Rational dilation and constrained algebras

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Abstract

If a set Ω is a spectral set for an operator T, is it necessarily a complete spectral set? That is, if the spectrum of T is contained in Ω, and von Neumann's inequality holds for T and rational functions with poles off of $\overline{\Omega}$, does it still hold for all such matrix valued rational functions? Equivalently, if Ω is a spectral set for T, does T have a dilation to a normal operator with spectrum in the boundary of Ω? This is true if Ω is the disk or the annulus, but has been shown to fail in many other cases. There are also multivariable versions of this problem. For example, it is known that rational dilation holds for the bidisk, though it has been recently shown to fail for a distinguished variety in the bidisk called the Neil parabola. The Neil parabola is naturally associated to a constrained subalgebra of the disk algebra, as are many other distinguished varieties.

We show that the rational dilation fails on certain distinguished varieties of the polydisk $\overline{\mathbb{D}}^N$ associated to the constrained subalgebra $\mathscr{A}_B := \mathbb{C} + B(z)\mathbb{A}(\mathbb{D})$. Here $\mathbb{A}(\mathbb{D})$ is the algebra of functions that are analytic on the open unit disk $\mathbb{D}$ and continuous on the closure of $\mathbb{D}$, and $B(z)$ is a finite Blaschke product of degree $N \geq 2$. To this end we identify and study the set of test functions Ψ_B for $H_B^\infty := \mathbb{C} + B(z)H^\infty(\mathbb{D})$. Among others, we show that Ψ_B is minimal (in a sense that there is no proper closed subset of Ψ_B is suffices).

Contents

Chapter 1

Introduction and known results

1.1 Introduction

The purpose of this thesis is to study the rational dilation problem on certain distinguished varieties of $\overline{\mathbb{D}}^N$ for $N \geq 2$ (that is, the intersection of a variety with the closed polydisk $\overline{\mathbb{D}^N}$ which intersects the boundary of $\overline{\mathbb{D}^N}$ in its distinguished boundary $\mathbb{T}^N$) associated to some constrained subalgebras of the disk algebra.

Let Ω be a compact subset of $\mathbb{C}^d$ and suppose that $T = (T_1, \ldots, T_d)$ is a $d-$tuple of commuting operators on a Hilbert space $\mathcal{H}$ with spectrum contained in Ω. Furthermore, let $\mathcal{R}(\Omega)$ be the algebra of rational functions with poles off Ω and assume that for every r in $\mathcal{R}(\Omega)$, the von Neumann inequality holds; that is,

$$\|r(T)\| \leq \|r\|_\Omega, \tag{1.1}$$

where $\|\cdot\|_\Omega$ is the supremum norm over Ω. When the von Neumann inequality (1.1) holds for an operator (or tuple of operators) T, we say that Ω is a **spectral set** for T. More about the von Neumann inequality and its generalizations can be found in [39]. Also, more recent improvements can be found in [9], [8], [6], [32]).

The **rational dilation problem** asks: If Ω is spectral set for a commuting d-tuple T operators on a Hilbert space $\mathcal{H}$, then does T dilate to a d-tuple of commuting normal operators $\mathcal{N} = (\mathcal{N}_1, \ldots, \mathcal{N}_d)$ with the joint spectrum in $\partial\Omega$, the Shilov (or distinguished) boundary of Ω? More precisely, does there exits a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a tuple of commuting normal operators $\mathcal{N}$ on Ω such that $\sigma_A(\mathcal{N}) \subset \partial\Omega$ and

$$r(T) = P_{\mathcal{H}} r(\mathcal{N})|_{\mathcal{H}} \tag{1.2}$$

for all $r \in \mathcal{R}(\Omega)$? Here $P_{\mathcal{H}}$ denotes the orthogonal projection from $\mathcal{K}$ to $\mathcal{H}$, $|_H$ is the restriction to H and $\sigma_A(\mathcal{N})$ is the joint spectrum of $\mathcal{N}$ (see the definition 3.3.1). If such a commuting normal d-tuple $\mathcal{N}$ exists, we say that rational dilation holds, and otherwise, that it fails.

The rational dilation problem is not yet fully solved, but operator theorists have discovered a great deal about this problem; see the next subsection. In this thesis, we give a negative answer for this question when Ω comes form a certain class of distinguished varieties.

This thesis consists of the following two main parts.

In the first part, we study the test functions of certain weak-$*$ closed, unital subalgebras of H^∞. Namely we find a set of test functions Ψ_B for $H^\infty_B := \mathbb{C} + BH^\infty(\mathbb{D})$, where B is a finite Blaschke product of degree 2 or more. We also show that Ψ_B is minimal, in the sense that no proper closed subset of Ψ_B is a set of test functions for H^∞_B . The interpolation problem of the algebras H^∞_B was already studied in [22, 43]. The first application of test functions appeared in the solution of the Pick interpolation problem on the bidisk in the unpublished work by Jim Agler [2], also stated in [4]. Subsequent work has expanded its use to other types of interpolation problem and rational dilation problems; see [7], [25], [33], [27], [42], [28]. Finding a set of test functions for H^∞_B is our starting point in dealing with the rational dilation problem.

In the second part, by using the set Ψ_B and applying a method from [24] we show that the constrained algebra $\mathscr{A}_B = \mathbb{C}+B\mathbb{A}(\mathbb{D})$ has a contractive representation which is not completely contractive. Here $\mathbb{A}(\mathbb{D})$ is the disk algebra, that is, the algebra of the analytic functions on the open unit disk $\mathbb{D}$ which are continuous on the closure of $\mathbb{D}$. Consequently, we show that the rational dilation problem fails on certain distinguished varieties associated to $\mathscr{A}_B$.

Our main tool for dealing with the rational dilation problem on a distinguished variety associated to $\mathscr{A}_B$ is a remarkable result of William Arveson [14]. It says that the n-tuple T dilates to a normal n-tuple N with spectrum in the (Shilov) boundary of Ω (relative to $\mathcal{R}(\Omega)$) if and only if π_T is completely contractive, where π_T is unital representation of $\mathcal{R}(\Omega)$ on H via $\pi_T = r(T)$. Note that the condition Ω is a spectral set for T is equivalent to the condition that the representation π_T is contractive.

1.2 Some known positive and negative cases

The first dilation theorem was proved by Sz.-Nagy [45] in the 1953.

Theorem (Sz.-Nagy's dilation theorem)**.** Every contraction T (i.e, an operator of norm less than or equal to 1 on a Hilbert space $\mathcal{H}$) dilates to a unitary operator. That is, there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a unitary operator U on K such that

$$f(T) = P_{\mathcal{H}} f(U)|_{\mathcal{H}}$$

for all $f \in \mathcal{R}(\overline{\mathbb{D}})$, the rational functions poles off of $\overline{\mathbb{D}}$, where $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$ and $|_{\mathcal{H}}$ is the restriction on $\mathcal{H}$.

One of the most important application of Sz.-Nagy's dilation theorem is the von Neumann inequality [50], though von Neumann's inequality appeared two years before the Sz.-Nagy's dilation theorem.

Theorem (von Neumann's inequality)**.** An operator T is a contraction if and only if $\|f(T)\| \leq \|f\|$ for all $f \in \mathcal{R}(\overline{\mathbb{D}})$, where $\|f\| = \sup_{\lambda \in \overline{\mathbb{D}}} |f(\lambda)|$ and the left hand norm is the usual operator norm.

Observe that there is the common condition that appears in both the Sz.-Nagy dilation theorem and von Neumann inequality: $\|T\| \leq 1$. The above theorems immediately implies that the rational dilation holds when $\Omega = \overline{\mathbb{D}}$. Berger (1963), Foias (1959) and Lebow (1963) extended the Sz.-Nagy dilation theorem to more general simply connected planar domains. Hence the rational dilation holds for any simply connected domain in the complex plane; see [39]. A deep result of J.Agler [3] shows that rational dilation holds when Ω is an annulus. A simplified proof can be found in [24]. In 1992, P.Chu [39] showed that if T is a tuple of commuting 2×2 matrices, then rational dilation holds-so no matter how badly behaved Ω is.

However, for (many) planar domains of higher connectivity rational dilation fails. For example, Agler, Harland and Raphael [5] have showed this by machine computation, in an example of a two holed domain in the complex plane. More generally, Dritschel and McCullough [26] showed that rational dilation fails when Ω is any triply connected domain with analytic boundaries in $\mathbb{C}$. Furthermore, Pickering [42] showed that rational dilation fails whenever Ω is a domain in $\mathbb{C}$ with n holes, satisfying a symmetry condition for $3 \leq n \leq \infty$. The approaches of [26] and [42] was to find a set of test functions Ψ so that $H^\infty(K_\Psi) = H^\infty(\Omega)$. Then they show that test functions can be used to characterize the contractive and completely contractive representation of $H^\infty(\Omega)$. Using these characterizations they showed that certain contractive representations of $H^\infty(\Omega)$ are not completely contractive.

The first multivariable positive answer to the rational dilation problem is given by Tsuyoshi Andô [13] in the 1963.

Theorem (Andô's dilation theorem)**.** Let $\mathcal{H}$ be a Hilbert space and assume that $T = (T_1, T_2)$ is a commuting pair of operators acting on $\mathcal{H}$. If $\|T_i\| \leq 1, i = 1, 2$, then T has a unitary dilation, i.e., there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a pair of commuting unitary operators $U = (U_1, U_2)$ acting on $\mathcal{K}$ such that

$$f(T_1, T_2) = P_{\mathcal{H}} f(U_1, U_2)|_{\mathcal{H}}$$

for all $f \in \mathcal{R}(\overline{\mathbb{D}^2})$, where $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$ and $|_{\mathcal{H}}$ is the restriction on $\mathcal{H}$.

Theorem (von Neumann inequality for the bidisk)**.** Let $\mathcal{H}$ be a Hilbert space and assume that $T = (T_1, T_2)$ is a commuting pair of contractions acting on $\mathcal{H}$, then

$$\|f(T_1, T_2)\| \leq \|f\|$$

for all $f \in \mathcal{R}(\overline{\mathbb{D}^2})$, where $\|f\| = \sup_{(\lambda_1,\lambda_2)\in\overline{\mathbb{D}^2}} |f(\lambda_1, \lambda_2)|$.

Andô's result implies that rational dilation holds for the closed bidisk. Another positive answer for the 2-variable case of the rational dilation problem is obtained when Ω is the closed symmetrized bidisk Γ [10, 17, 36], a domain in $\mathbb{C}^2$ defined by

$$\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1| \leq 1, |z_2| \leq 1\}. \tag{1.3}$$

However, Parrot showed by a counterexample [38, 39] that rational dilation fails on the closed tridisk $\overline{\mathbb{D}^3}$. Very recently, S. Pal [37] showed that rational dilation fails also when Ω is the closure of the tetrablock $\mathbb{E}$, a polynomially convex, non-convex and inhomogeneous domain in $\mathbb{C}^3$, defined as

$$\mathbb{E} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : 1 - zx_1 - wx_2 + zwx_3 \neq 0 \text{ whenever } |z| \leq 1, |w| \leq 1\}. \tag{1.4}$$

Note that there is a way of mapping an annulus to a distinguished variety of the closed bidisk, so rational dilation holding for annuli is equivalent to it holding for a certain family of distinguished varieties in $\overline{\mathbb{D}}^2$, see [24]. It has been shown in [24] that rational dilation holds for the distinguished variety $\{(z,w) \in \overline{\mathbb{D}}^2 : z^2 = w^2\}$. It is natural therefore to wonder if this is in some sense a legacy of what we know about rational dilation for $\overline{\mathbb{D}^2}$, and so perhaps rational dilation also holds for other distinguished varieties in $\overline{\mathbb{D}^2}$?

But this is too much to hope for. In [24], it was also proved that rational dilation fails for the Neil parabola $\mathscr{N}_{z^2} = \{(z,w) \in \mathbb{D}^2 : z^3 = w^2\}$. The method of proof is somewhat indirect. In this case, there is a complete isometry mapping $R(\mathscr{N}_{z^2})$ onto $\mathscr{A}_{z^2} = \mathbb{C} + z^2\mathbb{A}(\mathbb{D})$. It is shown that this algebra has a contractive representation which is not 2-contractive, and so not completely contractive.

1.3 Definitions and notation

Our approach to solving the rational dilation problem for a distinguished variety associated to $\mathscr{A}_B$ requires finding a family of so-called "test functions" for the algebra H^∞_B. For other purposes (such as solving interpolation problems), it is useful for this family to be in some sense minimal. This is technically the most challenging aspect of the problem, and once accomplished the method used in [24] can be readily modified to yield the desired examples on rational dilation.The minimality of the set of test functions also allows us to construct a representation which is not contractive, despite sending the generators of $\mathscr{A}_B$ to contractions. This is in the spirit of the example due to Kaijser and Varopolous [48]. We give a brief synopsis of the notion of test functions and their use in the solution of interpolation problems. We refer to [27] for further details.

Definition 1.3.1. A set Ψ of complex-valued functions on a set X is called a **set of test functions** if:

1. For any $x \in X, \sup_{\psi \in \Psi} |\psi(x)| < 1$; and
2. The elements of Ψ separates the points of X.

Definition 1.3.2. An $n \times n$ matrix $A = (a_{ij})$ is positive semidefinite ($A \geq 0$) if

$$\sum_{i,j=1}^{n} a_{ij}\overline{c}_i c_j \geq 0$$

for all $c_1, \ldots, c_n \in \mathbb{C}$.

For further properties of positive semidefinite matrices, see [16].

Definition 1.3.3. Let X be a set and $\mathbf{A}$ be a C^*-algebra. A function $k : X \times X \to \mathbf{A}$ is called a **kernel**. It is a **positive kernel** if for every finite set $\{x_1, \ldots, x_n\}$ of distinct points in X, the matrix $(k(x_i, x_j)) \in M_n(\mathbf{A})$ is positive semidefinite.

Every set of test functions defines a Banach algebra in the following sense.

Definition 1.3.4. Let X be a set. For a set of test functions Ψ, we define a set of positive semidefinite kernels, called the **admissible kernels**, by

$$\mathcal{K}_\Psi := \Big\{ k : X \times X \to \mathbb{C} : \Big((1 - \psi(x)\overline{\psi(y)})k(x,y) \Big) \geq 0 \ \ \forall \psi \in \Psi \Big\}.$$

Here $\geq$ indicates that the left-hand side is a positive semi-definite kernel. We use the admissible kernels to define a Banach algebra.

Definition 1.3.5. Let X be a set. Also let $\mathcal{K}_\Psi$ be the set of admissible kernels for a set of test functions Ψ on X. We define the Banach algebra $H^\infty(\mathcal{K}_\Psi)$ consisting of those functions $\varphi : X \to \mathbb{C}$ for which there is a finite constant $C \geq 0$ such that for all $k \in \mathcal{K}_\Psi$, the kernel

$$\big((C^2 - \varphi(x)\varphi(y)^*)k(x,y) \big) \tag{1.5}$$

is positive semidefinite.

We set

$$C_\varphi = \inf\{ C : \big((C^2 - \varphi(x)\varphi(y)^*)k(x,y) \big) \geq 0 \ \text{ for all } \ k \in \mathcal{K}_\Psi \}.$$

Then the norm is given by

$$\|\varphi\|_{H^\infty(\mathcal{K}_\Psi)} = C_\varphi$$

on $H^\infty(\mathcal{K}_\Psi)$. One can check that $(H^\infty(\mathcal{K}_\Psi), \| \cdot \|_{H^\infty(\mathcal{K}_\Psi)})$ is a Banach algebra, with pointwise addition and multiplication (see the Appendix A.1). Obviously, the test functions are in the unit ball of $H^\infty(\mathcal{K}_\Psi)$. Because the kernel $k(x,x) = 1$ for all x and $k(x,y) = 0$ if $x \neq y$ is an admissible kernel, the norm of $H^\infty(\mathcal{K}_\Psi)$ will always be greater than or equal to the supremum norm, and so $H^\infty(\mathcal{K}_\Psi)$ is weakly closed (that is, closed under pointwise convergence).

The algebra of all bounded continuous functions on Ψ with pointwise algebra operations, is denoted by $C_b(\Psi)$. If Ψ is compact, then the set $C(\Psi)$ of continuous functions from Ψ to $\mathbb{C}$ is equal to $C_b(\Psi)$ (see page 2, [35]). Let $C_b(\Psi)^*$ be the dual space of $C_b(\Psi)$. If Ψ is compact, then $C_b(\Psi)^*$ is the Borel measures on Ψ, see [31]. We assume that Ψ is endowed with a suitable topology so that for all $x \in X$, the functions $E(x) : \psi \in \Psi \mapsto \psi(x)$ are in $C_b(\Psi)$. In this case $E(x)^* : \psi \in \Psi \mapsto \psi(x)^*$ is also in $C_b(\Psi)$.

A key result in the study of algebras generated through test functions is the realization theorem [27], which gives several equivalent characterizations of membership in the closed unit ball of the algebra $H^\infty(\mathcal{K}_\Psi)$. We state the portion relevant to us here.

Theorem 1.3.6 (Realization theorem). *Let Ψ be a collection of test functions on a set X and $H^\infty(\mathcal{K}_\Psi)$ the associated function algebra. For $\varphi : X \to \mathbb{C}$, the following are equivalent:*

1. *$\varphi \in H^\infty(\mathcal{K}_\Psi)$ and $\|\varphi\|_{H^\infty(\mathcal{K}_\Psi)} \leq 1$.*
2. *(a) For each finite set $F \subset X$ there exists a positive kernel $\Gamma : F \times F \to C_b(\Psi)^*$ such that for all $x, y \in F$,*

$$1 - \varphi(x)\varphi(y)^* = \Gamma(x, y)\,(1 - E(x)E(y)^*) .$$

   (b) There exists a positive kernel $\Gamma : X \times X \to C_b(\Psi)^*$ such that for all $x, y \in X$,

$$1 - \varphi(x)\varphi(y)^* = \Gamma(x, y)\,(1 - E(x)E(y)^*) .$$

3. *If π is any unital representation of $H^\infty(\mathcal{K}_\Psi)$ such that $\|\pi(\psi)\| < 1$ for all $\psi \in \Psi$, then π is contractive.*

The proof of the realization theorem is the basis for the following interpolation theorem of [27].

Theorem 1.3.7 (Agler-Pick interpolation theorem). *Let Ψ be a collection of test functions on a set X and $H^\infty(\mathcal{K}_\Psi)$ the associated function algebra. Fix a finite subset $F \subset X$. For $f : F \to \mathbb{D}$, the following are equivalent:*

1. *There is a function $\varphi \in H^\infty(\mathcal{K}_\Psi)$ with $\|\varphi\|_{H^\infty(\mathcal{K}_\Psi)} \leq 1$ such that $\varphi|_F = f$.*
2. *For each $k \in \mathcal{K}_\Psi$, the kernel*

$$F \times F \ni (x, y) \to ((1 - f(x)f(y)^*)\,k(x, y))$$

is positive.

3. *There is a positive kernel $\Gamma : F \times F \to C_b(\Psi)^*$ so that for all $x, y \in F$*

$$1 - f(x)f(y)^* = \Gamma(x, y)\,(1 - E(x)E(y)^*)\ .$$

To sum up, given a set of test functions Ψ, we can construct a normed function algebra $H^\infty(\mathcal{K}_\Psi)$ via a set of admissible kernels $\mathcal{K}_\Psi$. But, for example, if we want to study interpolation problems or completely contractive representations of a given normed function algebra $\mathcal{A}$ on a domain Ω, then finding a set of test functions Ψ such that the function algebra $H^\infty(\mathcal{K}_\Psi)$ is isometrically isomorphic to $\mathcal{A}$ is more useful tool. A trivial solution to this problem is to take the set of test functions Ψ to be the open unit ball of $\mathcal{A}$. However, if we want Ψ be minimal, in the sense that there is no proper closed subset of Ψ such that $H^\infty(\mathcal{K}_\Psi)$ is isometrically isomorphic to $\mathcal{A}$, then this problem becomes harder. Because, if Ψ is not closed, then we may possibly remove a finite or a countable number of test functions from Ψ and still get a set of test functions $\hat{\Psi}$ such that $H^\infty(\mathcal{K}_{\hat{\Psi}})$ is isometrically isomorphic to $\mathcal{A}$. So we need an additional constraint on Ψ to be a norm closed (and hence compact) subset of $\mathcal{A}$. The minimal set of test functions will only be defined up to an automorphism (for example, see Lemma 2.1.1).

1.4 Main results

Let B be a finite Blaschke product. Write

$$B(z) = \left(\frac{z-\alpha_0}{1-\overline{\alpha_0}z}\right)^{t_0}\left(\frac{z-\alpha_1}{1-\overline{\alpha_1}z}\right)^{t_1}\cdots\left(\frac{z-\alpha_n}{1-\overline{\alpha_n}z}\right)^{t_n}, \tag{1.6}$$

where $\alpha_0, \ldots, \alpha_n$ are distinct complex numbers in the open unit disk $\mathbb{D}$, and $t_0, \ldots, t_n$ and n are non-negative positive integers with $t_0 + \cdots + t_n = N \geq 2$.
Let

$$H_B^\infty := \mathbb{C} + B(z)H^\infty(\mathbb{D}),$$

where $H^\infty(\mathbb{D})$ is the algebra of bounded holomorphic functions in the open unit disk. Note that a function $f \in H^\infty(\mathbb{D})$ is in H_B^∞ if and only if it satisfies the following two constraints

1. $f(\alpha_i) = f(\alpha_j)$ for $0 \leq i, j \leq n$;

2. $f^{(k)}(\alpha_i) = 0$ for $k = 1, \ldots, t_i - 1$ whenever $t_i \geq 2$.

For the proof of above statement see the introduction of the Chapter 3.

Let Ψ_B be a set consisting of functions of the form $\psi(z) = cB(z)R(z)$, where R is a Blaschke product with number of zeros between 0 and $N-1$, and $c = \prod_{j=1}^{k} \frac{1-\overline{\alpha_j}}{1-\alpha_j}$, where the α_j 's are the zeros of ψ and $N \leq k \leq 2N-1$. These form the set of test functions for H^∞_B in the following sense.

Theorem. 2.3.10 The Banach algebras $H^\infty(\mathcal{K}_{\Psi_B})$ and H^∞_B are isometrically isomorphic.

Obviously there could be other choices for the set of test functions for H^∞_B, for example we can simply take the unit ball of H^∞_B. We want the set of test functions to be minimal. There is a dual version of this for the set of kernel functions for this algebra; see [43]. The next theorem shows that the set Ψ_B is a minimal set of test functions for the algebra H^∞_B, in the sense that there is no proper closed subset of Ψ_B such that the realization theorem holds for all functions in the unit ball of H^∞_B (or $H^\infty(\mathcal{K}_\Psi)$.)

Theorem. 2.4.4 The set Ψ_B is a minimal set of test functions for the algebra H^∞_B.

This theorem covers the special case when $B(z) = z^2$, a result of Dritschel and Pickering [28]. As an application of Theorem 2.3.10 and with the reformulation of rational dilation problem by Arveson (see [39, Cor. 7.8] and [14]), we show that rational dilation does not hold on certain distinguished varieties of $\overline{\mathbb{D}}^N$ associated to the algebra $\mathscr{A}_B = \mathbb{C} + B\mathbb{A}(\mathbb{D})$, where $\mathbb{A}(\mathbb{D})$ is the disk algebra.

Theorem. 3.4.1 The algebra $\mathscr{A}_B$ has a contractive representation which is not completely contractive.

Outline of the proof of Theorem 3.4.1. First, let S be a finite subset of $\mathbb{D}$ and form the closed convex cone

$$C_{2,S} = \left(\int_{\Psi_B} (1 - \psi(x)\psi(y)^*) \, d\mu_{x,y}(\psi) \right)_{x,y \in S}, \tag{1.7}$$

where $\mu = (\mu_{x,y}) \in M_2^+(S)$ is a kernel taking its values $\mu_{x,y}$ in the 2×2 matrix valued such that for all Borel subset Ω of Ψ_B, the measure

$$\mu(\Omega) = (\mu_{x,y}(\Omega))_{x,y \in S} \in M_s(M_2(\mathbb{C})) \tag{1.8}$$

takes positive semidefinite values in $M_s(M_2(\mathbb{C}))$, where s is the cardinality of the set S. Second, we define the kernel

$$\Delta_{F,S} = (I_2 - F(x)F(y)^*)_{x,y\in S}$$

for $F = (F_{i,j})_{i,j=1}^2 \in M_2(\mathscr{A}_B)$.

If $\Delta_{F,S} \notin C_{2,S}$ for some $F \in M_2(\mathscr{A}_B)$, then by using a Hahn-Banach separation argument we can separate $\Delta_{F,S}$ from $C_{2,S}$ with a positive functional, and apply a GNS construction to get a contractive representation of $\mathscr{A}_B$ that is not completely contractive.

We fix an analytic function $F \in M_2(\mathscr{A}_B)$, which is unitary valued on $\mathbb{T}$ and non-diagonalizable. However, if we assume that $\Delta_{F,S} \in C_{2,S}$ for this fixed F, then there exists an $M_2(\mathbb{C})$-valued positive semidefinite measure μ such that

$$I_2 - F(x)F(y)^* = \int_{\Psi_B} (1 - \psi(x)\psi(y)^*)\ d\mu_{x,y}(\psi) \qquad \text{for} \qquad x, y \in S. \tag{1.9}$$

Relying on the concreteness of set of test functions Ψ_B we show that F must then be diagonalizable, giving a contradiction. $\square$

Let $\mathscr{A}_B^0$ be the subalgebra of $\mathscr{A}_B$ generated by $B(z)$ and $zB(z)$. Then the distinguished variety associated to $\mathscr{A}_B^0$ is given by

$$\mathscr{N}_B = \left\{ (x, y) \in \overline{\mathbb{D}}^2 : x \prod_{k=1}^{N} (x - \overline{\alpha}_k y) = \prod_{k=1}^{N} (y - \alpha_k x) \right\}.$$

See section 3.2 for further details.

Theorem. 3.3.12 Let $\mathbb{A}(\mathscr{N}_B)$ be the algebra of analytic functions on $\mathscr{N}_B$ which extends continuously to the boundary with the supremum norm. The algebra $\mathbb{A}(\mathscr{N}_B)$ is completely isometrically isomorphic to the algebra $\mathscr{A}_B^0$, which consists of those functions in $\mathscr{A}_B$ which do not have terms of the form $z^i B^j(z), j = 1, \ldots, N-2$ and $i = j+1, \ldots, N-1$. This algebra contains $B^{N-1}(z)\mathbb{A}(\mathbb{D})$, so in particular, when $N = 2, \mathscr{A}_B^0 = \mathscr{A}_B$.

In Lemma 3.1.1 we show that the algebra $\mathscr{A}_B$ is generated by $B(z), zB(z), \ldots,$ $z^{N-1}B(z)$. Then in section 3.2 we show that the associated distinguished variety associated to $\mathscr{A}_B$ is given by

$$\mathscr{V}_B = \left\{ x_1^3 - S_N(\alpha)x_1^2 + \sum_{k=0}^{N-1} \left(S_{N-k}(\overline{\alpha})x_1 - S_k(\alpha)\right) x_{N-k}x_2 = 0 : (x_1, \ldots, x_N) \in \overline{\mathbb{D}}^N \right\}.$$

Theorem. 3.3.13 The algebra $\mathcal{R}(\mathscr{V}_B)$ is completely isometrically isomorphic to the algebra $\mathscr{A}_B$.

As a consequence of Theorem 3.3.13 an Theorem 3.3.12 we have the following main result:

Theorem. 3.4.2 Rational dilation fails on the distinguished variety $\mathscr{V}_B$.

In particular, when $N = 2$ we have that $\mathscr{V}_B = \mathscr{N}_B = \{(x, y) \in \overline{\mathbb{D}}^2 : x(x - \overline{\alpha}y)(x - \overline{\beta}y) = (y - \alpha x)(y - \beta x)\}$, where α, β two zeros of B. This covers the special case when $B(z) = z^2$, which was been previously considered by Dritschel, Jury and McCullough [24].

Chapter 2

Test functions for constrained algebras

2.1 Test functions for H_B^∞

Let us turn our attention to the constrained algebra $\mathscr{A}_B$. We wish to construct a set of test functions Ψ_B for $\mathscr{A}_B$, or rather, for its weak-$*$ closure $H_B^\infty = \mathbb{C} + B \cdot H^\infty(\mathbb{D})$. To simplify the work, we assume that $\alpha_0 = 0$.

This assumption imposes no real restriction. For suppose that B is a Blaschke product with zeros $\{\alpha_0, \dots, \alpha_n\}$ such that $\alpha_0 \neq 0$. Composing B with the Möbius map $m_{-\alpha_0} = (z + \alpha_0)/(1 + \overline{\alpha_0} z)$, we get a Blaschke product $\tilde{B}$ with zeros $\{\tilde{\alpha}_j = m_{\alpha_0}(\alpha_j)\}_{j=0}^n$, hence $\tilde{\alpha}_0 = 0$. Obviously composing with m_{α_0} maps $\tilde{B}$ back to B. Since $m_{\pm\alpha_0}$ is an automorphism of $\mathbb{D}$, we find that $f \in H_B^\infty$ if and only if $\tilde{f} = f \circ m_{-\alpha_0} \in H_{\tilde{B}}^\infty$, and furthermore, $\|f\| = \|\tilde{f}\|$.

Lemma 2.1.1. *If Ψ and $\tilde{\Psi}$ are the set of complex valued functions such that $\Psi = \{\tilde{\psi} \circ m_{\alpha_0} : \tilde{\psi} \in \tilde{\Psi}\}$, then*

a) $\tilde{\Psi}$ is a set of test functions for $H_{\tilde{B}}^\infty$ if and only if Ψ is a set of test functions for H_B^∞. That is, the algebras $H^\infty(\mathcal{K}_{\tilde{\Psi}})$ and $H_{\tilde{B}}^\infty$ are isometrically isomorphic if and only if the algebras $H^\infty(\mathcal{K}_\Psi)$ and H_B^∞ are isometrically isomorphic.

b) The set $\tilde{\Psi}$ is minimal for $H_{\tilde{B}}^\infty$ if and only if Ψ is minimal for H_B^∞.

Proof. a) Suppose that $\tilde{\Psi}$ is a family of test functions for $H_{\tilde{B}}^\infty$. Since m_{α_0} is an automorphism of the unit disk and $|\psi(z)| < 1$ for $z \in \mathbb{D}$, the map $M_{\alpha_0}(\tilde{\psi}) = \tilde{\psi} \circ m_{\alpha_0}$ maps $\tilde{\Psi}$ injectively onto Ψ. This map has the inverse $M_{\alpha_0}^{-1}(\psi) = \psi \circ m_{-\alpha_0}$.

So M_{α_0} is an isomorphism of $\tilde{\Psi}$ and Ψ. Thus we can identify $C_b(\tilde{\Psi})$ and $C_b(\Psi)$. So we may identify the spaces $C_b(\tilde{\Psi})^*$ and $C_b(\Psi)^*$.

For $x \in \mathbb{D}$, set $\tilde{x} = m_{\alpha_0}(x)$. Then

$$E(x)(\psi) = \psi(x) = \tilde{\psi}(m_{\alpha_0}(x)) = \tilde{\psi}(\tilde{x}) = E(\tilde{x})(\tilde{\psi}).$$

Let $\varphi \in H^\infty_B$ and set $\tilde{\varphi} = \varphi \circ m_{-\alpha_0}$. Assume $\|\tilde{\varphi}\|(= \|\varphi\|) = 1$. By the realization theorem and the assumption that $\tilde{\Psi}$ is a family of test functions for $H^\infty_{\tilde{B}}$, there is a positive kernel $\tilde{\Gamma} : \mathbb{D} \times \mathbb{D} \to C_b(\tilde{\Psi})^*$ such that for all $x, y \in \mathbb{D}$, and $\tilde{x} = m_{\alpha_0}(x), \tilde{y} = m_{\alpha_0}(y)$ we have

$$\begin{aligned} 1 - \varphi(x)\varphi(y)^* &= 1 - \tilde{\varphi}(\tilde{x})\tilde{\varphi}(\tilde{y})^* \\ &= \tilde{\Gamma}(\tilde{x}, \tilde{y})(1 - E(\tilde{x})E(\tilde{y})^*) \\ &= \Gamma(x, y)(1 - E(x)E(y)^*), \end{aligned} \tag{2.1}$$

where $\Gamma(x, y) = \tilde{\Gamma}(m_{\alpha_0}(x), m_{\alpha_0}(y))$ is a positive kernel from $\mathbb{D} \times \mathbb{D}$ to $C_b(\Psi)^*$. We conclude that H^∞_B is in the algebra $H^\infty(\mathcal{K}_\Psi)$ induced by the test functions Ψ and φ is in the unit ball of $H^\infty(\mathcal{K}_\Psi)$. Since the norm of φ in $H^\infty(\mathcal{K}_\Psi)$ is greater than or equal to the supremum norm (the norm in H^∞_B), the norms must be equal. On the other hand, if φ is in the unit ball of $H^\infty(\mathcal{K}_\Psi)$, then by realization theorem there exist a positive kernel $\Gamma : \mathbb{D} \times \mathbb{D} \to C_b(\Psi)$ such that

$$1 - \varphi(x)\varphi(y)^* = \Gamma(x, y)(1 - E(x)E(y)^*).$$

Then the same computation as in (2.1) , for $\tilde{\varphi} = \varphi \circ m_{-\alpha_0}$ yields that

$$1 - \tilde{\varphi}(\tilde{x})\tilde{\varphi}(\tilde{y})^* = \tilde{\Gamma}(\tilde{x}, \tilde{y})(1 - \tilde{E}(\tilde{x})\tilde{E}(\tilde{y})^*),$$

where $\tilde{\Gamma}(\tilde{x}, \tilde{y}) = \Gamma(x, y)$. Hence $\tilde{\varphi} \in H^\infty(\mathcal{K}_{\tilde{\Psi}})$. By assumption $H^\infty(\mathcal{K}_{\tilde{\Psi}})$ is isometrically isomorphic to $H^\infty_{\tilde{B}}$. So $\tilde{\varphi} \in H^\infty_{\tilde{B}}$. This implies that $\varphi \in H^\infty_B$. Thus $H^\infty(\mathcal{K}_\Psi)$ is isometrically isomorphic to H^∞_B, and we conclude that Ψ is a family of test functions for H^∞_B.

b) It is suffices to show one direction. Suppose $\tilde{\Psi}$ is minimal for $H^\infty_{\tilde{B}}$ and let C be closed subset of Ψ, which is a set of test functions for H^∞_B. Let $\tilde{C} = M^{-1}_{\alpha_0}(C) =$

$\{\psi \circ m_{-\alpha_0} : \psi \in C\}$. Let $\{\psi_j\}_{j=1}^{\infty}$ be a sequence in $\tilde{C}$ which converges to a function $\tilde{\psi}$. Then $\{M_{\alpha_0}(\tilde{\psi}_j)\}_{j=1}^{\infty} = \{\tilde{\psi}_j \circ m_{\alpha_0}\}_{j=1}^{\infty}$ is a sequence in C, which has the the limit $\tilde{\psi} \circ m_{\alpha_0} \in \Psi$. Since C is a closed set in Ψ, we must have that $\tilde{\psi} \circ m_{\alpha_0} \in C$. Since m_{α_0} is an automorphism of the unit disk and $|\psi(z)| < 1$ for $z \in \mathbb{D}$, we have $M_{\alpha_0}^{-1}(\tilde{\psi} \circ m_{\alpha_0}) = \tilde{\psi} \circ m_{\alpha_0} \circ m_{-\alpha_0} = \tilde{\psi} \in \tilde{C}$. It follows that $\tilde{C}$ is a closed subset of $\tilde{\Psi}$. Since C is a set of test functions for H_B^{∞}, by part a) we see that $\tilde{C}$ is a set of test functions for $H^{\infty}(\mathcal{K}_{\tilde{\Psi}})$. By minimality of $\tilde{\Psi}$, $\tilde{C} = \tilde{\Psi}$. Hence $C = \tilde{C} \circ m_{\alpha_0} = \tilde{\Psi} \circ m_{\alpha_0} = \Psi$.

