Homological properties of Banach and C*-algebras of continuous fields

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Abstract

One concern in the homological theory of Banach algebras is the identification of projective algebras and projective closed ideals of algebras. Besides being of independent interest, this question is closely connected to the continuous Hochschild cohomology.

In this thesis we give necessary and sufficient conditions for the left projectivity and biprojectivity of Banach algebras defined by locally trivial continuous fields of Banach algebras. We identify projective C*-algebras $\mathcal{A}$ defined by locally trivial continuous fields $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ such that each C*-algebra A_t has a strictly positive element. We also identify projective Banach algebras $\mathcal{A}$ defined by locally trivial continuous fields $\mathcal{U} = \{\Omega, (K(E_t))_{t\in\Omega}, \Theta\}$ such that each Banach space E_t has an extended unconditional basis.

In particular, for a left projective Banach algebra $\mathcal{A}$ defined by locally trivial continuous fields $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ we prove that Ω is paracompact. We also show that the biprojectivity of $\mathcal{A}$ implies that Ω is discrete. In the case $\mathcal{U}$ is a continuous field of elementary C*-algebras satisfying Fell's condition (not nessecarily a locally trivial field) we show that the left projectivity of $\mathcal{A}$ defined by $\mathcal{U}$, under some additional conditions on $\mathcal{U}$, implies paracompactness of Ω.

For the above Banach algebras $\mathcal{A}$ we give applications to the second continuous Hochschild cohomology group $\mathcal{H}^2(\mathcal{A}, X)$ of $\mathcal{A}$ and to the strong splittability of singular extensions of $\mathcal{A}$.

"There was one picture in particular which bothered him. It had begun with a leaf caught in the wind, and it became a tree; and the tree grew, sending out innumerable branches, and thrusting out the most fantastic roots."

Leaf by Niggle - J.R.R. Tolkien

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Contents

1 Introduction

1.1 History and recent work

In a purely algebraic setting the homological theory of algebras was first introduced by Hochschild, MacLane and Cartan and Eilenberg during 1940-1960, see [3, 14, 15, 24]. The theory has had a big influence on many areas of pure mathematics.

In 1962 the first paper on continuous Hochschild cohomology appeared [17].
Kamowitz used the second Banach cohomology groups to solve a Wedderburn decomposition problem for commutative Banach algebras. Later the Homological theory of Banach and C*-algebras was developed somewhat independently by the Moscow school led by Prof A. Ya. Helemskii [12] and by several Western authors, particularly, by Prof B. E. Johnson of Newcastle, England [16].

In 1970 Helemskii adapted purely algebraic homology theory to the continuous version and defined ($\mathbb{C}$-relative) projective Banach modules. He showed that as in the purely algebraic version, one can compute cohomology groups by constructing projective or injective resolutions of the corresponding module and the algebra.

The theory of continuous homology of Banach algebras is now well developed. It has applications in many branches of mathematics, including extensions of Banach algebras [2] and automatic continuity [6], spectral theory [11], the theory of de Rham homology [4] and K-theory.

For example, a research monograph by Bade, Dales and Lykova [2] on algebraic and strong splittings of extensions of Banach algebras includes many examples of Banach algebras A for which it is proven that there exists an extension of A which does not split algebraically or strongly. See also a recent paper by Laustsen and Skillicorn [19].

We should mention the most recent papers on projective and injective modules. In [8] Dales and Polyakov characterize some homological properties such as flatness and injectivity of Banach left $L^1(G)$-modules, where G is a locally compact group. In [29], Racher gives more examples of projective and flat modules over the Banach algebra $L^1(G)$ and connections of those properties with the compactness and amenability of G. See also [7] and [37].

In [12] Helemskii describes the projective ideals of $C_0(\Omega)$. He shows that a closed left ideal of $C_0(\Omega)$ is projective if and only if its spectrum is paracompact. We will generalise this property to the Banach algebras defined by continuous fields of Banach algebras.

In [20] Lykova proved the projectivity of $K(E)$ for some Banach spaces E. In [28] Phillips and Raeburn showed that all $\sigma-$unital C*-algebras are left projective. In [23] Lykova proves, using different methods, that every ideal of a separable C*-algebra is left projective.

In this thesis we generalise these results for C*-algebras defined by locally trivial continuous fields $\mathcal{U} = \{\Omega, A_t, \Theta\}$ where each A_t is a σ-unital C*-algebra. We also prove results on the right projectivity of Banach algebras defined by locally trivial continuous fields $\mathcal{U} = \{\Omega, K(E_t), \Theta\}$ where each E_t is a Banach space with an extended unconditional basis.

Some main results of this thesis are already published in my joint paper with Lykova [5].

1.2 Main results

The main results are the following.

Theorem 4.16. *Let Ω be a Hausdorff locally compact space with the topological dimension* $\dim \Omega \leq \ell$, *for some $\ell \in \mathbb{N}$, let $\mathcal{U} = \{\Omega, A_x, \Theta\}$ be a locally trivial continuous field of σ-unital C^*-algebras, and let the C^*-algebra $\mathcal{A}$ be defined by $\mathcal{U}$. Then the following conditions are equivalent:*
(i) *Ω is paracompact;*
(ii) *$\mathcal{A}$ is right projective and $\mathcal{U}$ is a disjoint union of σ-locally trivial continuous fields of C^*-algebras with strictly positive elements.*

Theorem 5.25 *Let Ω be a Hausdorff locally compact space. Let $\mathcal{U} = \{\Omega, (K(E_x)), \Theta\}$ be an ℓ-locally trivial continuous field of Banach algebras, for some $\ell \in \mathbb{N}$. Suppose, for each $x \in \Omega$, E_x is a separable Banach space with a hyperorthogonal basis $(e^x_n)_{n\in\mathbb{N}} \subset E_x$. Let $\mathcal{A}$ be the Banach algebra generated by $\mathcal{U}$.*
Then the following conditions are equivalent:
(i) *Ω is paracompact;*
(ii) *$\mathcal{A}$ is right projective and $\mathcal{U}$ is a disjoint union of σ-locally trivial continuous fields of Ba-*

nach algebras.

Recall definitions from Dixmier's book "Les C^*-algèbres et leurs représentations" [9, Section 2].

Definition 1.1 (Fell's condition)**.** *Let Ω be a locally compact Hausdorff space, and $\mathcal{U} = \{\Omega, A_x, \Theta\}$ a continuous field of elementary C*-algebras. $\mathcal{U}$ is said to satisfy Fell's condition if, for every $x \in \Omega$, there exists a neighbourhood U_x of x and a vector field p of $\mathcal{U}$, defined and continuous in U_x, such that, for every $t \in U_x$, $p(t)$ is a projection of rank 1. Note that $\mathcal{U}|U_x \cong \mathcal{U}(\mathcal{H}_x)$.*

Definition 1.2. *Let $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ be a continuous field of Banach algebras over Ω. Let $\Lambda \subset \Theta$. Then Λ is said to be total if, for every $t \in \Omega$, the set $x(t)$, as x runs through Λ, is total in A_t. $\mathcal{U}$ is said to be separable if Θ has a countable total subset.*

Theorem 6.29. *Let Ω be a locally compact Hausdorff space of finite dimension and $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ be a separable continuous field of elementary C*-algebras, of rank $\aleph_0$, satisfying Fell's condition. Let $\mathcal{A}$ be the C*-algebra defined by $\mathcal{U}$. Then the following are equivalent*

(i) Ω is paracompact.

(ii) $\mathcal{U}$ is a disjoint union of continuous fields of elementary C-algebras that satisfies the σ-Fell condition and $\mathcal{A}$ is left projective .*

See appendix A for a definition of paracompactness as well as known topological properties of paracompact spaces.

1.3 Description of results by sections

In Section 2 we investigate the properties of left and right projective Banach algebras defined by locally trivial continuous fields $\mathcal{U} = \{\Omega, (A_x)_{x\in\Omega}, \Theta\}$ of Banach algebras. We prove the projectivity of the Banach algebras $A_x, x \in \Omega$.

Proposition 2.3. *Let Ω be a locally compact topological space, let $\mathcal{U} = \{\Omega, (A_x)_{x\in\Omega}, \Theta\}$ be a locally trivial continuous field of Banach algebras and let A be the Banach algebra defined by $\mathcal{U}$. Suppose that A is projective in A-mod. Then the Banach algebras $A_x, x \in \Omega$ are uniformly left projective.*

Proposition 2.4 *Let $\mathcal{U} = \{\Omega, (A_x)_x, A\}$ be a locally trivial continuous field of Banach algebras such that every A_x has an identity e_{A_x} such that $\sup_{x\in\Omega} \|e_{A_x}\| \leq C$ for some constant C. Suppose that Ω is paracompact. Let A be the Banach algebra defined by $\mathcal{U}$. Then A is projective in A-mod.*

In Section 3 we prove the following results on the topological properties of Ω.

Proposition 3.4. *Let Ω be a locally compact Hausdorff space, let $\mathcal{U} = \{\Omega, (A_x)_{x\in\Omega}, \Theta\}$ be a σ-locally trivial continuous field of Banach algebras, and let A be the Banach algebra defined by $\mathcal{U}$. Suppose that A is projective in A-mod. Then Ω is paracompact.*

We also give an example of a locally trivial continuous field of left projective Banach algebras such that the algebra defined by this field is not left projective.

In Sections 4 and 5 we show sufficient conditions for the left projectivity of $\mathcal{A}$ defined by locally trivial continuous fields of Banach algebras and give proofs of Theorems 4.16 and 5.25. In Section 6 we study projectivity of C*-algebras with Fell's condition and give a proof of Theorem 6.29. In Section 7 we apply the methods used in Sections 2 and 3 to investigate the biprojectivity of Banach algebras defined by locally trivial fields of Banach algebras. This produces the following results.

Theorem 7.4 *Let Ω be a locally compact Hausdorff space and let $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ be a locally trivial continuous field of Banach algebras. Let A denote the Banach algebra defined by $\mathcal{U}$. If A is biprojective then the Banach algebras $(A_t)_{t\in\Omega}$ are uniformly biprojective.*

Theorem 7.5 *Let Ω be a locally compact Hausdorff space and let $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ be a locally trivial continuous field of Banach algebras. Let A denote the Banach algebra defined by $\mathcal{U}$. If A is biprojective then Ω is discrete.*

We also give examples of families of biprojective Banach algebras such that continuous field of them are biprojective for discrete Ω.

1.4 Definitions and notations

The unit circle $\mathbb{T}$ is defined as

$$\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

Let A be a Banach algebra. In this report we consider homological properties of Banach A-modules. The modules we are interested in here are Banach modules. The following definition of Banach modules can be found in [13].

Let A be a Banach algebra and let X be a Banach space. We say that X is a *left Banach A-module* if there exists a bounded bilinear operator

$$\begin{aligned} m : A \times X &\longrightarrow X \\ (a, x) &\longmapsto a \cdot x \end{aligned}$$

such that, for every $a, b \in A$ and $x \in X$, we have

$$a \cdot (b \cdot x) = (a \cdot b) \cdot x.$$

Right Banach A-modules are defined similarly.

Let X be a Banach space. We say that X is a *Banach A-bimodule* if it is a left Banach A-module and a right Banach A-module and if the following relation is satisfied

$$(a \cdot x) \cdot b = a \cdot (x \cdot b),$$

for every $a, b \in A$ and $x \in X$.

Let P and Q be left Banach A-modules. Let $\rho : P \to Q$ be a bounded continuous linear map. We say that ρ is a *morphism of left Banach A-modules* if $\rho(ax) = a\rho(x)$ for all $a \in A$, $x \in P$.
For a Banach algebra A we will denote the category of all left Banach A modules and morphisms of left (right) Banach A-modules by A-mod (mod-A) and the category of all Banach A bimodules and morphisms of Banach A-bimodules by A-mod-A.

1.4.1 The projective tensor product

The following definitions of the tensor products of vector spaces can be found in [13]. Let E and F be complex vector spaces and let $E \otimes F$ be the algebraic tensor product of E and F.

Suppose A and B are algebras. We make $A \otimes B$ into an algebra by setting multiplication as $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$. The algebra $A \otimes B$ is called the *algebraic tensor product* of A and B.

Let E and F be Banach spaces. The projective norm on $E \otimes F$ is defined by

$$\|u\| = \inf_{u = \sum_{i=1}^{n} a_i \otimes b_i} \sum_{i=1}^{n} \|a_i\| \|b_i\|,$$

where the infimum is taken over all expressions of u of the form $u = \sum_{i=1}^{n} a_i \otimes b_i$ with $n \in \mathbb{N}$, $a_i \in E$ and $b_i \in F$ for $i = 1, \ldots, n$. We call the completion of $E \otimes F$ with respect to this norm *the projective tensor product* of E and F and we denote it by $E\hat{\otimes}F$. When A and B are Banach algebras then $A\hat{\otimes}B$ is also a Banach algebra. This fact can be found in [13].

We will need the following lemmas to approximate the norm of some elements of the projective tensor product of two Banach spaces.

Lemma 1.3. *Let ξ be a primary nth root of unity and let $j \in \mathbb{Z}$. Then*

$$\sum_{k=1}^{n} (\xi^j)^k = \begin{cases} n & \text{if } n|j \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose that $n|j$. Then $\xi^j = 1$ which gives

$$\sum_{k=1}^{n} (\xi^j)^k = \sum_{k=1}^{n} 1 = n.$$

Now suppose otherwise. Then $\xi^j \neq 1$ and so

$$\sum_{k=1}^{n} (\xi^j)^k = \frac{(\xi^j)^{n+1} - \xi^j}{\xi^j - 1} = \frac{\xi^j((\xi^j)^n - 1)}{\xi^j - 1} = \frac{\xi^j((\xi^n)^j - 1)}{\xi^j - 1} = 0.$$

□

Lemma 1.4. *Let X and Y be Banach spaces. Suppose an element $u \in X\hat{\otimes}Y$ can be represented as*

$$u = \sum_{k=1}^{n} x_k \otimes y_k$$

and that ξ is a primary n^{th} root of unity, then

$$\|u\|_{X\hat{\otimes}Y} \leq \frac{1}{n} \sum_{k=1}^{n} \left\| \sum_{i=1}^{n} \xi^{ki} x_i \right\|_X \left\| \sum_{j=1}^{n} \xi^{-kj} y_j \right\|_Y.$$

Proof. Consider $v \in X\hat{\otimes}Y$,

$$v = \frac{1}{n} \sum_{k=1}^{n} \left[\left(\sum_{i=1}^{n} \xi^{ki} x_i \right) \otimes \left(\sum_{j=1}^{n} \xi^{-kj} y_j \right) \right].$$

Then by definition of the norm in $X\hat{\otimes}Y$,

$$\|v\|_{X\hat{\otimes}Y} \leq \frac{1}{n}\sum_{k=1}^{n}\left\|\sum_{i=1}^{n}\xi^{ki}x_i\right\|_X \left\|\sum_{j=1}^{n}\xi^{-kj}y_j\right\|_Y .$$

Therefore it is enough to show that $u = v$.

By Lemma 1.3, we have, for $i = 1,..,n$ and $j = 1,..,n$,

$$\sum_{k=1}^{n}\left(\xi^{i-j}\right)^k x_i = \begin{cases} nx_j & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

We then have

$$\begin{aligned} v &= \frac{1}{n}\sum_{k=1}^{n}\left[\left(\sum_{i=1}^{n}\xi^{ki}x_i\right)\otimes\left(\sum_{j=1}^{n}\xi^{-kj}y_j\right)\right] \\ &= \frac{1}{n}\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\xi^{ki}x_i\right)\otimes\left(\xi^{-kj}y_j\right) \\ &= \frac{1}{n}\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\xi^{k(i-j)}x_i\right)\otimes\left(y_j\right) \\ &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\left(\xi^{i-j}\right)^k x_i\right)\otimes\left(y_j\right) \\ &= \frac{1}{n}\sum_{j=1}^{n}\left(nx_j\right)\otimes\left(y_j\right) \\ &= \sum_{j=1}^{n}x_j\otimes y_j \\ &= u. \end{aligned}$$

□

Lemma 1.5. *Let X and Y be Banach spaces. Suppose an element $u \in X\hat{\otimes}Y$ can be represented as*

$$u = \sum_{l=1}^{m}\sum_{k=1}^{n}x_k^l\otimes y_k^l, \text{ where } x_k^l \in X, y_k^l \in Y, \text{ with indices } k = 1,...,n \text{ and } l = 1,...,m;$$

and that ξ is a primary m^{th} root of unity and that η is a primary n^{th} root of unity. Then

$$\|u\|_{X\hat{\otimes}Y} \leq \frac{1}{mn}\sum_{l=1}^{m}\sum_{k=1}^{n}\left\|\sum_{t=1}^{m}\sum_{i=1}^{n}\xi^{lt}\eta^{ki}x_i^t\right\|_X \left\|\sum_{s=1}^{m}\sum_{j=1}^{n}\xi^{-ls}\eta^{-kj}y_j^s\right\|_Y .$$

Proof. Similar to above, consider $v \in X \hat{\otimes} Y$,

$$v = \frac{1}{mn} \sum_{l=1}^{m} \sum_{k=1}^{n} \left[\left(\sum_{t=1}^{m} \sum_{i=1}^{n} \xi^{lt} \eta^{ki} x_i^t \right) \otimes \left(\sum_{s=1}^{m} \sum_{j=1}^{n} \xi^{-ls} \eta^{-kj} y_j^s \right) \right].$$

Then by definition of the norm in $X \hat{\otimes} Y$

$$\|v\|_{X \hat{\otimes} Y} \leq \frac{1}{mn} \sum_{l=1}^{m} \sum_{k=1}^{n} \left\| \sum_{t=1}^{m} \sum_{i=1}^{n} \xi^{lt} \eta^{ki} x_i^t \right\|_X \left\| \sum_{s=1}^{m} \sum_{j=1}^{n} \xi^{-ls} \eta^{-kj} y_j^s \right\|_Y .$$

Therefore it is enough to show that $u = v$.

By Lemma 1.3, we have, for $s = 1, .., m$, $t = 1, ..., m$, $j = 1, ..., n$ and $k = 1, .., n$,

$$\sum_{l=1}^{m} \left(\xi^{t-s}\right)^l \left(\eta^{i-j}\right)^k x_i^t = \begin{cases} m \left(\eta^{i-j}\right)^k x_i^s & \text{if } s = t \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_{k=1}^{n} m \left(\eta^{i-j}\right)^k x_i^s = \begin{cases} mnx_j^s & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

We then have

$$\begin{aligned} v &= \frac{1}{mn} \sum_{l=1}^{m} \sum_{k=1}^{n} \left[\left(\sum_{t=1}^{m} \sum_{i=1}^{n} \xi^{lt} \eta^{ki} x_i^t \right) \otimes \left(\sum_{s=1}^{m} \sum_{j=1}^{n} \xi^{-ls} \eta^{-kj} y_j^s \right) \right] \\ &= \frac{1}{mn} \sum_{l=1}^{m} \sum_{k=1}^{n} \sum_{t=1}^{m} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{s=1}^{m} \left[\left(\left(\xi^{t-s}\right)^l \left(\eta^{i-j}\right)^k x_i^t \right) \otimes \left(y_j^s \right) \right] \\ &= \frac{1}{mn} \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{s=1}^{m} \left[\left(m \left(\eta^{i-j}\right)^k x_i^s \right) \otimes \left(y_j^s \right) \right] \\ &= \frac{1}{mn} \sum_{j=1}^{n} \sum_{s=1}^{m} \left[\left(mnx_j^s \right) \otimes \left(y_j^s \right) \right] \\ &= \sum_{s=1}^{m} \sum_{j=1}^{n} x_j^s \otimes y_j^s \\ &= u. \end{aligned}$$

□

1.4.2 Continuous fields $\mathcal{U} = \{\Omega, (A_t), \Theta\}$ of Banach and C*-algebras

The following can be found in [10].

Definition 1.6. *Let Ω be a Hausdorff space. We say that Ω is locally compact if every point of Ω has a compact neighbourhood.*

Definition 1.7. *Let Ω be a locally compact Hausdorff space. We say a function $f : \Omega \to \mathbb{C}$ vanishes at infinity if for every $\varepsilon > 0$ there exists a compact subset K of Ω such that $|f(t)| < \varepsilon$ for every $t \in \Omega \setminus K$. We write $f(t) \to 0$ as $t \to \infty$.*

In this thesis we consider homological properties of Banach algebras defined by continuous fields of C*-algebras and Banach algebras. All of the definitions below can be found in [9].

Definition 1.8. *Let $(A_t)_{t\in\Lambda}$ be a family of Banach algebras. We denote by $\Pi_{t\in\Lambda}A_t$ the product space of $(A_t)_{t\in\Lambda}$ equipped with pointwise operations and the sup norm. Every element of $\Pi_{t\in\Lambda}A_t$ is called a vector field. More generally, if $Y \subset \Lambda$, an element of $\Pi_{t\in Y}A_t$ is called a vector field over Y.*

Definition 1.9. *A continuous field $\mathcal{U}$ of Banach algebras is a triple $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ where Ω is a locally compact Hausdorff space, $(A_t)_{t\in\Omega}$ is a family of Banach algebras and Θ is a subalgebra of $\Pi_{t\in\Omega}A_t$ such that*

(i) for every $t \in \Omega$, the set $x(t)$ for $x \in \Theta$ is dense in A_t;

(ii) for every $x \in \Theta$, the function $t \mapsto \|x(t)\|_{A_t}$ is continuous on Ω;

(iii) whenever $x \in \Pi_{t\in\Omega}A_t$ and, for every $t \in \Omega$ and every $\varepsilon > 0$, there is an $x' \in \Theta$ such that $\|x(s) - x'(s)\|_{A_s} \leq \varepsilon$ throughout some neighbourhood of t, it follows that $x \in \Theta$.

The elements of Θ are called the continuous vector fields of $\mathcal{U}$.

Definition 1.10. *Let $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ be a continuous field of Banach algebras over a locally compact Hausdorff space Ω. Let $\mathcal{A}$ be the norm closed subalgebra of Θ. Then it can be shown that $\mathcal{A}$ equipped with the norm $\|x\| = \sup_{t\in\Omega} \|x(t)\|_{A_t}$ is a Banach algebra which we call the Banach algebra defined by $\mathcal{U}$.*

For every $t \in \Omega$, the map $\tau_t : \mathcal{A} \to A_t : a \mapsto a(t)$ is a Banach algebra homomorphism with dense image and norm less than or equal to 1.

Definition 1.11. *Let $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ be a continuous field of Banach algebras over a locally compact Hausdorff space Ω. Let $Y \subset \Omega$ and $t_0 \in Y$. A vector field x over Y is said to be continuous at t_0 if, for every $\varepsilon > 0$, there is an $x' \in \Theta$ such that $\|x(t) - x'(t)\|_{A_t} \leq \varepsilon$ throughout some neighbourhood of t_0. The vector field x is said to be continuous on Y if it is continuous at every point of Y.*

Let $\Theta|_Y$ be the set of continuous vector fields over Y. A triple $\{Y, (A_t)_{t\in Y}, \Theta|_Y\}$ is a continuous field of Banach algebras over Y, which is called the field induced by $\mathcal{U}$ on Y, and which is denoted by $\mathcal{U}|_Y$.

Definition 1.12. *Let Ω be a locally compact Hausdorff space, and let*

$$\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\} \quad \text{and} \quad \mathcal{U}' = \{\Omega, (A'_t)_{t\in\Omega}, \Theta'\}$$

be two continuous fields of Banach algebras (C^*-algebras) over Ω. Let $(\phi_t)_{t\in\Omega}$ be a family of maps such that each ϕ_t is an isometric ($*$-)isomorphism of Banach algebras A_t onto A'_t.

Define the map ϕ by

$$\begin{aligned} \phi : \Pi_{t\in\Lambda} A_t &\to \Pi_{t\in\Lambda} A'_t \\ (a(t))_{t\in\Omega} &\mapsto (\phi_t(a(t)))_{t\in\Omega}. \end{aligned}$$

If $\phi(\Theta) = \Theta'$ we say that $\phi = (\phi_t)_{t\in\Omega}$ isomorphism of $\mathcal{U}$ onto $\mathcal{U}'$.

Let $\mathcal{A}$ be the Banach algebra defined by $\mathcal{U}$ and $\mathcal{A}'$ be the Banach algebra defined by $\mathcal{U}'$. Then one can see that, for all $x \in \mathcal{A}$,

$$\|\phi(x)(t)\|_{A'_t} = \|\phi_t(x(t))\|_{\mathcal{A}'} = \|x(t)\|_{A_t} \to 0 \ \text{as} \ t \to \infty.$$

Thus $\phi(A) = \phi(A')$.

Remark 1.13. *Let A be a Banach algebra, let Ω be a locally compact Hausdorff space, and let Θ be the algebra of continuous mappings from Ω into A. For every $t \in \Omega$, put $A_t = A$. It can be shown that, $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ is a continuous field of Banach algebras over Ω.*

Definition 1.14. *Let A be a Banach algebra and let Ω be a locally compact Hausdorff space. The continuous field of Banach algebras $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$, where $A_t = A$ for every $t \in \Omega$, is called the constant field over Ω defined by A. A field isomorphic to the constant field is called* trivial.

If every point of Ω possesses a neighbourhood V such that $\mathcal{U}|_V$ is trivial, then $\mathcal{U}$ is said to be locally trivial.

Definition 1.15. *We say that a continuous field $\mathcal{U}$ of Banach algebras over Ω σ-*locally *(n-*locally*)* satisfies a condition Γ *if there is an open cover $\{U_\mu\}$, $\mu \in \mathcal{M}$, of Ω such that each $\mathcal{U}|_{U_\mu}$ satisfies the condition Γ and, in addition, there is a countable (cardinality n, respectively) open cover $\{V_j\}$ of Ω such that $\overline{V_j} \subset U_{\mu(j)}$ for each j and some $\mu(j) \in \mathcal{M}$.*

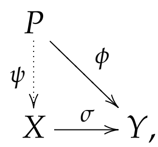
Remark 1.16. Let Ω be a Hausdorff locally compact space, and let $\mathcal{U} = \{\Omega, A_t, \Theta\}$ be a continuous field of Banach algebras which locally satisfies a condition Γ. Suppose Ω is σ-compact (compact) and Ω_0 is an open subset of Ω. Then $\mathcal{U}|_{\Omega_0}$ σ-locally (n-locally for some $n \in \mathbb{N}$, respectively) satisfies the condition Γ.

Definition 1.17. *Let Ω be a disjoint union of a family of open subsets $\{W_\mu\}$, $\mu \in \mathcal{M}$, of Ω. We say that $\mathcal{U} = \{\Omega, A_t, \Theta\}$* is a disjoint union of *$\mathcal{U}|_{W_\mu}$, $\mu \in \mathcal{M}$.*

Remark 1.18. Let Ω be a paracompact Hausdorff locally compact space, and let $\mathcal{U} = \{\Omega, A_t, \Theta\}$ be a continuous field of Banach algebras which locally satisfies a condition Γ. By [10, Theorem 5.1.27 and Problem 3.8.C(b)], the space Ω is the disjoint union of open-closed σ-compacts G_μ, $\mu \in \mathcal{M}$, of Ω. Suppose Ω_0 is an open subset of Ω. Then $\mathcal{U}|_{\Omega_0}$ is a disjoint union of $\mathcal{U}|_{G_\mu \cap \Omega_0}$, $\mu \in \mathcal{M}$, σ-locally satisfying the condition Γ.

1.4.3 Left projective Banach modules

Let A be a Banach algebra. Let P, X and Y be left Banach A-modules. Let $\sigma : X \to Y$ be an epimorphism of Banach A-modules. Let $\phi : P \to Y$ be a morphism of Banach A-modules. The map ψ is a *lifting* of ϕ if the following diagram commutes:



that is $\sigma \circ \psi = \phi$.
The *lifting problem* is whether or not there exists a lifting ψ which is a morphism of Banach A-modules. If such a morphism exists we say that the lifting problem has a positive solution. Otherwise it has a negative solution. A lifting problem is called admissible if σ is admissible, that is there exists a bounded linear operator $\alpha : Y \to X$ which is a right inverse to σ.

A left Banach A-module P is said to be *projective* in the category A-mod if every admissible lifting problem for P has a positive solution in A-mod. We say that a Banach algebra A is *left projective* if it is projective in the category A-mod.

We give an equivalent definition for projectivity which is the one that we will use most of the time. This can be found in [12]. We must first introduce a special morphism of modules. Let A be a Banach algebra and let X be a left Banach A-module.

We define the *unitisation* of A as $A_+ = A \oplus \mathbb{C}$ with multiplication given by $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$ for $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$ and $\|(a, \lambda)\| = \|a\| + |\lambda|$. The identity of A_+ is $e = (0, 1)$. We shall write $a + \lambda e$ for an element (a, λ).

Let X be a left Banach A-module. Then X is a left Banach A_+-module with multiplication defined by $(a, \lambda) \cdot x = a \cdot x + \lambda x$.

For $X \in A$-mod, consider the free left Banach A-module $A_+ \hat{\otimes} X$ with multiplication

$$a \cdot (b \otimes x) = ab \otimes x; \; a \in A, b \in A_+, x \in X.$$

The morphism of left Banach A-module

$$\begin{aligned} \pi_X : A_+ \hat{\otimes} X &\longrightarrow X \\ a \otimes x &\longmapsto a \cdot x. \end{aligned}$$

is known as the canonical projection. When it is obvious what module we are working with we will simply write π instead of π_X. It is known that π is admissible in A-mod since we can define the following bounded linear operator which is a right inverse to π:

$$\begin{aligned} \alpha : X &\longrightarrow A_+ \hat{\otimes} X \\ x &\longmapsto e \otimes x, \end{aligned}$$

where e is the identity in A_+.

Observe that

$$(\pi \circ \alpha)(x) = \pi(e \otimes x)e \cdot x = x$$

for every $x \in X$.

Note that α is not a morphism of left A-modules.

We now state the following theorem of Helemskii's:

Theorem 1.19. *[[13], p.168] Let P be a left Banach A-module. P is projective in A-mod if and only if there exists a morphism of left Banach A-modules $\rho : P \to A_+ \hat{\otimes} P$ such that $\pi \circ \rho$ is the identity on P.*

Proof. We give a sketch of a proof.

First suppose that P is projective in A-mod. Consider the following diagram

$$\begin{array}{ccc} P & & \\ & \searrow^{id_P} & \\ A_+ \hat{\otimes} P & \xrightarrow{\pi} & P. \end{array}$$

Since P is projective in A-mod there exists a morphism of modules ρ such that the following diagram commutes

$$\begin{array}{ccc} P & & \\ {\scriptstyle\rho}\,\vdots & \searrow^{id_P} & \\ A_+ \hat{\otimes} P & \xrightarrow{\pi} & P, \end{array}$$

Thus completing the claim.

Now suppose there exists a morphism of left Banach A-modules $\rho : P \to A_+ \hat{\otimes} P$ such that $\pi \circ \rho$ is the identity on P.

We show that $A_+ \hat{\otimes} P$ is projective in A-mod and the claim easily follows. Consider the following admissible lifting problem

$$\begin{array}{ccc} A_+ \hat{\otimes} P & & \\ {\scriptstyle\psi}\,\vdots & \searrow^{\phi} & \\ X & \xrightarrow{\sigma} & Y, \end{array}$$

where σ has a right inverse α. Let e be the identity of A_+. Defining $\psi(a \otimes x) = a(\alpha \circ \phi)(e \otimes x)$ makes the above diagram commute. $\square$

1.5 Cohomology groups

The following definitions can be found in [13, Ch 1 §3. Pages 71-72].

Let A be a Banach algebra, and let E be a Banach A-bimodule. The Banach space of bounded n-linear maps from $A \times \cdots \times A$ into E is denoted by $\mathcal{B}^n(A, E)$; the elements of $\mathcal{B}^n(A, E)$ are known as the the continuous n-cochains. We set $\mathcal{B}^0(A, E) = E$.

For $n \geq 1$ consider the map $\delta^n : \mathcal{B}^n(A, E) \to \mathcal{B}^{n+1}(A, E)$ given by $T \mapsto \delta^n T$, where

$$\delta^n T(a_1, ..., a_{n+1}) = a_1 \cdot T(a_2, ..., a_{n+1})$$

$$+ \sum_{k=1}^{n} (-1)^k T(a_1, ..., a_{k-1}, a_k a_{k+1}, .., a_{n+1})$$
$$+ (-1)^{n+1} T(a_1, ..., a_n) \cdot a_{n+1},$$

$a_i \in A, i = 1, \ldots, n+1$.

The map $\delta^0 : \mathcal{B}^0(A, E) \to \mathcal{B}^1(A, E)$ is defined by

$$\delta^0(x)(a) = a \cdot x - x \cdot a, a \in A, x \in E.$$

The spaces $\ker \delta^n$ and $\mathrm{im}\delta^{n-1}$ are denoted by $\mathcal{Z}^n(A, E)$ and $\mathcal{N}^n(A, E)$ respectively. Note that $\delta^{n+1}\delta^n = 0$ for all n and so $\mathcal{N}^n(A, E) \subset \mathcal{Z}^n(A, E)$.

The complex

$$0 \to \mathcal{B}^0(A, E) \xrightarrow{\delta^0} \mathcal{B}^1(A, E) \xrightarrow{\delta^1} \ldots$$

is called the standard cohomology complex.

Definition 1.20. *Let A be a Banach algebra and let E be a Banach A-bimodule. Let $n \in \mathbb{N}$. The n^{th} continuous cohomology group of A with coefficients in E is defined as*

$$\mathcal{H}^n(A, E) = \mathcal{Z}^n(A, E) / \mathcal{N}^n(A, E);$$

$$\mathcal{H}^0(A, E) = E.$$

2 Necessary conditions on $A_x, x \in \Omega$, for left projectivity of $\mathcal{A}$ defined by locally trivial fields $\mathcal{U} = \{\Omega, (A_t), \Theta\}$

In this section we prove results on a locally trivial continuous field of Banach algebras $\{\mathcal{U}, (A_t)_{t\in\Omega}, \Theta\}$ when the Banach algebra $\mathcal{A}$ defined by $\mathcal{U}$ is projective.

2.1 Left projectivity of $A_x, x \in \Omega$

Lemma 2.1. *Let Ω be a locally compact Hausdorff space, let $\mathcal{U} = \{\Omega, A_t, \Theta\}$ be a locally trivial continuous field of Banach algebras A_x and let $\mathcal{A}$ be the Banach algebra defined by $\mathcal{U}$. Let $y \in \Omega$. For every $a_y \in A_y$ there is $a \in \mathcal{A}$ such that $a(y) = a_y$.*

Proof. Fix $y \in \Omega$. By [18, Theorem 5.17], Ω is regular and, by [10, Theorem 3.3.1], Ω is a Tychonoff space. By assumption, $\mathcal{U}$ is locally trivial and so there are open neighbourhoods V_y and U_y of y such that $\overline{V_y} \subset U_y$, $\mathcal{U}|_{U_y}$ is trivial and V_y is relatively compact. Fix a continuous function $f_y \in C_0(\Omega)$ such that $0 \leq f_y \leq 1$, $f_y(y) = 1$ and $f_y|_{\Omega\setminus U_y} = 0$. Let $\phi = (\phi_x)_{x\in U_y}$ be an isomorphism of $\mathcal{U}|_{U_y}$ onto the trivial continuous field of Banach algebras over U_y where, for each $y \in \Omega$, ϕ_x is an isometric isomorphism of Banach algebras.

For an arbitrary element a_y, define a field a to be equal to $a(x) = f_y(x)(\phi_x^{-1} \circ \phi_y)(a_y)$, for every $x \in U_y$ and 0 otherwise. By property (iii) of Definition 1.9, the field a is continuous and $a \in \Theta$. Since $\|a(x)\|_{A_x} \to 0$ as $x \to \infty$, we have that $a \in \mathcal{A}$. □

Definition 2.2. *Let $(A_x)_{x\in\Omega}$ be a family of Banach algebras. We say that the Banach algebras A_x, $x \in \Omega$, are uniformly left projective if, for every $x \in \Omega$, there is a morphism of left Banach A_x-modules*

$$\rho_x : A_x \to (A_x)_+ \widehat{\otimes} A_x$$

such that $\pi_{A_x} \circ \rho_x = \mathrm{id}_{A_x}$ and $\sup_{x\in\Omega} \|\rho_x\|_{A_x} < \infty$.

Proposition 2.3. *Let Ω be a locally compact Hausdorff space, let $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ be a locally trivial continuous field of Banach algebras and let $\mathcal{A}$ be the Banach algebra defined by $\mathcal{U}$. Suppose that $\mathcal{A}$ is projective in $\mathcal{A}$-mod. Then the Banach algebras $(A_x)_{x\in\Omega}$ are uniformly left projective.*

Proof. Fix $x \in \Omega$. Since $\mathcal{U}$ is locally trivial, there exists an open neighbourhood $U_x \subset \Omega$ of x such that $\mathcal{U}|U_x$ is trivial. Let $\phi = \{\phi_t\}_{t\in U_x}$ be an isomorphism of $\mathcal{U}|U_x$ onto the trivial continuous field of Banach algebras $C_0(U_x, \tilde{A}_x)$ over U_x, where, for each $t \in U_x$,

$\phi_t : A_t \to \tilde{A}_x$ is an isometric isomorphism of Banach algebras.

By [18, Theorem 5.17], Ω is regular, and so there exists an open neighbourhood, $V_x \subset U_x$, of x such that $\overline{V_x} \subset U_x$. By [10, Theorem 3.3.1], Ω is Tychonoff and so there exists an $f_x \in C_0(\Omega)$ such that $0 \leq f_x \leq 1$, $f_x(x) = 1$ and $f_x|_{\Omega \setminus U_x} = 0$.

For $a_x \in A_x$ set $\tilde{a}_x(t) = f_x(t)(\phi_t^{-1} \circ \phi_x)(a_x)$, $t \in U_x$ and 0 otherwise. Then, by the proof of Lemma 2.1, $\tilde{a}_x \in \mathcal{A}$. It is clear that, for $a_x \in A_x$, we have $\tau_x(\tilde{a}_x) = a_x$.

Since $\mathcal{A}$ is projective in $\mathcal{A}$-mod, by Theorem 1.19, there exists a morphism of Banach left $\mathcal{A}$-modules $\rho : \mathcal{A} \to \mathcal{A}_+ \hat{\otimes} \mathcal{A}$ such that $\pi \circ \rho = 1_{\mathcal{A}}$. Now define

$$\begin{aligned} \tilde{\rho}_x : A_x &\longrightarrow (A_x)_+ \hat{\otimes} A_x \\ a_x &\longmapsto (\tau_{x_+} \otimes \tau_x)\rho(\tilde{a}_x) \end{aligned}$$

where $\tau_{x_+} : \mathcal{A}_+ \to (A_x)_+$ sends $a + \lambda e$ to $a(x) + \lambda e_x$, e and e_x are the adjoined identities in $\mathcal{A}_+$ and $(A_x)_+$ respectively. We now show that $\tilde{\rho}_x$ is a morphism of Banach left A_x-modules. We first show that it is linear. For $a_x, b_x \in A_x$, $\mu, \lambda \in \mathbb{C}$,

$$\begin{aligned} (\lambda a_x + \mu b_x)^{\sim} &= f_x(t)(\phi_t^{-1} \circ \phi_x)(\lambda a_x + \mu b_x) \\ &= \lambda f_x(t)(\phi_t^{-1} \circ \phi_x)(a_x) + \mu f_x(t)(\phi_t^{-1} \circ \phi_x)(b_x) \\ &= \lambda \tilde{a}_x + \mu \tilde{b}_x \end{aligned}$$

and so

$$\begin{aligned} \tilde{\rho}_x(\lambda a_x + \mu b_x) &= (\tau_{x_+} \otimes \tau_x)\rho((\lambda a_x + \mu b_x)^{\sim}) \\ &= (\tau_{x_+} \otimes \tau_x)\rho(\lambda \tilde{a}_x + \mu \tilde{b}_x) \\ &= (\tau_{x_+} \otimes \tau_x)(\lambda \rho(\tilde{a}_x) + \mu \rho(\tilde{b}_x)) \\ &= \lambda \tilde{\rho}_x(a_x) + \mu \tilde{\rho}_x(b_x). \end{aligned}$$

Let $a_x, b_x \in A_x$.

