Relating Formal Models of Concurrency for the Modelling of Asynchronous Digital Hardware

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Abstract

This Thesis investigates formal models of concurrency that are often used in the process of the design of asynchronous circuits, namely transition systems and Petri nets. The aim of the Thesis is to relate various classes of transition systems and nets, so that different models can be used at different design stages. We characterise three classes of transition systems: the sequential Semi-elementary Transition Systems, and two classes of step transition systems, where arcs are labelled by sets of concurrently executed events: TSENI and TSENI$_{apost}$ Transition Systems. All three classes can be employed to describe the behaviour of safe Petri nets used in circuit design. Semi-elementary Transition Systems are generated by Semi-elementary Net Systems, which are basically Elementary Net Systems with added self-loops. TSENI (TSENI$_{apost}$) Transition Systems are step transition systems generated by Elementary Net Systems with Inhibitor Arcs executed according to the a-priori (resp. a-posteriori) semantics, and called ENI-systems (resp. ENI$_{apost}$-systems). The relationship between each class of transition systems and nets is established via the notion of a region in the process of solving the synthesis problem for the appropriate class of nets. The Thesis compares the three classes of transition systems and gives examples of their use in the specification of asynchronous circuits behaviour.

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Chapter 1

Introduction

The last decade has seen a significant growth of interest in the design of asynchronous digital hardware systems [16], which offer a number of potential advantages for developing systems exhibiting high-performance and reliability. These include:

- *High modularity*: One can independently design, re-use and maintain system modules.
- *Avoiding problems with clock distribution*: The clocks generate their signals asynchronously and are not used as a global coordination mechanism; they are treated just as specific real-time components.
- *Operational scalability*: Even if one component of the system runs more slowly, the system itself does not fail; there is a significant degree of robustness to parameter variations, for example, temperature and power voltage.

One of the most active areas of research on asynchronous digital hardware is that of a formal support for their design and verification. Over the past decade, a number of different formal models have been proposed and studied for that purpose. The progress made was substantial, but in many cases the state of the art is still below that required of a fully developed formalism. In particular, there are several open problems when it comes to relating different models for representing behaviours of concurrent asynchronous digital hardware. Some models, such as Reachability Graphs and Transition Diagrams, are better suited for the representation of low-level operational behaviour of asynchronous digital hardware and its formal verification, while others, such as Petri Nets and Change Diagrams, are more adequate for representing the static structure of asynchronous digital hardware and ultimately for its automatic synthesis from high-level specifications. It is now generally accepted that rather than having a single model for the design and verification of asynchronous digital hardware, different models should be used at different stages of the design process. It is this context in which the relationship between different formal models of concurrent systems becomes a crucial issue.

1.1 Modelling Using Petri Nets

Petri nets[1] are a formalism for modelling systems with concurrent behaviour. They are usually employed as a syntax-level or system-level model. The user of such a model can specify causality, concurrency, choice and conflicts in terms of events and local conditions between events. Expressing all those and other relationships or characteristics locally cannot, however, guarantee that some global properties of the behaviour intended by the user will be automatically fulfilled by the system. Such properties, both safety and progress ones, are typically characterised at a semantical or behavioural level, namely the level of states and transitions. Transition systems[2] provide a means for explaining the operational semantics of system behaviour. It is often the case that the designer prefers to use a transition system to capture the intended behaviour. For example, in [45, 49] the key aspects of synchronisation between two pipelines with data flowing in two opposite directions were much easier to define in the form of a state graph. Since many methods and tools for asynchronous hardware design are based on Petri nets [18, 32, 33, 42], an important task is to synthesise a Petri net model from a state-based description. The pipeline example and other examples of circuit synthesis, which involve transformations between concurrency models and Petri nets were presented in [22]. Most of them require

[1] A Petri net system (or place/transition system) is a quadruple (S, T, W, M_{in}), where: S is a set of places, T is a set of transitions disjoint from S, $W : (S \times T) \cup (T \times S) \to N$ is a weight function and $M_{in} : S \to N$ is the initial marking. In the graphical representation for Petri nets, places are depicted by circles and transitions by boxes. An arrow is drawn from $s \in S$ to $t \in T$ (from t to s) if $W(s,t) > 0$ ($W(t,s) > 0$); it is labelled with the value of $W(s,t)$ ($W(t,s)$). The initial marking is distinguished by placing $M_{in}(s)$ tokens (dots) in every circle corresponding to $s \in S$. Places in a Petri net can be interpreted as local states. Global states are called markings and are given by any function $M : S \to N$. A transition t is enabled at a marking M if for all $s \in S$, $M(s) \geq W(s,t)$. An enabled transition can be executed (can fire), producing a new marking M' such that for every $s \in S$: $M'(s) = M(s) - W(s,t) + W(t,s)$. A Petri net is pure if for all $s \in S$ and $t \in T$, $W(s,t) \cdot W(t,s) = 0$, and bounded if there is $k \in N$ such that for every reachable marking M (obtained by firing a sequence of transitions starting from M_{in}) and for all $s \in S$, $M(s) \leq k$. A Petri net is safe if it is bounded with $k = 1$.

[2] Transition systems are usually defined as labelled graphs (S, E, T, s_{in}), where: S is a set of states, E is a set of events, $T \subseteq S \times E \times S$ is a set of transitions (not to be confused with transitions in Petri nets), and s_{in} is the initial state.

passing through an intermediate semantic level of transition systems.

1.2 The Synthesis Problem

In general, the synthesis problem for Petri nets is to construct a Petri net for a given transition system in such a way that the reachability graph of the net is isomorphic to the transition system. This problem was solved for the class of Elementary Net Systems[3] in [27], using the notion of a *region* which links nodes of transition systems (global states) with conditions in the corresponding nets (local states). In other words, regions in a transition system correspond to the places (conditions) of the synthesised net. The solution was later extended to pure and bounded Place/Transition Nets [13], general Petri Nets [37], Safe Nets [47], Flip-Flop Nets [43] and Elementary Net Systems with Inhibitor Arcs [17, 39, 40, 41]. It turned out that using all possible regions leads to exponential synthesis algorithms and, in [7], it was proved that the synthesis problem for the class of Elementary Net Systems is NP-complete. Other synthesis methods were discussed in [12] and [26], following the idea that not all the regions are actually needed. Practical algorithms for the synthesis problem were studied in, e.g. [5], [6] and [22]. It was stressed in [22] that the class of Petri nets normally used for hardware synthesis is restricted to Safe Nets. Such nets are closely related to Elementary Net Systems, whose transition systems have been studied in [38]. With a certain re-formulation, the results of [38] have been found amenable to symbolic manipulation (based on binary decision diagrams) yielding algorithms and software for hardware design [22].

[3]An elementary net system is a quadruple (B, E, F, c_{in}), where: B is a set of conditions (places), E is a set of events (transitions) disjoint from B, $F \subseteq (B \times E) \cup (E \times B)$ is a flow relation and $c_{in} \subseteq B$ is the initial case. In diagrams, conditions are depicted as circles, events as boxes, and elements of the flow relation as directed arcs. The initial case is indicated by single tokens placed in every $b \in c_{in}$. The global states of an elementary net system are called cases and are any subsets $c \subseteq B$. An event e is enabled at a case c if all its input conditions (pre-conditions) are in c and all of its output conditions (post-conditions) are not. An enabled event e can be executed and produce a new case by removing tokens from the pre-conditions of e and placing a token in every post-condition of e. An elementary net system is contact-free if for every case c, which can be reached from the initial case c_{in} by firing a sequence of events, and every event e the following hold: if all pre-conditions of e are marked (are in c) then none of its post-conditions is marked. The firing rule for the Elementary Net Systems differs from the one for the Petri Net Systems. Nevertheless, for every safe and pure Petri net one can construct an elementary net with the same behaviour. Conversely, any contact-free elementary net can be viewed as a safe net, and for a non contact-free one an equivalent safe net can be built using complementary places.

1.3 Synthesis Problems Considered in this Thesis

We will now outline the contributions made by this Thesis, and compare them with related work in the area of net synthesis.

We here solve the synthesis problem for various extensions of Elementary Net Systems which still remain in the class of safe nets. The first class, discussed in chapter 2, are the Semi-elementary Net Systems which extend the Elementary Net Systems by allowing self-loops.[4] The transition systems generated by them are the Semi-elementary Transition Systems, and are defined by a system of axioms which differs slightly from that introduced for the Elementary Transition Systems in [38]. For this extension, there was no need to modify the definition of a region and we use the one from [38].

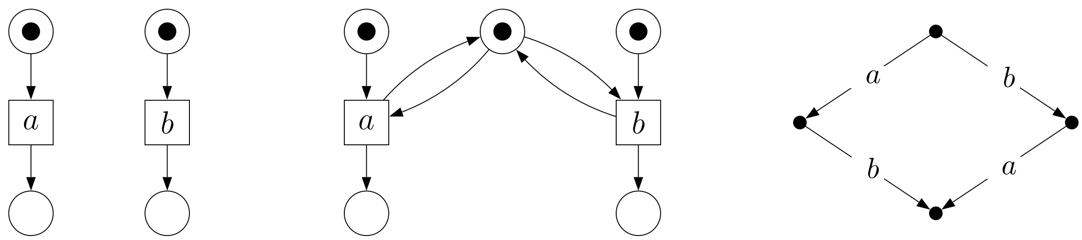
Semi-elementary Transition Systems are sequential transition systems (i.e. transitions are labelled with individual events) without self-loops.[5] The latter means that every transition changes the state of a system. Such a restriction is justified when designing asynchronous circuits. In the asynchronous interpretation of a system behaviour, we can only state that one transition takes place in a finite but unbounded amount of time after some other transition, but we cannot tell the exact time at which each of the transitions takes place. In the synchronous interpretation, each transition takes place during a tick of the clock, so that two consecutive transitions are executed during two consecutive clock ticks. Asynchronous systems can also be modelled as synchronous systems (see [2]), by adding 'wait' or 'null' transitions in the form of self-loops at some states in which the system remains for an arbitrary period of time. Therefore, transition systems without self-loops seem more appropriate for the design of asynchronous systems where empty (null) transitions are not needed and every event is treated as significant (changing the state of the system). Due to this restriction (no self-loops in the transition systems) Semi-elementary Net Systems cover almost (but not exactly) the class of Safe Nets. This restriction was dropped in [46] where the thus extended class of the Semi-elementary Net Systems coincided with the class of Safe Nets.

In [28], it was shown that Elementary Transition Systems, despite being sequential, contain all the necessary information about the concurrency in the systems they model.

[4] A self-loop in a semi-elementary net is a pair consisting of an event $e \in E$ and a condition $b \in B$ such that $(b, e), (e, b) \in F$. Such a b is sometimes called a side-condition.

[5] A self-loop in a transition system is created by a transition $(s, e, s) \in T$ with the same source and target.

More precisely, it is possible to deduce from the sequential graph of the system behaviour which two events can be executed concurrently. However, this is not the case for the Semi-elementary Transition Systems, and two semi-elementary net systems with different independence relation on events can generate the same semi-elementary transition system, as shown below. In the net depicted in (a), the events a and b are concurrent, while



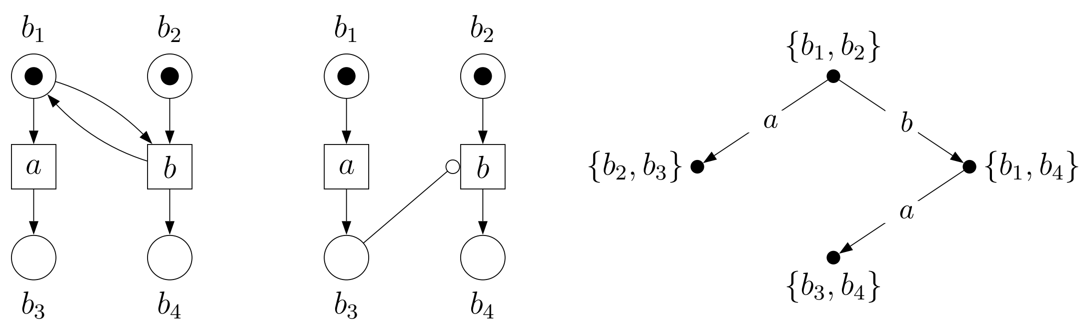
(a) (b) (c)

in the net depicted in (b), they are mutually exclusive. However, the transition systems produced by both nets are the same (see (c)), showing that the information about the concurrency between events in the system is lost when passing from nets to transition systems. In section 2.3 of chapter 2, we address this problem by introducing an additional stage of the design during which self-loops generated in the process of net synthesis can be re-interpreted. More precisely, some of the self-loops can be removed, if they create unnecessary constraints on the concurrency within the system. Other are left to act as standard self-loops, allowing an event connected to a condition by such a self-loop to consume the token from the condition, while being executed, and return it back afterwards. The third option for the re-interpretation of a self-loop in the synthesised net is to change it to a 'positive' context arc (see [36]), which means that an event connected in this way to a condition can be executed when the condition is marked, but without consuming the token. This interpretation is useful in the circuit design, as it captures the situation when an action modelled by an event is that of checking the state of the wires (it does not change this state, however).

A different approach to the concurrency aspect of the synthesis problem was presented in [47], where the information about concurrency between events was included in the definition of a transition system: the Asynchronous Transition Systems introduced there are sequential transition systems for which there is an explicit independence relation defined on events. This is presumably a model which is the closest to those presented in this Thesis in the sense that Asynchronous Transition Systems are synthesised to Safe Nets.

1.4 Step Transition Systems

Another way of overcoming the limitation of Semi-elementary Transition Systems with respect to the modelling of concurrency is to use step transition systems, whose transitions are labelled by sets of concurrently executed events. The rest of this Thesis (chapters 3-8) considers such transition systems. Moreover, we change the way of modelling non-elementarity in nets. Instead of using self-loops, we employ inhibitor arcs (arcs ending with a small circle), like the one between condition b_3 and event b in the diagram (b) below, which indicates that b can only be executed if b_3 is empty. It can be seen that the sequential behaviour of a net with a self-loop in (a) and the one with an inhibitor arc in (b) is the same (as shown in (c)).



(a) (b) (c)

Chapters 3-8 deal with the synthesis problem for the Elementary Net Systems with Inhibitor Arcs derived from step transition systems. According to [19], the non-sequential behaviour of nets with inhibitor arcs can be interpreted in two different ways which opens up the possibility to choose the semantics more suitable for the application at hand. The first interpretation, called the a-posteriori semantics, treats events as instantaneous (their occurrence takes zero time), while the second one, the a-priori semantics, assumes that events take some time to complete. Which of the two semantics should be applied depends on the properties of events which the net is supposed to model, and on the properties of the enabling mechanism (see [19, 29]). Although Elementary Net Systems with Inhibitor Arcs can be represented by Safe Nets with self-loops, the synthesis problem for these nets was only solved for some variants of the a-posteriori semantics [37, 47]. The main contribution of this Thesis is a solution of the synthesis problem for the Elementary Net Systems with Inhibitor Arcs executed under the a-priori semantics (ENI-systems) and a complete characterisation (axiomatisation) of the transition systems generated by them (TSENI Transition Systems). This is achieved in chapters 3-6. The a-posteriori semantics for nets with inhibitor arcs is then considered in chapter 7 (the nets are called ENI_{apost}-systems and their transition systems TSENI_{apost} Transition Systems). The two semantics are then compared in chapter 8.

The synthesis of nets from step transition systems was first considered in [37] to deal with the concurrent behaviour of the general Petri nets, and later in [3]. The transitions of the PN-transition Systems in [37] are labelled with finite multisets over the set of events E (empty steps and autoconcurrency are allowed). In [3], there are two definitions of step transition systems. The first one defines a step transition system as a sequential transition system equipped with a binary relation between states and finite multisets of events. Although the transitions are still labelled with single events like in Asynchronous Transition Systems, this additional relation carries information not only about the independence between events (like in Asynchronous Transition Systems) but also about states at which particular multisets of events are enabled. The second definition introduces step transition systems over a commutative monoid. These have transitions labelled with the elements of a monoid and in that they extend the original definition of step transition systems of [37] (the set of finite multisets over E used for labelling transitions in [37] constitutes the free commutative monoid generated by E). The step transition systems of [37] and [3] satisfy the 'intermediate state' property which states that every step can be split into two consecutive substeps:

$$s \xrightarrow{\alpha+\beta} s' \quad \Rightarrow \quad \exists s'' \in S \;:\; s \xrightarrow{\alpha} s'' \;\wedge\; s'' \xrightarrow{\beta} s'.$$

In [37], the above property follows from the two regional axioms defining PN-transition Systems, while in [3] it is a part of the definition. Step transition systems defined in this Thesis have their transitions labelled with finite sets of events. We do not allow empty steps, while autoconcurrency is ruled out by an axiom which demands that there are no self-loops in transition systems. Such an axiom means that in the synthesised (safe) inhibitor net every event has at least one pre-condition, which guarantees nonautoconcurrent behaviour of the event. Hence, sets of events are used as steps instead of multisets. In [47], steps are defined as sets as well, but there it was due to the fact that the independence relation determining which two events can be executed concurrently is irreflexive. To ensure that steps of TSENI and TSENI$_{apost}$ Transition Systems represent sets of concurrent events, we introduce step axioms which relate the enabledness of a step at some state with the enabledness of the events in the step at this state. For a more restrictive definition of a step in TSENI$_{apost}$ Transition Systems, an additional step axiom is needed which relates a step with its target state. None of the step axioms involves sequences of events. The 'intermediate state' property is not a part of the definition of TSENI nor TSENI$_{apost}$ Transition Systems, but for the latter it follows from the defining axioms. TSENI Transition Systems do not satisfy the 'intermediate state' property, and thus they are different from any other class considered in the literature.

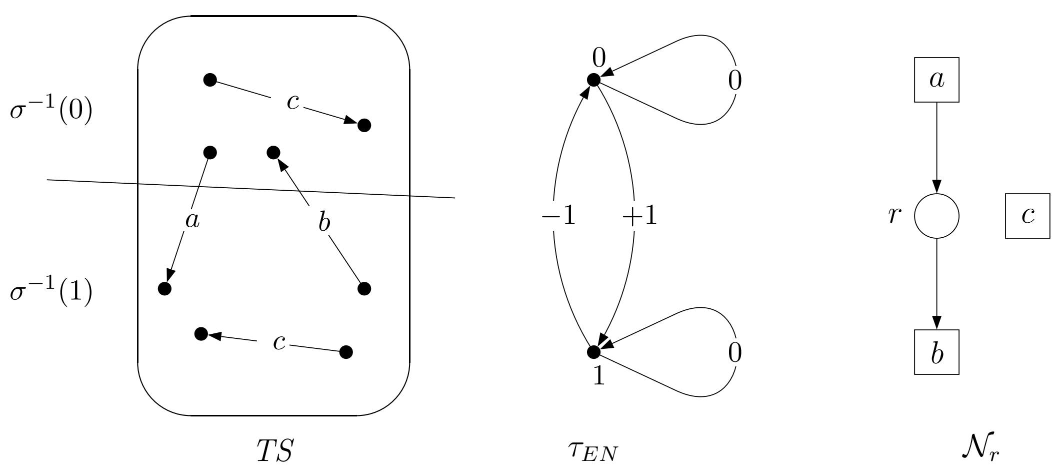
1.5 Regions

In order to synthesise inhibitor nets from step transition systems, we introduce a new definition of a region. The standard definition of a region for the Elementary Transition Systems [27, 38], which is adopted for the sequential Semi-elementary Transition Systems, states that a region is a set of states in a transition system with which all transitions labelled with the same event have the same 'crossing' relationship (either they all enter the region, or all exit it, or all do not cross its 'border'). We generalise this definition to cope with sets of events (steps). A region r is now a set of states such that the 'crossing' relationship of a step with respect to r depends on containing some special event. For a fixed step and a fixed region this special event is unique.

There are other approaches in the literature to define regions for more complex nets, and to deal with step transition systems. In the case of Asynchronous Transition Systems [47], a region (called a condition) is a subset of states (treated as idle transitions) and transitions. For the purpose of synthesising pure bounded Place/Transition Nets [13], the notion of a region was generalised to a multiset of states with which every transition carrying the same label has the same 'gradient' (each transition with the same label translates uniformly the multiplicities of the region). The standard definition of a region is then a special case, when multiplicities are 1 or 0, and can be used to determine whether a particular state is inside (1) or outside (0) a region treated as a set. Yet another definition of a region was used for the synthesis of general Petri Nets from step transition systems [37]. As previously, a region aims to characterise a place in a synthesised net, and is represented by two functions. The first function is defined on states of a transition system and specifies the number of tokens held in a place (corresponding to the region described by the function) under markings which correspond to the states. The second function is defined on the set of events and returns two values which represent the 'weights' of arcs in the synthesised net between the place (region) and events.

Regions as Morphisms

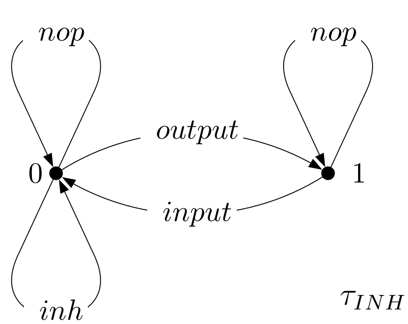
In [8], a uniform approach to the synthesis problem was proposed based on regions defined as morphisms from a given transition system into a classifying transition system - called a type of nets - which describes the behaviour of the class of nets under consideration. To synthesise a net from a sequential transition system $TS = (S, E, T, s_{in})$ one needs to find regions in the form of morphisms $(\sigma, \eta) : (S, E, T) \to \tau$, where τ is an uninitialised classifying transition system.[6] For Elementary Transition Systems, a region $r \subseteq S$ can be represented as a morphism (σ, η), where σ is a characteristic function $\sigma = \chi_r : S \to \{0, 1\}$ and $\eta : E \to \{-1, 0, 1\}$ is such that $\eta(e) = \sigma(s') - \sigma(s)$ for every transition $s \xrightarrow{e} s'$ in T. The classifying transition system is then $\tau_{EN} = (\{0, 1\}, \{-1, 0, 1\}, \{0 \xrightarrow{0} 0, 0 \xrightarrow{1} 1, 1 \xrightarrow{-1} 0, 1 \xrightarrow{0} 1\})$. The diagram below, taken from [8], illustrates the concept of regions as morphisms for Elementary Transition Systems.



[6]A morphism for uninitialised transition systems $(\sigma, \eta) : (S, E, T) \to (S', E', T')$ is a pair of maps $\sigma : S \to S'$ and $\eta : E \to E'$ such that $s \xrightarrow{e} s'$ in T implies $\sigma(s) \xrightarrow{\eta(e)} \sigma(s')$ in T'.

By identifying every place in the net being generated with such a morphism, one can build atomic nets $\mathcal{N}_r = (\{r\}, E, F, c_{in})$, where the flow relation F is given by η ($\eta(a) = +1$ means that a deposits a token in r; $\eta(b) = -1$ means that b consumes a token from r; and $\eta(c) = 0$ means that c does not make any changes in r) and the initial case, c_{in}, by σ (r is initially marked if $s_{in} \in r = \sigma^{-1}(1)$). Solving the synthesis problem amounts then to gluing a set of such atomic net systems (by identifying common events), provided that the set of regions is admissible, i.e. it contains witnesses for the satisfaction of every instance of two separation axioms. The first one, called the state separation property, ensures that there are enough regions to distinguish every two different states in the transition system. The second axiom, usually referred to as the event/state separation property, states that for every event e and every state s at which this event is not enabled, there is a region which disallows e at s. For all classes of nets, whatever the definition of a region, a solution to the synthesis problem requires some variant of these regional axioms to be fulfilled. When representing regions as morphisms, these two axioms can uniformly be expressed in terms of functions σ and η, for all types of nets τ.

A classifying transition system, τ, which is needed to define regions as morphisms, characterises the behaviour of synthesised nets. More precisely, it describes all possible 'quantitative' changes which may happen in a place of a net of that type. For pure nets (those without side-conditions), the definition of a type allows to determine the function η of a morphism (region) from the function σ. Since the values of σ represent the number of tokens in a place modelled by the morphism, in pure nets there is only one way of interpreting the changes in that place. The execution of an event can add tokens to the place, or take some away, or leave it unaffected. With the introduction of side-conditions, the uniqueness of η for a given σ is no longer true, as the same change in the number of tokens in a place can result from different combinations of tokens being added and removed at the same time. Since the function η is responsible for the modelling of relationships between events and the place modelled by (σ, η), care must be taken in order that one can distinguish between a place being a side-condition of an event, and not connected to that event at all. This is another occasion to observe that non-pureness may lead to ambiguity in interpreting the relationship between events and places and, as a consequence, to an ambiguity in interpreting the independence relation on events. As a result, the function η for non-pure nets is more complex than that needed for the regions of pure nets. Nevertheless, there is still a clear quantitative interpretation of the changes that can take place in side-conditions of non-pure nets. They can be viewed as two consecutive changes of consuming (removing) and producing (adding) tokens. This suggest that the values of η for non-pure regions should be represented by pairs of numbers rather than by single numbers. When it comes to the inhibitor nets, it is difficult to give a quantitative interpretation of the inhibitor arc between a place and an event. The execution of such an event requires the place to be empty and does not change the number of tokens. Thus, from the quantitative point of view, it behaves like an event which is not connected to the place. To distinguish between these two situations, the function η should be provided with more information, in case of returning 0 (or no change) for some e. We need to inspect all the transitions labelled by e in the transition system, and the values of σ for their sources and targets. Although the Elementary Net Systems with Inhibitor Arcs can be synthesised under the general framework where regions are represented as morphisms with the classifying transition system τ_{INH} (see below the diagram taken from [4]), there is still a difficulty with interpreting the action of inhibition in a quantitative sense. Hence the values of η, labelling arcs in the diagram of τ_{INH}, were given names (representing the relationships between a place of an inhibitor net and events, and so the possible changes in the place caused by the executions of the events) rather than numerical values. For an event e, $\eta(e) = \mathit{input}$ means that the place corresponding to (σ, η) is a pre-condition of e; $\eta(e) = \mathit{output}$ means that the place (σ, η) is a post-condition of e; $\eta(e) = \mathit{inh}$ means that the place (σ, η) is connected to e by an inhibitor arc; and $\eta(e) = \mathit{nop}$ states that there is no connection between e and the place modelled by (σ, η).



The type τ_{INH} describes the sequential firing rule of inhibitor nets (the changes in a place caused by single events). In [3], the idea of types of nets was generalised to extended or enriched types of nets in order to define regions as morphisms in step transition systems.

1.6 Minimisation of Synthesised Net

The basic net solution to the synthesis problem is constructed using all the regions as places (conditions), and the result is usually called a saturated net system [27, 38, 37, 39]. Such a solution is generally exponential in size of the original transition system. In [26], a method was presented for constructing all net solutions to the synthesis problem for the Elementary Net Systems, and especially those which are polynomial in the size of the transition system. This method is general enough to be applicable to other classes of transition systems and nets, and it is based on an idea of admissible sets of regions, which are sets of regions sufficient for building net solutions for a given synthesis problem (the transition systems generated by any net solution is isomorphic to the transition system generated by the saturated net). It was proved in [26] that the set of regions is admissible if it contains witnesses for the satisfaction of every instance of the regional separation axioms. A possible admissible set of regions of an elementary transition system is the set of minimal regions (minimal with respect to set inclusion). The properties of minimal regions and their use in the synthesis of Elementary Net Systems were investigated in [12]. In particular, it turned out that net systems constructed on the basis of minimal regions (minimal nets) are contact-free, and are decomposable into state machine components. The admissibility of the set of minimal regions was proved for the synthesis of Elementary Net Systems with Inhibitor Arcs as well [17, 40]. In [17], where the sequential behavior of Elementary Net Systems with Inhibitor Arcs was studied, the ability to decompose a minimal net into state machine components was used to detect superfluous inhibitor arcs. In this Thesis, we prove that minimal regions form an admissible set for the synthesis of Elementary Net Systems with Inhibitor Arcs from step transition systems. The elimination of superfluous inhibitor arcs is then done by taking advantage of properties of the non-minimal regions. The method based on non-minimal regions does not depend on the decomposability of a net into state machine components, and therefore can possibly be generalised to non-safe nets. Minimal regions were important for the synthesis of pure safe nets from Excitation-closed Transition Systems (a class wider than Elementary Transition Systems) [23]. The aim of that work was to design an efficient algorithm for the synthesis of Petri nets, and the structure of minimal regions was investigated to find the best way to generate minimal nets. The role of minimal regions was reflected in the form of the second regional axiom (the first one is not used) for Excitation-closed Transition Systems which, if satisfied, is automatically satisfied for the set of minimal regions. The problem of finding an irredundant set of regions (minimal admissible set of regions, where none of the regions is redundant), and eventually an optimal solution to the synthesis problem (generated on the basis of irredundant set of regions and fulfilling some additional requirements, e.g., having the minimal number of places or being contact-free), was considered in [26] for Elementary Transition Systems, and in [23] for Excitation-closed Transition Systems.