□

2.2 The Herglotz representation and finite measures

Recall that we are assuming that $\tilde{B}$ is Blaschke product of degree bigger than 1 with a zero at $\alpha_0 = 0$ of multiplicity at least 1. Write t_j for the multiplicity of the zero α_j of $\tilde{B}$.

Assume that $\varphi \in H_{\tilde{B}}^{\infty}$ which is non-constant. By subtracting $\varphi(0)$ we may assume $\varphi(0) = 0$. And considering $\varphi/\|\varphi\|_{\infty}$, we also may assume that $\|\varphi\|_{\infty} \leq 1$. Define $f = M \circ \varphi$, where $M(z) = \frac{1+z}{1-z}$ maps the unit disk to the right half plane $\mathbb{H}$. Then $Ref \geq 0$ and $f(0) = 1$. The map M has the inverse $M^{-1}(z) = \frac{z-1}{z+1}$, and so we get a one to one correspondence between the set of functions in $H_{\tilde{B}}^{\infty}$ which are zero at 0 and the set of holomorphic functions mapping the disk to the right half plane with non-negative real part and value 1 at 0.

By the Herglotz representation theorem, for any holomorphic function $f : \mathbb{D} \to \mathbb{H}$ with $Ref \geq 0$ and $f(0) = 1$, there is a unique probability measure μ on $\mathbb{T}$ such that

$$f(z) = \int_{\mathbb{T}} \frac{w+z}{w-z}\, d\mu(w),$$

and conversely, if μ is a probability measure on $\mathbb{T}$, then

$$f(z) := \int_{\mathbb{T}} \frac{w+z}{w-z}\, d\mu(w)$$

defines a holomorphic function on $\mathbb{D}$ to $\mathbb{H}$ with $Ref \geq 0$ and $f(0) = 1$.

Lemma 2.2.1. *Let φ be a non-constant function in $H_{\tilde{B}}^{\infty}$. Then there is a unique* *probability measure* μ_φ *on* $\mathbb{T}$ *such that*

$$\int_{\mathbb{T}} \frac{1}{(w-\alpha_j)^k}\, d\mu_\varphi(w) = 0, \qquad j > 0 \text{ and } k = 1, \dots, t_j. \tag{2.2}$$

and

$$\int_{\mathbb{T}} w^k\, d\mu_\varphi(w) = 0, \qquad k = 1, \dots, t_0 - 1, \tag{2.3}$$

whenever $t_0 > 1$*. Furthermore, let* f *be an analytic function on* $\mathbb{D}$ *with positive real part and has the property that*

$$f(\alpha_j) = 1 \text{ for } j = 0, \dots, n \text{ and } f^{(k)}(\alpha_j) = 0 \text{ for } j = 0, 1, \dots, n; 1 \le k \le t_j - 1. \tag{2.4}$$

If f *has the Herglotz representation with a probability measure* μ*, then this measure has the properties (2.2) and (2.3).*

Proof. As noted above we may assume that $\|\varphi\|_\infty \le 1$ and $\varphi(0) = 0$. It follows that we can define the map $f = M \circ \varphi$ from $\mathbb{D}$ to $\mathbb{H}$. Then the assumption $\varphi(0) = 0$ imply that $\operatorname{Re} f \ge 0$ and $f(0) = 1$. By the Herglotz representation theorem, there is a unique probability measure μ_φ on $\mathbb{T}$ such that

$$f(z) = \int_{\mathbb{T}} \frac{w+z}{w-z}\, d\mu_\varphi(w). \tag{2.5}$$

Recall that t_j is the multiplicity of zero of α_j of $\tilde{B}$, and we are assuming that $\alpha_0 = 0$. Then for $j > 0$ we have

$$1 = f(\alpha_j) = \int_{\mathbb{T}} \frac{w+\alpha_j}{w-\alpha_j}\, d\mu_\varphi(w) = \int_{\mathbb{T}} \left[1 + \frac{2\alpha_j}{w-\alpha_j}\right] d\mu_\varphi(w) = 1 + 2\alpha_j \int_{\mathbb{T}} \frac{1}{w-\alpha_j}\, d\mu_\varphi(w),$$

and thus

$$\int_{\mathbb{T}} \frac{1}{w-\alpha_j}\, d\mu_\varphi(w) = 0, \qquad j > 0.$$

Inductively we get

$$\begin{aligned} f^{(k)}(z) &= 2k! \int_{\mathbb{T}} \frac{w}{(w-z)^{k+1}}\, d\mu_\varphi(w) \\ &= 2k! \left[\int_{\mathbb{T}} \frac{1}{(w-z)^k}\, d\mu_\varphi(w) + \int_{\mathbb{T}} \frac{z}{(w-z)^{k+1}}\, d\mu_\varphi(w)\right]. \end{aligned}$$

If $t_j > 1$, then as neither M nor its derivatives have any zeros in $\mathbb{D}$, the Faà di Bruno formula implies that

$$f^{(k)}(\alpha_j) = 0, \quad j > 0 \text{ and } k = 1, \ldots, t_j - 1.$$

Thus

$$\int_{\mathbb{T}} \frac{1}{(w-\alpha_j)^k}\, d\mu_\varphi(w) = 0, \quad j > 0 \text{ and } k = 1, \ldots, t_j.$$

On the other hand, if $z = \alpha_0 = 0$ and $t_0 > 1$, then

$$\int_{\mathbb{T}} \frac{1}{w^k}\, d\mu_\varphi(w) = 0, \quad k = 1, \ldots, t_0 - 1.$$

Finally, suppose that $f(z) = \int_{\mathbb{T}} \frac{w+z}{w-z}\, d\mu(w)$ and that it has the property (2.4). First note that $1 = \int_{\mathbb{T}} d\mu(w) = f(0) = f(\alpha_0)$.

Next for $j = 1, \ldots, n$,

$$\begin{aligned} 1 &= f(\alpha_j) = \int_{\mathbb{T}} \frac{w+\alpha_j}{w-\alpha_j}\, d\mu(w) \\ &= \int_{\mathbb{T}} \frac{w-\alpha_j}{w-\alpha_j}\, d\mu(w) + 2\alpha_j \int_{\mathbb{T}} \frac{1}{w-\alpha_j}\, d\mu(w) \\ &= 1 + 2\alpha_j \int_{\mathbb{T}} \frac{1}{w-\alpha_j}\, d\mu(w). \end{aligned}$$

Since $\alpha_j \neq 0$ for $j = 1, \ldots, n$ we have

$$0 = \int_{\mathbb{T}} \frac{1}{w-\alpha_j}\, d\mu(w). \tag{2.6}$$

Thus by induction for $k = 1, \ldots, t_j - 1$, we get

$$\begin{aligned} 0 &= f^{(k)}(\alpha_j) = 2k! \int_{\mathbb{T}} \frac{w}{(w-\alpha_j)^{k+1}}\, d\mu(w) \\ &= 2k! \left[\int_{\mathbb{T}} \frac{1}{(w-\alpha_j)^k}\, d\mu(w) + \int_{\mathbb{T}} \frac{\alpha_j}{(w-\alpha_j)^{k+1}}\, d\mu(w) \right] \\ &= 2k! \int_{\mathbb{T}} \frac{\alpha_j}{(w-\alpha_j)^{k+1}}\, d\mu(w). \end{aligned}$$

Since $\alpha_j \neq 0$ for $j = 1, \ldots, n$ we have

$$0 = \int_{\mathbb{T}} \frac{1}{(w-\alpha_j)^k}\, d\mu(w) \text{ for } k = 2, \ldots, t_j. \tag{2.7}$$

Together with (2.6) and (2.7) gives (2.2).

If $t_0 > 1$, then by the assumption (2.4) we have

$$0 = f^{(k)}(\alpha_0) = 2k! \int_{\mathbb{T}} \frac{w}{(w-\alpha_0)^{k+1}}\, d\mu(w) = \int_{\mathbb{T}} \frac{1}{w^k}\, d\mu(w) = \int_{\mathbb{T}} \overline{w}^k\, d\mu(w)$$

for $k = 1, \ldots, t_0 - 1$. Taking conjugation in later equation gives (2.3). $\square$

Remark 2.2.2. Note that (2.2) and (2.3) impose a variety of constraints on the probability measure μ. For example, if $t_0 > 1$, then the first $t_0 - 1$ moments of μ are zero.

Lemma 2.2.3. *Let μ be a positive finite atomic measure on $\mathbb{T}$, $\mu = \{(\lambda_j, m_j)\}_{j=1}^n \subset \mathbb{T} \times \mathbb{R}_{>0}$, with f the function having Herglotz representation with this measure. Then $\varphi = M^{-1} \circ f$ is a unimodular constant muliple of a Blaschke product with n zeros, counting multiplicities, and $\varphi(0) \in \mathbb{R}$.*

Conversely, given a Blaschke product φ with n zeros $\{\alpha_j\}$ counting multiplicities such that $\varphi(0) \in \mathbb{R}$, there is a positive finite atomic measure μ on $\mathbb{T}$ such that $f(z) = M \circ \varphi(z)$ has a Herglotz representation with this measure. Furthermore, μ is probability measure if and only if $\varphi(0) = 0$.

Proof. We begin by introducing some notation. For $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, define $S_0(x) = 1$ and

$$S_k(x) = (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}, \quad k = 1, \ldots, n,$$

the k-th (signed) symmetric sum of the elements of x. Then

$$\prod_{j=1}^n (z - x_j) = \sum_{k=0}^n S_k(x) z^{n-k} \qquad \text{and} \qquad \prod_{j=1}^n (1 - \overline{x_j} z) = \sum_{k=0}^n S_k(\overline{x}) z^k, \tag{2.8}$$

where $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_N)$. We also define $S_0^{-i}(x) = 1$ and

$$S_k^{-i}(x) = S_k(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n), \qquad k = 1, \ldots, n.$$

Then $S_n^{-i}(x) = -S_n(x)$. For $\lambda \subset \mathbb{T}^n$, it is straightforward (see the remark at end of the proof) that

$$S_k(\lambda) = S_n(\lambda) S_{n-k}(\overline{\lambda}) \qquad \text{and} \qquad S_k^{-i}(\lambda) = -S_n^{-i}(\lambda) S_{n-k}^{-i}(\overline{\lambda}). \tag{2.9}$$

Let

$$f(z) = \int_{\mathbb{T}} \frac{w+z}{w-z}\, d\mu(w) = -\sum_{i=1}^{n} m_i \frac{z+\lambda_i}{z-\lambda_i},$$

a holomorphic function from $\mathbb{D}$ to $\mathbb{H}$. Set $m = \sum m_i$. Then

$$\begin{aligned} f(z) \mp 1 &= \frac{-\sum_i m_i(z+\lambda_i)\prod_{j\neq i}(z-\lambda_j) \mp \prod_{i=1}^n (z-\lambda_i)}{\prod_j (z-\lambda_j)} \\ &= \frac{\sum_{k=0}^n [-\sum_{i=1}^n m_i S_k^{-i}(\lambda) \mp S_k(\lambda)] z^{n-k}}{\prod_j (z-\lambda_j)}, \end{aligned}$$

and

$$\begin{aligned} \varphi(z) := (M^{-1}\circ f)(z) =& \frac{f(z)-1}{f(z)+1} \\ =& \frac{\sum_{k=0}^n \left[\sum_{i=1}^n m_i S_k^{-i}(\lambda) + S_k(\lambda)\right] z^{n-k}}{\sum_{k=0}^n \left[\sum_{i=1}^n m_i S_k^{-i}(\lambda) - S_k(\lambda)\right] z^{n-k}} \end{aligned}$$

is a holomorphic map of the disk to itself.

Since the coefficient of z^n in the numerator of φ is $1+m>0$, the numerator is a polynomial of degree n with complex roots $\alpha_1,\ldots,\alpha_n$. Express the numerator as $(1+m)\prod_j(z-\alpha_j)$. Then

$$(1+m)S_k(\alpha) = \sum_{i=1}^n m_i S_k^{-i}(\lambda) + S_k(\lambda),$$

and so the denominator can be expressed as

$$\begin{aligned} \sum_{k=0}^n \left[\sum_{i=1}^n m_i S_k^{-i}(\lambda) - S_k(\lambda)\right] z^{n-k} &= -S_n(\lambda)\sum_{k=0}^n \left[\sum_{i=1}^n m_i S_{n-k}^{-i}(\overline{\lambda}) + S_{n-k}(\overline{\lambda})\right] z^{n-k} \\ &= -S_n(\lambda)(1+m)\sum_{k=0}^n S_k(\overline{\alpha}) z^k \\ &= -S_n(\lambda)(1+m)\prod_{j=1}^n (1-\overline{\alpha_j}z). \end{aligned}$$

Hence

$$\varphi(z) = -S_n(\overline{\lambda})\prod_{j=1}^n \frac{z-\alpha_j}{1-\overline{\alpha_j}z}.$$

Since $f(0) = \sum_{i=1}^{n} m_i \in \mathbb{R}$, the same is obviously true for φ.

Conversely, assume that $\varphi = cB$, where c is a unimodular constant and B is a Blaschke product with n zeros $\alpha_1, \dots, \alpha_n$, counting multiplicity and $\varphi(0) \in \mathbb{R}$. Then

$$f(z) = \frac{1+\varphi(z)}{1-\varphi(z)} = \frac{\prod_j (1-\overline{\alpha_j} z) + c\prod_j (z-\alpha_j)}{\prod_j (1-\overline{\alpha_j} z) - c\prod_j (z-\alpha_j)} \tag{2.10}$$

is a holomorphic map from $\mathbb{D}$ to $\mathbb{H}$. In the denominator, the leading coefficient is $C = \overline{S_n(\alpha)} - c = c(cS_n(\alpha) - 1)$, which is non-zero since $|cS_n(\alpha)| < 1$. Thus the denominator has n zeros in $\mathbb{C}\backslash\mathbb{D}$, which we write as $\lambda_1, \dots, \lambda_n$, and we write the denominator as $C\prod_j (z - \lambda_j)$.

If the numerator and denominator have a common root w, then $\prod_j (w - \alpha_j) = 0$, implying $\lambda_k = \alpha_j \in \mathbb{D}$ for some k and j, which is a contradiction. The constant coefficient of the denominator equals $(1 - cS_n(\alpha))/C = -\bar{c}$, which has absolute value 1. Hence each $\lambda_j \in \mathbb{T}$.

Suppose that the denominator of f has a repeated root at some $\lambda_i \in \mathbb{T}$. Then the logarithmic derivative of φ,

$$\frac{\varphi'(z)}{\varphi(z)} = \sum_{k=1}^{n} \frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k} z)(z-\alpha_k)} = \frac{2f'(z)}{f(z)^2 - 1},$$

is zero at λ_i. On the other hand, since $\lambda_i \in \mathbb{T}$,

$$\frac{\varphi'(\lambda_i)}{\overline{\lambda_i}\varphi(\lambda_i)} = \sum_k \frac{1-|\alpha_k|^2}{|\lambda_i - \alpha_k|^2} > 0, \tag{2.11}$$

giving a contradiction. Hence we conclude that λ_i are all distinct for $i = 1, \dots, n$.

In the numerator of f, the leading coefficient is $\overline{S_n(\alpha)} + c = c(cS_n(\alpha)+1)$, which is non-zero. Thus the numerator of f has degree equal to n. Consequently, since the denominator of f has n simple roots, f has a partial fraction decomposition

$$f(z) = -\left(m + \sum_{k=1}^{n} m_k \frac{2\lambda_k}{z - \lambda_k}\right). \tag{2.12}$$

It remains to verify that each $m_k > 0$ and $m = \sum_k m_k$. This will then imply

$$f(z) = -\sum_{i=1}^{n} m_i \frac{z+\lambda_i}{z-\lambda_i},$$

and so f has a Herglotz representation with positive finite atomic measure $\mu = \{(\lambda_j, m_j)\}_{j=1}^n$ on $\mathbb{T}$ as claimed. Now, by (2.12)

$$\lim_{z\to\lambda_j}(z-\lambda_j)f(z) = -2m_j\lambda_j. \tag{2.13}$$

Also since $\varphi(\lambda_j) = 1$ for all $j = 1, \dots, n.$,

$$\begin{aligned}
\lim_{z\to\lambda_j}(z-\lambda_j)f(z) &= \lim_{z\to\lambda_j}(z-\lambda_j)\frac{1+\varphi(z)}{1-\varphi(z)} \\
&= \lim_{z\to\lambda_j}\frac{z-\lambda_j}{\varphi(\lambda_j)-\varphi(z)}(1+\varphi(z)) \\
&= -\frac{1+\varphi(\lambda_j)}{\varphi'(\lambda_j)} \\
&= -\frac{2\varphi(\lambda_j)}{\varphi'(\lambda_j)}.
\end{aligned}$$

Hence by 2.12 and 2.13 we see that

$$m_j = \frac{\overline{\lambda_j}\varphi(\lambda_j)}{\varphi'(\lambda_j)} > 0.$$

The assumptions that $c \in \mathbb{T}$ and $\varphi(0) = cS_n(\alpha) \in \mathbb{R}$, along with (2.10) and (2.12), imply that

$$-m = \lim_{z\to\infty} f(z) = \frac{\overline{S_n(\alpha)}+c}{\overline{S_n(\alpha)}-c} = -\frac{1+cS_n(\alpha)}{1-cS_n(\alpha)} = -f(0) = m - 2\sum_{k=1}^n m_k.$$

Hence $m = \sum_{k=1}^n m_k$. We see from this that if $\alpha_j = 0$ for some j, then $m = 1$, and so μ is a probability measure, and conversely, if μ is probability measure, then $cS_n(\alpha) = 0$, and so $\alpha_j = 0$ for some j. This completes the proof. □

Remark 2.2.4. Just for completeness we give a proof for the following: For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{T}^n$,

$$S_k(\lambda) = S_n(\lambda)S_{n-k}(\overline{\lambda}) \qquad \text{and} \qquad S_k^{-i}(\lambda) = -S_n^{-i}(\lambda)S_{n-k}^{-i}(\overline{\lambda}).$$

Proof. The first identity is clear. So we only need to prove the second identity.

Without loss of generality we may assume $i = 1$. Observe the following identity

$$\begin{aligned} S_k^{-1}(\lambda) &= S_k(-\lambda_1, \lambda_2, \dots, \lambda_n) = -\lambda_1(-1)S_{k-1}(\lambda_2, \dots, \lambda_n) + S_k(\lambda_2, \dots, \lambda_n) \\ &= \lambda_1 S_{k-1}(\lambda_2, \dots, \lambda_n) + S_k(\lambda_2, \dots, \lambda_n), \end{aligned} \tag{2.14}$$

for $1 \le k \le n-1$.

Next using the first identity for the points $\lambda_2, \dots, \lambda_n$ in (2.14) gives

$$\begin{aligned} S_k^{-1}(\lambda) &= \lambda_1(-1)^{n-1}\lambda_2 \cdots \lambda_n \cdot S_{n-1-(k-1)}(\overline{\lambda}_2, \dots, \overline{\lambda}_n) \\ &+ (-1)^{n-1}\lambda_2 \cdots \lambda_n S_{n-1-k}(\overline{\lambda}_2, \dots, \overline{\lambda}_n) \\ &= (-1)^{n-1}\lambda_1\lambda_2 \cdots \lambda_n \left(S_{n-k}(\overline{\lambda}_2, \dots, \overline{\lambda}_n) + \overline{\lambda}_1 S_{n-1-k}(\overline{\lambda}_2, \dots, \overline{\lambda}_n) \right). \end{aligned}$$

Finally, applying the identity (2.14) for the points $\overline{\lambda}_1, \dots, \overline{\lambda}_n$ in the last expression yields

$$S_k^{-1}(\lambda) = -S_n(\lambda) S_{n-k}^{-1}(\overline{\lambda}).$$

□

2.3 The Agler-Herglotz representation

In this section we give a concrete description of the set of test functions for the algebra $H^\infty_{\tilde{B}}$, and so for H^∞_B.

We recall some definitions from the theory of the convex analysis.

Definition 2.3.1. A subset $\mathcal{C}$ of a real vector space X is said to be **convex** if, given any collection of vectors $u_1, \dots, u_r$ in $\mathcal{C}$ and a collection of nonnegative real numbers $c_1, \dots, c_r$ with $c_1 + \cdots + c_r = 1$, then one has that the convex linear combination $\sum_{k=1}^r c_k u_k \in \mathcal{C}$.

Definition 2.3.2. If X is a real vector space and $\mathcal{W} \subseteq X$, we say $\mathcal{W}$ is a **wedge** if $a + b \in \mathcal{W}$ and $ta \in \mathcal{W}$ whenever $a, b \in \mathcal{W}$ and $t \ge 0$. A wedge $\mathcal{W}$ is a **cone** if $\mathcal{W} \cap -\mathcal{W} = \{0\}$, where 0 is the zero vector of X.

Note that by definition a wedge is a convex set.

Definition 2.3.3. A point $x \in \mathcal{W}$ called an **extreme point** for $\mathcal{W}$ if $x = (1-t)x_1 + tx_2$, with $x_1, x_2 \in \mathcal{W}$ and $0 < t < 1$, then $x_1 = x = x_2$.

In practice the following result is more useful than the definition of extreme points of the convex sets.

Lemma 2.3.4 ([15, Lemma 1.1]). *Let $\mathcal{C}$ be convex set in a real vector space X. The point $x \in \mathcal{C}$ is an extreme point of the convex set $\mathcal{C}$ if and only if the follwoing condition holds: whenever $y \in X$ is such that $x \pm y \in \mathcal{C}$, then $y = 0$.*

Definition 2.3.5. A nonzero vector x in $\mathcal{W}$ is an **extreme ray** in $\mathcal{W}$ if $x = x_1 + x_2$, with $x_1, x_2 \in \mathcal{W}$, then $x_1 = tx$ and $x_2 = sx$ for some $t, s \geq 0$. Extreme rays are also called **extreme directions** in [5].

Definition 2.3.6. A X topological space X is called **locally compact**, if every point of X has a compact neighborhood.

Let X be a compact Hausdorff space. We let denote $M_{\mathbb{R}}(X)$ the space of finite regular Borel measures on X and $C_{\mathbb{R}}(X)$ denote the space of real valued continuous functions on X with the norm topology. Let $M_{\mathbb{R}}^{+}(X)$ be the space of positive measures in $M_{\mathbb{R}}(X)$.

Define the following continuous (in fact holomorphic) functions on $\mathbb{T}$ given by

$$\begin{aligned} L_j(w) &:= w^j \text{ for } j = 1, \ldots, t_0 - 1 \text{ whenever } t_0 > 1 \\ &\textit{and} \\ L_{i,k}(w) &:= \frac{1}{(w - \alpha_i)^k} \text{ for } i = 1, \ldots, n; k = 1, \ldots, t_i. \end{aligned} \tag{2.15}$$

Clearly, $\mathrm{Re}L_j, \mathrm{Im}L_j, \mathrm{Re}L_{i,k}, \mathrm{Im}L_{i,k} \in C_{\mathbb{R}}(\mathbb{T})$. This will give us in total $2N - 2$ continuous real valued functions on $\mathbb{T}$, and for notational complexity we write them $\{h_j\}_{j=1}^{2N-2} \in C_{\mathbb{R}}(\mathbb{T})$. Then taking the real and imaginary parts in the constraints in (2.2) and (2.3) we have

$$\int_{\mathbb{T}} h_j(w)\, d\mu(w) = 0 \text{ for } j = 1, \ldots, 2N - 2, \tag{2.16}$$

for the measures in Lemma 2.2.1. Also we define the sets

$$M_{\tilde{B},\mathbb{R}}^{1}(\mathbb{T}) := \left\{\mu \in M_{\mathbb{R}}^{+}(\mathbb{T}) : \mu(\mathbb{T}) = 1 \text{ and (2.16) holds } \text{ for } \mu\right\} \tag{2.17}$$

and

$$M_{\tilde{B},\mathbb{R}}^{+}(\mathbb{T}) := \left\{\mu \in M_{\mathbb{R}}^{+}(\mathbb{T}) : \text{(2.16) holds } \text{ for } \mu\right\}. \tag{2.18}$$

In other words, the set $M^+_{\tilde{B},\mathbb{R}}(\mathbb{T})$ is the set of positive measures satisfying the constraints in (2.16). This is a weak-$*$ closed, convex, locally compact set in the Banach space of finite Borel measures $M_{\tilde{B},\mathbb{R}}(\mathbb{T}) = \overline{\bigvee M^+_{\tilde{B},\mathbb{R}}(\mathbb{T})}$, and additionally is a cone since it is closed under sums, positive scalar multiples, and $M^+_{\tilde{B},\mathbb{R}}(\mathbb{T}) \cap -M^+_{\tilde{B},\mathbb{R}}(\mathbb{T}) = \{0\}$.

The set $M^1_{\tilde{B},\mathbb{R}}(\mathbb{T})$ is weak*-closed, convex and forms a *base* for $M^+_{\tilde{B},\mathbb{R}}(\mathbb{T})$, because any $\tilde{\mu} \in M^+_{\tilde{B},\mathbb{R}}(\mathbb{T})$ is of the form $t\mu$ for some $\mu \in M^1_{\tilde{B},\mathbb{R}}(\mathbb{T})$ and $t \geq 0$. Hence $M^1_{\tilde{B},\mathbb{R}}(\mathbb{T})$ is in the closed unit ball of $M_{\tilde{B},\mathbb{R}}(\mathbb{T})$, the dual space of normed vector space $C_{\tilde{B},\mathbb{R}}(\mathbb{T})$. So by the Banach-Alaoglu theorem $M^1_{\tilde{B},\mathbb{R}}(\mathbb{T})$ is compact. Thus by the Krein-Milman Theorem [41], $M^1_{\tilde{B},\mathbb{R}}(\mathbb{T})$ is the closed convex hull of $ext\left(M^1_{\tilde{B},\mathbb{R}}(\mathbb{T})\right)$, which is the set of extreme points of $M^1_{\tilde{B},\mathbb{R}}(\mathbb{T})$. Henceforth, we fix the notation

$$\widehat{\Theta}_{\tilde{B}} = ext\left(M^1_{\tilde{B},\mathbb{R}}(\mathbb{T})\right).$$

It is an elementary observation that $\mu \in M^1_{\tilde{B},\mathbb{R}}(\mathbb{T})$ is an extreme point if and only if $\{t\mu : t \in \mathbb{R}^+\}$ is an extreme ray in $M^+_{\tilde{B},\mathbb{R}}(\mathbb{T})$ (see [5, Lemma 1.3.4]).

Note that the local compactness of $M^+_{\tilde{B},\mathbb{R}}(\mathbb{T})$ is equivalent to compactness of $M^1_{\tilde{B},\mathbb{R}}(\mathbb{T})$ (see [31, 13.C , Lemma 1] and [41, Proposition 11.6]).

We need the following general result from [15], we just state here scalar-valued case. To state this theorem we need to introduce some notations from this paper. Let X be a compact Hausdorff space. Given a collection $\phi = \{\phi_1, \ldots, \phi_r\}$ of m real valued continuous functions on X, we define a subset of $M^+_{\mathbb{R}}(X)$ given by

$$\begin{aligned} \mathcal{C}(X,1,\phi) = \Big\{\mu \in M^+_{\mathbb{R}}(X) : \mu(X) = 1, \text{ and} \\ \mu(\phi_r) := \int_X \phi_r(x)\, d\mu(x) = 0 \text{ for } r = 1, \ldots, m\Big\}. \end{aligned} \tag{2.19}$$

The space $\mathcal{C}(X,1,\phi)$ is a convex subset of the real Banach space $M_{\mathbb{R}}(X)$ which is compact in the weak-* topology on $M_{\mathbb{R}}(X)$ induced by its duality with respect to the real Banach space $C_{\mathbb{R}}(X)$. Note that the space $\mathcal{C}(X,1,\phi)$ can be empty, see for example [15, page 538]. But in our case it is always case that $\mathcal{C}(X,1,\phi) \neq \emptyset$.

Theorem 2.3.7 ([15, Theorem 2.2]). *Let X be a compact Hausdorff space. Suppose that $\mu \in M^+_{\mathbb{R}}(X)$ is an extreme point of the set $\mathcal{C}(X,1,\phi)$. Then there is a natural number k with $1 \leq k \leq (m+1)$, k distinct points $x_1, \ldots, x_k \in X$, and k positive* *real numbers* $m_1, \ldots, m_k$ *such that*

$$\sum_{i=1}^{k} m_i = 1, \quad \sum_{i=1}^{k} \phi_r(x_i) m_i = 1 \quad \textit{for } r = 1, \ldots m \tag{2.20}$$

and μ *is a positive finite atomic measure,i.e.*

$$\mu = \sum_{i=1}^{k} m_i \delta_{x_i},$$

where δ_{x_i} *is the scalar-valued measure equal to the unit point-mass at the point* x_i.

Next we turn to concretely characterizing elements of $\widehat{\Theta}_{\tilde{B}} = ext(M^1_{\tilde{B},\mathbb{R}}(\mathbb{T}))$.

Theorem 2.3.8. *Let* N *be the number of zeros of* $\tilde{B}$*, counting multiplicities. Then the extreme points of* $M^1_{\tilde{B},\mathbb{R}}(\mathbb{T})$ *are probability measures on* $\mathbb{T}$ *supported at* ℓ *points in* $\mathbb{T}$*, where* $N \leq \ell \leq 2N - 1$.

Proof. Let μ be an extreme point of $M^1_{\tilde{B},\mathbb{R}}(\mathbb{T})$. If we choose $X = \mathbb{T}$, $\phi = \{h_j\}_{j=1}^{2N-2}$ in (2.19), then we have $\mathcal{C}(\mathbb{T}, 1, \phi) = M^1_{\tilde{B},\mathbb{R}}(\mathbb{T})$, which is non-empty. Then by Theorem 2.3.7 we see that the support of μ is at most $2N - 1$ and this measure is a finite atomic measure.

Now consider the lower bound. Then first part of this proof implies that μ is a finite atomic measure on $\mathbb{T}$. Write

$$\mu = \sum_{i=1}^{\ell} m_i \delta_{\lambda_i},$$

where for all $i, m_i > 0$ and δ_{λ_i} is the point measure on $\mathbb{T}$ supported at λ_i. Consider the function

$$r(x) = \sum_{i=1}^{\ell} \frac{m_i}{\lambda_i - x},$$

with its derivatives

$$r^{(k)}(x) = k! \sum_{i=1}^{\ell} \frac{m_i}{(\lambda_i - x)^{k+1}} \quad k \in \mathbb{N}.$$

We conclude from equations (2.2) and (2.3) that r has roots at α_0 of at least multiplicity $t_0 - 1$ and at α_j of at least multiplicity $t_j, j = 1, \ldots, n$. Hence r has at least

$(\sum_{i=0}^{n} t_i) - 1 = N - 1$ roots. The same is then true for the polynomial

$$p(x) = r(x) \cdot \prod_{i=1}^{\ell} (\lambda_i - x) = \sum_{j=1}^{\ell} \prod_{j \neq i} (\lambda_i - x) m_j,$$

which has degree $\ell - 1$. We conclude that $\ell \geq N$. □

For $\hat{\theta} \in \widehat{\Theta}_{\tilde{B}}$, define

$$h_{\hat{\theta}}(z) := \int_{\mathbb{T}} \frac{w+z}{w-z} \, d\hat{\theta}(w), \tag{2.21}$$

an analytic function on $\mathbb{D}$ with positive real part and value 1 when $z = 0$. We then have, as in [28, Theorem 6], the so called *Agler- Herglotz representation.*

Theorem 2.3.9 (Agler-Herglotz representation)**.** *Let f be an analytic function on $\mathbb{D}$ with positive real part. Suppose further that*

$$f(\alpha_j) = 1, \quad j = 0, 1, \ldots, n \quad \text{and} \quad f^{(k)}(\alpha_j) = 0, \quad j = 0, 1, \ldots, n, 1 \leq k \leq t_j - 1. \tag{2.22}$$

Then there exists a probability measure ν on $\hat{\Theta}_{\tilde{B}}$ such that

$$f(z) = \int_{\widehat{\Theta}_{\tilde{B}}} h_{\hat{\theta}}(z) \, d\nu(\hat{\theta}). \tag{2.23}$$

Proof. Since $\operatorname{Re} f \geq 0$ and $f(\alpha_0) = f(0) = 1$, by the Herglotz representation theorem there is a unique probability measure μ such that

$$f(z) = \int_{\mathbb{T}} \frac{w+z}{w-z} \, d\mu(w).$$

By assumption f has property (2.22), and hence by Lemma 2.2.1, μ has the properties (2.2) and (2.3). Thus $\mu \in M^1_{\tilde{B},\mathbb{R}}(\mathbb{T})$. On the other hand, by the Choquet-Bishop-de Leeuw theorem [41], given any $\mu \in M^1_{\tilde{B},\mathbb{R}}(\mathbb{T})$, there is a probability measure ν_μ on $\widehat{\Theta}_{\tilde{B}}$ such that

$$\mu = \int_{\widehat{\Theta}_{\tilde{B}}} \hat{\theta} \, d\nu_\mu(\hat{\theta}).$$

Then we have

$$\begin{aligned} f(z) &= \int_{\mathbb{T}} \frac{w+z}{w-z}\, d\mu(w) \\ &= \int_{\mathbb{T}} \frac{w+z}{w-z} \left[\int_{\widehat{\Theta}_{\tilde{B}}} (\, d\hat{\theta}(w))\, d\nu_\mu(\hat{\theta}) \right] \\ &= \int_{\widehat{\Theta}_{\tilde{B}}} \left[\int_{\mathbb{T}} \frac{w+z}{w-z}\, d\hat{\theta}(w) \right] d\nu_\mu(\hat{\theta}) \\ &= \int_{\widehat{\Theta}_{\tilde{B}}} h_{\hat{\theta}}(z)\, d\nu_\mu(\hat{\theta}). \end{aligned}$$

This completes the proof. $\square$

We now translate our results on measures to statements about functions in the unit ball of $H^\infty_{\tilde{B}}$. Using a Cayley transform from the right half plane to the unit disk, for each $\mu \in M^{+,1}_{\tilde{B},\mathbb{R}}(\mathbb{T})$, define a map

$$\psi_\mu := \frac{h_\mu - 1}{h_\mu + 1}. \tag{2.24}$$

We have that

$$\begin{aligned} &1 - \psi_\mu(z)\psi_\mu(w)^* \\ &= 1 - \frac{h_\mu(z) - 1}{h_\mu(z) + 1} \frac{h_\mu(w)^* - 1}{h_\mu(w)^* + 1} \\ &= \frac{h_\mu(z)h_\mu(w)^* + h_\mu(z) + h_\mu(w)^* + 1 - h_\mu(z)h_\mu(w)^* + h_\mu(z) + h_\mu(w)^* - 1}{(h_\mu(z)+1)(h_\mu(w)^*+1)} \\ &= 2\frac{h_\mu(z) + h_\mu(w)^*}{(h_\mu(z)+1)(h_\mu(w)^*+1)} \end{aligned} \tag{2.25}$$

and so in particular, ψ_μ is a map of the unit disk to itself.