Set $\tilde{\tilde{a}}_x(t) = \sqrt{f_x(t)}(\phi_t^{-1} \circ \phi_x)(a_x)$, $t \in U_x$ and 0 otherwise. Likewise set $\tilde{\tilde{b}}_x(t) = \sqrt{f_x(t)}(\phi_t^{-1} \circ \phi_x)(b_x)$, $t \in U_x$ and 0 otherwise.

Note the following

$$\tau_x(\tilde{\tilde{a}}_x) = a_x,$$

$$\begin{aligned} \tau_x(\tilde{\tilde{b}}_x) &= b_x, \\ (a_x b_x)^{\sim} &= \tilde{\tilde{a}}_x \tilde{\tilde{b}}_x, \\ \tilde{a}_x \tilde{\tilde{b}}_x &= \tilde{\tilde{a}}_x \tilde{b}_x. \end{aligned}$$

Then

$$\begin{aligned} \tilde{\rho}_x(a_x b_x) &= (\tau_{x_+} \otimes \tau_x)\rho((a_x b_x)^{\sim}) \\ &= (\tau_{x_+} \otimes \tau_x)\rho(\tilde{\tilde{a}}_x \tilde{\tilde{b}}_x) \\ &= \tau_x(\tilde{\tilde{a}}_x)(\tau_{x_+} \otimes \tau_x)\rho(\tilde{\tilde{b}}_x) \\ &= \tau_x(\tilde{a}_x)(\tau_{x_+} \otimes \tau_x)\rho(\tilde{\tilde{b}}_x) \\ &= (\tau_{x_+} \otimes \tau_x)\rho(\tilde{a}_x \tilde{\tilde{b}}_x) \\ &= (\tau_{x_+} \otimes \tau_x)\rho(\tilde{\tilde{a}}_x \tilde{b}_x) \\ &= \tau_x(\tilde{\tilde{a}}_x)(\tau_{x_+} \otimes \tau_x)\rho(\tilde{b}_x) \\ &= a_x \tilde{\rho}_x(b_x). \end{aligned}$$

Let $a_x \in A_x$. Then

$$\|\tilde{a}_x\|_{\mathcal{A}} = \|f_x(\phi_\bullet^{-1} \circ \phi_x)(a_x)\|_{\mathcal{A}} \le \|f_x\| \|\phi_\bullet^{-1}\| \|\phi_x\| \|a_x\|_{A_x} \le \|a_x\|_{A_x}.$$

It is then clear that $\tilde{\rho}_x$ is bounded, since

$$\|\tilde{\rho}_x\|_{A_x} = \sup_{\|a_x\| \le 1} \|\tilde{\rho}_x(a_x)\|$$

$$= \sup_{\|\tilde{a}_x\| \le 1} \|(\tau_{x_+} \otimes \tau_x)\rho(\tilde{a}_x)\|_{A_{x_+} \hat{\otimes} A_x} \le \sup_{\|a_x\| \le 1} \|\tau_x\|^2 \|\rho\|_{\mathcal{A}} \|a_x\|_{\mathcal{A}} \le \|\rho\|_{\mathcal{A}}.$$

Therefore $\tilde{\rho}_x$ is a morphism of Banach left A_x-modules.

We now show that $\pi_{A_x} \circ \tilde{\rho}_x = 1_{A_x}$. We first show that

$$\pi_{A_x}((\tau_{x_+} \otimes \tau_x)u) = \tau_x(\pi(u))$$

for every $u \in \mathcal{A}_+ \hat{\otimes} \mathcal{A}$. By the linearity and boundness of π_{A_x}, τ_x and τ_{x_+} we only need to prove this when u is an elementary tensor. Let $a \in \mathcal{A}_+, b \in \mathcal{A}$. Then

$$\begin{aligned} \pi_{A_x}((\tau_{x_+} \otimes \tau_x)(a \otimes b)) &= \pi_{A_x}((\tau_{x_+}(a) \otimes \tau_x(b))) \\ &= \tau_{x_+}(a)\tau_x(b) \\ &= \tau_x(ab) \end{aligned}$$

$$= \tau_x(\pi(a \otimes b)).$$

Let $a_x \in A_x$. Then

$$\begin{aligned}(\pi_{A_x} \circ \tilde{\rho}_x)(a_x) &= \pi_{A_x}((\tau_{x_+} \otimes \tau_x)\rho(\tilde{a}_x)) \\ &= \tau_x(\pi_{\mathcal{A}}(\rho(\tilde{a}_x))) \\ &= \tau_x(\tilde{a}_x) \\ &= a_x.\end{aligned}$$

Therefore, by Theorem 1.19, the Banach algebras $(A_x)_{x\in\Omega}$ are uniformly left projective.

□

2.2 Banach algebras of continuous fields which are left projective

Proposition 2.4. *Let $\mathcal{U} = \{\Omega, (A_x)_{x\in\Omega}, \Theta\}$ be a locally trivial continuous field of Banach algebras and suppose that every A_x has an identity e_{A_x} such that $\sup_{x\in\Omega} \|e_{A_x}\|_{A_x} \leq C$ for some constant C. Suppose that Ω is paracompact. Let $\mathcal{A}$ be the Banach algebra defined by $\mathcal{U}$. Then $\mathcal{A}$ is projective in $\mathcal{A}$-mod.*

Proof. By assumption Ω is a paracompact locally compact Hausdorff space. Let $\mathcal{B} = \{V_\mu\}_{\mu\in\Lambda}$ be an open cover of Ω such that each point of Ω has a neighbourhood that intersects with no more than three sets of $\mathcal{B}$ as in [12]. By [18, Problem 5.W], since $\{V_\mu\}_{\mu\in\Lambda}$ is a locally finite open cover of the normal space Ω, it is possible to select a non-negative continuous function h_μ for each V_μ in $\mathcal{B}$ such that h_μ is 0 outside V_μ and is everywhere less than or equal to one, and

$$\sum_{\mu\in\Lambda} h_\mu(s) = 1 \text{ for all } s \in \Omega.$$

Set $g_\mu = \sqrt{h_\mu}$.

Consider a field $p \in \prod_{t\in\Omega} A_t$ such that $p(t) = e_{A_t}$. By assumption $\mathcal{U}$ is locally trivial. Therefore there is a neighbourhood U_t of t and $p' \in \Theta$ such that $p'(s) = p(s)$ for every $s \in U_t$ and so $p \in \Theta$ by Part (iii) of Definition 1.9. Since $g_\mu(t) \to 0$ as $t \to \infty$ we have that $g_\mu p \in \mathcal{A}$.

Order the finite subsets $N(\Lambda)$ of Λ by inclusion. Let $a \in \mathcal{A}$ and let $\lambda \in N(\Lambda)$. Define

$$y_{a,\lambda} = \sum_{\mu\in\lambda} g_\mu a \otimes g_\mu p \,.$$

Note that since $\|a(t)\| \to 0$ as $t \to \infty$, for any $\varepsilon > 0$, there exists a compact set K such that

$$\|a(t)\|_{A_t} < \frac{\varepsilon}{18C}$$

for all $t \notin K$.

Since $\{V_\mu\}_{\mu\in\Lambda}$ is a locally finite open cover of the normal space Ω, the compact set K intersects only with a finite number of sets $\{V_{\mu_1}, ..., V_{\mu_0}\}$ of $\mathcal{B}$. Take $\lambda_0 = (\mu_1, ..., \mu_{m_0}) \in N(\Lambda)$, then

$$\|g_\mu a\|_A < \frac{\varepsilon}{18C}$$

for every $\mu \notin \lambda_0$.

Let $\lambda_2 > \lambda_1 > \lambda_0$, where $\lambda_1 = (\mu_1, ..., \mu_{m_0}, ..., \mu_{m_1})$ and $\lambda_2 = (\mu_1, ..., \mu_{m_0}, ..., \mu_{m_1}, ...\mu_{m_2})$.

Note that

$$\begin{aligned}\|y_{a,\lambda_2} - y_{a,\lambda_1}\|_{\mathcal{A}\hat{\otimes}\mathcal{A}} =& \|y_{a,\lambda_2\backslash\lambda_0} + y_{a,\lambda_0} - (y_{a,\lambda_1\backslash\lambda_0} - y_{a,\lambda_0})\|_{\mathcal{A}\hat{\otimes}\mathcal{A}} \\ =& \|y_{a,\lambda_2\backslash\lambda_0} - y_{a,\lambda_1\backslash\lambda_0}\|_{\mathcal{A}\hat{\otimes}\mathcal{A}} \\ \leq& \|y_{a,\lambda_2\backslash\lambda_0}\|_{\mathcal{A}\hat{\otimes}\mathcal{A}} + \|y_{a,\lambda_1\backslash\lambda_0}\|_{\mathcal{A}\hat{\otimes}\mathcal{A}}.\end{aligned}$$

Let η be a primary $(m_2 - m_0)th$ root of unity. By Lemma 1.4,

$$\|y_{a,\lambda_2\backslash\lambda_0}\|_{\mathcal{A}\hat{\otimes}\mathcal{A}} \leq \frac{1}{m_2 - m_0} \sum_{t=1}^{m_2-m_0} \left\| \sum_{s=m_0+1}^{m_2} \eta^{ts} g_{\mu_s} a \right\|_{\mathcal{A}} \left\| \sum_{s=m_0+1}^{m_2} \eta^{-ts} g_{\mu_s} p \right\|_{\mathcal{A}}.$$

For any $x \in \Omega$ there are no more than three $\mu \in \Lambda$ such that $g_\mu(x) \neq 0$ which gives us

$$\left\| \sum_{s=m_0+1}^{m_2} \eta^{ts} g_{\mu_s} a \right\|_{\mathcal{A}} = \sup_{x\in\Omega} \left\| \sum_{s=m_0+1}^{m_2} \eta^{ts} g_{\mu_s}(x) a(x) \right\|_{A_x} \leq 3 \max_{\mu\in\lambda_2\backslash\lambda_0} \|g_\mu a\|_{\mathcal{A}} < \frac{\varepsilon}{6C}. \quad (2.1)$$

Similarly

$$\begin{aligned}\left\| \sum_{s=m_0+1}^{m_2} \eta^{-ts} g_{\mu_s} p \right\|_{\mathcal{A}} &= \sup_{x\in\Omega} \left\| \sum_{s=m_0+1}^{m_2} \eta^{-ts} g_{\mu_s}(x) p(x) \right\|_{A_x} \\ &\leq 3 \max_{\mu_s\in\lambda_2\backslash\lambda_0} \|g_{\mu_s} p\|_{\mathcal{A}} < 3C. \quad (2.2)\end{aligned}$$

This gives us

$$\|y_{a,\lambda_2\backslash\lambda_0}\|_{\mathcal{A}\hat{\otimes}\mathcal{A}} < \frac{\varepsilon}{2}.$$

Similarly

$$\|y_{a,\lambda_1\setminus\lambda_0}\|_{\mathcal{A}\hat{\otimes}\mathcal{A}} < \frac{\varepsilon}{2},$$

and so

$$\|y_{a,\lambda_2} - y_{a,\lambda_1}\|_{\mathcal{A}\hat{\otimes}\mathcal{A}} < \varepsilon.$$

This shows that $y_{a,\lambda}$ converges in $\mathcal{A}_+\hat{\otimes}\mathcal{A}$. We define the following map

$$\rho : \mathcal{A} \to \mathcal{A}_+\hat{\otimes}\mathcal{A}$$
$$a \mapsto \lim_{\lambda} y_{a,\lambda}.$$

Let us show that ρ is a morphism of left Banach $\mathcal{A}$-modules.
For $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$ we have

$$\begin{aligned}
\rho(\alpha a + \beta b) &= \lim_{\lambda} y_{\alpha a+\beta b,\lambda} \\
&= \lim_{\lambda} \sum_{\mu\in\lambda} g_\mu(\alpha a + \beta b) \otimes g_\mu p \\
&= \lim_{\lambda} \sum_{\mu\in\lambda} g_\mu \alpha a \otimes g_\mu p + \lim_{\lambda} \sum_{\mu\in\lambda} g_\mu \beta b \otimes g_\mu p \\
&= \lim_{\lambda} \alpha \sum_{\mu\in\lambda} g_\mu a \otimes g_\mu p + \lim_{\lambda} \beta \sum_{\mu\in\lambda} g_\mu b \otimes g_\mu p \\
&= \alpha\rho(a) + \beta\rho(b).
\end{aligned}$$

For all $a, b \in \mathcal{A}$,

$$\begin{aligned}
\rho(ab) &= \lim_{\lambda} y_{ab,\lambda} \\
&= \lim_{\lambda} \sum_{\mu\in\lambda} g_\mu ab \otimes g_\mu p \\
&= a \lim_{\lambda} \sum_{\mu\in\lambda} g_\mu b \otimes g_\mu p \\
&= a\rho(b).
\end{aligned}$$

Let $a \in \mathcal{A}$ and $\lambda = (\mu_1, ..., \mu_m) \in N(\Lambda)$. Let η be a primary mth root of unity. Another application of Lemma 1.4 yields,

$$\|y_{a,\lambda}\|_{\mathcal{A}\hat{\otimes}\mathcal{A}} \leq \frac{1}{m} \sum_{t=1}^{m} \left\| \sum_{s=1}^{m} \eta^{ts} g_{\mu_s} a \right\|_{\mathcal{A}} \left\| \sum_{s=1}^{m_2} \eta^{-ts} g_{\mu_s} p \right\|_{\mathcal{A}} \leq 9C\|a\|_{\mathcal{A}}.$$

Thus $\|\rho\| \leq 9C$ and so ρ is bounded. Therefore ρ is a morphism of left Banach $\mathcal{A}$-modules.

It remains to show that $\pi \circ \rho = 1_{\mathcal{A}}$ in order for $\mathcal{A}$ to be left projective. Let $a \in \mathcal{A}$,

$$\begin{aligned}(\pi \circ \rho)(a) &= \pi(\lim_{\lambda} y_{a,\lambda}) \\ &= \pi\left(\lim_{\lambda} \sum_{\mu \in \lambda} g_\mu a \otimes g_\mu p\right) \\ &= \lim_{\lambda} \sum_{\mu \in \lambda} g_\mu^2 a p \\ &= \lim_{\lambda} \left(\sum_{\mu \in \lambda} h_\mu\right) a p \\ &= a\end{aligned}$$

□

Below we will need the following result.

Theorem 2.5 ([32], Corollary 3.35)**.** *Let A be a Banach algebra and let B be a closed subalgebra of A which contains a right bounded approximate identity for A. If B is left projective then A is left projective.*

We now give a result on the projectivity of trivial bundles of Banach algebras.

Proposition 2.6. *Let Ω be a compact Hausdorff space and let B be a Banach algebra with a right bounded approximate identity. Let $A = C(\Omega, B)$. Then A is left projective if and only if B is left projective.*

Proof. If A is left projective we have that B is left projective from Proposition 2.3.

Suppose that B is left projective. Let $f_e \in C(\Omega)$ be the identity. For $b \in B$ define $f_e b \in A$ by $(f_e b)(t) = b$. Set $f_e B = \{f_e b : b \in B\}$.

Let us prove that $f_e B$ is closed in A.

Suppose that $\{u_n\} = \{f_e b_n\}$, where $b_n \in B$ for all n, is a sequence in $f_e B$ such that

$$\lim_{n\to\infty} u_n = a \in A \text{ with respect to } \|.\|.$$

Let $t \in \Omega$. Then

$$\|b_n - a(t)\|_B \le \|f_e b_n - a\|_A = \|u_n - a\|_A.$$

Then, for every $t \in \Omega$,

$$\lim_{n\to\infty} b_n = a(t) \in B.$$

Since the limit is unique in a normed space, for all $t \in \Omega$, $a(t) = b$ for some $b \in B$. Thus $a = f_e b \in f_e B$ and so $f_e B$ is closed in A and $f_e B \cong B$.

By Theorem 2.5, to show that A is left projective it is enough to show that $f_e B$ contains a right bounded approximate identity for A.
Let $(x_\lambda)_{\lambda\in\Lambda}$ be a right bounded approximate identity in B with $\|x_\lambda\|_B \le C < \infty$. Let $f_e x_\lambda$ be as above. We claim that $(f_e x_\lambda)_{\lambda\in\Lambda}$ is a right bounded approximate identity in A.
Let $\varepsilon > 0$ and let $a \in A$. Since a is continuous on Ω, for each $t \in \Omega$ there exists an open neighbourhood U_t of t such that

$$\|a(s) - a(t)\|_B < \frac{\varepsilon}{2(1+C)}$$

for all $s \in U_t$. Note that $\{U_t\}_{t\in\Omega}$ is an open cover of Ω. By assumption, Ω is compact, hence there exists an finite subcover $\{U_{t_i}\}_{i=1}^n$ of $\{U_t\}_{t\in\Omega}$. Since $(x_\lambda)_{\lambda\in\Lambda}$ is a right bounded approximate identity in B, for every $i = 1, \dots, n$, there exists a $\lambda_i \in \Lambda$ such that, for all $\lambda > \lambda_i$,

$$\|a(t_i)x_\lambda - a(t_i)\|_B < \frac{\varepsilon}{2},$$

Pick $\lambda_0 \in \Lambda$ such that $\lambda_0 > \lambda_i$ for each $i = 1, ..., n$.

Then, for all $\lambda > \lambda_0$ and for all $i = 1, ..., n$,

$$\|a(t_i)x_\lambda - a(t_i)\|_B < \frac{\varepsilon}{2}.$$

For $s \in U_{t_i}$ and all $\lambda > \lambda_0$, we have

$$\begin{aligned}
&\|a(s)x_\lambda - a(s)\|_B \\
=&\|(a(s) + a(t_i) - a(t_i))x_\lambda - a(s) + a(t_i) - a(t_i)\|_B \\
\le&\|(a(s) - a(t_i))x_\lambda\|_B + \|a(t_i) - a(s)\|_B + \|a(t_i)x_\lambda - a(t_i)\|_B \\
<&\frac{C\varepsilon}{2(1+C)} + \frac{\varepsilon}{2(1+C)} + \frac{\varepsilon}{2} \\
=&\varepsilon.
\end{aligned}$$

Therefore, for $s \in \Omega$, and all $\lambda > \lambda_0$, since $s \in U_{t_j}$ for some j, we have

$$\|a(s)x_\lambda - a(s)\|_B \leq \varepsilon$$

and so

$$\|af_e x_\lambda - a\|_A \leq \varepsilon.$$

Thus $f_e x_\lambda$ is a right bounded approximate identity in A, and so, by Corollary 2.5, A is left projective. □

2.3 An example of a Banach algebra defined by a continuous field which is not left projective

Example 2.7. We now consider an example of a continuous field of Banach algebras $\mathcal{U}$ such that, for all $t \in \Omega$, A_t is left projective, but $\mathcal{A}$ defined by $\mathcal{U}$ is not left projective. Let $\mathcal{U} = \{\mathbb{N}, (\ell_t^2)_{t\in\mathbb{N}}, \prod_{t\in\mathbb{N}} \ell_t^2\}$ where $\ell_t^2 = \{x = (x_1 \dots x_t) : \|x\|_{\ell_t^2} = (\sum_{i=1}^t |x_i|^2)^{\frac{1}{2}}\}$ is the Banach algebra with pointwise operations and product. Set $1_{\ell_t^2} = (1, \dots, 1)$. Let $\mathcal{A}$ be the Banach algebra defined by $\mathcal{U}$. Note that $\mathcal{A}$ is the c_0-sum of the Banach algebras $(\ell_t^2)_{t\in\mathbb{N}}$ Then $1_{\ell_t^2}$ is an identity for ℓ_t^2. Note that $\|1_{\ell_t^2}\|_{\ell_t^2} = \sqrt{t}$ and so $\|1_{\ell_t^2}\|_{\ell_t^2} \to \infty$ as $t \to \infty$.

All of the ℓ_t^2 are left and right projective since they have an identity. Let $\rho_t : \ell_t^2 \to (\ell_t^2)_+ \hat{\otimes} \ell_t^2$ be a morphism of modules such that $\pi_t \circ \rho_t = id_{\ell_t^2}$. It is clear that $\rho_t(\ell_t^2)$ belongs to $\ell_t^2 \hat{\otimes} \ell_t^2$.

Fix $t \in \mathbb{N}$. Define $e_n = (0, ..., 1, ..., 0) \in \ell_t^2$ with the 1 in the nth position and zeros elsewhere. Let $\rho_t(e_n) = \sum_{i=1}^\infty \lambda_i^n a_i^n \otimes b_i^n$.

Then, for every $n \leq t$,

$$\begin{aligned}
\rho_t(e_n) &= \rho_t(e_n^2) = e_n \rho(e_n) \\
&= e_n \sum_{i=1}^\infty \lambda_i^n a_i^n \otimes b_i^n \\
&= \sum_{i=1}^\infty \lambda_i^n e_n a_i^n \otimes b_i^n \\
&= e_n \otimes \left[\sum_{i=1}^\infty \lambda_i^n (a_i^n)_n b_i^n \right] \qquad (\text{since } \sum_{i=1}^\infty \lambda_i^n (a_i^n)_n \in \mathbb{C}) \\
&= e_n \otimes u^n \qquad (\text{where } u^n \in \ell_t^2).
\end{aligned}$$

Note that $e_n = (\pi \circ \rho_n)(e_n) = \pi(e_n \otimes u^n) = e_n u^n$ and so $(u^n)_n = 1$.

We now define the following linear functional:

$$V : \ell_t^2 \hat{\otimes} \ell_t^2 \to \mathbb{C}$$
$$(x_1, ..., x_t) \otimes (y_1, ..., y_t) \mapsto \sum_{i=1}^{t} x_i y_i.$$

Then

$$V(\rho_t(e_n)) = V(e_n \otimes u^n) = 1,$$

and so

$$V(\rho_t \left(\sum_{n=1}^{t} e_n \right)) = t.$$

Thus

$$t = |V(\rho_t \left(\sum_{n=1}^{t} e_n \right))| \leq \|V\| \|\rho_t\| \| \sum_{n=1}^{t} e_n\| = \|V\| \|\rho_t\| \sqrt{t},$$

which shows that $\|\rho_t\| \geq \frac{\sqrt{t}}{\|V\|}$. An application of the Cauchy-Schwarz inequality $\|V\| = 1$ so we have $\|\rho_t\| \to \infty$ as $t \to \infty$. Thus, by Proposition 2.3, $\mathcal{A}$ is not projective.

3 Topological properties of Ω

3.1 A necessary condition on Ω for the left projectivity of $\mathcal{A}$ defined by $\mathcal{U} = \{\Omega, (A_t), \Theta\}$

In [12, Theorem 4] Helemskii proved a commutative C^*-algebra $A = C_0(\Omega)$ is projective in *A-mod* if and only if its spectrum Ω is paracompact.

Lemma 3.1. *Let A be a left projective Banach algebra and $A \neq \{0\}$. Then $A^2 \neq 0$.*

Proof. Suppose that $A^2 = \{0\}$. Pick $a \in A$ such that $a \neq 0$. Let $\rho : A \to A_+ \hat{\otimes} A$ be a morphism of Banach A-modules such that $\pi \circ \rho = 1_A$. By [31, Theorem 3.6.4], $\rho(a) = \sum_{i=1}^{\infty}(a_i + \lambda_i e) \otimes (b_i)$ where e is the identity in A_+, $a_i \in A$, $b_i \in A$, $\lambda_i \in \mathbb{C}$ and so $a_i + \lambda_i e \in A_+$.

Then $a = (\pi \circ \rho)a = \pi(\sum_{i=1}^{\infty}(a_i + \lambda_i e) \otimes (b_i)) = \sum_{i=1}^{\infty}(a_i b_i + \lambda_i b_i) = \sum_{i=1}^{\infty} \lambda_i b_i$.

Therefore

$$\begin{aligned} 0 &= \rho(0) \\ &= \rho(a^2) \\ &= a\rho(a) \\ &= a(\sum_{i=1}^{\infty}(a_i + \lambda_i e) \otimes (b_i)) \\ &= \sum_{i=1}^{\infty}(aa_i + \lambda_i a) \otimes (b_i) \\ &= \sum_{i=1}^{\infty} \lambda_i a \otimes b_i \\ &= a \otimes \sum_{i=1}^{\infty} \lambda_i b_i \\ &= a \otimes a. \end{aligned}$$

This implies that $a = 0$, so we have a contradiction. □

We extend Helemskii's result to the case of Banach algebras defined by continuous fields of Banach algebras.

Proposition 3.2. *Let Ω be a locally compact Hausdorff space. Let $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ be a locally trivial continuous field of Banach algebras, let $\mathcal{A}$ be the Banach algebra defined by $\mathcal{U}$ and let $u \in \mathcal{A}\hat{\otimes}\mathcal{A}$. Define the function*

$$\begin{aligned} F_u : \Omega \times \Omega &\longrightarrow \mathbb{R} \\ (s,t) &\longmapsto \|(\tau_s \otimes \tau_t)(u)\|_{A_s\hat{\otimes}A_t}. \end{aligned}$$

Then

(i) F_u is continuous on $\Omega \times \Omega$,

(ii) For every compact K, $F_u(s,t) \to 0$ as $t \to \infty$ uniformly for $s \in K$,

(iii) For every compact K, $F_u(s,t) \to 0$ as $s \to \infty$ uniformly for $t \in K$,

(iv) If $\rho : \mathcal{A} \to \mathcal{A}_+\hat{\otimes}\mathcal{A}$ is a morphism of left Banach $\mathcal{A}$-modules such that $\pi \circ \rho = 1_{\mathcal{A}}$ then, for $a \in \overline{\mathcal{A}^2}$ we have that $F_{\rho(a)}(s,s) \geq \|\tau_s(a)\|_{A_s}$ for all $s \in \Omega$.

Proof. By [31, Theorem 3.6.4], every element u from $\mathcal{A}\hat{\otimes}\mathcal{A}$ can be written as $\sum_{i=1}^{\infty} \lambda_i\, a_i \otimes b_i$, $\lambda_i \in \mathbb{C}$, $a_i \in \mathcal{A}, b_i \in \mathcal{A}$, where $\sum_{i=1}^{\infty} |\lambda_i| < \infty$ and the sequences $\{a_i\}, \{b_i\}$ converge to zero in $\mathcal{A}$ as $i \to \infty$.

(i) We shall show that F_u continuous at an arbitrary point $(s_0, t_0) \in \Omega \times \Omega$. Let U_{s_0} and U_{t_0} be neighbourhoods of s_0 and t_0 respectively such that $\mathcal{U}|U_{s_0}$ and $\mathcal{U}|U_{t_0}$ are trivial. Therefore there exists Banach algebras B_{s_0}, B_{t_0} such that $\mathcal{U}|U_{s_0} \cong C_0(U_{s_0}, B_{s_0})$, $\mathcal{U}|U_{t_0} \cong C_0(U_{t_0}, B_{t_0})$, and there exist isometric isomorphisms of Banach algebras $\phi_s : A_s \to B_{s_0}$ and $\psi_t : A_t \to B_{t_0}$.

Let $\varepsilon > 0$. Then there exists an N such that

$$\sum_{i=N+1}^{\infty} |\lambda_i| \|a_i\|_{\mathcal{A}} \|b_i\|_{\mathcal{A}} < \frac{\varepsilon}{4}.$$

Let the sequences $(\|a_i\|_{\mathcal{A}})_{i=1}^{\infty}$ and $(\|b_i\|_{\mathcal{A}})_{i=1}^{\infty}$ be bounded by C_a and C_b respectively. Let $C = \max\{C_a, C_b\}$. Choose D such that $\sum_{i=1}^{\infty} |\lambda_i| < D$.

Note that for each i we have the following

$$\begin{aligned} &\|(\phi_s \otimes \psi_t)(\tau_s \otimes \tau_t)(a_i \otimes b_i) - (\phi_{s_0} \otimes \psi_{t_0})(\tau_{s_0} \otimes \tau_{t_0})(a_i \otimes b_i)\|_{B_{s_0}\hat{\otimes}B_{t_0}} \\ =&\|(\phi_s \otimes \psi_t)(a_i(s) \otimes b_i(t)) - (\phi_{s_0} \otimes \psi_{t_0})(a_i(s_0) \otimes b_i(t_0))\|_{B_{s_0}\hat{\otimes}B_{t_0}} \\ =&\|\phi(\bar{a}_i)(s) \otimes \psi(\bar{b}_i)(t) - \phi(\bar{a}_i)(s_0) \otimes \psi(\bar{b}_i)(t_0)\|_{B_{s_0}\hat{\otimes}B_{t_0}} \\ =&\|\phi(\bar{a}_i)(s) \otimes \psi(\bar{b}_i)(t) - \phi(\bar{a}_i)(s_0) \otimes \psi(\bar{b}_i)(t_0) \end{aligned}$$

$$
\begin{aligned}
&+\phi(\bar{a}_i)(s)\otimes\psi(\bar{b}_i)(t_0)-\phi(\bar{a}_i)(s)\otimes\psi(\bar{b}_i)(t_0)\|_{B_{s_0}\hat{\otimes}B_{t_0}}\\
\leq&\|\phi(\bar{a}_i)(s)\otimes\psi(\bar{b}_i)(t)-\phi(\bar{a}_i)(s)\otimes\psi(\bar{b}_i)(t_0)\|_{B_{s_0}\hat{\otimes}B_{t_0}}\\
+&\|\phi(\bar{a}_i)(s_0)\otimes\psi(\bar{b}_i)(t_0)-\phi(\bar{a}_i)(s)\otimes\psi(\bar{b}_i)(t_0)\|_{B_{s_0}\hat{\otimes}B_{t_0}}\\
\leq&\|\phi(\bar{a}_i)(s)\otimes(\psi(\bar{b}_i)(t)-\psi(\bar{b}_i)(t_0))\|_{B_{s_0}\hat{\otimes}B_{t_0}}\\
+&\|(\phi(\bar{a}_i)(s_0)-\phi(\bar{a}_i)(s))\otimes\psi(\bar{b}_i)(t_0)\|_{B_{s_0}\hat{\otimes}B_{t_0}}\\
\leq&\|\phi(\bar{a}_i)(s)\|_{B_{s_0}}\|(\psi(\bar{b}_i)(t)-\psi(\bar{b}_i)(t_0))\|_{B_{t_0}}\\
+&\|\psi(\bar{b}_i)(t_0)\|_{B_{s_0}}\|(\phi(\bar{a}_i)(s_0)-\phi(\bar{a}_i)(s))\|_{B_{s_0}}\\
\leq&\|a_i\|_{\mathcal{A}}\|(\psi(\bar{b}_i)(t)-\psi(\bar{b}_i)(t_0))\|_{B_{t_0}}+\|b_i\|_{\mathcal{A}}\|(\phi(\bar{a}_i)(s_0)-\phi(\bar{a}_i)(s))\|_{B_{s_0}}
\end{aligned}
$$

where $\bar{a}_i = a_i|U_{s_0}$ and $\bar{b}_i = b_i|U_{t_0}$ for $i = 1\ldots.$

Since ϕ is continuous, let $W^i_{s_0} \subset V_{s_0}$ be an open neighbourhood of s_0 such that for all $s \in W^i_{s_0}$ we have

$$\|(\phi(\bar{a}_i)(s_0)-\phi(\bar{a}_i)(s))\|_{B_{s_0}} < \frac{\varepsilon}{4CDN}.$$

Similarly let $Y^i_{t_0} \subset V_{t_0}$ be an open neighbourhood of t_0 such that for all $t \in Y^i_{t_0}$ we have

$$\|(\psi(\bar{b}_i)(t)-\psi(\bar{b}_i)(t_0))\|_{B_{t_0}} < \frac{\varepsilon}{4CDN}.$$

Then

$$
\begin{aligned}
&\|(\phi_s\otimes\psi_t)(\tau_s\otimes\tau_t)(\lambda_i a_i\otimes b_i)-(\phi_{s_0}\otimes\psi_{t_0})(\tau_{s_0}\otimes\tau_{t_0})(\lambda_i a_i\otimes b_i)\|_{B_{s_0}\hat{\otimes}B_{t_0}}\\
\leq&|\lambda_i|\|a_i\|_{\mathcal{A}}\|(\psi(\bar{b}_i)(t)-\psi(\bar{b}_i)(t_0))\|_{B_{t_0}}+|\lambda_i|\|b_i\|_{\mathcal{A}}\|(\phi(\bar{a}_i)(s_0)-\phi(\bar{a}_i)(s))\|_{B_{s_0}}\\
<&|\lambda_i|\|a_i\|_{\mathcal{A}}\cdot\frac{\varepsilon}{4CDN}+|\lambda_i|\|b_i\|_{\mathcal{A}}\cdot\frac{\varepsilon}{4CDN}\\
<&\frac{\varepsilon}{2N}
\end{aligned}
$$

for all $(s,t) \in W^i_{s_0} \times Y^i_{t_0}$.

Set $W_{s_0} \times Y_{t_0} = \bigcap_{i=1}^N W^i_{s_0} \times Y^i_{t_0}$. Note that $W_{s_0} \times Y_{t_0}$ is open.
Then for all $(s,t) \in W_{s_0} \times Y_{t_0}$ we have

$$
\begin{aligned}
&\|(\phi_s\otimes\psi_t)(\tau_s\otimes\tau_t)(\sum_{i=1}^{\infty}\lambda_i a_i\otimes b_i)-(\phi_{s_0}\otimes\psi_{t_0})(\tau_{s_0}\otimes\tau_{t_0})(\sum_{i=1}^{\infty}\lambda_i a_i\otimes b_i)\|_{B_{s_0}\hat{\otimes}B_{t_0}}\\
&\leq\|(\phi_s\otimes\psi_t)(\tau_s\otimes\tau_t)(\sum_{i=1}^{N}\lambda_i a_i\otimes b_i)-(\phi_{s_0}\otimes\psi_{t_0})(\tau_{s_0}\otimes\tau_{t_0})(\sum_{i=1}^{N}\lambda_i a_i\otimes b_i)\|_{B_{s_0}\hat{\otimes}B_{t_0}}+\frac{\varepsilon}{2}
\end{aligned}
$$

$$\leq \sum_{i=1}^{N} \|(\phi_s \otimes \psi_t)(\tau_s \otimes \tau_t)(\lambda_i a_i \otimes b_i) - (\phi_{s_0} \otimes \psi_{t_0})(\tau_{s_0} \otimes \tau_{t_0})(\lambda_i a_i \otimes b_i)\|_{B_{s_0}\hat{\otimes} B_{t_0}} + \frac{\varepsilon}{2}$$
$$< \frac{N\varepsilon}{2N} + \frac{\varepsilon}{2}$$
$$= \varepsilon.$$

Thus $F_u(s,t)$ is continuous.

(ii) Let $\varepsilon > 0$. Similar to part (i) above pick $N_0 \in \mathbb{N}$ such that

$$\sum_{i=N_0+1}^{\infty} |\lambda_i| \|a_i\|_{\mathcal{A}} \|b_i\|_{\mathcal{A}} < \frac{\varepsilon}{2}.$$

Recall that the sequences $\{a_i\}, \{b_i\}$ converge to 0 in A as $i \to \infty$ and so the sequences $\{a_i\}, \{b_i\}$ are also bounded. There exists a $C \in \mathbb{R}$ such that $\|a_i\| < C$ for all i. We can therefore pick a compact subset $K \subset \Omega$ such that for all $t \in \Omega \setminus K$

$$\|b_i(t)\|_{A_t} < \frac{\varepsilon}{2M}, \quad \text{where } M > C \sum_{i=1}^{N_0} |\lambda_i| \tag{3.1}$$

for all $i = 1, \ldots, N_0$.

Then, by (3.1),

$$\begin{aligned} \sup_{s\in\Omega} F_u(s,t) &\leq \sup_{s\in\Omega} \left(\sum_{i=1}^{N_0} |\lambda_i| \|a_i(s)\|_{A_s} \|b_i(t)\|_{A_t} \right) + \frac{\varepsilon}{2} \\ &\leq \sum_{i=1}^{N_0} |\lambda_i| \|a_i\|_{\mathcal{A}} \|b_i(t)\|_{A_t} + \frac{\varepsilon}{2} \\ &< \sum_{i=1}^{N_0} |\lambda_i| \|a_i\|_{\mathcal{A}} \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

for all $t \in \Omega \setminus K$.

(iii) This follows in the same way as above.

(iv) Note that for $a \otimes b \in \mathcal{A}\hat{\otimes}\mathcal{A}$ we have that

$$\|\pi(a \otimes b)\|_{\mathcal{A}} = \|ab\|_{\mathcal{A}} \leq \|a\|_{\mathcal{A}} \|b\|_{\mathcal{A}} = \|a \otimes b\|_{\mathcal{A}\hat{\otimes}\mathcal{A}}.$$

Note that when $a \in \overline{\mathcal{A}^2}$ we have that $\rho(a) \in \mathcal{A}\hat{\otimes}\mathcal{A}$. Let $a \in \overline{\mathcal{A}^2}$ and we can view $\rho(a) = \sum_{i=1}^{\infty} a_i \otimes b_i$.

Thus, for all $s \in \Omega$,

$$\begin{aligned}
F_{\rho(a)}(s,s) &= \|(\tau_s \otimes \tau_s)\rho(a)\|_{\mathcal{A}_s\hat{\otimes}\mathcal{A}_s} \\
&\geq \|\pi_s((\tau_s \otimes \tau_s)(\rho(a)))\|_{\mathcal{A}_s} \\
&= \left\|\pi_s((\tau_s \otimes \tau_s)\left(\sum_{i=1}^{\infty} a_i \otimes b_i\right))\right\|_{\mathcal{A}_s} \\
&= \left\|\pi_s\left(\sum_{i=1}^{\infty} \tau_s(a_i) \otimes \tau_s(b_i)\right)\right\|_{\mathcal{A}_s} \\
&= \left\|\sum_{i=1}^{\infty}(\tau_s(a_i)\tau_s(b_i))\right\|_{\mathcal{A}_s} \\
&= \left\|\sum_{i=1}^{\infty}(\tau_s(a_ib_i))\right\|_{\mathcal{A}_s} \\
&= \left\|\tau_s(\sum_{i=1}^{\infty} a_ib_i)\right\|_{\mathcal{A}_s} \\
&= \|\tau_s(a)\|_{\mathcal{A}_s}.
\end{aligned}$$

□

The following defintion is a partial definition of Definition 1.15.

Definition 3.3. *Let Ω be a locally compact topological space. We say that a continuous field of Banach algebras $\mathcal{U} = \{\Omega, (A_x)_{x\in\Omega}, \Theta\}$ is σ-locally trivial if there is an open cover $\{U_\alpha\}$ of Ω such that each $\mathcal{U}|U_\alpha$ is trivial and that there is a countable open cover $\{V_j\}$ of Ω such that $\overline{V_j} \subset U_{\alpha(j)}$ for each j and some $\alpha(j)$.*

Proposition 3.4. *Let Ω be a locally compact Hausdorff space, let $\mathcal{U} = \{\Omega, (A_x)_{x\in\Omega}, \Theta\}$ be a σ-locally trivial continuous field of Banach algebras, and let $\mathcal{A}$ be the Banach algebra defined by $\mathcal{U}$. Suppose that $\mathcal{A}$ is projective in $\mathcal{A}$-mod or in mod-$\mathcal{A}$. Then Ω is paracompact.*

Proof. We give a proof for $\mathcal{A}$ which is projective in $\mathcal{A}$-mod. The case mod-$\mathcal{A}$ is similar.