1.7 Other Formulations of the Synthesis Problem

We deal with the synthesis problem in its classical form, i.e. we aim to obtain a Petri net $\mathcal{N}$ from a transition system TS in such a way that the transition system generated by $\mathcal{N}$ is isomorphic to TS. However, there are also other ways of defining the synthesis problem. In [5], algorithms were given to solve the synthesis problem for bounded nets from transition systems as well as from regular languages. The synthesis problem from languages (the synthesised net should behave according to a given language) is weaker than that from transition systems, and requires only a variant of the event/state separation axiom to be fulfilled. Further generalisation of the above problem was introduced in [24], where (possibly) unbounded nets are synthesised from specifications given by two regular languages which constitute a lower and upper bound for the language of the synthesised net. The second problem considered in [24] was the synthesis of nets from deterministic context-free languages. In [21, 23, 46], the net realisation of a transition system is sought up to some form of bisimilarity rather than isomorphism. Again, this weaker definition of the synthesis problem allows one to ignore the state separation property and, in consequence, to consider wider class of transition systems.

1.8 A Summary of the Contributions

In general, the work presented in this Thesis can be seen as contributing to the theory of synthesis of concurrent systems, in particular asynchronous circuits, as well as the theory of relationship between different models of concurrency. On the technical level, the Thesis is concerned with two models of concurrency, namely transition systems and Petri nets.

The relationship between these models is developed using techniques derived from the theory of regions as well as category theory. The detailed contributions are as follows:

- The definition and axiomatisation of Semi-elementary Transition Systems. This class of transition systems is an extension of the Elementary Transition Systems which allows to model the behaviour of asynchronous circuits. (Chapter 2)
- The synthesis procedure for the Semi-elementary Net Systems. (Chapter 2)
- The definition of behavioural transformations for the Semi-elementary Transition Systems and the proof that these transformations do not lead outside the class of Semi-elementary Transition Systems. The transformations are based on identifying a subgraph in a transition system called a ladder. Ladder-like structures are typical for asynchronous hardware behaviours with conflicts. (Chapter 2)
- The definition of a region for step transition systems generated by nets with inhibitor arcs. (Chapter 3)
- The definition and axiomatisation of TSENI Transition Systems. TSENI Transition Systems are particularly useful for the modelling of asynchronous controller and arbiter behaviour. (Chapter 3)
- The synthesis procedure for the ENI-systems. (Chapter 3)
- The definition of behaviour preserving morphisms for TSENI Transition Systems and ENI-systems. (Chapter 4)
- The definition of the category of TSENI Transition Systems and the category of ENI-systems. (Chapter 5)
- The definition of functors between the category of ENI-systems and category of TSENI Transition Systems, and the proof that they form an adjunction (precisely coreflection). (Chapter 5)
- A proof that minimal regions constitute an admissible set for the synthesis of ENI-systems. This is an important result from the point of view of extending the Petrify tool [22] to incorporate inhibitor arcs. The algorithms used by the Petrify tool rely on the fact that minimal regions are sufficient. (Chapter 6)

- A novel method for eliminating superfluous inhibitor arcs from the synthesised nets. This method is more operational than the other one presented in the literature, because it does not depend on the decomposability of a synthesised net into state machine components. (Chapter 6)

- The definition and axiomatisation of TSENI$_{apost}$ Transition Systems. (Chapter 7)

- The synthesis procedure for the ENI$_{apost}$-systems. (Chapter 7)

Some of the results contained in this Thesis were presented at two international conferences, [39, 40], and published as a journal article [41].

1.9 Outline of the Thesis

In section 2.1, we introduce Semi-elementary Transition Systems and the corresponding class of nets, following the treatment presented in [38]. In section 2.2, we study various operations which can be performed on Semi-elementary Transition Systems and their impact on the structure of the corresponding nets. Section 2.3 discusses different ways in which self-loops in nets can be interpreted, depending on the degree of concurrency exhibited by the system. Section 2.4 provides an illustration of the use of the presented theoretical results in deriving a semi-elementary net system for a transition system defining the behaviour of a counterflow pipeline synchronisation from [45]. Chapter 2 closes with section 2.5 which discusses the possibility of allowing self-loops in the Semi-elementary Transition Systems.

Section 3.1 introduces TSENI Transition Systems, and in section 3.2, we establish their basic properties. Section 3.3 recalls the syntax and semantics of ENI-systems, and the next section shows that all the transition systems generated by ENI-systems are TSENI transition systems. In section 3.5, we provide a construction of an ENI-system for a given TSENI transition system. Section 3.6 contains the proof of consistency of translations between ENI-systems and TSENI Transition Systems.

Section 4.1 introduces behaviour preserving morphisms for the TSENI Transition Systems, and section 4.2 introduces a class of behaviour preserving morphisms for the ENI-systems.

In section 5.1, we define the category of ENI-systems ($\mathcal{CAT}_{ENI}$), and the category of

TSENI Transition Systems ($\mathcal{CAT}_{TSENI}$). In sections 5.2 and 5.3, functors between the categories $\mathcal{CAT}_{TSENI}$ and $\mathcal{CAT}_{ENI}$ are introduced, and section 5.4 contains the proof that the two functors form an adjunction.

Section 6.1 examines properties of regions and minimal regions of TSENI Transition Systems. In section 6.2, we define for a given TSENI transition system a net which uses only minimal regions ($\mathcal{N}_{Min}$), and prove that it constitutes an ENI-system. Section 6.3 examines the relationship between $\mathcal{N}_{Sat}$ (the ENI-system obtained from the original construction, which uses all the regions) and $\mathcal{N}_{Min}$, by defining a net morphism between the two nets. It is proved that the transition systems generated by both nets are isomorphic. Section 6.4 looks at the possibility of a further minimisation of $\mathcal{N}_{Min}$, by removing some of its inhibitor arcs.

Chapter 7 discusses TSENI$_{apost}$ Transition Systems and ENI$_{apost}$-systems in the same way as the transition systems and nets for the a-priori semantics were treated in chapter 3.

In chapter 8, we study the relationship between transition systems that arise from the a-priori and a-posteriori semantics.

Finally, chapter 9 discusses the directions for further research.

Chapter 2

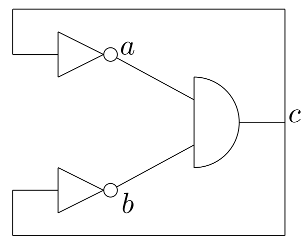
Synthesis of Nets with Self-loops

In this chapter we deal with *Semi-elementary Net Systems* which are basically the Elementary Net Systems of [38] with added self-loops. That such an extension is necessary can be shown in the following way. Imagine that an event is caused by (pre-)conditions some of which should remain true when the event is executed. In other words, the execution of an event in some operational case of an elementary net system should not necessarily make one of its pre-conditions false. A simple example of such an effect is shown in figure 2.1. Here, the state of the output of each gate[1] is represented by a pair of conditions, one for the True value and the other for False. Thus, the events labelled with rising (e.g., $a+$) and falling (e.g., $a-$) transitions have the above conditions as preconditions, which is depicted by single arcs directed from the conditions to events. On the other hand, each gate has inputs which are the outputs of other gates and the state of these inputs determines the circumstances under which the gate's output changes its state. In terms of the net model this means that each event must also include as its preconditions the conditions modelling the state of its inputs. However, since the change of the output of the gates does not affect the change of its inputs, it would be irrational to assume that such a precondition has to become False once the event has been executed. A consistent way to adequately model this situation would be to declare such a precondition also as a postcondition of the event. Note that in the net model of the circuit such relationship between conditions and events is depicted by two single-headed arcs - an obvious situation of a *self-loop* in the net[2].

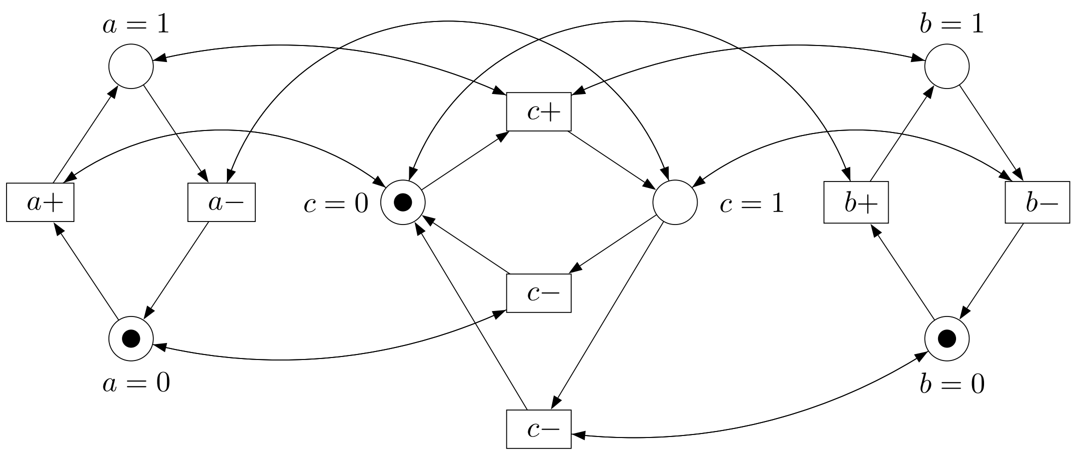
The above example, requiring the use of self-loops in nets (historically, Petri nets with self-loops have been called non-pure), can be easily extended to modelling situations where

[1] The list of gate symbols is given in the appendix.

[2] For simplicity, bidirectional arcs are often used to represent self-loops in Petri nets.



(a)



(b)

Figure 2.1: Example of a net with self-loops.

a circuit interacts with its environment through input and output signals. The transitions of input signals are associated with events which set to True the preconditions of events modelling the internal or output signal transitions. In some cases, these conditions have to remain True until some other event in the environment resets them to False. While self-loops are intuitively obvious at the syntactic level, it often happens that a behaviour of that sort is easier to capture at the transition system level [45, 49].

In the view of the above practical motivation, we present in this chapter a simple extension of the class of Elementary Net Systems to Semi-elementary Net Systems by allowing self-loops to be present[3]. Similarly, we formulate axioms for transition systems so as to check their semi-elementarity. Note that *Semi-elementary Transition Systems* are simply those generated by Semi-elementary Net Systems.

We aim at a characterisation of certain local transformations defined for transition systems which do not lead outside the class of Semi-elementary Transition Systems. The transformations we are interested in correspond to adding self-loops in the associated semi-elementary nets and, as a result, they are particularly easily implementable. The transformations are based on identifying a specific pattern (subgraph) in a transition system, called a *ladder*, from which it is possible to delete some (but not all) transitions, called *rungs*, without disturbing the rest of the transition system. It is important to stress that ladders are typical for asynchronous hardware behaviours with conflicts, such as arbitration and latches with independent clocks.

2.1 Semi-elementary Transition Systems and Nets

In this section we prepare the necessary formalism for the discussion in section 2.2. Essentially, we present a simple extension to the model of Elementary Transition Systems of [38] which allows one to consider non-pure nets (i.e. nets with self-loops). The resulting model is sufficient for our purposes although it is worth noting that there are other ways of representing non-pure nets using more expressive classes of transition systems, as discussed at the end of section 2.3. The presentation in this section closely follows that in [38]. The proofs are not included as they can easily be obtained by suitably modifying those found in [38].

[3]In fact, this is almost an extension to Safe Nets since for each place in a semi-elementary net system a unique complement place can be added without changing the net's behaviour.

2.1.1 Transition Systems

A *transition system* is a quadruple $TS = (S, E, T, s_{in})$, where S is a non-empty finite set of *states*, E is a finite set of *events*, $T \subseteq S \times E \times S$ is the *transition relation*, and $s_{in} \in S$ is the *initial state*. We assume TS satisfies the following three conditions (or *axioms*):

AX1 For every $(s, e, s') \in T$, $s \neq s'$.

AX2 For every $e \in E$ there are $s, s' \in S$ such that $(s, e, s') \in T$.

AX3 For every $s \in S \setminus \{s_{in}\}$ there are $(s_i, e_i, s_{i+1}) \in T$, for $i = 0, 1, \ldots, n$, such that $s_0 = s_{in}$ and $s_{n+1} = s$.

We will often write $s \xrightarrow{e} s'$ if $(s, e, s') \in T$, and $s \xrightarrow{e}$ (or $\xrightarrow{e} s$) if $(s, e, s') \in T$ (resp. $(s', e, s) \in T$), for some s'. Also, for $s \xrightarrow{e} s'$, we will call s the *source* and s' the *target* of this transition. Figure 2.2 shows a transition system represented as a labelled directed graph.

Note that we relaxed some of the constraints imposed on a transition system in [38]; in particular, axiom (A2) of [38] which did not allow multiple arcs between a pair of states. This restriction was introduced to avoid the so-called non-simple nets, where different events may have the same sets of pre- and post-conditions.

Let $TS = (S, E, T, s_{in})$ be a transition system fixed throughout the rest of this section.

Definition 2.1.1 *[38] A set of states $r \subseteq S$ is a* region *of TS if the following two conditions are satisfied:*

$$\begin{array}{lll} (s, e, s') \in T \ \wedge \ s \in r \ \wedge \ s' \notin r & \Rightarrow & \forall (s_1, e, s_1') \in T : \ \ s_1 \in r \ \wedge \ s_1' \notin r \\ (s, e, s') \in T \ \wedge \ s \notin r \ \wedge \ s' \in r & \Rightarrow & \forall (s_1, e, s_1') \in T : \ \ s_1 \notin r \ \wedge \ s_1' \in r. \end{array}$$

A region different from S and $\emptyset$ will be called a non-trivial *region.* □

If we denote by R_{TS} the set of non-trivial regions then by R_s we will mean the set of non-trivial regions containing a state $s \in S$,

$$R_s = \{r \in R_{TS} \mid s \in r\}.$$

The sets of *pre-regions* and *post-regions* of an event $e \in E$ are defined as follows:

$$\begin{array}{rcl} {}^\circ e & = & \{r \in R_{TS} \mid \exists (s, e, s') \in T : \ s \in r \ \wedge \ s' \notin r\} \\ e^\circ & = & \{r \in R_{TS} \mid \exists (s, e, s') \in T : \ s \notin r \ \wedge \ s' \in r\}. \end{array}$$

Regions of a transition system correspond to conditions (local states) in the corresponding net.

Proposition 2.1.1 *[38] A set of states $r \subseteq S$ is a region if and only if its complement $\overline{r} = S \setminus r$ is a region. Moreover, for every $e \in E$, $e^\circ = \{\overline{r} \mid r \in {}^\circ e\}$.* □

Proposition 2.1.2 *[38] If $s \xrightarrow{e} s'$ then $R_s \setminus R_{s'} = {}^\circ e$ and $R_{s'} \setminus R_s = e^\circ$. Moreover, ${}^\circ e \subseteq R_s$ and $e^\circ \cap R_s = \emptyset$ and $R_{s'} = (R_s \setminus {}^\circ e) \cup e^\circ$.* □

The next definition characterises the relationship between an event and a region in which a transition labelled by the event is completely (i.e., both its source and target states are) inside the region.

Let $r \in R_{TS}$ be a non-trivial region and e be an event. By

$$\mathcal{B}_r^e = \{(s, e, s') \in T \mid s \in r \ \wedge \ s' \in r\}$$

we will denote the set of all the arcs labelled by e which are 'buried' in r. The set of *co-regions* of an event $e \in E$ is then defined as follows:

$$\overset{\circ}{e} = \{r \in R_{TS} \mid \mathcal{B}_r^e \neq \emptyset \ \wedge \ \mathcal{B}_{\overline{r}}^e = \emptyset\}.$$

The co-regions of an event e correspond to those conditions in the associated net which form a self-loop with e.

The following proposition adds to the properties of regions associated with an event observed in propositions 2.1.1 and 2.1.2.

Proposition 2.1.3 *If $e \in E$ and $r \in \overset{\circ}{e}$ then $r \notin {}^\circ e \cup e^\circ$ and $\overline{r} \notin {}^\circ e \cup \overset{\circ}{e} \cup e^\circ$. Moreover, if $s \xrightarrow{e} s'$ then $\overset{\circ}{e} \subseteq R_s \cap R_{s'}$.* □

Transition system TS is said to be *semi-elementary* if it satisfies, in addition to (AX1)-(AX3), the following two regional axioms:

AX4 For all $s, s' \in S$, if $R_s = R_{s'}$ then $s = s'$.

AX5 For all $s \in S$ and $e \in E$, if ${}^\circ e \subseteq R_s$ and $\overset{\circ}{e} \subseteq R_s$ then $s \xrightarrow{e}$.

Figure 2.2 shows a semi-elementary transition system TS_0 which has 4 regions: $r_1 = \{s_{in}, s_1\}$, $r_2 = \{s_2, s_3\}$, $r_3 = \{s_{in}, s_2\}$ and $r_4 = \{s_1, s_3\}$. The pre-, post- and co-regions of a and b are as follows: ${}^\circ a = \{r_3\}$, $a^\circ = \{r_4\}$, $\overset{\circ}{a} = \emptyset$ and ${}^\circ b = \{r_1\}$, $b^\circ = \{r_2\}$, $\overset{\circ}{b} = \{r_3\}$.

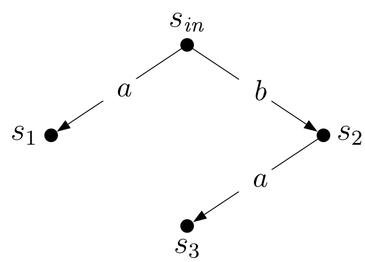


Figure 2.2: Semi-elementary transition system TS_0.

2.1.2 Nets

We now introduce the class of Petri nets which will be dealt with in this chapter.

A *net* is a triple $N = (B, E, F)$ where B is a finite set of *conditions*, E is a finite set of *events* disjoint from B, and $F \subseteq (B \times E) \cup (E \times B)$ is a *flow relation*. It is assumed that for every $x \in E$ there are $y, z \in B$ such that $(x, y), (z, x) \in F$ and $(y, x), (x, z) \notin F$.

Note that the above constraint excludes events having their connections with conditions **only** via self-loops. Note that this means that every event must have at least one 'pure' predecessor and one 'pure' successor condition (c.f. axiom (AX1)). By a *self-loop* in N we will mean a pair, $e \in E$ and $x \in B$, such that $(e, x) \in F$ and $(x, e) \in F$. Moreover, for every $x \in B \cup E$, we denote:

$$
\begin{array}{rcll}
{}^\bullet x & = & \{y \mid (y, x) \in F \ \wedge \ (x, y) \notin F\} & \text{(pre-elements)}, \\
x^\bullet & = & \{y \mid (x, y) \in F \ \wedge \ (y, x) \notin F\} & \text{(post-elements)}, \\
\overset{\bullet}{x} & = & \{y \mid (x, y) \in F \ \wedge \ (y, x) \in F\} & \text{(co-elements)}.
\end{array}
$$

A *semi-elementary net system* is a tuple $\mathcal{N} = (B, E, F, c_{in})$, where $N_\mathcal{N} = (B, E, F)$ is the underlying net and $c_{in} \subseteq B$ is the *initial case* (in general, a *case* is a subset of B). We use the standard way of graphical representation of nets, i.e, conditions are represented by circles, events by boxes and cases by tokens placed within circles. We will assume that $\mathcal{N}$ is fixed until the end of this section.

The semantics of $\mathcal{N}$ is given through the transition relation. We first define the transition relation, $\rightarrow_{N_\mathcal{N}}$, of the underlying net $N_\mathcal{N}$, as follows:

$$\rightarrow_{N_\mathcal{N}} = \{(c, e, c') \in 2^B \times E \times 2^B \mid c \setminus c' = {}^\bullet e \ \wedge \ c' \setminus c = e^\bullet \wedge \ \overset{\bullet}{e} \subseteq c \cap c'\}.$$

We then define the *state space* of $\mathcal{N}$, $C_\mathcal{N}$, which is the least subset of 2^B containing c_{in} which satisfies:

$$(c, e, c') \in \rightarrow_{N_\mathcal{N}} \wedge \, c \in C_\mathcal{N} \quad \Rightarrow \quad c' \in C_\mathcal{N},$$

and finally define $\rightarrow_{\mathcal{N}}$, the *transition relation* of $\mathcal{N}$, as $\rightarrow_{N_{\mathcal{N}}}$ restricted to $C_{\mathcal{N}} \times E \times C_{\mathcal{N}}$. Moreover,

$$E_{\mathcal{N}} = \{e \in E \mid \exists c, c' \in C_{\mathcal{N}} \ : \ (c, e, c') \in \rightarrow_{\mathcal{N}}\}$$

is the set of *active events* of $\mathcal{N}$.

We will often use $c \xrightarrow{e}$ whenever $(c, e, c') \in \rightarrow_{\mathcal{N}}$, for some c', and say that an event e is *enabled* at case c. The difference between elementary [38] and semi-elementary net systems is that the latter allow an event to be executed only if its co-conditions are true, but the state of these conditions cannot be changed by the event. Such co-conditions may however be pre- or post-conditions for some other event(s), which can change their state.

The following proposition is basically the same as the one of [38] except for its first statement, which requires co-conditions to be taken into account.

Proposition 2.1.4 *The following hold:*

1. $(\forall c \in C_{\mathcal{N}})(\forall e \in E)\ [c \xrightarrow{e} \Leftrightarrow ({}^{\bullet}e \cup \overset{\bullet}{e} \subseteq c \ \wedge \ e^{\bullet} \cap c = \emptyset)]$.
2. $(\forall (c, e, c') \in \rightarrow_{\mathcal{N}})\ [c' = (c \setminus {}^{\bullet}e) \cup e^{\bullet}]$.
3. $(\forall (c_1, e, c_2), (c_3, e, c_4) \in \rightarrow_{\mathcal{N}})\ [c_1 \setminus c_2 = c_3 \setminus c_4 \ \wedge \ c_2 \setminus c_1 = c_4 \setminus c_3]$.
4. $(c, e, c_1), (c, e, c_2) \in \rightarrow_{\mathcal{N}} \Rightarrow c_1 = c_2$. □

2.1.3 Relating Transition Systems and Nets

It is straightforward to construct a transition system for any semi-elementary net system. Let $\mathcal{N} = (B, E, F, c_{in})$ be a semi-elementary net system. Then

$$TS_{\mathcal{N}} = (C_{\mathcal{N}}, E_{\mathcal{N}}, \rightarrow_{\mathcal{N}}, c_{in})$$

is the transition system *generated* by $\mathcal{N}$.

Theorem 2.1.1 *$TS_{\mathcal{N}}$ is a semi-elementary transition system.* □

The definition of a net system associated with a transition system from [38] needs to be modified, as follows. Let $TS = (S, E, T, s_{in})$ be a semi-elementary transition system. Then $\mathcal{N}_{TS} = (R_{TS}, E, F_{TS}, R_{s_{in}})$ where

$$F_{TS} = \{(r, e) \mid e \in E \ \wedge \ r \in {}^{\circ}e \cup \overset{\circ}{e}\} \cup \{(e, r) \mid e \in E \ \wedge \ r \in e^{\circ} \cup \overset{\circ}{e}\}$$

is the net system *associated* with TS.

Theorem 2.1.2 $\mathcal{N}_{TS}$ *is a semi-elementary net system.* $\square$

For the semi-elementary transition system depicted in figure 2.2, the associated semi-elementary net system, $\mathcal{N}_{TS_0}$, is shown in figure 2.3.

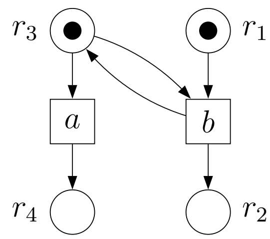


Figure 2.3: Semi-elementary net system $\mathcal{N}_{TS_0}$.

Proposition 2.1.5 *Let* $TS = (S, E, T, s_{in})$ *be a semi-elementary transition system and* $\mathcal{N} = (R_{TS}, E, F_{TS}, R_{s_{in}})$ *be a semi-elementary net system associated with it. Then*

1. $E_{\mathcal{N}} = E$,

2. $C_{\mathcal{N}} = \{R_s \mid s \in S\}$,

3. $\rightarrow_{\mathcal{N}} = \{(R_s, e, R_{s'}) \mid (s, e, s') \in T\}$. $\square$

We end with a result which states a basic consistency of the two translations defined in this section.

Theorem 2.1.3 *Let* TS *be a semi-elementary transition system and* $\mathcal{N}_{TS}$ *be a semi-elementary net system associated with it. Then* $TS_{\mathcal{N}_{TS}}$*, the transition system generated by* $\mathcal{N}_{TS}$*, is isomorphic to* TS*.* $\square$

2.2 Transforming Transition Systems

This section contains the main technical results of this chapter. We aim at a characterisation of certain local transformations defined for transition systems which do not lead outside the class of Semi-elementary Transition Systems. The transformations we are interested in correspond to adding self-loops in the associated semi-elementary nets and, as a result, they are particularly easily implementable. The transformations are based on identifying a specific pattern (subgraph) in a transition system, called a *ladder*, from which it is possible to delete some (but not all) transitions, called *rungs*, without disturbing the rest of the transition system.

2.2.1 Self-loops

We start by proving two results. The first captures the effect which adding of self-loops to a net has on the transition system generated by it.

For a transition system $TS = (S, E, T, s_{in})$ and a set of transitions $T_0 \subseteq T$, we will denote by $TS[T_0]$ the maximal transition system with the initial state s_{in} included in TS after removing the arcs T_0.

Proposition 2.2.1 *Let $TS = (S, E, T, s_{in})$ be a semi-elementary transition system and $\mathcal{N} = (R_{TS}, E, F_{TS}, R_{s_{in}})$ be the semi-elementary net system associated with it. Moreover, let $a \in E$ and $\emptyset \neq X \subseteq R_{TS}$ be such that $(a, x) \notin F_{TS}$ and $(x, a) \notin F_{TS}$, for all $x \in X$. Define*

$$\mathcal{N}' = (R_{TS}, E, F_{TS} \cup \bigcup_{x \in X} \{(a, x), (x, a)\}, R_{s_{in}}).$$

Then $\mathcal{N}'$ is a semi-elementary net system such that $TS_{\mathcal{N}'}$ is a semi-elementary transition system isomorphic to $TS[T_0]$ where

$$T_0 = \{(s, a, s') \in T \mid (s, a, s') \notin \bigcap_{x \in X} \mathcal{B}^a_x\}.$$

Proof: Clearly, $\mathcal{N}'$ is a semi-elementary net system. From $(x, a), (a, x) \notin F_{TS}$, for all $x \in X$, it follows that $\rightarrow_{\mathcal{N}'} \subseteq \rightarrow_{\mathcal{N}}$. This and proposition 2.1.5 means that it suffices to prove that for every $(s, e, s') \in T$,

$$(R_s, e, R_{s'}) \notin \rightarrow_{N_{\mathcal{N}'}} \Leftrightarrow e = a \wedge (s, e, s') \notin \bigcap_{x \in X} \mathcal{B}^a_x.$$

Note that $(s, e, s') \in T$ implies $(R_s, e, R_{s'}) \in \rightarrow_{\mathcal{N}}$ and consider three cases.
Case 1: $(s, e, s') \in T$ and $e \neq a$. Then, $(R_s, e, R_{s'}) \in \rightarrow_{N_{\mathcal{N}'}}$ follows from $(R_s, e, R_{s'}) \in \rightarrow_{\mathcal{N}}$ and the fact that the flow relation for $e \neq a$ is unchanged.
Case 2: $(s, a, s') \in T \setminus \bigcap_{x \in X} \mathcal{B}^a_x$. This means $(s, a, s') \notin \bigcap_{x \in X} \mathcal{B}^a_x$. Then there is $x \in X$ such that $(s, a, s') \notin \mathcal{B}^a_x$. Then, by $(x, a), (a, x) \notin F_{TS}$, $(s, a, s') \in \mathcal{B}^a_{S \setminus x}$. Hence $s \notin x$ and so $x \notin R_s$. But (x, a) is an arc in $\mathcal{N}'$. As a result, $(R_s, a, R_{s'}) \notin \rightarrow_{N_{\mathcal{N}'}}$.
Case 3: $(s, a, s') \in \bigcap_{x \in X} \mathcal{B}^a_x$. Then $x \in R_s \cap R_{s'}$, for all $x \in X$, which together with $(R_s, a, R_{s'}) \in \rightarrow_{\mathcal{N}}$ yields $(R_s, a, R_{s'}) \in \rightarrow_{N_{\mathcal{N}'}}$.

That $TS_{\mathcal{N}'}$ is semi-elementary follows directly from theorem 2.1.1. $\square$

The next result shows that if two semi-elementary transition systems 'differ' only by a set of a-labelled transitions then, provided that the shared a-labelled transitions are buried in a set of regions, the nets corresponding to the two transition systems 'differ' by a set of self-loops.