If $\hat{\theta}$ is an extremal measure in $M^{+,1}_{\tilde{B},\mathbb{R}}(\mathbb{T})$, then by Theorem 2.3.8, it is a finitely supported atomic probability measure on $\mathbb{T}$. It then follows from Lemma 2.2.3 that in this case there is a corresponding finite Blaschke product $\psi_{\hat{\theta}} = c_{\hat{\theta}}\tilde{B}R_{\hat{\theta}}$, where $c_{\hat{\theta}}$ is a unimodular constant and $R_{\hat{\theta}}$ is a Blaschke product with number of zeros between 0 and $N-1$. We write $\widehat{\Psi}_{\tilde{B}}$ for the collection $\{\psi_{\hat{\theta}} : \hat{\theta} \in \widehat{\Theta}_{\tilde{B}}\}$. The support of the measure $\hat{\theta}$ corresponds to the set $\psi_{\hat{\theta}}^{-1}(1)$ (see the proof of Lemma 2.2.3). Ultimately, we will use a subset of $\widehat{\Psi}_{\tilde{B}}$ as test functions. It is apparent from the realization theorem (Theorem 1.3.6) and equation (2.27), that we can replace any test function by a unimodular constant times the test function. So for convenience, we identify $\psi_{\hat{\theta}}(z)$ with $\overline{\psi_{\hat{\theta}}(1)}\psi_{\hat{\theta}}(z)$. This amounts then to having the point 1 as a support point for the measure $\hat{\theta}$. Let $\Theta_{\tilde{B}}$ be the subset of measures in $\widehat{\Theta}_{\tilde{B}}$ having 1 as a support point, and write $\Psi_{\tilde{B}}$ for the collection set $\{\psi_\theta : \theta \in \Theta_{\tilde{B}}\}$. Thus, clearly $\Psi_{\tilde{B}} \subseteq \widehat{\Psi}_{\tilde{B}}$.

Next theorem shows that $\Psi_{\tilde{B}}$ is a set of test functions for $H^\infty_{\tilde{B}}$.

Theorem 2.3.10. *The algebras $H^\infty(\mathcal{K}_{\Psi_{\tilde{B}}})$ and $H^\infty_{\tilde{B}}$ are isometrically isomorphic.*

Proof. It is enough to show that the unit ball $H^\infty_1(\mathcal{K}_{\Psi_{\tilde{B}}})$ of the algebra $H^\infty(\mathcal{K}_{\Psi_{\tilde{B}}})$ is same as the unit ball $H^\infty_{1,\tilde{B}}$ of the algebra $H^\infty_{\tilde{B}}$. Since any test function maps the open unit disk to itself, the Szegő kernel k_s is an admissible kernel for $H^\infty(\mathcal{K}_{\Psi_{\tilde{B}}})$. Hence for any function φ in the unit ball $H^\infty_1(\mathcal{K}_{\Psi_{\tilde{B}}})$, we conclude that $((1-\varphi(x)\varphi(y)^*)k_s(x,y))$ is positive kernel, and so $\varphi \in H^\infty_{1,\tilde{B}}$.

For the reverse containment, if $\varphi \in H^\infty_{1,\tilde{B}}$ with $\varphi(0) = 0$, and $f = M \circ \varphi$, then f is a function on the disk with positive real part and (2.22) holds, where $M(z) = \frac{1+z}{1-z}$. Also,

$$\varphi = \frac{f-1}{f+1},$$

and so

$$1 - \varphi(z)\varphi(w)^* = 2\frac{f(z) + f(w)^*}{(f(z)+1)(f(w)^*+1)}.$$

Applying the Agler-Herglotz representation (Theorem 2.3.9), there is a probability measure ν on $\widehat{\Theta}_{\tilde{B}}$ such that (2.23) holds. Thus, by (2.25) we have

$$\begin{aligned} 1 - \varphi(z)\varphi(w)^* &= \frac{2}{(f(z)+1)(f(w)^*+1)} \int_{\widehat{\Theta}_{\tilde{B}}} (h_{\hat{\theta}}(z) + h_{\hat{\theta}}(w)^*)\, d\nu(\hat{\theta}) \\ &= \int_{\widehat{\Theta}_{\tilde{B}}} H_{\hat{\theta}}(z)(1 - \psi_{\hat{\theta}}(z)\psi_{\hat{\theta}}(w)^*)H_{\hat{\theta}}(w)^*\, d\nu(\hat{\theta}), \end{aligned} \tag{2.26}$$

where $H_{\hat{\theta}}(z) = \frac{2}{(f(z)+1)(1-\psi_{\hat{\theta}}(z))} = \frac{1+h_{\hat{\theta}}(z)}{1+f(z)}$. Recall that we identify the test function $\psi_\theta := \overline{\psi_{\hat{\theta}}(1)}\psi_{\hat{\theta}}$ with $\psi_{\hat{\theta}}$ in $\widehat{\Psi}_{\tilde{B}}$. Thus $\psi_\theta(1) = 1$, and as noted above the corresponding measure θ is supported at 1. Thus $\theta \in \Theta_{\tilde{B}}$ and $\psi_\theta \in \Psi_{\tilde{B}}$. By (2.26) we have

$$\begin{aligned} 1 - \varphi(z)\varphi(w)^* &= \int_{\widehat{\Theta}_{\tilde{B}}} H_{\hat{\theta}}(z) \left(1 - \overline{\psi_{\hat{\theta}}(1)}\psi_{\hat{\theta}}(z) \left(\overline{\psi_{\hat{\theta}}(1)}\psi_{\hat{\theta}}(w)\right)^*\right) H_{\hat{\theta}}(w)^*\, d\nu(\hat{\theta}) \\ &= \int_{\Theta_{\tilde{B}}} H_\theta(z)\,(1 - \psi_\theta(z)\psi_\theta(w)^*)\, H_\theta(w)^*\, d\nu(\theta), \end{aligned} \tag{2.27}$$

where $H_\theta(z) = \frac{2}{(f(z)+1)(1-\psi_\theta(z))}$. It follows that

$$1 - \varphi(z)\varphi(w)^* = \Gamma(z,w)(1 - E(z)E(w)^*),$$

with $\Gamma : \mathbb{D} \times \mathbb{D} \to C_b(\Psi_{\tilde{B}})^*$ the positive kernel given by

$$\Gamma(z,w)g = \int_{\Theta_{\tilde{B}}} H_\theta(z) g(\psi) H_\theta(w)^* \, d\nu(\theta),$$

where $g \in C_b(\Psi_{\tilde{B}})$.

If $\varphi \in H^\infty_{1,\tilde{B}}$ with $\varphi(0) = c \neq 0$, then we consider the function

$$\varphi_0(z) = \frac{\varphi(z) - c}{1 - \bar{c}\varphi(z)}.$$

Thus we have

$$\begin{aligned} & 1 - \varphi_0(z)\varphi_0(w)^* \\ & = 1 - \frac{\varphi(z) - c}{1 - \bar{c}\varphi(z)} \frac{\varphi(w)^* - \bar{c}}{1 - c\varphi(w)^*} \\ & = \frac{1 - \bar{c}\varphi(z) - c\varphi(w)^* + c\bar{c}\varphi(z)\varphi(w)^* - \varphi(z)\varphi(w)^* + c\varphi(w)^* + \bar{c}\varphi(z) - c\bar{c}}{(1 - \bar{c}\varphi(z))(1 - c\varphi(w)^*)} \\ & = \frac{(1 - c\bar{c})(1 - \varphi(z)\varphi(w)^*)}{(1 - \bar{c}\varphi(z))(1 - c\varphi(w)^*)}. \end{aligned} \tag{2.28}$$

On the other hand, since $\varphi_0(0) = 0$ as in previous case we get the following

$$1 - \varphi_0(z)\varphi_0(w)^* = \int_{\Theta_{\tilde{B}}} H^0_\theta(z) \left(1 - \psi_\theta(z)\psi_\theta(w)^*\right) H^0_\theta(w)^* \, d\nu_0(\theta),$$

where $H^0_\theta(z) = \frac{2}{(f_0(z)+1)(1-\psi_\theta(z))}$ with $f_0 = M \circ \varphi_0$, and ν_0 is chosen as the probability measure associated to f_0 in the Agler-Herglotz representation. Then by (2.28) we get

$$1 - \varphi(z)\varphi(w)^* = \int_{\Theta_{\tilde{B}}} G_\theta(z) \left(1 - \psi_\theta(z)\psi_\theta(w)^*\right) G_\theta(w)^* \, d\nu_0(\theta), \tag{2.29}$$

where $G_\theta(z) = \frac{(1-\bar{c}\varphi(z))H^0_\theta(z)}{\sqrt{1-\bar{c}c}}$. Hence we get the following realization

$$1 - \varphi(z)\varphi(w)^* = \Gamma_0(z,w) \left(1 - E(z)E(w)^*\right),$$

with $\varGamma_0 : \mathbb{D} \times \mathbb{D} \to C_b(\varPsi_{\tilde{B}})^*$ the positive kernel given by

$$\varGamma_0(z, w)g = \int_{\varTheta_{\tilde{B}}} G_\theta(z) g(\psi) G_\theta(w)^* \, d\nu_0(\theta),$$

where $g \in C_b(\varPsi_{\tilde{B}})$. Finally, by the realization theorem (Theorem 1.3.6) we conclude that $\varphi \in H_1^\infty(\mathcal{K}_{\varPsi_{\tilde{B}}})$. □

Combining Theorem 2.3.8 with Lemma 2.2.3 and Theorem 2.3.10, we have shown the following.

Corollary 2.3.11. *Let $\varTheta_{\tilde{B}}$ be the set of extreme measures in $M^1_{\tilde{B},\mathbb{R}}(\mathbb{T})$ with 1 as a support point. Then $\varTheta_{\tilde{B}}$ consists of all such probability measures supported at $N \le k \le 2N-1$ points, where N is the number of zeros of $\tilde{B}$, counting multiplicity. Furthermore, the set $\varPsi_{\tilde{B}}$ is a collection of test functions for $H^\infty_{\tilde{B}}$, and consists of functions of the form $\psi_\theta = c\tilde{B}R_\theta$, where R_θ is a Blaschke product with number of zeros between 0 and $N-1$, and $c = \prod \frac{1-\overline{\alpha_j}}{1-\alpha_j} \in \mathbb{T}$.*

Let $\varPsi_B = \left\{ \tilde{\psi} \circ m_{\alpha_0} : \tilde{\psi} \in \varPsi_{\tilde{B}} \right\}$. The next result relates the algebra $H^\infty(\mathcal{K}_{\varPsi_B})$ to the constrained algebra H^∞_B.

Corollary 2.3.12. *The two algebras $H^\infty(\mathcal{K}_{\varPsi_B})$ and H^∞_B are isometrically isomorphic.*

Proof. Applying Lemma 2.1.1 to Theorem 2.3.10 gives the result. □

Finally we close this section with the following natural question: Is the set of test functions that we found minimal, in the sense that no proper closed subset of $\varPsi_B$ is a set of test functions for H^∞_B? In other words, if we take a closed subset of $\varPsi_B$, does Theorem 2.3.10 still hold, or more generally, is the realization theorem true? The next section is devoted to dealing with this question.

2.4 Minimality of the set of test functions

In this section we prove that there is no closed subset of $\varPsi_{\tilde{B}}$ which is a set of test functions for $H^\infty_{\tilde{B}}$. At this point Corollary 2.3.11 gives a fairly concrete description of the set of test functions. However, it is more useful for what follows to describe them in terms of the placement of the zeros rather than the support points for the measure in the Herglotz representation. Obviously, in writing any test function as a Blaschke product, changing the order of the zeros does not change the function. There is also the small problem that the number of zeros of a test function is between N and $2N-1$, where N is the number of zeros of $\tilde{B}$, so not all test functions necessarily have the same number of zeros. For this reason, we introduce the following order on elements of the one point compactification of the disk, ($\mathbb{D}_\infty$ for $\mathbb{D}\cup\infty$).

We order the points of $\mathbb{D}_\infty$ as follows: $\zeta_1 \preceq \zeta_2$ in $\mathbb{D}_\infty$ if either $|\zeta_1| < |\zeta_2|$ or $|\zeta_1| = |\zeta_2|$ and $\arg\zeta_1 \le \arg\zeta_2$. The point ∞ is the maximal element of $\mathbb{D}_\infty$ with respect to this order, and 0 the minimal element.

We can use this order to describe the set of test functions.
Let $\mathcal{Z}(\tilde{B}) = \{{\alpha'}_0, \dots, {\alpha'}_{N-1}\}$ be the (ordered) zeros of $\tilde{B}$, so $\tilde{B} = m_{{\alpha'}_0}\cdots m_{{\alpha'}_{N-1}}$, $m_{{\alpha'}_j}$ the Möbius map with zero ${\alpha'}_j$. If as an abuse of notation we let $m_\infty(z) = 1$, then any Blaschke product $B_\alpha = \tilde{B}R$ with a number of zeros between N and $2N-1$ can be written as

$$B_\alpha(z) = \prod_{j=0}^{2N-2} m_{\alpha_j},$$

where $\mathcal{Z}(B_\alpha) = \{0 = \alpha_0 \preceq \cdots \preceq \alpha_{2N-2}\}$, the ordered zeros of B_α in $\mathbb{D}_\infty$, contains the elements of $\mathcal{Z}(\tilde{B})$.

Define the set

$$\Psi^{\preceq} := \{cB_\alpha : \mathcal{Z}(B_\alpha) \text{ an ordered } 2N-1 \text{ tuple }, \mathcal{Z}(B_\alpha) \supseteq \mathcal{Z}(\tilde{B}) \text{ and } c = \overline{B_\alpha(1)}\}.$$

Then obviously, $\Psi_{\tilde{B}} \subseteq \Psi^{\preceq}$. Conversely, let $\varphi \in \Psi^{\preceq}$. Since $\mathcal{Z}(B_\alpha) \supseteq \mathcal{Z}(\tilde{B})$, we have $\varphi(0) = 0$. By Lemma 2.2.3 there is a unique probability finite atomic measure μ_φ supported at $N \le k \le 2N-1$ points on $\mathbb{T}$, write it $\mu_\varphi = \sum_{i=1}^k m_i\delta_{\lambda_i}$. Again the constraint $\mathcal{Z}(B_\alpha) \supseteq \mathcal{Z}(\tilde{B})$ implies that the measure μ_φ has to satisfy the constraint (2.16) (or the constraints (2.2) and (2.3)). Hence $\mu_\varphi \in M^1_{\tilde{B},\mathbb{R}}(\mathbb{T})$. On the other hand, we have that $\varphi(1) = \overline{B_\alpha(1)}B_\alpha(1) = 1$. Thus support of μ_φ contains the point 1 i.e. $1 \in \{\lambda_1, \dots, \lambda_k\}$. Also the proof of Lemma 2.2.3 implies that $m_i > 0$ for all $i = 1, \dots, k$. Hence we see that the measure μ_φ satisfies the condition of [15, Theorem 2.3]. This implies that $\mu_\varphi \in \Theta_{\tilde{B}}$. So $\varphi \in \Psi_{\tilde{B}}$. Hence we have the opposite containment. In summary, we have that $\Psi^{\preceq} = \Psi_{\tilde{B}}$.

With this identification, we view the measure in (2.27) as being on the set $\Psi_{\tilde{B}}$ in place of the set of extremal measures $\Theta_{\tilde{B}}$, so that

$$1 - \varphi(z)\varphi(w)^* = \int_{\Psi_{\tilde{B}}} H_\psi(z)(1-\psi(z)\psi(w)^*)H_\psi(w)^*\, d\nu(\psi). \tag{2.30}$$

To prove the minimality we need some results from [28].

Lemma 2.4.1 ([28]). *Let μ be a measure on a space C, H a separable Hilbert space, and $f \in B(H) \otimes L^1(\mu)$ with $f(x) \geq 0$ μ-almost everywhere. If $M \geq \int_C f(x)\, d\mu(x)$ for some $M \in B(H)$, then for all $\delta > 0$ there exists some subset $C_\delta \subseteq C$ and a constant $c_\delta > 0$, such that $\mu(C \backslash C_\delta) < \delta$ and $M \geq c_\delta f(x)$ for all $x \in C_\delta$.*

We also need the notion of *differentiating kernels* from [28]. Let $k(x,y) = \frac{1}{1-\overline{x}y}$ be the Szegő kernel on $H^2(\mathbb{D})$, then its differential is defined by

$$k(x^{(n)}, y) := n! \frac{y^n}{(1-\overline{x}y)^{n+1}} = \frac{\partial^n}{\partial \overline{x}^n} k(x,y). \tag{2.31}$$

For brevity, we write $k_x^{(i)}(\cdot)$ for the function $k(x^{(i)}, \cdot)$. If M_f is the multiplication operator of f on H^∞, then the differentiating kernels satisfy an analog of the reproducing property

$$M_f^* k_x^{(n)}(\cdot) = \sum_{j=0}^{n} \binom{n}{j} \overline{f^{(j)}(x)} k_x^{(n-j)}(\cdot). \tag{2.32}$$

For more details the reader is addressed to [28].

For k the Szegő kernel and for all function f in the unit ball of $H^\infty_{\tilde{B}}$, we define the positive kernel

$$\Phi_f(z,w) = (1 - f(z)f(w)^*)k(z,w).$$

The following is a generalization of some part of the proof of [28, Theorem 9].

Lemma 2.4.2. *Let $\mathcal{C}_{\tilde{B}}$ be some proper closed subset of $\Psi_{\tilde{B}}$, which is a set of test functions for $H^\infty_{\tilde{B}}$. Then the following hold:*

i) If $\psi \in \Psi_{\tilde{B}}$, then the kernel $\Phi_\psi(z,w)$ has rank at most $2N-1$.

ii) If $\psi_0 \in \Psi_{\tilde{B}} \backslash \mathcal{C}_{\tilde{B}}$, then there exists a measure μ on $\mathcal{C}_{\tilde{B}}$ and functions $h_{\psi,\ell} \in L^2(\mu)$ for $\ell = 1, \dots 2N$ such that

$$1 - \psi_0(z)\psi_0(w)^* = \int_{\mathcal{C}_{\tilde{B}}} \sum_{\ell=1}^{2N} h_{\psi,\ell}(z)(1 - \psi(z)\psi(w)^*) h_{\psi,\ell}(w)^*\, d\mu(\psi). \tag{2.33}$$

Moreover the following inequality holds,

$$\Phi_{\psi_0}(z,w) \geq \int_{\mathcal{C}_{\tilde{B}}} h_{\psi,\ell}(z) \Phi_\psi(z,w) h_{\psi,\ell}(w)^*\, d\mu(\psi) \tag{2.34}$$

for all $h_{\psi,\ell}(z) \in L^2(\mu)$, $\ell = 1, \ldots, 2N$.

Proof.

i) First note that any test function $\psi \in \Psi_{\tilde{B}}$ has at most $2N-1$ zeros. Suppose that $\mathcal{Z}(\psi) = \{\underbrace{\beta_1, \ldots, \beta_1}_{d_1}, \ldots, \underbrace{\beta_m, \ldots, \beta_m}_{d_m}\}$. Then, clearly $\sum_{j=1}^{m} d_j \leq 2N-1$. By Theorem A.2.1 the multiplication operator M_ψ is an isometry on $H^2(\mathbb{D})$, since ψ is a Blaschke product. Furthermore, by identity (2.32) we can see that $P_{\mathfrak{M}_\psi} := 1 - M_\psi M_\psi^*$ is the projection onto

$$\mathfrak{M}_\psi := \ker M_\psi^* = \operatorname{span}\{k_{\beta_1}, \ldots, k_{\beta_1}^{(d_1-1)}, \ldots, k_{\beta_m}, \ldots, k_{\beta_m}^{(d_m-1)}\}.$$

Since $\Phi_\psi(z,w) = \langle P_{\mathfrak{M}_\psi} k_w, k_z \rangle$ and $\ker M_\psi^* = P_{\mathfrak{M}_\psi} H^2$ (see [51, p.100]) we conclude that Φ_ψ has rank at most $2N-1$.

ii) Since $P_{\mathfrak{M}_\psi}$ is the projection from H^2 onto $\mathfrak{M}_\psi$ we have

$$\Phi_\psi(z,w) = \langle P_{\mathfrak{M}_\psi} k_w, k_z \rangle = \langle P_{\mathfrak{M}_\psi} k_w, P_{\mathfrak{M}_\psi} k_z \rangle.$$

Hence we can think $\Phi_\psi(z,w)$ as a holomorphic function in z and anti-holomorphic function in w. If we think of the anti-holomorphic function as being in the dual of H^2, then

$$\Phi_\psi \in H^2 \otimes (H^2)^* \cong B(H^2).$$

More explicitly, Φ_ψ defines an operator on H^2 as

$$\Phi_\psi f(z) := \int_{\mathbb{T}} \Phi_\psi(z,w) f(w)\, d\gamma(w),$$

where γ is the arc-length measure on $\mathbb{T}$. Let us calculate the following inner product

$$\begin{aligned}
\langle \Phi_\psi f, g\rangle &= \int_{\mathbb{T}}\int_{\mathbb{T}} \overline{g(z)}\Phi_\psi(z,w)f(w)\,d\gamma(w)\,d\gamma(z)\\
&= \int_{\mathbb{T}}\int_{\mathbb{T}} \overline{g(z)}\langle P_{\mathfrak{M}_\psi}k_w, P_{\mathfrak{M}_\psi}k_z\rangle f(w)\,d\gamma(w)\,d\gamma(z)\\
&= \int_{\mathbb{T}}\int_{\mathbb{T}} \langle f(w)P_{\mathfrak{M}_\psi}k_w, g(z)P_{\mathfrak{M}_\psi}k_z\rangle\,d\gamma(w)\,d\gamma(z)\\
&= \langle \int_{\mathbb{T}} f(w)P_{\mathfrak{M}_\psi}k_w\,d\gamma(w), \int_{\mathbb{T}} g(z)P_{\mathfrak{M}_\psi}k_z\,d\gamma(z)\rangle\\
&= \langle \int_{\mathbb{T}} f(w)P_{\mathfrak{M}_\psi}k_w\,d\gamma(w), \int_{\mathbb{T}} g(z)P_{\mathfrak{M}_\psi}k_z\,d\gamma(z)\rangle_{\mathfrak{M}_\psi}\\
&= \langle A_\psi f, A_\psi g\rangle\\
&= \langle A_\psi^* A_\psi f, g\rangle,
\end{aligned}$$

where $A_\psi : H^2 \to \mathfrak{M}_\psi$ is given by

$$A_\psi f := \int_{\mathbb{T}} f(w)P_{\mathfrak{M}_\psi}k_w\,d\gamma(w).$$

This gives the factorization

$$\Phi_\psi(z,w) = A_\psi^*(z)A_\psi(w). \tag{2.35}$$

Let $I_{\mathfrak{M}_\psi}$ be the embedding map of $\mathfrak{M}_\psi$ into H^2. We also consider the following inner product

$$\begin{aligned}
\langle A_\psi f, g\rangle &= \langle \int_{\mathbb{T}} f(w)P_{\mathfrak{M}_\psi}k_w\,d\gamma(w), g\rangle_{\mathfrak{M}_\psi}\\
&= \int_{\mathbb{T}} f(w)\langle P_{\mathfrak{M}_\psi}k_w, g\rangle_{\mathfrak{M}_\psi}\,d\gamma(w)\\
&= \int_{\mathbb{T}} f(w)\langle P_{\mathfrak{M}_\psi}k_w, I_{\mathfrak{M}_\psi}g\rangle\,d\gamma(w)\\
&= \int_{\mathbb{T}} f(w)\langle k_w, I_{\mathfrak{M}_\psi}g\rangle\,d\gamma(w)\\
&= \int_{\mathbb{T}} f(w)\overline{(I_{\mathfrak{M}_\psi}g)(w)}\,d\gamma(w)\\
&= \langle f, I_{\mathfrak{M}_\psi}g\rangle.
\end{aligned}$$

From this calculations we conclude that

$$A_{\psi}^{*} = I_{\mathfrak{M}_{\psi}}. \tag{2.36}$$

Let $F = \{z_1, \ldots, z_{2N}\}$ be an arbitrary finite subset of $\mathbb{D}$. Then we consider the classical Nevanlinna-Pick interpolation problem of finding a function φ in the closed unit ball of $H^{\infty}(\mathbb{D})$ such that $\varphi(z_i) = \psi_0(z_i)$ for $i = 1, \ldots, 2N$.

Since the operator $\Phi_{\psi_0}(z, w)$ has rank at most $2N - 1$ the $2N \times 2N$ matrix

$$\left(\left[1 - \psi_0(z_i)\overline{\psi_0(z_j)}\right] k(z_j, z_i)\right)_{i,j=1}^{2N}$$

must be singular, so the problem has a unique solution, namely $\varphi = \psi_0$.

Recall the assumption that $\mathcal{C}_{\tilde{B}}$ is a set of test functions for $H^{\infty}_{\tilde{B}}$. Hence by Theorem 1.3.7, there is a positive kernel $\Gamma : F \times F \to C_b(\mathcal{C}_{\tilde{B}})^*$ such that

$$1 - \psi_0(z_i)\overline{\psi_0(z_j)} = \Gamma(z_i, z_j)\left(1 - E(z_i)E(z_j)^*\right), \tag{2.37}$$

where $z_i, z_j \in F$ with $i, j = 1, \ldots, 2N$.
Moreover, by Theorem 1.3.6, the equation (2.37) must holds for all over $\mathbb{D}^2$. That is,

$$1 - \psi_0(z)\overline{\psi_0(w)} = \Gamma(z, w)\left(1 - E(z)E(w)^*\right), \tag{2.38}$$

for all $(z, w) \in \mathbb{D}^2$. We can rewrite this, in our case, by saying that there exists a measure μ on $\mathcal{C}_{\tilde{B}}$, and functions $h_{\psi,\ell}(z) \in L^2(\mu)$ (by Kolmogorov decomposition, see [7, Theorem 2.62]), for $\ell = 1, \ldots, 2N$ such that

$$1 - \psi_0(z)\psi_0(w)^* = \int_{\mathcal{C}_{\tilde{B}}} \sum_{\ell=1}^{2N} h_{\psi,\ell}(z)h_{\psi,\ell}(w)^* \left(1 - \psi(z)\psi(w)^*\right)\, d\mu(\psi). \tag{2.39}$$

Multiplying this equation by $k(z, w)$ gives

$$\Phi_{\psi_0}(z, w) = \int_{\mathcal{C}_{\tilde{B}}} \sum_{\ell=1}^{2N} h_{\psi,\ell}(z)\Phi_{\psi}(z, w)h_{\psi,\ell}(w)^*\, d\mu(\psi).$$

Since Φ_{ψ} is positive kernel and a positive operator, when seen as an operator on H^2, as above we have that for all $\ell = 1, ..., 2N$,

$$\Phi_{\psi_0}(z,w) \geq \int_{\mathcal{C}_{\tilde{B}}} h_{\psi,\ell}(z)\Phi_\psi(z,w)h_{\psi,\ell}(w)^*\,d\mu(\psi).$$

This completes the proof. □

Remark 2.4.3. Let $\mathcal{C}_{\tilde{B}}$ be a proper closed subset of $\Psi_{\tilde{B}}$. Then there exists a function $\psi_0 = c_0\tilde{B}\prod_{j=1}^{N-1} m_{\beta_j}$ in $\Psi_{\tilde{B}} \setminus \mathcal{C}_{\tilde{B}}$ such that $\beta_j \notin \mathcal{Z}(\tilde{B})$ and $\beta_i \neq \beta_j$ for $1 \leq i < j \leq N-1$, where we write $\mathcal{Z}(\tilde{B})$ for the zero set of $\tilde{B}$, including all the multiplicities i.e. $\mathcal{Z}(\tilde{B}) = \{0 = \beta_{i_0}, \ldots, \beta_{i_{N-1}}\}$. To see this, note that the set $\Psi_{\tilde{B}} \setminus \mathcal{C}_{\tilde{B}}$ is relatively open, so we can perturb the zeros of ψ_0 small enough which are not roots of $\tilde{B}$, so that without increasing the norm of ψ_0. This is because of the following argument. Let $\psi_{0,\epsilon} = c_0\tilde{B}\prod_{j=1}^{N-1} m_{\beta_j+\epsilon_j}$. Then

$$\begin{aligned}
&|\psi_0 - \psi_{0,\epsilon}| = |c_0\tilde{B}| \cdot \left|\prod_{j=0}^{N-1} m_{\beta_j} - \prod_{j=0}^{N-1} m_{\beta_j+\epsilon_j}\right| \\
&= \left|\tilde{B}\right| \cdot \Bigg|(m_{\beta_1} - m_{\beta_1+\epsilon_1})\prod_{j=2}^{N-2} m_{\beta_j} + \sum_{k=2}^{N-1}\prod_{i=1}^{k-1} m_{\beta_i+\epsilon_i}\left(m_{\beta_k} - m_{\beta_k+\epsilon_k}\right)\prod_{j=k+1}^{N-1} m_{\beta_j} \\
&+ \prod_{i=1}^{N-1} m_{\beta_i+\epsilon_i}\left(m_{\beta_{N-1}} - m_{\beta_{N-1}+\epsilon_{N-1}}\right)\Bigg| \\
&\leq \left|\tilde{B}\right| \Bigg(\left|(m_{\beta_1} - m_{\beta_1+\epsilon_1})\prod_{j=2}^{N-2} m_{\beta_j}\right| + \sum_{k=2}^{N-1}\left|\prod_{i=1}^{k-1} m_{\beta_i+\epsilon_i}\left(m_{\beta_k} - m_{\beta_k+\epsilon_k}\right)\prod_{j=k+1}^{N-1} m_{\beta_j}\right| \\
&+ \left|\prod_{i=1}^{N-1} m_{\beta_i+\epsilon_i}\left(m_{\beta_{N-1}} - m_{\beta_{N-1}+\epsilon_{N-1}}\right)\right|\Bigg),
\end{aligned}$$

which tends to 0 when $\epsilon_1 \to 0, \ldots, \epsilon_{N-1} \to 0$. Additionally, to prove the minimality of $\Psi_{\tilde{B}}$ we can assume also that $\beta_j \neq \infty$ for all $j = 1, \ldots, \infty$. Since $\psi_0 \in \Psi_{\tilde{B}}$ we must have the ordering on the numbers $\beta_1, \ldots, \beta_{N-1}$, so we can assume without loss of generality that $\beta_1 \preceq \cdots \preceq \beta_{N-1}$. Hence if $\beta_k = \infty$, then $\beta_\ell = \infty$ for all $\ell \geq k$. Consider the following closed subsets of $\Psi_{\tilde{B}}$ given by

$$\Psi^1 = \{\psi \in \Psi_{\tilde{B}} : \beta_1 = \infty\} = \{c\tilde{B}\}, \Psi^k = \{\psi \in \Psi_{\tilde{B}} : \beta_k = \infty\} \text{ for } 2 \leq k \leq N-1.$$

Since all $\Psi^k, 1 \leq k \leq N-1$ are closed, the finite union $\mathcal{C} := \cup_{k=1}^{N-1}\Psi^k \cup \mathcal{C}_{\tilde{B}}$ is a proper closed subset in $\Psi_{\tilde{B}}$. Now if we prove that $\mathcal{C}$ is not a minimal set of test functions for $H^\infty_{\tilde{B}}$, then this automatically implies that $\mathcal{C}_{\tilde{B}}$ is also not minimal.

Theorem 2.4.4. *The set $\Psi_{\tilde{B}}$ is a minimal set of test functions for the algebra $H^\infty_{\tilde{B}}$.*

Proof. The set $\Psi_{\tilde{B}}$ is norm closed in $H^\infty(\mathbb{D})$, and we endow it with the relative topology. Suppose that some proper closed subset $C_{\tilde{B}}$ of $\Psi_{\tilde{B}}$ is a set of test functions for $H^\infty_{\tilde{B}}$. Then $\Psi_{\tilde{B}}\backslash\mathcal{C}_{\tilde{B}}$ is relatively open, and there exists $\psi_0 = c_0\prod_{j=0}^{2N-2} m_{\tilde{\alpha}_j}$ in this set, where $\mathcal{Z}(\tilde{B}) = \{0 = \tilde{\alpha}_{j_0}, \tilde{\alpha}_{j_1}\dots, \tilde{\alpha}_{j_{N-1}}\} = \{\tilde{\alpha}_1 \dots, \tilde{\alpha}_N\}$ are the zeros of $\tilde{B}$. Since $\Psi_{\tilde{B}}\backslash\mathcal{C}_{\tilde{B}}$ is relatively open, by Remark 2.4.3 we can assume without loss of generality that no $\tilde{\alpha}_j = \infty$ and that any root which is not a root for $\tilde{B}$ is distinct from the roots of $\tilde{B}$ and all such roots are distinct from each other.

Let $\psi = c\prod_{k=0}^{2N-2} m_{\alpha_j}$ be in $\mathcal{C}_{\tilde{B}}$, and $\mathcal{Z}(\tilde{B}) = \{0 = \alpha_0, \alpha_{k_1}\dots, \alpha_{k_{N-1}}\}$ be the zeros of $\tilde{B}$. For any α_k in α which occurs only once, set $k_{\alpha_j}(z) = 1/(1-\overline{\alpha_j}z)$, the Szegő kernel, where $k_\infty := 0$. More generally, if a root other than ∞ is repeated, it is understood that we use the kernels $k^{(i)}_\alpha(z) = i!z^i/(1-\overline{\alpha}z)^{i+1}$ instead, where i runs from 0 to one less than the multiplicity of the root, though we do not write this explicitly to avoid notational complexity. We define $k_{\tilde{\alpha}_j}$ in an identical manner. By assumption the inequality (2.34) holds, hence by Lemma 2.4.1 for any $\delta > 0$, there is a set $C_\delta \subseteq C_{\tilde{B}}$ and a constant c_δ such that $\mu(C_{\tilde{B}}\backslash C_\delta) < \delta$ and

$$\Delta_{\psi_0}(z,w) \geq c_\delta h_{\psi,\ell}(z)\Delta_\psi(z,w)\overline{h_{\psi,\ell}(w)}$$

for all $\psi \in C_\delta$. Then from the factorization (2.35) we have

$$A_{\psi_0}(z)^* A_{\psi_0}(w) \geq c_\delta h_{\psi,\ell}(z) A_\psi(z)^* A_\psi(w)\overline{h_{\psi,\ell}(w)}$$

for all $\psi \in C_\delta$. It follows that by Douglas' lemma, the range of $h_{\psi,\ell}(\cdot)A^*_\psi$ is contained in the range of $A^*_{\psi_0}$. Hence by (2.36) there exists constants c_{jk} such that

$$h_{\psi,\ell} k_{\alpha_n} = \sum_{j=0}^{2N-2} c_{nj}\, k_{\tilde{\alpha}_j}. \tag{2.40}$$

for $n = 0, \dots, 2N-2$. Taking the limit for $\delta \to 0$ we see that above equations hold for $\mu-$almost all $\psi \in C_{\tilde{B}}$. In particular, taking $n = 0$ gives

$$h_{\psi,\ell} = \sum_{j=0}^{2N-2} c_{0j}\, k_{\tilde{\alpha}_j},$$

and so plugging this back into (2.40), we have

$$k_{\alpha_n} \sum_{j=0}^{2N-2} c_{0j}\, k_{\tilde{\alpha}_j} = \sum_{j=0}^{2N-2} c_{nj}\, k_{\tilde{\alpha}_j}. \tag{2.41}$$

The kernels extend to meromorphic functions on the Riemann sphere, as then does $h_{\psi,\ell}$.

We use (2.41) to eliminate some of the terms and to eventually solve for $h_{\psi,\ell}$. Consider $0 \neq \alpha_n \in \mathcal{Z}(\tilde{B})$. Then $\alpha_n = \tilde{\alpha}_j$ for some j. If this is a zero of order 1 for ψ_0, then the right side of (2.41) has a pole of at most order 1 at $1/\overline{\tilde{\alpha}_j}$, while the left side has a pole of order 2 at this point if $c_{0j} \neq 0$. Hence we must have $c_{0j} = 0$.