By assumption, $\mathcal{U}$ is a σ-locally trivial continuous field. By Definition 3.3, there is an open cover $\{U_\alpha\}$ of Ω such that each $\mathcal{U}|_{U_\alpha}$ is trivial and, in addition, there is a countable open cover $\{V_j\}$ of Ω such that $\overline{V_j} \subset U_{\alpha(j)}$ for each j.

We shall split the proof into the following lemmas:

Lemma 3.5. *If $\overline{V_j}$ is paracompact for every j then Ω is paracompact.*

Proof. Let $\mathcal{B}$ be an arbitrary open cover of Ω. For each $j \in \mathbb{N}$, the family $\mathcal{B}_j = \{B \cap \overline{V_j} : B \in \mathcal{B}\}$ is an open cover of $\overline{V_j}$. By assumption, $\overline{V_j}$ is paracompact and so $\mathcal{B}_j$ has an open locally finite refinement $\mathcal{D}_j$ that is also a cover of $\overline{V_j}$. The family of open subsets $\mathcal{D}'_j = \{D \cap V_j : D \in \mathcal{D}_j\}$ is locally finite in Ω and is a refinement of $\mathcal{B}$. Furthermore, since $\Omega = \bigcup_{j \in \mathbb{N}} V_j$, the family $\mathcal{D} = \bigcup_{j \in \mathbb{N}} \mathcal{D}'_j$ is an open σ-locally finite cover of Ω. By [18, Theorem 5.28], Ω is paracompact. □

Therefore to prove Proposition 3.4 it is enough to show that, for every $j \in \mathbb{N}$, the topological space $\overline{V_j}$ is paracompact. Let us fix j and prove that $\overline{V_j}$ is paracompact.

Since $\mathcal{A}$ is left projective, there exists a morphism of left Banach $\mathcal{A}$-modules $\rho : \mathcal{A} \to \mathcal{A}_+ \widehat{\otimes} \mathcal{A}$ such that $\pi_{\mathcal{A}} \circ \rho = \mathrm{id}_{\mathcal{A}}$.

By Definition 3.3, for $\overline{V_j}$ there exists $\alpha(j)$ such that $\overline{V_j} \subset U_{\alpha(j)}$ such that $\mathcal{U}|_{U_{\alpha(j)}}$ is trivial. Let $\mathcal{U}|_{U_\alpha(j)} \cong \{U_{\alpha(j)}, B_{\alpha(j)}, C(U_{\alpha(j)}, B_{\alpha(j)})\}$ with the family of isometric isomorphisms $\phi = \{\phi_t\}_{t \in U_\alpha(j)}$. By Proposition 2.3, $B_{\alpha(j)}$ is left projective. By Lemma 3.1, there exists $x_0, y_0 \in B_{\alpha(j)}$ such that $x_0 y_0 \neq 0$. For $t \in U_\alpha(j)$ set

$$x(t) = \phi_t^{-1}(x_0),$$
$$y(t) = \phi_t^{-1}(y_0).$$

Then x and y are continuous vector fields on $U_{\alpha(j)}$ such that $p(t) = x(t)y(t) \neq 0$ for every $t \in U_{\alpha(j)}$.

By [10, Theorem 3.3.1], Ω is a Tychonoff space and so, for every $s \in \overline{V_j} \subset U_{\alpha(j)}$, there is $f_s \in C_0(\Omega)$ such that $0 \leq f_s \leq 1$, $f_s(s) = 1$ and $f_s(t) = 0$ for all $t \in \Omega \setminus U_{\alpha(j)}$. By Property (iii) of Definition 1.9 and [9, Proposition 10.1.9], the field $f_s p$ is continuous and $\|f_s(t)p(t)\| \to 0$ as $t \to \infty$, so we have $f_s p \in \mathcal{A}$.

For every $s \in \overline{V_j} \subset U_{\alpha(j)}$ and $t \in \Omega$, we set

$$\Phi(s,t) = F_{\rho(f_s p)}(s,t)$$

$f_s \in C_0(\Omega)$ such that $f_s(s) = 1$ and $f_s(t) = 0$ for all $t \in \Omega \setminus U_{\alpha(j)}$ and the function F is defined in Proposition 3.2.

Lemma 3.6. *The function Φ is well defined.*

Proof. Let $s \in \overline{V_j} \subset U_{\alpha(j)}$ and $f_s, g_s \in C_0(\Omega)$ such that $f_s(s) = g_s(s) = 1$ and $f_s(t) = g_s(t) = 0$ for all $t \in \Omega \setminus U_{\alpha(j)}$.

We have, for $t \in \Omega$,

$$\begin{aligned} F_{\rho(f_s\ p)}(s,t) &= \|(\tau_s \otimes \tau_t)\rho(f_s\ p)\|_{A_s \widehat{\otimes} A_t} \\ &= \|\sqrt{f_s(s)}x(s)(\tau_s \otimes \tau_t)\rho(\sqrt{f_s}\ y)\|_{A_s \widehat{\otimes} A_t} \\ &= \|(\tau_s \otimes \tau_t)\rho(\sqrt{g_s}\sqrt{f_s}\ xy)\|_{A_s \widehat{\otimes} A_t} \\ &= \|\sqrt{f_s(s)}x(s)(\tau_s \otimes \tau_t)\rho(\sqrt{g_s}\ y)\|_{A_s \widehat{\otimes} A_t} \\ &= \|(\tau_s \otimes \tau_t)\rho(\sqrt{g_s}\ p)\|_{A_s \widehat{\otimes} A_t} \\ &= \|(\tau_s \otimes \tau_t)\rho(g_s\ p)\|_{A_s \widehat{\otimes} A_t} \\ &= F_{\rho(g_s p)}(s,t). \end{aligned}$$

Thus Φ is independent of the choice of f_s. □

Lemma 3.7. *The function Φ is a continuous function on $\overline{V_j} \times \Omega$.*

Proof. Let $(s_0, t_0) \in \overline{V_j} \times \Omega$ and $f_{s_0} \in C_0(\Omega)$ such that $0 \le f_{s_0} \le 1$, $f_{s_0}(s_0) = 1$ and $f_{s_0}(t) = 0$ for all $t \in \Omega \setminus U_{\alpha(j)}$. Consider the neighbourhood $V = U \times \Omega$ of the point (s_0, t_0) where $U = \{s \in \overline{V_j} : f_{s_0}(s) \neq 0\}$. Then, for $(s,t) \in V$,

$$\begin{aligned} \Phi(s,t) &= F_{\rho\left(\frac{f_{s_0}}{f_{s_0}(s)}\ p\right)}(s,t) \\ &= \|(\tau_s \otimes \tau_t)\rho\left(\frac{f_{s_0}}{f_{s_0}(s)}\ p\right)\|_{A_s \widehat{\otimes} A_t} \\ &= \frac{1}{f_{s_0}(s)}\|(\tau_s \otimes \tau_t)\rho(f_{s_0}\ p)\|_{A_s \widehat{\otimes} A_t} \\ &= \frac{1}{f_{s_0}(s)}\ F_{\rho(f_{s_0}\ p)}(s,t). \end{aligned}$$

Hence Φ is the ratio of two continuous functions on V, so it is continuous at (s_0, t_0). □

Lemma 3.8. *For every compact $K \subset \overline{V_j}$, the function $\Phi(s,t) \to 0$ as $t \to \infty$ in Ω uniformly for $s \in K$.*

Proof. By [10, Theorem 3.1.7], since Ω is a Tychonoff space, for a compact subset $K \subset \overline{V_j} \subset \Omega$ and for a closed subset $\Omega \setminus U_{\alpha(j)} \subset \Omega \setminus K$, there is $f_K \in C_0(\Omega)$ such that $0 \leq f_K \leq 1$, $f_K(s) = 1$ for all $s \in K$ and $f_K(t) = 0$ for all $t \in \Omega \setminus U_{\alpha(j)}$. Note that $f_K p \in \mathcal{A}$.

By Proposition 3.2, the function $F_{\rho(f_K\, p)}(s,t) \to 0$ as $t \to \infty$ in Ω uniformly for $s \in \Omega$.

Thus the function $\Phi(s,t) = F_{\rho(f_K p)}(s,t)$ on $K \times \Omega \subset \overline{V_j} \times \Omega$ tends to 0 as $t \to \infty$ in Ω uniformly for $s \in K$. $\square$

Conclusion of the proof of Proposition 3.4

For $(s,t) \in \overline{V_j} \times \overline{V_j}$, we set

$$E(s,t) = \Phi(s,t)/\|p(s)\|_{A_s}.$$

By Proposition 3.2, $\Phi(s,s) \geq \|p(s)\|_{A_s}$ for every $s \in \overline{V_j}$. Therefore $E(s,s) \geq 1$ for every $s \in \overline{V_j}$.

For $(s,t) \in \overline{V_j} \times \overline{V_j}$, we also set

$$G(s,t) = \min\{E(s,t), 1\} \min\{E(t,s), 1\}.$$

By Lemmas 3.6, 3.7 and 3.8, the function $G(s,t)$ has the following properties:

(i) $G(s,t)$ is continuous on $\overline{V_j} \times \overline{V_j}$,

(ii) for every compact $K \subset \overline{V_j}$, $G(s,t) \to 0$ as $t \to \infty$ uniformly for $s \in K$,

(iii) for every compact $K \subset \overline{V_j}$, $G(s,t) \to 0$ as $s \to \infty$ uniformly for $t \in K$,

(iv) $G(s,s) = 1$ for all $s \in \overline{V_j}$.

By [13, Theorem A.12, Appendix A], $\overline{V_j}$ is paracompact. By Lemma 3.5, Ω is paracompact.

$\square$

4 Projectivity of C*-algebras defined by continuous fields of σ-unital C*-algebras

4.1 Projectivity of $C_0(\Omega, A)$ for paracompact Ω and for a σ-unital C*-algebra A

In the next two sections we give sufficient conditions for Banach algebras $\mathcal{A}$ defined by locally trivial fields $\mathcal{U} = \{\Omega, (A_x)_{x\in\Omega}, \Theta\}$ to be left and/or right projective.

Recall that a C*-algebra is said to be *σ-unital* if it contains a sequential bounded approximate identity. We generalise the following theorem.

Theorem 4.1 (Lykova [21], Phillips and Raeburn [28]). *Let A be a σ-unital C*-algebra. Then A is both right and left projective.*

Definition 4.2. *Let A be a C*-algebra and let a be a positive element in A. We say that a is strictly positive if $f(a) > 0$ for every non-zero positive linear functional f.*

Theorem 4.3 ([27], 3.10.5). *Let A be a C*-algebra. Then A is σ-unital if and only if it possesses a strictly positive element.*

The following results are from Phillips and Raeburn [28].

Lemma 4.4 ([28], Lemma 2.1). *Let $A \cong C_0(\Omega)$ be a commutative C*-algebra with a sequential approximate identity $\{u_n\}$. Then there is an increasing sequence $\{f_n\}$ in A such that*

(1) $0 \le f_n \le 1$ for all n,

(2) $\|f_n u_n - u_n\|_A \le \frac{1}{n}$ for all n, and

(3) $f_n f_m = f_m$ if $n > m$.

Proof. Let $f_0 = 0$ and suppose we have chosen $f_0, f_1, ..., f_{n-1}$ in A, compactly supported on Ω, satisfying (1),(2) and (3). Define $K_n = \{x \in \Omega | \hat{u_n}(x) \ge \frac{1}{n}\}$ and let $K = K_n \bigcup supp \hat{f}_{n-1}$. Now choose f_n such that $\hat{f}_n | K = 1$, $\hat{f}_n$ is compactly supported, and $0 \le f_n \le 1$. Clearly, $\{f_0, f_1, ..., f_n\}$ satisfy (1), (2) and (3). □

Lemma 4.5 ([28], Lemma 2.2). *Let A be a commutative C*-algebra and let $\{f_n\}$ be an increasing sequence in A satisfying properties (1) and (3) of Lemma 4.4. For each n, set $e_n = f_n - f_{n-1}$ and let $\hat{e}_n$ be the Gelfand transform of e_n. Then*

1. *$0 \le e_n \le 1$ for all n,*

2. $e_n e_m = 0$ *if* $|m - n| > 1$, *and*

3. $\left\|\sum_{j=1}^{n} \eta_k e_k^{\frac{1}{2}}\right\|_A \leq \sqrt{2}$ *for any* $\{\eta_1, ..., \eta_n\} \subset \mathbb{T}$.

Proof. (1) is clear.

To see (2) suppose that m>n+1. Then,

$$\begin{aligned} e_n e_m &= (f_n - f_{n-1})(f_m - f_{m-1}) = (f_n - f_{n-1})f_m - (f_n - f_{n-1})f_{m-1} \\ &= (f_n - f_{n-1}) - (f_n - f_{n-1}) = 0. \end{aligned}$$

The following proof of (3) is due to N. Lausten. For the original proof see [28].

Let $k < m < n \in \mathbb{N}$. Then $n - k \geq 2$, so by (2) $0 = \widehat{e_n e_k} = \widehat{e_n}\widehat{e_k}$. That is either $\widehat{e_k} = 0$ or $\widehat{e_n} = 0$. In particular $|\{k \in \mathbb{N} : \widehat{e_k}(x) \neq 0\}| \leq 2$, if it is 2, then $\{k \in \mathbb{N} : \widehat{e_k}(x) \neq 0\} = k_0, k_{0+1}$ for some $k_0 \in \mathbb{N}$.

Choose $x_0 \in \Omega$ such that

$$\left\|\sum_{j=1}^{n} \eta_k e_k^{\frac{1}{2}}\right\|_A = \left|\sum_{j=1}^{n} \eta_k \widehat{e}_k^{\frac{1}{2}}(x_0)\right|.$$

By above there exists a j such that $\hat{e}_k(x_0) = 0$ for $k \neq j, j+1$.

Let $f_j(x_0) = a$ and $f_{j+1}(x_0) = b$. Then $0 = \widehat{e}_{j+2}(x_0) = \widehat{f}_{j+2}(x_0) - b$. Hence $b = \widehat{f}_{j+2}(x_0)$. Note that $0 \leq a \leq 1$ and $0 \leq b \leq 1$.

By Lemma 4.4(3),

$$b^2 = \widehat{f}_{j+2}(x_0)\widehat{f}_{j+1}(x_0) = \widehat{f}_{j+1}(x_0) = b.$$

Therefore $b = 0$ or $b = 1$. By Lemma 4.4(3), $ab = a$. Thus if $b = 0$ then $a = 0$.

Then

$$\left\|\sum_{j=1}^{n} \eta_k e_k^{\frac{1}{2}}\right\|_A = \left|\sum_{j=1}^{n} \eta_k \widehat{e}_k^{\frac{1}{2}}(x_0)\right| \leq \left|\eta_j a^{\frac{1}{2}} + \eta_{j+1}(b-a)^{\frac{1}{2}}\right|.$$

If $b = 0$ then

$$\left\|\sum_{j=1}^{n} \eta_k e_k^{\frac{1}{2}}\right\|_A = 0.$$

If $b = 1$ then

$$\left\| \sum_{j=1}^{n} \eta_k e_k^{\frac{1}{2}} \right\|_A = \left| \eta_j a^{\frac{1}{2}} + \eta_{j+1}(1-a)^{\frac{1}{2}} \right| \le a^{\frac{1}{2}} + (1-a)^{\frac{1}{2}} \le \sqrt{2}.$$

□

Lemma 4.6. *Let A be a commutative C*-algebra with a sequential approximate identity $\{u_n\}$. Let $\{f_n\}$ be an increasing sequence in A satisfying properties* (1),(2) *and* (3) *of Lemma* 4.4. *Then $\{f_n\}$ is a sequential approximate identity in A.*

Proof. Let $a \in A$ and $\varepsilon > 0$. For $n \in \mathbb{N}$ note that

$$\begin{aligned} \|f_n a - a\| &= \|f_n a - a + f_n u_n a - f_n u_n a + u_n a - u_n a\| \\ &\le \|u_n a - a\| + \|f_n a - f_n u_n a\| + \|f_n u_n a - u_n a\| \\ &\le \|u_n a - a\| + \|f_n\| \|a - u_n a\| + \|a\| \|f_n u_n - u_n\| \\ &\le 2\|u_n a - a\| + \frac{\|a\|}{n}. \end{aligned}$$

Pick N_1 such that $\frac{\|a\|}{N_1} < \frac{\varepsilon}{3}$. Pick N_2 such that $\|u_n - u_n a\| < \frac{\varepsilon}{3}$ for every $n \ge N_2$. Set $N = \max\{N_1, N_2\}$. Then for every $n \ge N$ we have that

$$\|f_n a - a\| \le 2\|u_n a - a\| + \frac{\|a\|}{n} < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

□

Theorem 4.7 ([1], Theorem 1)**.** *Let A be a C*-algebra with a strictly positive element. Then there is a commutative C*-subalgebra B of A which contains a sequential approximate identity for A.*

Theorem 4.8. *Let A be a C*-algebra with a strictly positive element and let Ω be a locally compact Hausdorff space which is paracompact. Then $C_0(\Omega, A)$ is right projective.*

Proof. By Theorem 4.7, there is a commutative C*-subalgebra B of A which contains a sequential approximate identity $\{u_n\}$ in A. As in Lemma 4.5 we use $\{u_n\}$ to construct an increasing sequence $\{f_n\}$ in B satisfying properties $(1) - (3)$ of Lemma 4.4. Note that, by Lemma 4.6, $\{f_n\}$ is a bounded approximate identity for B. As in Lemma 4.5, for each n, set $e_n = f_n - f_{n-1}$. We define $f_0 = 0$. We claim that $\{f_n\}$ is a bounded approximate identity for A.

Let $a \in A$ and $\varepsilon > 0$. Similar to the proof of Lemma 4.6, for $n \in \mathbb{N}$,

$$\|f_n a - a\| = \|f_n a - a + f_n u_n a - f_n u_n a + u_n a - u_n a\|$$

$$\leq \|u_n a - a\| + \|f_n a - f_n u_n a\| + \|f_n u_n a - u_n a\|$$
$$\leq \|u_n a - a\| + \|f_n\| \|a - u_n a\| + \|a\| \|f_n u_n - u_n\|$$
$$\leq 2\|u_n a - a\| + \frac{\|a\|}{n}.$$

Pick N_1 such that $\frac{\|a\|}{N_1} < \frac{\varepsilon}{3}$. Pick N_2 such that $\|u_n - u_n a\| < \frac{\varepsilon}{3}$ for every $n \geq N_2$. Set $N = \max\{N_1, N_2\}$. Then for every $n \geq N$ we have that

$$\|f_n a - a\| \leq 2\|u_n a - a\| + \frac{\|a\|}{n} < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Similarly, using $f_n u_n = u_n f_n$,

$$\|a f_n - a\| \leq 2\|a u_n - a\| + \frac{\|a\|}{n}.$$

Pick N_3 such that $\|u_n - a u_n\| < \frac{\varepsilon}{3}$ for every $n \geq N_3$. Set $N' = \max\{N_1, N_3\}$. Then for every $n \geq N'$ we have

$$\|a f_n - a\| < \varepsilon.$$

Thus $\{f_n\}$ is a bounded approximate identity for A.

By assumption Ω is a paracompact locally compact Hausdorff space. Let $\mathcal{B} = \{V_\mu\}_{\mu \in \Lambda}$ be an open cover of Ω such that each point of Ω has a neighbourhood that intersects with no more than three sets of $\mathcal{B}$ as in [12]. By [18, Problem 5.W], since $\{V_\mu\}_{\mu \in \Lambda}$ is a locally finite open cover of the normal space Ω, it is possible to select a non-negative continuous function h_μ for each V_μ in $\mathcal{B}$ such that h_μ is 0 outside V_μ and is everywhere less than or equal to one, and

$$\sum_{\mu \in \Lambda} h_\mu(s) = 1 \text{ for all } s \in \Omega.$$

Set $g_\mu = \sqrt{h_\mu}$.

For $a \in C_0(\Omega, A)$, $\lambda = (\mu_1, ..., \mu_m) \in N(\Lambda)$ and $n \in \mathbb{N}$ define

$$y_{\lambda,n,a} = \sum_{i=1}^{m} \sum_{j=1}^{n} g_{\mu_i} e_j^{\frac{1}{2}} \otimes g_{\mu_i} e_j^{\frac{1}{2}} a.$$

Define $N(\Lambda) \times \mathbb{N}$ as a directed set with $(\lambda', n) \preccurlyeq (\lambda'', m)$ if and only if $\lambda' \subset \lambda''$ and $n \leq m$.

For each $a \in C_0(\Omega, A)$, we wish to show that $(y_{\lambda,n,a})_{N(\Lambda) \times \mathbb{N}}$ is a Cauchy net. We break this up into the following lemmas.

Lemma 4.9. *1. Let $a \in C_0(\Omega, A)$, $n \in \mathbb{N}$, $\lambda_0 = (\mu_1, ..., \mu_{m_0}) \in N(\Lambda)$ and $\lambda_1 = (\mu_1, ..., \mu_{m_0}, ..., \mu_{m_1}) \in N(\Lambda)$. Then we have that*

$$\|y_{\lambda_1 \setminus \lambda_0, n, a}\|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+} \leq 18 \max_{\mu \in \lambda_1 \setminus \lambda_0} \|g_\mu a\|_{C_0(\Omega,A)}.$$

2. *Let $a \in C_0(\Omega, A)$. Then for any $\varepsilon > 0$, there exists a $\lambda_0 \in N(\Lambda)$ such that for all $\lambda > \lambda_0$ we have*
$$\sup_{\mu \in \lambda \setminus \lambda_0} \|g_\mu a\|_{C_0(\Omega,A)} < \frac{\varepsilon}{54}.$$

3. *Let $a \in C_0(\Omega, A)$, $\lambda = (\mu_1, ..., \mu_m) \in N(\Lambda)$ and $n_2 > n_1$. Let the sequence $(f_n) \subset A$ be defined as in Lemma 4.4. Then*

$$\|y_{\lambda,n_2,a} - y_{\lambda,n_1,a}\|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+} \leq 18 \sup_{t\in\Omega} \|a(t) - f_{n_1-1}a(t)\|_A.$$

4. *Let $a \in C_0(\Omega, A)$. For any $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have that*
$$\|a - f_n a\|_{C_0(\Omega,A)} = \sup_{t\in\Omega} \|a(t) - f_n a(t)\|_A < \frac{\varepsilon}{54}.$$

Proof. 1. Let η be a primary $(m_1 - m_0)th$ root of unity and let ξ be a primary nth root of unity. By Lemma 1.5, we have

$$\|y_{\lambda_1\setminus\lambda_0,n,a}\|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+} \leq$$
$$\frac{1}{n(m_1-m_0)} \sum_{l=1}^{n} \sum_{k=1}^{m_1-m_0} \left\| \sum_{t=1}^{n} \sum_{i=m_0+1}^{m_1} \xi^{lt} \eta^{ki} g_{\mu_i} e_t^{\frac{1}{2}} \right\|_{C_0(\Omega,A)} \left\| \sum_{t=1}^{n} \sum_{i=m_0+1}^{m_1} \xi^{-lt} \eta^{-ki} g_{\mu_i} e_t^{\frac{1}{2}} a \right\|_{C_0(\Omega,A)}.$$

Since for every $x \in \Omega$ there are at most 3 values of μ such that $g_\mu(x) \neq 0$ we have the following, for $k = 1, ..., m_1 - m_0, l = 1, ..., n$,

$$\begin{aligned}
&\left\| \sum_{t=1}^{n} \sum_{i=m_0+1}^{m_1} \xi^{lt} \eta^{ki} g_{\mu_i} e_t^{\frac{1}{2}} \right\|_{C_0(\Omega,A)} \\
=&\sup_{x\in\Omega} \left\| \sum_{i=m_0+1}^{m_1} \eta^{ki} g_{\mu_i}(x) \sum_{t=1}^{n} \xi^{lt} e_t^{\frac{1}{2}} \right\|_A \\
\leq&3 \max_{\mu\in\lambda_1\setminus\lambda_0} \sup_{x\in\Omega} \left\| g_\mu(x) \sum_{t=1}^{n} \xi^{lt} e_t^{\frac{1}{2}} \right\|_A \\
\leq&3 \left\| \sum_{t=1}^{n} \xi^{lt} e_t^{\frac{1}{2}} \right\|_A
\end{aligned}$$

$$\leq 3\sqrt{2} \quad (by\ \ Lemma\, 4.5).$$

Similarly, for $k = 1, ..., m_1 - m_0, l = 1, ..., n$,

$$\begin{aligned}
&\left\| \sum_{t=1}^{n} \sum_{i=m_0+1}^{m_1} \xi^{-lt} \eta^{-ki} g_{\mu_i} e_t^{\frac{1}{2}} a \right\|_{C_0(\Omega, A)} \\
=& \sup_{x\in\Omega} \left\| \sum_{i=m_0+1}^{m_1} \eta^{-ki} g_{\mu_i}(x) \sum_{t=1}^{n} \xi^{-lt} e_t^{\frac{1}{2}} a(x) \right\|_A \\
\leq& 3 \max_{\mu\in\lambda_1\setminus\lambda_0} \sup_{x\in\Omega} \left\| g_\mu(x) a(x) \sum_{t=1}^{n} \xi^{-lt} e_t^{\frac{1}{2}} \right\|_A \\
\leq& 3 \max_{\mu\in\lambda_1\setminus\lambda_0} \sup_{x\in\Omega} \| g_\mu(x) a(x) \|_A \sup_{x\in\Omega} \left\| \sum_{t=1}^{n} \xi^{-lt} e_t^{\frac{1}{2}} \right\|_A \\
\leq& 3\sqrt{2} \max_{\mu\in\lambda_1\setminus\lambda_0} \sup_{x\in\Omega} \| g_\mu(x) a(x) \|_A \\
\leq& 3\sqrt{2} \max_{\mu\in\lambda_1\setminus\lambda_0} \| g_\mu a \|_{C_0(\Omega, A)} .
\end{aligned}$$

Therefore

$$\| y_{\lambda_1\setminus\lambda_0, n, a} \|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)} \leq 18 \max_{\mu\in\lambda_1\setminus\lambda_0} \| g_\mu a \|_{C_0(\Omega,A)}.$$

2. Let $\varepsilon > 0$. Since $a \in C_0(\Omega, A)$ there is a compact set $K \subset \Omega$ such that $\|a(t)\|_A < \frac{\varepsilon}{54}$ for every $t \in \Omega \setminus K$. Since compact sets only intersect a finite number of elements of $\{V_\mu\}_{\mu\in\Lambda}$. We can find a finite set $\lambda_0 \in N(\Lambda)$ such that for $t \in K$ we have

$$\sup_{\mu\in\lambda\setminus\lambda_0} \| g_\mu(t) a(t) \|_A = 0.$$

Therefore

$$\sup_{\mu\in\lambda\setminus\lambda_0} \| g_\mu a \|_{C_0(\Omega,A)} \leq \frac{\varepsilon}{54}.$$

3. Recall for $b \in A$ and $a \in C_0(\Omega, A)$ we define the element $ba \in C_0(\Omega, A)$ by $(ba)(t) = b \cdot a(t)$.

Then

$$\begin{aligned}
&\| y_{\lambda,n_2,a} - y_{\lambda,n_1,a} \|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+} \\
=& \left\| \sum_{i=1}^{m} \sum_{j=n_1+1}^{n_2} g_{\mu_i} e_j^{\frac{1}{2}} \otimes g_{\mu_i} e_j^{\frac{1}{2}} a \right\|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+}
\end{aligned}$$

$$= \left\| \left(\sum_{i=1}^{m} \sum_{j=n_1+1}^{n_2} g_{\mu_i} e_j^{\frac{1}{2}} \otimes g_{\mu_i} e_j^{\frac{1}{2}} \right) \cdot \left(a - \sum_{j=1}^{n_1-1} e_j a \right) \right\|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+}$$

$$(since\, e_m e_n = 0\, for\, |m-n| > 1)$$

$$\leq \left\| \sum_{i=1}^{m} \sum_{j=n_1+1}^{n_2} g_{\mu_i} e_j^{\frac{1}{2}} \otimes g_{\mu_i} e_j^{\frac{1}{2}} \right\|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+} \left\| a - \sum_{j=1}^{n_1-1} e_j a \right\|_{C_0(\Omega,A)}$$

$$= \left\| \sum_{i=1}^{m} \sum_{j=n_1+1}^{n_2} g_{\mu_i} e_j^{\frac{1}{2}} \otimes g_{\mu_i} e_j^{\frac{1}{2}} \right\|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+} \sup_{t\in\Omega} \|a(t) - f_{n_1-1} a(t)\|_A$$

$$(since \sum_{j=1}^{n_1-1} e_j = f_{n_1-1}).$$

In a silmilar method to the proof of part (1), one can show that,

$$\left\| \sum_{i=1}^{m} \sum_{j=n_1+1}^{n_2} g_{\mu_i} e_j^{\frac{1}{2}} \otimes g_{\mu_i} e_j^{\frac{1}{2}} \right\|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+} \leq 18.$$

Thus

$$\|y_{\lambda,n_2,a} - y_{\lambda,n_1,a}\|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+} \leq 18 \sup_{t\in\Omega} \|a(t) - f_{n-1} a(t)\|_A.$$

4. Let $\varepsilon > 0$. Since a vanishes at infinity we can pick a compact set $K \subset \Omega$ such that for every $x \in \Omega \setminus K$ we have that

$$\|a(x)\|_A \leq \frac{\varepsilon}{108}.$$

Then, for every $x \in \Omega \setminus K$, we have that

$$\|a(x) - f_n a(x)\|_A \leq \|a(x)\|_A + \|f_n\|_A \|a(x)\|_A \leq \frac{\varepsilon}{108} + 1 \cdot \frac{\varepsilon}{108} = \frac{\varepsilon}{54}.$$

Since a is continuous, for every $x \in \Omega$, there exists an open set $U_x \subset \Omega$ such that, for each $y \in U_x$,

$$\|a(x) - a(y)\|_A < \frac{\varepsilon}{216}.$$

Then $\{U_x\}_{x\in K}$ is an open covering of K. Let $\{U_{x_i}\}_{i=1}^{m}$ be a finite subcover. Let $i \in \{1, ..., m\}$. Since $\{f_n\}$ is an approximate identity in A there exists an n_i such that for every $n > n_i$ we have that

$$\|a(x_i) - f_n a(x_i)\|_A < \frac{\varepsilon}{108}.$$

Then set $n_0 = \max_{1\le i\le m} n_i$. Then, for every $x \in K$, there exists an i such that $x \in U_{x_i}$ and, for $n > n_0$, we have

$$\begin{aligned}
&\|a(x) - f_n a(x)\|_A \\
=&\|a(x) - a(x_i) + a(x_i) - f_n a(x) - f_n a(x_i) + f_n a(x_i)\|_A \\
\le&\|a(x_i) - f_n a(x_i)\|_A + \|a(x) - a(x_i)\|_A + \|f_n a(x_i) - f_n a(x)\|_A \\
\le&\|a(x_i) - f_n a(x_i)\|_A + \|a(x) - a(x_i)\|_A + \|f_n\|_A \|a(x) - a(x_i)\|_A \\
\le&\frac{\varepsilon}{108} + \frac{\varepsilon}{216} + 1 \cdot \frac{\varepsilon}{216} \\
=&\frac{\varepsilon}{54}.
\end{aligned}$$

Therefore

$$\sup_{t\in\Omega} \|a(t) - f_n a(t)\|_A = \max\{\sup_{t\in K} \|a(t) - f_n a(t)\|_A, \sup_{t\notin K} \|a(t) - f_n a(t)\|_A\} < \frac{\varepsilon}{54}.$$

□

Lemma 4.10. *For each $a \in C_0(\Omega, A)$, $(y_{\lambda,n,a})_{(\lambda,n)}$ is a Cauchy net in $C_0(\Omega, A)\hat{\otimes}C_0(\Omega, A)_+$.*

Proof. Let $\varepsilon > 0$. Let λ_0 and n_0 be as in Lemma 4.9 (2) and (4) and let $(\lambda_2, n_2) > (\lambda_1, n_1) > (\lambda_0, n_0)$ such that $n_1 - 1 > n_0$. Note that

$$y_{\lambda_2,n_2,a} = y_{\lambda_2\setminus\lambda_0,n_2,a} + y_{\lambda_0,n_2,a},$$

$$y_{\lambda_1,n_1,a} = y_{\lambda_1\setminus\lambda_0,n_1,a} + y_{\lambda_0,n_1,a}.$$

So we have

$$\begin{aligned}
&\|y_{\lambda_2,n_2,a} - y_{\lambda_1,n_1,a}\|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+} \\
\le& \|y_{\lambda_2\setminus\lambda_0,n_2,a}\|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+} + \|y_{\lambda_1\setminus\lambda_0,n_1,a}\|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+} \\
&+\|y_{\lambda_0,n_2,a} - y_{\lambda_0,n_1,a}\|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+}.
\end{aligned}$$

By parts (1) and (2) of Lemma 4.9, since $\lambda_1, \lambda_2 > \lambda_0$

$$\|y_{\lambda_2\setminus\lambda_0,n_2,a}\|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+} \le 18 \sup_{\mu\in\lambda_2\setminus\lambda_0} \|g_\mu a\|_{C_0(\Omega,A)} < \frac{\varepsilon}{3}$$

and

$$\|y_{\lambda_1\setminus\lambda_0,n_1,a}\|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+} \le 18 \sup_{\mu\in\lambda_1\setminus\lambda_0} \|g_\mu a\|_{C_0(\Omega,A)} < \frac{\varepsilon}{3}.$$

By parts (3) and (4) of Lemma 4.9,

$$\|y_{\lambda_0,n_2,a} - y_{\lambda_0,n_1,a}\|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+} \le 18 \sup_{t\in\Omega} \|a(t) - f_{n_1-1}a(t)\|_A < \frac{\varepsilon}{3}.$$

Therefore

$$\|y_{\lambda_2,n_2,a} - y_{\lambda_1,n_1,a}\|_{C_0(\Omega,A)\hat{\otimes}C_0(\Omega,A)_+} < \varepsilon.$$

□

Conclusion of the proof of Theorem 4.8.

Note that $C_0(\Omega, A)\hat{\otimes}C_0(\Omega, A)_+$ is complete, hence $\lim_{(\lambda,n)} y_{\lambda,n,a}$ exists for every $a \in A$. We now set

$$\rho : C_0(\Omega, A) \to C_0(\Omega, A)\hat{\otimes}C_0(\Omega, A)_+$$
$$a \mapsto \lim_{(\lambda,n)} y_{\lambda,n,a}.$$

It is clear that ρ is linear since, for all $a, b \in C_0(\Omega, A)$ and for all $\alpha, \beta \in \mathbb{C}$,

$$\rho(\alpha a + \beta b) = \lim_{(\lambda,n)} y_{\lambda,n,\alpha a+\beta b} = \lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{i=1}^{n} g_\mu e_i^{\frac{1}{2}} \hat{\otimes} g_\mu e_i^{\frac{1}{2}} (\alpha a + \beta b)$$

$$= \alpha \lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{i=1}^{n} g_\mu e_i^{\frac{1}{2}} \hat{\otimes} g_\mu e_i^{\frac{1}{2}} a + \beta \lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{i=1}^{n} g_\mu e_i^{\frac{1}{2}} \hat{\otimes} g_\mu e_i^{\frac{1}{2}} b = \alpha\rho(a) + \beta\rho(b).$$

Similarly $\rho(ab) = \rho(a)b$ since

$$\rho(ab) = \lim_{(\lambda,n)} y_{\lambda,n,ab} = \lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{i=1}^{n} g_\mu e_i^{\frac{1}{2}} \hat{\otimes} g_\mu e_i^{\frac{1}{2}} (ab)$$

$$= \left(\lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{i=1}^{n} g_\mu e_i^{\frac{1}{2}} \hat{\otimes} g_\mu e_i^{\frac{1}{2}} a \right) b = \rho(a)b.$$

By part (1) of the proof of Lemma 4.9, we have that $\|\rho(a)\| \leq 18\|a\|$. Therefore ρ is a morphism of right Banach $C_0(\Omega, A)$-modules.

It remains to show that $\pi \circ \rho = 1_{C_0(\Omega,A)}$. Recall that for each $x \in \Omega$ we have that $\sum_{\mu\in\Lambda} h_\mu(x) = 1$ since h is a partition of unity. Thus

$$\pi(\rho(a)) = \lim_{(\lambda,n)} (\pi(y_{\lambda,n,a})) = \lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{j=1}^{n} h_\mu e_j a.$$

We wish to show that $\lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{j=1}^{n} h_\mu e_j a = a$. Consider it acting on an element $x \in \Omega$. Then

$$\lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{j=1}^{n} h_\mu(x) e_j a(x)$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \sum_{j=1}^{n} e_j a(x) \\
&= \lim_{n\to\infty} f_n a(x) \\
&= a(x).
\end{aligned}
$$

□

We also obtain the following theorem.

Theorem 4.11. *Let A be a C*-algebra with a strictly positive element and let Ω be a locally compact Hausdorff space which is paracompact. Then $C_0(\Omega, A)$ is left projective.*

Proof. This follows in the same way as in the proof of Theorem 4.8. □

4.2 Projectivity of A defined by locally trivial continuous fields of σ-unital C*-algebras

We now generalise these results to locally trivial continuous fields.

As in [10, Theorem 7.2.4], for a normal topological space Ω, we say that the topological dimension of Ω is less than or equal to ℓ if the following condition is satisfied: every locally finite open cover of Ω possesses an open locally finite refinement of order ℓ.

Theorem 4.12. *Let $\mathcal{U} = \{\Omega, (A_x), \Theta\}$ be a locally trivial continuous field of σ-unital C*-algebras. Let $\mathcal{A}$ be the C*-algebra generated by $\mathcal{U}$. Suppose that Ω is paracompact and has finite dimension. Then $\mathcal{A}$ is right projective.*

Proof. Let ℓ be the dimension of Ω. By assumption, $\mathcal{U}$ is locally trivial and so, for each $s \in \Omega$, there is an open neighbourhood U_s of s such that $\mathcal{U}|_{U_s}$ is trivial. Since Ω is paracompact, the open cover $\{U_s\}_{s\in\Omega}$ of Ω has an open locally finite refinement $\{W_\nu\}$ that is also a cover of Ω.

By [10, Theorem 5.1.5], the paracompactness of Ω implies that Ω is a normal topological space. By [10, Theorem 7.2.4], for the normal space Ω, the topological dimension $\dim \Omega \leq \ell$ implies that the locally finite open cover $\{W_\nu\}$ of Ω possesses an open locally finite refinement $\{V_\mu\}_{\mu\in\Lambda}$ of order ℓ that is also a cover of Ω. By [18, Problem 5.W], since $\{V_\mu\}_{\mu\in\Lambda}$ is a locally finite open cover of the normal space Ω, it is possible to select a non-negative continuous function h_μ for each V_μ in $\mathcal{B}$ such that h_μ is 0 outside V_μ and is everywhere less than or equal to one, and

$$\sum_{\mu\in\Lambda} h_\mu(s) = 1 \text{ for all } \ s \in \Omega.$$

Note that in the equality $\sum_{\mu\in\Lambda} h_\mu(s) = 1$, for any $s \in \Omega$, there are no more than ℓ nonzero terms. Set $g_\mu = \sqrt{h_\mu}$.

Let $\phi^\mu = (\phi^\mu_x)_{x\in V_\mu}$ be an isomorphism of $\mathcal{U}|V_\mu$ onto the constant field $\{V_\mu, \tilde{A}_\mu, C(V_\mu, \tilde{A}_\mu)\}$ over V_μ defined by A_μ.

Let $\mu \in \lambda$. By Theorem 4.7, there is a commutative C*-subalgebra B_μ of $\tilde{A}_\mu$ which contains a sequential approximate identity $\{u^\mu_n\}$ for $\tilde{A}_\mu$. As in Lemma 4.5 use $\{u^\mu_n\}$ to construct an increasing sequence $\{f^\mu_n\}$ in B_μ satisfying properties $(1)-(3)$ of Lemma 4.4. Note that, by Lemma 4.6, $\{f^\mu_n\}$ is a bounded approximate identity for B_μ. One can show that $\{f^\mu_n\}$ is a bounded approximate identity in $\tilde{A}_\mu$, as in the proof of Theorem 4.8. As in Lemma 4.5, for each n, set $e^\mu_n = f^\mu_n - f^\mu_{n-1}$. We define $f^\mu_0 = 0$.