Proposition 2.2.2 *Let $TS = (S, E, T, s_{in})$ be a transition system, $a \in E$ be an event, and $TS' = (S, E, T', s_{in})$ be a semi-elementary transition system obtained from TS by adding at least one transition labelled by a.*

If there is a non-empty set of regions $R \subseteq R_{TS'}$ such that

$$\bigcap_{x \in R} \mathcal{B}_x^a = \{(s, a, s') \mid (s, a, s') \in T \cap T'\}$$

then TS is generated by $\mathcal{N}$ obtained from $\mathcal{N}_{TS'}$ by adding two arcs, (a, x) and (x, a), for every $x \in R$, unless such arcs are already in $\mathcal{N}_{TS'}$.

Proof: Since $a \in E$ and TS is a transition system (see (AX2)), $\bigcap_{x \in R} \mathcal{B}_x^a \neq \emptyset$ in TS'. Moreover, since we added at least one arc, $\mathcal{B}_{S \setminus x}^a \neq \emptyset$ in TS', for some (at least one) $x \in R$. Let $X \subseteq R$ be the set of all such $x \in R$. Hence there is no arc (in either direction) between x and a in $\mathcal{N}_{TS'}$ ($(a, x) \notin F_{TS'}$ and $(x, a) \notin F_{TS'}$), for $x \in X$. Thus TS' and X satisfy the assumptions of proposition 2.2.1 (with TS' playing the role of TS). Hence after adding (a, x) and (x, a) to $\mathcal{N}_{TS'}$, for all $x \in X$, we obtain $\overline{\mathcal{N}}$ which generates transition system $\overline{TS}$ isomorphic to $TS'[T'_0]$ where $T'_0 = \{(s, a, s') \in T' \mid (s, a, s') \notin \bigcap_{x \in X} \mathcal{B}_x^a\}$ (note that $T'_0 = \{(s, a, s') \in T' \mid (s, a, s') \notin T \cap T'\}$ since $\bigcap_{x \in X} \mathcal{B}_x^a = \bigcap_{x \in R} \mathcal{B}_x^a$). One can now see that $\overline{TS}$ is isomorphic to TS, because TS was a transition system and from axiom (AX3) we know that every state was reachable from the initial state, so the arcs removed from TS' are exactly those which were added to TS. $\square$

2.2.2 Ladders and Rungs

We now turn our attention to special sub-structures of transition systems which, as we already mentioned, can provide a basis for local transformations on transition systems.

Definition 2.2.1 *Let $TS = (S, E, T, s_{in})$ be a transition system.*

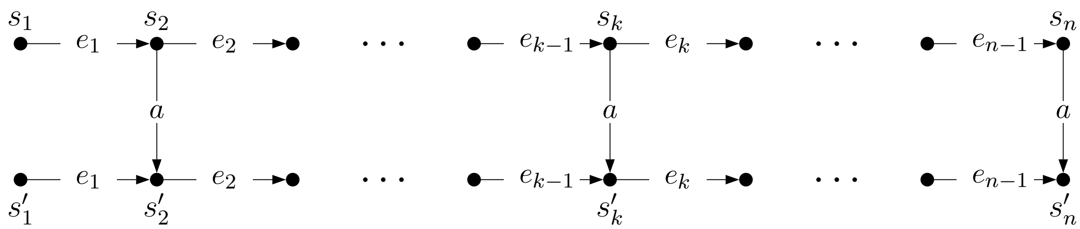
1. *A* path *in TS is a sequence of states and events $\sigma = s_1 e_1 s_2 \ldots s_{n-1} e_{n-1} s_n$ such that $n \geq 1$ and $(s_i, e_i, s_{i+1}) \in T$ for $1 \leq i < n$. We will denote $states(\sigma) = \{s_1, \ldots, s_{n-1}, s_n\}$.*

2. *Let* $\sigma = s_1 e_1 s_2 \dots s_{n-1} e_{n-1} s_n$ *and* $\sigma' = s'_1 e_1 s'_2 \dots s'_{n-1} e_{n-1} s'_n$ *be two paths in* TS *and* $a \in E$ *be such that* $|states(\sigma)| = |states(\sigma')| = n$, $states(\sigma) \cap states(\sigma') = \emptyset$ *and* $(s_k, a, s'_k) \in T$, *for some* $1 \le k \le n$. *Then the triple*

$$ldd = (\sigma, a, \sigma')$$

is called a ladder *in* TS. *We will also denote* $I(ldd) = \{i \mid (s_i, a, s'_i) \in T\}$ *and* $rungs(ldd) = \{(s_i, a, s'_i) \mid i \in I(ldd)\}$. □

A schematic representation of a ladder from the last definition is shown below:



Note that the ladder can have 'missing' rungs and also that we do not make any assumptions about other transitions labelled by a in the graph of TS.

Our first result states that if (σ, a, σ') is a ladder then each pre-region of a contains all the states of path σ and, similarly, each post-region of a contains all the states of path σ'.

Proposition 2.2.3 *Let* $ldd = (\sigma, a, \sigma')$ *be a ladder, and* r *be a region in a transition system* TS.

1. *If* $r \in {}^\circ a$ *then* $states(\sigma) \subseteq r$.

2. *If* $r \in a^\circ$ *then* $states(\sigma') \subseteq r$.

Proof: (1) Let ldd and TS be as in definition 2.2.1 (see also figure 2.4 for illustration). From $r \in {}^\circ a$ and $(s_k, a, s'_k) \in rungs(ldd)$ we have $s_k \in r$ and $s'_k \notin r$. Suppose $k - 1 \ge 1$. From $\xrightarrow{e_{k-1}} s_k$ and $\xrightarrow{e_{k-1}} s'_k$ and $s_k \in r$ and $s'_k \notin r$ we have $s_{k-1} \in r$ and $s'_{k-1} \notin r$. We can continue the same procedure: if $k-2 \ge 1$ then from $\xrightarrow{e_{k-2}} s_{k-1}$ and $\xrightarrow{e_{k-2}} s'_{k-1}$ and $s_{k-1} \in r$ and $s'_{k-1} \notin r$ we have $s_{k-2} \in r$ and $s'_{k-2} \notin r$ etc. Hence $\{s_1, \dots, s_{k-1}\} \subseteq r$. Similarly, one can show that $\{s_{k+1}, \dots, s_n\} \subseteq r$.
(2) The proof of this part is similar to that of (1). □

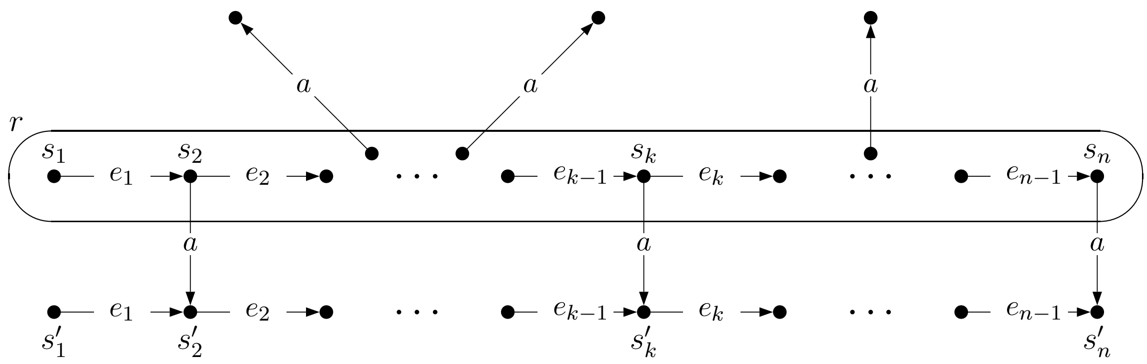


Figure 2.4: An illustration for proposition 2.2.3.

Recall that our aim is to be able to delete certain transitions from a transition system in such a way that the corresponding transformation on a semi-elementary net would simply consist in adding one or more self-loops. In particular, deleting a transition should neither create nor destroy any region (condition) in the associated net. The next proposition shows that the rungs of a ladder in a transition system do possess the latter property.

Proposition 2.2.4 *Let ldd be a ladder in a transition system* $TS = (S, E, T, s_{in})$. *If* $TS' = (S, E, T', s_{in})$ *is a transition system obtained from* TS *by deleting some (but not all) arcs from* $rungs(ldd)$ *then* $R_{TS} = R_{TS'}$.

Proof: Let ldd and TS be as in definition 2.2.1. It suffices to show that the result holds after deleting a single arc $(s_k, a, s'_k) \in rungs(ldd)$. Note that $T' = T \setminus \{(s_k, a, s'_k)\}$ in such a case.

Showing that $R_{TS} \subseteq R_{TS'}$ is straightforward. We prove that $R_{TS'} \subseteq R_{TS}$ by assuming that there is $r \in R_{TS'}$ such that $r \notin R_{TS}$. From the definition of region we know that there are arcs $(s, e, s') \in T$ and $(\widehat{s}, e, \widehat{s}') \in T$ which have different 'crossing relationship' with r. We consider two cases.

Case 1: $e \neq a$. The arcs (s, e, s') and $(\widehat{s}, e, \widehat{s}')$ belong to TS', so r cannot be a region in TS', a contradiction.

Case 2: $e = a$. Since r is a region in TS' we can assume, without loss of generality, that (s_k, a, s'_k) is $(\widehat{s}, a, \widehat{s}')$. According to the assumptions, not all arcs in *rungs(ldd)* were deleted. Suppose (s_m, a, s'_m), where $1 \leq m \leq n$ and $m \neq k$, is still in TS'.

If (s_m, a, s'_m) has different crossing relationship with r than (s, a, s') we have a contradiction, because they both belong to TS', so r cannot be a region in TS'.

Assume that (s_m, a, s'_m) has the same crossing relationship with r as (s, a, s'). Suppose $s, s_m \in r$ and $s', s'_m \notin r$. We have three cases to consider:

1. $s_k \notin r$ and $s'_k \in r$ (see figure 2.5 for illustration);

2. $s_k \in r$ and $s'_k \in r$; and

3. $s_k \notin r$ and $s'_k \notin r$.

In all three cases it is easy to show that there exist $(s_i, e_i, s_{i+1}) \in T$ and $(s'_i, e_i, s'_{i+1}) \in T$ on the paths σ and σ' respectively which has different crossing relationships with r (since $states(\sigma) \cap states(\sigma') = \emptyset$) and they are not removed from TS'. Hence r cannot be a region in TS'. All other cases, $s, s_m \notin r$ and $s', s'_m \in r$ etc. are similar.

Hence $R_{TS} = R_{TS'}$. $\square$

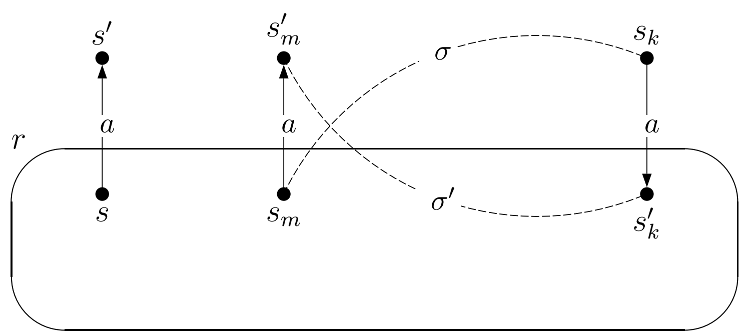


Figure 2.5: An illustration for proposition 2.2.4.

The next result complements the previous one in that it shows that different rungs in a ladder can be separated by a region such that one of the rungs is buried in it and the other not. Intuitively, this means that we can 'deactivate' one of the two rungs by adding a suitable self-loop in the corresponding net.

Proposition 2.2.5 *Let* $ldd = (\sigma, a, \sigma')$ *be a ladder in a semi-elementary transition system* TS*. Then for every two distinct arcs* $\tau, \tau' \in rungs(ldd)$ *there is a region* $r \in R_{TS}$ *such that* $\tau \in \mathcal{B}^a_r$ *and* $\tau' \notin \mathcal{B}^a_r$*.*

Proof: Let ldd be as in definition 2.2.1. Suppose $i \neq j \in I(ldd)$ are such that for all $r \in R_{TS}$, $(s_i, a, s'_i) \in \mathcal{B}^a_r \Leftrightarrow (s_j, a, s'_j) \in \mathcal{B}^a_r$. From proposition 2.2.3 it follows that: $(\forall r \in {}^\circ a)\ s_i, s_j \in r \ \wedge\ (\forall r \in a^\circ)\ s_i, s_j \notin r$. As a result, every region $r \in R_{TS}$ either contains both s_i and s_j or none of them. Hence $R_{s_i} = R_{s_j}$ and, by (AX4), we obtain $s_i = s_j$. This, however, contradicts $i \neq j$ and $|states(\sigma)| = n$. $\square$

We now can prove the main result of this chapter, theorem 2.2.1. It characterises situations under which deleting rungs from a ladder has no effect on being a semi-elementary transition system.

Theorem 2.2.1 *Let $TS = (S, E, T, s_{in})$ be a semi-elementary transition system and ldd be a ladder such that $rungs(ldd)$ are the only a-labelled arcs in TS.*
If $TS' = (S, E, T', s_{in})$ is a transition system obtained from TS by deleting all but one arc from $rungs(ldd)$ then TS' is semi-elementary.

Proof: Let ldd be as in definition 2.2.1 (see also figure 2.6 for illustration). TS' satisfies the assumptions of proposition 2.2.4, so $R_{TS} = R_{TS'}$. Hence, since (AX4) was true for TS, it is true for TS' as the sets of states and non-trivial regions are the same. We now prove that (AX5) is satisfied. Suppose $\tau = (s_k, a, s'_k)$ is the only arc from *rungs(ldd)* which belongs to TS'. From proposition 2.2.5 it follows that:

$$(\forall j \in I(ldd) \setminus \{k\})\ (\exists r_j \in R_{TS})\ \tau \in \mathcal{B}^a_{r_j}\ \wedge\ (s_j, a, s'_j) \notin \mathcal{B}^a_{r_j}$$

which implies

$$(*) \quad (\forall j \in I(ldd) \setminus \{k\})\ (\exists r_j \in R_{TS})\ \tau \in \mathcal{B}^a_{r_j}\ \wedge\ r_j \notin R_{s_j}.$$

Axiom (AX5) is satisfied for all $e \neq a$, because it was satisfied for TS, and the set of states is unchanged, the set of regions is unchanged, and the set of e-labelled arcs is unchanged. What we need to show is that in TS': $(\forall s \in S)\ {}^\circ a \subseteq R_s \wedge\ \mathring{a} \subseteq R_s \Rightarrow s \xrightarrow{a}$. We first observe that ${}^\circ a$ (in TS) is the same as ${}^\circ a$ (in TS') since $R_{TS} = R_{TS'}$. Moreover, $\mathring{a}$ (in TS) is a subset of $\mathring{a}$ (in TS'). Thus, the only property we need to check is:

$$(**) \quad (\forall j \in I(ldd) \setminus \{k\})\ \ \mathring{a} \not\subseteq R_{s_j}\ \text{(in } TS'\text{)}.$$

But we know that in TS': $\mathring{a} = \{r \mid r \in R_{TS'} \wedge (s_k, a, s'_k) \in \mathcal{B}^a_r\}$. This and (*) yields:

$$(\forall j \in I(ldd) \setminus \{k\})\ (\exists r_j \in R_{TS} = R_{TS'})\ r_j \in \mathring{a}\ \wedge r_j \notin R_{s_j}\ \text{(in } TS'\text{)}.$$

Hence (**) holds. $\square$

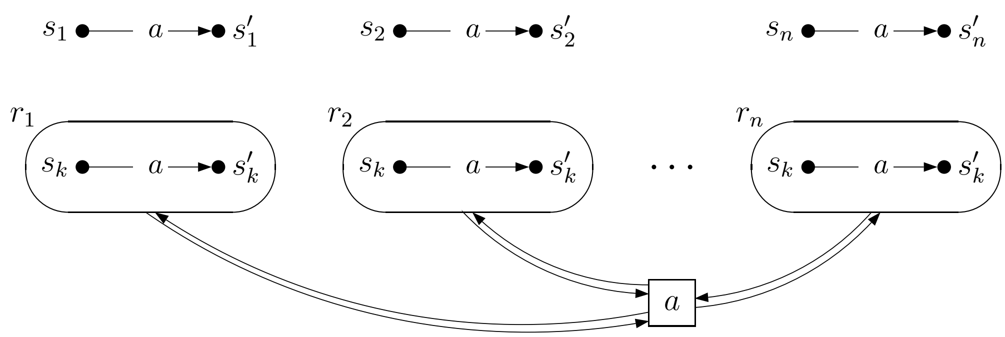


Figure 2.6: An illustration for theorem 2.2.1.

We can further observe that by applying propositions 2.2.1 and 2.2.2 to the two transition systems, TS and TS', in theorem 2.2.1 we can also relate the corresponding nets. More precisely, the construction can proceed by taking the regions r_j used in the proof of theorem 2.2.1, and connecting these by means of a self-loop with the event a in the net corresponding to TS. To conclude, local transformations of semi-elementary transition systems dealt with in theorem 2.2.1 correspond to adding self-loops on the level of the associated semi-elementary nets.

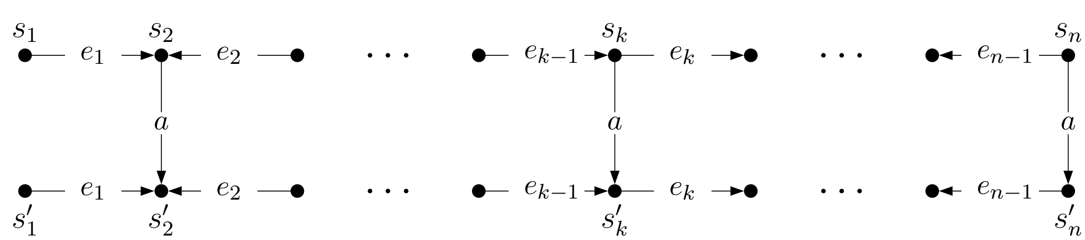
Our final observation in this section is that ladders, in the form introduced in definition 2.2.1, are typical behavioural structures for asynchronous circuits with conflicts. However, it is important to add that, from the technical point of view, their definition can be generalised by not demanding that σ and σ' be directed paths (the paths in definition 2.2.1 are true directed paths), but only that for all $1 \leq i < n$ either

$$(s_i, e_i, s_{i+1}) \in T \quad \text{and} \quad (s'_i, e_i, s'_{i+1}) \in T,$$

or

$$(s_{i+1}, e_i, s_i) \in T \quad \text{and} \quad (s'_{i+1}, e_i, s'_i) \in T.$$

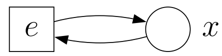
A schematic diagram for such a generalised ladder is given below.



It is worth noting that such a generalisation has no effect on the results proved in section 2.2.2, i.e. propositions 2.2.3, 2.2.4, 2.2.5 and theorem 2.2.1 still hold.

2.3 Self-loops and Contextual Nets

Semi-elementary nets defined in this chapter allow a self-loop such as



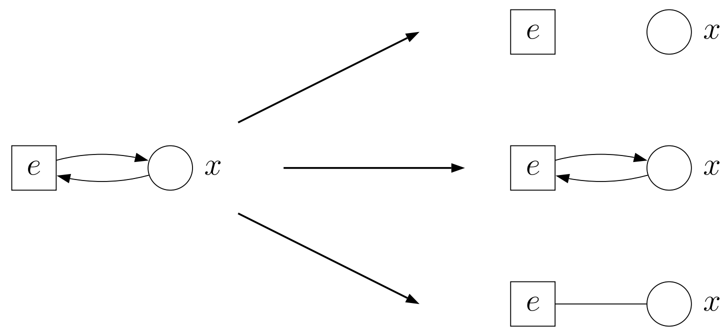
to be interpreted in the classical way: x is both pre- and post-condition of an event e. Such an x must be marked if e is to be executed and in operational terms, the execution of e implies taking the token from x and placing it back again, effectively preventing any other event which has x as its pre-condition from being executed.

A semi-elementary net $\mathcal{N} = (R_{TS}, E, F_{TS}, R_{s_{in}})$ associated with a semi-elementary transition system $TS = (S, E, T, s_{in})$ is saturated with conditions (i.e. uses all the non-trivial regions) and also may have many redundant self-loops joining conditions and events. The number of conditions can be reduced by restricting the net only to minimal regions (see, for example, [12, 22]). When it comes to self-loops, some of them can create unnecessary constraints on the degree of concurrency present in the system, and in the design process one may need to take further steps aimed at remedying the situation[4]. More specifically, once a satisfactory semi-elementary net has been constructed, the designer can build the final version of the net by suitably re-interpreting some of its self-loops, in at least three different ways:

- a self-loop between x and e is redundant,
- a self-loop between x and e is a classical one, i.e. the token is taken by e from x and then put back again when e is executed,
- a self-loop between x and e is like a positive context relationship of [36] where the token in x is needed for e to be enabled but is not consumed by e when it is executed and hence can be shared with another event.

[4]Note, however, that a semi-elementary transition system does not in general provide enough information needed for such a step (in particular, information about the independence relation on events) so it has to come from outside the specification.

The three different ways of reinterpreting a self-loop are illustrated below:



A class of nets which allow the last two ways of interpreting self-loops within a single model are contextual nets of [36]. In our notation, a *contextual net* is a tuple

$$\mathcal{CN} = (B, E, F, PC, c_{in})$$

such that (B, E, F, c_{in}) is a semi-elementary net (as defined in section 2.1) and $PC \subseteq B \times E$ is a *positive context* relation such that $(F \cup F^{-1}) \cap PC = \emptyset$. We also denote, for every $e \in E$, $\widehat{e} = \{x \mid (x, e) \in PC\}$. Note that [36] also allows *negative contexts* (similar to inhibitor arcs in Petri nets) which are omitted here since we do not generate them in the process of re-interpreting self-loops.

A crucial property of contextual nets is the way in which they define concurrent execution of sets of events. A case $c \subseteq B$ enables a non-empty set of events $G \subseteq E$ if, for every $e \in G$, ${}^\bullet e \cup \overset{\bullet}{e} \cup \widehat{e} \subseteq c$ and $e^\bullet \cap c = \emptyset$ and, furthermore, for all $e \neq f \in G$, $({}^\bullet e \cup e^\bullet \cup \overset{\bullet}{e}) \cap ({}^\bullet f \cup f^\bullet \cup \overset{\bullet}{f}) = \emptyset$ and $\widehat{e} \cap ({}^\bullet f \cup \overset{\bullet}{f}) = \widehat{f} \cap ({}^\bullet e \cup \overset{\bullet}{e}) = \emptyset$. Note, however, that it is not necessarily the case that $\widehat{e} \cap \widehat{f} = \emptyset$.

In operational terms, the above enabling rule means that a positive context relation between event e and condition x requires x to be marked for e to be enabled, but the execution of e does not prevent other events which have x as a positive context from being executed concurrently with e.

Using the above definitions, the task of synthesis of Petri net from state based specification can be split into two stages:

- construct a semi-elementary net from a semi-elementary transition system,

- turn a semi-elementary net into a contextual net by reinterpreting some of its self-loops.

The two stages are illustrated below:

$$\boxed{TS} \longrightarrow \boxed{\mathcal{N}_{TS}} \longrightarrow \boxed{\mathcal{CN}}$$

It is not difficult to see that $\mathcal{CN}$ will always be a contextual net since, by construction, $(F \cup F^{-1}) \cap PC = \emptyset$. Note also that the resulting contextual net $\mathcal{CN}$ will always generate the same transition system when restricted to single event executions, but the level of concurrency may be different (see [36] for detailed discussion of contextual nets and, in particular, the relationship between positive contexts and concurrency).

To summarise the discussion in this section, from the designer's point of view, there are two ways in which concurrency aspects of the system being synthesised can be dealt with. In the first approach, concurrency is present right from the start, which in our framework means that transition systems should be capable to explicitly model concurrency (or independence relation) among events. There are models of transition systems which could be used here, for example, step transition systems of [37] and asynchronous transition systems of [11, 44, 47]. The former represent concurrency by labelling transitions with sets of events executed concurrently, while the latter employs an explicit independence relation on events, from which concurrency on the operational level can be retrieved. The other approach, and one which we adopted in this chapter, deliberately delays the consideration of issues related to concurrency to a later stage of the design. In our case this means that only after deriving a net specification of the system will the designer consider such issues as independence among events. No matter, however, which of the approaches is taken, the final result will always be the same (or equivalent).

More recently, in [46], it was proved for the Asynchronous Transition Systems (the extension of the Semi-elementary Transition Systems by allowing self-loops) that once the solution to the synthesis problem is found (by the procedure which ignores the independence relation), one can always build a net which is still a solution to the synthesis problem for a given transition system, but preserves its independence relation.

2.4 Application of Semi-elementary Transition Systems and Nets

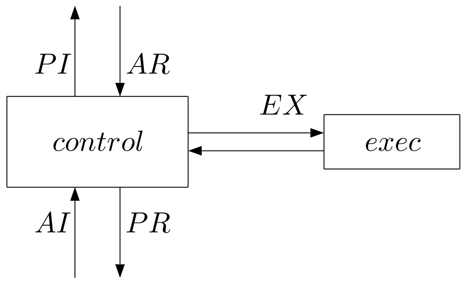
In this section we briefly illustrate how the theory developed in the previous sections can be applied in synthesising a net model from an initial specification of a system by means of a transition system. Our example originates from [45], where a counterflow pipeline processor (CFPP), now called the Sproull's processor, is described. In a CFPP there are two mutually synchronised pipelines connecting an instruction fetch unit and a register file. Instructions flow in one direction, along the instruction pipeline, while results are propagated in the opposite direction (counterflow) through the result pipeline. The instructions and results interact as they pass according to the rules specified in [45]. When an instruction and result are present in the same pipe stage the instruction may use the data from the result pipeline, produce further result and update the data in the result pipeline. The key part in the distributed control structure of the processor is played by a device which provides mutual synchronisation between the two pipelines. This device is supposed to be placed into each stage of the counterflow pipeline. The original idea of synchronisation is due to Charles Molnar, who described it in the form of a transition system, which is reproduced in figure 2.7(a,b). This figure also gives the meaning of the signals and the corresponding events and the states of the stage control. There are five possible states for each pipeline stage:

1. E: neither an instruction nor a result is present,
2. I: only an instruction is present,
3. R: only a result is present,
4. F: both an instruction and a result are present,
5. C: the pipeline rules have been enforced, and both instruction and result are free to move on.

The transition system depicted in figure 2.7(b) is not a semi-elementary transition system because $R_F = R_C$ thus violating (AX4). In [49], it was shown how to transform this initial transition system to the 'asymmetric' one, shown in figure 2.7(c), or the 'symmetric' one, shown in figure 2.7(d). Both transformations involve inserting a 'dummy' event and yield semi-elementary transition systems. Note that in the symmetric case there was a possibility to preserve the trace equivalence (up to the hiding the dummy event). In the asymmetric case, however, the system cannot execute event **AR** after executing **PR** from state **C** - the original model allowed that possibility. Both solutions were studied and explained in [49]. For our purposes, let us look further at the asymmetric case.

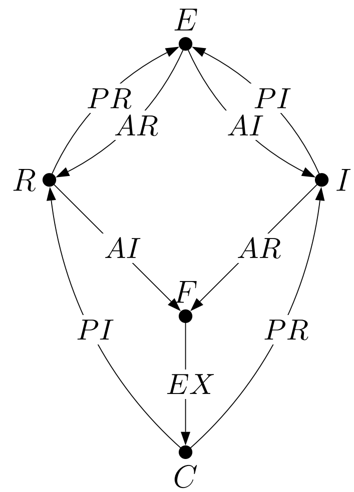
The transition system shown in figure 2.7(c) can be converted into a semi-elementary net. This is shown in figure 2.8. In the (a) part of this figure we showed all the regions which give rise to the conditions in the net depicted in the (b) part. Note that to avoid excessive cluttering of arcs we use double-headed arcs to indicate self-loops in parts (b) and (c). The (c) part of figure 2.8 shows the contextual net version of the net model, which is simpler. Note that the transformation to the contextual net involves certain interpretation of self-loops. Firstly, we can simplify the net by removing region r_8, which is redundant since the semi-elementarity axioms (AX4-AX5) are satisfied for all events without r_8, and r_8 is not minimal since $r_8 = r_4 \cup r_5$. Together with r_8, we remove the arcs incident to it. Secondly, we can remove self-loops between r_6 and **AI** (and r_7 and **PI**) as they are redundant since axiom (AX5) is satisfied for both these events without them. Thirdly, we keep the self-loops between r_6 and **AR** (and r_7 and **PR**) as classical ones, which means that the transitions labelled **AR** and **PR** are meant to be in conflict with **EX** or ϵ (dummy) and **AI**, respectively. This would allow us to refine them further into some subnets detailing their functionality. The fact that a token is effectively removed by **AR** and **PR** from r_6 and r_7, respectively, would mean that their corresponding refined actions are in the critical sections, thus preventing them from being executed concurrently. Finally, the pair of self-loops (r_3,**EX**) and (r_1, ϵ) can be put into a positive context relationship, indicating that neither **EX** nor ϵ have effect on their respective co-regions. Even if **EX** and ϵ are further refined into sub-actions, the latter would inherit their contextual relationship with r_3 and r_1.

