\mathsf{R}| + 1 = \mathrm{rank}\big(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N\big) + 1.$$

As with earlier proofs of this kind, we will again compute ranks in Diagram (6.20) using the property from Remark 4.3.3 that for a short exact sequence

$$0 \to A \to B \to C \to 0,$$

we know that

$$\mathrm{rank}(B) = \mathrm{rank}(A) + \mathrm{rank}(C).$$

Recall from Equations (6.17) and (6.21) the ranks of the free groups $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')$ and F. Also recall that by assumption

$$\mathrm{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N) = |\mathsf{R}|.$$

Now, from the bottom short exact sequence in Diagram (6.20), we get that

$$\mathrm{rank}(N) = |\mathsf{S}|.$$

Also, since the gluing process adds at most $|\mathsf{S}'|$ generators to N in order to form N', we see that

$$\mathrm{rank}(N') \leq |\mathsf{S}| + |\mathsf{S}'|,$$

and so the left vertical short exact sequence gives

$$\mathrm{rank}(\mathrm{Ker}\,\nu') \leq |\mathsf{S}'|.$$

Well, the top short exact sequence then gives that $\mathrm{rank}(\mathrm{Ker}\,\lambda') \geq 1$, but since we know by Lemma 6.2.10 that $\mathrm{Ker}\,\lambda'$ is cyclic, we must have

$$\mathrm{rank}(\mathrm{Ker}\,\lambda') = 1.$$

Finally, calculating ranks using the right vertical short exact sequence, we obtain

$$\mathrm{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N') = |\mathsf{R}| + 1,$$

as required. It remains to show that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ is free. Indeed, since $\mathrm{Ker}\,\lambda'$ is cyclic and of rank one, it must be free. Then, since both the first and final objects in the right vertical short exact sequence are free, the sequence must be split exact. Thus, $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N' \cong \mathrm{Ker}\,\lambda' \oplus \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$, and since both summands are free, $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ must also be free.

Case 2. (*All cells are even and all spokes in* R *have the same tagging.*) We now prove the other case of Statement 1 from the theorem.

Base: Assume $|\mathsf{R_s}| > 0$, $|\mathsf{R_a}| = 0$ and that every spoke has the same tagging. If $|\mathsf{R_s}| = 1$, then C is the only cell, and so we assume that it is even, however if $|\mathsf{R_s}| > 1$ then assume that every wedge cell in C is even. In either case, by Theorem 6.2.12, we know that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on $|\mathsf{R}| + 1$ generators.

Induction: Assume now that $|\mathsf{R}_a| > 0$, that all cells in the triangulation are even, that all spokes in R have the same tagging and that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian on $|\mathsf{R}| + 1$ generators.

Step: We now implement the gluing process, and glue on an even cell to the existing polygon, constructing the cluster tilting subcategory T', as defined in (6.16). We show that gluing on this even cell has increased the rank of the quotient group by one; that is,

$$\mathrm{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N') = |\mathsf{R}| + 2 = \mathrm{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N) + 1.$$

We will compute ranks in Diagram (6.20) using the same method as before. In order to do this, we will first show that $\mathrm{rank}(N') \leq |\mathsf{S}| + |\mathsf{S}'| - 1$. Recall from Equation (6.17) that $\mathrm{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})) = |\mathsf{R}| + |\mathsf{S}|$ and that by assumption $\mathrm{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N) = |\mathsf{R}| + 1$. Thus, the bottom short exact sequence in Diagram (6.20) tells us that

$$\mathrm{rank}(N) = |\mathsf{S}| - 1,$$

and so there must be some nontrivial linear relation between the generators $n(s)$ of N. Denote this relation by

$$\sum_{s \in \mathrm{indec}\,\mathsf{S}} \alpha(s) n(s) = 0, \tag{6.42}$$

where each $\alpha(s) \in \mathbb{Z}$, and not all $\alpha(s)$ are zero. In order to show that $\mathrm{rank}(N') \leq |\mathsf{S}| + |\mathsf{S}'| - 1$, we must show that there is also a nontrivial relation between the generators $n'(s')$ of N'.