Lemma 4.13. *For any $a \in \mathcal{A}$ and for any $\lambda = \{\mu_1, \dots, \mu_N\}$, $n, m \in \mathbb{N}$, $l, k \in \mathbb{N}$,*

$$\left\| \sum_{t=1}^{n} \sum_{i=1}^{N} \xi^{-lt} \eta^{-ki} g_{\mu_i} (\phi^{\mu_i}_\bullet)^{-1} \left(\sqrt{e^{\mu_i}_t} \right) a \right\|_{\mathcal{A}} \leq \ell\sqrt{2} \max_{\mu\in\lambda} \|g_\mu a\|_{\mathcal{A}} \tag{4.1}$$

and

$$\left\| \sum_{t=1}^{n} \sum_{i=1}^{N} \xi^{lt} \eta^{ki} g_{\mu_i} (\phi^{\mu_i}_\bullet)^{-1} \left(\sqrt{e^{\mu_i}_t} \right) \right\|_{\mathcal{A}} \leq \ell\sqrt{2} \tag{4.2}$$

where ξ is a primary n-th root of unity and η is a primary N-th root of unity in $\mathbb{C}$, and

$$\left\| \sum_{i=1}^{N} \sum_{j=n+1}^{m} g_{\mu_i} (\phi^{\mu_i}_\bullet)^{-1} \left(\sqrt{e^{\mu_i}_j} \right) \otimes \sqrt{g_{\mu_i}} (\phi^{\mu_i}_\bullet)^{-1} \left(\sqrt{e^{\mu_i}_j} \right) \right\|_{\mathcal{A}\hat{\otimes}\mathcal{A}_+} \leq 2\ell^2 \tag{4.3}$$

and

$$\left\| \sum_{i=1}^{N} \sum_{j=n+1}^{m} g_{\mu_i} (\phi^{\mu_i}_\bullet)^{-1} \left(\sqrt{e^{\mu_i}_j} \right) \otimes g_{\mu_i} (\phi^{\mu_i}_\bullet)^{-1} \left(\sqrt{e^{\mu_i}_j} \right) a \right\|_{\mathcal{A}\hat{\otimes}\mathcal{A}_+}$$
$$\leq 2\ell^3 \max_{\mu\in\lambda} \left\| \phi^\mu_\bullet(a) - f^\mu_{n-1} \phi^\mu_\bullet(a) \right\|_{C_0(V_\mu, A_\mu)}. \tag{4.4}$$

Proof. Since for every $x \in \Omega$ there are at most ℓ values of μ such that $g_\mu(x) \neq 0$ we have the following

$$
\begin{aligned}
&\left\| \sum_{t=1}^{n} \sum_{i=1}^{N} \zeta^{lt} \eta^{ki} g_{\mu_i} (\phi_\bullet^{\mu_i})^{-1} \left(\sqrt{e_t^{\mu_i}} \right) \right\|_{\mathcal{A}} \\
= &\sup_{x\in\Omega} \left\| \sum_{t=1}^{n} \sum_{i=1}^{N} \zeta^{lt} \eta^{ki} g_{\mu_i}(x) (\phi_x^{\mu_i})^{-1} \left(\sqrt{e_t^{\mu_i}} \right) \right\|_{A_x} \\
\leq &\ell \max_{\mu\in\lambda} \sup_{x\in\Omega} \left\| \sum_{t=1}^{n} \zeta^{lt} g_{\mu}(x) (\phi_x^{\mu})^{-1} \left(\sqrt{e_t^{\mu}} \right) \right\|_{A_x} \\
\leq &\ell \max_{\mu\in\lambda} \left\| \sum_{t=1}^{n} \zeta^{lt} g_{\mu} (\phi_\bullet^{\mu})^{-1} \left(\sqrt{e_t^{\mu}} \right) \right\|_{\mathcal{A}} \\
\leq &\ell \max_{\mu\in\lambda} \left\| \sum_{t=1}^{n} \zeta^{lt} (\phi_\bullet^{\mu})^{-1} \left(\sqrt{e_t^{\mu}} \right) \right\|_{A_x} \\
\leq &\ell\sqrt{2} \quad \textit{by Lemma } 4.5 \textit{ part } (3).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\left\| \sum_{t=1}^{n} \sum_{i=1}^{N} \zeta^{-lt} \eta^{-ki} g_{\mu_i} (\phi_\bullet^{\mu_i})^{-1} \left(\sqrt{e_t^{\mu_i}} \right) a \right\|_{\mathcal{A}} \\
= &\sup_{x\in\Omega} \left\| \sum_{t=1}^{n} \sum_{i=1}^{N} \zeta^{-lt} \eta^{-ki} g_{\mu_i}(x) (\phi_x^{\mu_i})^{-1} \left(\sqrt{e_t^{\mu_i}} \right) a(x) \right\|_{A_x} \\
\leq &\ell \max_{\mu\in\lambda} \sup_{x\in\Omega} \left\| \sum_{t=1}^{n} \zeta^{-lt} g_{\mu}(x) (\phi_x^{\mu})^{-1} \left(\sqrt{e_t^{\mu}} \right) a(x) \right\|_{A_x} \\
\leq &\ell \max_{\mu\in\lambda} \sup_{x\in\Omega} \left\| \sum_{t=1}^{n} \zeta^{-lt} (\phi_x^{\mu})^{-1} \left(\sqrt{e_t^{\mu}} \right) \right\|_{A_x} \sup_{x\in\Omega} \left\| g_\mu(x) a(x) \right\|_{A_x} \\
\leq &\ell\sqrt{2} \max_{\mu\in\lambda} \sup_{x\in\Omega} \left\| g_\mu(x) a(x) \right\|_{A_x} \\
= &\ell\sqrt{2} \max_{\mu\in\lambda} \left\| g_\mu a \right\|_{\mathcal{A}} .
\end{aligned}
$$

Thus the inequalities (4.1) and (4.2) hold.

Let γ be a primary $(m-n)$-th root of unity.

$$
\left\| \sum_{i=1}^{N} \sum_{j=n+1}^{m} g_{\mu_i} (\phi_\bullet^{\mu_i})^{-1} \left(\sqrt{e_j^{\mu_i}} \right) \otimes \sqrt{g_{\mu_i}} (\phi_\bullet^{\mu_i})^{-1} \left(\sqrt{e_j^{\mu_i}} \right) \right\|_{\mathcal{A} \hat{\otimes} \mathcal{A}_+}
$$

$$
\begin{aligned}
\leq & \frac{1}{(m-n)N} \sum_{l=n+1}^{m} \sum_{k=1}^{N} \left\| \sum_{t=n+1}^{m} \sum_{i=1}^{N} \gamma^{lt} \eta^{ki} g_{\mu_i} (\phi_{\bullet}^{\mu_i})^{-1} \left(\sqrt{e_t^{\mu_i}} \right) \right\|_{\mathcal{A}} \\
& \times \left\| \sum_{t=n+1}^{m} \sum_{i=1}^{N} \gamma^{-lt} \eta^{-ki} g_{\mu_i} (\phi_{\bullet}^{\mu_i})^{-1} \left(\sqrt{e_t^{\mu_i}} \right) \right\|_{\mathcal{A}} \\
\leq & \frac{1}{(m-n)N} \sum_{l=n+1}^{m} \sum_{k=1}^{N} 2\ell^2 \quad (\text{by Lemma } 4.5) \\
= & 2\ell^2.
\end{aligned}
$$

Hence inequality (4.3) holds.

Note that, for $u \in \mathcal{A}_+ \hat{\otimes} \mathcal{A}, b \in \mathcal{A}$, $\|ub\|_{\mathcal{A}_+ \hat{\otimes} \mathcal{A}} \leq \|u\|_{\mathcal{A}_+ \hat{\otimes} \mathcal{A}} \|b\|_{\mathcal{A}}$. Then

$$
\begin{aligned}
& \left\| \sum_{i=1}^{N} \sum_{j=n+1}^{m} g_{\mu_i} (\phi_{\bullet}^{\mu_i})^{-1} \left(\sqrt{e_j^{\mu_i}} \right) \otimes g_{\mu_i} (\phi_{\bullet}^{\mu_i})^{-1} \left(\sqrt{e_j^{\mu_i}} \right) a \right\|_{\mathcal{A} \hat{\otimes} \mathcal{A}_+} \\
= & \left\| \sum_{i=1}^{N} \left(\sum_{j=n+1}^{m} g_{\mu_i} (\phi_{\bullet}^{\mu_i})^{-1} \left(\sqrt{e_j^{\mu_i}} \right) \otimes \sqrt{g_{\mu_i}} (\phi_{\bullet}^{\mu_i})^{-1} \left(\sqrt{e_j^{\mu_i}} \right) \right) \right. \\
& \times \left. \left(\sqrt{g_{\mu_i}} a - \sum_{j=1}^{n-1} \sqrt{g_{\mu_i}} (\phi_{\bullet}^{\mu_i})^{-1} \left(\sqrt{e_j^{\mu_i}} \right) a \right) \right\|_{\mathcal{A} \hat{\otimes} \mathcal{A}_+} \\
\leq & \left\| \sum_{i=1}^{N} \sum_{j=n+1}^{m} g_{\mu_i} (\phi_{\bullet}^{\mu_i})^{-1} \left(\sqrt{e_j^{\mu_i}} \right) \otimes \sqrt{g_{\mu_i}} (\phi_{\bullet}^{\mu_i})^{-1} \left(\sqrt{e_j^{\mu_i}} \right) \right\|_{\mathcal{A} \hat{\otimes} \mathcal{A}_+} \\
& \times \left\| \sum_{i=1}^{N} (\sqrt{g_{\mu_i}} a - \sum_{j=1}^{n-1} \sqrt{g_{\mu_i}} (\phi_{\bullet}^{\mu_i})^{-1} \left(e_j^{\mu_i} \right) a) \right\|_{\mathcal{A}} \quad (\text{by Lemma } 4.5) \\
\leq & 2\ell^2 \left\| \sum_{i=1}^{N} (\sqrt{g_{\mu_i}} a - \sum_{j=1}^{n-1} \sqrt{g_{\mu_i}} (\phi_{\bullet}^{\mu_i})^{-1} \left(e_j^{\mu_i} \right) a) \right\|_{\mathcal{A}} \quad (\text{by inequality } (4.3)).
\end{aligned}
$$

Recall that for every $x \in \Omega$ there are at most ℓ values of μ such that $g_\mu(x) \neq 0$. Therefore

$$
\begin{aligned}
& 2\ell^2 \left\| \sum_{i=1}^{N} (\sqrt{g_{\mu_i}} a - \sum_{j=1}^{n-1} \sqrt{g_{\mu_i}} (\phi_{\bullet}^{\mu_i})^{-1} \left(e_j^{\mu_i} \right) a) \right\|_{\mathcal{A}} \\
\leq & 2\ell^3 \max_{\mu \in \lambda} \sup_{x \in V_\mu} \left\| \sqrt{g_\mu}(x) a(x) - \sum_{j=1}^{n-1} \sqrt{g_\mu}(x) (\phi_x^{\mu})^{-1} \left(e_j^{\mu} \right) a(x) \right\|_{A_x} \\
\leq & 2\ell^3 \max_{\mu \in \lambda} \sup_{x \in V_\mu} \left\| a(x) - \sum_{j=1}^{n-1} (\phi_x^{\mu})^{-1} \left(e_j^{\mu} \right) a(x) \right\|_{A_x}
\end{aligned}
$$

$$=2\ell^3 \max_{\mu\in\lambda} \sup_{x\in V_\mu} \left\| a(x) - (\phi_x^\mu)^{-1}\left(f_{n-1}^\mu\right) a(x) \right\|_{A_x}$$
$$=2\ell^3 \max_{\mu\in\lambda} \|\phi_\bullet^\mu(a) - f_{n-1}^\mu \phi_\bullet^\mu(a)\|_{C_0(V_\mu, \tilde{A}_\mu)}.$$

Hence the inequality (4.4) holds. □

For $a \in \mathcal{A}$, $n \in \mathbb{N}$, $\lambda \in N(\Lambda)$ we define the following element $y_{\lambda,n,a}$ in $\mathcal{A}\hat{\otimes}\mathcal{A}$,

$$y_{a,\lambda,n} = \sum_{\mu\in\lambda}\sum_{j=1}^{n} g_\mu(\phi_\bullet^\mu)^{-1}\left(\sqrt{e_j^\mu}\right) \otimes g_\mu(\phi_\bullet^\mu)^{-1}\left(\sqrt{e_j^\mu}\right) a.$$

Define $N(\Lambda)\times\mathbb{N}$ as a directed set with $(\lambda', n) \preccurlyeq (\lambda'', m)$ if and only if $\lambda' \subset \lambda''$ and $n \le m$.

Lemma 4.14. *For any $a \in \mathcal{A}$, the net $(y_{a,\lambda,n})_{\lambda,n}$ converges in $\mathcal{A}\widehat{\otimes}\mathcal{A}$.*

Proof. Note that any compact $K \subset \Omega$ intersects only a finite number of sets in the locally finite covering $\{V_\mu\}$ and, for any $a \in \mathcal{A}$, $\|a(t)\| \to 0$ as $t \to \infty$. Let $\varepsilon > 0$. There is a finite set $\lambda \in N(\Lambda)$ such that for $\mu \notin \lambda$ we have

$$\|g_\mu a\|_{\mathcal{A}} < \frac{\varepsilon}{6\ell^2}.$$

For $\lambda \subset \lambda' \subset \lambda''$ and $m \ge n$, we have

$$\begin{aligned}\|y_{a,\lambda'',m} - y_{a,\lambda',n}\|_{\mathcal{A}\hat{\otimes}\mathcal{A}} &= \|y_{a,\lambda''\setminus\lambda,m} + y_{a,\lambda,m} - y_{a,\lambda'\setminus\lambda,n} - y_{a,\lambda,n}\|_{\mathcal{A}\hat{\otimes}\mathcal{A}} \\ &\le \|y_{a,\lambda''\setminus\lambda,m}\|_{\mathcal{A}\hat{\otimes}\mathcal{A}} + \|y_{a,\lambda'\setminus\lambda,n}\|_{\mathcal{A}\hat{\otimes}\mathcal{A}} + \|y_{a,\lambda,m} - y_{a,\lambda,n}\|_{\mathcal{A}\hat{\otimes}\mathcal{A}}.\end{aligned}$$

By Lemma 1.5, for $\tilde{\lambda} = \{\mu_1, \dots, \mu_m\}$, $\tilde{n} \in \mathbb{N}$,

$$\begin{aligned}&\|y_{\tilde{\lambda},\tilde{n},a}\|_{\mathcal{A}\hat{\otimes}\mathcal{A}_+} \le \\ &\quad \frac{1}{m\tilde{n}}\sum_{l=1}^{\tilde{n}}\sum_{k=1}^{m}\left\|\sum_{t=1}^{\tilde{n}}\sum_{i=1}^{m} \xi^{lt}\eta^{ki} g_{\mu_i}(\phi_\bullet^{\mu_i})^{-1}\left(\sqrt{e_t^{\mu_i}}\right)\right\|_{\mathcal{A}} \\ &\quad\quad \times \left\|\sum_{t=1}^{\tilde{n}}\sum_{i=1}^{m} \xi^{-lt}\eta^{-ki} g_{\mu_i}(\phi_\bullet^{\mu_i})^{-1}\left(\sqrt{e_t^{\mu_i}}\right) a\right\|_{\mathcal{A}}.\end{aligned}$$

where ξ is a primary m-th root of unity and η is a primary $\tilde{n}$-th root of unity in $\mathbb{C}$.

By inequalities (4.1) and (4.2) from Lemma 4.13,

$$\|y_{a,\lambda''\setminus\lambda,m}\|_{\mathcal{A}\hat{\otimes}\mathcal{A}} \le 2\ell^2 \max_{\mu\in\lambda''\setminus\lambda} \|g_\mu a\|_{\mathcal{A}} \le \frac{\varepsilon}{3}$$

and

$$\|y_{a,\lambda'\setminus\lambda,n}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} \leq 2\ell^2 \max_{\mu\in\lambda'\setminus\lambda} \|g_\mu a\|_{\mathcal{A}} \leq \frac{\varepsilon}{3}.$$

By inequality (4.4) from Lemma 4.13, for $\lambda = \{\mu_1, \ldots, \mu_N\}$,

$$\begin{aligned}\|y_{a,\lambda,m} - y_{a,\lambda,n}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} &= \left\| \sum_{i=1}^{N} \sum_{j=n+1}^{m} g_{\mu_i} (\phi_\bullet^{\mu_i})^{-1} \left(\sqrt{e_j^{\mu_i}}\right) \otimes \sqrt{g_{\mu_i}} (\phi_\bullet^{\mu_i})^{-1} \left(\sqrt{e_j^{\mu_i}}\right) a \right\|_{\mathcal{A}\hat{\otimes}\mathcal{A}_+} \\ &\leq 2\ell^3 \max_{1\leq p\leq N} \left\| \phi_\bullet^{\mu_p}(a) - \phi_\bullet^{\mu_p}\left(f_{n-1}^{\mu_p} a\right)\right\|_{C_0(V_{\mu_p}, \tilde{A_{\mu_p}})}.\end{aligned}$$

By Part 4 of Lemma 4.9, for every p,

$$\left\|\phi_\bullet^{\mu_p}(a) - f_{n-1}^{\mu_p}\phi_\bullet^{\mu_p}(a)\right\|_{C_0(V_{\mu_p}, \tilde{A_{\mu_p}})} \to 0 \text{ as } i \to \infty.$$

Hence

$$\|y_{a,\lambda,m} - y_{a,\lambda,n}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} \to 0 \text{ as } n, m \to \infty.$$

Thus, in view of the completeness of $\mathcal{A}\widehat{\otimes}\mathcal{A}$, for any $a \in \mathcal{A}$, the net $y_{a,\lambda,n}$ converges in $\mathcal{A}\widehat{\otimes}\mathcal{A}$. □

Let us complete the proof of Theorem 4.12.

Set

$$\begin{aligned}\rho : \mathcal{A} &\to \mathcal{A}\hat{\otimes}\mathcal{A}_+ \\ a &\mapsto \lim_{(\lambda,n)} y_{\lambda,n,a}.\end{aligned}$$

We claim that ρ is a morphism of right Banach $\mathcal{A}$-modules and that $\pi \circ \rho = 1_{\mathcal{A}}$.

Let $a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned}\rho(\alpha a + \beta b) &= \lim_{(\lambda,n)} y_{\lambda,n,\alpha a+\beta b} \\ &= \lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{j=1}^{n} g_\mu (\phi_\bullet^\mu)^{-1} \left(\sqrt{e_j^\mu}\right) \otimes g_\mu (\phi_\bullet^\mu)^{-1} \left(\sqrt{e_j^\mu}\right) (\alpha a + \beta b) \\ &= \lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{j=1}^{n} g_\mu (\phi_\bullet^\mu)^{-1} \left(\sqrt{e_j^\mu}\right) \otimes g_\mu (\phi_\bullet^\mu)^{-1} \left(\sqrt{e_j^\mu}\right) \alpha a \\ &+ \lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{j=1}^{n} g_\mu (\phi_\bullet^\mu)^{-1} \left(\sqrt{e_j^\mu}\right) \otimes g_\mu (\phi_\bullet^\mu)^{-1} \left(\sqrt{e_j^\mu}\right) \beta b\end{aligned}$$

$$
\begin{aligned}
&= \alpha \lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{j=1}^{n} g_\mu(\phi_\bullet^\mu)^{-1}\left(\sqrt{e_j^\mu}\right) \otimes g_\mu(\phi_\bullet^\mu)^{-1}\left(\sqrt{e_j^\mu}\right) a \\
&+ \beta \lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{j=1}^{n} g_\mu(\phi_\bullet^\mu)^{-1}\left(\sqrt{e_j^\mu}\right) \otimes g_\mu(\phi_\bullet^\mu)^{-1}\left(\sqrt{e_j^\mu}\right) b \\
&= \alpha \lim_{(\lambda,n)} y_{\lambda,n,a} + \beta \lim_{(\lambda,n)} y_{\lambda,n,b} \\
&= \alpha\rho(a) + \beta\rho(b)
\end{aligned}
$$

and

$$
\begin{aligned}
\rho(ab) &= \lim_{(\lambda,n)} y_{\lambda,n,ab} \\
&= \lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{j=1}^{n} g_\mu(\phi_\bullet^\mu)^{-1}\left(\sqrt{e_j^\mu}\right) \otimes g_\mu(\phi_\bullet^\mu)^{-1}\left(\sqrt{e_j^\mu}\right) ab \\
&= \left(\lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{j=1}^{n} g_\mu(\phi_\bullet^\mu)^{-1}\left(\sqrt{e_j^\mu}\right) \otimes g_\mu(\phi_\bullet^\mu)^{-1}\left(\sqrt{e_j^\mu}\right) a \right) b \\
&= \left(\lim_{(\lambda,n)} y_{\lambda,n,a} \right) b \\
&= \rho(a)b.
\end{aligned}
$$

By inequalities (4.1) and (4.2) of Lemma 4.13, we have

$$\|\rho(a)\|_{\mathcal{A}\hat{\otimes}\mathcal{A}_+} \leq 2\ell^2 \|a\|.$$

Thus ρ is a morphism of right Banach $\mathcal{A}$-modules.

It remains to show that $\pi \circ \rho = 1_{\mathcal{A}}$. Let $a \in \mathcal{A}$. Then

$$
\begin{aligned}
(\pi\circ\rho)(a) &= \pi\left(\lim_{(\lambda,n)} y_{\lambda,n,a} \right) \\
&= \pi\left(\lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{j=1}^{n} g_\mu(\phi_\bullet^\mu)^{-1}\left(\sqrt{e_j^\mu}\right) \otimes g_\mu(\phi_\bullet^\mu)^{-1}\left(\sqrt{e_j^\mu}\right) a \right) \\
&= \lim_{(\lambda,n)} \pi\left(\sum_{\mu\in\lambda} \sum_{j=1}^{n} g_\mu(\phi_\bullet^\mu)^{-1}\left(\sqrt{e_j^\mu}\right) \otimes g_\mu(\phi_\bullet^\mu)^{-1}\left(\sqrt{e_j^\mu}\right) a \right) \\
&= \lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{j=1}^{n} \pi\left(g_\mu(\phi_\bullet^\mu)^{-1}\left(\sqrt{e_j^\mu}\right) \otimes g_\mu(\phi_\bullet^\mu)^{-1}\left(\sqrt{e_j^\mu}\right) a \right) \\
&= \lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{j=1}^{n} h_\mu(\phi_\bullet^\mu)^{-1}\left(e_j^\mu\right) a.
\end{aligned}
$$

Note that, for all $s \in \Omega$, there exists a μ_s such that $s \in V_{\mu_s}$.

Since (f_n^μ) is a bounded approximate identity in $\tilde{A}_\mu$ we have

$$\begin{aligned} a(s) &= \lim_{n\to\infty} (\phi_s^{\mu_s})^{-1} (f_n^{\mu_s})\, a(s) \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} (\phi_s^{\mu_s})^{-1} \left(e_j^{\mu_s}\right) a(s). \end{aligned}$$

Recall that $\sum_{\mu_s\in\Lambda} h_{\mu_s}(s) = 1$ for all $s \in \Omega$ and $h_\mu(s) = 0$ for $s \notin V_\mu$.

Therefore,

$$\begin{aligned} (\pi\circ\rho)(a)(s) &= \lim_{(\lambda,n)} \sum_{\mu_s\in\lambda} \sum_{j=1}^{n} h_{\mu_s}(s)(\phi_s^{\mu_s})^{-1} \left(e_j^{\mu_s}\right) a(s) \\ &= \lim_{\lambda} \sum_{\mu\in\lambda} h_\mu(s) \lim_{n\to\infty} \sum_{j=1}^{n} (\phi_s^{\mu_s})^{-1} \left(e_j^{\mu_s}\right) a(s) \\ &= \lim_{\lambda} \sum_{\mu_s\in\lambda} h_{\mu_s}(s) a(s) \\ &= a(s) \end{aligned}$$

for every $s \in \Omega$.

Thus $(\pi\circ\rho)(a) = a$. □

Lemma 4.15. *Let Ω be a Hausdorff locally compact space and let Ω be a disjoint union of a family of open subsets $\{W_\mu\}$, $\mu \in \mathcal{M}$, of Ω. Suppose, for every $\mu \in \mathcal{M}$, W_μ is paracompact. Then Ω is paracompact.*

Proof. Let $\mathcal{V} = \{V_\alpha\}$ be an arbitrary open cover of Ω. For each $\mu \in \mathcal{M}$, the family $V_\mu = \{V \cap W_\mu : V \in \mathcal{V}\}$ is an open cover of W_μ. Since W_μ is paracompact, V_μ has an open locally finite refinement $\mathcal{N}_\mu$ that is also a cover of W_μ. Hence $\mathcal{V}$ has an open locally finite refinement $\mathcal{N} = \cup_{\mu\in\mathcal{M}}\mathcal{N}_\mu$ of Ω. Therefore Ω is paracompact. □

Theorem 4.16. *Let Ω be a Hausdorff locally compact space with the topological dimension* $\dim\Omega \le \ell$, *for some $\ell \in \mathbb{N}$, let $\mathcal{U} = \{\Omega, A_x, \Theta\}$ be a locally trivial continuous field of σ-unital C^*-algebras, and let the C^*-algebra $\mathcal{A}$ be defined by $\mathcal{U}$. Then the following conditions are equivalent:*
(i) *Ω is paracompact;*
(ii) *$\mathcal{A}$ is right projective and $\mathcal{U}$ is a disjoint union of σ-locally trivial continuous fields of C^*-algebras with strictly positive elements.*

Proof. By Theorem 4.12, the fact that Ω is paracompact with the topological dimension $\dim \Omega \leq \ell$ implies right projectivity of $\mathcal{A}$. By Remark 1.18, since Ω is paracompact, $\mathcal{U}$ is a disjoint union of σ-locally trivial continuous fields of C^*-algebras.

By Proposition 3.4 and Lemma 4.15, conditions (ii) implies paracompactness of Ω. Thus (ii) $\Longleftrightarrow$ (i). $\square$

5 Projectivity and continuous fields of Banach algebras of compact operators

5.1 Projectivity of $K(E)$

The following definition and theorem is from [34].

Definition 5.1. *Let E be a Banach space and let $\{e_\alpha\}_{\alpha\in\Lambda}$ be a family of elements in E. We say that $\{e_\alpha\}_{\alpha\in\Lambda}$ is an extended unconditional basis for E if for every $x \in E$ there exists a unique family of scalars $\{\xi_i\}_{i\in\Lambda}$ such that*

$$x = \lim_{\lambda\in N(\Lambda)} \sum_{i\in\lambda} \xi_i e_i$$

where $N(\Lambda)$ is the collection of all finite subsets of Λ ordered by inclusion. If Λ is countable then we say that $\{e_\alpha\}_{\alpha\in\Lambda}$ is an unconditional basis for E.

Example 5.2. We give some examples of some Banach spaces with an unconditional or extended unconditional basis.

1. The sequence spaces c_0 and ℓ^p, for every $1 \le p < \infty$, have an unconditional basis.
2. Every separable Hilbert space has an unconditional basis and every non-separable Hilbert space has an extended unconditional basis.
3. Let Λ be an uncountable set equipped with the discrete topology. Then $C_0(\Lambda)$ has an extended unconditional basis.

Theorem 5.3 ([34], Theorem 17.5). *Let E be a Banach space and let $\{e_\alpha\}_{\alpha\in\Lambda}$ be a family of elements in E. Then the following are equivalent:*

(i) *$\{e_\alpha\}_{\alpha\in\Lambda}$ is an extended unconditional basis of E.*

(ii) *There exists a constant K such that*

$$\|\sum_{i\in\lambda} \eta_i e_i\|_E \le K\|\sum_{i\in\lambda} \gamma_i e_i\|_E$$

for any $\lambda \in N(\Lambda)$ and any $\{\eta_i\}_{i\in\lambda}, \{\gamma_i\}_{i\in\lambda} \in \mathbb{C}$ with $|\eta_i| \le |\gamma_i|$ for every $i \in \lambda$.

Definition 5.4. *Let E be a Banach space and let $\{e_\alpha\}_{\alpha\in\Lambda}$ be a family of elements in E. We say that $\{e_\alpha\}_{\alpha\in\Lambda}$ is a hyperorthogonal extended basis for E if*

$$\|\sum_{i\in\lambda} \eta_i e_i\|_E \le \|\sum_{i\in\lambda} \gamma_i e_i\|_E$$

for any $\lambda \in N(\Lambda)$ and any $\{\eta_i\}_{i\in\lambda}, \{\gamma_i\}_{i\in\lambda} \in \mathbb{C}$ with $|\eta_i| \le |\gamma_i|$ for every $i \in \lambda$.

Definition 5.5. *Let E be a Banach space with an extended unconditional basis* $\{e_\alpha\}_{\alpha\in\Lambda}$*. For each* $\theta \in \Lambda$ *define the following linear functional*

$$f_\theta : E \to \mathbb{C}$$
$$\lim_{\lambda\in N(\Lambda)} \sum_{\theta\in\lambda} \xi_\theta e_\theta \mapsto \xi_\theta.$$

The family $\{f_\theta\}$ *are known as the associated linear functionals to* $\{e_\theta\}$*.*

Definition 5.6. *Let E be a Banach space with an extended unconditional basis* $\{e_\theta\}_{\theta\in\Lambda}$ *with the associated linear functionals* $\{f_\theta\}_{\theta\in\Lambda}$*. We say that E is shrinking if* $\{f_\theta\}_{\theta\in\Lambda}$ *is an extended unconditional basis in* E^**.*

Example 5.7. The sequence spaces c_0 and ℓ^p, for every $1 < p < \infty$ are shrinking. Note that ℓ^1 is not shrinking.

The following lemma is a slight strengthening of Lemma 3 in [22].

Lemma 5.8. *Let E be a Banach space with and extended unconditional basis* $\{e_\theta\}_{\theta\in\Lambda}$ *with the associated linear functionals* $\{f_\theta\}_{\theta\in\Lambda}$*. Then there exists a constant M such that*

$$\sup_{\lambda\in N(\Lambda)} \|\sum_{\theta\in\lambda} f_\theta(x)e_\theta\|_E \le M\|x\|_E$$

and in particular the linear functionals f_θ *are continuous on E.*

Proof. Consider the linear space E_1 of sets $\xi = \{\xi_\theta \in \mathbb{C}, \theta \in \Lambda\}$ such that the family $\xi_\theta e_\theta$ is summable in E. We put the following norm on E_1

$$\|\xi\|_{E_1} = \sup_{\lambda\in N(\Lambda)} \|\sum_{\theta\in\lambda} \xi_\theta e_\theta\|_E.$$

It can be shown that this is a valid norm and that E_1 is complete with respect to this norm. We can then apply the inverse mapping theorem to show that the following operator is bounded

$$P : E \to E_1$$
$$x \mapsto \{f_\theta(x)\}.$$

This means that

$$\sup_{\lambda\in N(\Lambda)} \|\sum_{\theta\in\lambda} f_\theta(x)e_\theta\| \le \|P\|\,\|x\|.$$

We then have that

$$|f_\theta| = \|f_\theta(x)e_\theta\|/\|e_\theta\| \le (\|P\|/\|e_\theta\|)\|x\|$$

and so the f_θ are continuous. □

Let E be a Banach space with an extended unconditional basis $\{e_\theta\}_{\theta\in\Lambda}$. Let $\{f_\theta\}$ be the associated linear functionals to $\{e_\theta\}$. For $\theta \in \Lambda$ define

$$\begin{aligned} e_{\theta\theta} : E &\to E \\ x &\mapsto f_\theta(x)e_\theta. \end{aligned}$$

Note that $e_{\theta\theta} \in K(E)$ for each $\theta \in \Lambda$.

Let

$$\{C_\lambda = \sum_{\theta\in\lambda} e_{\theta\theta}\}_{\lambda\in N(\Lambda)}.$$

Lemma 5.9 ([22], Lemma 4)**.** *Let E be a Banach space with an extended unconditional basis $\{e_\theta\}_{\theta\in\Lambda}$. Let $\{f_\theta\}$ be the associated linear functionals to $\{e_\theta\}$. Then $\{C_\lambda\}_{\lambda\in N(\Lambda)}$ is a left bounded approximate identity in $K(E)$.*

Proof. Let $\varepsilon > 0$ and let $a \in K(E)$. Set $S = \{x \in E : \|x\| \leq 1\}$. Since a is compact we have that $\overline{a(S)}$ is compact.
Let M be the constant in Lemma 5.8. Let $y \in E$. Since E has an extended unconditional basis there exists $\lambda_y \in N(\Lambda)$, such that for all $\lambda \geq \lambda_y$ we have

$$\|\sum_{\theta\in\lambda} f_\theta(y)e_\theta - y\|_E < \frac{\varepsilon}{1+M}.$$

Now set

$$U_y = \{y' \in E : \|(\sum_{\theta\in\lambda_y} e_{\theta\theta} - id_E)(y')\|_E < \frac{\varepsilon}{1+M}\}.$$

The family $\{U_y\}_{y\in E}$ is an open cover of $\overline{a(S)}$ and so we can find a finite open subcover $\{U_{y_i}\}_{i=1,2\ldots n}$. Thus for every $y \in \overline{a(S)}$ there is an element $\lambda_{y_{i_0}}$ from $\{\lambda_{y_i}\}$ such that

$$\|\sum_{\theta\in\lambda_{y_{i_0}}} f_\theta(y)e_\theta - y\|_E < \frac{\varepsilon}{1+M}.$$

By Lemma 5.8 we have

$$\sup_{\lambda\in N(\lambda)} \|\sum_{\theta\in\lambda} f_\theta\left(\sum_{\theta'\in\lambda_{y_{i_0}}} f_{\theta'}(y)e_{\theta'} - y\right)e_\theta\|_E \leq \frac{M\varepsilon}{1+M},$$

and so

$$\sup_{\lambda\in N(\lambda);\lambda\cap\lambda_{y_{i_0}}=\emptyset} \|\sum_{\theta\in\lambda} f_\theta(y)e_\theta\|_E \leq \frac{M\varepsilon}{1+M}.$$

Let $\lambda_0 = \bigcup_{i=1}^n \lambda_{y_i}$. Then, for every $\lambda \geq \lambda_0$, and $y \in \overline{a(S)}$ we have

$$\begin{aligned}
&\|\sum_{\theta\in\lambda} f_\theta(y)e_\theta - y\|_E \\
\leq&\|\sum_{\theta\in\lambda_{y_{i_0}}} f_\theta(y)e_\theta - y\|_E + \|\sum_{\theta\in\lambda} f_\theta(y)e_\theta - \sum_{\theta\in\lambda_{y_{i_0}}} f_\theta(y)e_\theta\|_E \\
<&\frac{\varepsilon(1+M)}{(1+M)} \\
=&\varepsilon.
\end{aligned}$$

This shows that

$$\|C_\lambda a - a\|_E < \varepsilon$$

for all $\lambda \geq \lambda_0$. From Lemma 5.8 we have that C_λ is bounded. □

Lemma 5.10. *Let E be a Banach space with an extended unconditional basis $\{e_\theta\}_{\theta\in\Lambda}$ such that E is shrinking. Then $\{C_\lambda\}_{\lambda\in N(\Lambda)} = \{\sum_{\theta\in\lambda} e_{\theta\theta}\}_{\lambda\in N(\Lambda)}$ is a right bounded approximate identity in $K(E)$.*

Proof. Let $\varepsilon > 0$ and $a \in K(E)$. By [36, 7.2] the adjoint operator a^* is a compact operator from $K(E^*)$. By assumption E^* has an extended unconditional basis $\{f_\theta\}$. Then, by Lemma 5.9, there exists a $\lambda_0 \in N(\Lambda)$ such that

$$\|aC_\lambda - a\| = \|(aC_\lambda - a)^*\| = \|C_\lambda^* a^* - a^*\| < \varepsilon$$

for every $\lambda > \lambda_0$. □

We now generalise a result of Lykova from [20].

Theorem 5.11. *Let E be a Banach space with an extended unconditional basis $\{e_\theta\}_{\theta\in\Lambda}$. Then $K(E)$ is right projective.*

Proof. Let $\{f_\theta\}$ be the associated linear functionals to $\{e_\theta\}$ and set $e_{\theta\theta} = f_\theta e_\theta \in K(E)$. By Lemma 5.9, the family $\{\sum e_{\theta\theta}\}$ is a left bounded approximate identity for $K(E)$, so there exists a constant C_1 such that for all $\lambda \in N(\Lambda)$

$$\|\sum_{\theta\in\lambda} e_{\theta\theta}\|_{K(E)} \leq C_1.$$

Since $\{e_\theta\}_{\theta\in\Lambda}$ is an extended unconditional basis there exists a constant C_2 such that for all $\lambda = (\theta_1, ..., \theta_m) \in N(\Lambda)$, $(\eta_1, ..., \eta_m) \in \mathbb{T}$ and $a \in K(E)$ we have

$$\|\sum_{t=1}^m \eta_t e_{\theta_t\theta_t} a\|_{K(E)} \leq C_2 \|\sum_{t=1}^m e_{\theta_t\theta_t} a\|_{K(E)}.$$

Let $\lambda \in N(\Lambda)$ and $a \in K(E)$. We then define

$$y_{\lambda,a} = \sum_{\theta \in \lambda} e_{\theta\theta} \otimes e_{\theta\theta} a.$$

We wish to show that $(y_{\lambda,a})_{\lambda \in N(\Lambda)}$ converges in $K(E)\hat{\otimes}K(E)$. We break this into the following lemmas.

Lemma 5.12. *Let $a \in K(E)$ and let $\lambda > \lambda_0 \in N(\Lambda)$. Let $\lambda_0 = (\theta_1, ..., \theta_{m_0})$ and $\lambda = (\theta_1, ..., \theta_{m_0}, ..., \theta_{m_1})$. Then*

$$\|y_{\lambda \setminus \lambda_0, a}\|_{K(E)\hat{\otimes}K(E)} \leq C_1 C_2^2 \| \sum_{s=m_0+1}^{m_1} e_{\theta_s\theta_s} a \|_{K(E)}.$$

Proof. Let η be a primary $(m_1 - m_0)$th root of unity. From Lemma 1.4 we have that

$$\begin{aligned}
\|y_{\lambda \setminus \lambda_0, a}\|_{K(E)\hat{\otimes}K(E)} &\leq \frac{1}{m_1 - m_0} \sum_{t=1}^{m_1 - m_0} \left\| \sum_{s=m_0+1}^{m_1} \eta^{t(s-m_0)} e_{\theta_s\theta_s} \right\|_{K(E)} \left\| \sum_{s=m_0}^{m_1} \eta^{-t(s-m_0)} e_{\theta_s\theta_s} a \right\| \\
&\leq \frac{C_2^2}{m_1 - m_0} \sum_{t=1}^{m_1 - m_0} \left\| \sum_{s=m_0+1}^{m_1} e_{\theta_s\theta_s} \right\|_{K(E)} \left\| \sum_{s=m_0+1}^{m_1} e_{\theta_s\theta_s} a \right\|_{K(E)} \\
&\leq C_1 C_2^2 \left\| \sum_{t=m_0+1}^{m_1} e_{\theta_t\theta_t} a \right\|_{K(E)}.
\end{aligned}$$

□

Lemma 5.13. *For each $a \in K(E)$ and $\varepsilon > 0$ there exists $\lambda_0 \in N(\Lambda)$ such that for all $\lambda \in N(\Lambda)$,*

$$\left\| \sum_{\theta \in \lambda_0 \setminus \lambda} e_{\theta\theta} a \right\|_{K(E)} < \frac{\varepsilon}{2 C_1 C_2^2}.$$

Proof. This follows directly from Lemma 5.9.