What is also important is that this transition system has a ladder structure; for example, with respect to events **AR** or **PR**. It is easy to see that the ladder satisfies the conditions of theorem 2.2.1. We could therefore remove some of the ladder rungs labelled with **AR** and **PR** up to the point when only a single rung of each of them still remains. For example, if we consider the case shown in figure 2.8(a), the rungs labelled with **PR** will be buried in region r_7 which is the union of r_2 and r_5. Now, removing the **PR** rung

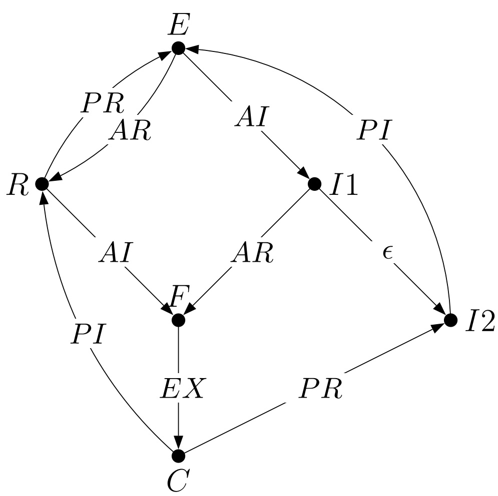


States :	*Events* :
E – *Empty*	AI – *accept instruction*
I – *Instruction*	PI – *pass instruction*
R – *Result*	AR – *accept result*
F – *Full*	PR – *pass result*
C – *Complete*	EX – *execute instr.*

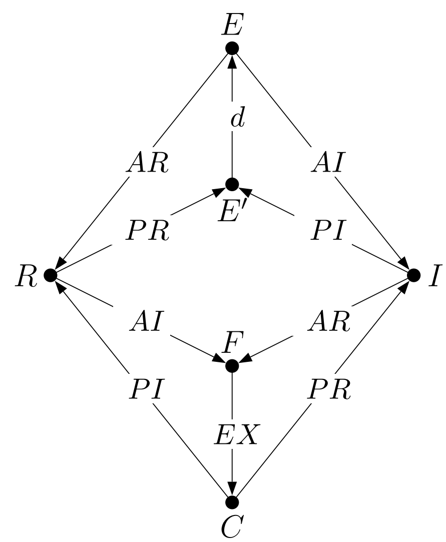
(a)



(b) Original transition system.

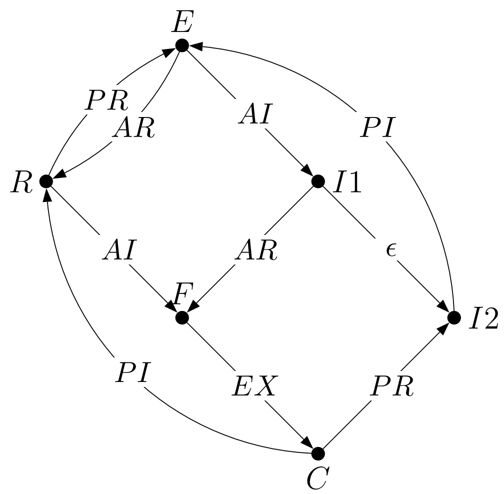


(c) Semi-elementary transition system.



(d) Semi-elementary transition system.

Figure 2.7: Counterflow pipeline example: constructing semi-elementary transition systems.



$Regions:$

$r_1 = \{E, I1, I2\}$

$r_2 = \{R, E\}$

$r_3 = \{R, F, C\}$

$r_4 = \{F, I1\}$

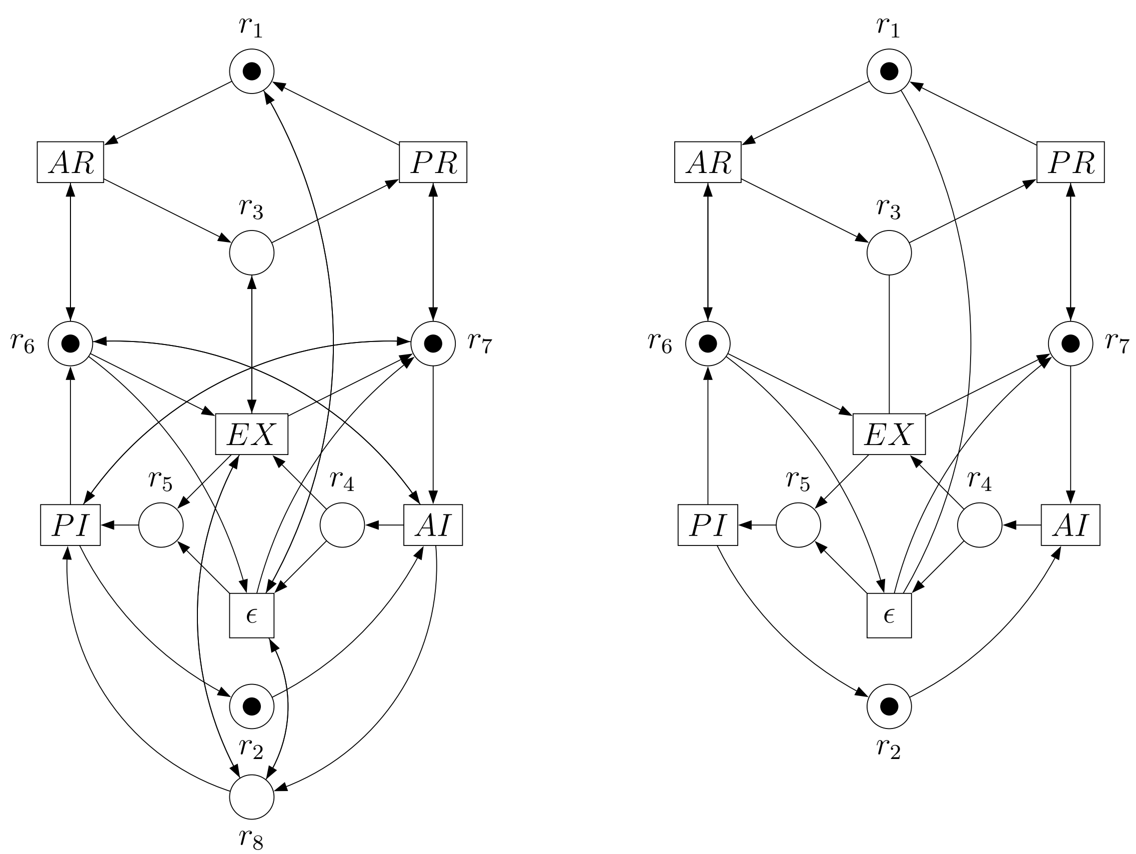
$r_5 = \{C, I2\}$

$r_6 = r_2 \cup r_4$

$r_7 = r_2 \cup r_5$

$r_8 = r_4 \cup r_5$

(a)



(b) (c)

Figure 2.8: Counterflow pipeline example: constructing semi-elementary and contextual nets from ladder transition system.

between **C** and **I2** makes the remaining **PR** rung between **R** and **E** buried in regions r_7, r_2 and r_6. In terms of the associated net, this would correspond to adding the connections with double-headed arcs between event **PR** and conditions r_2 or r_6. All such nets remain semi-elementary, with the same structure, except that they differ in the self-loop conditions, which constrain concurrency of some events. With the circuit design techniques available from Safe Nets [18, 22, 33], one can adjust the specification at the semantic level by changing the structure of ladders. This can be advantageous since adding some ordering constraints (by rung removal) often helps to satisfy timing or mutual exclusion requirements.

2.5 Remark on Self-loops in Transition Systems

It should be noted that our allowing of self-loops in net models is not semantically concerned with having self-loops in the transition systems. Self-loops (an arc labelled with some event leading to the same state) in transition systems have a different interpretation from the one we intended to capture. With our extension we would like to stay within the limits of modelling systems where each event is significant in the sense that it changes the state of the system. Such an assumption is perfectly acceptable for modelling asynchronous systems. As could be noted from our examples, the effect of our co-regions and self-loop conditions on events is always made in conjunction with pre- and post-conditions.

However, the theory can be easily extended to allow self-loops in the transition systems. It would mean dropping axiom (AX1), and changing the definition of the Semi-elementary Net Systems in such a way that some events can be connected with the rest of the net by means of self-loops only or, in the extreme, be isolated. While we can agree for the former, the latter would mean that we cannot control the behaviour of the disconnected events and introduce autoconcurrency. In [46], where the solution for the synthesis of the nets with self-loops presented here was extended to allow self-loops in the transition systems, the above problem was dealt with by adding to the set of places of the synthesised net a special place (marked under the initial case) to which all potentially isolated events are attached by self-loops. The class of Semi-elementary Transition Systems with self-loops is exactly the class of transition systems generated by the general Safe Nets (as proved in [46]).

Chapter 3

Synthesis of Nets with Inhibitor Arcs

In this chapter, we investigate the relationship between *Elementary Net Systems with Inhibitor Arcs (ENI-systems)* [29] and transition systems [2, 31]. ENI-systems are the Elementary Net Systems of [38] with added *inhibitor arcs*, as shown in figure 3.1. The meaning of all the elements of $\mathcal{N}$ is standard except for the inhibitor arcs between, e.g., the condition b_4 and event e, which is represented by an edge ending with a small circle, and indicates that e can only be fired if b_4 is empty. This has a clear interpretation if one

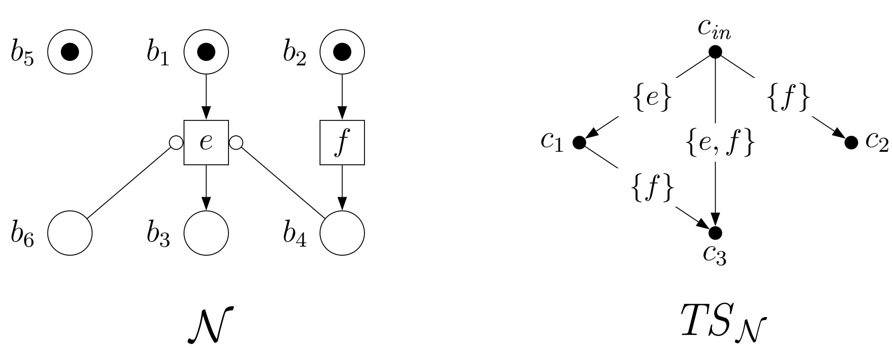


Figure 3.1: An ENI-system $\mathcal{N}$, and the TSENI transition system it generates, $TS_{\mathcal{N}}$.

considers purely interleaving net semantics: $\mathcal{N}$ can execute e or f or ef (i.e., e followed by f). However, when we consider a non-interleaving semantics based on step sequences, then there is a problem whether or not the concurrent step $\{e, f\}$ should be allowed. Basically, both interpretations are possible, as discussed in [19]. The one in which it is possible to execute $\{e, f\}$ is called there the *a-priori* semantics, and that in which $\{e, f\}$ is disallowed is called the *a-posteriori* semantics. In the a-posteriori interpretation [20, 36], the resulting semantics is a variant of the standard *causal partial order* model of behaviour. In the a-priori interpretation [14, 30], inhibitor arcs force one to use a more general view of causality in system's behaviour in which the classical causality is augmented with *weak causality* (see [29] for details). In the a-priori semantics, one can interpret events as taking some time to complete. For example, when the event f in $\mathcal{N}$ is being executed, no token is placed in b_4 immediately, giving a chance to execute e at the same time as f. In the a-posteriori semantics, the occurrence of an event can be understood as taking zero time. Under this semantics, the execution of f places a token in b_4 at the same moment as the token of b_2 is removed, blocking immediately any event of $\mathcal{N}$ for which b_4 is an inhibitor condition. In such a case e and f cannot be executed at the same time. In this chapter we will interpret all inhibitor arcs using the a-priori semantics. Chapter 7 deals with the a-posteriori semantics, where the nets are called ENI_{apost}-systems to distinguish them from the ones considered here.

The class of transition systems generated by ENI-systems will be called *Transition Systems of Elementary Nets with Inhibitor Arcs (TSENI)*. They constitute a subclass of *step transition systems* [37] (however, they are different from any known to us class of transition systems considered previously in the literature). In such transition systems, concurrency is represented explicitly since the moves between different states are labelled by *steps* (sets) of concurrently executed actions rather than by single events. As we will see, this is in general unavoidable if one wants to faithfully model the non-interleaving semantics. We will define translations between ENI-systems and TSENI Transition Systems which preserve their behavioural properties. The translation from transition systems to nets (net synthesis) is based on the notion of a region [9, 13, 27, 37, 38, 47] which is suitably modified to match the operational meaning of inhibitor arcs. For the ENI-system $\mathcal{N}$, the corresponding transition system is shown in figure 3.1.

3.1 TSENI Transition Systems

In this section, we introduce TSENI Transition Systems which are the class of transition systems generated by ENI-systems. We approach the final definition gradually, by introducing the six axioms characterising TSENI Transition Systems and proving some of their properties.

Let $\mathcal{E}$ be a non-empty set of *events* fixed throughout the rest of this thesis. A *transition system* is a quadruple $TS = (S, U, T, s_{in})$, where:

TS1 S is a non-empty finite set of *states*.

TS2 $U \subseteq 2^{\mathcal{E}}$ is a finite set of *steps*; every $u \in U$ is finite and non-empty.

TS3 $T \subseteq S \times U \times S$ is the *transition relation*.

TS4 $s_{in} \in S$ is the *initial state*.

We assume that TS satisfies the following three axioms:

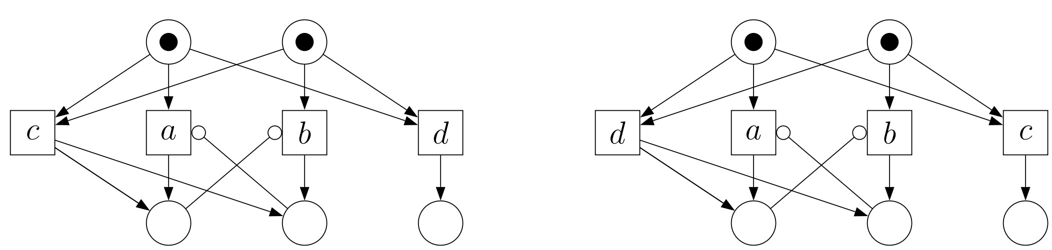
A1 For every $(s, u, s') \in T$, $s \neq s'$.

A2 For every $u \in U$, there are $s, s' \in S$ such that $(s, u, s') \in T$.

A3 For every $s \in S \setminus \{s_{in}\}$, there are $(s_0, u_0, s_1), (s_1, u_1, s_2), \ldots, (s_{n-1}, u_{n-1}, s_n) \in T$ such that $s_0 = s_{in}$ and $s_n = s$.

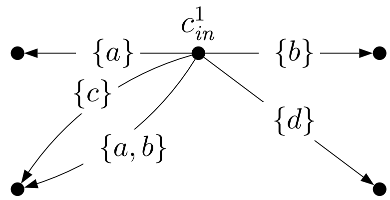
The first axiom excludes transition systems with self-loops, while the second ensures that all the steps in U are used as labels of transitions in TS. We do not require that U be subset closed as this will be a property dealt with later, in proposition 3.2.3. The third axiom implies that all the states in TS are *reachable* from the initial state. Throughout the rest of this section, the transition system TS will be fixed. We will use $s \xrightarrow{u} s'$ to denote $(s, u, s') \in T$, and respectively call s the *source* and s' the *target* of this transition. Moreover, $E_{TS} = \bigcup_{u \in U} u$ will denote all the events appearing in steps labelling transitions in TS.

Our transition systems are essentially a subset of the general *step transition systems* of [37]. A major remark which we need to make now is that, unlike [38], we cannot base our transition systems on arcs labelled by single events. For example, consider the following two ENI-systems:

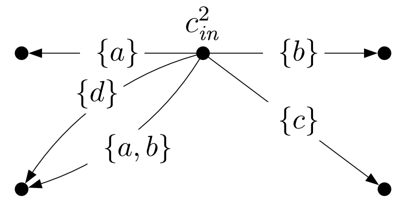


$\mathcal{N}_1$ $\mathcal{N}_2$

They generate the following transition systems, according to their a-priori semantics:



$TS_{\mathcal{N}_1}$



$TS_{\mathcal{N}_2}$

If we were to exclude transitions labelled with non-singleton steps then the two transition systems would become indistinguishable, although they correspond to nets with different behavioural properties. We could try to remedy the situation by augmenting the two transition systems with information about the (in)dependency of events a and b (similarly as it is done in the Asynchronous Transition Systems of [11, 44, 47]). This could, of course, provide us with information that in both transition systems a and b can be executed concurrently at the initial states. However, this would not be enough to determine the states resulting from executing $\{a, b\}$. As a result, to capture faithfully the behaviour of $\mathcal{N}_1$ and $\mathcal{N}_2$, we would have to provide such an information essentially amounting to the inclusion of the whole transition labelled $\{a, b\}$ in the description of the transition systems.

We now introduce a notion central to the whole approach as it links nodes of transition system (global states) with conditions in the corresponding net (local states).

Definition 3.1.1 *A set of states $r \subseteq S$ is a* region *if the following two conditions are satisfied:*

1. *If $s \xrightarrow{u} s'$ and $s \in r$ and $s' \notin r$ then there is $e \in u$ such that*
   (a) *if $u' \subseteq u \setminus \{e\}$ and $s \xrightarrow{u'} s''$ then $s'' \in r$,*
   (b) *if $q \xrightarrow{v} q'$ and $e \in v$ then $q \in r$ and $q' \notin r$.*
2. *If $s \xrightarrow{u} s'$ and $s \notin r$ and $s' \in r$ then there is $e \in u$ such that*
   (a) *if $u' \subseteq u \setminus \{e\}$ and $s \xrightarrow{u'} s''$ then $s'' \notin r$,*
   (b) *if $q \xrightarrow{v} q'$ and $e \in v$ then $q \notin r$ and $q' \in r$.* □

Intuitively, a region is a subset of states with which all steps containing certain event have the same 'crossing' relationship. As we will show in proposition 3.1.3, the event e appearing in definition 3.1.1 is unique. Such an event will be called *r-crossing* in u.

The set of *non-trivial* regions (i.e. those different from S and $\emptyset$) will be denoted by R_{TS}. Moreover, for every state $s \in S$, we will denote by R_s the set of non-trivial regions containing s,

$$R_s = \{r \in R_{TS} \mid s \in r\}.$$

The sets of pre-regions, $^\circ u$, and post-regions, u°, of a step $u \in U$ are defined as:

$$\begin{array}{rcl} ^\circ u & = & \{r \in R_{TS} \mid \exists (s, u, s') \in T : s \in r \wedge s' \notin r\} \\ u^\circ & = & \{r \in R_{TS} \mid \exists (s, u, s') \in T : s \notin r \wedge s' \in r\}. \end{array}$$

We will use $^\circ e$ and e° instead of respectively $^\circ\{e\}$ and $\{e\}^\circ$, for every $e \in E_{TS}$.

Consider the transition system TS_0 in figure 3.2. It has four non-trivial regions: $r_1 = \{s_{in}, s_2\}$, $r_2 = \{s_{in}, s_1\}$, $r_3 = \{s_1, s_3\}$ and $r_4 = \{s_2, s_3\}$. Moreover, the pre- and post-regions of a, b and $\{a, b\}$ are: $^\circ a = \{r_1\}$, $^\circ b = \{r_2\}$, $^\circ\{a, b\} = \{r_1, r_2\}$, $a^\circ = \{r_3\}$, $b^\circ = \{r_4\}$ and $\{a, b\}^\circ = \{r_3, r_4\}$.

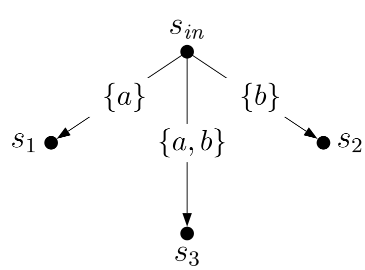


Figure 3.2: Transition system TS_0 - a running example.

Being a pre- or post-region of a step u is a global property, in the following sense:

Proposition 3.1.1 *If $s \xrightarrow{u} s'$ then*

1. *$r \in {}^\circ u$ implies $s \in r$ and $s' \notin r$,*

2. *$r \in u^\circ$ implies $s \notin r$ and $s' \in r$.*

Proof: (1) $r \in {}^\circ u$ means that there is $p \xrightarrow{u} p'$ such that $p \in r$ and $p' \notin r$. From definition 3.1.1(1b) it follows that if we take $v = u$, $q = s$ and $q' = s'$, we obtain $s \in r$ and $s' \notin r$.

(2) The proof is similar, using definition 3.1.1(2b) instead of definition 3.1.1(1b). □

We say that a step $u \in U$ is *enabled* at a state $s \in S$ if there is $s' \in S$ such that $s \xrightarrow{u} s'$. We will denote this by $s \xrightarrow{u}$. We say that a step $u \in U$ *leads* to a state $s' \in S$ if there is $s \in S$ such that $s \xrightarrow{u} s'$. We will denote this by $\xrightarrow{u} s'$.

In what follows, we will assume the transition system TS satisfies a fourth axiom:

A4 If $s \xrightarrow{u}$ and $e \in u$ then $s \xrightarrow{\{e\}}$.

(A4) means that a step u cannot be enabled at a state if any of its events is disabled. This will later be generalised to a stronger property that none of the non-empty subsets of u is disabled. However, to prove it we will need another axiom, (A6).

Corollary 3.1.1 *For every $e \in E_{TS}$, $\{e\} \in U$.*

Proof: Follows directly from (A2) and (A4). □

The above corollary is important in that it ensures that $^\circ e$ and e° are defined for all $e \in E_{TS}$. In particular, this means that the set V_{TS} below is well defined:

$$V_{TS} = \{u \subseteq E_{TS} \mid u \neq \emptyset \ \wedge \ \forall\, e, f \in u : (e \neq f \Rightarrow (\,^\circ e \cup e^\circ) \cap (\,^\circ f \cup f^\circ) = \emptyset)\}.$$

Intuitively, V_{TS} comprises sets of events which are potential steps in the transition system as they share neither pre- nor post-regions. For TS_0 in figure 3.2, we have $V_{TS_0} = \{\{a\}, \{b\}, \{a, b\}\}$.

Corollary 3.1.2 *For every $r \in R_{TS}$, there is $e \in E_{TS}$ such that $r \in {}^\circ e$ or $r \in e^\circ$.*

Proof: Follows directly from (A3), (A4) and definition 3.1.1. □

The pre- and post-regions of a step can be represented as the union of respectively pre- and post-regions of events it comprises.

Proposition 3.1.2 *If $u \in U$ then $^\circ u = \bigcup_{e \in u} {}^\circ e$ and $u^\circ = \bigcup_{e \in u} e^\circ$.*

Proof: Let $r \in {}^\circ u$. This means that there is $p \xrightarrow{u} p'$ such that $p \in r$ and $p' \notin r$. By definition 3.1.1(1b), there is $e \in u$ such that if $q \xrightarrow{v} q'$ and $e \in v$ then $q \in r$ and $q' \notin r$. From (A4) it follows that there is $p'' \in S$ such that $p \xrightarrow{\{e\}} p''$. Hence $p \in r$ and $p'' \notin r$ and, as a result, $r \in {}^\circ e$.

Suppose now that $e \in u$ and $r \in {}^\circ e$. This means that there is $p \xrightarrow{\{e\}} p'$ such that $p \in r$ and $p' \notin r$. From definition 3.1.1(1b) we have that if $q \xrightarrow{v} q'$ and $e \in v$ then $q \in r$ and $q' \notin r$. We know that $u \in U$. From (A2) it follows that there are $s, s' \in S$ such that $s \xrightarrow{u} s'$. Hence, since $e \in u$, $s \in r$ and $s' \notin r$. As a result, $r \in {}^\circ u$.

The second part of this proposition can be proved similarly, using definition 3.1.1(2b) instead of 3.1.1(1b). □

The next four results state some basic properties of TS. The first shows that event e appearing in definition 3.1.1 is always unique. Intuitively, this corresponds to the property of Petri nets that the sets of tokens consumed by concurrently executed events are disjoint. After that we prove that the definition of the set of potential steps of TS is consistent with the definition of U. In proposition 3.1.5 we re-establish some of the properties of regions formulated in [38] within our present setting. More precisely, we demonstrate that the complement of a region is also a region, and that post-regions of a step u are the complements of pre-regions of u. We also show how regions containing the target or source state of a transition are related using the pre- and post-regions of the step labelling this transition. Finally, in proposition 3.1.6, we prove the property which ensures that the synthesised nets are contact-free [38].

Proposition 3.1.3 *Event $e \in u$ which satisfies definition 3.1.1(1) (or 3.1.1(2)) is unique.*

Proof: Assume that there are $e_1 \neq e_2$ which can play the role of e in definition 3.1.1(1). From (A4) it follows that $s \xrightarrow{\{e_1\}} s_1$ and $s \xrightarrow{\{e_2\}} s_2$, for some s_1, s_2. Since definition 3.1.1(1a) holds for $e = e_1$, we can take $u' = \{e_2\} \subseteq u \setminus \{e_1\}$ and have $s_2 \in r$. Moreover, since definition 3.1.1(1b) holds for $e = e_2$, we can take $v = \{e_2\}$ and have $s_2 \notin r$. Hence we obtained a contradiction.

Similarly, we can prove the uniqueness of e in definition 3.1.1(2). □

Proposition 3.1.4 $U \subseteq V_{TS}$.

Proof: Let $u \in U$ and $e \neq f \in u$. By (A2), there is a transition $s \xrightarrow{u} s'$.

We show that ${}^\circ e \cap {}^\circ f = \emptyset$ by contradiction (the case $e^\circ \cap f^\circ = \emptyset$ is similar). Suppose that $r \in {}^\circ e \cap {}^\circ f$. This and (A4) and proposition 3.1.1(1) implies that there are $s^e, s^f \notin r$ such that $s \xrightarrow{\{e\}} s^e$ and $s \xrightarrow{\{f\}} s^f$. By proposition 3.1.2, we have $r \in {}^\circ u$, so, by proposition 3.1.1(1), $s \in r$ and $s' \notin r$. Hence, by proposition 3.1.3, there is a unique $g \in u$ such that $s \xrightarrow{\{g\}} s''$ and $s'' \notin r$, for some s''. But this produces a contradiction with the already established properties of e and f.

We then prove that $e^\circ \cap {}^\circ f = \emptyset$ by contradiction (the case $f^\circ \cap {}^\circ e = \emptyset$ is symmetric). Suppose that $r \in e^\circ \cap {}^\circ f$. From (A4) it follows that $s \xrightarrow{\{e\}} s^e$ and $s \xrightarrow{\{f\}} s^f$, for some $s^e, s^f \in S$. On the one hand, by $r \in e^\circ$ and proposition 3.1.1(2), $s \notin r$. On the other hand, by $r \in {}^\circ f$ and proposition 3.1.1(1), $s \in r$. We obtained a contradiction.

Thus $u \in V_{TS}$. □

Proposition 3.1.5 *The following hold:*

1. *$r \subseteq S$ is a region if and only if $S \setminus r$ is a region.*

2. *If $u \in U$ then $u^\circ = \{S \setminus r \mid r \in {}^\circ u\}$.*

3. *If $s \xrightarrow{u} s'$ then $R_s \setminus R_{s'} = {}^\circ u$ and $R_{s'} \setminus R_s = u^\circ$.*
   Moreover, ${}^\circ u \subseteq R_s$ and $u^\circ \cap R_s = \emptyset$ and $R_{s'} = (R_s \setminus {}^\circ u) \cup u^\circ$.

Proof: (1) follows from definition 3.1.1, and (2) is obvious. To show (3) suppose that $r \in R_s \setminus R_{s'}$. Then $s \in r$ and $s' \notin r$. This and $s \xrightarrow{u} s'$ means that $r \in {}^\circ u$. Now, let $r \in {}^\circ u$. From proposition 3.1.1 it follows that $s \in r$ and $s' \notin r$. Hence $r \in R_s \setminus R_{s'}$. We have proved that $R_s \setminus R_{s'} = {}^\circ u$. Similarly, we can show $R_{s'} \setminus R_s = u^\circ$. The second part of (3) is also easy to show. □

Proposition 3.1.6 *Let $s \in S$ and $e \in E_{TS}$ be such that ${}^\circ e \subseteq R_s$. Then $e^\circ \cap R_s = \emptyset$.*

Proof: Assume $e^\circ \cap R_s \neq \emptyset$. Let $r \in e^\circ \cap R_s$. Then $s \in r$. From $r \in e^\circ$ and proposition 3.1.5(2) we have $S \setminus r \in {}^\circ e$. But ${}^\circ e \subseteq R_s$, so $S \setminus r \in R_s$. This means $s \notin r$, a contradiction. □

The next axiom TS must fulfil is usually called the *state separation property* [9, 38].