More generally, suppose that ψ_0 has a zero of order $m > 1$ at $\alpha_n \in \mathcal{Z}(\tilde{B})$ (where now α_n may be 0). Let $\tilde{\alpha}_j = \cdots = \tilde{\alpha}_{j+m-1}$ be the m repeated zeros. If $\alpha_n \neq 0$, $k_{\tilde{\alpha}_{j+i}}$, $0 \leq i \leq m-1$, have poles of order between 1 and m, and so no term on the right side of (2.41) has a pole of order more than m at $1/\overline{\tilde{\alpha}_j}$. On the left side, if we choose k_{α_n} to have a pole of order m, and if any of c_{0j} to $c_{0,j+m-1}$ are nonzero, the corresponding term has a pole of order bigger than m. Hence each of these coefficients must be zero.

Things are slightly different when $\alpha_n = 0$. In this case, $j = 0$ and $k_{\tilde{\alpha}_i}$, $1 \leq i \leq m-1$, have poles of order between 1 and $m-1$ at ∞ (we take $k_{\tilde{\alpha}_0} = 1$). So reasoning as before, no term on the right of (2.41) has a pole of order bigger than $m-1$ at ∞, while if we choose k_{α_n} to have a pole of order $m-1$ there, the left side has a pole of order at least m at ∞ if any of c_{01} to $c_{0,m-1}$ are nonzero. So all of these coefficients must also be zero.

Combining these observations, we conclude that with the possible exception of c_{00}, all coefficients $c_{0,j}$ corresponding to $\tilde{\alpha}_j \in \mathcal{Z}(\tilde{B})$ are zero. Hence we have

$$h_{\psi,\ell} = c_{00} + \sum_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\backslash\mathcal{Z}(\tilde{B})} c_{0j} k_{\tilde{\alpha}_j} = r \prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\backslash\mathcal{Z}(\tilde{B})} k_{\tilde{\alpha}_j}.$$

Recall that we were able to choose the elements of $\mathcal{Z}(\psi_0)\backslash\mathcal{Z}(\tilde{B})$ so that they are distinct and none are repeats of elements of $\mathcal{Z}(\tilde{B})$. Consequently,

$$r(z) = c_{00} \prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\backslash\mathcal{Z}(\tilde{B})} (1 - \overline{\tilde{\alpha}_j} z) + \sum_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\backslash\mathcal{Z}(\tilde{B})} c_{0j} \prod_{\tilde{\alpha}_n \in \mathcal{Z}(\psi_0)\backslash\mathcal{Z}(\tilde{B}),\, n \neq j} (1 - \overline{\tilde{\alpha}_n} z)$$

is a polynomial of degree at most $N-1$. So (2.41) becomes

$$rk_{\alpha_n} \prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\setminus\mathcal{Z}(\tilde{B})} k_{\tilde{\alpha}_j} = \sum_{j=0}^{2N-2} c_{nj}\, k_{\tilde{\alpha}_j}. \tag{2.42}$$

Now we turn to $\alpha_n \in \mathcal{Z}(\psi)\setminus\mathcal{Z}(\tilde{B})$, where we write $\mathcal{Z}(\psi)$ for the zero set of ψ, but including all the multiplicities. Let m' be the multiplicity of α_n as a root of $\tilde{B}$ (which may be 0) and m the multiplicity of α_n as a root of ψ. The right side of (2.42) has a pole of order m' at $1/\overline{\alpha_n}$, so r must have a zero of order $m-m'$ at $1/\overline{\alpha_n}$. Running over all $\alpha_n \in \mathcal{Z}(\psi)\setminus\mathcal{Z}(\tilde{B})$, we conclude that

$$h_{\psi,\ell} = g_{\psi,\ell} \frac{\prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\setminus\mathcal{Z}(\tilde{B})} k_{\tilde{\alpha}_j}}{\prod_{\alpha_n \in \mathcal{Z}(\psi)\setminus\mathcal{Z}(\tilde{B})} k_{\alpha_n}}, \tag{2.43}$$

where, since the poles of $\prod_{\alpha_n \in \mathcal{Z}(\psi)\setminus\mathcal{Z}(\tilde{B})} k_{\alpha_n}$ are roots of r,

$$g_{\psi,\ell} = r \prod_{\alpha_n \in \mathcal{Z}(\psi)\setminus\mathcal{Z}(\tilde{B})} k_{\alpha_n}$$

is a polynomial of degree at most $2N-1-\deg\psi$, where r_ψ is the cardinality of $\mathcal{Z}(\psi)\setminus\mathcal{Z}(\tilde{B})$.

Substitute the formula for $h_{\psi,\ell}$ into (2.33) and multiply by $\prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\setminus\mathcal{Z}(\tilde{B})}(1 - \overline{\tilde{\alpha}_j} z)(1-\overline{\tilde{\alpha}_j} w)^*$ we get

$$\begin{aligned}
&\prod_{i=1}^{N-1}(1-\overline{\tilde{\alpha}_i}z)(1-\overline{\tilde{\alpha}_i}w)^* - \tilde{B}(z)\tilde{B}(w)^* \prod_{i=1}^{N-1}(z-\tilde{\alpha}_i)(w-\tilde{\alpha}_i)^* \\
&= \int_{\mathcal{C}_{\tilde{B}}} \sum_{\ell=1}^{2N} g_{\psi,\ell}(z) g_{\psi,\ell}(w)^* \left(\prod_{i=1}^{r_\psi}(1-\overline{\alpha_{\psi,i}}z)(1-\overline{\alpha_{\psi,i}}w)^* \right. \\
&\left. -\tilde{B}(z)\tilde{B}(w)^* \prod_{i=1}^{r_\psi}(z-\alpha_{\psi,i})(w-\alpha_{\psi,i})^* \right) d\nu(\psi),
\end{aligned} \tag{2.44}$$

where $\alpha_i \in \mathcal{Z}(\psi_0) \setminus \mathcal{Z}(\tilde{B})$ and $\alpha_{\psi,i} \in \mathcal{Z}(\psi) \setminus \mathcal{Z}(\tilde{B})$. We can expand this more precisely,

$$\sum_{m,n=0}^{N-1}\Big[z^m\overline{w}^n S_m(\tilde{\overline{\alpha}})S_n(\tilde{\overline{\alpha}})^* - \tilde{B}(z)\tilde{B}(w)^* z^{N-1-m}\overline{w}^{N-1-n}S_m(\tilde{\alpha})S_n(\tilde{\alpha})^*\Big]$$
$$= \int_{\mathcal{C}_{\tilde{B}}}\sum_{\ell=1}^{2N}\sum_{m,n=0}^{N-1}\Bigg(z^m\overline{w}^n \sum_{s=\max\{0,m-r_\psi\}}^{\min\{m,N-1-r_\psi\}}\ \sum_{t=\max\{0,n-r_\psi\}}^{\min\{n,N-1-r_\psi\}} g_{\psi,\ell,s}g^*_{\psi,\ell,t}S_{m-s}(\overline{\alpha_\psi})S_{n-t}(\overline{\alpha_\psi})^*$$
$$- \tilde{B}(z)\tilde{B}(w)^* z^{N-1-m}\overline{w}^{N-1-n}\sum_{s=\max\{0,N-1-m-r_\psi\}}^{\min\{N-1-m,N-1-r_\psi\}}\ \sum_{t=\max\{0,N-1-n-r_\psi\}}^{\min\{N-1-n,N-1-r_\psi\}}$$
$$g_{\psi,\ell,s}g^*_{\psi,\ell,t}S_{r_\psi-N+1+m+s}(\alpha_\psi)S_{r_\psi-N+1+n+t}(\alpha_\psi)^*\Big)\ d\nu(\psi). \tag{2.45}$$

Here we use the symmetric sum notation from the proof of Lemma 2.2.3 and as a shorthand notation, write $\tilde{\alpha}$ for $\mathcal{Z}(\psi_0)\backslash\mathcal{Z}(\tilde{B})$ and α_ψ for $\mathcal{Z}(\psi)\backslash\mathcal{Z}(\tilde{B})$. Observe that the coefficients of $z^m\overline{w}^n$ and $\tilde{B}(z)\tilde{B}(w)^*z^{N-1-m}\overline{w}^{N-1-n}$ are complex conjugates of each other, and so in particular are equal when $m = n$.

For $n, m = 0, \ldots, N-1$, define vectors in $L^2(\nu)\otimes\mathbb{C}^{2N}$ by

$$\begin{aligned} v_m^1 &= \left(\sum_{s=\max\{0,m-r_\psi\}}^{\min\{m,N-r_\psi-1\}} g_{\psi,\ell,s}S_{m-s}(\overline{\alpha_\psi})\right),\\ v_n^2 &= \left(\sum_{t=\max\{0,N-1-n-r_\psi\}}^{\min\{N-1-n,N-1-r_\psi\}} g_{\psi,\ell,t}S_{r_\psi-N+1+n+t}(\alpha_\psi)\right). \end{aligned} \tag{2.46}$$

Hence (2.45) becomes

$$\begin{aligned}&\sum_{m,n=0}^{N-1}\Big[z^m\overline{w}^n S_m(\tilde{\overline{\alpha}})S_n(\tilde{\overline{\alpha}})^* - \tilde{B}(z)\tilde{B}(w)^* z^{N-1-m}\overline{w}^{N-1-n}S_m(\tilde{\alpha})S_n(\tilde{\alpha})^*\Big]\\ &= \sum_{m,n=0}^{N-1}\Big(z^m\overline{w}^n\left\langle v_m^1, v_n^1\right\rangle - \tilde{B}(z)\tilde{B}(w)^*z^{N-1-m}\overline{w}^{N-1-n}\left\langle v_m^2, v_n^2\right\rangle\Big).\end{aligned} \tag{2.47}$$

Looking at the coefficients of $z^m\overline{w}^m$, $z^n\overline{w}^n$ and $z^m\overline{w}^n$ in (2.47), we get

$$\|v_m^1\|^2\|v_n^1\|^2 = |S_m(\tilde{\overline{\alpha}})|^2|S_n(\tilde{\overline{\alpha}})^*|^2 = |S_m(\tilde{\overline{\alpha}})S_n(\tilde{\overline{\alpha}})^*|^2 = \left|\left\langle v_m^1, v_n^1\right\rangle\right|^2. \tag{2.48}$$

It then follows from the Cauchy-Schwarz inequality that the vectors v_m^1 and v_n^1 are collinear. Identical reasoning shows that v_m^2 and v_n^2 are collinear as well.

Looking at the terms $z^0\overline{w}^0$ and $\tilde{B}(z)\tilde{B}(w)^* z^{N-1}\overline{w}^{N-1}$ in (2.47), we see that the vectors $v_0^1 = (g_{\psi,\ell,0})$ and $v_0^2 = (g_{\psi,\ell,N-1-r_\psi})$ both have norm equal to $|S_0(\tilde{\alpha})| = 1$, so are non-zero. Furthermore, it follows from (2.47) that $\langle v_n^1, v_0^1\rangle = \langle v_0^2, v_n^2\rangle = S_n(\overline{\tilde{\alpha}})$, and so

$$v_n^1 = S_n(\overline{\tilde{\alpha}})v_0^1 \quad \text{and} \quad v_n^2 = S_n(\tilde{\alpha})v_0^2, \qquad n = 1, \dots, N-1. \tag{2.49}$$

This implies that ν-a.e. $\psi \in \mathcal{C}_{\tilde{B}}$,

$$\begin{aligned} &\sum_{s=\max\{0,n-r_\psi\}}^{\min\{n,N-r_\psi-1\}} g_{\psi,\ell,s} S_{n-s}(\overline{\alpha_\psi}) = S_n(\overline{\tilde{\alpha}}) g_{\psi,\ell,0} \quad \text{and} \\ &\sum_{t=\max\{0,N-1-n-r_\psi\}}^{\min\{N-1-n,N-1-r_\psi\}} g_{\psi,\ell,t} S_{r_\psi-N+1+n+t}(\alpha_\psi) = S_n(\tilde{\alpha}) g_{\psi,\ell,N-1-r_\psi} \end{aligned} . \tag{2.50}$$

When $r_\psi = N-1$, $g_{\psi,\ell} = g_{\psi,\ell,0} = g_{\psi,\ell,N-1-r_\psi}$ is constant. From the first equation of (2.50) we have $g_{\psi,\ell,0}S_n(\overline{\alpha_\psi}) = S_n(\overline{\tilde{\alpha}})g_{\psi,\ell,0}$. Thus by considering the coefficients of $z^j \cdot \overline{w}^0$ in (2.45), we see that

$$\begin{aligned} |g_{\psi,\ell,0}|^2 \prod_{\alpha_j \in \mathcal{Z}(\psi)\setminus\mathcal{Z}(\tilde{B})} (1-\overline{\alpha_j}z) &= |g_{\psi,\ell,0}|^2 \sum_{n=0}^{N-1} S_n(\overline{\alpha_\psi})z^n = |g_{\psi,\ell,0}|^2 \sum_{n=0}^{N-1} S_n(\overline{\tilde{\alpha}})z^n \\ &= |g_{\psi,\ell,0}|^2 \prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\setminus\mathcal{Z}(\tilde{B})} (1-\overline{\tilde{\alpha}_j}z). \end{aligned}$$

Since $\mathcal{C}_{\tilde{B}}$ is a closed set not containing ψ_0, this gives a contradiction unless $g_{\psi,\ell,0}$, and hence $g_{\psi,\ell}$ is zero.

For $r_\psi < N-1$, $g_{\psi,\ell}$ is a non-constant polynomial. Factor it as

$$g_{\psi,\ell}(z) = g_{\psi,\ell,0} \prod_{i=1}^{N-1-r_\psi} (z-\beta_i) = g_{\psi,\ell,0} \sum_{s=0}^{N-1-r_\psi} S_s(\beta)z^s.$$

So $g_{\psi,\ell,s} = g_{\psi,\ell,0}S_s(\beta)$ for $s = 0, \dots, N-1-r_\psi$. Then (2.44) becomes

$$\prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\setminus\mathcal{Z}(\tilde{B})} (1-\overline{\tilde{\alpha}_j}z)(1-\overline{\tilde{\alpha}_j}w)^* - \tilde{B}(z)\tilde{B}(w)^* \prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\setminus\mathcal{Z}(\tilde{B})} (z-\tilde{\alpha}_j)(w-\tilde{\alpha}_j)^*$$
$$= \prod_{i=1}^{N-1-r_\psi} (z-\beta_i)(w-\beta_i)^* \left(\prod_{\alpha_{\psi,j}\in\mathcal{Z}(\psi)\setminus\mathcal{Z}(\tilde{B})} (1-\overline{\alpha_{\psi,j}}z)(1-\overline{\alpha_{\psi,j}}w)^* \left\langle v_0^1, v_0^1 \right\rangle \right.$$
$$\left. -\tilde{B}(z)\tilde{B}(w)^* \prod_{\alpha_{\psi,j}\in\mathcal{Z}(\psi)\setminus\mathcal{Z}(\tilde{B})} (z-\alpha_{\psi,j})(w-\alpha_{\psi,j})^* \left\langle v_0^2, v_0^2 \right\rangle \right).$$

Since $\langle v_0^1, v_0^1\rangle = 1, \langle v_0^2, v_0^2\rangle = 1$, the later equation becomes

$$\prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\setminus\mathcal{Z}(\tilde{B})} (1-\overline{\tilde{\alpha}_j}z)(1-\overline{\tilde{\alpha}_j}w)^* - \tilde{B}(z)\tilde{B}(w)^* \prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\setminus\mathcal{Z}(\tilde{B})} (z-\tilde{\alpha}_j)(w-\tilde{\alpha}_j)^*$$
$$= \prod_{i=1}^{N-1-r_\psi} (z-\beta_i)(w-\beta_i)^* \left(\prod_{\alpha_{\psi,j}\in\mathcal{Z}(\psi)\setminus\mathcal{Z}(\tilde{B})} (1-\overline{\alpha_{\psi,j}}z)(1-\overline{\alpha_{\psi,j}}w)^* \right.$$
$$\left. -\tilde{B}(z)\tilde{B}(w)^* \prod_{\alpha_{\psi,j}\in\mathcal{Z}(\psi)\setminus\mathcal{Z}(\tilde{B})} (z-\alpha_{\psi,j})(w-\alpha_{\psi,j})^* \right). \tag{2.51}$$

Considering the coefficients of $z^j \cdot \overline{w}^0$ in (2.51), we have that

$$S_{N-1-r_\psi}(\overline{\beta}) \prod_{i=1}^{N-1-r_\psi} (z-\beta_i) \prod_{\alpha_{\psi,j}\in\mathcal{Z}(\psi)\setminus\mathcal{Z}(\tilde{B})} (1-\overline{\alpha_{\psi,j}}z) = \prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\setminus\mathcal{Z}(\tilde{B})} (1-\overline{\tilde{\alpha}_j}z),$$

and so the union of $\beta^{-1} := (1/\overline{\beta_1}, \dots, 1/\overline{\beta_{N-1-r_\psi}})$ and $\mathcal{Z}(\psi)\setminus\mathcal{Z}(\tilde{B})$ equals $\mathcal{Z}(\psi_0)\setminus\mathcal{Z}(\tilde{B})$. In particular, $1/\beta_i \in \mathbb{D}$ for all i.

In a similar manner, but now using the coefficients of $z^n\tilde{B}(z)\tilde{B}(w)^*$ in (2.51), we find that

$$S_{N-1-r_\psi}(\overline{\beta}) \prod_{i=1}^{N-1-r_\psi} (z-\beta_i) \prod_{\alpha_{\psi,j}\in Z(\psi)\setminus\mathcal{Z}(\tilde{B})} (z-\alpha_{\psi,j}) \prod_{\alpha_{\psi,j}\in\mathcal{Z}(\psi)\setminus\mathcal{Z}(\tilde{B})} (-\overline{\alpha_{\psi,j}})$$
$$= \prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\setminus\mathcal{Z}(\tilde{B})} (z-\tilde{\alpha}_j)(-\overline{\tilde{\alpha}_j}).$$

Hence the union of $\beta = (\beta_1, \dots, \beta_{N-1-r_\psi})$ and $\mathcal{Z}(\psi)\backslash\mathcal{Z}(\tilde{B})$ equals $\mathcal{Z}(\psi_0)\backslash\mathcal{Z}(\tilde{B})$, implying that $\beta_i \in \mathbb{D}$ for all i. Thus we again have a contradiction unless $g_{\psi,\ell} = 0$.

We conclude therefore that $g_{\psi,\ell} = 0$ ν-a.e. This then implies that

$$\prod_{\tilde{\alpha}_j \in \mathcal{Z}(\psi_0)\backslash\mathcal{Z}(\tilde{B})} (z - \tilde{\alpha}_j) = 0,$$

which since $\mathcal{Z}(\psi_0)\backslash\mathcal{Z}(\tilde{B}) \neq \emptyset$, is an absurdity. We conclude that $\mathcal{C}_{\tilde{B}}$ cannot have been a set of test functions for the algebra $H^\infty_{\tilde{B}}$, and so it follows that $\Psi_{\tilde{B}}$ is a minimal set of test functions. $\square$

As a consequence of Theorem 2.4.4 and Lemma 2.1.1 we have the following general result:

Theorem 2.4.5. *The set Ψ_B is a minimal set of test functions for the algebra H^∞_B.*

Remark 2.4.6. At least in some simple cases, it is possible to describe the geometry of the set of test functions. First, if $\deg B = 2$, then the minimal set of test functions is given by

$$\Psi_B = \left\{ B(z)\frac{z-\zeta}{1-\overline{\zeta}z} : \zeta \in \mathbb{D}_\infty \right\}.$$

Then it is clear that Ψ_B is homeomorphic to the Riemann sphere. This covers the special case in [28].

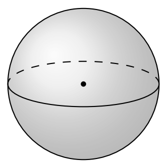

Figure 2.1: Ψ_B when $N = 2$

Next we consider when $B(z)$ is a Blaschke product degree 3, so

$$\Psi_B = \left\{ B(z)\frac{z-\zeta_1}{1-\overline{\zeta_1}z}\frac{z-\zeta_2}{1-\overline{\zeta_2}z} : (\zeta_1, \zeta_2) \in \mathbb{D}_\infty \times \mathbb{D}_\infty \ \text{ with } \ \zeta_1 \preceq \zeta_2 \right\}. \tag{2.52}$$

It is interesting to note that the set Ψ_B is homeormorphic to a closed ball in $\mathbb{R}^3$ with a smaller open ball tangent to the boundary removed from the interior. To see this, we have chosen the arguments to lie in the interval $(0, 2\pi]$. By rotating, we can always assume that one of the points (ζ_1 or ζ_2) has argument 2π. For this reason, we assume that ζ_2 is either ∞ or in the interval $[0,1)$. With $\zeta_1 \preceq \zeta_2$ in our order for $\zeta_2 > 0$, the set of $\zeta_1 \preceq \zeta_2$ is a closed disk of radius $|\zeta_2|$. The resulting set of points (ζ_1, ζ_2) with ζ_2 in the interval $[0,1)$ is thus homeomorphic to a solid truncated cone which contains the boundary along the side of the cone, but not the boundary at the top. To the top we attach the points of the form (ζ_1, ∞), when ζ_1 in the open unit disk. The resulting object then looks like a closed truncated cone, except that the rim of the cone is not included. To finish things off, the rim is identified with the point (∞, ∞), and then the parametrizing set is closed. It is homeomorphic to a closed ball in $\mathbb{R}^3$ with a smaller open ball tangent to the boundary removed from the interior.

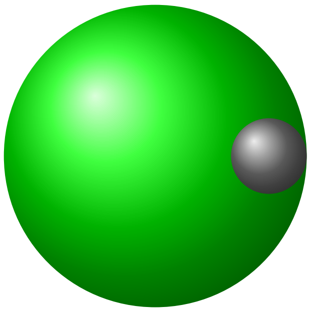

Figure 2.2: Ψ_B when $N = 3$

Chapter 3

Rational dilation on distinguished varieties

3.1 Rational dilation on the distinguished varieties associated to $\mathscr{A}_B$ and $\mathscr{A}_B^0$

Let $\mathbb{D}$ denote the open unit disk in the complex plane and its closure. The disk algebra, $\mathbb{A}(\mathbb{D})$, consists of the functions that are analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ its closure. One should note that by the maximum modulus principle the supremum of such a function over $\overline{\mathbb{D}}$ is attained on $\mathbb{T}$, the boundary of $\mathbb{D}$. Thus, we may regard $\mathbb{A}(\mathbb{D})$ as a closed subalgebra of $C(\mathbb{T})$, the space of continuous function on $\mathbb{T}$ with the supremum norm.

The goal of this section is to study rational dilation problem on the distinguished varieties associated to $\mathscr{A}_B$ (see (3.1)) and $\mathscr{A}_B^0$, where $\mathscr{A}_B^0$ is the subalgebra of $\mathscr{A}_B$, generated by $B(z), zB(z)$.

For $N = t_0 + \cdots + t_n \geq 2$, let

$$B(z) = \prod_{i=0}^{n} \left(\frac{z - \alpha_i}{1 - \overline{\alpha}_i z} \right)^{t_i}$$

be a Blaschke products at the distinct points $\alpha_0, \ldots, \alpha_n$ in the open unit disk $\mathbb{D}$ and $t_0, \ldots, t_n, n$ are non-negative integers.
Let us consider the following subalgebra of the disk algebra,

$$\mathscr{A}_B = \mathbb{C} + B(z)\mathbb{A}(\mathbb{D}). \tag{3.1}$$

We write $\alpha_1, \dots, \alpha_N$ for the zeros of B, including any repeated roots, i.e., the complex numbers $\underbrace{\alpha_0, \dots, \alpha_0}_{t_0}, \dots, \underbrace{\alpha_n, \dots, \alpha_n}_{t_n}$. Hence

$$B(z) = \prod_{j=1}^{N} \frac{z - \alpha_j}{1 - \overline{\alpha_j} z}.$$

A function $f \in \mathscr{A}_B$ if and only if it satisfies the following two constraints

1. $f(\alpha_i) = f(\alpha_j)$ for $0 \leq i, j \leq n$;

2. $f^{(k)}(\alpha_i) = 0$ for $k = 1, \dots, t_i - 1$ whenever $t_i \geq 2$.

Since $\mathbb{A}(\mathbb{D}) \subset H^\infty(\mathbb{D})$, we show the above statement for $H^\infty(\mathbb{D})$. Hence If $f \in H^\infty(\mathbb{D})$ and $f(\alpha_i) = f(\alpha_j)$ for distinct points $\alpha_i, \alpha_j \in \mathbb{D}$, then the function $f - f(\alpha_i)$ vanishes at both α_i and α_j. Hence, $f - f(\alpha_i) = Bg$, where B is the Blaschke product with zeros at α_i, α_j and $g \in H^\infty(\mathbb{D})$. On the other hand, if $f^{(\ell)}(\alpha_k) = 0$ for some $\alpha_k \in \mathbb{D}$ and $\ell = 0, \dots, t - 1$, then $f = Bg$, where $g \in H^\infty(\mathbb{D})$ (if $f(\alpha_k)$ is not zero, then we consider $f - f(\alpha_k)$) and B is the Blaschke product with zero at α_k with multiplicity at least t. Thus, this algebra is of the form $\mathbb{C} + BH^\infty(\mathbb{D})$. (see also [43].) In the following lemma we prove that the algebra $\mathscr{A}_B$ is finitely generated.

Lemma 3.1.1. *Let $N \geq 2$. The functions $f_0 := B(z), f_1 := zB(z), \dots, f_{N-1} := z^{N-1}B(z)$ generate the algebra $\mathscr{A}_B$.*

Proof. Since the polynomials on $\mathbb{D}$ are dense in $\mathbb{A}(\mathbb{D})$, we can see that $\mathbb{A}(\mathbb{D})$ is the closure of analytic polynomials in $\mathbb{C}(\overline{\mathbb{D}})$. Let $P(\mathbb{D})$ be the set of all polynomials on $\mathbb{D}$. We claim that $\mathscr{A}_B = \overline{\mathbb{C} + B(z)P(\mathbb{D})}$. It is straightforward to see that $\mathscr{A}_B \supseteq \overline{\mathbb{C} + B(z)P(\mathbb{D})}$. For the other containment, let $c + B(z)f(z)$ be in $\mathscr{A}_B$. Then $f \in \mathbb{A}(\mathbb{D})$. Hence by Runge's theorem there exists a sequence of polynomials $\{p_j\}_{j \geq 1}$ on $\mathbb{D}$ such that $p_j \to f$, and so $c + Bp_j \to c + Bf$. It follows that $c + Bf \in \overline{\mathbb{C} + B(z)P(\mathbb{D})}$. This completes the proof of the claim. Consequently, to complete the proof, for any $p \in P(\mathbb{D})$ it is enough to show that

$$c + Bp = q(f_0, f_1, \dots, f_{N-1}),$$

where q is a polynomial over $\mathbb{D}^N$. To prove this, observe that

$$f_k = z^k f_0 \quad \text{for} \quad 1 \leq k \leq N - 1, \tag{3.2}$$

and

$$f_k = z^{k-1} f_1 \quad \text{for} \quad 1 \le k \le N-1. \tag{3.3}$$

Since $f_0(z) = B(z)$ we see that

$$(1 - \overline{\alpha_1} z) \cdots (1 - \overline{\alpha_N} z) f_0 = (z - \alpha_1) \cdots (z - \alpha_N)$$

Using the identities (2.8) in the proof of Lemma 2.2.3, we have

$$\sum_{k=0}^{N} \overline{S_k(\alpha)} z^k f_0 = \sum_{k=0}^{N} S_k(\alpha) z^{N-k}, \tag{3.4}$$

where $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{D}^N$. Multiplying both sides of equation (3.4) by f_0 and using (3.2) and (3.3) we get

$$f_0^2 + \sum_{k=1}^{N} \overline{S_k(\alpha)} f_{k-1}(z) f_1(z) = f_N(z) + \sum_{k=1}^{N} S_k(\alpha) f_{N-k}(z), \tag{3.5}$$

where $f_N(z) = z^N f_0(z)$. From (3.5) we find that

$$f_N(z) = f_0^2 + \sum_{k=1}^{N} \left(\overline{S_k(\alpha)} f_{k-1}(z) - S_k(\alpha) f_{N-k}(z) \right). \tag{3.6}$$

Based on the (3.6), since all $f_0, \ldots, f_{N-1}$ are finite Blaschke products on $\mathbb{D}$, we have

$$f_N(z) = P\left(f_0(z), \ldots, f_{N-1}(z)\right), \tag{3.7}$$

where P is a multi-variable polynomial over the polydisk $\mathbb{D}^N$. Thus inductively we see that $z^k f_0(z)$ can be expressed as a polynomial of $f_0(z), \ldots, f_{N-1}(z)$ for $k \ge N$. Hence we conclude that $f_0(z) Q(z) \in \mathscr{A}_B$ can be expressed as a polynomial of $f_0, \ldots, f_{N-1}$, where $Q(z)$ is arbitrary polynomial with $\deg Q \ge N$. This completes the proof.

□

3.2 The distinguished varieties associated to the algebras $\mathscr{A}_B$ and $\mathscr{A}_B^0$

We begin this section with the following definition first given by Jim Agler and John McCarthy [9] and later reformulated by Greg Knese [32]. Note that distinguished varieties go back at least to Rudin's paper [44].

Definition 3.2.1. A non-empty set V in $\overline{\mathbb{D}}^2$ is a **distinguished variety** if there exists a polynomial p in $\mathbb{C}[x, y]$ such that

$$V = \left\{(x, y) \in \overline{\mathbb{D}}^2 : p(x, y) = 0\right\} \tag{3.8}$$

and such that V exits the bidisk through the distinguished boundary:

$$\overline{V} \cap \partial(\overline{\mathbb{D}}^2) = \overline{V} \cap \mathbb{T}^2. \tag{3.9}$$

Here $\overline{V}$ is the closure of V in $\overline{\mathbb{D}^2}$.

The following theorem was proved in [9], with an easier proof given in [32].

Theorem 3.2.2. *Let V be a distinguished variety, defined as the zero set of a polynomial $p \in \mathbb{C}[x, y]$ of minimal degree (n, m). Then, there is an $(m+n)\times(m+n)$ unitary matrix U which we write in block form as*

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathbb{C}^m \oplus \mathbb{C}^n \to: \mathbb{C}^m \oplus \mathbb{C}^n\,,$$

such that

i) D has no unimodular eigenvalues

ii) $p(x, y)$ is a constant multiple of

$$\det \begin{pmatrix} A - yI_m & zB \\ C & xD - I_n \end{pmatrix},$$

and

iii) defining the following rational matrix-valued inner functions:

$$\Psi(x) = A + xB\,(I_n - xD)^{-1}\,C,$$

we have

$$V = \{(x, y) \in \overline{\mathbb{D}}^2 : \det(yI_m - \Psi(x)) = 0\}.$$

Conversely, if Ψ is a matrix-valued rational inner function on $\overline{\mathbb{D}}$, then

$$\{(x, y) \in \overline{\mathbb{D}}^2 : \det(yI_m - \Psi(x)) = 0\}$$

is a distinguished variety.

Note that the roles of x and y can be reversed in the above theorem.
Let us recall the functions $f_0(z) = B(z), f_1(z) = zB(z)$ generate the algebra $\mathscr{A}_B^0$. Observe the following identity,

$$(1 - \overline{\alpha}_0 z)^{t_0} \cdots (1 - \overline{\alpha}_n z)^{t_n} f_0 = (z - \alpha_0)^{t_0} \cdots (z - \alpha_n)^{t_n}. \tag{3.10}$$

Multiplying both sides of the above equation by f_0^N we get

$$(f_0 - \overline{\alpha}_0 f_1)^{t_0} \cdots (f_0 - \overline{\alpha}_n f_1)^{t_n} f_0 = (f_1 - \alpha_1 f_0)^{t_0} \cdots (f_1 - \alpha_n f_0)^{t_n}. \tag{3.11}$$

Since f_0, f_1 are Blaschke products, when we run z over $\mathbb{D}$, the ranges of $f_0(z), f_1(z)$ are in $\mathbb{D}$. Thus we can make the substitution $x = f_0(z), y = f_1(z)$ for $(x, y) \in \mathbb{D}^2$. Then we have

$$(x - \overline{\alpha}_0 y)^{t_0} \cdots (x - \overline{\alpha}_n y)^{t_n} x = (y - \alpha_0 x)^{t_0} \cdots (y - \alpha_n x)^{t_n} \tag{3.12}$$

From (3.12)

$$x = \left(\frac{y/x - \alpha_0}{1 - \overline{\alpha}_0 y/x}\right)^{t_0} \cdots \left(\frac{y/x - \alpha_n}{1 - \overline{\alpha}_n y/x}\right)^{t_n}. \tag{3.13}$$

So if $|x| = 1$ then $|y| = |y/x|$. Taking the modulus at both sides of equation (3.13) we get that

$$1 = \left|\frac{y/x - \alpha_0}{1 - \overline{\alpha}_0 y/x}\right|^{t_0} \cdots \left|\frac{y/x - \alpha_n}{1 - \overline{\alpha}_n y/x}\right|^{t_n}. \tag{3.14}$$

If $|y| = |y/x| < 1$, then the right hand side of (3.14) is strictly less than 1, giving a contradiction. Hence we conclude that $|y| = 1$.
Likewise if $|y| = 1$, then $|x| = |x/y|$. From (3.12) we have

$$\left(\frac{x/y - \alpha_0}{1 - \overline{\alpha}_0 x/y}\right)^{t_0} \cdots \left(\frac{x/y - \alpha_n}{1 - \overline{\alpha}_n x/y}\right)^{t_n} x = 1. \tag{3.15}$$

If $|x| = |x/y| < 1$, then the modulus of the left hand side of (3.15) is strictly less than 1, which is a contradiction. Hence $|x| = 1$.
Consequently, $|x| = 1$ if and only if $|y| = 1$. Therefore the variety

$$\mathscr{N}_B = \left\{(x, y) \in \overline{\mathbb{D}}^2 : P(x, y) = 0\right\} \tag{3.16}$$

is a distinguished variety associated to $\mathscr{A}_B^0$, where

$$P(x, y) = (x - \overline{\alpha}_0 y)^{t_0} \cdots (x - \overline{\alpha}_n y)^{t_n} x - (y - \alpha_0 x)^{t_0} \cdots (y - \alpha_n x)^{t_n}.$$

Note that, when $n = 0, t_0 = 2$ and $\alpha_0 = 0$ we have the well known Neil parabola

$$\mathscr{N}_{z^2} = \left\{(x, y) \in \overline{\mathbb{D}}^2 : x^3 - y^2 = 0\right\}.$$

Recall that we are writing $\alpha_1, \ldots, \alpha_N$ for the zeros of B, including any repeated roots, i.e., the complex numbers $\underbrace{\alpha_0, \ldots, \alpha_0}_{t_0}, \ldots, \underbrace{\alpha_n, \ldots, \alpha_n}_{t_n}$. Hence

$$\mathscr{N}_B = \left\{(x, y) \in \overline{\mathbb{D}}^2 : x \prod_{k=1}^{N} (x - \overline{\alpha}_k y) = \prod_{k=1}^{N} (y - \alpha_k x)\right\}.$$

According to Theorem 3.2.2 we must have a determinantal representation $\mathscr{N}_B$.