□

Lemma 5.14. *Let $a \in K(E)$ and $\varepsilon > 0$. Then there exists a $\lambda_0 \in N(\Lambda)$ such that for all $\lambda_2 > \lambda_1 > \lambda_0 \in N(\Lambda)$ we have*

$$\|y_{\lambda_2, a} - y_{\lambda_1, a}\| < \varepsilon.$$

Proof. Let λ_0 be as in Lemma 5.13. Then we have

$$\begin{aligned}\|y_{\lambda_2,a} - y_{\lambda_1,a}\| &= \|y_{\lambda_2\backslash\lambda_0,a} + y_{\lambda_0,a} - y_{\lambda_1\backslash\lambda_0,a} - y_{\lambda_0,a}\| \\ &= \|y_{\lambda_2\backslash\lambda_0,a} - y_{\lambda_1\backslash\lambda_0,a}\| \\ &\leq \|y_{\lambda_2\backslash\lambda_0,a}\| + \|y_{\lambda_1\backslash\lambda_0,a}\| \\ &< \frac{C_1C_2^2\varepsilon}{2C_1C_2^2} + \frac{C_1C_2^2\varepsilon}{2C_1C_2^2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

□

We are now ready to show that $K(E)$ is right projective. We define the following morphism of Banach $K(E)-$modules

$$\begin{aligned}\rho : K(E) &\to K(E)\hat{\otimes}K(E) \\ a &\mapsto \lim_{\lambda} y_{\lambda,a}.\end{aligned}$$

We now show that ρ is a morphism of modules and that $\pi \circ \rho = 1_{K(E)}$.
The map ρ is linear since for $a, b \in K(E), \alpha, \beta \in \mathbb{C}$

$$\begin{aligned}\rho(\alpha a + \beta b) &= \lim_{\lambda\in\Lambda} y_{\lambda,\alpha a+\beta b} \\ &= \lim_{\lambda\in\Lambda}\left(\sum_{\theta\in\lambda} e_{\theta\theta}\otimes e_{\theta\theta}(\alpha a + \beta b)\right) \\ &= \lim_{\lambda\in\Lambda}\left(\alpha\sum_{\theta\in\lambda} e_{\theta\theta}\otimes e_{\theta\theta}a + \beta\sum_{\theta\in\lambda} e_{\theta\theta}\otimes e_{\theta\theta}b\right) \\ &= \alpha\lim_{\lambda\in\Lambda}\left(\sum_{\theta\in\lambda} e_{\theta\theta}\otimes e_{\theta\theta}a\right) + \beta\lim_{\lambda\in\Lambda}\left(\sum_{\theta\in\lambda} e_{\theta\theta}\otimes e_{\theta\theta}b\right) \\ &= \alpha\lim_{\lambda\in\Lambda} y_{\lambda,a} + \beta\lim_{\lambda\in\Lambda} y_{\lambda,b} \\ &= \alpha\rho(a) + \beta\rho(b)\end{aligned}$$

and

$$\begin{aligned}\rho(ab) &= \lim_{\lambda\in\Lambda} y_{\lambda,ab} \\ &= \lim_{\lambda\in\Lambda}\sum_{\theta\in\lambda} e_{\theta\theta}\otimes e_{\theta\theta}ab\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\lambda\in\Lambda}\left(\sum_{\theta\in\lambda} e_{\theta\theta}\otimes e_{\theta\theta}a\right) b \\
&= \lim_{\lambda\in\Lambda} y_{\lambda,a} b \\
&= \rho(a)b.
\end{aligned}
$$

The map ρ is bounded as $\|\rho(a)\| \leq C_1^2 C_2^2 \|a\|$ from above.
We finally show that $\pi\circ\rho = 1_{K(E)}$,

$$
\begin{aligned}
\pi\circ\rho(a) &= \pi(\lim_{\lambda\in\Lambda} y_{\lambda,a}) \\
&= \pi\left(\lim_{\lambda\in\Lambda}\sum_{\theta\in\lambda} e_{\theta\theta}\otimes e_{\theta\theta}a\right) \\
&= \lim_{\lambda\in\Lambda}\pi\left(\sum_{\theta\in\lambda} e_{\theta\theta}\otimes e_{\theta\theta}a\right) \\
&= \lim_{\lambda\in\Lambda}\left(\sum_{\theta\in\lambda} e_{\theta\theta}a\right) \\
&= a
\end{aligned}
$$

This completes the proof. □

5.2 Projectivity of $C_0(\Omega, K(E))$

Theorem 5.15. *Let Ω be a locally compact Hausdorff space which is also paracompact and let E be a Banach space with an extended unconditional basis. Then $C_0(\Omega, K(E))$ is projective in mod-$C_0(\Omega, K(E))$.*

Proof. Let $\{e_\theta\}_{\theta\in\Phi}$ be an extended unconditional basis for E. Let $\{f_\theta\}_{\theta\in\Phi}$ be the associated linear functionals to $\{e_\theta\}_{\theta\in\Phi}$. Set $e_{\theta\theta} = f_\theta e_\theta \in K(E)$.

By assumption Ω is a paracompact locally compact Hausdorff space. Let $\mathcal{B} = \{V_\mu\}_{\mu\in\Lambda}$ be an open cover of Ω such that each point of Ω has a neighbourhood that intersects with no more than three sets of $\mathcal{B}$ as in [12]. By [18, Problem 5.W], since $\{V_\mu\}_{\mu\in\Lambda}$ is a locally finite open cover of the normal space Ω, it is possible to select a non-negative continuous function h_μ for each V_μ in $\mathcal{B}$ such that h_μ is 0 outside V_μ and is everywhere less than or equal to one, and

$$\sum_{\mu\in\Lambda} h_\mu(s) = 1 \text{ for all } \; s\in\Omega.$$

Set $g_\mu = \sqrt{h_\mu}$.

Since $\{e_\theta\}$ is an extended unconditional basis there exists a constant C_1 such that for all $\sigma = (\theta_1, ..., \theta_m) \in N(\Phi)$, $(\eta_1, ..., \eta_m) \in \mathbb{T}$ and $a \in K(E)$ we have

$$\| \sum_{t=1}^{m} \eta_t e_{\theta_t \theta_t} a \|_{K(E)} \leq C_1 \| \sum_{t=1}^{m} e_{\theta_t \theta_t} a \|_{K(E)}.$$

Since $\{\sum e_{\theta\theta}\}_{\theta \in \Phi}$ is a bounded approximate identity in $K(E)$ there exists a constant C_2 such that for all $\sigma \in N(\Phi)$

$$\| \sum_{\theta \in \sigma} e_{\theta\theta} \|_{K(E)} \leq C_2.$$

For $a \in C_0(\Omega, K(E))$, $(\lambda, \sigma) \in N(\Lambda) \times N(\Phi)$ we define

$$y_{\lambda,\sigma,a} = \sum_{\mu \in \lambda} \sum_{\theta \in \sigma} g_\mu e_{\theta\theta} \otimes g_\mu e_{\theta\theta} a.$$

We wish to show that $y_{\lambda,\sigma,a}$ is a Cauchy net. We break this up into the following lemmas.

Lemma 5.16. *1. Let $a \in C_0(\Omega, K(E))$, $\sigma \in N(\Phi)$ and $\lambda' > \lambda \in N(\Lambda)$. Let $\lambda = (\mu_1, ..., \mu_{m_0})$ and $\lambda' = (\mu_1, ..., \mu_{m_0}, ..., \mu_{m_1})$. Then*

$$\|y_{\lambda' \setminus \lambda, \sigma, a}\|_{C_0(\Omega, K(E)) \hat{\otimes} C_0(\Omega, K(E))_+} \leq 9 C_1^2 C_2^2 \max_{\mu \in \lambda' \setminus \lambda} \|g_\mu a\|_{C_0(\Omega, K(E))}.$$

2. Let $a \in C_0(\Omega, K(E))$. Then for any $\varepsilon > 0$, there exists a $\lambda_0 \in N(\Lambda)$ such that for all $\lambda > \lambda_0$ we have

$$\sup_{\mu \in \lambda \setminus \lambda_0} \|g_\mu a\|_{C_0(\Omega, K(E))} < \frac{\varepsilon}{36 C_1^2 C_2^2}.$$

3. Let $a \in C_0(\Omega, K(E))$, $\lambda = (\mu_1, ..., \mu_m) \in N(\Lambda)$ and $\sigma' > \sigma \in N(\Phi)$. Let $\sigma = (\theta_1, ..., \theta_{m_0})$ and $\sigma' = (\theta_1, ..., \theta_{m_0}, ..., \theta_{m_1})$. Then

$$\|y_{\lambda, \sigma' \setminus \sigma, a}\|_{C_0(\Omega, K(E)) \hat{\otimes} C_0(\Omega, K(E))_+} \leq 9 C_1^2 C_2 \sup_{x \in \Omega} \left\| \sum_{t=n_0+1}^{n_1} e_{\theta_t \theta_t} a(x) \right\|_{K(E)}.$$

4. For any $\varepsilon > 0$ there exists a $\sigma_0 \in N(\Phi)$ such that for all $\sigma > \sigma_0$ we have that

$$\sup_{x \in \Omega} \left\| \sum_{\theta \in \sigma \setminus \sigma_0} e_{\theta\theta} a(x) \right\|_{K(E)} < \frac{\varepsilon}{36 C_1^2 C_2}.$$

Proof. 1. Let η be a primary $(m_1 - m_0)th$ root of unity and let ξ is be a primary nth root of unity. By Lemma 1.5 we have

$$\|y_{\lambda'\setminus\lambda,\sigma,a}\|_{C_0(\Omega,K(E))\hat{\otimes}C_0(\Omega,K(E))_+} \leq \frac{1}{n(m_1-m_0)}\sum_{l=1}^{n}\sum_{k=1}^{m_1-m_0}\left\|\sum_{t=1}^{n}\sum_{i=m_0+1}^{m_1}\xi^{lt}\eta^{ki}g_{\mu_i}e_{\theta_t\theta_t}\right\|_{C_0(\Omega,K(E))} \times\left\|\sum_{t=1}^{n}\sum_{i=m_0+1}^{m_1}\xi^{-lt}\eta^{-ki}g_{\mu_i}e_{\theta_t\theta_t}a\right\|_{C_0(\Omega,K(E))}.$$

Since for every $x\in\Omega$ there are at most 3 values of μ such that $g_\mu(x)\neq 0$ we have the following, for $k=1,...,m_1-m_0, l=1,...,n$,

$$\begin{aligned}
&\left\|\sum_{t=1}^{n}\sum_{i=m_0+1}^{m_1}\xi^{lt}\eta^{ki}g_{\mu_i}e_{\theta_t\theta_t}\right\|_{C_0(\Omega,K(E))}\\
=&\sup_{x\in\Omega}\left\|\sum_{t=1}^{n}\sum_{i=m_0+1}^{m_1}\xi^{lt}\eta^{ki}g_{\mu_i}(x)e_{\theta_t\theta_t}\right\|_{K(E)}\\
\leq&3\max_{\mu\in\lambda'\setminus\lambda}\sup_{x\in\Omega}\left\|\sum_{t=1}^{n}\xi^{lt}g_{\mu}(x)e_{\theta_t\theta_t}\right\|_{K(E)}\\
\leq&3\left\|\sum_{t=1}^{n}\xi^{lt}e_{\theta_t\theta_t}\right\|_{K(E)}\\
\leq&3C_1\left\|\sum_{t=1}^{n}e_{\theta_t\theta_t}\right\|_{K(E)} \quad (\textit{by Theorem } 5.3)\\
\leq&3C_1C_2. \qquad (\textit{by Lemma } 5.8)
\end{aligned}$$

Similarly

$$\begin{aligned}
&\left\|\sum_{t=1}^{n}\sum_{i=m_0+1}^{m_1}\xi^{-lt}\eta^{-ki}g_{\mu_i}e_{\theta_t\theta_t}a\right\|_{C_0(\Omega,K(E))}\\
=&\sup_{x\in\Omega}\left\|\sum_{t=1}^{n}\sum_{i=m_0+1}^{m_1}\xi^{-lt}\eta^{-ki}g_{\mu_i}(x)e_{\theta_t\theta_t}a(x)\right\|_{K(E)}\\
\leq&3\max_{\mu\in\lambda'\setminus\lambda}\sup_{x\in\Omega}\left\|\sum_{t=1}^{n}\xi^{-lt}g_{\mu}(x)e_{\theta_t\theta_t}a(x)\right\|_{K(E)}\\
\leq&3\max_{\mu\in\lambda'\setminus\lambda}\left\|\sum_{t=1}^{n}\xi^{-lt}e_{\theta_t\theta_t}\right\|_{K(E)}\sup_{x\in\Omega}\|g_\mu(x)a(x)\|_{K(E)}
\end{aligned}$$

$$\leq 3C_1C_2 \max_{\mu\in\lambda'\setminus\lambda} \sup_{x\in\Omega} \left\| g_\mu(x)a(x) \right\|_{K(E)} \text{ (by Theorem 5.3 and Lemma 5.8)}$$
$$\leq 3C_1C_2 \max_{\mu\in\lambda'\setminus\lambda} \left\| g_\mu a \right\|_{C_0(\Omega,K(E))}.$$

Therefore

$$\|y_{\lambda'\setminus\lambda,\sigma,a}\|_{C_0(\Omega,K(E))\hat{\otimes}C_0(\Omega,K(E))_+} \leq 9C_1^2C_2^2 \max_{\mu\in\lambda'\setminus\lambda} \|g_\mu a\|_{C_0(\Omega,K(E))}.$$

2. Since $a \in C_0(\Omega, K(E))$, for every $\varepsilon > 0$, there is a compact set $K \subset \Omega$ such that $\|a(t)\|_{K(E)} < \frac{\varepsilon}{36C_1^2C_2^2}$ for every $t \in \Omega \setminus K$. Compact sets intersect with a finite number of elements of $\{V_\mu\}_{\mu\in\Lambda}$. Set $\lambda_0 = \mu : V_\mu \cap K = \emptyset$, then for each $t \in K$ we have
$$\sup_{\mu\in\lambda\setminus\lambda_0} \|g_\mu(t)a(t)\|_{K(E)} = 0.$$
Therefore
$$\sup_{\mu\in\lambda\setminus\lambda_0} \|g_\mu(t)a(t)\|_{C_0(\Omega,K(E))} < \frac{\varepsilon}{36C_1^2C_2^2}.$$

3. Let η be an mth root of unity and let ξ is be an $(n_0 - n_1)th$ root of unity. By Lemma 1.5 we have

$$\|y_{\lambda,\sigma'\setminus\sigma,a}\| \leq$$
$$\frac{1}{m(n_1-n_0)} \sum_{l=1}^{n_1-n_0} \sum_{k=1}^{m} \left\| \sum_{t=n_0+1}^{n_1} \sum_{i=1}^{m} \xi^{lt}\eta^{ki} g_{\mu_i} e_{\theta_t\theta_t} \right\| \left\| \sum_{t=n_0+1}^{n_1} \sum_{i=1}^{m} \xi^{-lt}\eta^{-ki} g_{\mu_i} e_{\theta_t\theta_t} a \right\|.$$

From part (1) we have that

$$\left\| \sum_{t=n_0+1}^{n_1} \sum_{i=1}^{m} \xi^{lt}\eta^{ki} g_{\mu_i} e_{\theta_t\theta_t} \right\|_{C_0(\Omega,K(E))} \leq 3C_1C_2.$$

We then have

$$\left\| \sum_{t=n_0+1}^{n_1} \sum_{i=1}^{m} \xi^{-lt}\eta^{-ki} g_{\mu_i} e_{\theta_t\theta_t} a \right\|_{C_0(\Omega,K(E))}$$
$$= \sup_{x\in\Omega} \left\| \sum_{t=n_0+1}^{n_1} \sum_{i=1}^{m} \xi^{-lt}\eta^{-ki} g_{\mu_i}(x) e_{\theta_t\theta_t} a(x) \right\|_{K(E)}$$
$$\leq 3 \max_{\mu\in\lambda} \sup_{x\in\Omega} \left\| \sum_{t=n_0+1}^{n_1} \xi^{-lt} g_\mu(x) e_{\theta_t\theta_t} a(x) \right\|_{K(E)}$$

$$\leq 3 \sup_{x\in\Omega} \left\| \sum_{t=n_0+1}^{n_1} \xi^{-lt} e_{\theta_t\theta_t} a(x) \right\|_{K(E)}$$
$$\leq 3C_1 \sup_{x\in\Omega} \left\| \sum_{t=n_0+1}^{n_1} e_{\theta_t\theta_t} a(x) \right\|_{K(E)}.$$

Therefore

$$\|y_{\lambda,\sigma'\setminus\sigma,a}\|_{C_0(\Omega,K(E))+\hat{\otimes}C_0(\Omega,K(E))} \leq 9C_1^2C_2 \sup_{x\in\Omega} \left\| \sum_{t=n_0+1}^{n_1} e_{\theta_t\theta_t} a(x) \right\|_{K(E)}.$$

4. Since a vanishes at infinity we can pick a compact set $K \subset \Omega$ such that for every $x \in \Omega \setminus K$ we have that
$$\|a(x)\|_{K(E)} \leq \frac{\varepsilon}{36C_1^2C_2^2}.$$
Therefore, for $x \in \Omega \setminus K$ and $\sigma \in N(\Phi)$,
$$\| \sum_{\theta\in\sigma} e_{\theta\theta} a(x)\|_{K(E)} \leq \frac{\varepsilon}{36C_1^2C_2}. \tag{5.1}$$
Since a is continuous, for every $\varepsilon > 0$ and every $x \in \Omega$, there exists an open set $U_x \subset \Omega$ such that $\|a(x) - a(y)\| < \frac{\varepsilon}{72C_1^2C_2^2}$ for every $y \in U_x$. Then $\{U_x\}_{x\in K}$ is an open cover of the compact set K.

Let $\{U_{x_i}\}_{i=1}^m$ be a finite subcover of $\{U_x\}_{x\in K}$. Let $i \in \{1, ..., m\}$. Then $a(x_i)$ is a compact operator in $K(E)$. Therefore since $\{\sum e_{\sigma\sigma}\}$ is an approximate identity in $K(E)$ we can pick a finite set σ_i such that for every $\sigma > \sigma_i$ we have that
$$\left\| \sum_{\theta\in\sigma\setminus\sigma_i} e_{\theta\theta} a(x_i) \right\|_{K(E)} \leq \frac{\varepsilon}{72C_1^2C_2}.$$
Set $\sigma_0 = \bigcup_{i=1}^m \sigma_i$.

Let $x \in K$. Since $\{U_{x_j}\}_{j=1}^m$ is a cover of K there exists an i such that $x \in U_{x_i}$. Then for $\sigma > \sigma_0$ we have that
$$\left\| \sum_{\theta\in\sigma\setminus\sigma_0} e_{\theta\theta} a(x) \right\|_{K(E)}$$

$$
\begin{aligned}
&= \left\| \sum_{\theta \in \sigma \setminus \sigma_0} e_{\theta\theta}(a(x) - a(x_i) + a(x_i)) \right\|_{K(E)} \\
&\leq \left\| \sum_{\theta \in \sigma \setminus \sigma_0} e_{\theta\theta}(a(x) - a(x_i)) \right\|_{K(E)} + \left\| \sum_{\theta \in \sigma \setminus \sigma_0} e_{\theta\theta} a(x_i) \right\|_{K(E)} \\
&< \left\| \sum_{\theta \in \sigma \setminus \sigma_0} e_{\theta\theta} \right\|_{K(E)} \|a(x) - a(x_i)\|_{K(E)} + \frac{\varepsilon}{72C_1^2C_2} \\
&< C_2 \frac{\varepsilon}{72C_1^2C_2^2} + \frac{\varepsilon}{72C_1^2C_2} \\
&= \frac{\varepsilon}{36C_1^2C_2}.
\end{aligned}
$$

Therefore, for $x \in K$ and $\sigma \in N(\Phi)$,

$$\| \sum_{\theta \setminus \theta_0 \in \sigma} e_{\theta\theta} a(x) \|_{K(E)} \leq \frac{\varepsilon}{36C_1^2C_2}. \tag{5.2}$$

Thus by (5.1) and (5.2),

$$\sup_{x \in \Omega} \left\| \sum_{\theta \in \sigma \setminus \sigma_0} e_{\theta\theta} a(x) \right\|_{K(E)} < \frac{\varepsilon}{36C_1^2C_2}.$$

□

Lemma 5.17. *For each $a \in C_0(\Omega, K(E))$, $(y_{\lambda,n,a})_{(\lambda,n)}$ is a Cauchy net in $C_0(\Omega, K(E)) \hat{\otimes} C_0(\Omega, K(E))_+$.*

Proof. Let $\varepsilon > 0$, $a \in C_0(\Omega, K(E))$. Let λ_0 and σ_0 be as in Lemma 5.16 parts (2) and (4) respectively.

Then for $(\lambda_2, \sigma_2) > (\lambda_1, \sigma_1) > (\lambda_0, \sigma_0)$. Note that

$$y_{\lambda_2,\sigma_2,a} = y_{\lambda_2 \setminus \lambda_0,\sigma_2,a} + y_{\lambda_0,\sigma_2 \setminus \sigma_0,a} + y_{\lambda_0,\sigma_0,a},$$

$$y_{\lambda_1,\sigma_1,a} = y_{\lambda_1 \setminus \lambda_0,\sigma_1,a} + y_{\lambda_0,\sigma_1 \setminus \sigma_0,a} + y_{\lambda_0,\sigma_0,a}.$$

Thus

$$
\begin{aligned}
&\|y_{\lambda_2,\sigma_2,a} - y_{\lambda_1,\sigma_1,a}\|_{C_0(\Omega,K(E)) \hat{\otimes} C_0(\Omega,K(E))_+} \\
= &\|y_{\lambda_2 \setminus \lambda_0,\sigma_2,a} + y_{\lambda_0,\sigma_2 \setminus \sigma_0,a} + y_{\lambda_0,\sigma_0,a} - y_{\lambda_1 \setminus \lambda_0,\sigma_1,a} - y_{\lambda_0,\sigma_1 \setminus \sigma_0,a} - y_{\lambda_0,\sigma_0,a}\|_{C_0(\Omega,K(E)) \hat{\otimes} C_0(\Omega,K(E))_+}
\end{aligned}
$$

$$\leq \|y_{\lambda_2\setminus\lambda_0,\sigma_2,a}\|_{C_0(\Omega,K(E))\hat{\otimes}C_0(\Omega,K(E))_+} + \|y_{\lambda_1\setminus\lambda_0,\sigma_1,a}\|_{C_0(\Omega,K(E))\hat{\otimes}C_0(\Omega,K(E))_+}$$
$$+\|y_{\lambda_0,\sigma_1\setminus\sigma_0,a}\|_{C_0(\Omega,K(E))\hat{\otimes}C_0(\Omega,K(E))_+} + \|y_{\lambda_0,\sigma_2\setminus\sigma_0,a}\|_{C_0(\Omega,K(E))\hat{\otimes}C_0(\Omega,K(E))_+}.$$

By parts (1) and (2) of Lemma 5.16,

$$\|y_{\lambda_2\setminus\lambda_0,\sigma_2,a}\|_{C_0(\Omega,K(E))\hat{\otimes}C_0(\Omega,K(E))_+} < \frac{\varepsilon}{4}$$

and

$$\|y_{\lambda_1\setminus\lambda_0,\sigma_1,a}\|_{C_0(\Omega,K(E))\hat{\otimes}C_0(\Omega,K(E))_+} < \frac{\varepsilon}{4}.$$

By parts (3) and (4) of Lemma 5.16,

$$\|y_{\lambda_0,\sigma_1\setminus\sigma_0,a}\|_{C_0(\Omega,K(E))\hat{\otimes}C_0(\Omega,K(E))_+} < \frac{\varepsilon}{4}.$$

and

$$\|y_{\lambda_0,\sigma_2\setminus\sigma_0,a}\|_{C_0(\Omega,K(E))\hat{\otimes}C_0(\Omega,K(E))_+} < \frac{\varepsilon}{4}.$$

Therefore

$$\|y_{\lambda_2,\sigma_2,a} - y_{\lambda_1,\sigma_1,a}\|_{C_0(\Omega,K(E))\hat{\otimes}C_0(\Omega,K(E))_+} < \varepsilon.$$

□

Conclusion of the proof of Theorem 5.15.
Note that $C_0(\Omega, A)\hat{\otimes}C_0(\Omega, K(E))_+$ is complete. By Lemma 5.17, $y_{\lambda,\sigma,a}$ is a Cauchy net. Therefore $\lim_{(\lambda,n)} y_{\lambda,n,a}$ exists for every $a \in C_0(\Omega, K(E))$. Set

$$\rho : C_0(\Omega, K(E)) \to C_0(\Omega, K(E))\hat{\otimes}C_0(\Omega, K(E))$$
$$a \mapsto \lim_{(\lambda,\sigma)} y_{\lambda,\sigma,a}.$$

It is clear that ρ is linear since, for all $a, b \in C_0(\Omega, A)$ and for all $\alpha, \beta \in \mathbb{C}$,

$$\rho(\alpha a + \beta b) = \lim_{(\lambda,\sigma)} y_{\lambda,\sigma,\alpha a+\beta b} = \lim_{(\lambda,\sigma)} \sum_{\mu\in\lambda}\sum_{\theta\in\sigma} g_\mu e_{\theta\theta} \otimes g_\mu e_{\theta\theta}(\alpha a + \beta b)$$

$$= \alpha \lim_{(\lambda,\sigma)} \sum_{\mu\in\lambda}\sum_{\theta\in\sigma} g_\mu e_{\theta\theta} \otimes g_\mu e_{\theta\theta} a + \beta \lim_{(\lambda,\sigma)} \sum_{\mu\in\lambda}\sum_{\theta\in\sigma} g_\mu e_{\theta\theta} \otimes g_\mu e_{\theta\theta} b = \alpha\rho(a) + \beta\rho(b).$$

Similarly $\rho(ab) = \rho(a)b$ since

$$\rho(ab) = \lim_{(\lambda,\sigma)} y_{\lambda,\sigma,ab} = \lim_{(\lambda,\sigma)} \sum_{\mu\in\lambda}\sum_{\theta\in\sigma} g_\mu e_{\theta\theta} \otimes g_\mu e_{\theta\theta} ab = \rho(a)b.$$

By part (1) of Lemma 5.16, we have that $\|\rho(a)\| \leq 9C_1^2C_1^2\|a\|$. Thus ρ is a morphism of right Banach $C_0(\Omega, K(E))$-modules.

It remains to show that $\pi \circ \rho = 1_{C_0(\Omega, K(E))}$. Recall that for each $x \in \Omega$ we have that $\sum_{\mu \in \Lambda} h_\mu(x) = 1$ since h is partition of unity. Thus, since π is continuous,

$$\begin{aligned}
\pi(\rho(a)) &= \pi(\lim_{(\lambda,\sigma)} y_{\lambda,\sigma,a}) \\
&= \lim_{(\lambda,\sigma)} \pi(y_{\lambda,\sigma,a}) \\
&= \lim_{(\lambda,\sigma)} \pi(\sum_{\mu\in\lambda} \sum_{\theta\in\sigma} g_\mu e_{\theta\theta} \otimes g_\mu e_{\theta\theta} a) \\
&= \lim_{(\lambda,\sigma)} \sum_{\mu\in\lambda} \sum_{\theta\in\sigma} h_\mu e_{\theta\theta} a.
\end{aligned}$$

We wish to show that $\lim_{(\lambda,\sigma)} \sum_{\mu\in\lambda} \sum_{\theta\in\sigma} h_\mu e_{\theta\theta} a = a$. Consider it acting on an element $x \in \Omega$

$$\begin{aligned}
&\lim_{(\lambda,\sigma)} \sum_{\mu\in\lambda} \sum_{\theta\in\sigma} h_\mu(x) e_{\theta\theta} a(x) \\
=&\lim_{\sigma} \lim_{\lambda} \sum_{\mu\in\lambda} \sum_{\theta\in\sigma} h_\mu(x) e_{\theta\theta} a(x) \\
=&\lim_{\sigma} \sum_{\theta\in\sigma} e_{\theta\theta} a(x) \\
=&a(x).
\end{aligned}$$

□

As to the left projectivity of $C_0(\Omega, K(E))$ we have the following theorem.

Theorem 5.18. *Let Ω be a locally compact Hausdorff space which is also paracompact and let X be a Banach space with an extended unconditional basis which is shrinking. Then $C_0(\Omega, K(X))$ is left projective.*

Proof. Let $\{e_\theta\}_{\theta\in\Phi}$ be an extended unconditional basis for X. Let $\{f_\theta\}_{\theta\in\Phi}$ be the associated linear functionals to $\{e_\theta\}_{\theta\in\Phi}$. Set $e_{\theta\theta} = f_\theta e_\theta \in K(X)$.

In Lemma 5.10 we showed that $\{C_\lambda\}_{\lambda\in N(\Lambda)} = \{\sum_{\theta\in\lambda} e_{\theta\theta}\}_{\lambda\in N(\Lambda)}$ is a right bounded approximate identity in $K(E)$. The rest of the proof is similar to the proof of Theorem 5.15.

Let $\mathcal{B}$ be an open cover of Ω such that each point of Ω has a neighbourhood that intersects with no more than three sets of $\mathcal{B}$.

By [18, Problem 5.W], since $\{U_\mu\}_{\mu\in\Lambda}$ is a locally finite open cover of the normal space Ω, it is possible to select a non-negative continuous function h_μ for each U_μ in $\mathcal{U}$ such that h_μ is 0 outside U_μ and is everywhere less than or equal to one, and

$$\sum_{\mu\in\Lambda} h_\mu(s) = 1 \text{ for all } \ s \in \Omega.$$

Set $g_\mu = \sqrt{h_\mu}$.

For $a \in C_0(\Omega, K(E))$, $(\lambda, \sigma) \in N(\Lambda) \times N(\Phi)$ we define

$$y_{\lambda,\sigma,a} = \sum_{\mu\in\lambda} \sum_{\theta\in\sigma} a g_\mu e_{\theta\theta} \otimes g_\mu e_{\theta\theta}.$$

We show that $y_{\lambda,\sigma,a}$ converges in the following lemma

Lemma 5.19. *(i) Let $a \in C_0(\Omega, K(X))$, $\sigma \in N(\Phi)$ and $\lambda_1 > \lambda_0 \in N(\Lambda)$. Let $\lambda_0 = (\mu_1, ..., \mu_{m_0})$ and $\lambda_1 = (\mu_1, ..., \mu_{m_0}, ..., \mu_{m_1})$. We then have that*

$$\|y_{\lambda_1\setminus\lambda_0,\sigma,a}\|_{A\hat{\otimes}A} \leq 9C_1^2C_2^2 \max_{\mu\in\lambda_1\setminus\lambda_0} \|g_\mu a\|_{C_0(\Omega,K(X))}.$$

(ii) Let $a \in C_0(\Omega, A)$. Then for any $\varepsilon > 0$, there exists a $\lambda_0 \in N(\Lambda)$ such that for all $\lambda > \lambda_0$ we have that

$$\sup_{\mu\in\lambda\setminus\lambda_0} \|g_\mu a\|_{C_0(\Omega,K(X))} < \frac{\varepsilon}{36C_1^2C_2^2}.$$

(iii) Let $a \in C_0(\Omega, K(X))$, $\lambda \in N(\Lambda)$ and $\sigma_1 > \sigma_0 \in N(\Phi)$. Let $\sigma_0 = (\theta_1, ..., \theta_{m_0})$ and $\sigma_1 = (\theta_1, ..., \theta_{m_0}, ..., \theta_{m_1})$. We then have that

$$\|y_{\lambda,\sigma_1\setminus\sigma_0,a}\|_{A\hat{\otimes}A} \leq 9C_1^2C_2 \sup_{x\in\Omega} \left\| \sum_{t=n_0+1}^{n_1} a(x) e_{\theta_t\theta_t} \right\|.$$

(iv) For any $\varepsilon > 0$ there exists a $\sigma_0 \in N(\Phi)$ such that for all $\sigma > \sigma_0$ we have that

$$\sup_{x\in\Omega} \left\| \sum_{t\in\sigma\setminus\sigma_0} a(x) e_{\theta_t\theta_t} \right\| < \frac{\varepsilon}{36C_1^2C_2}.$$

Proof. (i), (ii) and (iii) follow as in Theorem 5.15.

(iv) Since a vanishes at infinity we can pick a compact set $K \subset \Omega$ such that for every $x \in \Omega \setminus K$ we have that

$$\|a(x)\|_{K(X)} < \frac{\varepsilon}{36C_1^2C_2^2}.$$

Then for every $x \in \Omega \setminus K$ we have that

$$\left\| \sum_{t \in \sigma \setminus \sigma_0} a(x) e_{\theta_t \theta_t} \right\|_{K(X)} \leq C_2 \|a(x)\| < \frac{\varepsilon}{36 C_1^2 C_2}.$$

Since a is continuous we have that for every $\varepsilon > 0$ and $x \in \Omega$ there exists an open set $U_x \subset \Omega$ such that $\|a(x) - a(y)\| < \frac{\varepsilon}{72 C_1^2 C_2^2}$ for every $y \in U_x$. Then $\{U_x\}_{x \in K}$ is an open cover of K.

Let $\{U_{x_i}\}_{i=1}^m$ be a finite subcover of $\{U_x\}_{x \in K}$. By Lemma 5.10 $\{\sum e_{\sigma\sigma}\}$ is a right bounded approximate identity in $K(X)$ and so for every i we can pick an σ_i such that for every $\sigma > \sigma_i$ we have that

$$\left\| \sum_{t \in \sigma \setminus \sigma_i} a(x_i) e_{\theta_t \theta_t} \right\| \leq \frac{\varepsilon}{72 C_1^2 C_2}.$$

Set $\sigma_0 = \bigcup_{i=1}^m \sigma_i$.

Let $x \in K$. Since $\{U_{x_j}\}_{j=1}^m$ is a cover of K there exists an i such that $x \in U_{x_i}$. Then for $\sigma > \sigma_0$ we have that

$$\begin{aligned}
&\left\| \sum_{t \in \sigma \setminus \sigma_0} a(x) e_{\theta_t \theta_t} \right\|_{K(E)} \\
=&\left\| \sum_{t \in \sigma \setminus \sigma_0} (a(x) - a(x_i) + a(x_i)) e_{\theta_t \theta_t} \right\|_{K(E)} \\
\leq&\left\| \sum_{t \in \sigma \setminus \sigma_0} (a(x) - a(x_i)) e_{\theta_t \theta_t} \right\|_{K(E)} + \left\| \sum_{t \in \sigma \setminus \sigma_0} a(x_i) e_{\theta_t \theta_t} \right\|_{K(E)} \\
\leq&\left\| \sum_{t \in \sigma \setminus \sigma_0} e_{\theta_t \theta_t} \right\|_{K(E)} \|a(x) - a(x_i)\|_{K(E)} + \frac{\varepsilon}{72 C_1^2 C_2} \\
<&C_2 \frac{\varepsilon}{72 C_1^2 C_2^2} + \frac{\varepsilon}{72 C_1^2 C_2} \\
=&\frac{\varepsilon}{36 C_1^2 C2}.
\end{aligned}$$

Thus

$$\sup_{x \in \Omega} \left\| \sum_{t \in \sigma \setminus \sigma_0} a(x) e_{\theta_t \theta_t} \right\|_{K(E)} < \frac{\varepsilon}{36 C_1^2 C_2}.$$

□

Set $\rho(a) = \lim_{\lambda,\sigma} y_{\lambda,\sigma,a}$. As in Thereom 5.15 ρ converges. The fact that ρ is a morphism of modules and that $\pi \circ \rho = 1_{C_0(\Omega,K(E))}$ follows in an almost identical manner as in Theorem 5.15. Therefore $C_0(\Omega, K(E))$ is left projective. □

5.3 Projectivity of $\mathcal{A}$ defined by locally trivial continuous fields of compact operators

We now generalise these results to continuous fields of these algebras with E separable.

Theorem 5.20 ([33], 20.2). *Let E be a Banach space and $\{e_n\}$ a complete sequence in E such that $x_n \neq 0 (n = 1, 2, ...)$. The following statements are equivalent:*

1. *$\{e_n\}$ is a hyperorthogonal basis of E.*
2. *The relations $\{\alpha_n\}$, $\{\gamma_n\} \subset \mathbb{C}$, $|\gamma_n| \leq |\alpha_n|$ $(n = 1, 2, ...)$, $\sum_{i=1}^{\infty} \alpha_i e_i \in E$ imply $\sum_{i=1}^{\infty} \gamma_i e_i \in E$ and*
$$\left\| \sum_{i=1}^{\infty} \gamma_i e_i \right\| \leq \left\| \sum_{i=1}^{\infty} \alpha_i e_i \right\|.$$

Let $(E_x)_{x \in \Omega}$ be a family of separable Banach spaces such that each E_x has a hyperorthogonal basis $\{e^x_n\}_{n \in \mathbb{N}}$. For each $x \in \Omega$ let $\{f^x_n\}$ be the associated linear functionals to $\{e^x_n\}$. Define

$$\begin{aligned} e^x_{nn} : E_x &\to E_x \\ t &\mapsto f^x_n(t) e^x_n \end{aligned}$$

Lemma 5.21. *Let E be a Banach space with a hyperorthogonal basis $(e_n)_{n \in \mathbb{N}}$. For any $n < m \in \mathbb{N}$ and any $\{\eta_i\}_{i \in \lambda} \in \mathbb{T}$. The following holds*

1. $$\left\| \sum_{i=n+1}^{m} \eta_i e_i \right\|_E \leq \left\| \sum_{i=n+1}^{m} e_i \right\|_E ; \tag{5.3}$$
2. $$\left\| \sum_{i=n+1}^{m} e_{ii} \right\|_{K(E)} \leq 1. \tag{5.4}$$

Proof. 1. Follows directly by Theorem 5.20.

2.

$$\begin{aligned}\left\|\sum_{i=n+1}^{m} e_{ii}\right\|_{K(E)} &= \sup_{\|y\|\le 1}\left\|\sum_{i=n+1}^{m} e_{ii}(y)\right\|_E \\ &= \sup_{\|y\|\le 1}\left\|\sum_{i=n+1}^{m} e_{ii}\left(\sum_{j=1}^{\infty} f_j(y)e_j\right)\right\|_E \\ &= \sup_{\|y\|\le 1}\left\|\sum_{i=n+1}^{m} f_i(y)e_i\right\|_E \\ &\le \sup_{\|y\|\le 1}\left\|\sum_{i=1}^{\infty} f_i(y)e_i\right\|_E \quad (\textit{by Theorem } 5.20) \\ &= \sup_{\|y\|\le 1}\|y\|_E = 1.\end{aligned}$$

□

Theorem 5.22. *Let $\mathcal{U} = \{\Omega, (K(E_x)), \Theta\}$ be an $\ell-$locally trivial continuous field of Banach algebras where, for $x \in \Omega$, E_x is a separable Banach space with a hyperorthogonal basis $(e_n^x)_{n\in\mathbb{N}} \subset E_x$. Let $\mathcal{A}$ be the Banach algebra generated by $\mathcal{U}$. Suppose that Ω is paracompact then $\mathcal{A}$ is projective in mod-$\mathcal{A}$.*

Proof. By assumption there is an open cover $\{W_\alpha\}$, $\alpha \in \mathcal{M}$, of Ω such that each $\mathcal{U}|_{W_\alpha}$ is trivial and, in addition, there is an open cover $\{B_j\}$ of cardinality ℓ of Ω such that $\overline{B_j} \subset W_{\alpha(j)}$ for each $j = 1, \dots, \ell$, and some $\alpha(j) \in \mathcal{M}$. By [12, Lemma 2.1], for any paracompact locally compact space Ω there exists an open cover $\mathcal{U} = \{U_\nu\}$ of relatively compact sets such that each point in Ω has a neighbourhood which intersects no more than three sets in $\mathcal{U}$. Consider an open cover $\{B_j \cap U_\nu : U_\nu \in \mathcal{U}, j = 1, \dots, \ell\}$ of Ω. Denote this cover by $\{V_\mu\}$. Note that $\{V_\mu\}$ is an open locally finite cover of Ω of order 3ℓ.