Consider the generators $n'(s')$ for each $s' \in \mathrm{indec}\,\mathsf{S}'$. These are the generators corresponding to the S' diagonals in the glued cell P'' in Figure 6.16. They are given by the following set:

$$\{n'(s') | s' \in \mathrm{indec}\,\mathsf{S}'\} = \{s'_2 - r', s'_3 - s'_1, s'_4 - s'_2, \dots, s'_k - s'_{k-2}, s'_{k-1}\}$$

where k is odd, and so $k-1$ even. Consider the following linear combination of these generators:

$$s'_{k-1} - (s'_{k-1} - s'_{k-3}) - \cdots - (s'_4 - s'_2) - (s'_2 - r') = r',$$

which is

$$n'(s'_k) - n'(s'_{k-2}) - \cdots - n'(s'_3) - n'(s'_1) = r'. \tag{6.43}$$

This shows $r' \in N'$.

Now consider the following linear combination of generators of N' coming from the indecomposables in S:

$$x = \sum_{s \in \operatorname{indec} \mathsf{S}} \alpha(s) n'(s) \in N'. \tag{6.44}$$

Then, using (6.19) and Diagram (6.20), we know

$$\begin{aligned}
\kappa' \varphi'(x) &= \varphi \nu'(x) \\
&= \sum_{s \in \operatorname{indec} \mathsf{S}} \alpha(s) \nu'(n'(s)) \\
&= \sum_{s \in \operatorname{indec} \mathsf{S}} \alpha(s) n(s) \\
&= 0,
\end{aligned}$$

where the third $=$ is by virtue of (6.19) and the final $=$ is due to the relation in (6.42). This shows that $\varphi'(x) \in \operatorname{Ker} \kappa'$, and so there must be some $y \in F$ such that $\mu(y) = \varphi'(x)$. Now, φ' is an inclusion, and so $\varphi'(x) = x \in \mathrm{K}_0^{\text{split}}(\mathsf{T}')$. Consider $x \in \mathrm{K}_0^{\text{split}}(\mathsf{T}')$ in terms of the generators of $\mathrm{K}_0^{\text{split}}(\mathsf{T}')$; that is, in terms of the elements of $\operatorname{indec} \mathsf{R} \cup \operatorname{indec} \mathsf{S} \cup \{r'\} \cup \operatorname{indec} \mathsf{S}'$. Well, by construction, the coefficients of the generators coming from $\operatorname{indec} \mathsf{S}'$ must be zero as the summation in (6.44) for x is taken over $s \in \operatorname{indec} \mathsf{S}$. But F is the free group generated by the s_i' and r', so $x = \mu(y)$ implies that y has the form

$$y = \alpha r'$$

for some $\alpha \in \mathbb{Z}$. So,

$$\varphi(x) = \mu(\alpha r'). \tag{6.45}$$

Since $x \in N'$ and $r' \in N'$, we know $x - \alpha r' \in N'$. Computing $\varphi'(x - \alpha r')$, we obtain

$$\begin{aligned}
\varphi'(x - \alpha r') &= \varphi'(x) - \alpha \varphi'(r') \\
&= \alpha r' - \alpha r' \\
&= 0,
\end{aligned}$$

where the second $=$ uses the equality in (6.45), as well as the fact that $\mu(\alpha r') = \alpha r'$, as μ is just an inclusion. Since $\varphi'(x - \alpha r') = 0$, it follows that $x - \alpha r' = 0$, and so we have found a linear relations between the generators of N'. We should now check that this relation is indeed nontrivial; that is, not all coefficients are zero. Well, substituting into our relation $x - \alpha r'$ using the equalities in (6.43) and (6.44) gives:

$$\sum_{s \in \operatorname{indec} \mathsf{S}} \alpha(s) n'(s) - \alpha\Big(n'(s_k') - n'(s_{k-2}) - \cdots - n'(s_3') - n'(s_1')\Big) = 0.$$