A5 For all $s, s' \in S$, if $R_s = R_{s'}$ then $s = s'$.

It essentially means that TS is deterministic, by excluding transition systems like TS_1 in figure 3.3. Formally, we have the following result.

Proposition 3.1.7 *If $s \xrightarrow{u} s'$ and $s \xrightarrow{u} s''$ then $s' = s''$.*

Proof: From proposition 3.1.5(3) we have $R_{s'} = (R_s \setminus {}^\circ u) \cup u^\circ$ and $R_{s''} = (R_s \setminus {}^\circ u) \cup u^\circ$, which means that $R_{s'} = R_{s''}$. Hence, by (A5), $s' = s''$. □

All the notions that we have introduced so far were essentially related to the ordinary arcs appearing in ENI-systems. The next definition is different in that it attempts to capture, for each event e, those regions (conditions in the corresponding net) which are linked to e by means of an inhibitor arc. We start with an auxiliary definition. Let $e \in E_{TS}$ be an event, and $r \in R_{TS}$ be a non-trivial region. Then

$$\mathcal{B}^e_r = \{(s, \{e\}, s') \in T \mid s \in r \wedge s' \in r\}$$

is the set of all the transitions labelled by $\{e\}$ which are inside r. Having introduced $\mathcal{B}^e_r$, the set of *inhibitor-regions* (I-regions) of e is defined as follows:

$$\overset{\square}{e} = \{r \in R_{TS} \mid \mathcal{B}^e_r = \emptyset \wedge \mathcal{B}^e_{S \setminus r} \neq \emptyset\}.$$

We can extend the last notion to any set of events $u \in U$, through $\overset{\square}{u} = \bigcup_{e \in u} \overset{\square}{e}$. Referring to the transition system TS_0 in figure 3.2, one can see that $\overset{\square}{a} = \{r_4\}$, $\overset{\square}{b} = \{r_3\}$ and, according to the last definition, $\overset{\square}{\{a, b\}} = \{r_3, r_4\}$.

Proposition 3.1.8 *If* $s \xrightarrow{\{e\}} s'$ *then* $r \in \overset{\square}{e}$ *implies* $s, s' \notin r$.

Proof: $r \in \overset{\square}{e}$ means that there is $p \xrightarrow{\{e\}} p'$ such that $p, p' \notin r$, and $s \notin r$ or $s' \notin r$. Suppose $s \notin r$ and $s' \in r$. This means that $r \in e^\circ$. Hence, by proposition 3.1.1 and $p \xrightarrow{\{e\}} p'$, $p \notin r$ and $p' \in r$, which contradicts $p, p' \notin r$. Similarly, we obtain a contradiction if we assume that $s \in r$ and $s' \notin r$. □

It is straightforward to show that a step can be executed at a state only if the I-regions of the former do not comprise the latter.

Proposition 3.1.9 *If* $s \xrightarrow{u} s'$ *then* $\overset{\square}{u} \cap R_s = \emptyset$.

Proof: Suppose that $r \in \overset{\square}{u} \cap R_s \neq \emptyset$. Then there is $e \in u$ such that $r \in \overset{\square}{e}$. Hence, by proposition 3.1.8, if $p \xrightarrow{\{e\}} p'$ then $p, p' \notin r$. In particular, by (A4) and $s \xrightarrow{u} s'$ and $e \in u$, $s \notin r$. On the other hand, by $r \in R_s$, $s \in r$, a contradiction. □

We can now define our desired class of transition systems. The transition system TS is a *TSENI transition system* if it satisfies, in addition to (A1)-(A5), the following axiom:

A6 Let $s \in S$ and $u \in V_{TS}$ be such that, for every $e \in u$, ${}^{\circ}e \subseteq R_s$ and $\overset{\square}{e} \cap R_s = \emptyset$. Then $s \xrightarrow{u}$.

The last axiom is a variation of the *forward closure property* [38] or the *event/state separation property* [9]. In particular, it can be used to prove that the set of steps U is (almost) subset closed (cf. proposition 3.2.3), and also excludes transition systems like TS_2 in figure 3.3 (to make TS_2 a valid TSENI transition system one must add transition $s^2_{in} \xrightarrow{\{a,b\}} s_3$).

It is easy to check that the transition system TS_0 in figure 3.2 is indeed a TSENI transition system.

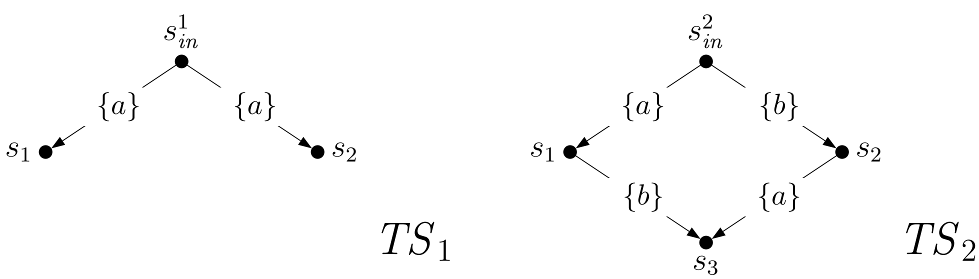


Figure 3.3: Transition systems which are not TSENI.

3.2 Properties of TSENI Transition Systems

We now formulate and prove some useful properties of the TSENI transition system TS.

Proposition 3.2.1 *For every $e \in E_{TS}$, ${}^{\circ}e$ and e° are non-empty sets and ${}^{\circ}e$, e° and $\overset{\square}{e}$ are mutually disjoint sets.*

Proof: Let $e \in E_{TS}$. From (A1), (A2) and corollary 3.1.1 it follows that there are $s \neq s' \in S$ such that $s \xrightarrow{\{e\}} s'$. From proposition 3.1.5(3) we have $R_s \setminus R_{s'} = {}^{\circ}e$ and $R_{s'} \setminus R_s = e^{\circ}$, so $e^{\circ} \cap {}^{\circ}e = \emptyset$. Moreover, by (A5), we have $R_s \neq R_{s'}$. Thus, without loss of generality, we may assume that ${}^{\circ}e \neq \emptyset$. Then there is $r \in {}^{\circ}e$, and from proposition 3.1.5(2) we get that $S \setminus r \in e^{\circ}$. Hence $e^{\circ} \neq \emptyset$. We now show that $(e^{\circ} \cup {}^{\circ}e) \cap \overset{\square}{e} = \emptyset$. Suppose that $r \in \overset{\square}{e}$. Then there exists $s \xrightarrow{\{e\}} s'$ such that $s \notin r$ and

$s' \notin r$. Hence $r \notin {}^\circ e \cup e^\circ$ because from proposition 3.1.1 we have that $r \in {}^\circ e$ would imply $s \in r$ and $s' \notin r$, and $r \in e^\circ$ would imply $s \notin r$ and $s' \in r$. □

Proposition 3.2.2 *For every $u \in U$, ${}^\circ u$ and u° are non-empty disjoint sets.*

Proof: Follows from propositions 3.1.2, 3.1.4 and 3.2.1. □

The next result implies that the set of steps U is subset closed, if we only ignore the empty subset.

Proposition 3.2.3 *If $s \xrightarrow{u}$ and $\emptyset \neq v \subset u$ then $s \xrightarrow{v}$.*

Proof: Let $s \xrightarrow{u}$. By propositions 3.1.5(3) and 3.1.9, we have ${}^\circ u \subseteq R_s$ and $\overset{\Box}{u} \cap R_s = \emptyset$. From corollary 3.1.1 it follows that $\{e\} \in U$, for all $e \in u$. Using proposition 3.1.2, we obtain that ${}^\circ e \subseteq R_s$ and $\overset{\Box}{e} \cap R_s = \emptyset$, for all $e \in u$. Also, by proposition 3.1.4, $u \in V_{TS}$. Thus all the conditions in axiom (A6) are satisfied for v. Hence $s \xrightarrow{v}$. □

It is worth noting that TSENI Transition Systems do not enjoy the 'intermediate state' property which is true of other classes of transition systems considered in the literature [37]. This property states that if a non-singleton step u is enabled at state s then for every partition v, w of u there are states q and r such that:

$$s \xrightarrow{v} q \xrightarrow{w} r.$$

That such a property does not hold for TSENI Transition Systems is easy to show. For example, we can take TS_0 in figure 3.2 with $u = \{a, b\}$ and $s = s_{in}$. Note that in this way we have also shown that the TSENI Transition Systems are not covered by any of the classes of transition systems generated by ordinary Petri nets.

Although, in general, we cannot split up a step into two consecutive steps, we still have two properties close to the 'diamond' property of transition systems [28].

Proposition 3.2.4 *Let u and v be disjoint sets of events such that $u \cup v \in V_{TS}$ and $s \in S$. If $s \xrightarrow{u}$ and $s \xrightarrow{v}$ then $s \xrightarrow{u \cup v}$.*

Proof: Follows easily from (A6) and propositions 3.1.5(3) and 3.1.9. □

Proposition 3.2.5 *If $s \xrightarrow{u} s' \xrightarrow{v} s''$ and $s \xrightarrow{v} s'''$ then $s \xrightarrow{u \cup v} s''$.*

Proof: We first show that $u \cup v \in V_{TS}$. Since $u \in V_{TS}$ and $v \in V_{TS}$, we only consider pairs of events e, f such that $e \in u$ and $f \in v$.

Suppose that $r \in {}^\circ e \cap {}^\circ f$. Then $r \in {}^\circ u \cap {}^\circ v$. From $s \xrightarrow{u} s'$ and $r \in {}^\circ u$ and proposition 3.1.1, we have $s' \notin r$. But from $s' \xrightarrow{v} s''$ and $r \in {}^\circ v$ and proposition 3.1.1, we have $s' \in r$, a contradiction. Hence ${}^\circ e \cap {}^\circ f = \emptyset$.

Suppose that $r \in e^\circ \cap f^\circ$. Then $r \in u^\circ \cap v^\circ$. From $s \xrightarrow{u} s'$ and $r \in u^\circ$ and proposition 3.1.1, we have $s' \in r$. But from $s' \xrightarrow{v} s''$ and $r \in v^\circ$ and proposition 3.1.1, we have $s' \notin r$, a contradiction. Hence $e^\circ \cap f^\circ = \emptyset$.

Suppose that $r \in {}^\circ e \cap f^\circ$ (the case $r \in {}^\circ f \cap e^\circ$ is symmetric). Then $r \in {}^\circ u \cap v^\circ$. From $s \xrightarrow{u} s'$ and $r \in {}^\circ u$ and proposition 3.1.1, we have $s \in r$. But from $s \xrightarrow{v} s'''$ and $r \in v^\circ$ and proposition 3.1.1, we have $s \notin r$, a contradiction. Hence ${}^\circ e \cap f^\circ = \emptyset$.

We have shown that $u \cup v \in V_{TS}$. Hence, by proposition 3.2.4, there is $p \in S$ such that $s \xrightarrow{u \cup v} p$. From proposition 3.1.5(3) it follows that $R_{s'} = (R_s \setminus {}^\circ u) \cup u^\circ$ and $R_{s''} = (R_{s'} \setminus {}^\circ v) \cup v^\circ$. Hence $R_{s''} = (((R_s \setminus {}^\circ u) \cup u^\circ) \setminus {}^\circ v) \cup v^\circ$. Also, by $u \cup v \in V_{TS}$, we have ${}^\circ v \cap u^\circ = \emptyset$. Hence $R_{s''} = (R_s \setminus {}^\circ(u \cup v)) \cup (u \cup v)^\circ$. Moreover, by proposition 3.1.5(3), we have $R_p = (R_s \setminus {}^\circ(u \cup v)) \cup (u \cup v)^\circ$. Hence $R_p = R_{s''}$ and, by (A5), we obtain $p = s''$. □

3.3 ENI-systems

In this section we recall (with only few notational adjustments) the definition of ENI-systems from [29]. We first define their syntax.

A *net with inhibitor arcs* is a tuple $N = (B, E, F, I)$ such that B and $E \subseteq \mathcal{E}$ are finite disjoint sets, $F \subseteq (B \times E) \cup (E \times B)$ and $I \subseteq B \times E$. The meaning and graphical representation of B (conditions), E (events) and F (flow relation) is the same as in the standard net theory. An *inhibitor* arc $(b, e) \in I$ means that e can be enabled only if b is not marked (in the diagrams, it is represented by an edge ending with a small circle). We denote, for every $x \in B \cup E$,

$$\begin{array}{rcll} {}^\bullet x &=& \{y \mid (y, x) \in F\} & \text{(pre-elements)}, \\ x^\bullet &=& \{y \mid (x, y) \in F\} & \text{(post-elements)}, \\ \overset{\blacksquare}{x} &=& \{y \mid (x, y) \in I \cup I^{-1}\} & \text{(I-elements)}. \end{array}$$

The dot-notation extends in the usual way to sets, for example, ${}^\bullet X = \bigcup_{x \in X} {}^\bullet x$. It is assumed that for every $e \in E$,

$$e^\bullet \neq \emptyset \neq {}^\bullet e \quad \text{and} \quad e^\bullet \cap {}^\bullet e = {}^\bullet e \cap \overset{\blacksquare}{e} = e^\bullet \cap \overset{\blacksquare}{e} = \emptyset. \tag{3.1}$$

An *elementary net system with inhibitor arcs* (ENI-system) is a tuple

$$\mathcal{N} = (B, E, F, I, c_{in})$$

such that $N_{\mathcal{N}} = (B, E, F, I)$ is the (underlying) net with inhibitor arcs and $c_{in} \subseteq B$ is the *initial case* (in general, any subset of B is a *case*). We will assume that $\mathcal{N}$ is fixed until the end of this section.

The concurrency semantics of ENI-systems will be based on steps of simultaneously executed events. We first define *valid steps*:

$$V_{\mathcal{N}} = \{u \subseteq E \mid u \neq \emptyset \ \wedge \ \forall\, e, f \in u \ : \ (e \neq f \Rightarrow ({}^\bullet e \cup e^\bullet) \cap ({}^\bullet f \cup f^\bullet) = \emptyset)\}. \tag{3.2}$$

The transition relation of $N_{\mathcal{N}}$, denoted by $\rightarrow_{N_{\mathcal{N}}}$, is given by:

$$\rightarrow_{N_{\mathcal{N}}} = \{(c, u, c') \in 2^B \times V_{\mathcal{N}} \times 2^B \mid c \setminus c' = {}^\bullet u \ \wedge \ c' \setminus c = u^\bullet \wedge \ \overset{\scriptscriptstyle\blacksquare}{u} \cap c = \emptyset\}. \tag{3.3}$$

The *state space* of $\mathcal{N}$, denoted by $C_{\mathcal{N}}$, is the least subset of 2^B containing c_{in} such that if $c \in C_{\mathcal{N}}$ and $(c, u, c') \in \rightarrow_{N_{\mathcal{N}}}$ then $c' \in C_{\mathcal{N}}$. The *transition relation* of $\mathcal{N}$, denoted by $\rightarrow_{\mathcal{N}}$, is then defined as $\rightarrow_{N_{\mathcal{N}}}$ restricted to $C_{\mathcal{N}} \times V_{\mathcal{N}} \times C_{\mathcal{N}}$. The set of *active steps* of $\mathcal{N}$ is given by

$$U_{\mathcal{N}} = \{u \in V_{\mathcal{N}} \mid \exists c, c' \in C_{\mathcal{N}} : \ (c, u, c') \in \rightarrow_{\mathcal{N}}\}.$$

We will use $c \xrightarrow{u}_{\mathcal{N}} c'$ to denote that $(c, u, c') \in \rightarrow_{\mathcal{N}}$. Also, $c \xrightarrow{u}_{\mathcal{N}}$ if $(c, u, c') \in \rightarrow_{\mathcal{N}}$, for some c'. Similarly, we will write $\xrightarrow{u}_{\mathcal{N}} c$ if $(c', u, c) \in \rightarrow_{\mathcal{N}}$, for some c'.

A *step sequence* of $\mathcal{N}$ is a sequence $\varrho = u_1 \ldots u_n$ of sets in $U_{\mathcal{N}}$ for which there are cases $c_1, \ldots, c_n$ satisfying $c_{in} \xrightarrow{u_1}_{\mathcal{N}} c_1$, $c_1 \xrightarrow{u_2}_{\mathcal{N}} c_2$, $\ldots$, $c_{n-1} \xrightarrow{u_n}_{\mathcal{N}} c_n$. We will denote this by $c_{in}[\varrho\rangle c_n$. For the ENI-system $\mathcal{N}$ in figure 3.1, we have the following:

$$\begin{array}{ll} \{b_1, b_2, b_5\}\,[\,\{e\}\,\rangle\,\{b_2, b_3, b_5\} & \qquad \{b_1, b_2, b_5\}\,[\,\{f\}\,\rangle\,\{b_1, b_4, b_5\} \\ \{b_1, b_2, b_5\}\,[\,\{e\}\{f\}\,\rangle\,\{b_3, b_4, b_5\} & \qquad \{b_1, b_2, b_5\}\,[\,\{e, f\}\,\rangle\,\{b_3, b_4, b_5\}. \end{array}$$

The above definition of the semantics of $\mathcal{N}$ is what is referred to as the *a-priori* semantics in [19]. The *a-posteriori* semantics will be discussed in chapter 7.

Proposition 3.3.1 *The following hold:*

1. *Let $c \in C_{\mathcal{N}}$ and $u \in V_{\mathcal{N}}$. Then $c \xrightarrow{u}_{\mathcal{N}}$ if and only if ${}^\bullet u \subseteq c$ and $(u^\bullet \cup \overset{\scriptscriptstyle\blacksquare}{u}) \cap \ c = \emptyset$.*

2. *If $c \xrightarrow{u}_{\mathcal{N}} c'$ then $c' = (c \setminus {}^\bullet u) \cup u^\bullet$.*

3. *If* $c \xrightarrow{u}_{\mathcal{N}} c'$ *and* $d \xrightarrow{u}_{\mathcal{N}} d'$ *then* $c \setminus c' = d \setminus d'$ *and* $c' \setminus c = d' \setminus d$.

4. *If* $c \xrightarrow{u}_{\mathcal{N}} c'$ *and* $c \xrightarrow{u}_{\mathcal{N}} c''$ *then* $c' = c''$.

Proof: (2), (3) and (4) follow easily from definitions. To show (1), suppose $c \xrightarrow{u}_{\mathcal{N}}$. Then there is $c' \in C_{\mathcal{N}}$ such that $c \xrightarrow{u}_{\mathcal{N}} c'$. From (3.3), ${}^\bullet u \subseteq c$ and $u^\bullet \cap c = \emptyset$ and $\overset{\blacksquare}{u} \cap c = \emptyset$. Suppose now that ${}^\bullet u \subseteq c$ and $(u^\bullet \cup \overset{\blacksquare}{u}) \cap c = \emptyset$. Define $c' = (c \setminus {}^\bullet u) \cup u^\bullet$. It is easy to show that $c \setminus c' = {}^\bullet u$ and $c' \setminus c = u^\bullet$. Hence, by (3.3) and $c \in C_{\mathcal{N}}$, $c \xrightarrow{u}_{\mathcal{N}} c'$ and thus $c \xrightarrow{u}_{\mathcal{N}}$. □

Figure 3.4 shows an example of ENI-system $\mathcal{N}_0$.

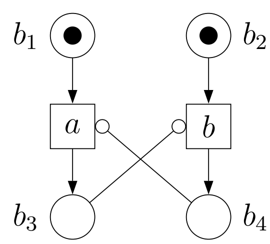


Figure 3.4: ENI-system $\mathcal{N}_0$.

Note that $\{b_1, b_2\} \xrightarrow{\{a,b\}}_{\mathcal{N}_0} \{b_3, b_4\}$ and $\{b_1, b_2\} \xrightarrow{\{a\}}_{\mathcal{N}_0} \{b_2, b_3\}$, but $\{b_2, b_3\} \xrightarrow{\{b\}}_{\mathcal{N}_0}$ does not hold.

3.4 Transition Systems of ENI-systems

The construction of a transition system for a given ENI-system is straightforward. Let $\mathcal{N} = (B, E, F, I, c_{in})$ be an ENI-system. Then

$$TS_{\mathcal{N}} = (C_{\mathcal{N}}, U_{\mathcal{N}}, \rightarrow_{\mathcal{N}}, c_{in})$$

is the *transition system generated* by $\mathcal{N}$.

Figure 3.5 shows the transition system generated by the ENI-system $\mathcal{N}_0$ in figure 3.4. Note that it is isomorphic to the TSENI transition system TS_0 in figure 3.2. Hence $TS_{\mathcal{N}_0}$ is a TSENI transition system and, as it turns out, this is true in general.

Theorem 3.4.1 *$TS_{\mathcal{N}}$ is a TSENI transition system.*

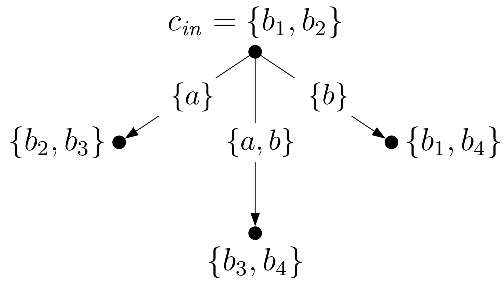


Figure 3.5: Transition system of $\mathcal{N}_0$, $TS_{\mathcal{N}_0}$.

Proof: Clearly, $TS_{\mathcal{N}}$ is a transition system. What we need to prove is that it satisfies (A1)-(A6).

(A1) Suppose $c \xrightarrow{u}_{\mathcal{N}} c'$ and $c = c'$. Then, by (3.3), $u^\bullet = {}^\bullet u = \emptyset$, contradicting (3.1).

(A2) and **(A3)** follow directly from the definition of $C_{\mathcal{N}}$ and $U_{\mathcal{N}}$.

(A4) Suppose $c \xrightarrow{u}_{\mathcal{N}}$ and $e \in u$. By proposition 3.3.1(1), ${}^\bullet u \subseteq c$ and $(u^\bullet \cup {}^{\blacksquare}u) \cap c = \emptyset$. We also have ${}^\bullet e \subseteq {}^\bullet u$, $e^\bullet \subseteq u^\bullet$ and ${}^{\blacksquare}e \subseteq {}^{\blacksquare}u$, so ${}^\bullet e \subseteq c$ and $(e^\bullet \cup {}^{\blacksquare}e) \cap c = \emptyset$. Thus, from proposition 3.3.1(1) it follows that $c \xrightarrow{\{e\}}_{\mathcal{N}}$.

Before proving (A5) and (A6) we show that, for every $b \in B$, $r_b = \{c \in C_{\mathcal{N}} \mid b \in c\}$ is (possibly trivial) region in $TS_{\mathcal{N}}$. Moreover,

$$\emptyset \neq r_b \neq C_{\mathcal{N}} \Rightarrow r_b \in R_{TS_{\mathcal{N}}}. \tag{3.4}$$

Suppose $c \xrightarrow{u}_{\mathcal{N}} c'$, where $c \in r_b$ and $c' \notin r_b$. Then $b \in c$ and $b \notin c'$. By (3.3), $c \setminus c' = {}^\bullet u$ and $c' \setminus c = u^\bullet$. Hence $b \in {}^\bullet u$ and $b \notin u^\bullet$, and we can choose $e \in u$ such that $b \in {}^\bullet e$. We now observe that if $d \xrightarrow{v}_{\mathcal{N}} d'$ and $e \in v$ then $d \in r_b$ and $d' \notin r_b$ (since, by (3.3), $b \in d$ and $b \notin d'$). Moreover, if $v \subseteq u \setminus \{e\}$ and $c \xrightarrow{v}_{\mathcal{N}} c''$ then $c'' \in r_b$, since by (3.2), $b \notin v^\bullet \cup {}^\bullet v$. The second part of definition 3.1.1 can be shown in a similar way. Hence r_b is a region in $TS_{\mathcal{N}}$. Clearly, if $\emptyset \neq r_b \neq C_{\mathcal{N}}$ then r_b is a non-trivial region and (3.4) holds.

(A5) Suppose that $c \neq c' \in C_{\mathcal{N}}$. Without loss of generality we may assume that there is $b \in c \setminus c'$. Hence $c \in r_b$ and $c' \notin r_b$. Thus, by (3.4) and $r_b \in R_c \setminus R_{c'}$, (A5) holds.

(A6) Suppose that $c \in C_{\mathcal{N}}$ and $u \in V_{TS_{\mathcal{N}}}$ are such that, for every $e \in u$, ${}^\circ e \subseteq R_c$ and ${}^{\square}e \cap R_c = \emptyset$. We first show that $c \xrightarrow{\{e\}}_{\mathcal{N}}$, for every $e \in u$. Let $e \in u$. Since $e \in E_{TS_{\mathcal{N}}}$ and (A4) and (A2) hold, there are $d, d' \in C_{\mathcal{N}}$ such that $d \xrightarrow{\{e\}}_{\mathcal{N}} d'$.

Consider any $b \in {}^\bullet e$. Then $b \in d$ and $b \notin d'$, and so $d \in r_b$ and $d' \notin r_b$. Hence, by (3.4), $r_b \in R_{TS_{\mathcal{N}}}$ and $r_b \in {}^\circ e$. From ${}^\circ e \subseteq R_c$ we have $r_b \in R_c$ which means $b \in c$. As a result, ${}^\bullet e \subseteq c$.

Consider now any $b \in e^\bullet$. Then $b \notin d$ and $b \in d'$, and so $d \notin r_b$ and $d' \in r_b$. Hence, by (3.4), $r_b \in e^\circ$. This and $e^\circ \cap R_c = \emptyset$ (follows from ${}^\circ e \subseteq R_c$ and proposition 3.1.6) means that $r_b \notin R_c$, and so $b \notin c$. Hence $e^\bullet \cap c = \emptyset$.

Suppose that $\overset{\blacksquare}{e} \cap c \neq \emptyset$. Then there is $b \in \overset{\blacksquare}{e}$ such that $b \in c$, and so $c \in r_b$. By (3.3) and $\overset{\blacksquare}{e} \cap e^\bullet = \emptyset$, $b \notin d$ and $b \notin d'$. Thus $d \notin r_b$ and $d' \notin r_b$. As a result, by (3.4), $r_b \in R_{TS_\mathcal{N}}$ and $d, d' \in C_\mathcal{N} \setminus r_b$. Hence $\mathcal{B}^e_{C_\mathcal{N} \setminus r_b} \neq \emptyset$. Suppose now that $f \xrightarrow{\{e\}}_\mathcal{N} f'$ belongs to $\mathcal{B}^e_{r_b}$. This means $f, f' \in r_b$ and we have $b \in f$ and $b \in f'$. But this and (3.3) contradict $b \in \overset{\blacksquare}{e}$. Hence $\mathcal{B}^e_{r_b} = \emptyset$ and, as a result, $r_b \in \overset{\square}{e}$. Since $\overset{\square}{e} \cap R_c = \emptyset$, $r_b \notin R_c$ which means $b \notin c$, a contradiction with $b \in \overset{\blacksquare}{e} \cap c$. Hence $\overset{\blacksquare}{e} \cap c = \emptyset$ which, together with ${}^\bullet e \subseteq c$ and $e^\bullet \cap c = \emptyset$, yields $c \xrightarrow{\{e\}}_\mathcal{N}$.