By the direct calculation we see that

$$\mathscr{N}_B = \det\left(x \begin{pmatrix} 1 & -\alpha_1 & 0 & \ldots & 0 \\ 0 & 1 & -\alpha_2 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & 1 & -\alpha_N \\ 0 & \ldots & \ldots & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -y & 0 & \ldots & 0 \\ 0 & \overline{\alpha_1} y & -y & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & \overline{\alpha_{N-1}} y & -y \\ (-1)^N & \ldots & \ldots & 0 & \overline{\alpha_N} y \end{pmatrix} \right)$$

$$= \det\left(xI_{N+1} - \begin{pmatrix} 0 & -y & 0 & \ldots & 0 \\ 0 & \overline{\alpha_1} y & -y & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & \overline{\alpha_{N-1}} y & -y \\ (-1)^N & \ldots & \ldots & 0 & \overline{\alpha_N} y \end{pmatrix} S^{-1} \right) \det(S)$$

$$= \det\left(xI_{N+1} - \begin{pmatrix} 0 & -y & 0 & \dots & 0 \\ 0 & \overline{\alpha_1}y & -y & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & \overline{\alpha_{N-1}}y & -y \\ (-1)^N & \dots & \dots & 0 & \overline{\alpha_N}y \end{pmatrix} S^{-1} \right),$$

where

$$S = \begin{pmatrix} 1 & -\alpha_1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -\alpha_2 & & \vdots & \vdots \\ 0 & 0 & 1 & \ddots & & \\ \vdots & \ddots & & \ddots & \ddots & 0 \\ & \dots & & \ddots & \ddots & -\alpha_N \\ 0 & \dots & \dots & & 0 & 1 \end{pmatrix}$$

and

$$S^{-1} = \begin{pmatrix} 1 & \alpha_1 & \alpha_1\alpha_2 & \dots & \alpha_1\cdots\alpha_{N-2} & \alpha_1\cdots\alpha_{N-1} & \alpha_1\cdots\alpha_N \\ 0 & 1 & \alpha_2 & \alpha_2\alpha_3 & \dots & \alpha_2\dots\alpha_{N-1} & \alpha_2\dots\alpha_N \\ 0 & 0 & 1 & \alpha_3 & \alpha_3\alpha_4 & \dots & \alpha_3\dots\alpha_N \\ & \dots & & \ddots & \ddots & \dots & \vdots \\ & \dots & & \ddots & \ddots & \dots & \vdots \\ \vdots & \ddots & & \ddots & \ddots & & \alpha_{N-1}\alpha_N \\ & \dots & & \ddots & 0 & 1 & \alpha_N \\ 0 & \dots & \dots & & 0 & 0 & 1 \end{pmatrix}$$

with $\det(S) = \det(S^{-1}) = 1$. Hence $\mathscr{N}_B = \Big\{(x,y) \in \overline{\mathbb{D}}^2 : \det(xI_{N+1} - \Psi(y)) = 0\Big\}$, where

$$\Psi(y) = \begin{pmatrix} 0 & -y & 0 & \dots & 0 \\ 0 & \overline{\alpha_1}y & -y & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & \overline{\alpha_{N-1}}y & -y \\ (-1)^N & \dots & \dots & 0 & \overline{\alpha_N}y \end{pmatrix} S^{-1}.$$

The geometric characterization of distinguished varieties was studied by Vegulla in [49]. He proved that every bounded planar domain with finitely many piecewise analytic boundary curves is a distinguished variety. Note that by Theorem 3.2.2

every distinguished variety of closed bidsik $\overline{\mathbb{D}}^2$ has a determinantal representation, but for the higher dimensional distinguished variety of the polydisk $\mathbb{D}^d$ not yet known.

Now we turn to describe a variety associated to $\mathscr{A}_B$. Define $x_j = z^{j-1}B(z), j = 1, \dots, N$. Then $(x_1, \dots, x_N) \in \overline{\mathbb{D}}^N$ and by Lemma 3.1.1 $x_1, \dots, x_N$ generates the algebra $\mathscr{A}_B$. Since

$$B(z)^3 \prod_{k=1}^{N}(1 - \overline{\alpha_k} z) = B(z)^2 \prod_{k=1}^{N}(z - \alpha_k),$$

we have

$$\begin{aligned}
&\sum_{k=1}^{N} S_k(\overline{\alpha})[z^{k-1}B(z)][zB(z)]B(z) + B(z)^3 \\
&= \sum_{k=0}^{N-1} S_k(\alpha)[z^{N-k-1}B(z)][zB(z)] + S_N(\alpha)B(z)^2,
\end{aligned}$$

where $\alpha = \mathscr{Z}(B)$. Hence the N-tuple $(x_1, \dots, x_N)$ satisfies the multi-variable polynomial $Q(x_1, \dots, x_N) = 0$, where

$$\begin{aligned}
Q(x_1, \dots, x_N) &= \sum_{k=1}^{N} S_k(\overline{\alpha}) x_k x_2 x_1 + x_1^3 - \left(\sum_{k=0}^{N-1} S_k(\alpha) x_{N-k} x_2 + S_N(\alpha) x_1^2 \right) \\
&= \sum_{k=0}^{N-1} \left(S_{N-k}(\overline{\alpha}) x_1 - S_k(\alpha) \right) x_{N-k} x_2 + x_1^3 - S_N(\alpha) x_1^2.
\end{aligned}$$

The locus described by $Q(x_1, \dots, x_N) = 0$ defines a variety in $\mathbb{C}^N$. Fix $j \in \{1, \dots, N\}$. Since $|x_j| = |z^{j-1}B(z)| = 1$ if and only if $|z| = 1$. Hence $|x_j| = 1$ if and only if $|x_k| = 1$ for all $k = 1, \dots, N$. It follows that this variety intersects the boundary of $\overline{\mathbb{D}}^N$ in $\mathbb{T}^N$, which is the Shilov (or distinguished) boundary of $\overline{\mathbb{D}}^N$. Thus, we conclude that the variety

$$\mathscr{V}_B = \left\{ (x_1, \dots, x_N) \in \overline{\mathbb{D}}^N : Q(x_1, \dots, x_N) = 0 \right\} \tag{3.17}$$

is a distinguished variety in $\overline{\mathbb{D}}^N$, which is associated to $\mathscr{A}_B$.

3.3 Rational dilation on the distinguished varieties $\mathscr{N}_B$ and $\mathscr{V}_B$

The following definitions are taken from Arveson [14].

Definition 3.3.1 ([14]). Let $T = (T_1, \dots, T_d)$ be a d-tuple of commuting bounded operators on a Hilbert space. Then the **joint spectrum** of T, denoted by $\sigma_A(T)$, is the set of all complex d-tuples $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d$ such that $p(\lambda)$ belongs to the spectrum of $p(T)$ for every multivariate polynomial $p \in \mathbb{C}[z_1, \dots, z_d]$.

A more general treatment of this spectrum can be found in [21]. In that paper the joint spectrum $\sigma_A(T)$ is called algebraic joint spectrum (in fact, Curto defined this joint spectrum for unital commutative Banach algebra), see in particular, [21, Proposition 1.2] and [14, Corollary 2]. The algebraic spectrum σ_A of d-tuples T of commuting operators on a Hilbert space is defined as the complement of the set

$$\{\lambda \in \mathbb{C}^d : \exists S = (S_1, \dots, S_d) \in B(\mathcal{H})^d \text{ such that } (T_1 - \lambda_1)S_1 + \cdots + (T_d - \lambda_d)S_d = I\}.$$

There are other notions of the joint spectrum of d-tuples $T = (T_1, \dots, T_d)$, for example the Taylor joint spectrum, see for example [7, 21, 46] and [47]). But in the rest of this thesis, we will only work with the joint spectrum $\sigma_A(T)$ of d-tuples $T = (T_1, \dots, T_d)$ of commuting operators on a Hilbert space.

Definition 3.3.2. Given a set $X \subset \mathbb{C}^d$, a function $f : X \to \mathbb{C}$ is **holomorphic** on X if for every $x \in X$, there is an open neighborhood of x to which f extends analytically.

Formally, a function $f : X \to \mathbb{C}$ is **holomorphic** on a set X in $\mathbb{C}^d$ if, at every point x in X , there is a non-empty ball $B(x, \epsilon)$ centered at x and an analytic mapping of d variables defined on $B(x, \epsilon)$ that agrees with f on $B(x, \epsilon) \cap X$.

Given a compact subset X of $\mathbb{C}^d$, let $\mathcal{R}(X)$ denote the algebra of rational functions with poles off of X with the norm

$$\|r\|_X = \sup_{x \in X} |r(x)|.$$

Theorem 3.3.3 ([14]). *Let $T = (T_1, \dots, T_d)$ be a d-tuple of commuting bounded operators on a Hilbert space and let $X = \sigma_A(T)$. Then* *i) The set X is non-empty;*

ii) $\sigma_A(r(T)) = r(\sigma_A(T))$ for every $r \in \mathcal{R}(X)$.

Definition 3.3.4. A compact set X in $\mathbb{C}^d$ is a **spectral set** for a d-tuple of commuting bounded operators $T = (T_1, \dots, T_d)$ defined on a Hilbert space $\mathcal{H}$ if $\sigma_A(T)$ lies in X and

$$\|r(T)\| \leq \|r\|_X \text{ for all } r \in R(X),$$

where the left hand norm is the usual operator norm (that is, a version of the von Neumann inequality holds).

Definition 3.3.5. A commuting d-tuple of operators T on a Hilbert space $\mathcal{H}$ having X as a spectral set, is said to have a **rational dilation** or **normal ∂X-dilation** if there exist a Hilbert space $\mathcal{K}$, an isometry $V : \mathcal{H} \to \mathcal{K}$ and a d-tuple of commuting normal operators $\mathcal{N}$ on $\mathcal{K}$ with $\sigma_A(\mathcal{N}) \subseteq \partial X$ such that

$$r(T) = V^* r(\mathcal{N}) V \text{ for all } r \in \mathcal{R}(X).$$

Here ∂X denotes the Shilov or distinguished boundary of X (see appendix A.3). The d-tuple $\mathcal{N}$ is referred to as a ***normal boundary dilation***.

Note that we can interpret the von Neumann inequality (or spectral set condition) as saying the T induces a contractive unital representation π_T of $R(\mathscr{N}_B)$ on $\mathcal{H}$ via

$$\pi_T(r) = r(T). \tag{3.18}$$

Theorem 3.3.6 (Arveson[14]). *Let X be a compact subset of $\mathbb{C}^d$ which is a spectral set for a commuting d-tuple of operators T on a Hilbert space $\mathcal{H}$. Then T has a normal boundary dilation if and only if the representation π_r of $\mathcal{R}(X)$ is completely contractive.*

Assume that a normal boundary dilation $\mathcal{N}$ of a commuting d-tuple T exists. Then since $\mathcal{N}_i\mathcal{N}_j = \mathcal{N}_j\mathcal{N}_i$ the Putnam-Fuglede theorem implies that $\mathcal{N}_i^*\mathcal{N}_j = \mathcal{N}_j\mathcal{N}_i^*$ and $\mathcal{N}_j^*\mathcal{N}_i = \mathcal{N}_i\mathcal{N}_j^*$ for $1 \leq i, j \leq d$. It follows that $p(\mathcal{N})$ commutes with $p(\mathcal{N})^*$ for $p \in \mathbb{C}[z_1, \dots, z_d]$. Hence $p(\mathcal{N})$ is a normal operator on $\mathcal{K}$. Consequently, by the spectral mapping theorem

$$\|p(\mathcal{N})\| \leq \sup_{\lambda \in \sigma_A(\mathcal{N})} |p(\lambda)| \text{ for } p \in \mathbb{C}[z_1, \dots, z_d].$$

The assumption implies also that $\sigma_A(\mathcal{N}) \subset \partial X$. Hence

$$\|p(\mathcal{N})\| \leq \sup_{\lambda \in \partial X} |p(\lambda)| = \|p\|_{\partial X} = \|p\|_X \text{ for } p \in \mathbb{C}[z_1, \dots, z_d], \tag{3.19}$$

where the last equality follows by the maximum modulus principle.

Definition 3.3.7 ([12, 21])**.** Let Ω be a compact subset of $\mathbb{C}^d$. The **polynomially convex hull** of Ω is

$$\hat{H}(\Omega) := \{z \in \mathbb{C}^d : |p(z)| \leq \sup_{\Omega} |p(z)| \text{ for all } p \in \mathbb{C}[z_1, \dots, z_d]\}.$$

Note that the set $\hat{H}$ is compact and contains Ω by definition.

Definition 3.3.8 ([12, 21])**.** A compact subset Ω of $\mathbb{C}^d$ is **polynomially convex** if $\hat{H}(\Omega) = \Omega$.

Remark 3.3.9. Showing that a domain in $\mathbb{C}^d$ is polynomially convex is not an easy task. We list here some well known polynomially domains:

- A compact simply connected domain in the complex plane is polynomially convex [12, Lemma 7.2], [29, Lemma 13], [21, Example 1.5];
- The polydisk $\overline{\mathbb{D}}^d$ is polynomially convex [21, Example 1.5], more generally any compact and convex subsets of $\mathbb{C}^d$ is polynomially convex [29, page 67];
- The symmetrized bidisk and tetrablock as defined in (1.3) and (1.4) are polynomially convex [11, Theorem 2.3], [1, Theorem 2.9], respectively.

Theorem 3.3.10 (Oka-Weil [34, Theorem 24.12], [12, Theorem 7.3],[20])**.** *Let X be a compact, polynomially convex set in $\mathbb{C}^d$. Then for every function f holomorphic in some neighborhood of X, we can find a sequence p_j of polynomials in $\mathbb{C}[z_1, \dots, z_d]$ with $p_j \to f$ uniformly on X.*

Remark 3.3.11. Polynomial convexity might only play a role if we were trying to prove a positive result on rational dilation. Hence we do not need have to concern ourselves about the polynomial convexity of the domains we consider.

Under the assumption that existence of normal boundary dilation d-tuple $\mathcal{N}$ of d-tuple T and $X \subset \mathbb{C}^d$ is a polynomially convex domain, the Oka-Weil theorem implies that there exists a sequence p_j of polynomials in $\mathbb{C}[z_1, \ldots, z_d]$ such that $p_j \to f$ uniformly on X.

$$\|r(\mathcal{N})\| = \| \lim_{j\to\infty} p_j(\mathcal{N})\| \le \lim_{j\to\infty} \sup_X |p_j(z)| = \sup_X |r(z)|,$$

where inequality by (3.19). Thus we conclude that X is spectral set for $\mathcal{N}$. Then by the maximum modulus principle we see that

$$\| \Big(r_{ij}(\mathcal{N})\Big)_{i,j}^{k} \| \le \| \Big(r_{ij}\Big)_{i,j}^{k} \|$$

for $r_{ij} \in \mathcal{R}(X)$ and for all $k \in \mathbb{N}$. It follows that if T has a normal boundary dilation, then

$$\|r(T)\| = \|V^* r(\mathcal{N})V\| \le \|r(\mathcal{N})\| \text{ for all } r \in \mathcal{R}(X),$$

so it is also the case that $\|r(T)\| \le \|r\|$ for $r \in \mathcal{R}(X) \otimes M_k(\mathbb{C}), k \in \mathbb{N}$. In other words, when rational dilation holds, contractive representations of $\mathcal{R}(X)$ are completely contractive. A theorem of Arveson's shows the converse is also true [14]. Thus a strategy for showing that rational dilation fails is to find a contractive representation of $\mathcal{R}(X)$ which is not completely contractive.

Our goal in this section is to study the rational dilation problem on the distinguished variety $\mathscr{V}_B$ in $\overline{\mathbb{D}}^N$ for $\deg B = N \ge 2$. In particular when $N = 2$ we show that rational dilation fails for $\mathscr{V}_B = \mathscr{N}_B = \{(x, y) \in \overline{\mathbb{D}}^2 : x(x - \overline{\alpha}y)(x - \overline{\beta}y) = (y - \alpha x)(y - \beta x)\}$, where $\mathcal{Z}(B) = \{\alpha, \beta\}$.

Theorem 3.3.12. *Let $\mathbb{A}(\mathscr{N}_B)$ be the algebra of analytic functions on $\mathscr{N}_B$ which extends continuously to the boundary with the supremum norm. The algebra $\mathbb{A}(\mathscr{N}_B)$ is completely isometrically isomorphic to the algebra $\mathscr{A}_B^0$, which consists of those functions in $\mathscr{A}_B$, which do not have the terms of the form $z^i B^j(z), j = 1, \ldots, N-2$ and $i = j+1, \ldots, N-1$. This algebra contains $B^{N-1}(z)\mathbb{A}(\mathbb{D})$, so in particular, when $N = 2, \mathscr{A}_B^0 = \mathscr{A}_B$.*

Proof. Define a map $\rho : \mathcal{R}(\mathscr{N}_B) \to \mathscr{A}_B^0$ by

$$\rho(p/q) = \frac{p(B(z), zB(z))}{q(B(z), zB(z))} \text{ where } p, q \text{ polynomials}$$

and extending linearly. If it were the case that $q(B(\xi), \xi B(\xi)) = 0$ for some $\xi \in \overline{\mathbb{D}}$, we would have for $(x_0, y_0) = (B(\xi), \xi B(\xi)) \in \mathscr{N}_B$ so that $q(x_0, y_0) = 0$, and so p/q

cannot be in $\mathcal{R}(\mathscr{N}_B)$. So the image of ρ is in $\mathbb{A}(\mathbb{D})$. Hence, about any point in $\mathscr{N}_B \cap \overline{\mathbb{D}}^2$, $1/q(\eta,\zeta)$ has a power series expansion $\sum_{i,j=0} c_{ij}\eta^i\zeta^j$ since it is analytic. Thus $1/q(x,y) = 1/q(B(z), zB(z)) = \sum_{i,j=0} c_{i,j}B(z)^i(zB(z))^j$. So the image of ρ is generated by $B(z), zB(z)$, which equals $\mathscr{A}_B^0$. For $f \in \mathcal{R}(\mathscr{N}_B)$, the maximum modulus principle holds for $\rho(f(x,y)) = f(B(z), zB(z))$. Since $(x,y) \in \mathscr{N}_B \cap \mathbb{T}^2$ if and only if the associated z is in $\mathbb{T}$, f achieves its maximum modulus on $(x,y) \in \mathscr{N}_B \cap \mathbb{T}^2$. Hence the map is isometric. The same reasoning shows that the map is a complete isometry. Since $\mathcal{R}(\mathscr{N}_B)$ is dense in $\mathbb{A}(\mathscr{N}_B)$ the complete isometry extends to a complete isometric homomorphism from $\mathbb{A}(\mathscr{N}_B)$ to $\mathscr{A}_B^0$.

Now we turn to the description of $\mathscr{A}_B^0$. Suppose for the time being that B has three or more zeros, and that there is some $f \in \mathbb{A}(\mathscr{N}_B)$ such that $\rho(f) = z^2B(z) \in \mathscr{A}_B^0$. Then $\rho(xf) = z^2B^2(z) = \rho(y^2)$. Since the map ρ is isometric, this implies that $xf = y^2$ in an open neighborhood U of $(0,0)$. Fix any non-zero complex number t and let $C_t = \{(x,y) \in \mathbb{C}^2 : x = ty^2\}$. For y_0 small enough and non-zero, $(x_0,y_0) \neq (0,0)$ is in $C_t \cap U$. Evaluating at (x_0,y_0) gives $f(x_0,y_0) = t^{-1}$. Hence f cannot be analytic, and so $z^2B(z)$ is not in $\mathscr{A}_B^0$. The same argument shows that any term of the form $z^iB(z)^j, j = 1,\dots,N-2$ and $i = j+1,\dots,N-1$ of $\mathscr{A}_B$ is not in $\mathscr{A}_B^0$. Obviously anything of this form where j is arbitrary and $i \leq j$ can be written as a product of powers of $B(z)$ and $zB(z)$.

Now suppose B has $N \geq 2$ zeros and let $j = N-1$. Then

$$z^NB(z)^{N-1} = \left(\prod_{j=1}^{N}(1-\overline{\alpha_j}z)B(z) - g\right)B(z)^{N-1} = \left(\sum_{j=0}^{N} S_j(\overline{\alpha})z^jB(z) - g\right)B(z)^{N-1},$$

where $\deg g \leq N-1$. All terms have the form $cz^iB(z)^j$, c a constant and $i \leq j$, and hence are in $\mathscr{A}_B^0$. Also,

$$z^{N+k}B(z)^{N-1} = z^k\left(\sum_{0}^{N} S_j(\overline{\alpha})z^jB(z) - g\right)B(z)^{N-1},$$

so by an induction argument, we find that all of these are in $\mathscr{A}_B^0$ as well. Hence, $\mathscr{A}_B^0 \supset B(z)^{N-1}\mathbb{A}(\mathbb{D})$. In particular, if B has only two zeros, $B(z)$ and $zB(z)$ generate the algebra $\mathscr{A}_B$, and in this case ρ is onto. □

Mimicking the proof part i) of Theorem 3.3.12 we have the following result.

Theorem 3.3.13. *The algebra $\mathcal{R}(\mathscr{V}_B)$ is completely isometrically isomorphic to the algebra $\mathscr{A}_B$.*

Proof. Define a map $\rho : \mathcal{R}(\mathscr{V}_B) \to \mathscr{A}_B$ by

$$\rho(p/q) = \frac{p(B(z), zB(z), \dots, z^{N-1}B(z))}{q(B(z), zB(z), \dots, z^{N-1}B(z))} \text{ where } p, q \text{ polynomials}$$

and extend linearly. If it were the case that $q(B(\xi), \xi B(\xi), \dots \xi^{N-1}B(\xi)) = 0$ for some $\xi \in \overline{\mathbb{D}}$, we would have for $(x_1^0, \dots, x_N^0) = (B(\xi), \xi B(\xi), \dots, \xi^{N-1}B(\xi)) \in \mathscr{V}_B$ so that $q(x_1^0, \dots, x_N^0) = 0$, and so p/q cannot be in $\mathcal{R}(\mathscr{V}_B)$. So the image of ρ is in $\mathbb{A}(\mathbb{D})$. Hence, about any point in $\mathscr{V}_B \cap \overline{\mathbb{D}}^N$, $1/q(y_1, \dots, y_N)$ has a power series expansion $\sum_{i_1,\dots,i_N} c_{i_1,\dots,i_N} y_1^{i_1} \dots y_N^{i_N}$ since it is analytic. Thus

$$1/q(x_1, \dots, x_N) = \sum_{i_1,\dots,i_N} c_{i_1,\dots,i_N} (B(z))^{i_1} \dots (z^{N-1}B(z))^{i_N}.$$

So the image of ρ is generated by $B(z), zB(z), \dots, z^{N-1}B(z)$. On the other hand, by Lemma 3.1.1 the algebra $\mathscr{A}_B$ is generated by $B(z), zB(z), \dots, z^{N-1}B(z)$, so the map ρ is onto. For $f \in \mathcal{R}(\mathscr{V}_B)$, the maximum modulus principle holds for

$$\rho(f(x_1, \dots, x_N)) = f(B(z), zB(z), \dots, z^{N-1}B(z)).$$

Since $(x_1, \dots, x_N) \in \mathscr{V}_B \cap \mathbb{T}^N$ if and only if the associated z is in $\mathbb{T}$, f achieves its maximum modulus on $(x, y) \in \mathscr{V}_B \cap \mathbb{T}^N$. Hence the map is isometric. The same reasoning shows that the map is a complete isometry. This completes the proof. $\square$

3.4 Completely contractive representations of $\mathscr{A}_B$

This section inherits much of its structure from section 1 and 2 of [24], and in particular, almost all the results in this section are analogues of results from that paper. Proofs are included for completeness.

A unital representation $\tau : \mathscr{A}_B \to \mathcal{B}(\mathcal{H})$ on $\mathcal{H}$ is **contractive** if $\|\tau(f)\| \leq \|f\|$ for all $f \in \mathscr{A}_B$, where $\|f\|$ represents the norm of f as an element of $C(\overline{\mathbb{D}})$ and $\|\tau(f)\|$ is the operator norm of $\tau(f)$.

The norm $\|F\|$ of an element $F = (f_{j,\ell})$ in $M_n(\mathscr{A}_B)$ is the supremum of the set $\{\|F(z)\| : z \in \mathbb{D}\}$, where $\|F(z)\|$ is the operator norm of the $n \times n$ matrix $F(z)$.

Applying τ to each entry of F,

$$\tau^{(n)}(F) = 1_n \otimes \tau(F) = (\tau(f_{j,\ell}))$$

produces an operator on the Hilbert space $\oplus_1^n \mathcal{H}$ and $\|\tau^{(n)}(F)\|$ is then its operator norm. The mapping τ is **completely contractive** if for each n and $F \in M_n(\mathscr{A}_B)$,

$$\|\tau^{(n)}(F)\| \leq \|F\|.$$

The following theorem is the main result of this section.

Theorem 3.4.1. *The algebra $\mathscr{A}_B$ has a contractive representation which is not completely contractive.*

In fact, we show that there exists a finite dimensional Hilbert space H and a unital contractive representation $\tau : \mathscr{A}_B \to B(H)$ which is not 2 contractive. This is done by showing that $\|\tau^{(2)}(F)\| > 1$ for some rational inner matrix function $F \in M_2(\mathscr{A}_B)$ with $\|F\| \leq 1$.

Consequently, combining Theorem 3.3.13 and Theorem 3.4.1 we have the following failure of rational dilation on $\mathscr{V}_B$ when the Blaschke product has two or more zeros.

Theorem 3.4.2. *Rational dilation fails on the distinguished variety $\mathscr{V}_B$. In particular rational dilation fails for $\mathscr{V}_B = \mathscr{N}_B = \{(x,y) \in \overline{\mathbb{D}}^2 : x(x-\overline{\alpha}y)(x-\overline{\beta}y) = (y-\alpha x)(y-\beta x)\}$, where $\mathcal{Z}(B) = \{\alpha, \beta\}$.*

The following theorem characterizes the completely contractive representations of $\mathscr{A}_B$. The case $B(z) = z^2$ has been proved in [19] and [24]. Mimicking the proof of Theorem 2.1 in [24] we prove the general case.

Theorem 3.4.3. *A unital representation $\pi : \mathscr{A}_B \to \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space, is completely contractive if and only if there is a unitary operator U acting on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that for all $k \in \mathbb{Z}^+$,*

$$\pi\left(z^k B(z)\right) = P_{\mathcal{H}}\left(U^k B(U)\right)|_{\mathcal{H}}, \tag{3.20}$$

where $\mathbb{Z}^+ := \{0, 1, 2, \dots\}$.

Proof. Let $\pi : \mathscr{A}_B \to \mathcal{B}(\mathcal{H})$ be unital, completely contractive representation. Let $\mathscr{A}_B^* \subseteq C(\mathbb{T})$ denote the set of complex conjugates of functions in $\mathscr{A}_B$. Then $\mathscr{A}_B + \mathscr{A}_B^*$ is an operator system and $\rho : \mathscr{A}_B + \mathscr{A}_B^* \to \mathcal{B}(\mathcal{H})$ given by

$$\rho(f + g^*) = \pi(f) + \pi(g)^*$$

is well defined (Proposition 2.12 in [39]). Since π is unital and $\mathscr{A}_B \cap \mathscr{A}_B^* = \mathbb{C}1$, ρ is completely positive (Proposition 3.5 in [39]). By the Arveson extension theorem, ρ extends to a unital, completely positive (ucp) map $\sigma : C(\mathbb{T}) \to \mathcal{B}(\mathcal{H})$. By the Stinespring theorem there is a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$, a unital $*$-homomorphism $\tilde{\sigma} : C(\mathbb{T}) \to \mathcal{B}(\mathcal{K})$, and a bounded operator $V : \mathcal{H} \to \mathcal{K}$ with $\|\sigma(1)\| = \|V\|^2$ such that

$$\sigma(a) = V^* \tilde{\sigma}(a) V.$$

Now since σ is unital and V is isometry, we may identify $V\mathcal{H}$ with $\mathcal{H}$. With this identification, V^* becomes the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$, $P_{\mathcal{H}}$. Setting $\tilde{\sigma}(z) = U$, where z is coordinate function, and since $z\overline{z} = \overline{z}z = 1$ we have that U is unitary and that

$$\sigma(z^k) = P_{\mathcal{H}} \tilde{\sigma}(z^k)|_{\mathcal{H}} = P_{\mathcal{H}} U^k|_{\mathcal{H}} \quad \text{for} \quad \text{all} \quad k \in \mathbb{Z}^+.$$

With this $U \in \mathcal{B}(\mathcal{K})$ for all $k \in \mathbb{Z}^+$ we have

$$\sigma\left(z^k B(z)\right) = P_{\mathcal{H}}\left(U^k B(U)\right)|_{\mathcal{H}}.$$

Since

$$\pi\left(z^k B(z)\right) = \sigma\left(z^k B(z)\right),$$

for all nonnegative integer k, one direction follows.

Conversely, suppose that there is a unitary operator $U \in \mathcal{B}(\mathcal{K})$ such that for all $k \geq 0$,

$$\pi\left(z^k B(z)\right) = P_{\mathcal{H}}\left(U^k B(U)\right)|_{\mathcal{H}},$$

Then $\tilde{\pi}$ defined as

$$\tilde{\pi}(z) = U$$

defines a completely contractive representation of $C(\mathbb{T})$ (hence $\tilde{\pi}\left(z^k B(z)\right) = U^k B(U), k \in \mathbb{Z}^+$). So $\tilde{\pi}$ restricted to the operator system $\mathscr{A}_B \cap \mathscr{A}_B^*$ is completely positive, as is ρ, its compression to $\mathcal{H}$, by the Stinespring dilation theorem. Since unital completely positive maps are completely contractive, $\pi = \rho|_{\mathscr{A}_B}$ is completely contractive. $\square$

By using Theorem 3.3.12, the same arguments work to give a dilation theorem for the algebra $\mathscr{A}_B^0$.

Theorem 3.4.4. *A unital representation $\pi : \mathscr{A}_B^0 \to \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space, is completely contractive if and only if there is a unitary operator U acting on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that for $1 \leq j \leq N-2$ and $1 \leq i \leq j$, and for $j = N-1$ and $i \in \mathbb{N}$,*

$$\pi(z^i B^j) = P_{\mathcal{H}} U^i B(U)^j|_{\mathcal{H}}.$$

As in [24], it happens that even though there is a contraction $T := P_{\mathcal{H}} U|_{\mathcal{H}}$, for neither algebra is it necessarily the case that $\pi(B) = B(T)$ and $\pi(zB) = TB(T)$. To see this, we find the complex annihilator of $\mathscr{A}_B$. Recall that a function f in $\mathscr{A}_B$ can be written as

$$f(z) = c + \prod_{j=1}^{N} (z - \alpha_j) g(z),$$

for some $g \in \mathbb{A}(\mathbb{D})$, where $\alpha_j \in \mathcal{Z}(B)$. By [31, Theorem 1H], the annihilator $(\mathscr{A}_B)^\perp$ of $\mathscr{A}_B$ is isometrically isomorphic to the dual of $\mathbb{A}(\mathbb{D})/\mathscr{A}_B$. On the other hand, the space $\mathbb{A}(\mathbb{D})/\mathscr{A}_B$ is spanned by $z^k + \mathscr{A}_B, k = 1, \ldots, N-1$ and so has dimension $N-1$. So the dimension of $(\mathscr{A}_B)^\perp$ is also $N-1$. The kernel functions

$$k_\alpha^{(i)}(z) = i! \frac{z^i}{(1 - \overline{\alpha} z)^{i+1}} \tag{3.21}$$

have the property that $\langle f(z), k_\alpha^{(i)}(z) \rangle = f^{(i)}(\alpha)$, the i-th dervative of f evaluated at $\alpha \in \mathbb{D}$ (consider the Taylor series of the functions $f(z)$ and $k_\alpha^{(i)}(z)$). So for $0 \leq j \leq n, 1 \leq i \leq t_j - 1$ and $f \in H_B^\infty$,

$$\langle f(z), k_{\alpha_j}^{(i)}(z) \rangle = f^{(i)}(\alpha_j) = 0.$$

This accounts for $\sum_{j=0}^{n} (t_j - 1) = N - (n+1)$ linearly independent functions. Fix α_ℓ. Then for $j = 0, \ldots, n$ and $j \neq \ell$,

$$\langle f(z), k_{\alpha_\ell}^{(0)}(z) - k_{\alpha_j}^{(0)}(z) \rangle = c - c.$$

These n functions along with the previous $N - (n+1)$ functions then form a linearly independent set, and hence a basis for the complex annihilator of $\mathscr{A}_B$. We write $\{g_k\}_{k=1}^{N-1}$ for the list of these functions.

Proposition 3.4.5. *For both $\mathscr{A}_B$ and $\mathscr{A}_B^0$, there is a completely contractive representation π in $\mathcal{B}(\mathcal{H})$ for which there is no operator $T \in \mathcal{B}(\mathcal{H})$ such that $\pi(B) = B(T)$ and $\pi(zB) = TB(T)$.*

Proof. Consider $\mathscr{A}_B$ to begin with. Recall the functions $g_1, \dots, g_{N-1}$ defined in terms of the kernel functions $k_{\alpha_j}^{(i)}$. By definition, $k_{\alpha_j}^{(i)}$ is divisible by z^i (and no higher power of z) and a simple calculation shows that likewise, the functions $k_{\alpha_\ell}^{(0)}(\cdot) - k_{\alpha_j}^{(0)}(\cdot)$, $j \neq \ell$ are divisible by z but no higher power of z. Each g_j is in $H^2(\mathbb{D})$, the functions in $L^2(\mathbb{T})$ (with normalized Lebesgue measure) where the coefficients of z^j are zero when $j < 0$.

Define $\mathcal{H} \subset H^2(\mathbb{D})$ to be the orthogonal complement of the span of g_i, where either $g_i = k_{\alpha_j}^{(1)}$ for some j or $k_{\alpha_\ell}^{(0)}(\cdot) - k_{\alpha_j}^{(0)}(\cdot)$. Since B has degree at least 2, there is always one such g_i. Since ran B is orthogonal to the span of g_i, $\mathcal{H}$ is invariant under multiplication by both B and zB.

Let U be the bilateral shift on $L^2(\mathbb{T})$, which is unitary. Then $\mathcal{H}$ is invariant under both $B(U)$ and $UB(U)$. Hence by Theorem 3.4.3, the representation π of $\mathscr{A}_B$ defined by $\pi(z^jB) = P_{\mathcal{H}} U^j B(U)|_{\mathcal{H}}$, $j \in \mathbb{N}$, is completely contractive.

Furthermore, $U^* g_j \in H^2(\mathbb{D})$, and z does not divide $U^* g_j$. Since each g_j is divisible by z, this implies that $U^* g_j$ is not in the annihilator of $\mathscr{A}_B$.

Suppose that there exists $T \in \mathcal{B}(\mathcal{H})$ such that $\pi(B) = B(T) = B(U)|_{\mathcal{H}}$ and $\pi(zB) = TB(T) = UB(U)|_{\mathcal{H}}$. Since B is inner, both $\pi(B)$ and $\pi(zB)$ are isometries. The quotient space $\hat{\mathcal{H}} = H^2(\mathbb{D})/\bigvee g_i$ is isometrically isomorphic to $\mathcal{H}$. Let q be the quotient map. Since $\mathcal{H}$ is invariant under U, T passes to a contraction operator $\hat{T}$ on the quotient space and $\hat{T}^j B(\hat{T})$ are isometries, $j = 0, 1$. Also, there is an isometry $V : \hat{\mathcal{H}} \to L^2(\mathbb{T})$ such that $\hat{T}^j B(\hat{T}) = V^* U^j B(U) V$, and so this induces a completely contractive representation $\hat{\pi}$ of $\mathscr{A}_B$ into $\mathcal{B}(\hat{\mathcal{H}})$.

Since $U(U^* g_i) = g_i$, $\hat{T} q(U^* g_i) = 0$. As we saw, the map $\hat{T}$ is isometric, and so it follows that $q(U^* g_i) = 0$. But as was noted, $U^* g_i$ is not in the annihilator of $\mathscr{A}_B \supset \bigvee g_i$, so $q(U^* g_i)$ cannot be 0, giving a contradiction.