By [18, Problem 5.W], since $\{V_\mu\}_{\mu\in\Lambda}$ is a locally finite open cover of the normal space Ω, it is possible to select a non-negative continuous function h_μ for each V_μ such that h_μ is 0 outside V_μ and is everywhere less than or equal to one, and

$$\sum_{\mu\in\Lambda} h_\mu(s) = 1 \text{ for all } s \in \Omega.$$

Note that in the equality $\sum_{\mu\in\Lambda} h_\mu(s) = 1$, for any $s \in \Omega$, there are no more than 3ℓ nonzero terms. Set $g_\mu = \sqrt{h_\mu}$.

Let $\phi^\mu = (\phi^\mu_x)_{x\in V_\mu}$ be an isomorphism of $\mathcal{U}|_{V_\mu}$ onto the trivial continuous field of Banach algebras $\{V_\mu, (K(\widetilde{E_\mu})), C_0(V_\mu, K(\widetilde{E_\mu}))\}$ where ϕ^μ_x is an isomorphism of Banach algebras $\phi^\mu_x : K(E_x) \to K(\widetilde{E_\mu})$.

For each $x \in \Omega$ let $\{f^x_n\}$ be the associated linear functionals to $\{e^x_n\}$.

Lemma 5.23. *For any $a \in \mathcal{A}$ and for any $\lambda = \{\mu_1, \dots, \mu_N\}$, $n, N \in \mathbb{N}, l, k \in \mathbb{N}$,*

$$\left\| \sum_{t=1}^{n} \sum_{i=1}^{N} \xi^{lt} \eta^{ki} g_{\mu_i} (\phi^{\mu_i}_\bullet)^{-1} (e^{\mu_i}_{tt}) \right\|_{\mathcal{A}} \le 3\ell \tag{5.5}$$

and

$$\left\| \sum_{t=1}^{n} \sum_{i=1}^{N} \xi^{-lt} \eta^{-ki} g_{\mu_i} (\phi^{\mu_i}_\bullet)^{-1} (e^{\mu_i}_{tt}) a \right\|_{\mathcal{A}} \le 3\ell \max_{\mu\in\lambda} \|g_\mu a\|_{\mathcal{A}} \tag{5.6}$$

where ξ is a primary n-th root of unity and η is a primary N-th root of unity in $\mathbb{C}$, and

$$\left\| \sum_{t=n+1}^{m} \sum_{i=1}^{N} \gamma^{-lt} \eta^{-ki} g_{\mu_i} (\phi^{\mu_i}_\bullet)^{-1} (e^{\mu_i}_{tt}) a \right\|_{\mathcal{A}}$$

$$\le 3\ell \max_{1\le p\le N} \sup_{x\in V_p} \left\| \sum_{t=n+1}^{m} (\phi^{\mu_p}_x)^{-1} (e^{\mu_p}_{tt}) a(x) \right\|_{K(E_x)} \tag{5.7}$$

where γ is a primary $(m-n)$th root of unity.

Proof. Since for every $x \in \Omega$ there are at most 3ℓ values of μ such that $g_\mu(x) \ne 0$ we have the following,

$$\begin{aligned}
&\left\| \sum_{t=1}^{n} \sum_{i=1}^{N} \xi^{lt} \eta^{ki} g_{\mu_i} (\phi^{\mu_i}_\bullet)^{-1} (e^{\mu_i}_{tt}) \right\|_{\mathcal{A}} \\
=&\sup_{x\in\Omega} \left\| \sum_{t=1}^{n} \sum_{i=1}^{N} \xi^{lt} \eta^{ki} g_{\mu_i}(x) (\phi^{\mu_i}_x)^{-1} (e^{\mu_i}_{tt}) \right\|_{K(E_x)} \\
\le&3\ell \max_{\mu\in\lambda} \sup_{x\in\Omega} \left\| g_\mu(x) \sum_{t=1}^{n} \xi^{lt} (\phi^{\mu}_x)^{-1} (e^{\mu}_{tt}) \right\|_{K(E_x)} \\
\le&3\ell \max_{\mu\in\lambda} \sup_{x\in\Omega} |g_\mu(x)| \left\| \sum_{t=1}^{n} \xi^{lt} (\phi^{\mu}_x)^{-1} (e^{\mu}_{tt}) \right\|_{K(E_x)} \\
\le&3\ell \max_{\mu\in\lambda} \left\| \sum_{t=1}^{n} \xi^{lt} e^{\mu}_{tt} \right\|_{K(\widetilde{E_\mu})}
\end{aligned}$$

$$
\begin{aligned}
&\leq 3\ell \max_{\mu\in\lambda} \left\| \sum_{t=1}^{n} e_{tt}^{\mu} \right\|_{K(\widetilde{E_\mu})} \quad (\textit{by inequality } (5.3)) \\
&\leq 3\ell \quad (\textit{by inequality } (5.4)).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\left\| \sum_{t=1}^{n} \sum_{i=1}^{N} \xi^{-lt} \eta^{-ki} g_{\mu_i} (\phi_{\bullet}^{\mu_i})^{-1} (e_{tt}^{\mu_i}) a \right\|_{\mathcal{A}} \\
&= \sup_{x\in\Omega} \left\| \sum_{t=1}^{n} \sum_{i=1}^{N} \xi^{-lt} \eta^{-ki} g_{\mu_i}(x) (\phi_x^{\mu_i})^{-1} (e_{tt}^{\mu_i}) a(x) \right\|_{K(E_x)} \\
&\leq 3\ell \max_{\mu\in\lambda} \sup_{x\in\Omega} \left\| \sum_{t=1}^{n} \xi^{-lt} g_{\mu}(x) (\phi_x^{\mu})^{-1} (e_{tt}^{\mu}) a(x) \right\|_{K(E_x)} \\
&\leq 3\ell \max_{\mu\in\lambda} \left\| \sum_{t=1}^{n} \xi^{-lt} e_{tt}^{\mu} \right\|_{K(\widetilde{E_\mu})} \sup_{x\in\Omega} \left\| g_{\mu}(x) a(x) \right\|_{K(E_x)} \\
&\leq 3\ell \max_{\mu\in\lambda} \left\| \sum_{t=1}^{n} e_{tt}^{\mu} \right\|_{K(\widetilde{E_\mu})} \sup_{x\in\Omega} \left\| g_{\mu}(x) a(x) \right\|_{K(E_x)} \quad (\textit{by inequality } (5.3)) \\
&\leq 3\ell \max_{\mu\in\lambda} \sup_{x\in\Omega} \left\| g_{\mu}(x) a(x) \right\|_{K(E_x)} \quad (\textit{by inequality } (5.4)) \\
&= 3\ell \max_{\mu\in\lambda} \left\| g_{\mu} a \right\|_{\mathcal{A}}.
\end{aligned}
$$

Thus the inequalities (5.6) and (5.5) hold.

$$
\begin{aligned}
&\left\| \sum_{t=n+1}^{m} \sum_{i=1}^{N} \gamma^{-lt} \eta^{-ki} g_{\mu_i} (\phi_{\bullet}^{\mu_i})^{-1} (e_{tt}^{\mu_i}) a \right\|_{\mathcal{A}} \\
&= \sup_{x\in\Omega} \left\| \sum_{t=n+1}^{m} \sum_{i=1}^{N} \gamma^{-lt} \eta^{-ki} g_{\mu_i}(x) (\phi_x^{\mu_i})^{-1} (e_{tt}^{\mu_i}) a(x) \right\|_{K(E_x)} \\
&\leq 3\ell \max_{1\leq p\leq N} \sup_{x\in V_{\mu_p}} \left\| \sum_{t=n+1}^{m} \gamma^{-lt} g_{\mu_p}(x) (\phi_x^{\mu_p})^{-1} (e_{tt}^{\mu_p}) a(x) \right\|_{K(E_x)} \\
&\leq 3\ell \max_{1\leq p\leq N} \sup_{x\in V_{\mu_p}} \left\| \sum_{t=n+1}^{m} \gamma^{-lt} (\phi_x^{\mu_p})^{-1} (e_{tt}^{\mu_p}) a(x) \right\|_{K(E_x)} \\
&= 3\ell \max_{1\leq p\leq N} \sup_{x\in V_{\mu_p}} \left\| (\phi_x^{\mu_p})^{-1} \left(\sum_{t=n+1}^{m} \gamma^{-lt} e_{tt}^{\mu_p} \phi_x^{\mu_p}(a(x)) \right) \right\|_{K(E_x)}
\end{aligned}
$$

$$
\begin{aligned}
&=3\ell \max_{1\le p\le N} \sup_{x\in V_{\mu_p}} \left\| \sum_{t=n+1}^{m} \gamma^{-lt} e_{tt}^{\mu_p} \phi_x^{\mu_p}(a(x)) \right\|_{K(\tilde{E}_{\mu_p})} \\
&=3\ell \max_{1\le p\le N} \sup_{x\in V_{\mu_p}} \sup_{u\in \tilde{E}_{\mu_p}, \|u\|\le 1} \left\| \sum_{t=n+1}^{m} \gamma^{-lt} \left\langle f_t^{\mu_p}, \phi_x^{\mu_p}(a(x))u \right\rangle e_t^{\mu_p} \right\|_{\tilde{E}_{\mu_p}} \\
&\le 3\ell \max_{1\le p\le N} \sup_{x\in V_{\mu_p}} \sup_{u\in \tilde{E}_{\mu_p}, \|u\|\le 1} \left\| \sum_{t=n+1}^{m} \left\langle f_t^{\mu_p}, \phi_x^{\mu_p}(a(x))u \right\rangle e_t^{\mu_p} \right\|_{\tilde{E}_{\mu_p}} \\
&=3\ell \max_{1\le p\le N} \sup_{x\in V_{\mu_p}} \left\| \sum_{t=n+1}^{m} e_{tt}^{\mu_p} \phi_x^{\mu_p}(a(x)) \right\|_{K(\tilde{E}_{\mu_p})} \\
&=3\ell \max_{1\le p\le N} \sup_{x\in V_{\mu_p}} \left\| (\phi_x^{\mu_p})^{-1} \left(\sum_{t=n+1}^{m} e_{tt}^{\mu_p} \phi_x^{\mu_p}(a(x)) \right) \right\|_{K(E_x)} \\
&=3\ell \max_{1\le p\le N} \sup_{x\in V_{\mu_p}} \left\| \sum_{t=n+1}^{m} (\phi_x^{\mu_p})^{-1}(e_{tt}^{\mu_p}) a(x) \right\|_{K(E_x)} .
\end{aligned}
$$

Hence inequality (5.7) holds. □

For $a \in \mathcal{A}, n \in \mathbb{N}, \lambda \in N(\Lambda)$ we define the following element $y_{a,\lambda,n}$ in $\mathcal{A}\hat{\otimes}\mathcal{A}$

$$
y_{\lambda,n,a} = \sum_{\mu\in\lambda} \sum_{i=1}^{n} g_\mu (\phi_\bullet^\mu)^{-1}(e_{ii}^\mu) \otimes g_\mu (\phi_\bullet^\mu)^{-1}(e_{ii}^\mu) a.
$$

Define $N(\Lambda) \times \mathbb{N}$ as a directed set with $(\lambda', n) \preccurlyeq (\lambda'', m)$ if and only if $\lambda' \subset \lambda''$ and $n \le m$.

Lemma 5.24. *For any $a \in \mathcal{A}$, the net $(y_{a,\lambda,n})_{\lambda,n}$ converges in $\mathcal{A}\widehat{\otimes}\mathcal{A}$.*

Proof. Note that any compact $K \subset \Omega$ intersects only a finite number of sets in the locally finite covering $\{V_\mu\}$ and, for any $a \in \mathcal{A}$, $\|a(t)\| \to 0$ as $t \to \infty$. Let $\varepsilon > 0$. There is a finite set $\lambda \in N(\Lambda)$ such that for $\mu \notin \lambda$ we have

$$
\|g_\mu a\|_{\mathcal{A}} < \frac{\varepsilon}{27\ell^2}.
$$

For $\lambda \subset \lambda' \subset \lambda''$ and $m \ge n$, we have

$$
\begin{aligned}
\|y_{a,\lambda'',m} - y_{a,\lambda',n}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} &= \|y_{a,\lambda''\setminus\lambda,m} + y_{a,\lambda,m} - y_{a,\lambda'\setminus\lambda,n} - y_{a,\lambda,n}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} \\
&\le \|y_{a,\lambda''\setminus\lambda,m}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} + \|y_{a,\lambda'\setminus\lambda,n}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} + \|y_{a,\lambda,m} - y_{a,\lambda,n}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}}.
\end{aligned}
$$

By Lemma 1.5, for $\tilde{\lambda} = \{\mu_1, \dots, \mu_m\}$, $\tilde{n} \in \mathbb{N}$,

$$
\|y_{a,\tilde{\lambda},\tilde{n}}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}_+} \le
$$

$$\frac{1}{m\tilde{n}}\sum_{l=1}^{\tilde{n}}\sum_{k=1}^{m}\left\|\sum_{t=1}^{\tilde{n}}\sum_{i=1}^{m}\xi^{lt}\eta^{ki}g_{\mu_i}(\phi_{\bullet}^{\mu_i})^{-1}\left(e_{tt}^{\mu_i}\right)\right\|_{\mathcal{A}} \times \left\|\sum_{t=1}^{\tilde{n}}\sum_{i=1}^{m}\xi^{-lt}\eta^{-ki}g_{\mu_i}(\phi_{\bullet}^{\mu_i})^{-1}\left(e_{tt}^{\mu_i}\right)a\right\|_{\mathcal{A}}$$

where ξ is a primary m-th root of unity and η is a primary $\tilde{n}$-th root of unity in $\mathbb{C}$.

By inequalities (5.6) and (5.5) from Lemma 5.23,

$$\|y_{a,\lambda''\setminus\lambda,m}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} \le 9\ell^2 \max_{\mu\in\lambda''\setminus\lambda}\|g_\mu a\|_{\mathcal{A}} \le \frac{\varepsilon}{3}$$

and

$$\|y_{a,\lambda'\setminus\lambda,n}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} \le 9\ell^2 \max_{\mu\in\lambda'\setminus\lambda}\|g_\mu a\|_{\mathcal{A}} \le \frac{\varepsilon}{3}.$$

By inequality (5.7) from Lemma 5.23, for $\lambda = \{\mu_1,\ldots,\mu_N\}$,

$$\begin{aligned}\|y_{a,\lambda,m}-y_{a,\lambda,n}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} &= \left\|\sum_{i=1}^{N}\sum_{j=n+1}^{m} g_{\mu_i}(\phi_{\bullet}^{\mu_i})^{-1}\left(e_{jj}^{\mu_i}\right)\otimes g_{\mu_i}(\phi_{\bullet}^{\mu_i})^{-1}\left(e_{jj}^{\mu_i}\right)a\right\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}_+}\\ &\le 9\ell \max_{1\le p\le N}\sup_{x\in V_{\mu_p}}\left\|\sum_{t=n+1}^{m}(\phi_x^{\mu_p})^{-1}(e_{tt}^{\mu_p})a(x)\right\|_{K(E_x)}\end{aligned}$$

By Part 4 of Lemma 5.16, for every μ_p, $1\le p\le N$,

$$\sup_{x\in V_{\mu_p}}\left\|\sum_{t=n}^{m}(\phi_x^{\mu_p})^{-1}(e_{tt}^{\mu_p})a(x)\right\|_{K(E_x)} = \sup_{x\in V_{\mu_p}}\left\|\sum_{t=n}^{m}e_{tt}^{\mu_p}\phi_x^{\mu_p}(a(x))\right\|_{K(\tilde{E}_{\mu_p})} \to 0 \text{ as } i\to\infty.$$

Hence

$$\|y_{a,\lambda,m}-y_{a,\lambda,n}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} \to 0 \text{ as } n,m\to\infty.$$

Thus, in view of the completeness of $\mathcal{A}\widehat{\otimes}\mathcal{A}$, for any $a\in\mathcal{A}$, the net $y_{a,\lambda,n}$ converges in $\mathcal{A}\widehat{\otimes}\mathcal{A}$. □

Let us complete the proof of Theorem 5.22

Set

$$\begin{aligned}\rho : \mathcal{A} &\to \mathcal{A}\widehat{\otimes}\mathcal{A}_+\\ a &\mapsto \lim_{(\lambda,n)} y_{a,\lambda,n}.\end{aligned}$$

We claim that ρ is a morphism of right Banach $\mathcal{A}$-modules and that $\pi \circ \rho = 1_{\mathcal{A}}$.

Let $a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned}\rho(\alpha a + \beta b) &= \lim_{(\lambda,n)} y_{\alpha a+\beta b,\lambda,n} \\ &= \lim_{(\lambda,n)} \sum_{\mu\in\lambda}\sum_{i=1}^{n} g_\mu(\phi_\bullet^\mu)^{-1}(e_{ii}^\mu) \otimes g_\mu(\phi_\bullet^\mu)^{-1}(e_{ii}^\mu)(\alpha a + \beta b) \\ &= \alpha \lim_{(\lambda,n)} \sum_{\mu\in\lambda}\sum_{i=1}^{n} g_\mu(\phi_\bullet^\mu)^{-1}(e_{ii}^\mu) \otimes g_\mu(\phi_\bullet^\mu)^{-1}(e_{ii}^\mu)a \\ &+ \beta \lim_{(\lambda,n)} \sum_{\mu\in\lambda}\sum_{i=1}^{n} g_\mu(\phi_\bullet^\mu)^{-1}(e_{ii}^\mu) \otimes g_\mu(\phi_\bullet^\mu)^{-1}(e_{ii}^\mu)b \\ &= \alpha\rho(a) + \beta\rho(b)\end{aligned}$$

and

$$\begin{aligned}\rho(ab) &= \lim_{(\lambda,n)} y_{ab,\lambda,n} \\ &= \lim_{(\lambda,n)} \sum_{\mu\in\lambda}\sum_{i=1}^{n} g_\mu(\phi_\bullet^\mu)^{-1}(e_{ii}^\mu) \otimes g_\mu(\phi_\bullet^\mu)^{-1}(e_{ii}^\mu)ab \\ &= \rho(a)b\end{aligned}$$

Let $a \in \mathcal{A}$. Then

$$\|\rho(a)\| \leq 9\ell^2\|a\|$$

by inequalities 5.6 and 5.5.

Thus ρ is a morphism of right Banach $\mathcal{A}$-modules.

Let $a \in \mathcal{A}$. Then

$$\begin{aligned}(\pi\circ\rho)a &= \pi(\lim_{(\lambda,n)} y_{a,\lambda,n}) \\ &= \pi\left(\lim_{(\lambda,n)} \sum_{\mu\in\lambda}\sum_{i=1}^{n} g_\mu(\phi_\bullet^\mu)^{-1}(e_{ii}^\mu) \otimes g_\mu(\phi_\bullet^\mu)^{-1}(e_{ii}^\mu)a\right) \\ &= \lim_{(\lambda,n)} \sum_{\mu\in\lambda}\sum_{i=1}^{n} \pi(g_\mu(\phi_\bullet^\mu)^{-1}(e_{ii}^\mu) \otimes g_\mu(\phi_\bullet^\mu)^{-1}(e_{ii}^\mu)a)\end{aligned}$$

$$= \lim_{(\lambda,n)} \sum_{\mu\in\lambda} \sum_{i=1}^{n} h_\mu (\phi_\bullet^\mu)^{-1}(e_{ii}^\mu)a$$
$$= a.$$

□

Theorem 5.25. *Let Ω be a Hausdorff locally compact space. Let $\mathcal{U} = \{\Omega, (K(E_x)), \Theta\}$ be an ℓ-locally trivial continuous field of Banach algebras, for some $\ell \in \mathbb{N}$. Suppose, for each $x \in \Omega$, E_x is a separable Banach space with a hyperorthogonal basis $(e_n^x)_{n\in\mathbb{N}} \subset E_x$. Let $\mathcal{A}$ be the Banach algebra generated by $\mathcal{U}$.*

Then the following conditions are equivalent:
(i) *Ω is paracompact;*
(ii) *$\mathcal{A}$ is right projective and $\mathcal{U}$ is a disjoint union of σ-locally trivial continuous fields of Banach algebras.*

Proof. By Theorem 5.22, the fact that Ω is paracompact implies right projectivity of $\mathcal{A}$. By Remark 1.18, since Ω is paracompact, $\mathcal{U}$ is a disjoint union of σ-locally trivial continuous fields of Banach algebras.

By Proposition 3.4 and Lemma 4.15, conditions (ii) implies paracompactness of Ω. Thus (ii) $\Longleftrightarrow$ (i). □

Theorem 5.26. *Let $\mathcal{U} = \{\Omega, (K(E_x)), \Theta\}$ be an $\ell-$locally trivial continuous field of Banach algebras where, for $x \in \Omega$, E_x is a separable Banach space with a shrinking hyperorthogonal basis $(e_n^x)_{n\in\mathbb{N}} \subset E_x$. Let $\mathcal{A}$ be the Banach algebra generated by $\mathcal{U}$. Suppose that Ω is paracompact. Then $\mathcal{A}$ is left projective.*

Proof. By assumption there is an open cover $\{W_\alpha\}$, $\alpha \in \mathcal{M}$, of Ω such that each $\mathcal{U}|_{W_\alpha}$ is trivial and, in addition, there is an open cover $\{B_j\}$ of cardinality ℓ of Ω such that $\overline{B_j} \subset W_{\alpha(j)}$ for each $j = 1, \ldots, \ell$, and some $\alpha(j) \in \mathcal{M}$. By [12, Lemma 2.1], for any paracompact locally compact space Ω there exists an open cover $\mathcal{U} = \{U_\nu\}$ of relatively compact sets such that each point in Ω has a neighbourhood which intersects no more than three sets in $\mathcal{U}$. Consider an open cover $\{B_j \cap U_\nu : U_\nu \in \mathcal{U}, j = 1, \ldots, \ell\}$ of Ω. Denote this cover by $\{V_\mu\}$. Note that $\{V_\mu\}$ is an open locally finite cover of Ω of order 3ℓ.

By [18, Problem 5.W], since $\{V_\mu\}_{\mu\in\Lambda}$ is a locally finite open cover of the normal space Ω, it is possible to select a non-negative continuous function h_μ for each V_μ such that h_μ is 0 outside V_μ and is everywhere less than or equal to one, and

$$\sum_{\mu\in\Lambda} h_\mu(s) = 1 \text{ for all } s \in \Omega.$$

Note that in the equality $\sum_{\mu\in\Lambda} h_\mu(s) = 1$, for any $s \in \Omega$, there are no more than 3ℓ nonzero terms. Set $g_\mu = \sqrt{h_\mu}$.

Let $\phi^\mu = (\phi^\mu_x)_{x\in V_\mu}$ be an isomorphism of $\mathcal{U}|_{V_\mu}$ onto the trivial continuous field of Banach algebras V_μ where ϕ^μ_x is an isomorphism of Banach algebras $\phi^\mu_x : K(E_x) \to K(\widetilde{E_\mu})$.

For each $x \in \Omega$ let $\{f^x_n\}$ be the associated linear functionals to $\{e^x_n\}$.

Lemma 5.27. *For any $a \in \mathcal{A}$ and for any $\lambda = \{\mu_1, \dots, \mu_N\}$, $n, m \in \mathbb{N}, l, k \in \mathbb{N}$,*

$$\left\| \sum_{t=1}^{n} \sum_{i=1}^{N} \xi^{-lt} \eta^{-ki} g_{\mu_i} (\phi^{\mu_i}_\bullet)^{-1} (e^{\mu_i}_{tt}) \right\|_{\mathcal{A}} \leq 3\ell \tag{5.8}$$

and

$$\left\| \sum_{t=1}^{n} \sum_{i=1}^{N} \xi^{lt} \eta^{ki} g_{\mu_i} a (\phi^{\mu_i}_\bullet)^{-1} (e^{\mu_i}_{tt}) \right\|_{\mathcal{A}} \leq 3\ell \max_{\mu\in\lambda} \|g_\mu a\|_{\mathcal{A}} \tag{5.9}$$

where ξ is a primary n-th root of unity and η is a primary N-th root of unity in $\mathbb{C}$, and

$$\left\| \sum_{t=n+1}^{m} \sum_{i=1}^{N} \gamma^{lt} \eta^{ki} g_{\mu_i} a (\phi^{\mu_i}_\bullet)^{-1} (e^{\mu_i}_{tt}) \right\|_{\mathcal{A}}$$

$$\leq 3\ell \max_{1\leq p\leq N} \sup_{x\in V_p} \left\| \sum_{t=n+1}^{m} a(x) (\phi^{\mu_p}_x)^{-1} (e^{\mu_p}_{tt}) \right\|_{K(E_x)} \tag{5.10}$$

where γ is a primary $(m-n)$th root of unity.

Proof. It follows as in the proof of Lemma 5.23. □

For $a \in \mathcal{A}, n \in \mathbb{N}, \lambda \in N(\Lambda)$ we define the following element $y_{\lambda,n,a}$ in $\mathcal{A}\hat{\otimes}\mathcal{A}$

$$y_{a,\lambda,n} = \sum_{\mu\in\lambda} \sum_{i=1}^{n} g_\mu a (\phi^\mu_\bullet)^{-1} (e^\mu_{ii}) \otimes g_\mu (\phi^\mu_\bullet)^{-1} (e^\mu_{ii}).$$

Define $N(\Lambda) \times \mathbb{N}$ as a directed set with $(\lambda', n) \preccurlyeq (\lambda'', m)$ if and only if $\lambda' \subset \lambda''$ and $n \leq m$.

Lemma 5.28. *For any $a \in \mathcal{A}$, the net $(y_{a,\lambda,n})_{\lambda,n}$ converges in $\mathcal{A}\widehat{\otimes}\mathcal{A}$.*

Proof. Note that any compact $K \subset \Omega$ intersects only a finite number of sets in the locally finite covering $\{V_\mu\}$ and, for any $a \in \mathcal{A}$, $\|a(t)\| \to 0$ as $t \to \infty$. Let $\varepsilon > 0$. There is a finite set $\lambda \in N(\Lambda)$ such that for $\mu \notin \lambda$ we have

$$\|g_\mu a\|_{\mathcal{A}} < \frac{\varepsilon}{27\ell^2}.$$

For $\lambda \subset \lambda' \subset \lambda''$ and $m \geq n$, we have

$$\|y_{a,\lambda'',m} - y_{a,\lambda',n}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} = \|y_{a,\lambda''\setminus\lambda,m} + y_{a,\lambda,m} - y_{a,\lambda'\setminus\lambda,n} - y_{a,\lambda,n}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}}$$
$$\leq \|y_{a,\lambda''\setminus\lambda,m}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} + \|y_{a,\lambda'\setminus\lambda,n}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} + \|y_{a,\lambda,m} - y_{a,\lambda,n}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}}.$$

By Lemma 1.5, for $\tilde{\lambda} = \{\mu_1, \ldots, \mu_m\}$, $\tilde{n} \in \mathbb{N}$,

$$\|y_{a,\tilde{\lambda},\tilde{n}}\|_{\mathcal{A}\hat{\otimes}\mathcal{A}_+} \leq$$
$$\frac{1}{m\tilde{n}} \sum_{l=1}^{\tilde{n}} \sum_{k=1}^{m} \left\| \sum_{t=1}^{\tilde{n}} \sum_{i=1}^{m} \xi^{lt} \eta^{ki} g_{\mu_i} a (\phi_\bullet^{\mu_i})^{-1} \left(e_{tt}^{\mu_i}\right) \right\|_{\mathcal{A}}$$
$$\times \left\| \sum_{t=1}^{\tilde{n}} \sum_{i=1}^{m} \xi^{-lt} \eta^{-ki} g_{\mu_i} (\phi_\bullet^{\mu_i})^{-1} \left(e_{tt}^{\mu_i}\right) \right\|_{\mathcal{A}}.$$

where ξ is a primary m-th root of unity and η is a primary $\tilde{n}$-th root of unity in $\mathbb{C}$.

By inequalities (5.9) and (5.8) from Lemma 5.27,

$$\|y_{a,\lambda''\setminus\lambda,m}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} \leq 9\ell^2 \max_{\mu\in\lambda''\setminus\lambda} \|g_\mu a\|_{\mathcal{A}} \leq \frac{\varepsilon}{3}$$

and

$$\|y_{a,\lambda'\setminus\lambda,n}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} \leq 9\ell^2 \max_{\mu\in\lambda'\setminus\lambda} \|g_\mu a\|_{\mathcal{A}} \leq \frac{\varepsilon}{3}.$$

By inequality (5.10) from Lemma 5.27, for $\lambda = \{\mu_1, \ldots, \mu_N\}$,

$$\|y_{a,\lambda,m} - y_{a,\lambda,n}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} = \left\| \sum_{i=1}^{N} \sum_{j=n+1}^{m} g_{\mu_i} a (\phi_\bullet^{\mu_i})^{-1} \left(e_{jj}^{\mu_i}\right) \otimes g_{\mu_i} (\phi_\bullet^{\mu_i})^{-1} \left(e_{jj}^{\mu_i}\right) \right\|_{\mathcal{A}\hat{\otimes}\mathcal{A}_+}$$
$$\leq 9\ell \max_{1\leq p\leq N} \sup_{x\in V_{\mu_p}} \left\| \sum_{t=n+1}^{m} a(x) (\phi_x^{\mu_p})^{-1} (e_{tt}^{\mu_p}) \right\|_{K(E_x)}$$

By Part 4 of Lemma 5.19, for every μ_p, $1 \leq p \leq N$,

$$\sup_{x\in V_{\mu_p}} \left\| \sum_{t=n}^{m} a(x) (\phi_x^{\mu_p})^{-1} (e_{tt}^{\mu_p}) \right\|_{K(E_x)} \to 0 \text{ as } i \to \infty,$$

since for every $x \in V_{\mu_p}$, E_x is a separable Banach space with a shrinking hyperorthogonal basis $(e_n^x)_{n\in\mathbb{N}}$. Hence

$$\|y_{a,\lambda,m} - y_{a,\lambda,n}\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} \to 0 \text{ as } n, m \to \infty.$$

Thus, in view of the completeness of $\mathcal{A}\widehat{\otimes}\mathcal{A}$, for any $a \in \mathcal{A}$, the net $y_{a,\lambda,n}$ converges in $\mathcal{A}\widehat{\otimes}\mathcal{A}$. □

Set

$$\rho : \mathcal{A} \to \mathcal{A}\hat{\otimes}\mathcal{A}_+$$
$$a \mapsto \lim_{(\lambda,n)} y_{a,\lambda,n}.$$

We claim that ρ is a morphism of right Banach $\mathcal{A}$-modules and that $\pi \circ \rho = 1_{\mathcal{A}}$. The rest of our proof is similar to the proof of Theorem 5.22. □

Theorem 5.29. *Let Ω be a Hausdorff locally compact space. Let $\mathcal{U} = \{\Omega, (K(E_x)), \Theta\}$ be an ℓ-locally trivial continuous field of Banach algebras, for some $\ell \in \mathbb{N}$. Suppose, for each $x \in \Omega$, E_x is a separable Banach space with a shrinking hyperorthogonal basis $(e_n^x)_{n\in\mathbb{N}} \subset E_x$. Let $\mathcal{A}$ be the Banach algebra generated by $\mathcal{U}$.*
Then the following conditions are equivalent:
(i) Ω *is paracompact;*
(ii) $\mathcal{A}$ *is left projective and* $\mathcal{U}$ *is a disjoint union of σ-locally trivial continuous fields of Banach algebras.*

Proof. By Theorem 5.26, the fact that Ω is paracompact implies left projectivity of $\mathcal{A}$. By Remark 1.18, since Ω is paracompact, $\mathcal{U}$ is a disjoint union of σ-locally trivial continuous fields of Banach algebras.

By Proposition 3.4 and Lemma 4.15, condition (ii) implies paracompactness of Ω. Thus (ii) $\iff$ (i). □

6 On the Projectivity of C*-algebras with Fell's condition

6.1 Definitions and notation

The following definitions can be found in Dixmier's book "Les C^*-algebres et leurs representations" [9, Section 2].

Definition 6.1. *Let $(H, \langle . \rangle_H)$ and $(K, \langle . \rangle_K)$ be Hilbert spaces. Define the following inner product for $x_1, x_2 \in H$ and $y_1, y_2 \in K$*

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle_H \langle y_1, y_2 \rangle_K .$$

Set $H \otimes_{\mathcal{H}} K$ to be the completion of $H \otimes K$ with respect to the norm generated by this inner product. $H \otimes_{\mathcal{H}} K$ is known as the Hilbert tensor product of H and K.

Remark 6.2. Let H and K be Hilbert spaces. We can view $B(H) \otimes B(K)$ as a subset of $B(H \otimes_{\mathcal{H}} K)$ via $(T \otimes S)(h \otimes k) = T(h) \otimes S(k)$ for $T \in B(H), S \in B(K)$.

Definition 6.3 ([9], 2.12.15)**.** *Let A and B be C*-algebras and let $\pi : A \to B(H)$, $\tau : B \to B(K)$ be faithful representations of A and B respectively. We denote by $A \otimes_{C*} B$ the C*-algebra of operators on $H \otimes_{\mathcal{H}} K$ generated by $\pi(a) \otimes \tau(b)$ where $a \in A, b \in B$. The C*-algebra $A \otimes_{C*} B$ is known as the C*-tensor product of A and B and does not depend on the choice of representations π and τ.*

Remark 6.4. *Let A and B be C*-algebras. By [30, 1.22.2] the following map*

$$\begin{aligned} i : A \hat{\otimes} B &\to A \otimes_{C^*} B, \\ \text{such that} \quad a \otimes b &\mapsto a \otimes b, \quad a \in A,\ b \in B, \end{aligned}$$

is well defined and

$$\|i(u)\|_{A \otimes_{C^*} B} \leq \|u\|_{A \hat{\otimes} B}.$$

Theorem 6.5 ([13], Proposition II.2.50)**.** *Let Ω be a locally compact Hausdorff space. Then the map*

$$\begin{aligned} \gamma : C_0(\Omega) \hat{\otimes} C_0(\Omega) &\to C_0(\Omega \times \Omega) \\ \text{such that} \quad a \otimes b &\mapsto a \otimes b, \quad a \in C_0(\Omega),\ b \in C_0(\Omega), \end{aligned}$$

is not a topological isomorphism.

Note that by [13, Remark II.2.10] γ is not surjective.

Definition 6.6. *A C*-algebra A is said to be elementary if there is a Hilbert space H such that the C*-algebras A and $K(H)$ are isomorphic as C*-algebras.*

In this section we investigate continuous fields of C*-algebras which satisfy a property known as Fell's condition. We first review some results from [9, Section 10.7].

Theorem 6.7 ([9],10.6.1). *Let H be a Hilbert space. For $\xi, \eta \in H \setminus \{0\}$ we denote by $\theta_{\xi,\eta}$ the operator*

$$\begin{aligned} \theta_{\xi,\eta} : H &\to H \\ \zeta &\mapsto \langle \zeta, \eta \rangle \xi. \end{aligned}$$

The operator $\theta_{\xi,\eta}$ is of rank ≤ 1 and every operator of rank ≤ 1 is of this type. If $\xi', \eta' \in H$ we have

$$\theta_{\xi,\eta}\theta_{\xi',\eta'} = \langle \xi', \eta \rangle \theta_{\xi,\eta'},$$

and

$$\theta^*_{\xi,\eta} = \theta_{\eta,\xi}.$$

Remark 6.8 ([9], 10.6.2). Let $\xi_1(t), ..., \xi_{2n}(t)$ be variable vectors of H. By [9, 3.5.6], if $\langle \xi_i(t), \xi_j(t) \rangle$ converges to $\langle \eta_i, \eta_j \rangle$ for any i and j then
$\|\theta_{\xi_1(t),\xi_2(t)} + ... + \theta_{\xi_{2n-1}(t),\xi_{2n}(t)}\|_{B(H)}$ converges to $\|\theta_{\eta_1,\eta_2} + ... + \theta_{\eta_{2n-1},\eta_{2n}}\|_{B(H)}$.
In other words $\|\theta_{\xi_1(t),\xi_2(t)} + ... + \theta_{\xi_{2n-1}(t),\xi_{2n}(t)}\|_{B(H)}$ is a continuous function of the scalar products $\langle \xi_i(t), \xi_i(t) \rangle$.

Definition 6.9. *A continuous field $\mathcal{H}$ of Hilbert spaces is a triple $\mathcal{H} = \{\Omega, (H_t)_{t\in\Omega}, \Gamma\}$ where Ω is a locally compact Hausdorff space, $(H_t)_{t\in\Omega}$ is a family of Hilbert spaces and Γ is a subspace of $\Pi_{t\in\Omega} H_t$ such that*

(i) for every $t \in \Omega$, the set $x(t)$ for $x \in \Gamma$ is dense in H_t;

(ii) for every $x \in \Gamma$, the function $t \mapsto \|x(t)\|_{H_t}$ is continuous on Ω, where $\|x(t)\|_{H_t} = \langle x(t), x(t) \rangle^{\frac{1}{2}}_{H_t}$ is the norm induced by the inner product of H_t;

(iii) whenever $x \in \Pi_{t\in\Omega} H_t$ and, for every $t \in \Omega$ and every $\varepsilon > 0$, there is an $x' \in \Gamma$ such that $\|x(s) - x'(s)\|_{H_s} \leq \varepsilon$ throughout some neighbourhood of t, it follows that $x \in \Gamma$.

Remark 6.10 ([9], 10.7.1). Let $\mathcal{H} = \{\Omega, (H_t)_{t\in\Omega}, \Gamma\}$ be a continuous field of Hilbert spaces. If $x, y \in \Gamma$ then the function $t \mapsto \langle x(t), y(t) \rangle_t$ is continuous by the inequality

$$4 \langle x(t), y(t) \rangle_{H_t} = \|x(t) + y(t)\|^2_{H_t} - \|x(t) - y(t)\|^2_{H_t} + i\|x(t) + iy(t)\|^2_{H_t} - i\|x(t) - iy(t)\|^2_{H_t}.$$

Remark 6.11 ([9], 10.7.2)**.** Let $\mathcal{H} = \{\Omega, (H_t)_{t\in\Omega}, \Gamma\}$ be a continuous field of Hilbert spaces. For each $t \in \Omega$, let $A_t = K(H_t)$. For $x, y \in \Gamma$ define $\theta_{x,y} \in \Pi_{t\in\Omega} A_t$ by the formula $\theta_{x,y}(t) = \theta_{x(t),y(t)}$. By above we have that

$$\theta^*_{x,y} = \theta_{y,x}, \qquad \theta_{x,y}\theta_{x',y'} = \theta_{z,y'}$$

with $z(t) = \langle x'(t), y(t)\rangle\, x(t)$. Then the set Λ of the vector fields

$$\theta_{x_1,x_2} + \theta_{x_3,x_4} + ... + \theta_{x_{2n-1},x_{2n}},$$

where $x_1, ..., x_{2n} \in \Gamma$ is an involutive subalgebra of $\Pi_{t\in\Omega} A_t$. The set $\Lambda(t)$ is dense in the set of operators of finite rank in H_t, and is therefore dense in A_t. By Remark 6.8,

$$\|\theta_{x_1,x_2}(t) + ... + \theta_{x_{2n-1},x_{2n}}(t)\|_{H_t}$$

is a continuous function of the $\langle x_i(t), x_j(t)\rangle$, and therefore of t. Then by [9, Proposition 10.3.2], there exists a unique set $\Theta \subset \Pi_{t\in\Omega} A_t$ such that $\mathcal{U} = \{\Omega, A_t, \Theta\}$ is a continuous field of elementary C*-algebras. *The triple $\mathcal{U}$ is said to be associated to $\mathcal{H}$. It will be denoted by $\mathcal{U}(\mathcal{H})$.*

We now consider the correspondence betwen continuous fields of Hilbert spaces and continuous fields of elementary C*-algberas. Again the following can be found in [9, Section 10.7].