We proved that $c \xrightarrow{\{e\}}_\mathcal{N}$, for every $e \in u$. Moreover, we have already shown that $b \in {}^\bullet e$ implies $r_b \in {}^\circ e$, and $b \in e^\bullet$ implies $r_b \in e^\circ$, for all $e \in u$. This and $u \in V_{TS_\mathcal{N}}$ means that $u \in V_\mathcal{N}$. Hence $c \xrightarrow{u}_\mathcal{N}$. □

3.5 ENI-systems of TSENI Transition Systems

The translation from TSENI Transition Systems to ENI-systems is based on the pre- post- and I-regions of events appearing in a transition system. Let $TS = (S, U, T, s_{in})$ be a TSENI transition system. The net system *associated* with TS is defined as

$$\mathcal{N}_{TS} = (R_{TS}, E_{TS}, F_{TS}, I_{TS}, R_{s_{in}})$$

where F_{TS} and I_{TS} are defined thus:

$$\begin{aligned} F_{TS} &= \{(r,e) \in R_{TS} \times E_{TS} \mid r \in {}^\circ e\} \cup \{(e,r) \in E_{TS} \times R_{TS} \mid r \in e^\circ\} \\ I_{TS} &= \{(r,e) \in R_{TS} \times E_{TS} \mid r \in \overset{\square}{e}\}. \end{aligned} \tag{3.5}$$

Directly from the definition of $\mathcal{N}_{TS}$ we obtain that, for every $e \in E_{TS}$,

$${}^\circ e = {}^\bullet e \quad \text{and} \quad e^\circ = e^\bullet \quad \text{and} \quad \overset{\square}{e} = \overset{\blacksquare}{e}\,. \tag{3.6}$$

Theorem 3.5.1 *$\mathcal{N}_{TS}$ is an ENI-system.*

Proof: We can always assume that $R_{TS} \cap E_{TS} = \emptyset$. From (3.5) we have $F_{TS} \subseteq R_{TS} \times E_{TS} \cup E_{TS} \times R_{TS}$ and $I \subseteq R_{TS} \times E_{TS}$. Hence to prove that $\mathcal{N}_{TS}$ is an ENI-system it suffices to show that (3.1) holds. By proposition 3.2.1 and (3.6), ${}^\bullet e$ and $e^\bullet$ and $\overset{\blacksquare}{e}$ are mutually disjoint sets, for all $e \in E_{TS}$. Moreover, again by proposition 3.2.1 and (3.6), ${}^\bullet e \neq \emptyset \neq e^\bullet$. Hence (3.1) holds. □

The above construction would produce a net which is saturated both with conditions and inhibitor arcs. Notice however that, due to corollary 3.1.2 and (3.6), $\mathcal{N}_{TS}$ does not contain any isolated conditions nor trivial inhibitor arcs (i.e. the inhibitor arcs (b, e) such that ${}^\bullet b = \emptyset = b^\bullet$).

3.6 Consistency of the Translations

In this section we show that the ENI-system associated with a TSENI transition system TS generates a transition system which is isomorphic to TS.

Proposition 3.6.1 *Let $TS = (S, U, T, s_{in})$ be a TSENI transition system and $\mathcal{N} = \mathcal{N}_{TS}$ be the ENI-system associated with it.*

1. $C_{\mathcal{N}} = \{R_s \mid s \in S\}$.
2. $\rightarrow_{\mathcal{N}} = \{(R_s, u, R_{s'}) \mid (s, u, s') \in T\}$.

Proof: We first note that from the definition of $C_{\mathcal{N}}$, every $c \in C_{\mathcal{N}}$ is reachable from c_{in} in $\mathcal{N}$, and from axiom (A3), that every $s \in S$ is reachable from s_{in} in TS.

We first show that if $c \xrightarrow{u}_{\mathcal{N}} c'$ and $c = R_s$, for some $s \in S$, then there is $s' \in S$ such that $s \xrightarrow{u} s'$ and $c' = R_{s'}$. We have that $c \setminus c' = {}^\bullet u$ and $c' \setminus c = u^\bullet$ and $\overset{\blacksquare}{u} \cap c = \emptyset$. This means ${}^\bullet e \subseteq c$ and $\overset{\blacksquare}{e} \cap c = \emptyset$, for all $e \in u$. This and (3.6) implies that ${}^\circ e \subseteq c$ and $\overset{\square}{e} \cap c = \emptyset$, for all $e \in u$. Hence ${}^\circ e \subseteq R_s$ and $\overset{\square}{e} \cap R_s = \emptyset$, for all $e \in u$. Moreover, by $u \in V_{\mathcal{N}}$ and (3.6), we have $u \in V_{TS}$. Hence from (A6) it follows that $s \xrightarrow{u} s'$, for some $s' \in S$. Then, by proposition 3.1.5(3), $R_{s'} = (R_s \setminus {}^\circ u) \cup u^\circ$. At the same time, from proposition 3.3.1(2), $c' = (c \setminus {}^\bullet u) \cup u^\bullet$. Hence, by (3.6) and proposition 3.1.2 and $c = R_s$, $c' = R_{s'}$.

As a result, we have shown (note that $c_{in} = R_{s_{in}} \in \{R_s \mid s \in S\}$) that $C_{\mathcal{N}} \subseteq \{R_s \mid s \in S\}$ and $\rightarrow_{\mathcal{N}} \subseteq \{(R_s, u, R_{s'}) \mid (s, u, s') \in T\}$.

We now will prove that $\{R_s \mid s \in S\} \subseteq C_{\mathcal{N}}$. By definition, $R_{s_{in}} \in C_{\mathcal{N}}$. What needs to be shown is that if $s \xrightarrow{u} s'$ and $R_s \in C_{\mathcal{N}}$ then $R_{s'} \in C_{\mathcal{N}}$. By propositions 3.1.5(3) and 3.1.9, we have ${}^\circ u \subseteq R_s$ and $(u^\circ \cup \overset{\square}{u}) \cap R_s = \emptyset$. So, using (3.6) and proposition 3.1.2, ${}^\bullet u \subseteq R_s$ and $(u^\bullet \cup \overset{\blacksquare}{u}) \cap R_s = \emptyset$. Moreover, from proposition 3.1.4 and (3.6) we obtain that u is a valid step in $\mathcal{N}$. Hence, by proposition 3.3.1(1), we have $R_s \xrightarrow{u}_{\mathcal{N}}$. This implies

$(R_s \setminus {}^\bullet u) \cup u^\bullet \in C_\mathcal{N}$. On the other hand, by proposition 3.1.5(3) and $s \xrightarrow{u} s'$, we have $R_{s'} = (R_s \setminus {}^\circ u) \cup u^\circ$. Hence, by (3.6) and proposition 3.1.2, $R_{s'} \in C_\mathcal{N}$.

What remains to be shown is that $\{(R_s, u, R_{s'}) \mid (s, u, s') \in T\} \subseteq \rightarrow_\mathcal{N}$. Suppose $s \xrightarrow{u} s'$. From propositions 3.1.5(3) and 3.1.9 it follows that $R_s \setminus R_{s'} = {}^\circ u$, $R_{s'} \setminus R_s = u^\circ$ and $\overset{\square}{u} \cap R_s = \emptyset$. We have already proved that $C_\mathcal{N} = \{R_s \mid s \in S\}$. So there are $c, c' \in C_\mathcal{N}$ such that $c = R_s$ and $c' = R_{s'}$. From (3.6) and proposition 3.1.2 it follows that $c \setminus c' = {}^\bullet u$ and $c' \setminus c = u^\bullet$ and $\overset{\blacksquare}{u} \cap c = \emptyset$. Since $s \xrightarrow{u} s'$, from proposition 3.1.4 and (3.6), it follows that u is a valid step. Hence, by (3.3), $c \xrightarrow{u}_\mathcal{N} c'$. $\square$

Theorem 3.6.1 *Let $TS = (S, U, T, s_{in})$ be a TSENI transition system and $\mathcal{N} = \mathcal{N}_{TS}$ be the ENI-system associated with it. Then $TS_\mathcal{N}$ is isomorphic to TS.*

Proof: Let $\psi : S \to C_\mathcal{N}$ be a mapping given by $\psi(s) = R_s$, for all $s \in S$ (note that, by proposition 3.6.1(1), ψ is well defined). We will show that ψ is an isomorphism for TS and $TS_\mathcal{N}$.

Note that $\psi(s_{in}) = R_{s_{in}}$. From proposition 3.6.1(1) it follows that ψ is onto. Moreover, by (A5), ψ is injective. Hence ψ is a bijection. We then observe that, by proposition 3.6.1(2), $(s, u, s') \in T$ if and only if $(\psi(s), u, \psi(s')) \in \rightarrow_\mathcal{N}$. Hence ψ is an isomorphism for TS and $TS_\mathcal{N}$. $\square$

3.7 A-priori Behaviour in Asynchronous Circuits

Asynchronous circuits, unlike synchronous ones, are not regulated by any special clock which controls the entire circuit. As a consequence, signal delays must be taken into account in any asynchronous circuit theory. The survey [15] presented the historical development of the subject of asynchronous circuits theory, and discussed those aspects of the theory that deal with the behaviour of circuits under the assumption that there are (unbounded - Part I, or bounded - Part II) delays in the circuit components and wires. In Part I, several models of the behaviour of asynchronous sequential circuits[1] were discussed. They differ in the assumptions made about the delays present in a circuit. The 'gate-and-wire-delay' model associates a delay with every gate and every wire in the circuit, while the 'gate-delay' model only assumes delays in gates. In the 'feedback-delay' model,

[1]A circuit is called sequential when its output cannot be determined by the input signals only (like in combinational circuits) but depends as well on the past history of the circuit.

the delays are associated with 'feedback wires', i.e. the wires whose removal deletes all the feedback loops from the circuit. The 'cutting' of feedback wires produces a circuit which can be analysed like a combinational circuit where the output depends on the input signals and the feedback set (the set of variables associated with the 'cut' feedback wires). The feedback set is not uniquely defined for a given circuit.

The following example, taken from [15] and first introduced in [34], explains the behaviour of a circuit depicted in figure 3.6 under the assumptions of the feedback-delay model. The state of a circuit is determined by seven state variables: one input variable,

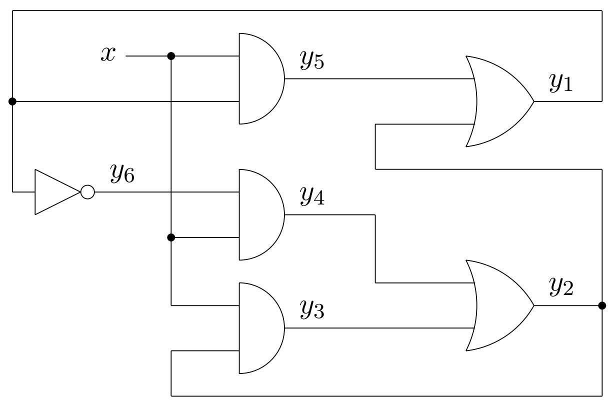


Figure 3.6: Langdon's circuit.

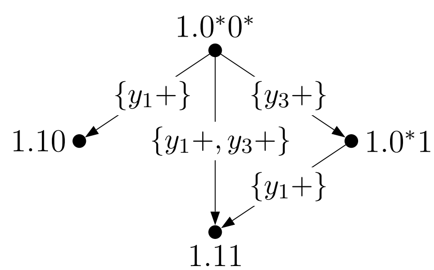
x, and six delay variables: $y_1, \ldots, y_6$. The changes of the input value introduce changes of other variables in the circuit. We will denote by Y_i the *new* (next) value of y_i $(i = 1, \ldots, 6)$ which are the functions of all input and delay variables. The state $(x, y_1, \ldots, y_6)$ of the circuit is *stable* if $Y_i(x, y_1, \ldots, y_6) = y_i$, for every $i = 1, \ldots, 6$. For the Langdon's circuit the functions Y_i are as follows:

$$
\begin{array}{rcrcl}
Y_1 & = & y_2 & \vee & y_5 \\
Y_2 & = & y_3 & \vee & y_4 \\
Y_3 & = & x & \wedge & y_2 \\
Y_4 & = & x & \wedge & y_6 \\
Y_5 & = & x & \wedge & y_1 \\
Y_6 & = & \neg y_1 & &
\end{array}
$$

Suppose that the feedback set is $\{y_1, y_3\}$. The next values of the state variables, expressed in terms of x, y_1, y_3, are:

$$\begin{array}{rcrcl} Y_1 & = & x & \vee & y_3 \\ Y_2 & = & y_3 & \vee & (x \wedge (\neg y_1)) \\ Y_3 & = & x & \wedge & (y_3 \vee (\neg y_1)) \\ Y_4 & = & x & \wedge & (\neg y_1) \\ Y_5 & = & x & \wedge & y_1 \\ Y_6 & = & \neg y_1 & & \end{array}$$

Starting in the state 0.000001 (the full stop is used to separate the values of the input and delay variables) and changing the input to 1 gives the state 1.0*10*101 in which both y_1 and y_3 are unstable (indicated by an asterisk written after the value of the variable). If y_1 changes first, the circuit reaches the state 1.100010 which is stable. Although y_3 was unstable for some time, it is no longer unstable. It has 'lost the race' to y_1. If y_3 changes first, the circuit reaches the state 1.0*11101 in which y_1 is unstable. This instability will lead to the next change producing the stable state 1.111010. If y_1 and y_3 respond to the change of x at 'exactly' the same time, the state 1.111010 is reached directly from 1.0*10*101. The diagram below shows the transition system of the circuit behaviour described above projected on the subset of signals $\{x, y_1, y_3\}$. The states are represented by triples of the form $x.y_1y_3$. Note that the transition system depicted below



is a TSENI transition system. It should be added that although the set of stable states is independent from the choice of a feedback set, the transitions among the states are not. The choice of a different feedback set in the example above, e.g. $\{y_1, y_2\}$, would produce a different transition system.

More recently, in [48], the precise modelling of asynchronous controller and arbiter behaviour was discussed. The stress was put on the specification of all possible causal relationships between events (rising and falling of input and output signals) in the circuit. Firstly, step transition systems were found more appropriate for the modelling purposes, as their graphs show which concurrent events might accidentally occur simultaneously. Secondly, one of the introduced relationships between events, called biased concurrency (b-concurrency) has the form of 'weak causality' of [29]. Formally, b-concurrency is defined as follows (see [48]). An input signal IS is b-concurrent to an output signal OS if and only if

1. the environment generates the IS independently of the OS, and

2. the IS prevents the OS if the OS has not yet occurred.

At the net level, modelling b-concurrency requires nets with inhibitor arcs equipped with the a-priori concurrent semantics. The b-concurrency can be useful for the 'pessimistic' design. In general, however, in the models which specify required and desired behaviour, the designer does not wish the OS to be prevented. For that reason, [48] introduces another relation based on b-concurrency, called time-constrained b-concurrency (tcb-concurrency), which assumes that OS has to be generated before or at the latest simultaneously with IS.

Chapter 4

Morphisms for ENI and TSENI

In this chapter we introduce morphisms for the ENI-systems, and for the TSENI Transition Systems. The *net morphisms* we define are of the form $(\alpha, \beta) : \mathcal{N}_1 \to \mathcal{N}_2$, where α is a partial function mapping conditions of $\mathcal{N}_2$ into conditions of $\mathcal{N}_1$ and β is a partial function which maps events of $\mathcal{N}_1$ into events of $\mathcal{N}_2$, and are similar to the N-morphisms of [38]. Net morphisms preserve the environments of events and initial cases, in the sense that conditions of $\mathcal{N}_1$ and $\mathcal{N}_2$ related by α either both belong or both do not belong to their respective initial cases. The crucial difference is due to the presence of inhibitor arcs, and we require that $\mathcal{N}_2$ exhibits at least the same degree of concurrency as $\mathcal{N}_1$. Our net morphisms, unlike the N-morphisms of [38], do not enjoy the property of being uniquely determined by the way events are mapped. That is, two net morphisms $(\alpha_i, \beta_i) : \mathcal{N}_1 \to \mathcal{N}_2$ $(i = 1, 2)$ which satisfy $\beta_1 = \beta_2$ can be different. This is due to the fact that we do not require nets to be simple and allow isolated conditions. *Transition system morphisms* are of the form $(\sigma, \eta) : TS_1 \to TS_2$, where σ is a total function mapping states of TS_1 into states of TS_2 and η is a partial function which maps events of TS_1 into events of TS_2, and are similar to the transition system morphisms defined in [47]. Transition system morphisms preserve initial states, and step transitions if a step in TS_1 is mapped into a non-empty set of events of TS_2. The partiality of η allows for some events of TS_1 to be treated as internal, and such events are not required to be mapped into any events of TS_2. Labelling transitions with steps, rather than with single events, can be viewed as embedding an independence relation on events explicitly in the graph of the transition system. Morphisms between TSENI transition systems enjoy the property of being uniquely determined by the way events are mapped. That is, two transition system morphisms $(\sigma_i, \eta_i) : \mathcal{N}_1 \to \mathcal{N}_2$ $(i = 1, 2)$ which satisfy $\eta_1 = \eta_2$ are the same.

4.1 Transition Systems Morphisms

In this section we introduce morphisms between TSENI transition systems. Below, for any (partial or total) function $f : X \to Y$ we will denote by $dom(f)$ the domain of f, by $codom(f)$ the codomain of f, and by $\widehat{f}$ the lifting of f to a total function $\widehat{f} : 2^X \to 2^Y$ defined, for every $X' \subseteq X$, by

$$\widehat{f}(X') = f(X' \cap dom(f)).$$

Definition 4.1.1 *Let $TS_i = (S_i, U_i, T_i, s^i_{in})$ (for $i = 1, 2$) be TSENI transition systems. A* transition system morphism *from TS_1 to TS_2 is a pair of functions $f = (\sigma, \eta) : TS_1 \to TS_2$ such that the following hold.*

MTS1 *$\sigma : S_1 \to S_2$ is a total function satisfying $\sigma(s^1_{in}) = s^2_{in}$.*

MTS2 *$\eta : E_{TS_1} \to E_{TS_2}$ is a partial function, which is injective on every $u \in U_1$.*

MTS3 *For every $(s, u, s') \in T_1$, either $\widehat{\eta}(u) = \emptyset$ and $\sigma(s) = \sigma(s')$, or $(\sigma(s), \widehat{\eta}(u), \sigma(s')) \in T_2$.* □

Transition system morphisms defined above are similar to the ones defined in [47]. They preserve the initial states, and step transitions if a step in TS_1 is mapped into a non-empty set of events of TS_2. The partiality of η allows for some events of TS_1 to be treated as internal, and such events are not required to be mapped into any events of TS_2. Directly from (MTS3), (A2), and proposition 3.1.4, we obtain the following.

Corollary 4.1.1 *For every $u \in U_1$, $\widehat{\eta}(u) \in V_{TS_2} \cup \{\emptyset\}$.*

This means that transition system morphisms preserve the independence of events locally within the steps, due to (MTS3) where steps are used instead of individual events. There is no need to assume this separately, like it was done for the morphisms of Asynchronous Transition Systems in [47]. Notice that labelling transitions with steps, rather than with individual events only, can be viewed as embedding an independence relation on events explicitly in the graph of a transition system. The first result we prove shows that a transition system morphism $f : TS_1 \to TS_2$ is determined by the way in which steps in TS_1 have been transformed into steps in TS_2.

Proposition 4.1.1 *Let TS_1 and TS_2 be TSENI transition systems, and $f = (\sigma_f, \eta_f)$ and $g = (\sigma_g, \eta_g)$ be transition system morphisms from TS_1 to TS_2 such that $\eta_f = \eta_g$. Then $f = g$.*

Proof: We prove that $\sigma_f(s) = \sigma_g(s)$, for every $s \in S_1$, by induction on the smallest number k of transitions it takes to reach s from s^1_{in} (such an induction is valid due to (A3)).

The base case is $k = 0$. Then $s = s^1_{in}$ and we have, by (MTS1), $\sigma_f(s^1_{in}) = s^2_{in} = \sigma_g(s^1_{in})$.

Suppose that $k > 0$. Let ϱ be a path of length k defined by the following sequence of transitions in T_1:

$$(s^1_{in}, u_1, s_1), (s_1, u_2, s_2), \ldots, (s_{k-2}, u_{k-1}, s_{k-1}), (s_{k-1}, u_k, s).$$

By the induction hypothesis, $\sigma_f(s_{k-1}) = \sigma_g(s_{k-1})$. We consider two cases (recall that $\widehat{\eta}_f = \widehat{\eta}_g$).

Case 1: $\widehat{\eta}_f(u_k) = \widehat{\eta}_g(u_k) = \emptyset$. Then from (MTS3) it follows that

$$\sigma_f(s) = \sigma_f(s_{k-1}) = \sigma_g(s_{k-1}) = \sigma_g(s).$$

Case 2: $\widehat{\eta}_f(u_k) = \widehat{\eta}_g(u_k) \neq \emptyset$. Then from (MTS3) it follows that

$$(\sigma_f(s_{k-1}), \widehat{\eta}_f(u_k), \sigma_f(s)) \in T_2 \quad \text{and} \quad (\sigma_g(s_{k-1}), \widehat{\eta}_g(u_k), \sigma_g(s)) \in T_2.$$

As TS_2 is a TSENI transition system, it is deterministic, by proposition 3.1.7. Hence, by $\sigma_f(s_{k-1}) = \sigma_g(s_{k-1})$ and $\widehat{\eta}_f(u_k) = \widehat{\eta}_g(u_k)$, we obtain $\sigma_f(s) = \sigma_g(s)$. $\square$

In the next proposition we show how transition system morphisms preserve regions.

Proposition 4.1.2 *Let TS_1 and TS_2 be TSENI transition systems and $f = (\sigma, \eta) : TS_1 \to TS_2$ be a transition system morphism from TS_1 to TS_2. If $r \subseteq S_2$ is a region in TS_2 then $\sigma^{-1}(r)$ is a region in TS_1. Moreover, the following hold.*

1. *For all $u \in U_1$,* $\quad \sigma^{-1}(r) \in {}^\circ u \Leftrightarrow \widehat{\eta}(u) \neq \emptyset \wedge r \in {}^\circ\widehat{\eta}(u)$
   $\quad \sigma^{-1}(r) \in u^\circ \Leftrightarrow \widehat{\eta}(u) \neq \emptyset \wedge r \in \widehat{\eta}(u)^\circ$.
2. *For all $e \in dom(\eta)$,* $\quad \sigma^{-1}(r) \in \overset{\Box}{e} \Leftarrow r \in \eta(\overset{\Box}{e}) \wedge \sigma^{-1}(r) \neq \emptyset$.

Proof: We first show that $r' = \sigma^{-1}(r)$ is a region in TS_1.

Suppose that $(s, u, s') \in T_1$, $s \in r'$ and $s' \notin r'$. Then $\sigma(s) \in r$ and $\sigma(s') \notin r$. Hence $\sigma(s) \neq \sigma(s')$ and so, by (MTS3), $\widehat{\eta}(u) \neq \emptyset$ and $(\sigma(s), \widehat{\eta}(u), \sigma(s')) \in T_2$. Let $d \in \widehat{\eta}(u)$ be the r-crossing event in $\widehat{\eta}(u)$. From (A4) it follows that $\sigma(s) \xrightarrow{\{d\}}$ in TS_2. Let $e \in u$ be the unique event[1] such that $d = \eta(e)$. Again, from (A4) it follows that $s \xrightarrow{\{e\}}$ in TS_1. We will show that e is the r'-crossing event in u and thus that r' is a region (since the argument is symmetric if $s \notin r'$ and $s' \in r'$).

Consider $w \subseteq u \setminus \{e\}$ such that $(s, w, q) \in T_1$. To show that $q \in r'$ we consider two cases:

Case 1: $\widehat{\eta}(w) = \emptyset$. Then, by (MTS3), $\sigma(s) = \sigma(q)$. Hence $\sigma(q) \in r$ and so $q \in r'$.

Case 2: $\widehat{\eta}(w) \neq \emptyset$. Then, by (MTS3), $(\sigma(s), \widehat{\eta}(w), \sigma(q)) \in T_2$. Since η is injective on steps in TS_1, $d \notin \widehat{\eta}(w)$. Hence, since r is a region and d is the r-crossing event in $\widehat{\eta}(u)$, $\sigma(q) \in r$. Thus $q \in r'$.

Consider now $(q, v, q') \in T_1$ such that $e \in v$. We need to show that $q \in r'$ and $q' \notin r'$. Since $\eta(e) = d \in \widehat{\eta}(v) \neq \emptyset$, by (MTS3), $(\sigma(q), \widehat{\eta}(v), \sigma(q')) \in T_2$. Hence, since r is a region and d is the r-crossing event in $\widehat{\eta}(u)$, $\sigma(q) \in r$ and $\sigma(q') \notin r$. Thus $q \in r'$ and $q' \notin r'$.

We now move on to the second part of the proposition. Let $u \in U_1$. To show the ($\Rightarrow$) implication, we proceed as follows. By $\sigma^{-1}(r) \in {}^\circ u$, there exists $(s, u, s') \in T_1$ such that $s \in \sigma^{-1}(r)$ and $s' \notin \sigma^{-1}(r)$. Hence $\sigma(s) \in r$ and $\sigma(s') \notin r$ which means $\sigma(s) \neq \sigma(s')$. Thus, by (MTS3), $(\sigma(s), \widehat{\eta}(u), \sigma(s')) \in T_2$. Hence $\widehat{\eta}(u) \neq \emptyset$ and $r \in {}^\circ\widehat{\eta}(u)$.

To show the reverse ($\Leftarrow$) implication, assume that $\widehat{\eta}(u) \neq \emptyset$ and $r \in {}^\circ\widehat{\eta}(u)$. By $u \in U_1$ and (A2), there is a transition $(q, u, q') \in T_1$. From (MTS3) it follows that $(\sigma(q), \widehat{\eta}(u), \sigma(q')) \in T_2$. Moreover, from proposition 3.1.1(1) and $r \in {}^\circ\widehat{\eta}(u)$, we have $\sigma(q) \in r$ and $\sigma(q') \notin r$. Hence $q \in \sigma^{-1}(r)$ and $q' \notin \sigma^{-1}(r)$, and thus $\sigma^{-1}(r) \in {}^\circ u$. A similar argument applies to post-regions.

Finally, we will prove that, for every $e \in dom(\eta)$, $r \in \overset{\square}{\eta(e)}$ and $\sigma^{-1}(r) \neq \emptyset$ imply $\sigma^{-1}(r) \in \overset{\square}{e}$. Let $d = \eta(e)$. From (A2) and corollary 3.1.1, we have $(s, \{e\}, s') \in T_1$, for some s and s'. Hence, by (MTS3), $(\sigma(s), \{d\}, \sigma(s')) \in T_2$. This, proposition 3.1.8, and $r \in \overset{\square}{d}$ yield $\sigma(s), \sigma(s') \notin r$. Hence $s, s' \notin \sigma^{-1}(r)$. Moreover, $\sigma^{-1}(r) \neq \emptyset$, so $\sigma^{-1}(r)$ is non-trivial and $\mathcal{B}^e_{S_1 \setminus \sigma^{-1}(r)} \neq \emptyset$. Suppose now that $\mathcal{B}^e_{\sigma^{-1}(r)} \neq \emptyset$. Then there is $(s, \{e\}, s') \in T_1$ such that $s, s' \in \sigma^{-1}(r)$ and, consequently, $\sigma(s), \sigma(s') \in r$. Moreover, from $(s, \{e\}, s') \in T_1$ and (MTS3) it follows that $(\sigma(s), \{d\}, \sigma(s')) \in T_2$. Hence $\mathcal{B}^d_r \neq \emptyset$, which contradicts $r \in \overset{\square}{d}$. As a result, $\mathcal{B}^e_{\sigma^{-1}(r)} = \emptyset$, and so $\sigma^{-1}(r) \in \overset{\square}{e}$. □

[1]Notice that the injectivity of η on every step $u \in U_1$ guarantees the uniqueness of e.

The condition $\sigma^{-1}(r) \neq \emptyset$ in the last implication of proposition 4.1.2 is needed to guarantee that region $\sigma^{-1}(r)$ is non-trivial. This problem is discussed in example 4.1.2.