The representation $\hat{\pi}$ of $\mathscr{A}_B$ constructed above restricts to a completely contractive representation of $\mathscr{A}_B^0$. Since there is no operator $\hat{T}$ such that $\hat{T}^j B(\hat{T})$, $j = 0, 1$, and these latter are in $\mathscr{A}_B^0$, the claim holds for $\mathscr{A}_B^0$ as well. $\square$

3.5 The cone generated by the test functions

Recall that by Corollary 2.3.12 the set of test functions for H_B^∞,

$$\Psi_B = \{\psi_\lambda = B(z)D_\lambda(z) : \lambda = (\lambda_1, \dots, \lambda_{N-1}) \in \mathbb{D}_\infty^{N-1} \text{ with } \lambda_1 \preceq \cdots \preceq \lambda_{N-1}\}, \tag{3.22}$$

where D_λ is the finite Blaschke product with zeros $\lambda_1, \dots, \lambda_{N-1}$. Note that Ψ_B inherits the topology and the Borel structure from $\mathbb{D}_\infty^{N-1}$. Moreover, it is clear that Ψ_B separates the points of $\mathbb{D}$ and $\sup_{\psi_\lambda \in \Psi_B} |\psi_\lambda(z)| < 1$ for all $z \in \mathbb{D}$.

Definition 3.5.1 ([26]). Let X be a compact Hausdorff space. A $k\times k$ matrix-valued measure

$$\mu = (\mu_{i,j})_{i,j=1}^k$$

is a $k \times k$ matrix whose entries $\mu_{i,j}$ are complex valued Borel measures on X. The measure μ is positive ($\mu \geq 0$) if for each function $f : X \to \mathbb{C}^k$,

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix},$$

we have

$$0 \leq \int_X f^* \, d\mu f := \sum_{i,j=1}^k \int_X \overline{f_j} f_i \, d\mu_{i,j}.$$

The positive measure μ is bounded by $C > 0$ if

$$CI_k - (\mu_{i,j}(X)) \geq 0$$

is positive semi-definite, where I_k is the $k \times k$ identity matrix.

Lemma 3.5.2 ([26, Lemma 5.3]). *The $k \times k$ matrix valued measure μ is positive if and only if for each Borel set Ω the $k \times k$ matrix*

$$(\mu_{i,j}(\Omega))_{i,j=1}^k$$

is positive semi-definite.

Further, if there is a κ so that each diagonal entry $\mu_{i,i}(X) \leq \kappa$, then each entry $\mu_{i,j}$ of μ has total mass at most κ. Particularly, if μ is bounded by C, then each entry has total variation at most C.

Lemma 3.5.3 ([26, Lemma 5.4]). *If μ^n is a sequence of positive $k \times k$ matrix-valued measures on X which are all bounded above by C, then μ^n has a weak-$*$ convergent sub-sequence. That is, there exists a positive $k \times k$ matrix-valued measure μ, such that for each pair of continuous functions f, $g : X \to \mathbb{C}^k$,*

$$\int_X g^* \, d\mu^{n_\ell} f = \sum_{i,j} \int_X f_i \overline{g_j} d\mu_{ij}^{n_\ell} \to \sum_{i,j} \int_X f_i \overline{g_j} d\mu_{ij} = \int_X g^* \, d\mu f.$$

Lemma 3.5.4 ([26, Lemma 5.5]). *If μ is a positive $k \times k$ matrix-valued measure on X, then the diagonal entries, μ_{jj} are positive measures. Further, with $\nu = \sum_{j=1}^k \mu_{jj}$, there exists a $k \times k$ matrix-valued function $\Delta : X \to M_k(\mathbb{C})$ so that $\Delta(x)$ is positive semi-definite for each $x \in X$ and $d\mu = \Delta d\nu$; that is, for each pair of continuous functions f, $g : X \to \mathbb{C}^k$,*

$$\sum_{i,j} \int_X \overline{g_i} f_j d\mu_{ij} = \sum_{i,j} \int_X \overline{g_i} \, \Delta_{ij} f_j d\nu \, .$$

Let $M(\Psi_B)$ be the space of finite Borel measures on Ψ_B. For every fixed subset Y of $\mathbb{D}$, we define the set

$$M^+(Y) = \{\mu : Y \times Y \to M(\Psi_B) : \mu \geq 0\}.$$

We write $\mu_{x,y}$ for the value of μ at the pair (x, y). The kernel $\mu \in M^+(Y)$ is positive if for all finite sets $Z \subseteq Y$ and all Borel sets $\Omega \subseteq \Psi_B$, the matrix

$$\mu(\Omega) = (\mu_{x,y}(\Omega))_{x,y \in Z} \tag{3.23}$$

is positive semidefinite.

The following example illustrates what it means for μ to be a positive $M^+(Y)$-valued kernel.

Example 3.5.5. If $\mu_{x,y}$ is identically equal to a fixed positive measure ν, or more generally is of the form $\mu_{x,y} = f(x)f(y)^*\nu$ for a fixed positive measure ν and bounded measurable function $f : \mathbb{D} \to \mathbb{C}$, or more generally still is a finite sum of such terms, then $\mu = (\mu_{x,y})$ is positive.

Proposition 3.5.6. *Let f be a function in $\mathbb{A}(\mathbb{D})$. Then $f \in \mathscr{A}_B$ and $\|f\|_\infty \leq 1$ if* *and only if there is a positive kernel* $\mu \in M^+(\mathbb{D})$ *such that*

$$1 - f(x)f(y)^* = \int_{\Psi_B} (1 - \psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi) \tag{3.24}$$

for all $x, y \in \mathbb{D}$.

Proof. Suppose $f \in \mathscr{A}_B$ and $\|f\|_\infty \leq 1$. Then the function $\tilde{f} = f \circ m_{-\alpha_0}$ is in the algebra $\mathscr{A}_{\tilde{B}}$ and $\|\tilde{f}\|_\infty \leq 1$. Hence by (2.27) (or (2.29)), we have

$$1 - \tilde{f}(x)\tilde{f}(y)^* = \int_{\Theta_{\tilde{B}}} H_\theta(x)\left(1 - \tilde{\psi}_\theta(x)\tilde{\psi}_\theta(y)^*\right) H_\theta(y)^*\, d\tilde{\nu}(\theta). \tag{3.25}$$

Since $\Psi_{\tilde{B}} = \{\tilde{\psi}_\theta : \theta \in \Theta_{\tilde{B}}\}$ we view the measure in (3.25) as being on the set $\Psi_{\tilde{B}}$ in place of the set of extremal measures $\Theta_{\tilde{B}}$, so that

$$1 - \tilde{f}(x)\tilde{f}(y)^* = \int_{\Psi_{\tilde{B}}} H_{\tilde{\psi}}(x)\left(1 - \tilde{\psi}(x)\tilde{\psi}(y)^*\right) H_{\tilde{\psi}}(y)^*\, d\tilde{\nu}(\tilde{\psi}).$$

This is equivalent to

$$1-f(m_{-\alpha_0}(x))f(m_{-\alpha_0}(y))^* = \int_{\Psi_B} H_\psi(x)\left(1 - \psi(m_{-\alpha_0}(x))\psi(m_{-\alpha_0}(y))^*\right) H_\psi(y)^*\, d\nu(\psi),$$

where $\Psi_B = \left\{\psi := \tilde{\psi} \circ m_{\alpha_0} : \tilde{\psi} \in \Psi_{\tilde{B}}\right\}$. Since $m_{-\alpha_0}$ is the automorphism of $\mathbb{D}$, we have

$$1 - f(x)f(y)^* = \int_{\Psi_B} (1 - \psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi), \tag{3.26}$$

where $\mu_{x,y}(\psi) = H_\psi(x)\nu(\psi)H_\psi(y)^*$.

Conversely, suppose $f \in \mathbb{A}(\mathbb{D})$ and there is a positive kernel $\mu \in M^+(\mathbb{D})$ such that (3.24) holds. By the realization theorem (Theorem 1.3.6) we have that $f \in H_1^\infty(\mathcal{K}_{\Psi_B})$. By Corollary 2.3.12 we conclude that $f \in H^\infty_{1,B}$. Hence the assumption $f \in \mathbb{A}(\mathbb{D})$ implies that $f \in \mathscr{A}_B$. This completes the proof. $\square$

3.6 A closed matrix cone and the separation argument

Let $M_2(\mathbb{C})$ denote the 2×2 matrices with entries from $\mathbb{C}$. In order to study the action of representations on $M_2(\mathscr{A}_B)$, we consider a finite subset $S \subseteq \mathbb{D}$.

Let $\mathcal{K}_{2,S}$ denote the set of all kernels $K : S \times S \to M_2(\mathbb{C})$ and $\mathcal{L}_{2,S} = \{F \in \mathcal{K}_{2,S} : F(x,y)^* = F(y,x)\}$ the set of all self-adjoint kernels in $\mathcal{K}_{2,S}$. Finally, write $\mathcal{C}_{2,S}$ for the cone in $\mathcal{L}_{2,S}$ of elements of the form

$$\left(\int_{\Psi_B} (1 - \psi(x)\psi(y)^*) \; d\mu_{x,y}(\psi) \right)_{x,y \in S}, \tag{3.27}$$

where $\mu = (\mu_{x,y}) \in M_2^+(S)$ is a kernel taking its values $\mu_{x,y}$ in the 2×2 matrix valued measure on Ψ_B such that for every Borel subset Ω of Ψ_B the measures

$$\mu(\Omega) = (\mu_{x,y}(\Omega))_{x,y \in S} \tag{3.28}$$

takes positive semidefinite values in $M_s(M_2(\mathbb{C}))$, where s is the cardinality of the set S. Given $f : S \to \mathbb{C}^2$, the kernel $(f(x)f(y)^*)_{x,y \in S}$ is called a *square*.

Lemma 3.6.1 ([24, Lemma 3.3]). *The cone $\mathcal{C}_{2,S}$ is closed and contains the squares.*

Proof. By definition

$$\sup_{\psi \in \Psi_B} |\psi(x)| < 1 \text{ for } x \in S.$$

Hence as S is finite, there exists a $0 < \kappa \leq 1$ such that for all $x \in S$ and $\psi \in \Psi_B$

$$1 - \psi(x)\psi(x)^* \geq \kappa.$$

Consequently, for the kernel

$$K(x,y) = \int_{\Psi_B} (1 - \psi(x)\psi(y)^*) \; d\mu_{x,y}(\psi) \in \mathcal{C}_{2,S}$$

we have

$$\frac{1}{\kappa} K(x,x) \succeq \mu_{x,x}(\Psi_B),$$

where the inequality is in the sense of the positive semidefinite matrices in $M_2(\mathbb{C})$.

Let $\{K_n\}_{n \geq 1}$ be a sequence in $\mathcal{C}_{2,S}$ which converges to some K. So for each n there is a positive measure μ^n such that

$$K_n(x,y) = \int_{\Psi_B} (1 - \psi(x)\psi(y)^*) \; d\mu^n_{x,y}(\psi).$$

Hence there exists a $\tilde{\kappa} > 0$ such that $\tilde{\kappa} \cdot I_2 \geq \Gamma_n(x,x)$ for all n and all $x \in S$ (because

S is finite), and so

$$\frac{\tilde{\kappa}}{\kappa} \cdot I_2 \geq \mu^n_{x,x}(\Psi_B)$$

for all n and all $x \in S$. By Lemma 3.5.2 we see that positivity of the μ^ns implies that the measures $\mu^n_{x,y}$ are uniformly bounded. Hence by Lemma 3.5.3 there exists a subsequence μ^{n_j} and a measure μ such that μ^{n_j} converges weak-$*$ to μ, which therefore is positive. Thus the positive kernel K is given by

$$K(x,y) = \int_{\Psi_B} (1 - \psi(x)\psi(y)^*) \; d\mu_{x,y}(\psi)$$

is in $\mathcal{C}_{2,S}$. We conclude that $\mathcal{C}_{2,S}$ is a closed cone.

Fix a test function ψ_0 in Ψ_B. Let $f : S \to \mathbb{C}^2$ be given. Let δ_0 denote the unit scalar point mass at ψ_0. Then

$$\mu_{x,y}(\Omega) = f(x)\frac{1}{1 - \psi_0(x)\psi_0(y)^*}\delta_0(\Omega)f(y)^*,$$

defines a positive $M_s(\mathbb{C})$-valued measure, where Ω is a Borel subset of Ψ_B. Thus

$$\left(f(x)f(y)^*\right) = \left(\int_{\Psi_B} (1 - \psi_0(x)\psi_0(y)^*) \; d\mu_{x,y}(\psi)\right) \in \mathcal{C}_{2,S}.$$

$\square$

Proposition 3.6.2. *If $a \in \mathscr{A}_B$ is analytic in a neighborhood of the closure of the open unit disk with $\|a\|_\infty \leq 1$ and $f : S \to \mathbb{C}^2$, then*

$$\left(f(x)\left(1 - a(x)a(y)^*\right)f(y)^*\right)_{x,y \in S} \in \mathcal{C}_{2,S}.$$

Proof. By assumption and Proposition 3.5.6 there exist a positive kernel $\nu \in M^+(\mathbb{D})$ such that

$$1 - a(x)a(y)^* = \int_{\Psi_B} (1 - \psi(x)\psi(y)^*) \; d\nu_{x,y}(\psi) \tag{3.29}$$

for all $x, y \in \mathbb{D}$. Since $\nu \in M^+(\mathbb{D})$, we have

$$\nu(\Omega) = \left(\nu_{x,y}(\Omega)\right)_{x,y \in S}$$

is positive semidefinite for every Borel subset Ω of Ψ_B, and each $\nu_{x,y}$ is a scalar valued measure. Then, since $f : S \to \mathbb{C}^2$, for each Ω a Borel subset of Ψ_B,

$$\mu(\Omega) := \left(\mu_{x,y}(\Omega)\right)_{x,y\in S} = \left(f(x)\nu_{x,y}(\Omega)f(y)^*\right)_{x,y\in S}$$

defines an $M_s(M_2(\mathbb{C}))$ -valued positive measure. Thus, by (3.29) we conclude that

$$\left(f(x)\left(1 - a(x)a(y)^*\right)f(y)^*\right)_{x,y\in S} = \left(\int_{\Psi_B} \left(1 - \psi(x)\psi(y)^*\right) \, d\mu_{x,y}(\psi)\right)_{x,y\in S} \in \mathcal{C}_{2,S}.$$

□

Let I_B denote the ideal of functions in $\mathscr{A}_B$ which vanish on S; i.e.,

$$I_B = \{\varphi \in \mathscr{A}_B : \varphi(x) = 0 \text{ for all } x \in S\}.$$

Since S is finite set, the ideal I_B is closed. The canonical quotient map $q : \mathscr{A}_B \to \mathscr{A}_B/I_B$ is completely contractive. We denote by F^t the transpose of the matrix function $F \in M_2(\mathscr{A}_B)$. Thus , $F^t(z) = F(z)^t$. Clearly, $F^t \in M_2(\mathscr{A}_B)$ and $\|F\|_\infty = \|F^t\|_\infty$ whenever $F \in M_2(\mathscr{A}_B)$. Given $F \in M_2(\mathscr{A}_B)$, let $\Delta_{F,S}$ denote the kernel

$$\Delta_{F,S} = (I_2 - F(x)F(y)^*)_{x,y\in S}. \tag{3.30}$$

Proposition 3.6.3 ([24, Proposition 3.5])**.** *Let $q : \mathscr{A}_B \to \mathscr{A}_B/I_B$ be the canonical quotient map. If $F \in M_2(\mathscr{A}_B)$ and $\|F\|_\infty \leq 1$, but $\Delta_{F,S} \notin \mathcal{C}_{2,S}$, then there exists a Hilbert space $\mathcal{H}$ and representation $\tau : \mathscr{A}_B/I_B \to B(\mathcal{H})$ such that for all $a \in \mathscr{A}_B$,*

(i) $\|\tau(q(a))\| \leq 1$ *whenever* $\|a\| \leq 1$*; but*

(ii) $\|\tau^{(2)}(q^{(2)}(F^t))\| > 1$.

Therefore the representation $\tau \circ q$ is contractive, but not completely contractive.

Proof.

(i) We use the Hahn-Banach cone separation with a GNS construction to get a linear functional that separates $\Delta_{F,S}$ from $\mathcal{C}_{2,S}$.

By Lemma 3.6.1 the cone $\mathcal{C}_{2,S}$ is closed and by assumption $\Delta_{F,S} \notin \mathcal{C}_{2,S}$. Hence by the separation theorem there exists a nonconstant $\mathbb{R}$-linear functional $\Lambda : \mathcal{L}_{2,S} \to \mathbb{R}$ such that $\Lambda(\mathcal{C}_{2,S}) \geq 0$, but $\Lambda(\Delta_{F,S}) < 0$. By Lemma 3.6.1 for given $f : S \to \mathbb{C}^2$ the square $ff^* := (f(x)f(y)^*)_{x,y\in S}$ is in the cone $\mathcal{C}_{2,S}$. Thus $\Lambda(ff^*) \geq 0$. For any kernel K in $\mathcal{K}_{2,S}$, there exist unique kernels $U_K, V_K \in \mathcal{L}_{2,S}$ such that $K = U_K + iV_K$, where

$$U_K = \frac{1}{2}(K + K^*), V_K = \frac{1}{2i}(K - K^*)$$

So there exists a unique $L : \mathcal{K}_{2,S} \to \mathbb{C}$ linearly extending Λ. Let $\mathcal{H}$ denote the Hilbert space obtained by giving $(\mathbb{C}^2)^S$ the pre-inner product

$$\langle f, g\rangle = L(fg^*)$$

and passing to the quotient by the space of null vectors (those f for which $L(ff^*) = 0$). Since S is finite, the quotient will be complete.
Define a representation ρ of $\mathscr{A}_B$ on $B(\mathcal{H})$ by

$$\rho(a)f(x) = f(x)a(x), \tag{3.31}$$

where the scalar valued a multiplies the vector valued f entrywise.

Indeed ρ is unital homomorphism, since

$$\begin{aligned}
\rho(1)f(x) &= f(x)\cdot 1 = f(x),\\
\rho(a+b)f(x) &= f(x)(a(x)+b(x)) = f(x)a(x) + f(x)b(x) = \rho(a)f(x) + \rho(b)f(x),\\
\rho(ab)f(x) &= f(x)a(x)b(x) = \rho(a)f(x)b(x) = \rho(a)\rho(b)f(x).
\end{aligned}$$

We also have that ρ is $*$-homomorphism, because of finiteness of S implies that

$$\begin{aligned}
\langle \rho(a^*)f, g\rangle &= \langle f(x)a^*, g\rangle\\
&= L(fa^*g^*) = L(f(ga)^*)\\
&= \langle f, ga\rangle = \langle f, \rho(a)g\rangle\\
&= \langle \rho(a)^* f, g\rangle.
\end{aligned}$$

If $a \in \mathscr{A}_B$, is analytic in a neighborhood of closed unit disk and $\|a\|_\infty \leq 1$, then by Proposition 3.6.2, $(f(x)(1 - a(x)a(y)^*)f(y)^*)_{x,y\in S} \in \mathcal{C}_{2,S}$. Thus,

$$\langle f, f\rangle - \langle \rho(a)f, \rho(a)f\rangle = L\left((f(x)(1-a(x)a(y)^*)f(y)^*)_{x,y\in S}\right) \geq 0. \tag{3.32}$$

Hence, if $\|a\|_\infty \leq 1$, then $\|\rho(a)\| \leq 1$. That is, ρ is a contractive representation of

$\mathscr{A}_B$. Since the definition of ρ depends only on the values of a on S, if $a \in I_B$, then

$$\rho(a)f(x) = f(x)a(x) = f(x) \cdot 0 = 0.$$

Thus, $\rho(a) = 0$ whenever $a \in I_B$. Hence $I_B \subseteq \ker \rho$. By [18, Theorem 2.3.5] ρ descends to a contractive representation $\tau : A/I_B \to B(\mathcal{H})$ given by

$$\rho = \tau \circ q.$$

This completes the proof of (i).

(ii) Let $\{e_1, e_2\}$ denote the standard basis for $\mathbb{C}^2$ and let $[e_j] : S \to \mathbb{C}^2$ be the constant function $[e_j](x) = e_j$. Note that $\{e_i e_j^*\}_{i,j=1}^2$ are a system of 2×2 matrix units. We find

$$\rho^{(2)}(F^t)([e_1] \oplus [e_2]) = \begin{pmatrix} \rho(F_{1,1})e_1 + \rho(F_{2,1})e_2 \\ \rho(F_{1,2})e_1 + \rho(F_{2,2})e_2 \end{pmatrix}.$$

Since

$$\begin{aligned} & (\rho(F_{1,1})e_1 + \rho(F_{2,1})e_2)\,(\rho(F_{1,1})e_1 + \rho(F_{2,1})e_2)^* \\ = \;& \rho(F_{1,1}F_{1,1}^*)e_1e_1^* + \rho(F_{2,1}F_{1,1}^*)e_2e_1^* + \rho(F_{1,1}F_{2,1}^*)e_1e_2^* + \rho(F_{2,1}F_{2,1}^*)e_2e_2^* \\ = \;& \begin{pmatrix} \rho(F_{1,1}F_{1,1}^*) & \rho(F_{1,1}F_{2,1}^*) \\ \rho(F_{2,1}F_{1,1}^*) & \rho(F_{2,1}F_{2,1}^*) \end{pmatrix}. \end{aligned}$$

and

$$\begin{aligned} & (\rho(F_{1,2})e_1 + \rho(F_{2,2})e_2)(\rho(F_{1,2})e_1 + \rho(F_{2,2})e_2)^* \\ = \;& \rho(F_{1,2}F_{1,2}^*)e_1e_1^* + \rho(F_{2,2}F_{1,2}^*)e_2e_1^* + \rho(F_{1,2}F_{2,2}^*)e_1e_2^* + \rho(F_{2,2}F_{2,2}^*)e_2e_2^* \\ = \;& \begin{pmatrix} \rho(F_{1,2}F_{1,2}^*) & \rho(F_{1,2}F_{2,2}^*) \\ \rho(F_{2,2}F_{1,2}^*) & \rho(F_{2,2}F_{2,2}^*) \end{pmatrix}, \end{aligned}$$

it follows that

$$\begin{aligned} \left\langle \rho^{(2)}(F^t)([e_1] \oplus [e_2]), \rho^{(2)}(F^t)([e_1] \oplus [e_2]) \right\rangle &= L\left(\begin{pmatrix} F_{1,1}F_{1,1}^* + F_{1,2}F_{1,2}^* & F_{1,1}F_{2,1}^* + F_{1,2}F_{2,2}^* \\ F_{2,1}F_{1,1}^* + F_{2,2}F_{1,2}^* & F_{2,1}F_{2,1}^* + F_{2,2}F_{2,2}^* \end{pmatrix} \right) \\ &= L(FF^*), \end{aligned}$$

Since $L(\varDelta_{F,S}) < 0$,

$$\left\langle \left(I_2 - \rho^{(2)}\left(F^t\right)\rho^{(2)}\left(F^t\right)^*\right)[e_1] \oplus [e_2], [e_1] \oplus [e_2] \right\rangle < 0.$$

We conclude that $\|\rho^{(2)}(F^t)\| > 1$. Since $\|F\|_\infty \leq 1$ and q is completely contractive , the representation ρ is not 2-contractive. Thus $\rho = \tau \circ q$ is contractive but not completely contractive. $\square$

3.7 Preliminary results

Lemma 3.7.1 ([24, Lemma 4.2])**.** *Let X be a set and $\varSigma$ a σ-algebra over X. Suppose $\mu_{i,j}$ are 2×2 matrix-valued measures on the measure space $(X, \varSigma)$ for $i, j = 0, 1$. If $\mu_{i,j}(X) = I_2$ for all i, j and if for each $\varOmega \in \varSigma$ the 4×4 matrix-valued measure (block 2×2 matrix with 2×2 matrix entries)*

$$(\mu_{i,j}(\varOmega))_{i,j=0}^{1}$$

is positive semidefinite, then $\mu_{i,j} = \mu_{0,0}$ for each $i, j = 0, 1$.

Lemma 3.7.2. *Let $U \in M_2(\mathbb{C})$ be a unitary matrix. Given distinct points $p_1, p_2 \in \mathbb{D} \setminus \{0\}$, let B_1, B_2 be finite Blaschke products such that $p_1 \in \mathcal{Z}(B_1) \setminus \mathcal{Z}(B_2)$ and $p_2 \in \mathcal{Z}(B_2) \setminus \mathcal{Z}(B_1)$. Let*

$$G = \begin{pmatrix} B_1 & 0 \\ 0 & 1 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & B_2 \end{pmatrix} \tag{3.33}$$

Then there exists unimodular constants s and t such that

$$G = \begin{pmatrix} sB_1 & 0 \\ 0 & tB_2 \end{pmatrix} \tag{3.34}$$

if and only if there exists unitaries V and W in $M_2(\mathbb{C})$ such that

$$G = V \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} W^*. \tag{3.35}$$

Proof. Since

$$\begin{pmatrix} sB_1 & 0 \\ 0 & tB_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$$

choosing $V = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$, $W^* = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$ yields the forward implication.

For the converse, evaluating at p_2 gives

$$G(p_2) = V \begin{pmatrix} B_1(p_2) & 0 \\ 0 & 0 \end{pmatrix} W^* = \begin{pmatrix} B_1(p_2) & 0 \\ 0 & 1 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since V is unitary we have that

$$\begin{pmatrix} B_1(p_2) & 0 \\ 0 & 0 \end{pmatrix} W^* = V^* \begin{pmatrix} B_1(p_2) & 0 \\ 0 & 1 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.36}$$

Assume $W = \begin{pmatrix} w_{11} & w_{21} \\ w_{21} & w_{22} \end{pmatrix}, V = \begin{pmatrix} v_{11} & v_{21} \\ v_{21} & v_{22} \end{pmatrix}, U = \begin{pmatrix} u_{11} & u_{21} \\ u_{21} & u_{22} \end{pmatrix} \in M_2(\mathbb{C})$. Then (3.36) becomes

$$\begin{pmatrix} B_1(p_2) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{w}_{11} & \overline{w}_{21} \\ \overline{w}_{12} & \overline{w}_{22} \end{pmatrix} = \begin{pmatrix} \overline{v}_{11} & \overline{v}_{21} \\ \overline{v}_{12} & \overline{v}_{22} \end{pmatrix} \begin{pmatrix} B_1(p_2) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{21} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This simplifies to

$$\begin{pmatrix} B_1(p_2)\overline{w}_{11} & B_1(p_2)\overline{w}_{21} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B_1(p_2)\overline{v}_{11}u_{11} + \overline{v}_{21}u_{21} & 0 \\ B_1(p_2)\overline{v}_{22}u_{11} + \overline{v}_{22}u_{22} & 0 \end{pmatrix}.$$

Since $B_1(p_2) \neq 0$, we have $w_{21} = 0$. It follows that

$$W^*W = \begin{pmatrix} |w_{11}|^2 & \overline{w}_{11}w_{12} \\ \overline{w}_{12}w_{11} & |w_{12}|^2 + |w_{22}|^2. \end{pmatrix}$$

On the other hand, W is unitary, so $|w_{11}|^2 = 1$. Hence $w_{11} \neq 0$. This imply that $w_{12} = 0$. So W is diagonal matrix with entries $w_{11}, w_{22} \in \mathbb{T}$. A similar argument shows that V is diagonal matrix with entries $v_{11}, v_{22} \in \mathbb{T}$. Finally, with the choice $s = v_{11}\overline{w_{11}}$ and $t = v_{22}\overline{w_{22}}$ gives the desired result. □

Lemma 3.7.3 ([24, Lemma 4.3])**.** *Let $U \in M_2(\mathbb{C})$ be a unitary matrix with non-zero* *entries. Given distinct points $p_1, p_2 \in \mathbb{D} \setminus \{0\}$, let k_{p_1} and k_{p_2} be the Szegő kernels. If G is given as in (3.33) with $B_j(z) = m_{p_j}(z)$ for $j = 1, 2$, the Möbius map at the points p_1, p_2, then there exists linearly independent vectors $v_1, v_2 \in \mathbb{C}^2$ and, for any finite subset S of $\mathbb{D}$, functions $a, b : S \to \mathbb{C}^2$ in the span of $\{k_{p_1}(x)v_1, k_{p_2}(x)v_2\}$ such that*

$$\frac{I_2 - G(x)G(y)^*}{1 - xy^*} = a(x)a(y)^* + b(x)b(y)^*. \tag{3.37}$$

Proof. Let e_1, e_2 denote the standard basis for $\mathbb{C}^2$ and let M_G denote the operator multiplication by G on $H^2_{\mathbb{C}^2}$, the Hardy space of $\mathbb{C}^2$-valued functions on the disk. First, we claim that the operator M_G is an isometry on $\mathbb{T}$. To prove this, we need to show $\langle M_G f, M_G f\rangle = \langle f, f\rangle$ for $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in H^2_{\mathbb{C}^2}$. Since

$$\|M_G f\|^2 = \lim_{r\to 1} \int_{\mathbb{T}} \|G(r\lambda) f(r\lambda)\|^2 \, d\sigma = \lim_{r\to 1} \int_{\mathbb{T}} \|f(r\lambda)\|^2 \, d\sigma = \|f\|^2,$$

and so the claim is proved.

It is well known that $M_G^* k_\lambda v = G^*(\lambda) k_\lambda v$ for $v \in \mathbb{C}^2$ (see section A.2 in the Appendix). Hence the first and third factors in $G^*(\lambda)$ have one dimensional kernels. Since U^* is unitary in $M_2(\mathbb{C})$, it has zero kernel. Thus we conclude that the dimension of the kernel of M_G^* is at most two.

Observe that for $v_1 = e_1$,

$$\begin{aligned} M_G^* k_{p_1} e_1 = G^*(p_1) k_{p_1} e_1 &= \begin{pmatrix} 1 & 0 \\ 0 & \overline{m}_{p_2}(p_1) \end{pmatrix} U^* \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k_{p_1} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \overline{m}_{p_2}(p_1) \end{pmatrix} U^* \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Thus $k_{p_1} v_1$ is in the kernel of M_G^*.

Choose a unit vector v_2 in $\mathbb{C}^2$ with entries α and $\beta \neq 0$ such that

$$\begin{pmatrix} \overline{m}_{p_1}(p_2) & 0 \\ 0 & 1 \end{pmatrix} v_2 = \begin{pmatrix} \overline{m}_{p_1}(p_2) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \overline{m}_{p_1}(p_2) \\ \beta \end{pmatrix} = U e_2.$$

That such a choice of α and $\beta \neq 0$ is possible follows from the assumption that $p_1 \neq p_2$, which ensures that $\overline{m}_{p_1}(p_2) \neq 0$, and the assumption that U has no nonzero entries, giving $\beta \neq 0$. Further, with this choice of v_2 we have that

$$M_G^* k_{p_2} v_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U^* \begin{pmatrix} \overline{m}_{p_1}(p_2) & 0 \\ 0 & 1 \end{pmatrix} v_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U^* U e_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which shows that $k_{p_2} v_2$ is in the kernel of M_G^*. Hence, the dimension of the kernel of M_G^* is two. Since M_G is isometry, $I - M_G M_G^*$ is the projection onto the kernel of M_G^*.

Choose an orthonormal basis $\{a, b\}$ for the kernel of M_G^* so that

$$I - M_G M_G^* = aa^* + bb^*.$$

It now follows that for vectors $u, w \in \mathbb{C}^2$,

$$\begin{aligned}
\left\langle \frac{I_2 - G(x)G(y)^*}{1 - xy^*} u, w \right\rangle_{H^2_{\mathbb{C}^2}} &= \langle (I_2 - G(x)G(y)^*) k_y(x) u, w \rangle \\
&= \langle k_y(x) u, w \rangle - \langle G(x)G(y)^* k_y(x) u, w \rangle \\
&= \langle k_y u, k_x w \rangle - \langle G(x)G(y)^* k_y u, k_x w \rangle \\
&= \langle k_y u, k_x w \rangle - \langle G(y)^* k_y u, G(x)^* k_x w \rangle \\
&= \langle k_y u, k_x w \rangle - \langle M_G^* k_y u, M_G^* k_x w \rangle \\
&= \langle (I - M_G M_G^*) k_y u, k_x w \rangle_{H^2_{\mathbb{C}^2}} \\
&= \langle (aa^* + bb^*) k_y u, k_x w \rangle .
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\langle (aa^* + bb^*) k_y u, k_x w \rangle &= \langle aa^* k_y u, k_x w \rangle + \langle bb^* k_y u, k_x w \rangle \\
&= \langle a \langle k_y u, a \rangle, k_x w \rangle + \langle b \langle k_y u, b \rangle, k_x w \rangle \\
&= \langle k_y u, a \rangle \langle a, k_x w \rangle + \langle k_y u, b \rangle \langle b, k_x w \rangle \qquad (3.38) \\
&= \overline{\langle a, k_y u \rangle} \langle a, k_x w \rangle + \overline{\langle b, k_y u \rangle} \langle b, k_x w \rangle \\
&= \overline{\langle a(y), u \rangle} \langle a(x), w \rangle + \overline{\langle b(y), u \rangle} \langle b(x), w \rangle .
\end{aligned}$$

Since $a(x), a(y), b(x), b(y), u, w \in \mathbb{C}^2$, we have

$$\begin{aligned} \overline{\langle a(y), u\rangle}\,\langle a(x), w\rangle &= (\overline{a_1(y)}u_1 + \overline{a_2(y)}u_2)(a_1(x)\overline{w}_1 + a_2(x)\overline{w}_2) \\ &= \left\langle \begin{pmatrix} a_1(x)\overline{a_1(y)} & a_1(x)\overline{a_2(y)} \\ a_2(x)\overline{a_1(y)} & a_2(x)\overline{a_2(y)} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle \\ &= \langle a(x)a(y)^* u, w\rangle. \end{aligned} \tag{3.39}$$

and likewise

$$\overline{\langle b(y), u\rangle}\,\langle b(x), w\rangle = \langle b(x)b(y)^* u, w\rangle. \tag{3.40}$$

Hence by (3.38), (3.39) and (3.40),

$$\left\langle \frac{I_2 - G(x)G(y)^*}{1 - xy^*} u, w \right\rangle_{H^2_{\mathbb{C}^2}} = \langle (a(x)a(y)^* + b(x)b^*(y))\, u, w\rangle.$$

This completes the proof. $\square$

3.8 Construction of the counterexample

Let m_ζ be the Möbius map on $\zeta \in \mathbb{D}$. Fix distinct points $p_1, p_2 \in \mathbb{D} \setminus \mathcal{Z}(B)$, where $\mathcal{Z}(B) = \{\alpha_1, \ldots, \alpha_N\}$ the zero set of $B(z)$. Fix a finite subset S of $\mathbb{D}$ containing $p_1, p_2, \mathcal{Z}(B)$ and consisting of at least $2N + 4$ distinct points. Recall that the set of test functions Ψ_B is given by

$$\Psi_B = \{\psi_\lambda(z) = B(z)D_\lambda(z) : \lambda = (\lambda_1, \ldots, \lambda_{N-1}) \in \mathbb{D}_\infty^{N-1}\},$$

with the ordering $\lambda_1 \preceq \cdots \preceq \lambda_{N-1}$, where $D_\lambda(z) = \prod_{j=1}^{N-1} m_{\lambda_j}(z)$. Recall also that we take $m_\infty(z) = 1$ in Ψ_B for all $\lambda_j \in \mathbb{D}_\infty$. Fix $j \in 1, \ldots, N-1$. Then the above ordering implies that if $\lambda_j = \infty$, then $\lambda_k = \infty$ for all $k = j+1, \ldots, N-1$. So in the rest of this section we fix the following notation

$$\psi^0(z) := \psi_{(\infty,\ldots,\infty)}(z) = B(z) \tag{3.41}$$

and

$$\psi^i_\lambda(z) := \psi_{(\lambda_1,\ldots,\lambda_i,\infty,\ldots,\infty)}(z), \tag{3.42}$$

where $\lambda_1, \ldots, \lambda_i \in \mathbb{D}$ for $i = 1, \ldots, N-2$. Also, we write ∞^{N-1} for the $(N-1)$-tuple $(\infty, \ldots, \infty)$ in $\mathbb{D}_\infty^{N-1}$.