This is clearly a nontrivial relation as we know by assumption that at least one $\alpha(s)$ is nonzero. Since N' is generated by $|\mathsf{S}| + |\mathsf{S}'|$ generators, which we have found a relation between, we can conclude that

$$\operatorname{rank}(N') \leq |\mathsf{S}| + |\mathsf{S}'| - 1.$$

We now compute $\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N')$ using the short exact sequences in Diagram (6.20). Recall that we already know each of the following ranks:

$$\begin{aligned}
\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N) &= |\mathsf{R}| + 1, \\
\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})) &= |\mathsf{R}| + |\mathsf{S}|, \\
\operatorname{rank}(N) &= |\mathsf{S}| - 1, \\
\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')) &= |\mathsf{R}| + |\mathsf{S}| + |\mathsf{S}'| + 1, \\
\operatorname{rank}(N') &\leq |\mathsf{S}| + |\mathsf{S}'| - 1, \\
\operatorname{rank}(F) &= |\mathsf{S}'| + 1.
\end{aligned}$$

Computing ranks in the left vertical short exact sequence gives

$$\operatorname{rank}(\operatorname{Ker}\nu') \leq |\mathsf{S}'|.$$

Then, using the top short exact sequence, we see that $\operatorname{rank}(\operatorname{Ker}\lambda') \geq 1$, however since $\operatorname{Ker}\lambda'$ is cyclic by Lemma 6.2.10, we must have

$$\operatorname{rank}(\operatorname{Ker}\lambda') = 1.$$

The right vertical short exact sequence then gives us the desired result:

$$\operatorname{rank}(\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N') = |\mathsf{R}'| + 2.$$

The final thing to verify is that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ is cyclic. Well, $\operatorname{Ker}\lambda'$ is cyclic and of rank one, so must be free. Then, since the first and last terms of the right vertical short exact sequence in Diagram (6.20) are both free, the sequence splits. Thus, $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N' \cong \operatorname{Ker}\lambda' \oplus \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$, which is the direct sum of two free abelian groups, so $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T}')/N'$ must itself be free. This concludes the proof of part 1 of Theorem 6.2.15.

We now verify that everything required for part 2 of Theorem 6.2.15 has already been proved. Consider first for $|\mathsf{R}_\mathsf{s}| = 0$ the "otherwise" case in the theorem; that is, the case when at least one of the two following statements is false:

1. The central region C of P is odd.

2. Every cell is even.

Assume first that $|\mathsf{R_a}| = 0$ and that the only cell C is even. Then, by Theorem 6.2.12, $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on $1 = |\mathsf{R}| + 1$ generators. Then, for an induction, assume that for some $|\mathsf{R_a}| \geq 0$ the quotient group $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on $|\mathsf{R}| + 1$ generators. Now, we will glue a cell on to the polygon and show that this has increased the rank of the corresponding quotient group by one. This can be done by the same arguement as in the induction step of Case 1 for part 1 of this theorem. The aforementioned induction step also verifies that the resulting quotient group is free.

Consider now for $|\mathsf{R_s}| = 0$ the case when the central region C of P is odd and all other cells are even. Assume first that $|\mathsf{R_a}| = 0$, and that the only cell C is odd. Then, by Theorem 6.2.12, the quotient $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on $2 = |\mathsf{R}| + 2$ generators. For induction, assume now that for some $|\mathsf{R_a}| \geq 0$, where the central region C of P is odd and all other cells are even, that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$ is the finitely generated free abelian group on $|\mathsf{R}| + 2$ generators. Now, we employ the gluing procedure by gluing an even cell on to the existing polygon, and show that this process increases the rank of the corresponding quotient group by one. This can be done by the same arguement as in the induction step for Case 2 of part 1 of this theorem. The arguement also proves that the resulting quotient group is free, as required. □

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