Example 4.1.1 *The implication in proposition 4.1.2(2) cannot be reversed. Figure 4.1 shows two transitions systems such that for a suitable morphism and* $e \in dom(\eta)$, $\sigma^{-1}(r) \in \overset{\square}{e}$ *but* $r \notin \eta(\overset{\square}{e})$. *The details of this counterexample are as follows. The morphism* $\widetilde{f} = (\widetilde{\sigma}, \widetilde{\eta}) : TS_1 \to TS_2$ *is defined by:*

$$\begin{array}{rclcrclcrcl} \widetilde{\sigma}(s_{in}) & = & s'_{in} & & \widetilde{\sigma}(s_1) & = & s'_1 & & \widetilde{\sigma}(s_2) & = & s'_2 \\ \widetilde{\sigma}(s_3) & = & s'_3 & & \widetilde{\eta}(a) & = & a' & & \widetilde{\eta}(b) & = & b'. \end{array}$$

The regions in TS_1 *are:*

$$\begin{array}{rclcrcl} r_1 & = & \{s_{in}, s_1\} & & r_2 & = & \{s_{in}, s_2\} \\ r_3 & = & \{s_2, s_3\} & & r_4 & = & \{s_1, s_3\} \end{array}$$

and the pre-regions, post-regions and I-regions of events are given by:

$$\begin{array}{rclcrclcrcl} {}^\circ a & = & \{r_2\} & & a^\circ & = & \{r_4\} & & \overset{\square}{a} & = & \{r_3\} \\ {}^\circ b & = & \{r_1\} & & b^\circ & = & \{r_3\} & & \overset{\square}{b} & = & \{r_4\}. \end{array}$$

The regions in TS_2 *are:*

$$\begin{array}{rclcrcl} r'_1 & = & \{s'_{in}, s'_1\} & & r'_2 & = & \{s'_{in}, s'_2\} \\ r'_3 & = & \{s'_2, s'_3\} & & r'_4 & = & \{s'_1, s'_3\} \end{array}$$

and the pre-regions, post-regions and I-regions of events are given by:

$$\begin{array}{rclcrclcrcl} {}^\circ a' & = & \{r'_2\} & & a'^\circ & = & \{r'_4\} & & \overset{\square}{a'} & = & \emptyset \\ {}^\circ b' & = & \{r'_1\} & & b'^\circ & = & \{r'_3\} & & \overset{\square}{b'} & = & \emptyset. \end{array}$$

Since $\widetilde{\sigma}^{-1}(r'_i) = r_i$, *for* $i = 1, 2, 3, 4$, *to produce a counterexample one can take* $e = a$ *and* $r = r'_3$, *or* $e = b$ *and* $r = r'_4$. □

Example 4.1.2 *Proposition 4.1.2 states that, if* $r \subseteq S_2$ *is a region in* TS_2 *then* $\sigma^{-1}(r)$ *is a region in* TS_1. *Notice that it does not assume that* r *nor* $\sigma^{-1}(r)$ *are non-trivial regions. It may happen that for a non-trivial* r, $\sigma^{-1}(r)$ *will be trivial. Consider, for example, the transition systems in figure 4.1, and the transition system morphism* $f = (\sigma, \eta) : TS_1 \to TS_2$ *defined as follows:*

$$\begin{array}{rclcrclcrcl} \sigma(s_{in}) & = & s'_{in} & & \sigma(s_1) & = & s'_1 & & \sigma(s_2) & = & s'_{in} \\ \sigma(s_3) & = & s'_1 & & \eta(a) & = & a' & & \eta(b) & - & \textit{not defined}. \end{array}$$

In the above example σ *is not injective. Observe that* $\sigma^{-1}(r'_1) = S_1$ *and* $\sigma^{-1}(r'_3) = \emptyset$. □

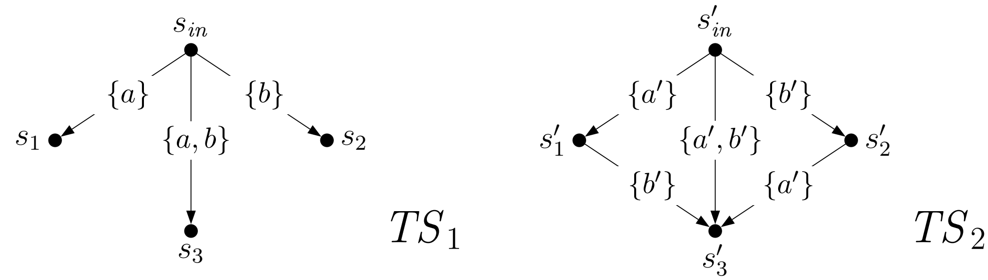


Figure 4.1: Transition systems for examples 4.1.1 and 4.1.2.

4.2 Inhibitor Nets and their Morphisms

In this section we will introduce a class of morphisms for the ENI-systems.

Definition 4.2.1 *Let $\mathcal{N}_i = (B_i, E_i, F_i, I_i, c^i_{in})$ $(i = 1, 2)$ be ENI-systems. A* net morphism *from $\mathcal{N}_1$ to $\mathcal{N}_2$ is a pair $(\alpha, \beta) : \mathcal{N}_1 \to \mathcal{N}_2$ such that the following hold.*

MENI1 *$\alpha : B_2 \to B_1$ is a partial function.*

MENI2 *$\beta : E_1 \to E_2$ is a partial function.*

MENI3 *For every $b \in dom(\alpha)$, $\alpha(b) \in c^1_{in}$ if and only if $b \in c^2_{in}$.*

MENI4 *For every $e \in E_1 \setminus dom(\beta)$, $\alpha^{-1}({}^\bullet e) = \emptyset = \alpha^{-1}(e^\bullet)$.*

MENI5 *For every $e \in dom(\beta)$:*

$$\begin{aligned}
\alpha^{-1}({}^\bullet e) &= {}^\bullet\beta(e), \\
\alpha^{-1}(e^\bullet) &= \beta(e)^\bullet, \\
\beta(\overset{\blacksquare}{e}) \cap \mathcal{M}_{(\alpha,\beta)} &\subseteq \alpha^{-1}(\overset{\blacksquare}{e}),
\end{aligned}$$

where $\mathcal{M}_{(\alpha,\beta)} = \{b \in B_2 \mid b \in c^2_{in} \ \vee \ \exists e \in dom(\beta) \ : \ b \in \beta(e)^\bullet\}$. □

In the above definition $\mathcal{M}_{(\alpha,\beta)}$ denotes a set of all the conditions of a net $\mathcal{N}_2$ which are potentially marked by at least one case reachable from c^2_{in} when $\mathcal{N}_1$ is simulated by $\mathcal{N}_2$ according to (α, β). The net morphisms defined above are similar to the N-morphisms of [38]. They preserve initial cases, in the sense that $\alpha^{-1}(c^1_{in}) \subseteq c^2_{in}$, and the environments of events. The crucial difference is due to the presence of inhibitor arcs. The condition $\beta(\overset{\blacksquare}{e}) \cap \mathcal{M}_{(\alpha,\beta)} \subseteq \alpha^{-1}(\overset{\blacksquare}{e})$ means that, in the net $\mathcal{N}_2$, we can have more concurrency (less inhibition). Notice that we only take into account I-conditions of $\mathcal{N}_2$ which can potentially disable events, i.e. those which can potentially be marked when $\mathcal{N}_2$ simulates $\mathcal{N}_1$. The net version of proposition 4.1.1 is not true, i.e. two net morphisms $(\alpha_i, \beta_i) : \mathcal{N}_1 \to \mathcal{N}_2$ $(i = 1, 2)$ which satisfy $\beta_1 = \beta_2$ can be different[2]. This is due to the fact that we do not require nets to be simple (see [38]) and allow isolated conditions.

Example 4.2.1 *Figure 4.2 shows two ENI-system. We can define a net morphism* $g = (\alpha, \beta) : \mathcal{N}_1 \to \mathcal{N}_2$ *between them in the following way:*

$$\alpha(b'_1) \; = \; b_1 \qquad \alpha(b'_3) \; = \; b_3 \qquad \beta(e) \; = \; e'.$$

Notice that $\alpha(b'_2)$, $\alpha(b'_4)$, $\alpha(b'_5)$ *and* $\beta(f)$ *are not defined. The pre-conditions, post-conditions and I-conditions of events in* $\mathcal{N}_1$ *are given by:*

$$\begin{array}{rclrclrcl} {}^\bullet e & = & \{b_1\} & e^\bullet & = & \{b_3\} & \overset{\blacksquare}{e} & = & \emptyset \\ {}^\bullet f & = & \{b_2\} & f^\bullet & = & \{b_4\} & \overset{\blacksquare}{f} & = & \{b_3\}. \end{array}$$

The pre-conditions, post-conditions and I-conditions of events in $\mathcal{N}_2$ *are given by:*

$$\begin{array}{rclrclrcl} {}^\bullet e' & = & \{b'_1\} & e'^\bullet & = & \{b'_3\} & \overset{\blacksquare}{e'} & = & \{b'_4, b'_5\} \\ {}^\bullet f' & = & \{b'_2\} & f'^\bullet & = & \{b'_4\} & \overset{\blacksquare}{f'} & = & \{b'_3\}. \end{array}$$

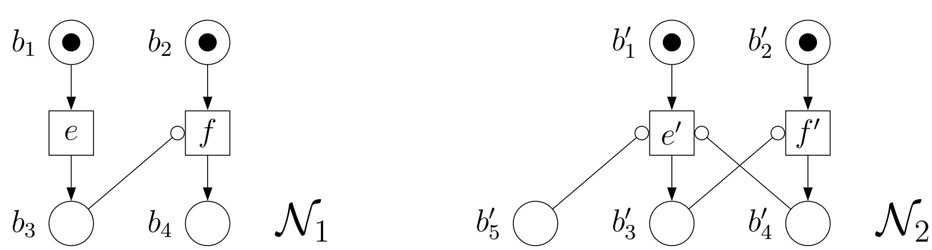


Figure 4.2: ENI-systems for example 4.2.1.

It is straightforward to check that g *is well defined net morphism. Observe that for* e*:* $\alpha^{-1}(\overset{\blacksquare}{e}) = \emptyset$ *and* $\beta(\overset{\blacksquare}{e}) = \{b'_4, b'_5\}$ *and* $\mathcal{M}_{(\alpha,\beta)} = \{b'_1, b'_2, b'_3\}$*, so the inclusion in (MENI5) holds. Notice that although* $\mathcal{N}_2$ *has more inhibitor arcs than* $\mathcal{N}_1$*, it has no more* active inhibitor arcs *when it simulates* $\mathcal{N}_1$*. Both inhibitor conditions of* e' *in* $\mathcal{N}_2$ *give rise to* passive inhibitor arcs*:* b'_5 *is never marked in* $\mathcal{N}_2$*, and* b'_4 *is never marked in* $\mathcal{N}_2$ *when* $\mathcal{N}_1$ *is simulated (*f' *will never be executed as it is not an image of any event in* $\mathcal{N}_1$*).* □

[2]The N-morphisms of [38] enjoy the property of being uniquely determined by the way events are mapped.

Proposition 4.2.1 *Let $(\alpha, \beta) : \mathcal{N}_1 \to \mathcal{N}_2$ be a net morphism between ENI-systems $\mathcal{N}_1$ and $\mathcal{N}_2$. Then, for every $X \subseteq B_1$,*

$$\alpha^{-1}(X) \cap (c^2_{in} \setminus \alpha^{-1}(c^1_{in})) = \emptyset.$$

Proof: Suppose that there is d such that: (i) $d \in \alpha^{-1}(X)$, and (ii) $d \in (c^2_{in} \setminus \alpha^{-1}(c^1_{in}))$. From (i) it follows that $d \in dom(\alpha)$. From (ii) it follows that $d \in c^2_{in}$ which means, by (MENI3), that $\alpha(d) \in c^1_{in}$. The latter in turn gives $d \in \alpha^{-1}(c^1_{in})$ contradicting (ii). $\square$

The following proposition is similar to the result obtained for the Elementary Net Systems in [38].

Proposition 4.2.2 *Let $\mathcal{N}_i = (B_i, E_i, F_i, I_i, c^i_{in})$ $(i = 1, 2)$ be ENI-systems and $(\alpha, \beta) : \mathcal{N}_1 \to \mathcal{N}_2$ be a net morphism. Moreover, let $f_\alpha : C_{\mathcal{N}_1} \to 2^{B_2}$ be a mapping such that, for every $c \in C_{\mathcal{N}_1}$,*

$$f_\alpha(c) = \alpha^{-1}(c) \cup (c^2_{in} \setminus \alpha^{-1}(c^1_{in})).$$

Then the following hold:

1. *For every $c \in C_{\mathcal{N}_1}$, $f_\alpha(c) \in C_{\mathcal{N}_2}$.*
2. *If $(c, u, c') \in \to_{\mathcal{N}_1}$ and $\widehat{\beta}(u) = \emptyset$ then $f_\alpha(c) = f_\alpha(c')$.*
3. *If $(c, u, c') \in \to_{\mathcal{N}_1}$ and $\widehat{\beta}(u) \neq \emptyset$ then $(f_\alpha(c), \widehat{\beta}(u), f_\alpha(c')) \in \to_{\mathcal{N}_2}$.*

Proof: (1) Let $c \in C_{\mathcal{N}_1}$ and ϱ be a shortest step sequence of $\mathcal{N}_1$ such that $c^1_{in}[\varrho\rangle c$. The proof proceeds by induction on the length k of ϱ.

The base case is $k = 0$. Then $c = c^1_{in}$ and, by (MENI3), we have $\alpha^{-1}(c^1_{in}) \subseteq c^2_{in}$. Hence

$$f_\alpha(c^1_{in}) = \alpha^{-1}(c^1_{in}) \cup (c^2_{in} \setminus \alpha^{-1}(c^1_{in})) = c^2_{in} \in C_{\mathcal{N}_2}.$$

In the induction step, $k > 0$. Let $\varrho = \varrho' u$ and $c^1_{in}[\varrho'\rangle c' \xrightarrow{u}_{\mathcal{N}_1} c$ in $\mathcal{N}_1$. By the induction hypothesis, $c'' = f_\alpha(c') \in C_{\mathcal{N}_2}$. From $c' \xrightarrow{u}_{\mathcal{N}_1} c$ and proposition 3.3.1(2) it follows that $c = (c' \setminus {}^\bullet u) \cup u^\bullet$ in $\mathcal{N}_1$, and we consider two cases.
Case 1: $\widehat{\beta}(u) = \emptyset$. Then

$$\begin{array}{rcl} f_\alpha(c) & = & \alpha^{-1}(c) \cup (c^2_{in} \setminus \alpha^{-1}(c^1_{in})) \\ & = & \alpha^{-1}((c' \setminus {}^\bullet u) \cup u^\bullet) \cup (c^2_{in} \setminus \alpha^{-1}(c^1_{in})) \\ & \overset{(MENI4)}{=} & f_\alpha(c') \in C_{\mathcal{N}_2}. \end{array}$$

Case 2: $\widehat{\beta}(u) \neq \emptyset$. Denote $w = \widehat{\beta}(u)$. We will show that w is enabled at $c'' \in C_{\mathcal{N}_2}$. First, knowing that $u \in V_{\mathcal{N}_1}$, we will prove that $w \in V_{\mathcal{N}_2}$. Clearly, for every pair $e_2 \neq f_2 \in w$ we can find a pair $e_1 \neq f_1 \in u$ such that $\beta(e_1) = e_2$ and $\beta(f_1) = f_2$. Then

$$\begin{array}{rcl} ({}^\bullet e_2 \cup e_2^\bullet) \cap ({}^\bullet f_2 \cup f_2^\bullet) & \overset{(MENI5)}{=} & (\alpha^{-1}({}^\bullet e_1) \cup \alpha^{-1}(e_1^\bullet)) \cap (\alpha^{-1}({}^\bullet f_1) \cup \alpha^{-1}(f_1^\bullet)) \\ & = & \alpha^{-1}(({}^\bullet e_1 \cup e_1^\bullet) \cap ({}^\bullet f_1 \cup f_1^\bullet)) \\ & \overset{(3.2)}{=} & \emptyset. \end{array}$$

Hence $w \in V_{\mathcal{N}_2}$. Since $c'' \in C_{\mathcal{N}_2}$, we can use proposition 3.3.1(1) to show that w is enabled at c''. We need to prove that ${}^\bullet w \subseteq c''$, $w^\bullet \cap c'' = \emptyset$ and $\overset{\blacksquare}{w} \cap c'' = \emptyset$. By $w \in V_{\mathcal{N}_2}$, it suffices to prove that if $f \in w$ then ${}^\bullet f \subseteq c''$, $f^\bullet \cap c'' = \emptyset$ and $\overset{\blacksquare}{f} \cap c'' = \emptyset$. Let $e \in u$ be such that $\beta(e) = f$.

We first show that

$${}^\bullet f \subseteq c'' = f_\alpha(c') = \alpha^{-1}(c') \cup (c_{in}^2 \setminus \alpha^{-1}(c_{in}^1)).$$

From (MENI5) it follows that $\alpha^{-1}({}^\bullet e) = {}^\bullet f$. Hence what we need to show is that

$$\alpha^{-1}({}^\bullet e) \subseteq \alpha^{-1}(c') \cup (c_{in}^2 \setminus \alpha^{-1}(c_{in}^1)).$$

Now, from $c' \xrightarrow{u}_{\mathcal{N}_1} c$ it follows that ${}^\bullet e \subseteq c'$. Hence $\alpha^{-1}({}^\bullet e) \subseteq \alpha^{-1}(c')$ and so ${}^\bullet f \subseteq c''$.

We next show that $f^\bullet \cap c'' = \emptyset$. By (MENI5), $\alpha^{-1}(e^\bullet) = f^\bullet$. From $c' \xrightarrow{u}_{\mathcal{N}_1} c$ it follows that $e^\bullet \cap \; c' = \emptyset$, so $\alpha^{-1}(e^\bullet) \cap \alpha^{-1}(c') = \emptyset$. Consequently, $f^\bullet \cap \alpha^{-1}(c') = \emptyset$. Moreover, $f^\bullet \cap (c_{in}^2 \setminus \alpha^{-1}(c_{in}^1)) = \emptyset$ which follows from $f^\bullet = \alpha^{-1}(e^\bullet)$ and proposition 4.2.1. Hence $f^\bullet \cap c'' = \emptyset$.

We now show that $\overset{\blacksquare}{f} \cap c'' = \emptyset$. From (MENI5) we have $\overset{\blacksquare}{f} \cap \mathcal{M}_{(\alpha,\beta)} \subseteq \alpha^{-1}(\overset{\blacksquare}{e})$, and from $c' \xrightarrow{u}_{\mathcal{N}_1} c$ it follows that $\overset{\blacksquare}{e} \cap c' = \emptyset$. Hence

$$\overset{\blacksquare}{f} \cap \mathcal{M}_{(\alpha,\beta)} \cap \alpha^{-1}(c') \subseteq \alpha^{-1}(\overset{\blacksquare}{e}) \cap \alpha^{-1}(c') = \emptyset.$$

Consequently, $\overset{\blacksquare}{f} \cap \mathcal{M}_{(\alpha,\beta)} \cap \alpha^{-1}(c') = \emptyset$. Moreover, $\overset{\blacksquare}{f} \cap \mathcal{M}_{(\alpha,\beta)} \cap (c_{in}^2 \setminus \alpha^{-1}(c_{in}^1)) = \emptyset$ which follows from $\overset{\blacksquare}{f} \cap \mathcal{M}_{(\alpha,\beta)} \subseteq \alpha^{-1}(\overset{\blacksquare}{e})$ and proposition 4.2.1. Hence

$$\overset{\blacksquare}{f} \cap \mathcal{M}_{(\alpha,\beta)} \cap (\alpha^{-1}(c') \cup (c_{in}^2 \setminus \alpha^{-1}(c_{in}^1))) = \emptyset.$$

We now show that $c'' = \alpha^{-1}(c') \cup (c_{in}^2 \setminus \alpha^{-1}(c_{in}^1)) \subseteq \mathcal{M}_{(\alpha,\beta)}$. Suppose $b \in c''$. It implies that $b \in c_{in}^2 \setminus \alpha^{-1}(c_{in}^1)$ from which it immediately follows that $b \in \mathcal{M}_{(\alpha,\beta)}$, or $b \in \alpha^{-1}(c')$ which means $\alpha(b) \in c'$. But $c' \in C_{\mathcal{N}_1}$, so $\alpha(b) \in c_{in}^1$ which by (MENI3) means $b \in c_{in}^2$, or there exists $g \in E_1$ such that $\alpha(b) \in g^\bullet$ (so $b \in \alpha^{-1}(g^\bullet)$ and $\alpha^{-1}(g^\bullet) \neq \emptyset$). From (MENI4) and (MENI5) it follows that $g \in dom(\beta)$ and $\alpha^{-1}(g^\bullet) = \beta(g)^\bullet$. So $b \in \beta(g)^\bullet$ for some $g \in dom(\beta)$ and, consequently, $b \in \mathcal{M}_{(\alpha,\beta)}$. So $\overset{\blacksquare}{f} \cap c'' = \emptyset$. Hence we have shown that $c'' \xrightarrow{w}_{\mathcal{N}_2}$.

Let $\widetilde{c} = (c'' \setminus {}^\bullet w) \cup w^\bullet$ in $\mathcal{N}_2$. We have $\widetilde{c} \in C_{\mathcal{N}_2}$ and it suffices to show that $f_\alpha(c) = \widetilde{c}$ in order to prove that $f_\alpha(c) \in C_{\mathcal{N}_2}$. We proceed as follow:

$$
\begin{aligned}
f_\alpha(c) &= f_\alpha((c' \setminus {}^\bullet u) \cup u^\bullet)\\
&= \alpha^{-1}((c' \setminus {}^\bullet u) \cup u^\bullet) \cup (c^2_{in} \setminus \alpha^{-1}(c^1_{in}))\\
&= \alpha^{-1}(c' \setminus {}^\bullet u) \cup \alpha^{-1}(u^\bullet) \cup (c^2_{in} \setminus \alpha^{-1}(c^1_{in}))\\
&= (\alpha^{-1}(c') \setminus \alpha^{-1}({}^\bullet u)) \cup \alpha^{-1}(u^\bullet) \cup (c^2_{in} \setminus \alpha^{-1}(c^1_{in}))\\
&\overset{(prop.\ 4.2.1)}{=} ((\alpha^{-1}(c') \cup (c^2_{in} \setminus \alpha^{-1}(c^1_{in}))) \setminus \alpha^{-1}({}^\bullet u)) \cup \alpha^{-1}(u^\bullet)\\
&= (f_\alpha(c') \setminus \alpha^{-1}({}^\bullet u)) \cup \alpha^{-1}(u^\bullet)\\
&= (c'' \setminus \alpha^{-1}({}^\bullet u)) \cup \alpha^{-1}(u^\bullet)\\
&\overset{(MENI4,MENI5)}{=} (c'' \setminus {}^\bullet w) \cup w^\bullet\\
&= \widetilde{c}.
\end{aligned}
$$

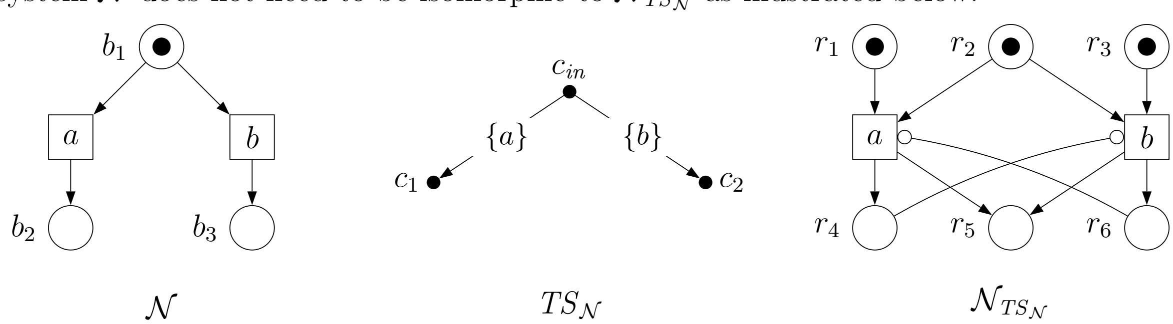
(2,3) These parts were proved while showing (1). □

Chapter 5

Categories of ENI and TSENI

Category Theory provides notions and methodology for relating different mathematical models. In this approach models are treated as categories where, for example, systems (or their behaviours) are represented as objects of the category, and the relationship between systems is given in the form of morphisms. Different models are 'categorically' related by the use of functors. The properties of functors can indicate which of the related models is more abstract (gives less information about the behaviour of the systems, concentrating on some selective features), and which one is more concrete (more behavioural features are expressed in the model).

In this chapter, we define two functors to relate ENI-systems and TSENI Transition Systems, and prove that they form an adjunction. The first functor (the left adjoint) embeds the TSENI Transition Systems in ENI-systems, while the second functor (the right adjoint) gives the behaviour of an ENI-system in the form of a TSENI transition system. The embedding of the TSENI Transition Systems in ENI-systems is full and faithful (TS is isomorphic to $TS_{\mathcal{N}_{TS}}$), making the adjunction a coreflection. ENI-systems, on the other hand, are not embedded in the TSENI Transition Systems. The right adjoint is not full and faithful proving that the adjunction is not a reflection. Indeed, an ENI-system $\mathcal{N}$ does not need to be isomorphic to $\mathcal{N}_{TS_{\mathcal{N}}}$ as illustrated below.



As a consequence, ENI-systems and TSENI Transition Systems are not equivalent models. TSENI Transition Systems, as a more abstract model, are embedded in the net model.

The categories of various other classes of nets and transition systems were defined and studied in, for example, [11, 37, 38], while [47] presents a comprehensive study on how to relate different models of concurrency by the use of categorical notions.

In the rest of this chapter, we will denote the composition of two functions or partial functions by "$\circ$". For two partial functions $f : X \to Y$ and $g : Y \to Z$, the composition $g \circ f : X \to Z$ will be understood as follows. For all $x \in X$,

$$g \circ f(x) = \begin{cases} z & \text{if there is } y \in Y \text{ such that } f(x) = y \text{ and } g(y) = z \\ \textit{undefined} & \text{otherwise.} \end{cases}$$

5.1 Categories $\mathcal{CAT}_{ENI}$ and $\mathcal{CAT}_{TSENI}$

We start by recalling some basic definitions concerning categories and functors from [1], [10] and [35].

Definition 5.1.1 *A category $\mathcal{K}$ comprises a collection of objects of $\mathcal{K}$, called $\mathcal{K}_0$, together with, for each pair A, B of objects of $\mathcal{K}$, a distinct (possibly empty) collection of morphisms from A to B, called $\mathcal{K}_1$, subject to the conditions (C1) and (C2) below. We write $f : A \to B$ to indicate that f is a morphism from A to B, and then refer to A as the source of f and to B as the target of f. For two morphisms, f and g, such that the target of f is the source of g, there is the composite morphism denoted by $g \circ f$. The source of $g \circ f$ is the source of f, and the target of $g \circ f$ is the target of g. For every object A of $\mathcal{K}$, we will denote by id_A a distinguished morphism from A to A, called the identity of the object A.*

C1 *$(h \circ g) \circ f = h \circ (g \circ f)$ whenever either side of the equality is defined.*

C2 *If $f : A \to B$, then $f \circ id_A = id_B \circ f = f$.* □

Definition 5.1.2 *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is a pair of functions $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ and $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ such that the following hold.*

F1 *If $f : A \to B$ in $\mathcal{C}$, then $F_1(f) : F_0(A) \to F_0(B)$ in $\mathcal{D}$.*

F2 *For every object A of $\mathcal{C}$, $F_1(id_A) = id_{F_0(A)}$.*

F3 *If $g \circ f$ is defined in $\mathcal{C}$, then $F_1(g) \circ F_1(f)$ is defined in $\mathcal{D}$ and $F_1(g \circ f) = F_1(g) \circ F_1(f)$.*

Whenever it does not lead to ambiguity, we will denote F_0 and F_1 by F. □

Definition 5.1.3 *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.*

1. *F is* faithful (or an embedding) *if for every pair A, B of objects of $\mathcal{C}$ and for every pair $f : A \to B$ and $g : A \to B$ of morphisms of $\mathcal{C}$, the inequality $f \neq g$ implies $F(f) \neq F(g)$.*

2. *F is* full *if for every pair A, B of objects of $\mathcal{C}$ and for every morphism $g : F(A) \to F(B)$ in $\mathcal{D}$, there exists a morphism $f : A \to B$ in $\mathcal{C}$ such that $F(f) = g$.* □

We now define two categories: the category of TSENI Transition Systems with morphisms defined as in section 4.1, and the category of ENI-systems with morphisms defined as in section 4.2. To define these categories, we need to say what are the identity morphisms, and define the compositions of two morphisms. Let $TS = (S, U, T, s_{in})$ be a TSENI transition system, and $\sigma_{id} : S \to S$ and $\eta_{id} : E_{TS} \to E_{TS}$ be total identity functions. Then $id_{TS} = (\sigma_{id}, \eta_{id})$ will denote an identity morphism $id_{TS} : TS \to TS$.

Let $TS_i = (S_i, U_i, T_i, s^i_{in})$ (for $i = 1, 2, 3$) be TSENI transition systems, and $f = (\sigma_f, \eta_f) : TS_1 \to TS_2$ and $g = (\sigma_g, \eta_g) : TS_2 \to TS_3$ be two transition system morphisms. Then the composition of the morphisms is defined by $g \circ f = (\sigma_g \circ \sigma_f, \eta_g \circ \eta_f) : TS_1 \to TS_3$, where $\sigma_g \circ \sigma_f$ is a total function composition and $\eta_g \circ \eta_f$ is a partial function composition. It is straightforward to prove that the TSENI Transition Systems with transition system morphisms form a category. We will denote it by $\mathcal{CAT}_{TSENI}$.

Let $\mathcal{N} = (B, E, F, I, c_{in})$ be an ENI-system, and $\alpha_{id} : B \to B$ and $\beta_{id} : E \to E$ be two total identity functions. Then $id_{\mathcal{N}} = (\alpha_{id}, \beta_{id})$ will denote an identity morphism $id_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$.

Let $\mathcal{N}_i = (B_i, E_i, F_i, I_i, c^i_{in})$ (for $i = 1, 2, 3$) be ENI-systems, and $f = (\alpha_f, \beta_f) : \mathcal{N}_1 \to \mathcal{N}_2$ and $g = (\alpha_g, \beta_g) : \mathcal{N}_2 \to \mathcal{N}_3$ be two net morphisms. Then the composition of the morphisms is defined as follows: $g \circ f = (\alpha_f \circ \alpha_g, \beta_g \circ \beta_f) : \mathcal{N}_1 \to \mathcal{N}_3$, where both $\alpha_f \circ \alpha_g$ and $\beta_g \circ \beta_f$ are partial function compositions. Notice that α's and β's are composed in different order.