Let

$$\Pi = \begin{pmatrix} m_{p_1} & 0 \\ 0 & 1 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & m_{p_2} \end{pmatrix}, \tag{3.43}$$

where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

In particular Π is a 2×2 matrix inner function with $\det \Pi(z) = 0$ at the points p_1, p_2 and degree at most 2.

Let

$$F(z) = B(z)\Pi(z), \tag{3.44}$$

where F is a rational inner function in $M_2(\mathscr{A}_B)$ with $\|F\|_\infty = 1$.

Theorem 3.8.1. *Let $F = (F_{i,j})_{i,j=1}^2 \in M_2(\mathscr{A}_B)$ be defined as in (3.44). Then $\Delta_{F,S} \notin \mathcal{C}_{2,S}$.*

Consequently this will establish the proofs of and Theorem 3.4.1 and Theorem 3.4.2.

The proof of Theorem 3.4.1. By Theorem 3.8.1, we have that $\Delta_{F,S} \notin \mathcal{C}_{2,S}$. Hence by Proposition 3.6.3 there exists a contractive representation of $\mathscr{A}_B$, which is not completely contractive. □

The proof of Theorem 3.4.2. By Theorem 3.4.1, there exists a contractive representation of $\mathscr{A}_B$, which is not completely contractive. Then by Theorem 3.3.13 there exists a contractive representation of $\mathcal{R}(\mathscr{V}_B)$, which not completely contractive. Hence by Theorem 3.3.6, rational dilation fails for the N- distinguished variety $\mathscr{V}_B$. This completes the proof. □

The proof of Theorem 3.8.1 goes by a contradiction. More precisely, we assume that $\Delta_{F,S}$ lies in the cone $\mathcal{C}_{2,S}$; that is, there exists an $M_2(\mathbb{C})$-valued positive measure μ such that

$$I_2 - F(x)F(y)^* = \int_{\Psi_B} (1 - \psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi) \quad \text{for} \quad x, y \in S, \tag{3.45}$$

for $x, y \in S$. We will restrict the measures $\mu_{x,y}$ in (3.45) by a sequence of lemmas.

Multiplying (3.45) by the Szegő kernel $s(x,y) = (1-xy^*)^{-1}$ we get

$$\left(\frac{I_2 - F(x)F(y)^*}{1-xy^*}\right)_{x,y\in S} = \left(\int_{\Psi_B}\left(\frac{1-\psi(x)\psi(y)^*}{1-xy^*}\right)d\mu_{x,y}(\psi)\right)_{x,y\in S}. \tag{3.46}$$

Since $F(x) = B(x)\Pi(x)$, then

$$\frac{I_2 - F(x)F(y)^*}{1-xy^*} = \frac{1-B(x)B(y)^*}{1-xy^*}I_2 + B(x)B(y)^*\left(\frac{I_2 - \Pi(x)\Pi(y)^*}{1-xy^*}\right). \tag{3.47}$$

Similarly, for the test functions $\psi_\lambda(x) = B(x)D_\lambda(x)$ for $\lambda \in \mathbb{D}_\infty^{N-1}$, we have that

$$\frac{1-\psi_\lambda(x)\psi_\lambda(y)^*}{1-xy^*} = \frac{1-B(x)B(y)^*}{1-xy^*} + B(x)B(y)^*\left(\frac{1-D_\lambda(x)D_\lambda(y)^*}{1-xy^*}\right). \tag{3.48}$$

Let

$$k_\zeta(x) = \frac{\sqrt{1-|\zeta|^2}}{1-\overline{\zeta}x} \tag{3.49}$$

denote the normalized Szegő kernel at $\zeta \in \mathbb{D}_\infty$. We take $k_\infty = 0$ when $\zeta = \infty$ (this is because we identify the infinity point with the boundary of $\mathbb{D}$). A direct calculation verifies that

$$\frac{1-m_\zeta(x)m_\zeta(y)^*}{1-xy^*} = k_\zeta(x)k_\zeta(y)^*, \tag{3.50}$$

for all $\zeta \in \mathbb{D}_\infty$. Let $\mathscr{B}$ be a Blaschke product with zeros $\xi_1,\ldots,\xi_\ell$ in $\mathbb{D}_\infty$ such that $\xi_1 \preceq \cdots \preceq \xi_\ell$. Then

$$\frac{1-\mathscr{B}(x)\mathscr{B}(y)^*}{1-xy^*} = \frac{1-m_{\xi_1}(x)m_{\xi_1}(y)^*}{1-xy^*} + \sum_{k=2}^{\ell}\prod_{i=1}^{k-1} m_{\xi_i}(x)\frac{1-m_{\xi_k}(x)m_{\xi_k}(y)^*}{1-xy^*}\prod_{i=1}^{k-1} m_{\xi_i}(y)^*$$

Using 3.50 we get

$$\frac{1-\mathscr{B}(x)\mathscr{B}(y)^*}{1-xy^*} = k_{\xi_1}(x)k_{\xi_1}(y)^* + \sum_{k=2}^{\ell}\prod_{i=1}^{k-1} m_{\xi_i}(x)k_{\xi_k}(x)k_{\xi_k}(y)^*\prod_{i=1}^{k-1} m_{\xi_i}(y)^* \tag{3.51}$$

Define

$$K_\xi(x) = \begin{pmatrix} k_{\xi_1}(x) & m_{\xi_1}(x)k_{\xi_2}(x) & \ldots & \prod_{i=1}^{\ell-1} m_{\xi_i}(x)k_{\xi_\ell}(x)\end{pmatrix} \in M_{1\times\ell}(\mathbb{C}). \tag{3.52}$$

Then identity (3.51) becomes

$$\frac{1-\mathscr{B}(x)\mathscr{B}(y)^*}{1-xy^*} = K_\xi(x)K_\xi(y)^*. \tag{3.53}$$

It follows also that the kernel $\left(\frac{1-\mathscr{B}(x)\mathscr{B}(y)^*}{1-xy^*}\right)$ is positive semidefinite. Applying (3.53) with the choice $\mathscr{B} = D_\lambda$ in equation (3.48), we obtain

$$\frac{1-\psi_\lambda(x)\psi_\lambda(y)^*}{1-xy^*} = \frac{1-B(x)B(y)^*}{1-xy^*} + B(x)B(y)^*K_\lambda(x)K_\lambda(y)^*, \tag{3.54}$$

where

$$K_\lambda(x) = \begin{pmatrix} k_{\lambda_1}(x) & m_{\lambda_1}(x)k_{\lambda_2}(x) & \dots & \prod_{i=1}^{N-2} m_{\lambda_i}(x)k_{\lambda_{N-1}}(x)\end{pmatrix} \in M_{1\times(N-1)}(\mathbb{C}). \tag{3.55}$$

Using (3.47) and (3.54), rewrite (3.46) as follows

$$\begin{aligned}&\frac{1-B(x)B(y)^*}{1-xy^*}I_2 + B(x)B(y)^*\left(\frac{I_2 - \varPi(x)\varPi(y)^*}{1-xy^*}\right)\\ &= \frac{1-B(x)B(y)^*}{1-xy^*}\int_{\varPsi_B} d\mu_{x,y}(\psi) + B(x)B(y)^*\int_{\varPsi_B^0} K_\lambda(x)K_\lambda(y)^*\,d\mu_{x,y}(\psi).\end{aligned} \tag{3.56}$$

As pointed out at the beginning of the section if $\lambda_1 = \infty$, then we have $\lambda_2 = \infty, \dots, \lambda_{N-1} = \infty$, this implies that $k_{\lambda_j} = k_\infty = 0$ for all $j = 1, \dots, N-1$. Hence the second integral in (3.56) is restricted to $\varPsi_B^0 = \varPsi_B \setminus \{\psi^0\}$.

By Lemma 3.7.3, there exist linearly independent vectors v_1, v_2 in $\mathbb{C}^2$ and functions $f, g : S \to \mathbb{C}^2$ in the span of $\{k_{p_1}v_1, k_{p_2}v_2\}$ such that

$$\begin{aligned}&\frac{1-B(x)B(y)^*}{1-xy^*}I_2 + B(x)B(y)^*\left(f(x)f(y)^* + g(x)g(y)^*\right)\\ &= \frac{1-B(x)B(y)^*}{1-xy^*}\int_{\varPsi_B} d\mu_{x,y}(\psi) + B(x)B(y)^*\int_{\varPsi_B^0} K_\lambda(x)K_\lambda(y)^*\,d\mu_{x,y}(\psi).\end{aligned} \tag{3.57}$$

Let

$$\begin{aligned}A(x,y) &= \int_{\varPsi_B} d\mu_{x,y}(\psi);\\ R(x,y) &= B(x)B(y)^*\left(f(x)f(y)^* + g(x)g(y)^*\right); \quad \text{and}\\ \tilde{R}(x,y) &= B(x)B(y)^*\int_{\varPsi_B^0} K_\lambda(x)K_\lambda(y)^*\,d\mu_{x,y}(\psi),\end{aligned}$$

which are all positive kernels on S. Then (3.57) becomes

$$R(x,y) - \tilde{R}(x,y) = \frac{1 - B(x)B(y)^*}{1 - xy^*}\left(A(x,y) - I_2\right). \tag{3.58}$$

We next show that $\mu_{x,y}$ is independent of x and y. To do this, we define

$$\mathbb{K} := \{B(x)k_{p_1}(x)v_1, B(x)k_{p_2}(x)v_2\}, \tag{3.59}$$

where the points p_1, p_2 are fixed as before.

Lemma 3.8.2. *If $\Delta_{F,S} \in \mathcal{C}_{2,S}$ and for $x, y \in S$, then the following hold*

(i) The $M_2(\mathbb{C})$ valued kernel $\Big(A(x,y) - I_2\Big)$ is positive semidefinite ;

(ii) The $M_2(\mathbb{C})$ valued kernel $\Big(R(x,y) - \tilde{R}(x,y)\Big)$ is positive semidefinite with rank at most two;

(iii) The range of $\tilde{R}$ lies in the range of R, which is in the span of $\mathbb{K}$; and

(iv) Let s be the cardinality of the set S and let $[I_2]$ denote the $s \times s$ block matrix with all entries consisting of I_2. Then either

(a) The kernel $A - [I_2]$ has rank at most one; i.e., there is a function $u : S \to \mathbb{C}^2$ such that

$$A(x,y) = I_2 + u(x)u(y)^* \tag{3.60}$$

or

(b) There exist functions $u, v : S \to \mathbb{C}^2$ such that

$$A(x,y) = I_2 + u(x)u(y)^* + v(x)v(y)^*, \tag{3.61}$$

and a point $p \in S \setminus \mathcal{Z}(B)$ such that $u(p) = v(p) = 0$.

Proof. (i) Recall that $\alpha = \{\alpha_i\}$ is the zero set of B, so $\psi(\alpha_i) = 0$ for all $\psi \in \Psi_B$ and $1 \le i \le N$. It follows from (3.45),

$$I_2 = I_2 - F(\alpha_i)F(\alpha_j)^* = \int_{\Psi_B} (1 - \psi(\alpha_i)\psi(\alpha_j)^*)\, d\mu_{\alpha_i,\alpha_j}(\psi) = \int_{\Psi_B} d\mu_{\alpha_i,\alpha_j}(\psi) = A(\alpha_i, \alpha_j)$$

and

$$I_2 = I_2 - F(\alpha_i)F(y)^* = \int_{\Psi_B} (1 - \psi(\alpha_i)\psi(y)^*)\, d\mu_{\alpha_i,y}(\psi) = \int_{\Psi_B} d\mu_{\alpha_i,y}(\psi) = A(\alpha_i, y)$$

for all $y \in S$. The square matrix $(A(x,y))_{x,y\in S}$ is positive semidefinite, so there exists a matrix D such that $(A(x,y))_{x,y\in S} = D^*D$. Fix $\alpha_i \in \mathcal{Z}(B)$, then

$$0 \preceq \begin{pmatrix} \Big(A(x,y)\Big)_{x,y\in S} & \Big(A(x,\alpha_i)\Big)_{x\neq\alpha_i} \\ \Big(A(\alpha_i,y)\Big)_{y\neq\alpha_i} & \Big(A(\alpha_i,\alpha_j)\Big) \end{pmatrix} = \begin{pmatrix} D^*D & \begin{pmatrix} I_2 \\ \vdots \\ I_2 \end{pmatrix} \\ \begin{pmatrix} I_2 & \cdots & I_2 \end{pmatrix} & I_2 \end{pmatrix}.$$

So there is a contraction Z such that $\begin{pmatrix} I_2 \\ \vdots \\ I_2 \end{pmatrix} = (D^*D)^{1/2}Z$ (see [16, Proposition 1.3.2]). Hence,

$$\begin{aligned} \Big(A(x,y)\Big)_{x,y\in S} &= D^*D = (D^*D)^{1/2}(D^*D)^{1/2} \succeq (D^*D)^{1/2}ZZ^*(D^*D)^{1/2} \\ &= (D^*D)^{1/2}ZZ^*(DD^*)^{1/2} = \begin{pmatrix} I_2 \\ \vdots \\ I_2 \end{pmatrix} \begin{pmatrix} I_2 & \cdots & I_2 \end{pmatrix} = [I_2]. \end{aligned}$$

This completes the proof of (i).

(ii) Applying (3.53) with the choice $\mathscr{B} = B$ in (3.58) gives

$$\frac{1 - B(x)B(y)^*}{1 - xy^*} = K_\alpha(x)K_\alpha(y)^*,$$

where

$$K_\alpha(x) = \Big(k_{\alpha_1}(x) \quad m_{\alpha_1}(x)k_{\alpha_2}(x) \quad \dots \quad \prod_{i=1}^{N-1} m_{\alpha_i}(x)k_{\alpha_N}(x)\Big) \in M_{1\times N}(\mathbb{C}). \tag{3.62}$$

It follows that the matrix $\left(\frac{1-B(x)B(y)^*}{1-xy^*}\right)_{x,y\in S}$ is positive semidefinite. On the other hand, the Schur product of positive semidefinite matrices is positive semidefinite, so since $(A - [I_2])_{x,y\in S}$ is positive semidefinite, we have that

$$\left(\left(\frac{1 - B(x)B(y)^*}{1 - xy^*}\right)(A - [I_2])\right)_{x,y\in S}$$

is positive semidefinite. Hence by (3.58) we conclude that $\Big(R(x,y) - \tilde{R}(x,y)\Big)_{x,y\in S}$

is positive semidefinite.

Since $\operatorname{ran} R \subset \mathbb{K}$, R has rank at most 2. Hence, since $\tilde{R}$ is positive semidefinite, $\left(R(x,y) - \tilde{R}(x,y)\right)_{x,y\in S}$ has rank at most 2.

(iii) By item (ii) and Douglas' lemma, the range of $\tilde{R}$ is contained in the range of R. By Lemma 3.7.3, the range of R is spanned by the set $\mathbb{K}$ and (iii) follows.

(iv) First note that in any case equation (3.58) and item (ii) imply $A-[I_2]$ has at most rank two, because the matrix $\left(\frac{1-B(x)B(y)^*}{1-xy^*}\right)_{x,y\in S}$ is invertible. So there exists $u, v : S \to \mathbb{C}^2$ such that

$$A(x,y) - I_2 = u(x)u(y)^* + v(x)v(y)^*.$$

Hence by (3.58) and (3.62) we have

$$R(x,y) - \tilde{R}(x,y) = K_\alpha(x)K_\alpha(y)^* \left(u(x)u(y)^* + v(x)v(y)^*\right).$$

Fix $\alpha_1, \alpha_2 \in \mathcal{Z}(B)$, the first zeros of B that appears as in (3.62). Thus, all the functions $u(x)k_{\alpha_1}(x)$, $v(x)k_{\alpha_1}(x)$, $u(x)k_{\alpha_2}(x)m_{\alpha_1}(x)$, $v(x)k_{\alpha_2}(x)m_{\alpha_1}(x)$ lie in the range of R, which equals the span of $\mathbb{K}$. If u is nonzero at two points in S, then $u(x)k_{\alpha_1}(x)$ and $u(x)k_{\alpha_2}(x)m_{\alpha_1}(x)$ are linearly independent. Otherwise there exists distinct points $z_1, z_2 \in S$ such that $u(z_1) \neq 0, u(z_2) \neq 0$ so there exist complex numbers c_1, c_2 (at least one is nonzero) such that

$$c_1 u(x)k_{\alpha_1}(x) + c_2 u(x)k_{\alpha_2}(x)m_{\alpha_1}(x) = 0 \text{ for } x = z_1, z_2.$$

Hence we get $c_1 k_{\alpha_1}(x) + c_2 k_{\alpha_2}(x)m_{\alpha_1}(x) = 0$ for $x = z_1, z_2$. Multiplying the equation for z_1 by $k_{\alpha_1}(w)$ and the equation for z_2 by $k_{\alpha_1}(z)$, and taking the difference gives

$$k_{\alpha_2}(z_1)k_{\alpha_1}(z_2)m_{\alpha_1}(z_1) - k_{\alpha_2}(z_2)k_{\alpha_1}(z_1)m_{\alpha_1}(z_2) = 0. \tag{3.63}$$

Since $1 - \overline{\alpha}_1 x \neq 0$ and $1 - \overline{\alpha}_2 x \neq 0$ for $x = z_1, z_2$, clearing out denominator in (3.63) gives

$$(z_1 - \alpha_1)(1 - \overline{\alpha_2}z_2) - (z_2 - \alpha_1)(1 - \overline{\alpha_2}z_1) = 0.$$

This simplifies to

$$(z_1 - z_2)(1 - \alpha_1\overline{\alpha_2}) = 0.$$

Since $1 - \alpha_1\overline{\alpha_2} \neq 0$, we have $z_1 = z_2$, which contradicts our assumption. Thus

$u(x)k_{\alpha_1}(x)$ and $u(x)k_{\alpha_2}(x)m_{\alpha_1}(x)$ span the range of R. In this case, as both $v(x)k_{\alpha_1}$ and $v(x)k_{\alpha_2}(x)m_{\alpha_1}(x)$ are in the range of R there exists γ_j and β_j (for $j = 1, 2$) such that

$$\begin{aligned} v(x)k_{\alpha_1}(x) &= \gamma_1 u(x)k_{\alpha_1}(x) + \gamma_2 u(x)k_{\alpha_2}(x)m_{\alpha_1}(x) \\ v(x)k_{\alpha_2}(x)m_{\alpha_1}(x) &= \beta_1 u(x)k_{\alpha_1}(x) + \beta_2 u(x)k_{\alpha_2}(x)m_{\alpha_1}(x). \end{aligned}$$

Multiplying the first equation by $k_{\alpha_2}(x)m_{\alpha_1}(x)$ and the second by $k_{\alpha_1}(x)$, and taking the difference, we get

$$0 = v(x)k_{\alpha_1}(x)k_{\alpha_2}(x)m_{\alpha_1}(x) - v(x)k_{\alpha_1}(x)k_{\alpha_2}(x)m_{\alpha_1}(x) = p(x)u(x), \tag{3.64}$$

where $p(x) = \beta_1(k_{\alpha_1}(x))^2 + (\beta_2 - \gamma_1)k_{\alpha_1}(x)k_{\alpha_2}(x)m_{\alpha_1}(x) - \gamma_2(k_{\alpha_2}(x))^2(m_{\alpha_1}(x))^2$. If $\gamma_2 = 0$, then v is a multiple of u and case $(iv)(a)$ holds. Otherwise, in view of (3.64), u is zero except at two points (the two roots of $\beta_1(k_{\alpha_1}(x))^2 + (\beta_2 - \gamma_1)k_{\alpha_1}(x)k_{\alpha_2}(x)m_{\alpha_1}(x) - \gamma_2(k_{\alpha_2}(x))^2(m_{\alpha_1}(x))^2 = 0$). Thus u is zero at two points in S, one of which, say p, must be different from 0 (because all points in S are distinct). Since v must be zero when u is, $v(p) = 0$ too, and so $(iv)(b)$ holds. The same argument works if v is nonzero at two points in S.

Finally, there is only one possibility left that we need to check. That is, both u and v are nonzero at at most one point each and these may be distinct. In this case, the intersection of zero sets of u and v has the cardinality at least $2N$ (excluding p_1, p_2). Since all points in S are distinct points and $2N > |\mathcal{Z}(B)| = N$, there exists a point p such that $p \in S \setminus \mathcal{Z}(B)$ and $u(p) = v(p) = 0$. This proves (iv)(b). □

Lemma 3.8.3. *If $\Delta_{F,S} \in \mathcal{C}_{2,S}$, then $A(x, y) = I_2$ for all $x, y \in S$.*

Proof. By Lemma 3.7.3, the range of R is spanned by the set $\mathbb{K}$. According to Lemma 3.8.2 (iv), the matrix $\Big(A(x, y)\Big)_{x,y\in S}$ can be expressed in two ways. First we assume that

$$A(x, y) = I_2 + u(x)u(y)^*$$

as in (3.60). So in this case we need to show $u = 0$. Rewrite (3.58) in the following way

$$R(x, y) = \tilde{R}(x, y) + K_\alpha(x)K_\alpha(y)^* u(x)u(y)^*. \tag{3.65}$$

As in the proof of the previous lemma, $u(x)k_{\alpha_1}(x)$ and $u(x)k_{\alpha_2}(x)m_{\alpha_1}(x)$ are in the range of R; that is, both $u(x)k_{\alpha_1}(x)$ and $u(x)k_{\alpha_2}(x)m_{\alpha_1}(x)$ are in the span of $\mathbb{K}$. It follows that there exists γ_j and β_j $(j = 1, 2)$ such that

$$u(x)k_{\alpha_1}(x) = B(x)\sum_{j=1}^{2}\gamma_j k_{p_j}(x)v_j$$
$$u(x)k_{\alpha_2}(x)m_{\alpha_1}(x) = B(x)\sum_{j=1}^{2}\beta_j k_{p_j}(x)v_j.$$

Multiplying the first equation by $k_{\alpha_2}(x)m_{\alpha_1}(x)$, the second by $k_{\alpha_1}(x)$, and taking the difference gives

$$0 = B(x)\sum_{j=1}^{2}(\beta_j k_{\alpha_1}(x) - \gamma_j k_{\alpha_2}(x)m_{\alpha_1}(x))k_{p_j}(x)v_j. \tag{3.66}$$

Since the set $\{v_1, v_2\}$ is a basis for $\mathbb{C}^2$ (see Lemma 3.7.3), it has a dual basis $\{w_1, w_2\}$. Taking the inner product with w_ℓ $(\ell = 1, 2)$ in equation (3.66) gives, for $x \in S$,

$$0 = B(x)(\beta_\ell k_{\alpha_1}(x) - \gamma_\ell k_{\alpha_2}(x)m_{\alpha_1}(x))k_{p_\ell}(x). \tag{3.67}$$

Evaluating at $x = p_\ell \in S \setminus \mathcal{Z}(B)$ in (3.67), we get

$$\beta_\ell k_{\alpha_1}(p_\ell) - \gamma_\ell k_{\alpha_2}(\lambda_\ell)m_{\alpha_1}(p_\ell) = 0. \tag{3.68}$$

and at $x \in S \setminus \{p_1, p_2, \mathcal{Z}(B)\}$ gives,

$$\beta_\ell k_{\alpha_1}(x) - \gamma_\ell k_{\alpha_2}(x)m_{\alpha_1}(x) = 0, \tag{3.69}$$

because $k_{p_\ell}(p_\ell) \neq 0, k_{p_j}(x) \neq 0, B(p_\ell) \neq 0$ and $B(x) \neq 0$. Multiplying (3.68) by $k_{\alpha_1}(x)$ and (3.69) by $k_{\alpha_1}(\lambda_\ell)$ and taking the difference, we get

$$\gamma_\ell\left(k_{\alpha_2}(p_\ell)m_{\alpha_1}(p_\ell)k_{\alpha_1}(x) - k_{\alpha_2}(x)m_{\alpha_1}(x)k_{\alpha_1}(p_\ell)\right) = 0.$$

Since all $1 - \overline{\alpha}_1 x, 1 - \overline{\alpha}_1 p_\ell, 1 - \overline{\alpha}_2 x$ and $1 - \overline{\alpha}_2 p_\ell$ in the denominator of the last expression is non-zero, so the last equation simplifies to

$$\gamma_\ell[(p_\ell - \alpha_1)(1 - \overline{\alpha}_2 x) - (x - \alpha_1)(1 - \overline{\alpha}_2 p_\ell)] = 0,$$

which can be rewritten as

$$\gamma_\ell(x-p_\ell)(1-\alpha_1\overline{\alpha}_2)=0.$$

Since $x\neq p_\ell$ and $1-\alpha_1\overline{\alpha}_2\neq 0$, we have $\gamma_\ell=0$, and so $\beta_\ell=0$. We conclude that $u(x)=0$ for all $x\in S$.

Now assume $A(x,y)=I+u(x)u(y)^*+v(x)v(y)^*$ and there exists a point $p\in S\setminus\mathcal{Z}(B)$ such that $r(p)=s(p)=0$. In this case, from (3.58) , we have

$$R=\tilde{R}+K_\alpha(x)K_\alpha(y)^*\left(u(x)u(y)^*+v(x)v(y)^*\right). \tag{3.70}$$

Thus, $u(x)k_{\alpha_1}(x)$ and $v(x)k_{\alpha_1}(x)$ are in the range of R is spanned by $\mathbb{K}$. Hence there exists complex numbers γ_j and β_j such that

$$\begin{aligned} u(x)k_{\alpha_1}(x) &= B(x)\sum_{j=1}^{2}\gamma_j k_{p_j}(x)v_j \\ v(x)k_{\alpha_1}(x) &= B(x)\sum_{j=1}^{2}\beta_j k_{p_j}(x)v_j. \end{aligned} \tag{3.71}$$

Choosing $x=p$ and taking the inner product with w_ℓ in the first equation of (3.71) gives

$$0=\gamma_\ell k_{p_\ell}(p).$$

Since $k_{p_\ell}(p)\neq 0$, we have $\gamma_\ell=0$ for $\ell=1,2$. Similarly, from the second equation of (3.71), we have $\beta_\ell=0$ for $\ell=1,2$. Thus $u=v=0$. This completes the proof. □

Lemma 3.8.4 ([24, Lemma 5.5])**.** *If $\Delta_{F,S}\in\mathcal{C}_{2,S}$, then there exists a 2×2 matrix valued positive measure μ on Ψ_B such that $\mu(\Psi_B)=I_2$ and*

$$\frac{I-\Pi(x)\Pi(y)^*}{1-xy^*}=\int_{\Psi_B^0}K_\lambda(x)K_\lambda(y)^*\,d\mu(\psi). \tag{3.72}$$

for all $x,y\in S\setminus\mathcal{Z}(B)$.

Proof. Applying Lemma 3.8.3 to equation (3.56) we have

$$B(x)B(y)^*\frac{I_2-\Pi(x)\Pi(y)^*}{1-xy^*}=B(x)B(y)^*\int_{\Psi_B^0}K_\lambda(x)K_\lambda(y)^*\,d\mu_{x,y}(\psi).$$

Dividing both sides by $B(x)B(y)^*$ when $x, y \in S \setminus \mathcal{Z}(B)$ gives

$$\frac{I_2 - \Pi(x)\Pi(y)^*}{1 - xy^*} = \int_{\Psi_B^0} K_\lambda(x)K_\lambda(y)^* \, d\mu_{x,y}(\psi).$$

Again by Lemma 3.8.3 we have $\int_{\Psi_B} d\mu_{x,y}(\psi) = I_2$ for $x, y \in S \setminus \mathcal{Z}(B)$. So by Lemma 3.7.1, there exists a positive measure μ on Ψ_B such that $\mu = \mu_{x,y}$ for all $x, y \in S \setminus \mathcal{Z}(B)$. It follows that

$$\frac{I_2 - \Pi(x)\Pi(y)^*}{1 - xy^*} = \int_{\Psi_B^0} K_\lambda(x)K_\lambda(y)^* \, d\mu(\psi)$$

and

$$I_2 = \int_{\Psi_B} d\mu_{x,y}(\psi) = \int_{\Psi_B} d\mu(\psi) = \mu(\Psi_B).$$

□

In a view of equation (3.72), we see that the entries of μ are independent of x, y. The next step is to restrict the support of μ via Lemma 3.7.3. To do this, we need the following result.

Given a 2×2 matrix valued measure ν and a vector $\gamma \in \mathbb{C}^2$, let ν_γ denote the scalar measure defined by $\nu_\gamma(\Omega) = \gamma^*\nu(\Omega)\gamma$ for every Borel subset Ω of Ψ_B. Note that if ν is a positive measure (that is, takes positive semidefinite values), then each ν_γ is a positive measure.

Lemma 3.8.5 ([24, Lemma 4.5])**.** *Suppose ν is a 2×2 positive matrix-valued measure on $\Psi_B^0 = \Psi_B \setminus \{\psi^0\}$. For each $\gamma \in \mathbb{C}^2$ the measure ν_γ is a nonnegative linear combination of at most k point masses if and only if there exist (possibly not distinct) points $\eta_1, \dots, \eta_k \in \mathbb{D}_\infty^{N-1} \setminus \{\infty^{N-1}\}$ and positive semidefinite matrices $P_1, \dots, P_k$ in $M_2(\mathbb{C})$ such that*

$$\nu = \sum_{j=1}^{k} \delta_{\eta_j} P_j,$$

where $\delta_{\eta_1}, \dots, \delta_{\eta_k}$ are scalar unit point measures on Ψ_B^0 supported at $\psi_{\eta_1}, \dots, \psi_{\eta_k}$, respectively.

Proof. Assume that every ν_γ is a nonnegative linear combination of at most k point masses. Let $\nu = \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix} \in M_2(\mathbb{C})$ be a matrix valued measure on Ψ_B^0 with respect to the standard orthonormal basis e_1, e_2 of $\mathbb{C}^2$, where each entry ν_{ij} is a scalar valued measure on Ψ_B^0.

By Lemma 3.5.4 the measures ν_{11} and ν_{22} are positive because $\nu(\Omega)$ is positive matrix for every Borel subset Ω of Ψ_B^0. Also the positivity of ν implies that $\nu_{21} = \nu_{12}^*$. By Lemma 3.5.2, if $\nu_{ii}(\Omega) = 0$ for a Borel subset Ω of Ψ_B^0, then $\nu_{ij}(\Omega) = 0$. Hence the measures ν_{12} and ν_{21} are absolutely continuous with respect to both ν_{11} and ν_{22}. It follows that supp ν_{12} = supp $\nu_{21} \subseteq$ supp $\nu_{11} \cap$ supp ν_{22}, where supp ν_{ij} is the support of ν_{ij} for $i, j = 1, 2$.

Choosing $\gamma = e_1$ and $\gamma = e_2$, we have

$$\nu_\gamma = e_1^* \nu e_1 = \nu_{11} \text{ and } \nu_\gamma = e_2^* \nu e_2 = \nu_{22},$$

respectively. Then the assumption implies that supp ν_{ii} is finite, and so supp ν_{ij} is finite. Let $n_{ij} = |\text{supp } \nu_{ij}|$. By assumption $n_{ij} \leq k$ and there exists nonnegative real numbers $c_{\ell_1}^{1,1}, c_{\ell_2}^{2,2}$ such that

$$\nu_{11} = \sum_{\ell_1=1}^{n_{11}} c_{\ell_1}^{1,1} \delta_{\tau_{11,\ell_1}} \text{ and } \nu_{22} = \sum_{\ell_2=1}^{n_{22}} c_{\ell_2}^{2,2} \delta_{\tau_{22,\ell_2}}.$$

Thus, for $\gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \end{pmatrix}^t$,

$$\begin{aligned}
\nu_\gamma &= |\gamma_1|^2 \nu_{11} + \gamma_1 \overline{\gamma_2} \nu_{21} + \overline{\gamma_1} \gamma_2 \nu_{12} + |\gamma_2|^2 \nu_{22} \\
&= |\gamma_1|^2 \nu_{11} + |\gamma_2|^2 \nu_{22} + 2\mathrm{Re}(\gamma_1 \overline{\gamma_2} \nu_{21}) \\
&= |\gamma_1|^2 \sum_{\ell_1=1}^{n_{11}} c_{\ell_1}^{1,1} \delta_{\tau_{11,\ell_1}} + |\gamma_2|^2 \sum_{\ell_2=1}^{n_{22}} c_{\ell_2}^{2,2} \delta_{\tau_{22,\ell_2}} + 2\mathrm{Re}(\gamma_1 \overline{\gamma_2}) \sum_{\ell=1}^{n_{21}} c_\ell \delta_{\tau_\ell},
\end{aligned} \tag{3.73}$$

where supp ν_{12} = supp $\nu_{21} \subseteq$ supp $\nu_{11} \cap$ supp ν_{22} and $c_\ell \in \{c_1^{1,1}, \ldots, c_{n_{11}}^{1,1}\} \cap \{c_1^{2,2}, \ldots, c_{n_{22}}^{2,2}\}$ for all $\ell = 1, \ldots, n_{21}$.

Assuming $\gamma_1, \gamma_2 = |\gamma_2| e^{i\theta}$ are nonzero, there are at most two values of $\theta \in [0, 2\pi)$ such that $2\mathrm{Re}(\gamma_1 \overline{\gamma_2}) c_\ell = (\gamma_1 |\gamma_2| e^{i\theta} + \gamma_1 |\gamma_2| e^{-i\theta}) c_\ell = -|\gamma_1|^2 c_{\ell_1}^{1,1} - |\gamma_2|^2 c_{\ell_2}^{2,2}$. Running all over ℓ, there are at most a finite number of such θ. Choosing θ avoiding these points, it follows that supp ν_γ = supp $\nu_{11} \cup$ supp ν_{22}. By assumption, at most k of these points can be distinct, and hence ν has the form claimed.

Conversely, if $\nu = \sum_{j=1}^k \delta_{\eta_j} P_j$ with $\eta_1, \ldots, \eta_k$ and $P_1, \ldots, P_k$ as in the statement of lemma, then the scalar valued measure $\nu_\gamma = \gamma^* \nu \gamma$ is a nonnegative linear combination of at most k point masses. This completes the proof.

$\square$

Lemma 3.8.6 ([24, Lemma 5.6])**.** *Let μ be the measure as in the statement of Lemma 3.8.4. If $\Delta_{F,S} \in \mathcal{C}_{2,S}$, then the measure μ has the form*

$$\mu = \delta_1 P_1 + \delta_2 P_2 + \delta_{12} P_{12} + \delta_\infty P_\infty,$$

where $P_1, P_2, P_{12}, P_\infty$ are 2×2 positive matrices such that $P_1 + P_2 + P_{12} + P_\infty = I_2$, and $\delta_1, \delta_2, \delta_{12}$, and δ_∞ are unit scalar point masses of measures on Ψ_B supported at Bm_{p_1}, Bm_{p_2}, $Bm_{p_1}m_{p_2}$, and $\psi^0 = B$, respectively.

Proof. Let ν denote the restriction of μ to Ψ_B^0 (or eqiuvalently to $\mathbb{D}_\infty^{N-1} \setminus \{\infty^{N-1}\}$). For $\gamma \in \mathbb{C}^2$, define a scalar valued measure ν_γ on Ψ_B^0 given by $\nu_\gamma(\Omega) = \gamma^* \nu(\Omega)\gamma$ for any Borel subset $\Omega \subseteq \Psi_B^0$. An application of Lemma 3.7.3 to Π and Lemma 3.8.4 implies that

$$\begin{aligned} \gamma^*(f(x)f(y)^* + g(x)g(y)^*)\gamma &= \gamma^* \left(\int_{\Psi_B^0} K_\lambda(x) K_\lambda(y)^* \, d\mu(\psi) \right) \gamma \qquad (3.74) \\ &= \int_{\Psi_B^0} K_\lambda(x) K_\lambda(y)^* \, d\nu_\gamma(\psi), \end{aligned}$$

where f, g are as in (3.57).