Remark 6.12 ([9], 10.7.5)**.** Let $\mathcal{H} = \{\Omega, (H_t), \Gamma\}$ be a continuous field of Hilbert spaces over a locally compact Hausdorff space Ω, with a continuous vector field x such that $\|x(t)\|_{H_t} = 1$ for every t. Let $p = \theta_{x,x} \in \Theta$ and $\mathcal{U} = \mathcal{U}(\mathcal{H}) = \{\Omega, (A_t), \Theta\}$. Then p is a continuous vector field of $\mathcal{U}$ of projections of rank 1.

Remark 6.13 ([9], 10.7.5)**.** Let $\mathcal{U} = \{\Omega, A_t, \Theta\}$ be a continuous field of elementary C*-algebras over a locally compact Hausdorff space Ω, with a continuous field of projections $p \in \Theta$ of rank 1. Let H_t be the subspace $A_t p(t)$ of A_t. By Remark 10.6.4 of [9] H_t is a Hilbert space. Let Γ be the set of all $x \in \Theta$ such that $x(t) \in H_t$ for every $t \in \Omega$. Then $\mathcal{H} = \{\Omega, H_t, \Gamma\}$ is a continuous field of Hilbert spaces. Moreover let $x = p$. Then x is an element of Γ such that $\|x(t)\|_{H_t} = 1$ for every $t \in \Omega$.

Definition 6.14 (Fell's condition)**.** *Let Ω be a locally compact Hausdorff space, and $\mathcal{U} = \{\Omega, A_x, \Theta\}$ a continuous field of elementary C*-algebras. $\mathcal{U}$ is said to satisfy Fell's condition if, for every $x \in \Omega$, there exists a neighbourhood U_x of x and a vector field p of $\mathcal{U}$, such that, for every $t \in U_x$, $p(t)$ is a projection of rank 1. Note that $\mathcal{U}|U_x \cong \mathcal{U}(\mathcal{H}_x)$.*

Proposition 6.15 ([9], Proposition 10.7.7). *Let Ω be a locally compact Hausdorff space, and $\mathcal{U}$ a continuous field of elementary C*-algebras over Ω. The following conditions are equivalent*

1. *For every $t_0 \in \Omega$, there exists a neighbourhood V of t_0 and a continuous field $\mathcal{H}$ of non-zero Hilbert spaces over V, such that $\mathcal{U}|V$ is isomorphic to $\mathcal{U}(\mathcal{H})$.*

2. *$\mathcal{U}$ satisfies Fell's condition.*

Proof. (1)⇒(2). Suppose that condition (1) is satisfied. Let $t_0 \in \Omega$. Let V_0 be a neighbourhood of t_0 and $\mathcal{H} = \{V_0, (H_t), \Gamma\}$ be a continuous field of non-zero Hilbert spaces over V_0 such that $\mathcal{U}|V_0$ is isomorphic to $\mathcal{U}(\mathcal{H})$. Let ξ be a non-zero element of H_{t_0} and let x be a continuous vector field of $\mathcal{H}$ such that $x(t_0) = \xi$. The set V of the $t \in V_0$ such that $x(t) \neq 0$ is a neighbourhood of t_0; put $y(t) = \|x(t)\|_{H_t}^{-1} x(t)$ for $t \in V$. The vector field $\theta_{y,y}$ of $\mathcal{U}(\mathcal{H})$ is defined and continuous on V, and $\theta_{y,y}(t)$ is a projection of rank 1 for every $t \in V$. There therefore exists a vector field p of $\mathcal{U}$, defined and continuous on V, such that the $p(t)$ are projections of rank 1.

(2)⇒(1). Suppose that $\mathcal{U}$ satisfies Fell's condition. Let $t_0 \in \Omega$. There exists a neighbourhood V of t_0 and a continuous field p of projections of $\mathcal{U}$ of rank 1 defined on V. Then, by [9, Lemma 10.7.6], $\mathcal{U}|V$ is isomorphic to $\mathcal{U}(\mathcal{H})$. □

6.2 Necessary and sufficient conditions for the projectivity of C*-algebras with Fell's condition

Let Ω be a locally compact Hausdorff space, and let $\mathcal{U} = \{\Omega, (A_t), \Theta\}$ be a continuous field of elementary C*-algebras satisfying Fell's condition. We define $A(s,t) = A_s \otimes_{C*} A_t$.

Definition 6.16. *Let $\mathcal{U} = \{\Omega, (A_t), \Theta\}$ be a continuous field of C*-algebras. Let $\mathcal{A}$ be the C*-algebra defined by $\mathcal{U}$. For $(s,t) \in \Omega \times \Omega$, define*

$$\varphi_{(s,t)} : \mathcal{A} \hat{\otimes} \mathcal{A} \to A(s,t)$$
$$a \otimes b \mapsto a(s) \otimes b(t)$$

and extend by linearity.

Then for $v \in \mathcal{A} \hat{\otimes} \mathcal{A}$, define

$$G_v : \Omega \times \Omega \to \mathbb{R}^+$$
$$(s,t) \mapsto \|\varphi_{(s,t)}(v)\|_{A(s,t)}.$$

Lemma 6.17. *Let $w \in \mathcal{A}\hat{\otimes}\mathcal{A}$ such that $w = \sum_{i=1}^{n} a_i \otimes b_i$. Then $G_w(s,t)$ is continuous on $\Omega \times \Omega$.*

Proof. Let $\varepsilon > 0$ and $s_0, t_0 \in \Omega$. We wish to show that G_v is continuous at (s_0, t_0).

Since $\mathcal{A}$ satisfies Fell's condition, by Proposition 6.15, there exists a neighbourhood V_{s_0} of s_0 and a continuous field of Hilbert spaces $\mathcal{H}_{s_0} = \{V_{s_0}, (H_{s_0}(s))_{s\in V_{s_0}}, \Gamma_{s_0}\}$ such that $\mathcal{A}|V_{s_0}$ is isomorphic to $\mathcal{U}(\mathcal{H}_{s_0})$. Likewise there exists a neighbourhood V_{t_0} of t_0 and a continuous field of Hilbert spaces $\mathcal{U}_{t_0} = \{V_{t_0}, (H_{t_0}(t))_{t\in V_{t_0}}, \Gamma_{t_0}\}$ such that $\mathcal{A}|V_{t_0}$ is isomorphic to $\mathcal{U}(\mathcal{H}_{t_0})$.

Therefore $a_i|V_{s_0} \in \Gamma_{s_0}$ and $b_i|V_{t_0} \in \Gamma_{t_0}$ for each i and so, for all $(s,t) \in V_{s_0} \times V_{t_0}$,

$$\left(\sum_{i=1}^{n} a_i \otimes b_i\right)(s,t) \in \mathcal{K}(H_{s_0}(s) \otimes_{\mathcal{H}} H_{t_0}(t)).$$

Therefore, by Remark 6.11, there exists $x_j^n, y_j^n \in \Gamma_{s_0}$ and $w_j^n, z_j^n \in \Gamma_{t_0}$, $m_n \in \mathbb{N}$ such that

$$\left(\sum_{i=1}^{n} a_i \otimes b_i\right)(s,t) = \lim_{n\to\infty}\left(\sum_{j=1}^{m_n} \theta_{x_j^n, y_j^n}(s) \otimes \theta_{w_j^n, z_j^n}(t)\right),$$

where the limit is taken in the uniform topology over a neighbourhood of (s,t).

We note that for $\xi \in H_{s_0}(s)$, $\eta \in H_{t_0}(t)$, $x_j, y_j \in \Gamma_{s_0}$ and $w_j, z_j \in \Gamma_{t_0}$ we have

$$\begin{aligned}
&\theta_{x_j,y_j}(s) \otimes \theta_{w_j,z_j}(t)(\xi \otimes \eta)\\
=&\left\langle \xi, y_j(s)\right\rangle x_j(s) \otimes \left\langle \eta, z_j(t)\right\rangle w_j(t)\\
=&\left\langle \xi \otimes \eta, y_j(s) \otimes z_j(t)\right\rangle x_j(s) \otimes w_j(t)\\
=&\theta_{x_j(s)\otimes w_j(t), y_j(s)\otimes z_j(t)}(\xi \otimes \eta).
\end{aligned}$$

Therefore, by Remark 6.10, $(s,t) \mapsto \theta_{x_j,y_j}(s) \otimes \theta_{w_j,z_j}(t)(\xi \otimes \eta)$ is a continuous function on $V_{s_0} \times V_{t_0}$. By Remark 6.11, $\left\|\sum_{j=1}^{m_n} \theta_{x_j^n,y_j^n}(s) \otimes \theta_{w_j^n,z_j^n}(t)\right\|_{A(s,t)}$ is continuous on $V_{s_0} \times V_{t_0}$.
Therefore $\|\varphi_{(s,t)}(w)\|_{A(s,t)}$ is a limit of continuous functions in the locally uniform topology, and so is continuous on $\Omega \times \Omega$.

□

Lemma 6.18. *Let $v \in \mathcal{A}\hat{\otimes}\mathcal{A}$. Then the following are true*

(i) $G_v(s,t)$ *is continuous on* $\Omega \times \Omega$*;*

(ii) $G_v(s,t) \to 0$ *as* $t \to \infty$ *uniformly for* $s \in \Omega$*;*

(iii) $G_v(s,t) \to 0$ *as* $s \to \infty$ *uniformly for* $t \in \Omega$.

Proof. (i) Let $\varepsilon > 0$ and $(s_0, t_0) \in \Omega \times \Omega$. We wish to show that G_v is continuous at (s_0, t_0).

By [31, Theorem 3.6.4], every element v from $\mathcal{A}\hat{\otimes}\mathcal{A}$ can be written as $\sum_{i=1}^{\infty} \lambda_i\, a_i \otimes b_i,\ \lambda_i \in \mathbb{C},\ a_i \in \mathcal{A}, b_i \in \mathcal{A},$ where $\sum_{i=1}^{\infty} |\lambda_i| < \infty$ and the sequences $\{a_i\}, \{b_i\}$ converge to zero in $\mathcal{A}$ as $i \to \infty$.

Pick N such that
$$\sum_{i=N+1}^{\infty} |\lambda_i| \|a_i\| \|b_i\| < \frac{\varepsilon}{4}.$$

Let $w = \sum_{i=1}^{N} \lambda_i a_i \otimes b_i$. By Lemma 6.17, G_w is continuous on $\Omega \times \Omega$. Therefore there exists a neighbourhood, $U_{(s_0,t_0)}$ of (s_0, t_0), such that
$$|G_w(s,t) - G_w(s_0,t_0)| < \frac{\varepsilon}{2},$$

for all $(s,t) \in U_{(s_0,t_0)}$.

Note that,
$$\begin{aligned}\|\varphi_{(s_0,t_0)}(w)\|_{A(s_0,t_0)} =& \|\varphi_{(s_0,t_0)}(w) - \varphi_{(s_0,t_0)}(v-w) + \varphi_{(s_0,t_0)}(v-w)\|_{A(s_0,t_0)} \\ \leq& \|\varphi_{(s_0,t_0)}(w) + \varphi_{(s_0,t_0)}(v-w)\|_{A(s_0,t_0)} \\ &+ \|\varphi_{(s_0,t_0)}(v-w)\|_{A(s_0,t_0)}.\end{aligned}$$

and so,
$$\begin{aligned}-\|\varphi_{(s_0,t_0)}(w) + \varphi_{(s_0,t_0)}(v-w)\|_{A(s_0,t_0)} \leq& \|\varphi_{(s_0,t_0)}(v-w)\|_{A(s_0,t_0)} \\ &- \|\varphi_{(s_0,t_0)}(w)\|_{A(s_0,t_0)}.\end{aligned}$$

Then, for all $(s,t) \in U_{(s_0,t_0)}$,
$$\begin{aligned}&G_v(s,t) - G_v(s_0,t_0) \\ =&\|\varphi_{(s,t)}(v)\|_{A(s,t)} - \|\varphi_{(s_0,t_0)}(v)\|_{A(s_0,t_0)}\end{aligned}$$

$$
\begin{aligned}
&=\|\varphi_{(s,t)}(w)+\varphi_{(s,t)}(v-w)\|_{A(s,t)}-\|\varphi_{(s_0,t_0)}(w)+\varphi_{(s_0,t_0)}(v-w)\|_{A(s_0,t_0)}\\
&\leq\|\varphi_{(s,t)}(w)\|_{A(s,t)}+\|\varphi_{(s,t)}(v-w)\|_{A(s,t)}\\
&\quad+\|\varphi_{(s_0,t_0)}(v-w)\|_{A(s_0,t_0)}-\|\varphi_{(s_0,t_0)}(w)\|_{A(s_0,t_0)}\\
&=\|\varphi_{(s,t)}(v-w)\|_{A(s,t)}+\|\varphi_{(s_0,t_0)}(v-w)\|_{A(s_0,t_0)}+G_w(s,t)-G_w(s_0,t_0)\\
&<\frac{\varepsilon}{2}+G_w(s,t)-G_w(s_0,t_0)\\
&\leq\frac{\varepsilon}{2}+|G_w(s,t)-G_w(s_0,t_0)|\\
&<\varepsilon.
\end{aligned}
$$

Similarly, for all $(s,t)\in U_{(s_0,t_0)}$,

$$
\begin{aligned}
&G_v(s_0,t_0)-G_v(s,t)\\
&=\|\varphi_{(s_0,t_0)}(v)\|_{A(s_0,t_0)}-\|\varphi_{(s,t)}(v)\|_{A(s,t)}\\
&=\|\varphi_{(s_0,t_0)}(w)+\varphi_{(s_0,t_0)}(v-w)\|_{A(s_0,t_0)}-\|\varphi_{(s,t)}(w)+\varphi_{(s,t)}(v-w)\|_{A(s,t)}\\
&\leq\|\varphi_{(s_0,t_0)}(w)\|_{A(s_0,t_0)}+\|\varphi_{(s_0,t_0)}(v-w)\|_{A(s_0,t_0)}\\
&\quad+\|\varphi_{(s,t)}(v-w)\|_{A(s,t)}-\|\varphi_{(s,t)}(w)\|_{A(s,t)}\\
&=\|\varphi_{(s_0,t_0)}(v-w)\|_{A(s_0,t_0)}+\|\varphi_{(s,t)}(v-w)\|_{A(s,t)}+G_w(s_0,t_0)-G_w(s,t)\\
&<\frac{\varepsilon}{2}+G_w(s_0,t_0)-G_w(s,t)\\
&\leq\frac{\varepsilon}{2}+|G_w(s_0,t_0)-G_w(s,t)|\\
&<\varepsilon.
\end{aligned}
$$

We therefore have that

$$|G_v(s,t)-G_v(s_0,t_0)|<\varepsilon.$$

(ii) Let $\varepsilon>0$. For $t\in\Omega$, let $\tau_t:\mathcal{A}\to A_t$ be evaluation at t. We note that

$$\|\varphi_{(s,t)}(v)\|_{A(s,t)}\leq\|(\tau_s\otimes\tau_t)(v)\|_{A_s\hat{\otimes}A_t}.$$

Pick N_0 such that

$$\sum_{i=N_0+1}^{\infty}|\lambda_i|\|a_i\|\|b_i\|<\frac{\varepsilon}{2}.$$

Recall that the sequences $\{a_i\},\{b_i\}$ converge to 0 in $\mathcal{A}$ as $i\to\infty$ and so the sequences $\{a_i\},\{b_i\}$ are also bounded. There exists a $C\in\mathbb{R}$ such that $\|a_i\|<C$ for all i. Also we can therefore pick a compact subset $K\subset\Omega$ such that for all $t\in\Omega\setminus K$

$$\|\tau_t(b_i)\|_{A_t}<\frac{\varepsilon}{2M},\quad\text{where } M>C\sum_{i=1}^{\infty}|\lambda_i|$$

for all $i = 1, \dots, N_0$. We then have

$$\begin{aligned}\sup_{s\in\Omega} G_v(s,t) &\leq \sup_{s\in\Omega}\left(\sum_{i=1}^{N_0} |\lambda_i| \|\tau_s(a_i)\| \|\tau_t(b_i)\|_{A_t}\right) + \frac{\varepsilon}{2} \\ &\leq \sum_{i=1}^{N_0} |\lambda_i| \|a_i\| \|\tau_t(b_i)\|_{A_t} + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

for all $t \in \Omega \setminus K$.

(iii) Follows as in (ii).

□

Definition 6.19. *Let Ω be a locally compact topological space. We say that a continuous field of elementary C*-algebras $\mathcal{U} = \{\Omega, (A_x)_{x\in\Omega}, \Theta\}$, satisfies the σ-Fell condition if for every $x \in \Omega$, there exists a neighbourhood U_x of x and a vector field, p_x, such that $p_x(s)$ is a projection of rank 1 for each $s \in U_x$ and there is a countable open cover $\{V_j\}$ of Ω such that $\overline{V_j} \subset U_{x(j)}$ for each j and some $x(j) \in \Omega$.*

Definition 6.20. *Let $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ be a continuous field of Banach algebras. Let Ω be a disjoint union of a family of open subsets $\{W_\mu\}_{\mu\in\mathcal{M}}$ of Ω. We say that $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ is a disjoint union of $\mathcal{U}|_{W_\mu}, \mu \in \mathcal{M}$.*

Theorem 6.21. *Let Ω be a locally compact Hausdorff space, let $\mathcal{U} = \{\Omega, (A_t), \Theta\}$ be a disjoint union of continuous fields of elementary C*-algebras $\mathcal{U}|W_\mu, \mu \in \mathcal{M}$, satisfying the σ-Fell condition and let $\mathcal{A}$ be the C*-algebra defined by $\mathcal{U}$. Suppose that $\mathcal{A}$ is left or right projective. Then Ω is paracompact.*

Proof. By assumption, Ω is a disjoint union of the family of the open subsets $\{W_\mu\}_{\mu\in\mathcal{M}}$. We shall split the proof into the following lemmas. By Lemma 4.15, if W_μ is paracompact for each μ then Ω is paracompact.

Fix $\mu \in \mathcal{M}$. By assumption $\mathcal{U}|_{W_\mu}$ satisifes the $\sigma-$Fell condition. Therefore, for every $x \in W_\mu$, there exists a neighbourhood U_x of x and a vector field p_x such that $p_x(s)$ is a projection of rank 1 for each $s \in U_x$ and there is a countable open cover $\{V_j\}$ of W_μ such that $\overline{V_j} \subset U_{x(j)}$ for each j and some $x(j)$.

Lemma 6.22. *The paracompactness of all $\overline{V_j} \cap W_\mu, j \in \mathbb{N}$, implies the paracompactness of W_μ.*

Proof. Let $\mathcal{B}$ be an arbitrary open cover of W_μ. For each $j \in \mathbb{N}$, the family $\mathcal{B}_j = \{B \cap \overline{V_j} \cap W_\mu : B \in \mathcal{B}\}$ is an open cover of $\overline{V_j} \cap W_\mu$. By assumption, $\overline{V_j} \cap W_\mu$ is paracompact and so $\mathcal{B}_j$ has an open locally finite refinement $\mathcal{D}_j$ that is also a cover of $\overline{V_j} \cap W_\mu$. The family of open subsets $\mathcal{D}_j^{'} = \{D \cap V_j : D \in \mathcal{D}_j\}$ is locally finite in W_μ and is a refinement of $\mathcal{B}$. Furthermore, since $W_\mu = \bigcup_{j \in \mathbb{N}} V_j$, the family $\mathcal{D} = \bigcup_{j \in \mathbb{N}} \mathcal{D}_j^{'}$ is an open $\sigma-$locally finite cover of W_μ. By Kelley [18, Theorem 5.8], W_μ is paracompact. □

Fix $j \in \mathbb{N}$. We will prove that $\overline{V_j} \cap W_\mu$ is paracompact.

Suppose that $\mathcal{A}$ is left projective. Then there exists a morphism of modules $\rho : \mathcal{A} \to \mathcal{A}_+ \hat{\otimes} \mathcal{A}$ such that $\pi \circ \rho = 1_{\mathcal{A}}$.

By [10, Theorem 3.3.1], W_μ is a Tychonoff space and so, for every $s \in \overline{V_j} \subset U_{x(j)}$, there is $f_s \in C_0(\Omega)$ such that $0 \leq f_s \leq 1$, $f_s(s) = 1$ and $f_s(t) = 0$ for all $t \in \Omega \setminus U_{x(j)}$. By Property (iii) of Definition 1.9 and [9, Proposition 10.1.9], the field fp is continuous and $\|f(t)p(t)\| \to 0$ as $t \to \infty$, so we have $fp_x^2 \in \mathcal{A}$.

Note that $\rho(f_s p_x^2) \in \mathcal{A} \hat{\otimes} \mathcal{A}$ for every $s \in \overline{V_j} \subset U_{x(j)}$. For every $s \in \overline{V_j} \subset U_{x(j)}$ and $t \in \Omega$, we set

$$\Phi(s,t) = G_{\rho(f_s p_x^2)}(s,t)$$

where the function G is defined in Definition 6.16.

Lemma 6.23. *Let $s \in \overline{V_x} \cap W_\mu$. Then Φ is independent of the choice of f_s.*

Proof. Let $f_s, g_s \in C_0(\Omega)$ such that $0 \leq f_s, g_s \leq 1$, $f_s(s) = g_s(s) = 1$ and $f_s(t) = g_s(t) = 0$ for all $t \in \Omega \setminus U_{x(j)}$.

Let $\rho(\sqrt{f_s} p_x) = \sum_{i=1}^{\infty} a_i \otimes b_i$ and $\rho(\sqrt{g_s} p_x) = \sum_{i=1}^{\infty} c_i \otimes d_i$ where a_i, b_i, c_i and $d_i \in \mathcal{A}$.

Then

$$\begin{aligned}
G_{\rho(f_s p_x^2)}(s,t) &= \|\varphi_{(s,t)}(\rho(f_s p_x^2))\|_{A(s,t)} \\
&= \|\varphi_{(s,t)}(\sqrt{f_s} p_x \rho(\sqrt{f_s} p_x))\|_{A(s,t)} \\
&= \|\varphi_{(s,t)}(\sqrt{f_s} p_x \sum_{i=1}^{\infty} a_i \otimes b_i)\|_{A(s,t)} \\
&= \|\varphi_{(s,t)}(\sum_{i=1}^{\infty} \sqrt{f_s} p_x a_i \otimes b_i)\|_{A(s,t)}
\end{aligned}$$

$$
\begin{aligned}
&= \| \sum_{i=1}^{\infty} \sqrt{f_s(s)} p_x(s) a_i(s) \otimes b_i(t) \|_{A(s,t)} \\
&= \| \sum_{i=1}^{\infty} p_x(s) a_i(s) \otimes b_i(t) \|_{A(s,t)} \\
&= \| \sum_{i=1}^{\infty} \sqrt{g_s(s)} p_x(s) a_i(s) \otimes b_i(t) \|_{A(s,t)} \\
&= \| \varphi_{(s,t)} (\sum_{i=1}^{\infty} \sqrt{g_s} p_x a_i \otimes b_i) \|_{A(s,t)} \\
&= \| \varphi_{(s,t)} (\sqrt{g_s} p_x \sum_{i=1}^{\infty} a_i \otimes b_i) \|_{A(s,t)} \\
&= \| \varphi_{(s,t)} (\sqrt{g_s} p_x \rho(\sqrt{f_s} p_x)) \|_{A(s,t)} \\
&= \| \varphi_{(s,t)} (\rho(\sqrt{f_s} \sqrt{g_s} p_x^2)) \|_{A(s,t)} \\
&= \| \varphi_{(s,t)} (\sqrt{f_s} p_x \rho(\sqrt{g_s} p_x)) \|_{A(s,t)} \\
&= \| \varphi_{(s,t)} (\sqrt{f_s} p_x \sum_{i=1}^{\infty} c_i \otimes d_i) \|_{A(s,t)} \\
&= \| \varphi_{(s,t)} (\sum_{i=1}^{\infty} \sqrt{f_s} p_x c_i \otimes d_i) \|_{A(s,t)} \\
&= \| \sum_{i=1}^{\infty} \sqrt{f_s(s)} p_x(s) c_i(s) \otimes d_i(t) \|_{A(s,t)} \\
&= \| \sum_{i=1}^{\infty} p_x(s) c_i(s) \otimes d_i(t) \|_{A(s,t)} \\
&= \| \sum_{i=1}^{\infty} \sqrt{g_s(s)} p_x(s) c_i(s) \otimes d_i(t) \|_{A(s,t)} \\
&= \| \sqrt{g_s(s)} p_x(s) \sum_{i=1}^{\infty} c_i(s) \otimes d_i(t) \|_{A(s,t)} \\
&= \| \varphi_{(s,t)} (\sqrt{g_s} p_x \sum_{i=1}^{\infty} c_i \otimes d_i) \|_{A(s,t)} \\
&= \| \varphi_{(s,t)} (\sqrt{g_s} p_x \rho(\sqrt{g_s} p_x) \|_{A(s,t)} \\
&= \| \varphi_{(s,t)} (\rho(g_s p_x^2) \|_{A(s,t)} \\
&= G_{\rho(g_s p_x^2)}(s,t)
\end{aligned}
$$

Therefore Φ does not depend on the choice of f_s. □

Lemma 6.24. *Let $s \in \overline{V_x} \cap W_\mu$. Then $\Phi(s,s) > 0$.*

Proof. Since $p_x(s) \neq 0$, there exists $\xi, \eta \in \mathcal{H}_x(s)$ such that $\langle \xi, p_x(s)(\eta) \rangle \neq 0$.

Recall that the projective tensor norm is the largest cross norm on $A_s \otimes A_s$and the injective tensor norm is the least cross norm on $A_s \otimes A_s$. Therefore the following continuous linear map exists

$$\begin{aligned} \gamma_{(s,s)} : A_s \hat{\otimes} A_s &\to A_s \otimes_{C^*} A_s \\ x \otimes y &\mapsto x \otimes y \end{aligned}$$

Define the following continuous linear maps

$$\begin{aligned} i_s : \mathcal{H}_x(s) \hat{\otimes} \mathcal{H}_x(s) &\to \mathcal{H}_x(s) \otimes_H \mathcal{H}_x(s) \\ x \otimes y &\mapsto x \otimes y, \\ \phi : A(s,s) &\to A_s^{op} \otimes_{C*} A_s \\ x \otimes y &\mapsto x^* \otimes y, \end{aligned}$$

and the following continuous maps which are linear in the first term and conjugate linear in the second term

$$\begin{aligned} \psi_{\xi,\eta} : A_s \hat{\otimes} A_s &\to \mathcal{H}_x(s) \hat{\otimes} \mathcal{H}_x(s) \\ x \otimes y &\mapsto x^*(\xi) \otimes y(\eta), \\ \tilde{\psi}_{\xi,\eta} : A_s^{op} \otimes_{C*} A_s &\to \mathcal{H}_x(s) \otimes_H \mathcal{H}_x(s) \\ x \otimes y &\mapsto x(\xi) \otimes y(\eta). \end{aligned}$$

Note that i_s is injective.

Consider the following diagram

$$\begin{array}{ccccc} \mathcal{A} \hat{\otimes} \mathcal{A} & \overset{\varphi_{(s,s)}}{\searrow} & & & \\ \downarrow{\scriptstyle \tau_s \hat{\otimes} \tau_s} & & & & \\ A_s \hat{\otimes} A_s & \xrightarrow{\gamma_{(s,s)}} & A_s \otimes_{C^*} A_s & & \\ \downarrow{\scriptstyle \psi_{\xi,\eta}} & & & \overset{\phi}{\searrow} & \\ H_x(s) \hat{\otimes} H_x(s) & \xrightarrow{i_s} & H_x(s) \otimes_{\mathcal{H}} H_x(s) & \xleftarrow[\tilde{\psi}_{\xi,\eta}]{} & A_s^{op} \otimes_{C*} A_s. \end{array}$$

Let $a \otimes b \in \mathcal{A} \otimes \mathcal{A}$. Then

$$(\gamma_{(s,s)} \circ \tau_s \otimes \tau_s)(a \otimes b) = \gamma_{(s,s)}(a(s) \otimes b(s))$$

$$\begin{aligned} &= a(s) \otimes b(s) \\ &= \varphi_{(s,s)}(a \otimes b). \end{aligned}$$

Thus $\gamma_{(s,s)} \circ \tau_s \otimes \tau_s = \varphi_{(s,s)}$.

Let $x \otimes y \in A_s \otimes A_s$. Then

$$\begin{aligned} (\tilde{\psi}_{\xi,\eta} \circ \phi \circ \gamma_{(s,s)})(x \otimes y) &= (\tilde{\psi}_{\xi,\eta} \circ \phi)(x \otimes y) \\ &= \tilde{\psi}_{\xi,\eta}(x^* \otimes y) \\ &= x^*(\xi) \otimes y(\eta), \end{aligned}$$

and

$$\begin{aligned} (i_s \circ \psi_{\xi,\eta})(x \otimes y) &= i_s(x^*(\xi) \otimes y(\eta)) \\ &= x^*(\xi) \otimes y(\eta). \end{aligned}$$

Therefore $i_s \circ \psi_{\xi,\eta} = \gamma_{(s,s)} \circ \tau_s \otimes \tau_s$. Thus the above diagram commutes.

Therefore $G_{\rho(f_s p_x^2)}(s,s) = \|\varphi_{(s,s)}\rho(f_s p_x^2)\|_{A(s,s)} = \|\gamma_{(s,s)}(\tau_s \hat{\otimes} \tau_s)\rho(f_s p_x^2)\|_{A(s,s)}$. Since i_s is injective it is enough to show that $\psi_{\xi,\eta}(\tau_s \hat{\otimes} \tau_s)\rho(f_s p_x^2) \neq 0$.

Suppose $\rho(f_s p_x^2) = \sum_{i=1}^{\infty} a_i \otimes b_i$. Thus $p_x(s) = \sum_{i=1}^{\infty} a_i(s) b_i(s)$. So we have

$$\begin{aligned} &tr(\psi_{\xi,\eta}(\tau_s \hat{\otimes} \tau_s)\rho(f_s p_x^2)) \\ =&tr(\sum_{i=1}^{\infty} a_i^*(s)(\xi) \otimes b_i(s)(\eta)) \\ =&\sum_{i=1}^{\infty} \langle a_i^*(s)(\xi), b_i(s)(\eta) \rangle \\ =&\sum_{i=1}^{\infty} \langle \xi, a_i(s) b_i(s)(\eta) \rangle \\ =&\left\langle \xi, \sum_{i=1}^{\infty} a_i(s) b_i(s)(\eta) \right\rangle \\ =&\langle \xi, p_x(s)(\eta) \rangle \\ \neq& 0. \end{aligned}$$

□

Lemma 6.25. *The function $\Phi(s,t)$ is continuous on $(\overline{V_j} \cap W_\mu) \times \Omega$.*

Proof. Let $(s_0, t_0) \in \overline{V_j} \times \Omega$ and $f_{s_0} \in C_0(\Omega)$ such that $0 \leq f_{s_0} \leq 1$, $f_{s_0}(s_0) = 1$ and $f_{s_0}(t) = 0$ for every $t \in \Omega \setminus U_{x(j)}$.

Consider the neighbourhood $V = U \times \Omega$ of (s_0, t_0) where $U = \{s \in \overline{V_j} \cap W_\mu | f_{s_0}(s) \neq 0\}$.

Then, for $(s,t) \in V$,

$$\begin{aligned}
\Phi(s,t) &= G_{\rho\left(\frac{f_{s_0}}{f_{s_0}(s)} p_x^2\right)} \\
&= \left\| \varphi_{(s,t)} \left(\rho \left(\frac{f_{s_0}}{f_{s_0}(s)} p_x^2 \right) \right) \right\|_{A(s,t)} \\
&= \frac{1}{f_{s_0}(s)} \left\| \varphi_{(s,t)} \left(\rho \left(f_{s_0} p_x^2 \right) \right) \right\|_{A(s,t)} \\
&= \frac{1}{f_{s_0}(s)} G_{\rho(f_{s_0} p_x^2)}(s,t).
\end{aligned}$$

By Lemma 6.18 part (i), $G_{\rho(f_{s_0} p_x^2)}(s,t)$ is continuous. Therefore $G_{\rho(f_s p_x^2)}(s,t)$ is the ratio of two continuous functions and hence is continuous. □

Lemma 6.26. *For every compact $K \subset \overline{V_x} \cap W_\mu$, the function $\Phi(s,t) \to 0$ as $t \to \infty$ in Ω uniformly for $s \in K$.*

Proof. By [10, Theorem 3.1.7], since Ω is a Tychonoff space, for a compact subset $K \subset \overline{V_j} \cap W_\mu \subset \Omega$ and for a closed subset $\Omega \setminus U_{x(j)} \subset \Omega \setminus K$, there is $f_K \in C_0(\Omega)$ such that $0 \leq f_K \leq 1$, $f_K(s) = 1$ for all $s \in K$ and $f_K(t) = 0$ for all $t \in \Omega \setminus U_{x(j)}$.

By Lemma 6.18, the function $G_{\rho(f_K\, p)}(s,t) \to 0$ as $t \to \infty$ in Ω uniformly for $s \in \Omega$.

Thus the function $\Phi(s,t) = G_{\rho(f_K p)}(s,t)$ on $K \times \Omega \subset (\overline{V_j} \cap W_\mu) \times \Omega$ tends to 0 as $t \to \infty$ in Ω uniformly for $s \in K$. □

Conclusion of the proof of Theorem 6.21

For $(s,t) \in (\overline{V_j} \cap W_\mu) \times (\overline{V_j} \cap W_\mu)$, we set

$$E(s,t) = \Phi(s,t)/\Phi(s,s).$$

By Lemma 6.24, $\Phi(s,s) > 0$ for every $s \in \overline{V_j} \cap W_\mu$. Therefore, by Lemma 6.25, $E(s,t)$ is continuous at every $(s,t) \in (\overline{V_j} \cap W_\mu) \times (\overline{V_j} \cap W_\mu)$.

For $(s,t) \in (\overline{V_j} \cap W_\mu) \times (\overline{V_j} \cap W_\mu)$, we also set

$$F(s,t) = \min\{E(s,t), 1\} \min\{E(t,s), 1\}.$$

By Lemmas 6.25 and 6.26, the function $F(s,t)$ has the following properties:

(i) $F(s,t)$ is continuous on $(\overline{V_j} \cap W_\mu) \times (\overline{V_j} \cap W_\mu)$,

(ii) for every compact $K \subset \overline{V_j} \cap W_\mu$, $F(s,t) \to 0$ as $t \to \infty$ uniformly for $s \in K$,

(iii) for every compact $K \subset \overline{V_j} \cap W_\mu$, $F(s,t) \to 0$ as $s \to \infty$ uniformly for $t \in K$,

(iv) $F(s,s) = 1$ for all $s \in \overline{V_j} \cap W_\mu$.

By [13, Theorem A.12, Appendix A], $\overline{V_j} \cap W_\mu$ is paracompact. By Lemma 6.22, W_μ is paracompact. By Lemma 4.15, Ω is paracompact.

□

Definition 6.27. *Let $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ be a continuous field of Banach algebras over Ω. Let $\Lambda \subset \Theta$. Then Λ is said to be total if, for every $t \in \Omega$, the set $x(t)$, as x runs through Λ, is total in A_t. $\mathcal{U}$ is said to be separable if Θ has a countable total subset.*

We state the following theorem, [9, Theorem 10.8.8], without proof.

Theorem 6.28. *Let Ω be a paracompact space of finite dimension and $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ be a separable continuous field of elementary C*-algebras, of rank $\aleph_0$, satisfying Fell's condition. Then $\mathcal{U}$ is locally trivial.*

Theorem 6.29. *Let Ω be a locally compact Hausdorff space of finite dimension and $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ be a separable continuous field of elementary C*-algebras, of rank $\aleph_0$, satisfying Fell's condition. Let $\mathcal{A}$ be the C*-algebra defined by $\mathcal{U}$. Then the following are equivalent*

(i) Ω is paracompact.

(ii) $\mathcal{U}$ is a disjoint union of continuous fields of elementary C-algebras that satisfies the σ-Fell condition and $\mathcal{A}$ is left projective .*

Proof. $(i) \Rightarrow (ii)$. Suppose that Ω is paracompact. By Theorem 6.28, the continuous field $\mathcal{U}$ is locally trivial and so, by Theorem 5.26, $\mathcal{A}$ is left projective.

Since Ω is paracompact it is the disjoint union of $\sigma-$compact sets $\{W_\mu\}$, see [10, Theorem 5.1.27]. Since $\mathcal{U}$ satisfies the Fell condition, for every $x \in W_\mu$, there exists a neighbourhood U_x of x, and a vector field p_x such that $p_x(s)$ is a projection of rank 1 for each $s \in U_x$. Since W_μ is σ-compact and Hausdorff it is regular, see [26]. Therefore for every U_x there exists an open set V_x such that $\overline{V_x} \subset U_x$. The family $\{V_x\}$ is an open cover of W_μ. Therefore, since W_μ is σ-compact, there exists a countable subcover $\{V_j\}_{j \in \mathbb{N}}$ of $\{V_x\}$. Therefore, by Definition 6.19, $\mathcal{U}|_{W_\mu}$ satisifies the σ-Fell condition. Thus $\mathcal{U}$ is a disjoint union of continuous fields of elementary C*-algebras that satisfies the σ-Fell condition.

$(ii) \Rightarrow (i)$. Suppose that $\mathcal{U}$ is a disjoint union of continuous fields of elementary C*-algebras that satisfies the σ-Fell condition and $\mathcal{A}$ is left projective. By assumption Ω has finite topological dimension. Then, by Theorem 6.21, Ω is paracompact. □

7 Biprojectivity

7.1 Biprojective Banach algebras

The following definition of biprojectivity can be found in [13].

Definition 7.1. *Let A be a Banach algebra. We say that A is biprojective if there exists a morphism of bimodules, $\rho : A \to A\hat{\otimes}A$, such that $\pi \circ \rho$ is the identity operator on A where π is the canonical morphism.*

Definition 7.2. *Let $(A_x)_{x\in\Omega}$ be a family of Banach algebras. We say that the Banach algebras A_x, $x \in \Omega$, are uniformly biprojective if, for every $x \in \Omega$, there is a morphism of Banach A_x bimodules*

$$\rho_x : A_x \to A_x\widehat{\otimes}A_x$$

such that $\pi_{A_x} \circ \rho_x = \mathrm{id}_{A_x}$ and $\sup_{x\in\Omega} \|\rho_x\|_{A_x} < \infty$.

The following result of Helemskii's, on the biprojectivity of commutative C*-algebras, can be found in [13].

Theorem 7.3. *Let Ω be a locally compact Hausdorff space. The Banach algbera $C_0(\Omega)$ is biprojective if and only if Ω is discrete.*

We will generalise this result to describe the biprojectivity of Banach algebras defined by continuous fields.

7.2 Biprojectivity of Banach algebras defined by locally trivial fields

Theorem 7.4. *Let Ω be a locally compact Hausdorff space and let $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ be a locally trivial continuous field of Banach algebras. Let $\mathcal{A}$ be the Banach algebra defined by $\mathcal{U}$. If A is biprojective then the family $(A_t)_{t\in\Omega}$ are uniformly biprojective.*

Proof. Fix $x \in \Omega$. Since $\mathcal{U}$ is locally trivial, there exists an open neighbourhood $U_x \subset \Omega$ of x be such that $\mathcal{U}|U_x$ is trivial. Let $\phi = \{\phi_t\}_{t\in U_x}$ be an isomorphism of $\mathcal{U}|U_x$ onto the trivial continuous field of Banach algebras over U_x where, for each $t \in \Omega$, $\phi_t : A_t \to \tilde{A}_x$ is an isometric isomorphism of Banach algebras.