Fix a set of three non-zero points $\mathcal{X} = \{z_1, z_2, z_3\} \subset S \setminus \mathcal{Z}(B)$. Let $c : \mathcal{X} \to \mathbb{C}$ be a nonzero vector in the orthogonal complement of $\bigvee_{j=1,2} \begin{pmatrix} k_{p_j}(z_1) & k_{p_j}(z_2) & k_{p_j}(z_3) \end{pmatrix}$. Suppose that one of the entries of c is zero, without loss of generality we may take this to be $c(z_3)$. Then the vectors $\begin{pmatrix} k_{p_j}(z_1) & k_{p_j}(z_2) \end{pmatrix}, j = 1, 2$ are orthogonal to $\begin{pmatrix} c(z_1) & c(z_2) \end{pmatrix}$. Since c is nonzero, this implies that the vectors $\begin{pmatrix} k_{p_1}(z_1) & k_{p_1}(z_2) \end{pmatrix}$ and $\begin{pmatrix} k_{p_2}(z_1) & k_{p_2}(z_2) \end{pmatrix}$ are collinear. Hence $\begin{pmatrix} k_{p_1}(z_1) & k_{p_1}(z_2) \end{pmatrix} = C \begin{pmatrix} k_{p_2}(z_1) & k_{p_2}(z_2) \end{pmatrix}$ for some constant C. Let $s_j := \sqrt{1 - |p_j|^2} \neq 0, j = 1, 2$. Then $(\overline{p}_1 s_2 - s_1 C \overline{p}_2) z_j = s_2 - C s_1, j = 1, 2$. Since $z_1 \neq z_2$, we must have $s_2 - C s_1 = 0$ and so $\overline{p}_1 = \overline{p}_2$, a contradiction. Thus no entry of c is zero. For any $\gamma \in \mathbb{C}^2$ is in the span of the dual basis $\{w_1, w_2\}$ to $\{v_1, v_2\}$, which are vectors from (3.57), we have

$$\sum_{x,y \in \mathcal{X}} c(x) \gamma^* \left(f(x) f(y)^* + g(x) g(y)^* \right) \gamma c(y)^* = 0. \qquad (3.75)$$

Thus by (3.74), we have

$$0 = \int_{\Psi_B^0} \left(\sum_{x\in\mathcal{X}} K_\lambda(x)c(x) \right) \left(\sum_{x\in\mathcal{X}} K_\lambda(x)c(x) \right)^* d\nu_\gamma(\psi). \tag{3.76}$$

Consequently,

$$\sum_{x\in\mathcal{X}} K_\lambda(x)c(x) = 0_{1\times N-1} \text{ for } \nu_\gamma - \text{ a.e. on } \Psi_B^0. \tag{3.77}$$

Returning to the definition of K_λ in (3.55), the later equation implies that

$$\sum_{x\in\mathcal{X}} c(x) \prod_{i=1}^{k-1} m_{\lambda_i}(x) k_{\lambda_k}(x) = 0, \quad k = 1,\dots,N-1, \nu_\gamma - \text{ a.e. on } \Psi_B^0. \tag{3.78}$$

Here we take $\prod_{i=1}^{k-1} m_{\lambda_i}(x) = 1$ when $k=1$. For $k=1$ this gives

$$c(z_1)k_{\lambda_1}(z_1) + c(z_2)k_{\lambda_1}(z_2) + c(z_3)k_{\lambda_1}(z_3) = 0. \tag{3.79}$$

Hence we have $k_{\lambda_1}(z_j) = a_1 k_{p_1}(z_j) + a_2 k_{p_2}(z_j)$ for some constants a_1, a_2. Let $c_0 := \sqrt{1-|\lambda_1|^2}$. Then we have

$$\begin{aligned} & c_0(1-(\overline{p}_1+\overline{p}_2)z_j + \overline{p_1p_2}z_j^2) \\ & = a_1s_1(1-(\overline{\lambda}_1+\overline{p}_2)z_j+\overline{\lambda_1p_2}z_j^2) + a_2s_2(1-(\overline{\lambda}_1+\overline{p}_1)z_j+\overline{\lambda_1p_1}z_j^2). \end{aligned}$$

Equating coefficients, we find that

$$\begin{aligned} c_0 &= a_1s_1+a_2s_2 && (3.80) \\ c_0(\overline{p}_1+\overline{p}_2) &= (a_1s_1+a_2s_2)\overline{\lambda_1} + a_1s_1\overline{p}_2 + a_2s_2\overline{p}_1 && (3.81) \\ c_0\overline{p_1p_2} &= (a_1s_1\overline{p}_2 + a_2s_2\overline{p}_1)\overline{\lambda_1}. && (3.82) \end{aligned}$$

Using (3.82) and (3.80) into (3.81), we get

$$c_0(\overline{p}_1+\overline{p}_2) = \overline{\lambda_1}c_0 + c_0\frac{\overline{p_1p_2}}{\overline{\lambda_1}}. \tag{3.83}$$

Note that if $\lambda_1 = 0$, then this contradicts with (3.81). There is an obvious solution to (3.83), namely $c_0 = 0$. Equivalently $\lambda_1 = \infty$ (this is because we are identifying ∞ with $\mathbb{T}$, the boundary of the disk). Then our ordering on the set of test functions imply that $\lambda_j = \infty$ for all $j = 2,\dots,N-1$. So this solution corresponds to the test function $\psi^0 = B$. Thus for $c_0 \neq 0$, from (3.83) we get $\overline{\lambda_1}^2 - (\overline{p_1} + \overline{p_2})\overline{\lambda_1} + \overline{p_1 p_2} = 0$. Hence $\lambda_1 \in \{p_1, p_2\}$. If $N = 2$, we stop at this point.

Otherwise, we take $\lambda_1 = p_1$. By (3.78) with $k = 2$, we likewise have that vector $(m_{p_1}(z_j)k_{\lambda_2}(z_j))$ is orthogonal to c, and so there are constants a_1, a_2 such that

$$\frac{z_j - p_1}{1 - \overline{p_1} z_j} \frac{\sqrt{1 - |\lambda_2|^2}}{1 - \overline{\lambda_2} z_j} = \frac{a_1 s_1 (1 - \overline{p_2} z_j) + a_2 s_2 (1 - \overline{p_1} z_j)}{(1 - \overline{p_1} z_j)(1 - \overline{p_2} z_j)}.$$

So either $\lambda_2 = \infty$ or

$$\sqrt{1 - |\lambda_2|^2}(z_j - p_1)(1 - \overline{p_2} z_j) = a_1 s_1 (1 - \overline{p_2} z_j)(z - \overline{\lambda_2} z_j) + a_2 s_2 (1 - \overline{p_1} z_j)(1 - \overline{\lambda_2} z_j).$$

In the first case, the ordering implies that $\lambda_j = \infty$ for all $j = 3, \ldots, \infty$. For the second case, equating coefficients yields $\lambda_2 = p_2$ or $1/\overline{p_1}$. Since $1/\overline{p_1} \notin \mathbb{D}$, we must have $\lambda_2 = p_2$. In the same manner, if we had assumed that $\lambda_1 = p_2$, we could have $\lambda_2 = p_1$ or ∞. If $N = 3$, we stop.

If $N > 3$, by (3.78) with $k = 3$ we now have the vector $(m_{p_1}(z_j)m_{p_2}(z_j)k_{\lambda_2}(z_j))$ is orthogonal to c. Now, we have $\lambda_1 = p_1, \lambda_2 = p_2$ (or vice versa). Then a similar calculation yields $\lambda_3 = 1/\overline{p_1}$ or $1/\overline{p_2}$. These are both outside of $\mathbb{D}$, so are ruled out. Then only other alternative is $\lambda_3 = \infty$. It follows that ν_γ is supported at three points in Ψ_B^0; namely Bm_{p_1}, Bm_{p_2} and $Bm_{p_1}m_{p_2}$. Then by Lemma 3.8.5 there exists positive semidefinite matrices $P_1, P_2, P_{12} \in M_2(\mathbb{C})$ such that

$$\nu = \delta_1 P_1 + \delta_2 P_2 + \delta_{12} P_{12},$$

where $\delta_1, \delta_2, \delta_{12}$ are the unit scalar point masses of measures supported at Bm_{p_1}, Bm_{p_2}, $Bm_{p_1}m_{p_2}$, respectively. Letting $P_\infty = \mu(\{\psi^0\})$, gives that

$$\mu = \delta_1 P_1 + \delta_2 P_2 + \delta_{12} P_{12} + \delta_\infty P_\infty.$$

Finally, by Lemma 3.8.4, we have

$$I_2 = \mu(\Psi_B) = P_1 + P_2 + P_{12} + P_\infty.$$

□

The proof of Theorem 3.8.1. Assume by the contradiction that $\Delta_{F,S} \in \mathcal{C}_{2,S}$. Then by Lemma 3.8.4 we have

$$\frac{I_2 - \Pi(x)\Pi(y)^*}{1 - xy^*} = \int_{\Psi_B^0} K_\lambda(x) K_\lambda(y)^* \, d\mu(\psi)$$

for all $x, y \in S \setminus \mathcal{Z}(B)$. Multiplying both sides by $1 - xy^*$ and using (3.50) we get

$$I_2 - \Pi(x)\Pi(y)^* = \int_{\Psi_B^0} (1 - D_\lambda(x) D_\lambda(y)^*) \, d\mu(\psi).$$

for all $x, y \in S \setminus \mathcal{Z}(B)$. By Lemma 3.8.6, the measure μ only supported at functions Bm_{p_1}, Bm_{p_2}, $Bm_{p_1}m_{p_2}$ in Ψ_B^0, and there exists a positive semidefinite matrices $P_1, P_2, P_{12}, P_\infty \in M_2(\mathbb{C})$ such that $P_1 + P_2 + P_{12} + P_\infty = I_2$ and

$$\begin{aligned} I_2 - \Pi(x)\Pi(y)^* &= (1 - m_{p_1}(x) m_{p_1}(y)^*) P_1 + (1 - m_{p_2}(x) m_{p_2}(y)^*) P_2 \\ &\quad + (1 - m_{p_1}(x) m_{p_2}(x) m_{p_1}(y)^* m_{p_2}(y)^*) P_{12} \end{aligned} \tag{3.84}$$

for $x, y \in S \setminus \mathcal{Z}(B)$. This simplifies to

$$\begin{aligned} \Pi(x)\Pi(y)^* &= m_{p_1}(x) m_{p_1}(y)^* P_1 + m_{p_2}(x) m_{p_2}(y)^* P_2 \\ &\quad + m_{p_1}(x) m_{p_2}(x) m_{p_1}(y)^* m_{p_2}(y)^* P_{12} + P_\infty \end{aligned} \tag{3.85}$$

for $x, y \in S \setminus \mathcal{Z}(B)$. Decompose $P_x = T_x^* T_x, x = 1, 2, 12, \infty$, where

$$T_x = \begin{pmatrix} a_x & b_x \\ 0 & c_x \end{pmatrix}.$$

Let $C_1 = |m_{p_2}(p_1)|^2$ and $C_2 = |m_{p_1}(p_2)|^2$. From (3.85) we get

$$\begin{aligned} \Pi(p_1)\Pi(p_1)^* &= \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 + C_1 \end{pmatrix} \\ &= \begin{pmatrix} C_1 |a_2|^2 + |a_\infty|^2 & C_1 \overline{a_2} b_2 + \overline{a_\infty} b_\infty \\ C_1 a_2 \overline{b_2} + a_\infty \overline{b_\infty} & C_1(|b_2|^2 + |c_2|^2) + |b_\infty|^2 + |c_\infty|^2 \end{pmatrix}, \end{aligned} \tag{3.86}$$

$$\begin{aligned}\mathit{\Pi}(p_2)\mathit{\Pi}(p_2)^* &= \frac{1}{2}\begin{pmatrix} 1+C_2 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} C_2|a_1|^2 + |a_\infty|^2 & C_2\overline{a_1}b_2 + \overline{a_\infty}b_\infty \\ C_2a_1\overline{b_2} + a_\infty\overline{b_\infty} & C_2(|b_1|^2 + |c_1|^2) + |b_\infty|^2 + |c_\infty|^2 \end{pmatrix},\end{aligned} \tag{3.87}$$

$$\mathit{\Pi}(p_1)\mathit{\Pi}(p_2)^* = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ \overline{m_{p_1}}(p_2) & 1 \end{pmatrix} = P_\infty \tag{3.88}$$

From the first of these equations, we have $a_\infty = 0$, while the second gives $b_\infty = c_\infty = 0$, and so $P_\infty = 0$. But this contradicts with the last equation. By the way, positivity of P_∞ would require that $p_1 = p_2$, which is also contradicts our assumptions. This completes the proof. $\square$

Chapter 4

Future projects

4.1 Planar domains associated to the distinguished variety $\mathcal{N}_B$

The annuli are homeomorphic to the distinguished varieties determined by

$$z^2 = \frac{w - a_1}{1 - \overline{a_1}w} \frac{w - a_2}{1 - \overline{a_2}w}$$

for $a_1, a_2 \in \mathbb{D}$ and $(z, w) \in \overline{\mathbb{D}}^2$[44]. Also every bounded planar domain with finitely many piecewise analytic boundary curves corresponds to a distinguished variety [49]. Conversely, we pose the following: Is there a planar domain which is homeomorphic to the distinguished variety $\mathcal{N}_B$ for some B?

4.2 The rational dilation problem on more general distinguished varieties

An interesting example of distinguished varieties

It is interesting to know whether rational dilation holds on the distinguished varieties of the form

$$B_1(z) = B_2(w) \text{ for } (z, w) \in \overline{\mathbb{D}}^2, \tag{4.1}$$

where B_1, B_2 are finite Blaschke products. For instance, if all the zeros of B_1, B_2 are zero then we obtain varieties of the form

$$z^m = w^n \text{ for } (z,w) \in \overline{\mathbb{D}}^2,$$

where m, n are the degrees of B_1, B_2, respectively. In [24], it has been shown that rational dilation holds for the distinguished variety

$$z^2 = w^2 \text{ for } (z,w) \in \overline{\mathbb{D}}^2.$$

We conjecture that rational dilation also holds on the distinguished variety

$$z^k = w^k \text{ for } (z,w) \in \overline{\mathbb{D}}^2,$$

where $k \in \mathbb{N}$.

4.3 Intersection of algebras of the form $\mathbb{C}+BH^\infty(\mathbb{D})$

Let

$$H^\infty_{B_j} := \mathbb{C} + B_j(z)H^\infty(\mathbb{D})$$

for $j = 1, \dots, n$, where B_j are finite Blaschke products and $n \in \mathbb{N}$. Then we can consider the intersection

$$H^\infty_{\cap B_j} := \cap_{j=1}^n H^\infty_{B_j}.$$

The following list of questions are naturally posed:

1) What is a minimal set of test functions for $H^\infty_{\cap B_j}$?

2) What is the distinguished variety associated to the algebra $\mathscr{A}_{\cap B_j} := \cap_{j=1}^n \mathscr{A}_{B_j}$?, where $\mathscr{A}_{B_j} = \mathbb{C} + B_j(z)\mathbb{A}(\mathbb{D})$ for $j = 1, \dots, n$.

4) Does rational dilation holds for the distinguished variety associated to $\mathscr{A}_{\cap B_j}$?

4.3.1 Sum of algebras of the form $\mathbb{C} + B(z)H^\infty(\mathbb{D})$

Another interesting algebras would be of the form

$$H^\infty_{\sum B_j} := \mathbb{C} + B_1(z)H^\infty(\mathbb{D}) + \cdots + B_n(z)H^\infty(\mathbb{D}),$$

where $B_j, j = 1, \ldots, n$ are the finite Blaschke product and $n \in \mathbb{N}$. So we could ask the same questions as above.

4.3.2 Constrained subalgebras of $\mathbb{A}(\mathbb{D}^n)$

Other interesting algebras include $\mathbb{C}+z^2\mathbb{A}(\mathbb{D}^2)$ and $\mathbb{C}+z^2w^2\mathbb{A}(\mathbb{D}^2)$ or more generally

$$\mathscr{A}_{\prod B_j} := \mathbb{C} + \prod_{j=1}^{m} B_j(z_j)\mathbb{A}(\mathbb{D}^n),$$

where each $B_j(z_j)$ is a finite Blaschke product. Again, the same questions can be posed for these.

Appendix A

A.1 The Banach algebra $H^\infty(\mathcal{K}_\Psi)$

Recall that if have a set of test functions Ψ on a set X, then we form a set of admissible kernel $\mathcal{K}_\Psi = \{k : X \times X \to \mathbb{C} : ((1 - \psi(x)\psi(y)^*)k(x,y) \geq 0)\, \forall \psi \in \Psi\}$ associated to Ψ. Then we define a normed algebra $H^\infty(\mathcal{K}_\Psi)$ consisting of those functions $f : X \to \mathbb{C}$ for which there is a finite constant $C \geq 0$ such that for all $k \in \mathcal{K}_\Psi$, the kernel

$$\big((C^2 - f(x)f(y)^*)k(x,y)\big)$$

is positive semi-definite, and the norm of f is given by

$$\|f\|_{H^\infty(\mathcal{K}_\Psi)} = \inf\{C : \big((C^2 - f(x)f(y)^*)k(x,y)\big) \geq \text{ for all } k \in \mathcal{K}_\Psi\}.$$

Let $\varphi, \phi \in H^\infty(\mathcal{K}_\Psi)$. For convenience set $\|\varphi\|_{H^\infty(\mathcal{K}_\Psi)} = C_\varphi$ and $\|\phi\|_{H^\infty(\mathcal{K}_\Psi)} = C_\phi$. We prove the submultiplicativity :

$$\|\varphi\phi\|_{H^\infty(\mathcal{K}_\Psi)} \leq \|\varphi\|_{H^\infty(\mathcal{K}_\Psi)}\|\phi\|_{H^\infty(\mathcal{K}_\Psi)}.$$

Let $F \subset X$ be a finite set with $|F| = n$. Then by definition we have

$$\big((C_\varphi^2 - \varphi(x)\varphi(y)^*)k(x,y)\big)_{x,y\in F} \geq 0$$

and

$$\big((C_\phi^2 - \phi(x)\phi(y)^*)k(x,y)\big)_{x,y\in F} \geq 0,$$

for all $k \in \mathcal{K}_\Psi$ and all $(x, y) \in F \times F$. Consequently,

$$
\begin{aligned}
& \left(\left((C_\varphi C_\phi)^2 - \varphi(x)\phi(x)\left(\varphi(y)\phi(y)\right)^*\right) k(x,y)\right)_{x,y\in F} \\
= \;& \left(C_\phi^2 \left(C_\varphi^2 - \varphi(x)\varphi(y)^*\right) k(x,y)\right)_{x,y\in F} \\
+ \;& \left(\varphi(x)\varphi(y)^* \left(C_\phi^2 - \phi(x)\phi(y)^*\right) k(x,y)\right)_{x,y\in F} \geq 0,
\end{aligned}
$$

for all $k \in \mathcal{K}_\Psi$ and all $(x, y) \in F \times F$. Then by definition $\|\varphi\phi\|_{H^\infty(\mathcal{K}_\Psi)} \leq C_\varphi C_\phi$. For the further properties of this norm we refer to [27, 33, 23].

Remark A.1.1. The kernel

$$
k(x,y) = \begin{cases} 1 & \text{when} \quad x = y \\ 0 & \text{when} \quad x \neq y \end{cases} \tag{A.1}
$$

is an admissible kernel, since

$$
\left(\left(1 - \psi(x)\psi(y)^*\right) k(x,y)\right)_{x,y\in F} = \begin{pmatrix} 1 - \psi(z_1)\overline{\psi(z_1)} & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & & 1 - \psi(z_n)\overline{\psi(z_n)} \end{pmatrix} \geq 0,
$$

because $\sup_{\psi\in\Psi} |\psi(z)| < 1$ for all $z \in \mathbb{D}$. We also claim that the Szegö kernel $k_S(x, y)$ is an admissible kernel. To prove this, let $a \in \mathbb{C}^n$. Then

$$
\begin{aligned}
& \left\langle \left(\left(1 - \psi(x)\psi(y)^*\right) k_S(x,y)\right)_{x,y\in F} a, a \right\rangle \\
= \;& \sum_{i,j=1}^{n} \overline{a_i} a_j (1 - \psi(z_i)\overline{\psi(w_j)}) k_S(z_i, w_j) \\
= \;& \sum_{i,j=1}^{n} \overline{a_i} a_j (1 - \psi(z_i)\overline{\psi(w_j)}) \langle k_{w_j}, k_{z_i} \rangle \\
= \;& \sum_{i,j=1}^{n} \overline{a_i} a_j \langle k_{w_j}, k_{z_i} \rangle - \sum_{i,j=1}^{n} \overline{a_i} a_j \langle \overline{\psi(w_j)} k_{w_j}, \overline{\psi(z_i)} k_{z_i} \rangle \\
= \;& \sum_{i,j=1}^{n} \langle a_j k_{w_j}, a_i k_{z_i} \rangle - \sum_{i,j=1}^{n} \langle a_j \overline{\psi(w_j)} k_{w_j}, \overline{a_i \psi(z_i)} k_{z_i} \rangle \\
= \;& \sum_{i=1}^{n} \|a_i k_{z_i}\|^2 - \sum_{i=1}^{n} \|\overline{a_i \psi(z_i)} k_{z_i}\|.
\end{aligned}
$$

Let $m = \max_{1 \le i \le n} |\psi(z_i)|$. Then

$$\left\langle \left((1 - \psi(x)\psi(y)^*) \, k_S(x,y) \right)_{x,y \in F} a, a \right\rangle \ge \sum_{i=1}^{n} \|a_i k_{z_i}\|^2 (1 - m^2) \ge 0.$$

The next theorem shows that $H^\infty(\mathcal{K}_\Psi)$ is a complete space.

Theorem A.1.2 ([23, Lemma 2.15])**.** *The space $H^\infty(\mathcal{K}_\Psi)$ is complete in the norm topology. Furthermore, its norm closed unit ball $H_1^\infty(\mathcal{K}_\Psi)$ is closed in both the topology of pointwise convergence and the topology of uniformly convergence on compact subsets of X.*

A.2 Multiplication operators

It can be shown that the multipliers of the Hardy-Hilbert space $H^2_{\mathbb{C}^2}$, i.e. the functions ϕ such that ϕf is in $H^2_{\mathbb{C}^2}$ whenever f is in $H^2_{\mathbb{C}^2}$, are precisely $H^\infty(\mathbb{D})$, the bounded analytic functions on $\mathbb{D}$ ([7, Theorem 3.24]). Moreover, $\|M_\phi\| = \|\phi\|_{H^\infty(\mathbb{D})}$. Now we claim that evaluation at any point z in $\mathbb{D}$ is a continuous linear functional on $H^2_{\mathbb{C}^2}$. Recall that for every $z \in \mathbb{D}$, the linear evaluation functional $E_z : H^2_{\mathbb{C}^2} \to \mathbb{D}$ is defined by $E_z(f) = f(z)$. Indeed we compute

$$\begin{aligned} |E_z(f)| &= |\sum_{n=0}^{\infty} a_n z^n| \le \sum_{n=0}^{\infty} |a_n||z^n| \\ &\le \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} |z|^{2n} \right)^{1/2} = \|f\| \cdot \frac{1}{\sqrt{1-|z|^2}}. \end{aligned}$$

This shows that every power series in $H^2_{\mathbb{C}^2}$ converges to a function on the disk. Moreover the map E_z is bounded with $\|E_z\| \le \frac{1}{\sqrt{1-|z|^2}}$, and so claim is proved. (This also shows that $H^2_{\mathbb{C}^2}$ is a RKHS on D.)

For a point $\zeta \in \mathbb{D}$, note that

$$g(z) = \sum_{n=0}^{\infty} \overline{\zeta}^n z^n \in H^2_{\mathbb{C}^2}$$

and for any $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2_{\mathbb{C}^2}$ we have that

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \zeta^n = f(\zeta).$$

Thus, g is the reproducing kernel for ζ and so

$$K(z, \zeta) = k_\zeta(z) = g(z) = \sum_{n=0}^{\infty} \overline{\zeta}^n z^n = \frac{1}{1 - \overline{\zeta} z}.$$

So we have that

$$\langle f, k_\zeta \rangle = f(\zeta)$$

for all $\zeta \in \mathbb{D}$ for any $H^2_{\mathbb{C}^2}$.

The observation is that the kernel functions are *eigenvectors for the adjoints of multiplication operators*:

$$\langle f, M_\phi^* k_\zeta \rangle = \langle \phi f, k_\zeta \rangle = \phi(\zeta) f(\zeta) = \langle f, \overline{\phi(\zeta)} k_\zeta \rangle \qquad \forall f \in H^2_{\mathbb{C}^2}.$$

Hence $M_\phi^* k_\zeta = \overline{\phi(\zeta)} k_\zeta$.

Theorem A.2.1 ([40]). *Let $f \in H^\infty(\mathbb{D})$ be an inner function, then the multiplication operator $M_f : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is an isometry and the range of M_f is a reproducing kernel Hilbert space with the kernel $\frac{f(z)\overline{f(w)}}{1 - z\overline{w}}$.*

Proof. Since f is an inner function, for any $\varphi \in H^2(\mathbb{T})$, we have that

$$\|M_f \varphi\|^2 = \|f\varphi\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})\varphi(e^{it})|^2 \, dt = \frac{1}{2\pi} \int_0^{2\pi} |\varphi(e^{it})|^2 \, dt = \|\varphi\|^2,$$

and so M_f is an isometry. By [40, Proposition 6.2] the kernel function $\frac{f(z)\overline{f(w)}}{1 - z\overline{w}}$ is kernel for range of M_f. $\square$

A.3 The Shilov boundary

Let Ω be a domain in $\mathbb{C}^d$ and $\overline{\Omega}$ be its closure. Let also $C(\Omega)$ be the space of all continuous complex-valued functions on $\overline{\Omega}$, with supremum norm. Then a closed subalgebra $\mathcal{A}(\Omega) \subseteq C(\Omega)$ is called a **uniform algebra** if $1 \in \mathcal{A}(\Omega)$ and $\mathcal{A}(\Omega)$ separates the points of Ω.

The notion of Shilov or distinguished boundary is a useful boundary because of being the smallest boundary in the following sense:

Definition A.3.1 ([30, 1])**.** Let $\mathcal{A}(\Omega)$ be a uniform algebra on $\Omega \in \mathbb{C}^d$. A **boundary** for $\overline{\Omega}$ is a subset X of $\overline{\Omega}$ such that every function in $\mathcal{A}(\Omega)$ attains its maximum modulus on the set X. By definitin 3.3.8 and [41, Proposition 6.4] if $\overline{\Omega}$ is polynomially convex, then there is a smallest closed boundary of $\overline{\Omega}$ that is contained in all closed boundaries of $\overline{\Omega}$. We call this boundary the **Shilov** or **distinguished** boundary of $\overline{\Omega}$ and denote it by $\partial\overline{\Omega}$.

Example A.3.2. Let $\overline{\Omega}$ be the closed bidisk, i.e.

$$\overline{\Omega} = \overline{\mathbb{D}} \times \overline{\mathbb{D}} = \overline{\mathbb{D}}^2 = \{(z, w) \in \mathbb{C}^2 : |z| \leq 1, |w| \leq 1\}.$$

Its topological boundary is

$$\mathbb{T} \times \overline{\mathbb{D}} \cup \overline{\mathbb{D}} \times \mathbb{T} = \{(z, w) \in \mathbb{C}^2 : |z|, |w| \leq 1, |z| = 1 \text{ or } |w| = 1\}$$

whereas its Shilov boundary is

$$\partial\overline{\Omega} = \mathbb{T} \times \mathbb{T} = \{(z, w) \in \mathbb{C}^2 : |z| = |w| = 1\}.$$

The last statement is obtained by applying twice the maximum modulus principle with respect to each complex variable. Note that by exactly the same argument we can see that

$$\partial\overline{\mathbb{D}}^n = \mathbb{T}^n.$$

Bibliography

[1] A. A. Abouhajar, M. C. White, and N. J. Young. A Schwarz lemma for a domain related to μ-synthesis. *J. Geom. Anal.*, 17(4):717–750, 2007.

[2] Jim Agler. Some interpolation theorems of nevanlinna–pick type. *preprint.*

[3] Jim Agler. Rational dilation on an annulus. *Ann. of Math. (2)*, 121(3):537–563, 1985.

[4] Jim Agler. On the representation of certain holomorphic functions defined on a polydisc. 48:47–66, 1990.

[5] Jim Agler, John Harland, and Benjamin J. Raphael. Classical function theory, operator dilation theory, and machine computation on multiply-connected domains. *Mem. Amer. Math. Soc.*, 191(892):viii+159, 2008.

[6] Jim Agler, Z. Lykova, and N. Young. Geodesics, retracts, and the norm-preserving extension property in the symmetrized bidisc. *to appear in Mem. Amer. Math. Soc. ArXiv e-prints:1603.04030v2*, March 2016.

[7] Jim Agler and John E. McCarthy. *Pick interpolation and Hilbert function spaces*, volume 44 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[8] Jim Agler and John E. McCarthy. Norm preserving extensions of holomorphic functions from subvarieties of the bidisk. *Ann. of Math. (2)*, 157(1):289–312, 2003.

[9] Jim Agler and John E. McCarthy. Distinguished varieties. *Acta Math.*, 194(2):133–153, 2005.

[10] Jim Agler and N. J. Young. A commutant lifting theorem for a domain in $\mathbf{C}^2$ and spectral interpolation. *J. Funct. Anal.*, 161(2):452–477, 1999.

[11] Jim Agler and N. J. Young. The hyperbolic geometry of the symmetrized bidisc. *J. Geom. Anal.*, 14(3):375–403, 2004.

[12] Herbert Alexander and John Wermer. *Several complex variables and Banach algebras*, volume 35 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 1998.

[13] T. Andô. On a pair of commutative contractions. *Acta Sci. Math. (Szeged)*, 24:88–90, 1963.

[14] William Arveson. Subalgebras of C^*-algebras. II. *Acta Math.*, 128(3-4):271–308, 1972.

[15] Joseph A. Ball and Moisés D. Guerra Huamán. Convexity analysis and the matrix-valued Schur class over finitely connected planar domains. *J. Operator Theory*, 70(2):531–571, 2013.

[16] Rajendra Bhatia. *Positive definite matrices*. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2007.

[17] Tirthankar Bhattacharyya, Sourav Pal, and Subrata Shyam Roy. Dilations of Γ-contractions by solving operator equations. *Adv. Math.*, 230(2):577–606, 2012.

[18] David P. Blecher and Christian Le Merdy. *Operator algebras and their modules—an operator space approach*, volume 30 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, Oxford, 2004. Oxford Science Publications.

[19] Adam Broschinski. Eigenvalues of Toeplitz operators on the annulus and Neil algebra. *Complex Anal. Oper. Theory*, 8(5):1037–1059, 2014.

[20] Andrew Browder. *Introduction to function algebras*. W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[21] Raúl E. Curto. Applications of several complex variables to multiparameter spectral theory. In *Surveys of some recent results in operator theory, Vol. II*, volume 192 of *Pitman Res. Notes Math. Ser.*, pages 25–90. Longman Sci. Tech., Harlow, 1988.

[22] Kenneth R. Davidson, Vern I. Paulsen, Mrinal Raghupathi, and Dinesh Singh. A constrained Nevanlinna-Pick interpolation problem. *Indiana Univ. Math. J.*, 58(2):709–732, 2009.

[23] Michael A. Dritschel. Realizations via Preorderings with Application to the Schur Class. *ArXiv e-prints*, March 2015.

[24] Michael A. Dritschel, Michael T. Jury, and Scott McCullough. Dilations and constrained algebras. *Oper. Matrices*, 10(4):829–861, 2016.

[25] Michael A. Dritschel, Stefania Marcantognini, and Scott McCullough. Interpolation in semigroupoid algebras. *J. Reine Angew. Math.*, 606:1–40, 2007.

[26] Michael A. Dritschel and Scott McCullough. The failure of rational dilation on a triply connected domain. *J. Amer. Math. Soc.*, 18(4):873–918, 2005.

[27] Michael A. Dritschel and Scott McCullough. Test functions, kernels, realizations and interpolation. In *Operator theory, structured matrices, and dilations*, volume 7 of *Theta Ser. Adv. Math.*, pages 153–179. Theta, Bucharest, 2007.

[28] Michael A. Dritschel and James Pickering. Test functions in constrained interpolation. *Trans. Amer. Math. Soc.*, 364(11):5589–5604, 2012.

[29] Theodore W. Gamelin. *Uniform algebras*. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1969.

[30] Kenneth Hoffman. *Banach spaces of analytic functions*. Dover Publications, Inc., New York, 1988. Reprint of the 1962 original.

[31] Richard B. Holmes. *Geometric functional analysis and its applications*. Springer-Verlag, New York-Heidelberg, 1975. Graduate Texts in Mathematics, No. 24.

[32] Greg Knese. Polynomials defining distinguished varieties. *Trans. Amer. Math. Soc.*, 362(11):5635–5655, 2010.

[33] Scott McCullough and Saida Sultanic. Ersatz commutant lifting with test functions. *Complex Anal. Oper. Theory*, 1(4):581–620, 2007.

[34] Jorge Mujica. *Complex analysis in Banach spaces*, volume 120 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1986. Holomorphic functions and domains of holomorphy in finite and infinite dimensions, Notas de Matemática [Mathematical Notes], 107.

[35] Gerard J. Murphy. *C^*-algebras and operator theory*. Academic Press, Inc., Boston, MA, 1990.

[36] Sourav Pal. From Stinespring dilation to Sz.-Nagy dilation on the symmetrized bidisc and operator models. *New York J. Math.*, 20:645–664, 2014.

[37] Sourav Pal. The failure of rational dilation on the tetrablock. *J. Funct. Anal.*, 269(7):1903–1924, 2015.

[38] Stephen Parrott. Unitary dilations for commuting contractions. *Pacific J. Math.*, 34:481–490, 1970.

[39] Vern Paulsen. *Completely bounded maps and operator algebras*, volume 78 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002.

[40] Vern I. Paulsen and Mrinal Raghupathi. *An introduction to the theory of reproducing kernel Hilbert spaces*, volume 152 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016.

[41] Robert R. Phelps. *Lectures on Choquet's theorem*. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1966.

[42] James Pickering. Counterexamples to rational dilation on symmetric multiply connected domains. *Complex Anal. Oper. Theory*, 4(1):55–95, 2010.

[43] Mrinal Raghupathi. Nevanlinna-Pick interpolation for $\mathbb{C} + BH^\infty$. *Integral Equations Operator Theory*, 63(1):103–125, 2009.

[44] Walter Rudin. Pairs of inner functions on finite Riemann surfaces. *Trans. Amer. Math. Soc.*, 140:423–434, 1969.

[45] Béla Sz.-Nagy. Sur les contractions de l'espace de Hilbert. *Acta Sci. Math. Szeged*, 15:87–92, 1953.

[46] Joseph L. Taylor. The analytic-functional calculus for several commuting operators. *Acta Math.*, 125:1–38, 1970.

[47] Joseph L. Taylor. A joint spectrum for several commuting operators. *J. Functional Analysis*, 6:172–191, 1970.

[48] N. Th. Varopoulos. On an inequality of von Neumann and an application of the metric theory of tensor products to operators theory. *J. Functional Analysis*, 16:83–100, 1974.

[49] Prasada Vegulla. *Geometry of distinguished varieties.* ProQuest LLC, Ann Arbor, MI, 2007. Thesis (Ph.D.)–Washington University in St. Louis.

[50] Johann von Neumann. Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes. *Math. Nachr.*, 4:258–281, 1951.

[51] N. E. Wegge-Olsen. *K-theory and C^*-algebras.* Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993. A friendly approach.