By [18, Theorem 5.17], Ω is regular, and so there exists an open neighbourhood, $V_x \subset U_x$, of x such that $\overline{V_x} \in U_x$. By [10, Theorem 3.3.1], Ω is Tychonoff and so there exists an $f_x \in C_0(\Omega)$ such that $0 \leq f_x \leq 1$, $f_x(x) = 1$ and $f_x|_{\Omega\setminus V_x} = 0$.

For $a_x \in A_x$ set $\tilde{a}_x(t) = f_x(t)\phi_t^{-1}\phi_x(a_x)$, $t \in \Omega$. Then, by a similar way to the proof of Lemma 2.1, $\tilde{a}_x \in A$. It is clear that, for $a_x \in A_x$, we have $\tau_x(\tilde{a}_x) = a_x$.

Since A is biprojective in A-mod-A there exists a morphism of Banach A-bimodules, $\rho : A \to A\hat{\otimes}A$ such that $\pi \circ \rho = 1_A$ where π is the canonical morphism of Banach A-bimodules. Now define

$$\tilde{\rho_x} : A_x \longrightarrow A_x\hat{\otimes}A_x$$
$$a \longmapsto (\tau_x \otimes \tau_x)\rho(\tilde{a}_x)$$

where $\tau_x(a) = a(x)$. As in Proposition 2.3 it can be shown that $\tilde{\rho_x}$ is a morphism of Banach A-bimodules, $\pi_x \circ \tilde{\rho_x} = id_{A_x}$ and that $\sup_{x\in\Omega} \|\tilde{\rho}_x\|_{A_x} \leq \|\rho\|_{\mathcal{A}}$. □

Theorem 7.5. *Let Ω be a locally compact Hausdorff space and let $\mathcal{U} = \{\Omega, (A_t)_{t\in\Omega}, \Theta\}$ be a locally trivial continuous field of Banach algebras. Let $\mathcal{A}$ denote the Banach algebra defined by $\mathcal{U}$. If $\mathcal{A}$ is biprojective then Ω is discrete.*

Proof. Since $\mathcal{A}$ is biprojective, there exists a morphism of Banach $\mathcal{A}$-bimodules $\rho : \mathcal{A} \to \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\pi_{\mathcal{A}} \circ \rho = \mathrm{id}_{\mathcal{A}}$.

By [18, Theorem 5.17], a Hausdorff locally compact space is regular. Therefore, since $\mathcal{U}$ is locally trivial on Ω, there is an open cover $\{U_\mu\}$, $\mu \in \mathcal{M}$, of Ω such that each $\mathcal{U}|_{U_\mu}$ is trivial and, in addition, there is an open cover $\{V_\alpha\}$ of Ω such that $\overline{V_\alpha} \subset U_{\mu(\alpha)}$ for each α and some $\mu(\alpha) \in \mathcal{M}$.

Let us show that, for every α, V_α is discrete. By Lemmas 2.1 and 3.1, there are continuous vector fields x and y on $U_{\mu(\alpha)}$ such that $p(t) = x(t)y(t) \neq 0$ for every $t \in U_{\mu(\alpha)}$. By [10, Theorem 3.3.1], Ω is a Tychonoff space and so, for every $s \in V_\alpha$, there is $f_s \in C_0(\Omega)$ such that $0 \leq f_s \leq 1$, $f_s(s) = 1$ and $f_s(t) = 0$ for all $t \in \Omega \setminus U_{\mu(\alpha)}$. Note that $f_s p \in \mathcal{A}$. For every $s \in V_\alpha$ and $t \in \Omega$, we set

$$\Phi(s,t) = F_{\rho(f_s p)}(s,t)/\|p(s)\|_{A_s},$$

where the function $F_u(s,t) = \|(\tau_s \otimes \tau_t)u\|_{A_s\hat{\otimes}A_t}$ is as defined in Proposition 3.3. By Proposition 3.3, $\Phi(s,s) \geq 1$ for every $s \in V_\alpha$.

As in Proposition 3.4, the function $\Phi(s,t)$ is a positive continuous function on $V_\alpha \times \Omega$ and does not depend on the choice of f_s.

Further, for every $s,t \in V_\alpha$ such that $s \neq t$, there is $g_s \in C_0(\Omega)$ such that $0 \leq g_s \leq 1$, $g_s(s) = 1$ and $g_s(t) = 0$. Since ρ is a morphism of Banach $\mathcal{A}$-bimodules, we have

$$\Phi(s,t) = F_{\rho(g\, f_s\, p)}(s,t)/\|p(s)\|_{A_s}$$

$$= \|(\tau_s \otimes \tau_t)\rho(f_s\, xg_s\, y)\|_{A_s \widehat{\otimes} A_t} / \|p(s)\|_{A_s}$$
$$= \|(\tau_s \otimes \tau_t)\rho(f_s\, x)g_s(t)\, y(t)\|_{A_s \widehat{\otimes} A_t} / \|p(s)\|_{A_s} = 0.$$

Therefore, $\Phi(s,t) = 0$ for every $s,t \in V_\alpha$ such that $s \neq t$, and $\Phi(s,s) \geq 1$ for every $s \in V_\alpha$. For $s_0 \in V_\alpha$, because $\Phi(s,t)$ is a positive continuous function on $V_\alpha \times V_\alpha$, the function $G_{s_0}(t) = \Phi(s_0, t)$ is a positive continuous function on V_α. Note that $G_{s_0}^{-1}(\{0\}) = V_\alpha \setminus \{s_0\}$. Therefore singletons are open in V_α. This implies that V_α is discrete.

Recall that $\Omega = \bigcup_\alpha V_\alpha$ where, for each α, V_α is an open subset of Ω. Thus Ω is discrete. □

Example 7.6. We now use Selinov's examples, from [32], of some biprojective Banach algebras to construct different Banach algebras which are also biprojective

Let Ω be a topological space with the discrete topology. For every $t \in \Omega$, let E_t be an arbitrary Banach space of dimension $\dim E_t > 1$. Take a continuous linear functional $f_t \in E_t^*$, $\|f_t\| = 1$ and define on E_t the structure of a Banach algebra $A_{f_t}(E_t)$ with multiplication given by $ab = f_t(a)b$, $a,b \in A_{f_t}(E_t)$. For each $t \in \Omega$, choose $e_t \in E_t$ such that $f_t(e_t) = 1$ and $\|e_t\| \leq 2$. Then e_t is a left identity of $A_{f_t}(E_t)$ since for any $a \in E_t$ we have
$$e_t \cdot a = f(e_t) \cdot a = 1 \cdot a = a.$$
Consider the operator $\rho_t : A_{f_t}(E_t) \to A_{f_t}(E_t) \widehat{\otimes} A_{f_t}(E_t)$ defined $a \mapsto e_t \otimes a$. We show that ρ_t is an $A_{f_t}(E_t)$-bimodule morphism. Let $a,b,c \in E_t$, $\lambda, \mu \in \mathbb{C}$. Then

$$\rho_t(abc) = e_t \otimes abc = (e_t \otimes ab)c = (e_t \otimes f_t(a)b)c = (f_t(a)e_t \otimes b)c = a\rho_t(b)c,$$

$$\rho_t(\lambda a + \mu b) = e_t \otimes (\lambda a + \mu b) = \lambda e_t \otimes a + \mu e_t \otimes b = \lambda \rho_t(a) + \mu \rho_t(b).$$

We can see that ρ_t is bounded since

$$\|\rho_t(a)\| = \|e_t \otimes a\| = \|e_t\|\|a\| \leq 2\|a\|.$$

We finally show that $\pi_{A_{f_t}(E_t)} \circ \rho_t = \mathrm{id}_{A_{f_t}(E_t)}$. Let $a \in E_t$. Then

$$(\pi_{A_{f_t}(E_t)})(a) \circ \rho_t = \pi_{A_{f_t}(E_t)}(e_t \otimes a) = e_t a = f_t(e_t)a = a.$$

Thus $A_{f_t}(E_t)$ is a biprojective Banach algebra.

Consider the continuous field of Banach algebras $\mathcal{U} = \{\Omega, A_t, \Theta\}$ where A_t is the Banach algebra $A_{f_t}(E_t)$ with $f_t \in E_t^*$, $\|f_t\| = 1$. Let g_t be a continuous function on Ω such that $g_t(t) = 1$ and $g_t(s) = 0$ for all $s \neq t$. Since Ω has the discrete topology $\Theta = \prod_{t\in\Omega} A_t$, and so the field $e_t g_t$ such that $(e_t g_t)(s) = 0$ for all $s \neq t$ and $(e_t g_t)(t) = e_t$ belongs to the Banach algebra $\mathcal{A}$ defined by $\mathcal{U}$.

For $a \in \mathcal{A}$ we define $(g_t a) \in \mathcal{A}$ by

$$(g_t a)(s) = \begin{cases} a(t) & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases}.$$

Let $N(\Omega)$ be the set of finite subsets of Ω ordered by inclusion. For $a \in \mathcal{A}$ and $\lambda \in N(\Omega)$, define

$$y_{\lambda,a} = \sum_{t\in\lambda} e_t g_t \otimes g_t a.$$

By assumption, $\sup_{t\in\Omega} \|e_t\| \leq 2$.

Note that compact subsets of Ω have a finite number of elements.

Pick $\lambda_0 = (x_1, ..., x_{n_0}) \subset \Omega$ such that $\|a(t)\| < \frac{\varepsilon}{2}$ for $t \notin \lambda_0$. Let $\lambda_2 > \lambda_1 > \lambda_0$ where $\lambda_1 = (x_1, ..., x_{n_0}, ..., x_{n_1})$ and $\lambda_2 = (x_1, ..., x_{n_0}, ..., x_{n_1}, ...x_{n_2})$. Let η be a $(n_2 - n_1)^{th}$ root of unity.

We then have

$$\begin{aligned}
\|y_{\lambda_2,a} - y_{\lambda_1,a}\|_{\mathcal{A}_+\hat{\otimes}\mathcal{A}} &\leq \frac{1}{n_2 - n_1} \sum_{c=1}^{n_2-n_1} \| \sum_{s=n_1}^{n_2} \eta^{c(s-n_1-1)} g_{x_s} e_{x_s} \|_{\mathcal{A}} \| \sum_{s=n_1}^{n_2} \eta^{-c(s-n_1-1)} g_{x_s} a \|_{\mathcal{A}} \\
&\leq \frac{1}{n_2 - n_1} \sum_{c=1}^{n_2-n_1} \sup_{t\in\lambda_2\setminus\lambda_1} \|e_t\|_{E_t} \sup_{t\in\lambda_2\setminus\lambda_1} \|a(t)\|_{E_t} \\
&< \frac{1}{n_2 - n_1} (n_2 - n_1) \cdot 2 \cdot \frac{\varepsilon}{2} \\
&= \varepsilon,
\end{aligned}$$

which shows that $y_{\lambda,a}$ is a Cauchy net in $\mathcal{A}\hat{\otimes}\mathcal{A}$ for each $a \in \mathcal{A}$. Let us define $\rho : \mathcal{A} \to \mathcal{A}\widehat{\otimes}\mathcal{A}$ by setting, for every $a \in \mathcal{A}$,

$$\rho(a) = \lim_{\lambda} \sum_{t\in\lambda} g_t e_t \otimes g_t a.$$

Let us now show that ρ is a morphism of $\mathcal{A}$-bimodules. We first show that ρ is linear. Let $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned}\rho(\alpha a + \beta b) &= \lim_{\lambda} \sum_{t \in \lambda} g_t e_t \otimes g_t(\alpha a + \beta b) \\ &= \lim_{\lambda} \sum_{t \in \lambda} g_t e_t \otimes g_t \alpha a + \lim_{\lambda} \sum_{t \in \lambda} g_t e_t \otimes g_t \beta b \\ &= \alpha \lim_{\lambda} \sum_{t \in \lambda} g_t e_t \otimes g_t a + \beta \lim_{\lambda} \sum_{t \in \lambda} g_t e_t \otimes g_t b \\ &= \alpha \rho(a) + \beta \rho(b).\end{aligned}$$

Let $a, b \in \mathcal{A}$. Then

$$(ab)(t) = f_t(a(t))b(t),$$

and so

$$(g_t(ab))(s) = \begin{cases} f_t(a(t))b(t) & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases}.$$

Note that

$$((g_t a)b)(s) = \begin{cases} f_t(a(t))b(t) & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases}$$

also and so

$$g_t(ab) = (g_t a)b.$$

This shows that

$$\begin{aligned}\rho(ab) &= \lim_{\lambda} \sum_{t \in \lambda} g_t e_t \otimes g_t ab \\ &= (\lim_{\lambda} \sum_{t \in \lambda} g_t e_t \otimes g_t a)b \\ &= \rho(a)b.\end{aligned}$$

We note that

$$(f_t(a(t))g_t(b))(s) = \begin{cases} f_t(a(t))b(t) & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases},$$

so $g_t ab = f_t(a(t))g_t b$. Therefore

$$\begin{aligned}\rho(ab) &= \lim_{\lambda} \sum_{t \in \lambda} g_t e_t \otimes g_t ab \\ &= \lim_{\lambda} \sum_{t \in \lambda} g_t e_t \otimes f_t(a(t)) g_t b\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\lambda} \sum_{t\in\lambda} f_t(a(t)) g_t e_t \otimes g_t b \\
&= \lim_{\lambda} \sum_{t\in\lambda} a g_t e_t \otimes g_t b \\
&= \lim_{\lambda} a \sum_{t\in\lambda} g_t e_t \otimes g_t b \\
&= a\rho(b).
\end{aligned}
$$

From above $\|\rho(a)\| \leq \|e_t\|\|a\| \leq 2\|a\|$. Therefore ρ is a morphism of Banach $\mathcal{A}$-bimodules.
We now show that $\pi \circ \rho = 1_{\mathcal{A}}$. Let $a \in \mathcal{A}$. Then

$$
\begin{aligned}
\pi \circ \rho(a) &= \pi \left(\lim_{\lambda} \sum_{t\in\lambda} g_t e_t \otimes g_t a \right) \\
&= \lim_{\lambda} \pi \left(\sum_{t\in\lambda} g_t e_t \otimes g_t a \right) \\
&= \lim_{\lambda} \sum_{t\in\lambda} g_t e_t a \\
&= \lim_{\lambda} \sum_{t\in\lambda} g_t e_t a(t) \\
&= \lim_{\lambda} \sum_{t\in\lambda} g_t a(t) \\
&= a.
\end{aligned}
$$

Therefore $\mathcal{A}$ is biprojective.

Example 7.7. Let Ω be a topological space with the discrete topology. For every $t \in \Omega$, let $(E_t, F_t, \langle \cdot, \cdot \rangle_t)$ be a pair of Banach spaces with a non-degenerate continuous bilinear form $\langle x, y \rangle_t$, $x \in E_t$, $y \in F_t$, with $\|\langle \cdot, \cdot \rangle_t\| \leq 1$ and $\inf_{t\in\Omega} \|\langle \cdot, \cdot \rangle_t\| > 0$. The *tensor algebra* $E_t \widehat{\otimes} F_t$ *generated by the duality* $\langle \cdot, \cdot \rangle_t$ can be constructed on the Banach space $E_t \widehat{\otimes} F_t$ where the multiplication is defined by the formula

$$
(x_1 \otimes y_1)(x_2 \otimes y_2) = \langle x_2, y_1 \rangle_t x_1 \otimes y_2, \quad x_i \in E_t, y_i \in F_t.
$$

Choose $x_t^0 \in E_t, y_t^0 \in F_t$ such that $\langle x_t^0, y_t^0 \rangle_t = 1$, $\|y_t^0\| = 1$ and $\|x_t^0\| \leq 2$.

Consider the operator $\rho_t : E_t \widehat{\otimes} F_t \to (E_t \widehat{\otimes} F_t) \widehat{\otimes} (E_t \widehat{\otimes} F_t)$ defined on an elementary tensor by $x \otimes y \mapsto (x \otimes y_t^0) \otimes (x_t^0 \otimes y)$, $x \in E_t, y \in F_t$.

Note that

$$
\|\rho_t(x \otimes y)\| = \|(x \otimes y_t^0) \otimes (x_t^0 \otimes y)\| \leq 2\|x\|\|y\| = 2\|x \otimes y\|.
$$

Extend ρ_t linearly and on $E_t\hat{\otimes}F_t$ since ρ_t is bounded. We now show that ρ_t is an $E_t\widehat{\otimes}F_t$-bimodule morphism and that $\pi_{E_t\widehat{\otimes}F_t} \circ \rho_t = \mathrm{id}_{E_t\widehat{\otimes}F_t}$, so that $E_t\widehat{\otimes}F_t$ is a biprojective Banach algebra.

Let $a, b \in E_t\hat{\otimes}F_t$ and $\lambda, \mu \in \mathbb{C}$. We note that $\rho_t(\mu a + \lambda b) = \mu\rho_t(a) + \lambda\rho_t(b)$ by construction.

Let
$$a = \sum_{n=1}^{\infty} x_n^a \otimes y_n^a, \quad b = \sum_{n=1}^{\infty} x_n^b \otimes y_n^b.$$
We show that $\rho_t(ab) = a\rho_t(b) = \rho(a)b$.

Then
$$\begin{aligned} ab &= \left(\sum_{i=1}^{\infty} x_i^a \otimes y_i^a\right)\left(\sum_{j=1}^{\infty} x_j^b \otimes y_j^b\right) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(x_i^a \otimes y_i^a)(x_j^b \otimes y_j^b) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left\langle x_j^b, y_i^a\right\rangle(x_i^a \otimes y_j^b). \end{aligned}$$

Therefore
$$\rho_t(ab) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left\langle x_j^b, y_i^a\right\rangle(x_i^a \otimes y_t^0) \otimes (x_t^0 \otimes y_j^b).$$

Note that
$$\rho_t(a) = \sum_{n=1}^{\infty}(x_n^a \otimes y_t^0) \otimes (x_t^0 \otimes y_n^a),$$

and so
$$\rho_t(a)b = \sum_{n=1}^{\infty}(x_n^a \otimes y_t^0) \otimes \left((x_t^0 \otimes y_n^a)\left(\sum_{k=1}^{\infty} x_k^b \otimes y_k^b\right)\right).$$

Note that
$$(x_t^0 \otimes y_n^a)\left(\sum_{k=1}^{\infty} x_k^b \otimes y_k^b\right) = \sum_{k=1}^{\infty}(x_t^0 \otimes y_n^a)\left(x_k^b \otimes y_k^b\right) = \sum_{k=1}^{\infty}\left\langle x_k^b, y_n^a\right\rangle\left(x_t^0 \otimes y_k^b\right).$$

Combining this with above gives us
$$\rho_t(a)b = \sum_{n=1}^{\infty}(x_n^a \otimes y_t^0) \otimes \left((x_t^0 \otimes y_n^a)\left(\sum_{k=1}^{\infty} x_k^b \otimes y_k^b\right)\right)$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left\langle x_k^b, y_n^a \right\rangle (x_n^a \otimes y_t^0) \otimes (x_t^0 \otimes y_k^b)$$
$$= \rho_t(ab).$$

Similarly we have

$$\rho_t(b) = \sum_{n=1}^{\infty} (x_n^b \otimes y_t^0) \otimes (x_t^0 \otimes y_n^b)$$

and so

$$\begin{aligned} a\rho_t(b) &= \left(\sum_{k=1}^{\infty} x_k^a \otimes y_k^a \right) \left(\sum_{n=1}^{\infty} (x_n^b \otimes y_t^0) \otimes (x_t^0 \otimes y_n^b) \right) \\ &= \left(\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} x_k^a \otimes y_k^a \right) (x_n^b \otimes y_t^0) \otimes (x_t^0 \otimes y_n^b) \right) \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left\langle x_k^b, y_n^a \right\rangle (x_n^a \otimes y_t^0) \otimes (x_t^0 \otimes y_k^b) \\ &= \rho_t(ab). \end{aligned}$$

We now show that $\pi_{E_t \widehat{\otimes} F_t} \circ \rho_t = \mathrm{id}_{E_t \widehat{\otimes} F_t}$.

Let a be as above. Then

$$\begin{aligned} \pi_{E_t \widehat{\otimes} F_t} \circ \rho_t(a) &= \pi_{E_t \widehat{\otimes} F_t} \left(\sum_{n=1}^{\infty} (x_n^a \otimes y_t^0) \otimes (x_t^0 \otimes y_n^a) \right) \\ &= \sum_{n=1}^{\infty} (x_n^a \otimes y_i^0)(x_t^0 \otimes y_n^a) \\ &= \sum_{n=1}^{\infty} \left\langle x_t^0, y_t^0 \right\rangle (x_n^a \otimes y_n^a) \\ &= \sum_{n=1}^{\infty} (x_n^a \otimes y_n^a) \\ &= a. \end{aligned}$$

Consider the continuous field of Banach algebras $\mathcal{U} = \{\Omega, A_t, \Theta\}$ where A_t is the Banach algebra $E_t \widehat{\otimes} F_t$ with $\|\langle \cdot, \cdot \rangle_t\| \leq 1$. Let $\mathcal{A}$ be the Banach algebra defined by $\mathcal{U}$.

Since Ω has the discrete topology $\Theta = \prod_{t \in \Omega} A_t$, and so, for every $t \in \Omega$ and every $x_t \otimes y_t \in E_t \widehat{\otimes} F_t$, the field $g_t x_t \otimes y_t$, such that $(g_t x_t \otimes y_t)(s) = 0$ for all $s \neq t$ and $(g_t x_t \otimes y_t)(t) = x_t \otimes y_t$ belongs to $\mathcal{A}$. Let $N(\Omega)$ be the set of finite subsets of Ω ordered by inclusion. For every $t \in \Omega$, choose $x_t^0 \in E_t, y_t^0 \in F_t$ such that $\langle x_t^0, y_t^0 \rangle_t = 1$, $\|y_t^0\| = 1$ and $\|x_t^0\| \leq 2$. For $a = \{x(t) \otimes y(t)\}_{t \in \Omega} \in \mathcal{A}$, and $\lambda \in N(\Omega)$, define

$$y_{a,\lambda} = \sum_{t \in \lambda} (g_t x(t) \otimes y_t^0) \otimes (g_t x_t^0 \otimes y(t)),$$

and extend by linearality.

Let $\epsilon > 0$. Pick $\lambda_0 \subset \Omega$ compact such that $\|x(t)\| < \frac{\varepsilon}{2}$ and $\|y(t)\| \leq 1$ for $t \notin \lambda_0$. Let $\lambda_2 > \lambda_1 > \lambda_0$ where $\lambda_2 = (w_1, ..., w_{n_0}, ..., w_{n_1}, ...w_{n_2}), \lambda_1 = (w_1, ..., w_{n_0}, ..., w_{n_1})$ and $\lambda_0 = (w_1, ..., w_{n_0})$. We then have

$$\begin{aligned}
&\|y_{\lambda_2,a} - y_{\lambda_1,a}\|_{\mathcal{A}_+\hat{\otimes}\mathcal{A}} \\
&\leq \frac{1}{n_2 - n_1} \sum_{c=1}^{n_2-n_1} \|\sum_{s=n_1}^{n_2} \eta^{c(s-n_1-1)} (g_{w_s} x(w_s) \otimes y_{w_s}^0)\|_{\mathcal{A}} \|\sum_{s=n_1}^{n_2} \eta^{-c(s-n_1-1)} (g_{w_s} x_{w_s}^0 \otimes y(w_s))\|_{\mathcal{A}},
\end{aligned}$$

where η is a primary $(n_2 - n_1)th$ root of unity.

Then

$$\begin{aligned}
&\|\sum_{s=n_1}^{n_2} \eta^{c(s-n_1-1)} (g_{w_s} x(w_s) \otimes y_{w_s}^0)\|_{\mathcal{A}} \\
=& \sup_{t \in \Omega} \|\sum_{s=n_1}^{n_2} \eta^{c(s-n_1-1)} (g_{w_s}(t) x(w_s) \otimes y_{w_s}^0)\|_{A_t} \\
=& \max_{n_1 \leq s \leq n_2} \|x(w_s) \otimes y_{w_s}^0\|_{A_{w_s}} \\
\leq& \max_{n_1 \leq s \leq n_2} \|x(w_s)\|_{A_{w_s}} \\
<& \frac{\varepsilon}{2}.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\|\sum_{s=n_1}^{n_2} \eta^{-c(s-n_1-1)} (g_{w_s} x_{w_s}^0 \otimes y(w_s))\|_{\mathcal{A}} \\
=& \sup_{t \in \Omega} \|\sum_{s=n_1}^{n_2} \eta^{-c(s-n_1-1)} (g_{w_s}(t) x_{w_s}^0 \otimes y(w_s))\|_{A_t} \\
=& \max_{n_1 \leq s \leq n_2} \|x_{w_s}^0 \otimes y(w_s)\|_{A_{w_s}} \\
\leq& 2 \max_{n_1 \leq s \leq n_2} \|y(w_s)\|_{A_{w_s}}
\end{aligned}$$

$$\leq 2.$$

Therefore
$$\|y_{\lambda_2,a} - y_{\lambda_1,a}\|_{\mathcal{A}_+\hat{\otimes}\mathcal{A}} < \varepsilon.$$
Similarly, for $\lambda = (w_1, ..., w_n)$, we have

$$\begin{aligned}
&\|y_{\lambda,a}\|_{\mathcal{A}_+\hat{\otimes}\mathcal{A}} \\
&\leq \frac{1}{n}\sum_{c=1}^{n} \|\sum_{s=1}^{n} \eta^{cs}(g_{w_s}x(w_s) \otimes y^0_{w_s})\|_{\mathcal{A}} \|\sum_{s=1}^{n} \eta^{-cs}(g_{w_s}x^0_{w_s} \otimes y(w_s))\|_{\mathcal{A}} \\
&\leq 2 \max_{1\leq s\leq n} \|x(w_s)\|_{A_{w_s}} \max_{1\leq s\leq n} \|y(w_s)\|_{A_{w_s}} \\
&\leq 2\|a\|_{\mathcal{A}}.
\end{aligned}$$

for each $\lambda \in N(\Lambda)$ and $a \in \mathcal{A}$.

Therefore for any $a \in \mathcal{A}$, the net $(y_{a,\lambda})_\lambda$ converges in $\mathcal{A}\widehat{\otimes}\mathcal{A}$. Let us define $\rho : \mathcal{A} \to \mathcal{A}\widehat{\otimes}\mathcal{A}$ by the formula,

$$\rho(a) = \lim_{\lambda} \sum_{t\in\lambda} (g_t x(t) \otimes y^0_t) \otimes (g_t x^0_t \otimes y(t)),$$

on elementary tensors and extend by linearity and continuity on $\mathcal{A}\hat{\otimes}\mathcal{A}$.

We show that ρ is a morphism of modules. Since ρ is linear and continuous by construction and above we may prove this condition only for elementary tensors. Let $a = \{x(t) \otimes y(t)\}_{t\in\Omega}, b = \{u(t) \otimes v(t)\}_{t\in\Omega} \in \mathcal{A}$.

Note that
$$ab = \{\langle u(t), y(t)\rangle_t \, x(t) \otimes v(t)\}_{t\in\Omega},$$
and so
$$\rho(ab) = \lim_{\lambda} \sum_{t\in\lambda} (\langle u(t), y(t)\rangle_t \, g_t x(t) \otimes y^0_t) \otimes (g_t x^0_t \otimes v(t)),$$

$$\rho(a) = \lim_{\lambda} \sum_{t\in\lambda} (g_t x(t) \otimes y^0_t) \otimes (g_t x^0_t \otimes y(t)),$$

$$\rho(b) = \lim_{\lambda} \sum_{t\in\lambda} (g_t u(t) \otimes y^0_t) \otimes (g_t x^0_t \otimes v(t)),$$

$$\rho(a)b = \lim_{\lambda} \sum_{t\in\lambda} (g_t x(t) \otimes y^0_t) \otimes (\langle u(t), y(t)\rangle_t \, g_t x^0_t \otimes v(t)),$$

$$a\rho(b) = \lim_{\lambda} \sum_{t\in\lambda} (\langle u(t), y(t)\rangle_t\, g_t x(t) \otimes y_t^0) \otimes (g_t x_t^0 \otimes v(t)).$$

Hence

$$\rho(ab) = a\rho(b) = \rho(a)b.$$

We now show that $\pi \circ \rho = 1_A$. Let $a \in \mathcal{A}$. Then

$$(\pi_{\mathcal{A}} \circ \rho)(a) = \lim_{\lambda} \sum_{t\in\lambda} g_t(x(t) \otimes y_t^0)(x_t^0 \otimes y(t)) = \lim_{\lambda} \sum_{t\in\lambda} g_t(x(t) \otimes y(t)) = a.$$

Therefore $\mathcal{A}$ is biprojective.

8 Applications to $\mathcal{H}^2(A,X)$ and the splitting of extensions

In this section we look at the second continuous cohomology group. The following definitions can be found in [2, Section 2].

Let A be a Banach algebra, and let E be a Banach A-bimodule. Recall that $\mathcal{B}^n(A,E)$ is the Banach space of bounded n-linear maps from $A\times ...\times A$ into E and the elements of $\mathcal{B}^n(A,E)$ are the continuous n-cochains. We set $\mathcal{B}^0(A,E)=E$.

A map $T\in\mathcal{B}^2(A,E)$ is a 2-cocycle if

$$a\cdot T(b,c)-T(ab,c)+T(a,bc)-T(a,b)\cdot c=0$$

for all $a,b,c\in A$. Let $\mathcal{Z}^2(A,E)$ be the space of all 2-cocycles in $\mathcal{B}^2(A,E)$.

Recall that, for $S\in\mathcal{B}(A,E)$,

$$(\delta^1 S)(a,b)=a\cdot S(b)-S(ab)+S(a)\cdot b,\quad a,b,\in A.$$

Let $T\in\mathcal{B}^2(A,E)$. Then T is a 2-coboundary if there exists $S\in\mathcal{B}(A,E)$ such that $\delta^1 S=T$. Let $\mathcal{N}^2(A,E)$ be the space of all 2-coboundries in $\mathcal{B}^2(A,E)$.

The second continuous cohomology group of A with coefficients in E is defined as

$$\mathcal{H}^2(A,E)=\mathcal{Z}^2(A,E)/\mathcal{N}^2(A,E).$$

Definition 8.1. *Let A be a Banach algbera and let E be a Banach A-bimodule such that $x\cdot a=0$ for every $x\in E$, $a\in A$. We say that E is a right annihilator Banach A-bimodule.*

Definition 8.2. *Let A be a Banach algebra. An extension of A is a short exact sequence of Banach algebras and continuous homomorphisms*

$$\sum : 0\longrightarrow I\overset{\iota}{\longrightarrow} B\overset{\pi}{\longrightarrow} A\longrightarrow 0\qquad \sum=\sum(B;I).$$

If $I^2=\{0\}$ we say that the extension $\sum$ is singular. The extension $\sum$ is admissible if there is a continuous linear map $Q:A\to B$ such that $\pi\circ Q=i_A$. The extension splits strongly if there is a continuous homomorphism $\theta: A\to B$ such that $\pi\circ\theta=i_A$.

Let $\Sigma(B; I)$ be a singular extension of a Banach algebra A. Then I is not just a Banach B-bimodule, but also a Banach A-bimodule with respect to the operations

$$a \cdot x = bx, \quad x \cdot a = xb \quad (x \in I, a \in A),$$

where $b \in B$ is chosen such that $\pi(b) = a$; these operations are well defined exactly because $I^2 = \{0\}$. Conversely, let E be a non-zero Banach A-bimodule, and let $\Sigma = \Sigma(B; I)$ be a singular extension of A such that E is isomorphic to I as a Banach A-bimodule. Then the sequence Σ is a *singular extension of A by E.* Note that such an extension $\Sigma(B, I)$ always exists.

The following two theorems can be found in [13].

Theorem 8.3 ([13], Proposition IV.2.10(I))**.** *Let A be a Banach algebra. Then the following are equivalent:*

1. *$\mathcal{H}^2(A, E) = \{0\}$ for any right annihilator Banach A-bimodule E;*
2. *A is left projective.*

Theorem 8.4 ([13], Theorem I.1.10)**.** *Let A be a Banach algebra, and let E be a Banach A-bimodule. Then the following are equivalent:*

1. *$\mathcal{H}^2(A, E) = \{0\}$;*
2. *every singular, admissible extension of A by E splits strongly.*

We now apply the results of the above sections to the second cohomology group and the strong splittability of singular extensions of Banach algebras.

Proposition 8.5. *Let Ω be a locally compact Hausdorff space and let $\mathcal{U} = \{\Omega, (A_t), \Theta\}$ be a disjoint union of σ-locally trivial continuous fields $\mathcal{U}_{W_\mu}, \mu \in \mathcal{M}$ of Banach algebras. Suppose one of the following conditions hold:*

1. *Ω is not paracompact;*
2. *the Banach algebras $A_t, t \in \Omega$, are not uniformly left projective.*

Then for the Banach algebra $\mathcal{A}$ defined by $\mathcal{U}$

(i) there exists a Banach $\mathcal{A}$-bimodule X such that $\mathcal{H}^2(\mathcal{A}, X) \neq \{0\}$; and

(ii) there exists a strongly unsplittable singular admissible extension of the Banach algebra $\mathcal{A}$.

Proof. If condition (1) holds then by Proposition 3.4, $\mathcal{A}$ is not left projective. Similarly if condition (2) holds then, by Proposition 2.3, $\mathcal{A}$ is not left projective. Therefore, by Theorem 8.3, there exists a right annihilator Banach A-bimodule X such that $\mathcal{H}^2(\mathcal{A}, X) \neq \{0\}$. By Theorem 8.4, there exists a strongly unsplittable singular extension of the Banach algebra $\mathcal{A}$. □

Theorem 8.6. *Let $\mathcal{U} = \{\Omega, (K(E_x)), \Theta\}$ be an $\ell-$locally trivial continuous field of Banach algebras where, for $x \in \Omega$, E_x is a separable Banach space with a shrinking hyperorthogonal basis $(e_n^x)_{n\in\mathbb{N}} \subset E_x$. Let $\mathcal{A}$ be the Banach algebra generated by $\mathcal{U}$. Suppose that Ω is paracompact. Then $\mathcal{H}^2(A, E) = \{0\}$ for any right annihilator Banach A-bimodule E.*

Proof. By Theorem 5.26 $\mathcal{A}$ is left projective. Therefore, by Theorem 8.3, $\mathcal{H}^2(A, E) = \{0\}$ for any right annihilator Banach A-bimodule E. □

Theorem 8.7. *Let Ω be a locally compact Hausdorff space, let $\mathcal{U} = \{\Omega, (A_t), \Theta\}$ be a disjoint union of continuous fields of elementary C*-algebras $\mathcal{U}|W_\mu, \mu \in \mathcal{M}$, satisfying the σ-Fell condition and let $\mathcal{A}$ be the C*-algebra defined by $\mathcal{U}$. Suppose that Ω is not paracompact. Then*

(i) there exists a Banach $\mathcal{A}$-bimodule X such that $\mathcal{H}^2(\mathcal{A}, X) \neq \{0\}$; and

(ii) there exists a strongly unsplittable singular admissible extension of the Banach algebra $\mathcal{A}$.

Proof. By Theorem 6.21 $\mathcal{A}$ is not left projective. Therefore, by Theorem 8.3, there exists a right annihilator Banach A-bimodule X such that $\mathcal{H}^2(\mathcal{A}, X) \neq \{0\}$. By Theorem 8.4, there exists a strongly unsplittable singular extension of the Banach algebra $\mathcal{A}$. □

A Paracompact topological spaces

Here we have collected the topological material that we use throughout this thesis.

For a review of paracompact spaces see [18, p156-160].

Definition A.1. *Let $\mathcal{B}$ be a cover of a topological space Ω. $\mathcal{B}$ is said to be locally finite if every point in Ω has a neighbourhood intersecting only a finite number of set in $\mathcal{B}$.*

Definition A.2. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two systems of sets. $\mathcal{B}_1$ is said to be a refinement of $\mathcal{B}_2$ if every set from $\mathcal{B}_1$ is contained in some set from $\mathcal{B}_2$.*

Definition A.3. *A Hausdorff topological space Ω is said to be* paracompact *if every open cover of Ω has an open locally finite refinement that is also a cover of Ω.*

Example A.4. We give some examples of some paracompact spaces.

1. All compact spaces are paracompact.

2. All metric spaces are paracompact.

3. All locally compact Hausdorff second countable spaces are paracompact, see [25].

4. Let $\mathbb{R}_\ell$ be the real numbers equipped with the topology generated by the basis of all half-open intervals $[a, b)$, where $a, b \in \mathbb{R}$. The space $\mathbb{R}_\ell$ is call the Sorgenfrey line and is paracompact, see [35].

Example A.5. We give some examples of some spaces which are not paracompact.

1. Let $\beta\mathbb{N}$ be the Stone-Čech compactification of the natural numbers and let $p \in \beta\mathbb{N} \setminus \mathbb{N}$. Then $\beta\mathbb{N} \setminus \{p\}$ is not paracompact.

2. Let $\mathbb{R}_\ell$ be the Sorgenfrey line. In [35] it was shown that $R_\ell \times \mathbb{R}_\ell$, known as the Sorgenfrey plane, is not paracompact.

Note that some authors replace Hausdorff with regular in the above definition. In our case our topological spaces are always locally compact and Hausdorff and therefore regular.

Theorem A.6 ([26], Theorem 41.1)**.** *Every paracompact Hausdorff space Ω is normal.*

Theorem A.7 ([10], Theorem 7.2.4)**.** *Let Ω be a normal topological space Ω. The topological dimension of Ω is less than or equal to ℓ if every locally finite open cover of Ω possesses an open locally finite refinement of order ℓ.*

Theorem A.8 ([18], Theorem 5.28). *Let Ω be a regular topological space, then the following are equivalent:*

1. *the space Ω is paracompact;*
2. *each open cover of Ω has a locally finite refinement;*
3. *each open cover of Ω has a σ-locally finite refinement.*

Definition A.9. *Let $(U_\alpha)_{\alpha\in\Lambda}$ be an indexed open cover of a topological space Ω. An indexed family of continuous functions*

$$\mathcal{F} = \phi_\alpha : \Omega \to [0,1]$$

is said to be a partition of unity on Ω, subordinate $(U_\alpha)_{\alpha\in\Lambda}$, if

1. $\sum_{\alpha\in\Lambda} \phi_\alpha(x) = 1$ *for each x,*
2. *for each $x \in \Omega$ all but a finite number of $\mathcal{F}$ vanish outside some neighbourhood of x,*
3. *ϕ_α vanishes outside U_α for each $\alpha \in \Lambda$.*

Theorem A.10 ([26], Theorem 41.7). *Let Ω be a paracompact Hausdorff space; let $(U_\alpha)_{\alpha\in\Lambda}$ be an indexed open covering of Ω. Then there exists a partition of unity on Ω dominated by $(U_\alpha)_{\alpha\in\Lambda}$.*

Theorem A.11 ([13], Theorem A12). *Let Ω be a locally compact Hausdorff space such that there exists a continuous function $F : \Omega \times \Omega \to [0,1]$ satisfying the following properties*

(i) for every compact K, $F(s,t) \to 0$ as $t \to \infty$ uniformly for $s \in K$,

(ii) for every compact K, $F(s,t) \to 0$ as $s \to \infty$ uniformly for $t \in K$,

(iii) $F(s,s) = 1$ for every $s \in \Omega$.

Then Ω is paracompact.

By $F(s,t) \to 0$ as $t \to \infty$ uniformly for $s \in K$ we mean that for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \Omega$ such that $|F(s,t)| < \varepsilon$ for every $t \in \Omega \setminus K_\varepsilon$ and every $s \in K$.

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