We now show that $g \circ f$ is a net morphism. It is clear that $\alpha_f \circ \alpha_g : B_3 \to B_1$ and $\beta_g \circ \beta_f : E_1 \to E_3$ are partial functions. (MENI3), (MENI4) and the first parts of

(MENI5) are straightforward to show. We prove the last part of (MENI5), i.e. $\beta_g \circ \overset{\blacksquare}{\beta}_f(e) \cap \mathcal{M}_{(\alpha_f \circ \alpha_g, \beta_g \circ \beta_f)} \subseteq (\alpha_f \circ \alpha_g)^{-1}(\overset{\blacksquare}{e})$, for every $e \in dom(\beta_g \circ \beta_f)$. From the fact that f and g are net morphisms we have:

$$\forall\, e \in dom(\beta_g) \;:\, \overset{\blacksquare}{\beta}_g(e) \cap \mathcal{M}_{(\alpha_g, \beta_g)} \subseteq \alpha_g^{-1}(\overset{\blacksquare}{e}), \tag{5.1}$$

$$\forall\, e \in dom(\beta_f) \;:\, \overset{\blacksquare}{\beta}_f(e) \cap \mathcal{M}_{(\alpha_f, \beta_f)} \subseteq \alpha_f^{-1}(\overset{\blacksquare}{e}). \tag{5.2}$$

Consider $b \in \beta_g \circ \overset{\blacksquare}{\beta}_f(e) \cap \mathcal{M}_{(\alpha_f \circ \alpha_g, \beta_g \circ \beta_f)}$. This means $b \in \beta_g(\overset{\blacksquare}{\beta}_f(e))$, moreover, $b \in c^3_{in}$ or there exists $e' \in dom(\beta_g \circ \beta_f)$ such that $b \in \beta_g \circ \beta_f(e')^{\bullet}$. Thus $b \in \beta_g(\overset{\blacksquare}{\beta}_f(e))$ and, moreover, $b \in c^3_{in}$ or there exists $e'' = \beta_f(e') \in dom(\beta_g)$ such that $b \in \beta_g(e'')^{\bullet}$. From (5.1) we have $b \in \alpha_g^{-1}(\overset{\blacksquare}{\beta}_f(e))$ and then $\alpha_g(b) \in \overset{\blacksquare}{\beta}_f(e)$. So $b \in dom(\alpha_g)$ and, if $b \in c^3_{in}$ we have from the fact that g is a net morphism (MENI3), $\alpha_g(b) \in c^2_{in}$. If, on the other hand, there exists $e' \in dom(\beta_g \circ \beta_f)$ such that $b \in \beta_g(\beta_f(e'))^{\bullet}$, then from the fact that g is a net morphism (MENI5), there exists $e' \in dom(\beta_f)$ such that $b \in \alpha_g^{-1}(\beta_f(e')^{\bullet})$. Consequently, there exists $e' \in dom(\beta_f)$ such that $\alpha_g(b) \in \beta_f(e')^{\bullet}$. We have proved that $\alpha_g(b) \in \overset{\blacksquare}{\beta}_f(e)$ and, moreover, $\alpha_g(b) \in c^2_{in}$ or there exists $e' \in dom(\beta_f)$ such that $\alpha_g(b) \in \beta_f(e')^{\bullet}$. Since f is a net morphism we have, by (5.2), $\alpha_g(b) \in \alpha_f^{-1}(\overset{\blacksquare}{e})$, which means $\alpha_f \circ \alpha_g(b) \in \overset{\blacksquare}{e}$ and, finally, $b \in (\alpha_f \circ \alpha_g)^{-1}(\overset{\blacksquare}{e})$. It is now easy to see that the ENI-systems with net morphisms form a category. We will denote it by $\mathcal{CAT}_{ENI}$.

5.2 A Functor from $\mathcal{CAT}_{ENI}$ to $\mathcal{CAT}_{TSENI}$

To define a functor we need to show how the objects and morphisms of one category are mapped into objects and morphisms (respectively) of another category. In section 3.4, we have shown how to construct the TSENI transition system, $TS_{\mathcal{N}}$, for a given ENI-system, $\mathcal{N}$.

The next proposition defines the mapping between morphisms of $\mathcal{CAT}_{ENI}$ and morphisms of $\mathcal{CAT}_{TSENI}$.

Proposition 5.2.1 *Let $\mathcal{N}_i = (B_i, E_i, F_i, I_i, c^i_{in})$ $(i = 1, 2)$ be ENI-systems and (α, β) be a net morphism from $\mathcal{N}_1$ to $\mathcal{N}_2$. Moreover, let $f_\alpha : C_{\mathcal{N}_1} \to C_{\mathcal{N}_2}$ be a total function defined as follows:*

$$f_\alpha(c) = \alpha^{-1}(c) \cup (c^2_{in} \setminus \alpha^{-1}(c^1_{in}))$$

and $f_\beta : E_{TS_{\mathcal{N}_1}} \to E_{TS_{\mathcal{N}_2}}$ be a mapping defined by $f_\beta = \beta$. Then (f_α, f_β) is a transition system morphism from $TS_{\mathcal{N}_1}$ to $TS_{\mathcal{N}_2}$.

Proof: (MTS1) and (MTS3) follow directly from proposition 4.2.2. Note that $E_{TS_{\mathcal{N}_1}} \subseteq E_1$ and $\beta : E_1 \to E_2$ is a partial function. Moreover, from proposition 4.2.2(3) we have that, if $e \in E_{TS_{\mathcal{N}_1}}$ and $\beta(e)$ is defined, then $\beta(e) \in E_{TS_{\mathcal{N}_2}}$. Hence f_β is a well defined partial function. What is left to show is the injectivity of $f_\beta = \beta$ on steps. Suppose that $u \in U_{\mathcal{N}_1}$. We need to show that for all different $e, f \in u \cap dom(\beta)$, $\beta(e) \neq \beta(f)$. Assume that $\beta(e) = \beta(f)$. Then, by (MENI5), $\alpha^{-1}({}^\bullet e) = \alpha^{-1}({}^\bullet f) \neq \emptyset$. Thus there is $x \in \alpha^{-1}({}^\bullet e) \cap \alpha^{-1}({}^\bullet f) = \alpha^{-1}({}^\bullet e \cap {}^\bullet f)$. Hence $\alpha(x) \in {}^\bullet e \cap {}^\bullet f$ which contradicts $u \in U_{\mathcal{N}_1}$ (since $u \in V_{\mathcal{N}_1}$). $\square$

Now we are ready to define a functor from $\mathcal{CAT}_{ENI}$ to $\mathcal{CAT}_{TSENI}$.

Theorem 5.2.1 *Let $H : \mathcal{CAT}_{ENI} \to \mathcal{CAT}_{TSENI}$ be a mapping defined, for every ENI-system $\mathcal{N}$ and net morphism (α, β), by $H(\mathcal{N}) = TS_{\mathcal{N}}$ and $H(\alpha, \beta) = (f_\alpha, f_\beta)$. Then H is a functor.*

Proof: Let $\mathcal{N} = (B, E, F, I, c_{in})$ and $\mathcal{N}_i = (B_i, E_i, F_i, I_i, c^i_{in})$ $(i = 1, 2, 3)$ be ENI-systems. Let $id_{\mathcal{N}} = (\alpha_{id}, \beta_{id}) : \mathcal{N} \to \mathcal{N}$ be an identity morphism and $f = (\alpha_f, \beta_f) : \mathcal{N}_1 \to \mathcal{N}_2$ and $g = (\alpha_g, \beta_g) : \mathcal{N}_2 \to \mathcal{N}_3$ be two net morphisms. Then (F1) follows from proposition 5.2.1. We will show that $H(id_{\mathcal{N}}) = id_{TS_{\mathcal{N}}}$ (i.e. (F2)) and $H(g \circ f) = H(g) \circ H(f)$ (i.e. (F3)). To prove the former we observe that $H(\alpha_{id}, \beta_{id}) = (f_{\alpha_{id}}, f_{\beta_{id}})$ where, for every $c \in C_{\mathcal{N}}$ and $e \in E_{TS_{\mathcal{N}}}$,

$$\begin{array}{lclclcl} f_{\alpha_{id}}(c) & = & \alpha_{id}^{-1}(c) \cup (c_{in} \setminus \alpha_{id}^{-1}(c_{in})) & = & c \cup (c_{in} \setminus c_{in}) & = & c \\ f_{\beta_{id}}(e) & = & \beta_{id}(e) & & & = & e. \end{array}$$

The latter is proved in the following way. We have:

$$\begin{array}{rcl} H(g \circ f) & = & H((\alpha_g, \beta_g) \circ (\alpha_f, \beta_f)) \\ & = & H(\alpha_f \circ \alpha_g, \beta_g \circ \beta_f) \\ & = & (f_{\alpha_f \circ \alpha_g}, f_{\beta_g \circ \beta_f}) \\ H(g) \circ H(f) & = & H(\alpha_g, \beta_g) \circ H(\alpha_f, \beta_f) \\ & = & (f_{\alpha_g}, f_{\beta_g}) \circ (f_{\alpha_f}, f_{\beta_f}) \\ & = & (f_{\alpha_g} \circ f_{\alpha_f}, f_{\beta_g} \circ f_{\beta_f}). \end{array}$$

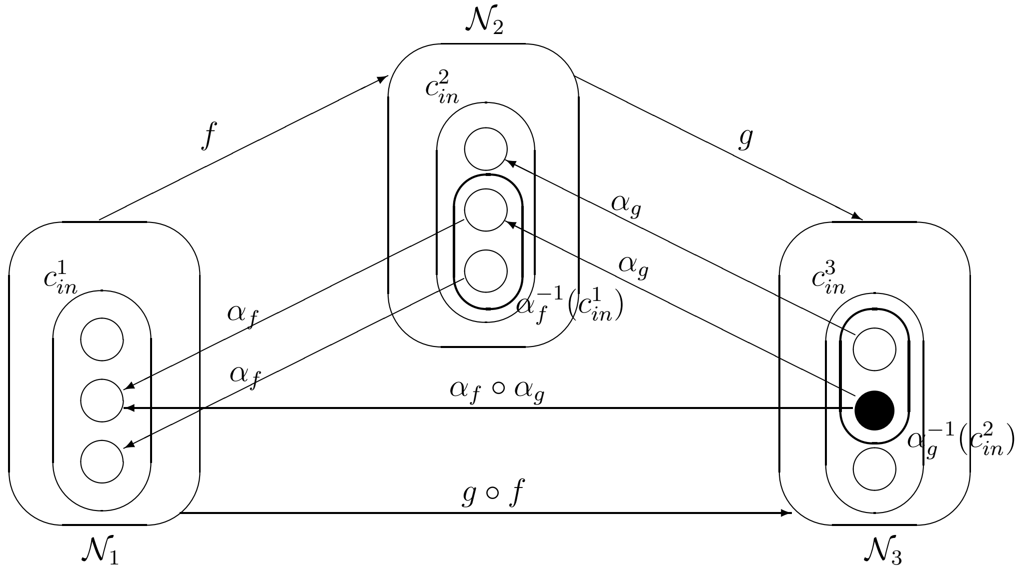


Figure 5.1: An illustration for theorem 5.2.1 where the black circle denotes $\alpha_g^{-1} \circ \alpha_f^{-1}(c^1_{in})$.

Since it is clear that $f_{\beta_g \circ \beta_f} = f_{\beta_g} \circ f_{\beta_f}$, it suffices to show that $f_{\alpha_f \circ \alpha_g} = f_{\alpha_g} \circ f_{\alpha_f}$[1].

We have, for every $c \in C_{\mathcal{N}_1}$,

$$\begin{aligned} f_{\alpha_f \circ \alpha_g}(c) &= (\alpha_f \circ \alpha_g)^{-1}(c) \cup \Big(c^3_{in} \setminus (\alpha_f \circ \alpha_g)^{-1}(c^1_{in})\Big) \\ &= \alpha_g^{-1} \circ \alpha_f^{-1}(c) \cup \Big(c^3_{in} \setminus \alpha_g^{-1} \circ \alpha_f^{-1}(c^1_{in})\Big). \end{aligned}$$

From $\alpha_f^{-1}(c^1_{in}) \subseteq c^2_{in}$ and $\alpha_g^{-1}(c^2_{in}) \subseteq c^3_{in}$ it follows that $\alpha_g^{-1} \circ \alpha_f^{-1}(c^1_{in}) \subseteq \alpha_g^{-1}(c^2_{in}) \subseteq c^3_{in}$. Hence, for every $c \in C_{\mathcal{N}_1}$,

$$\begin{aligned} f_{\alpha_g} \circ f_{\alpha_f}(c) &= f_{\alpha_g}(f_{\alpha_f}(c)) \\ &= f_{\alpha_g}\Big(\alpha_f^{-1}(c) \cup (c^2_{in} \setminus \alpha_f^{-1}(c^1_{in}))\Big) \\ &= \alpha_g^{-1}\Big(\alpha_f^{-1}(c) \cup (c^2_{in} \setminus \alpha_f^{-1}(c^1_{in}))\Big) \cup \Big(c^3_{in} \setminus \alpha_g^{-1}(c^2_{in})\Big) \\ &= \alpha_g^{-1}\Big(\alpha_f^{-1}(c)\Big) \cup \alpha_g^{-1}\Big(c^2_{in} \setminus \alpha_f^{-1}(c^1_{in})\Big) \cup \Big(c^3_{in} \setminus \alpha_g^{-1}(c^2_{in})\Big) \\ &= \alpha_g^{-1} \circ \alpha_f^{-1}(c) \cup \Big(\alpha_g^{-1}(c^2_{in}) \setminus \alpha_g^{-1} \circ \alpha_f^{-1}(c^1_{in})\Big) \cup \Big(c^3_{in} \setminus \alpha_g^{-1}(c^2_{in})\Big) \\ &= \alpha_g^{-1} \circ \alpha_f^{-1}(c) \cup (c^3_{in} \setminus \alpha_g^{-1} \circ \alpha_f^{-1}(c^1_{in})). \end{aligned}$$

Thus, for every $c \in C_{\mathcal{N}_1}$, $f_{\alpha_g} \circ f_{\alpha_f}(c) = f_{\alpha_f \circ \alpha_g}(c)$ which completes the proof. □

[1] We will prove this as an exercise, as $f_{\alpha_f \circ \alpha_g} = f_{\alpha_g} \circ f_{\alpha_f}$ follows directly from proposition 4.1.1 and $f_{\beta_g \circ \beta_f} = f_{\beta_g} \circ f_{\beta_f}$.

5.3 A Functor from $\mathcal{CAT}_{TSENI}$ to $\mathcal{CAT}_{ENI}$

We have shown how to construct a functor H from $\mathcal{CAT}_{ENI}$ to $\mathcal{CAT}_{TSENI}$. In this section, we will define a functor in the opposite direction. To build the ENI-system, $\mathcal{N}_{TS}$, for a given TSENI transition system, TS, we will use the construction from section 3.5.

The next proposition defines a mapping between morphisms of $\mathcal{CAT}_{TSENI}$ and morphisms of $\mathcal{CAT}_{ENI}$.

Proposition 5.3.1 *Let $TS_i = (S_i, U_i, T_i, s^i_{in})$ ($i = 1, 2$) be TSENI transition systems, and $(\sigma, \eta) : TS_1 \to TS_2$ be a transition system morphism. Moreover, let $f_\sigma : R_{TS_2} \to R_{TS_1}$ be a mapping such that $f_\sigma(r) = \sigma^{-1}(r)$, for every $r \in R_{TS_2}$ such that $\emptyset \neq \sigma^{-1}(r) \neq S_1$, and $f_\eta : E_{TS_1} \to E_{TS_2}$ be a mapping defined by $f_\eta = \eta$. Then (f_σ, f_η) is a net morphism from $\mathcal{N}_{TS_1}$ to $\mathcal{N}_{TS_2}$.*

Proof: Let $\mathcal{N}_{TS_i} = (R_{TS_i}, E_{TS_i}, F_{TS_i}, I_{TS_i}, R_{s^i_{in}})$, for $i = 1, 2$. We observe that (MENI1) holds since f_σ is a partial function from R_{TS_2} to R_{TS_1} (follows from proposition 4.1.2), and (MENI2) holds since f_η is a partial function from E_{TS_1} to E_{TS_2}. To show (MENI3), for every $r \in dom(f_\sigma)$ we need to demonstrate that $f_\sigma(r) \in R_{s^1_{in}} \Leftrightarrow r \in R_{s^2_{in}}$. This is equivalent to showing $s^1_{in} \in \sigma^{-1}(r) \Leftrightarrow s^2_{in} \in r$ which clearly holds since $\sigma(s^1_{in}) = s^2_{in}$. To prove (MENI4), for every $e \in E_{TS_1} \setminus dom(\eta)$ we need to show that $f_\sigma^{-1}({}^\bullet e) = \emptyset = f_\sigma^{-1}(e^\bullet)$. Assume that $r \in f_\sigma^{-1}({}^\bullet e) \neq \emptyset$. Then $f_\sigma(r) \in {}^\bullet e$ and so $\sigma^{-1}(r) \in {}^\bullet e$ (in $\mathcal{N}_{TS_1}$) which means $\sigma^{-1}(r) \in {}^\circ e$ (in TS_1). From corollary 3.1.1 and proposition 4.1.2(1) we have that $\eta(e)$ is defined, a contradiction. Hence $f_\sigma^{-1}({}^\bullet e) = \emptyset$. The same can be shown for $f_\sigma^{-1}(e^\bullet)$. Finally, to show (MENI5), we need to prove that, for every $e \in dom(\eta)$,

$$f_\sigma^{-1}({}^\bullet e) = {}^\bullet\eta(e) \quad \text{and} \quad f_\sigma^{-1}(e^\bullet) = \eta(e)^\bullet \quad \text{and} \quad \eta(\overset{\blacksquare}{e}) \cap \mathcal{M}_{(f_\sigma, f_\eta)} \subseteq f_\sigma^{-1}(\overset{\blacksquare}{e}).$$

The first equality can be proved as follows (note that by corollary 3.1.1, $\{e\} \in U_1$).

$$\begin{array}{rcl} r \in f_\sigma^{-1}({}^\bullet e) & \Leftrightarrow & f_\sigma(r) \in {}^\bullet e = {}^\circ e \\ & \Leftrightarrow & \sigma^{-1}(r) \in {}^\circ e \\ & \overset{(prop.\ 4.1.2(1))}{\Leftrightarrow} & r \in {}^\circ\eta(e) = {}^\bullet\eta(e). \end{array}$$

The second equality can be proved in a similar way.

To prove the last part of (MENI5) notice that, $r \in \mathcal{M}_{(f_\sigma, f_\eta)}$ means that $r \in R_{TS_2}$ and, moreover, $r \in R_{s^2_{in}}$ or there is $e \in dom(\eta)$ such that $r \in \eta(e)^\bullet$. If $r \in R_{s^2_{in}}$, then $s^2_{in} \in r$

and together with $\sigma(s^1_{in}) = s^2_{in}$ we have $s^1_{in} \in \sigma^{-1}(r)$. So $\sigma^{-1}(r) \neq \emptyset$. If $r \in \eta(e)^\bullet$, for some $e \in dom(\eta)$, then $\sigma^{-1}(r) \neq \emptyset$ follows from corollary 3.1.1 and proposition 4.1.2(1). The inclusion follows then from the fact that, for $r \in \eta(\overset{\blacksquare}{e}) = \eta(\overset{\square}{e})$ and $\sigma^{-1}(r) \neq \emptyset$, proposition 4.1.2(2) states that $\sigma^{-1}(r) \in \overset{\square}{e} = \overset{\blacksquare}{e}$. Hence $f_\sigma(r) \in \overset{\blacksquare}{e}$, and so $r \in f_\sigma^{-1}(\overset{\blacksquare}{e})$. $\square$

The next theorem defines a functor from $\mathcal{CAT}_{TSENI}$ to $\mathcal{CAT}_{ENI}$.

Theorem 5.3.1 *Let $J : \mathcal{CAT}_{TSENI} \to \mathcal{CAT}_{ENI}$ be a mapping defined, for every TSENI transition system TS and transition system morphism (σ, η), by $J(TS) = \mathcal{N}_{TS}$ and $J(\sigma, \eta) = (f_\sigma, f_\eta)$. Then J is a functor.*

Proof: Let $TS = (S, U, T, s_{in})$ and $TS_i = (S_i, U_i, T_i, s^i_{in})$ (for $i = 1, 2, 3$) be TSENI transition systems. Let $id_{TS} = (\sigma_{id}, \eta_{id}) : TS \to TS$ be an identity morphism and $f = (\sigma_f, \eta_f) : TS_1 \to TS_2$ and $g = (\sigma_g, \eta_g) : TS_2 \to TS_3$ be two transition system morphisms. Then (F1) follows from proposition 5.3.1. We now show that $J(id_{TS}) = id_{\mathcal{N}_{TS}}$ (i.e. (F2)) and $J(g \circ f) = J(g) \circ J(f)$ (i.e. (F3)) also hold.

The former follows from $f_{\sigma_{id}}(r) = \sigma_{id}^{-1}(r) = r$ and $f_{\eta_{id}}(e) = \eta_{id}(e) = e$, for $r \in R_{TS}$ and $e \in E_{TS}$. The latter can be shown as follows. We have:

$$\begin{array}{rcl} J(g \circ f) & = & J((\sigma_g, \eta_g) \circ (\sigma_f, \eta_f)) \\ & = & J(\sigma_g \circ \sigma_f, \eta_g \circ \eta_f) \\ & = & (f_{\sigma_g \circ \sigma_f}, f_{\eta_g \circ \eta_f}) \\ J(g) \circ J(f) & = & (f_{\sigma_g}, f_{\eta_g}) \circ (f_{\sigma_f}, f_{\eta_f}) \\ & = & (f_{\sigma_f} \circ f_{\sigma_g}, f_{\eta_g} \circ f_{\eta_f}). \end{array}$$

We then observe that $f_{\sigma_g \circ \sigma_f} = (\sigma_g \circ \sigma_f)^{-1} = \sigma_f^{-1} \circ \sigma_g^{-1} = f_{\sigma_f} \circ f_{\sigma_g}$ and $f_{\eta_g \circ \eta_f} = \eta_g \circ \eta_f = f_{\eta_g} \circ f_{\eta_f}$ which completes the proof. $\square$

5.4 An Adjunction between Functors H and J

The two functors $J : \mathcal{CAT}_{TSENI} \to \mathcal{CAT}_{ENI}$ and $H : \mathcal{CAT}_{ENI} \to \mathcal{CAT}_{TSENI}$ are closely related since, in categorical terms, they form an adjunction. We recall some definitions from [10] and [35].

Definition 5.4.1 *Given two functors $F, G : \mathcal{A} \to \mathcal{B}$, between categories $\mathcal{A}$ and $\mathcal{B}$, a* natural transformation *$\tau : F \to G$ is a function which assigns to each object A of $\mathcal{A}$ a*

$$
\begin{tikzcd}
F(A) \arrow[r, "\tau(A)"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] & A \arrow[d, "f"] \\
F(A') \arrow[r, "\tau(A')"] & G(A') & A'
\end{tikzcd}
$$

Figure 5.2: Natural transformation $\tau : F \to G$.

morphism $\tau(A) : F(A) \to G(A)$ *of* $\mathcal{B}$ *in such a way that every morphism* $f : A \to A'$ *in* $\mathcal{A}$ *yields a diagram as in figure 5.2 which is commutative. The morphism* $\tau(A)$ *for an object* A *is called the* component *of the natural transformation* τ *at* A. □

In the above definition, the commutativity of the diagram in figure 5.2 means that the equality $G(f) \circ \tau(A) = \tau(A') \circ F(f)$ holds.

Definition 5.4.2 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be categories. If* $F : \mathcal{A} \to \mathcal{B}$ *and* $G : \mathcal{B} \to \mathcal{A}$ *are functors, we say that* F *is a* left adjoint *to* G *and* G *is a* right adjoint *to* F *provided there is natural transformation* $\tau : id_{\mathcal{A}} \to G \circ F$[2] *such that for any objects* A *of* $\mathcal{A}$ *and* B *of* $\mathcal{B}$ *and any morphism* $f : A \to G(B)$, *there is a unique morphism* $g : F(A) \to B$ *such that* $f = G(g) \circ \tau(A)$. *The triple* (F, G, τ) *constitutes an* adjunction. *The natural transformation* τ *is called the* unit *of the adjunction.* □

We will use the term *reflection* for an adjunction in which the right adjoint is full and faithful, and *coreflection* for an adjunction in which the left adjoint is full and faithful (see [47]). It was proved in [35] that an adjunction is a coreflection (reflection) if every component of its unit (resp. counit) is an isomorphism.

Before proving the next theorem, we consider an ENI-system $\mathcal{N} = (B, E, F, I, c_{in})$ and the related transition system $H(\mathcal{N}) = TS_{\mathcal{N}} = (C_{\mathcal{N}}, U_{\mathcal{N}}, \to_{\mathcal{N}}, c_{in})$. By (3.4), for every $b \in B$, $r_b = \{c \in C_{\mathcal{N}} \mid b \in c\}$ is a (possibly trivial) region in $TS_{\mathcal{N}}$. We need to prove one more property of the regions in $H(\mathcal{N}) = TS_{\mathcal{N}}$ before presenting the main result of this section.

Proposition 5.4.1 *Let* $\mathcal{N} = (B, E, F, I, c_{in})$ *be an ENI-system and* $H(\mathcal{N}) = TS_{\mathcal{N}}$ *be the transition system generated by* $\mathcal{N}$. *Then, for all* $b \in B$ *and* $e \in E_{TS_{\mathcal{N}}}$, *the following hold:*

[2] $id_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ is an identity functor which maps objects and morphisms of $\mathcal{A}$ onto themselves.

1. $r_b \in {}^\circ e$ *(in* $H(\mathcal{N})$*)* $\Leftrightarrow b \in {}^\bullet e$ *(in* $\mathcal{N}$*).*

2. $r_b \in e^\circ$ *(in* $H(\mathcal{N})$*)* $\Leftrightarrow b \in e^\bullet$ *(in* $\mathcal{N}$*).*

Proof: (1) To show the ($\Rightarrow$) implication we proceed as follows. $r_b \in {}^\circ e$ (in $H(\mathcal{N})$) implies that there exists $c \xrightarrow{\{e\}}_{\mathcal{N}} d$ such that $c \in r_b$ and $d \notin r_b$. Hence $b \in c$ and $b \notin d$. From (3.3) it follows that $c \setminus d = {}^\bullet e$ and so $b \in {}^\bullet e$ (in $\mathcal{N}$).

To show the reverse implication, assume that $b \in {}^\bullet e$ (in $\mathcal{N}$). From (A2), corollary 3.1.1 and $e \in E_{TS_\mathcal{N}}$ it follows that there exist $d, d' \in C_\mathcal{N}$ such that $d \xrightarrow{\{e\}}_{\mathcal{N}} d'$. From $b \in {}^\bullet e$ and $d \setminus d' = {}^\bullet e$ we have that $b \in d$ and $b \notin d'$. Hence $d \in r_b$ and $d' \notin r_b$. So r_b is a non-trivial region in $H(\mathcal{N}) = TS_\mathcal{N}$ and $r_b \in {}^\circ e$.
(2) Can be proved in a similar way. $\square$

Theorem 5.4.1 *Let* $\tau : id_{\mathcal{CAT}_{TSENI}} \to H \circ J$ *be a function, where* $\tau(TS) : TS \to H \circ J(TS)$ *is a morphism defined as follows.*

$$\tau(TS) = ([\tau(TS)]_0, [\tau(TS)]_1)$$

where $[\tau(TS)]_0 : S \to C_{\mathcal{N}_{TS}}$ *and* $[\tau(TS)]_1 : E_{TS} \to E_{TS_{\mathcal{N}_{TS}}}$ *are total functions defined below.*

$$\begin{array}{lrcl} \forall\, s \in S : & [\tau(TS)]_0(s) & = & R_s \\ \forall\, e \in E_{TS} : & [\tau(TS)]_1(e) & = & e. \end{array} \tag{5.3}$$

Then, $J : \mathcal{CAT}_{TSENI} \to \mathcal{CAT}_{ENI}$ *and* $H : \mathcal{CAT}_{ENI} \to \mathcal{CAT}_{TSENI}$ *form an adjunction with* J *as left adjoint and* τ *as a unit (see figure 5.3).*

Proof: Clearly, τ is a natural transformation. We need to show that, for every TSENI transition system TS_1 in $\mathcal{CAT}_{TSENI}$ and every ENI-system $\mathcal{N}_2$ in $\mathcal{CAT}_{ENI}$, if there is a transition system morphism $f : TS_1 \to H(\mathcal{N}_2)$ then there is a unique net morphism $g : J(TS_1) \to \mathcal{N}_2$ such that

$$f = H(g) \circ \tau(TS_1). \tag{5.4}$$

Let $TS_1 = (S_1, U_1, T_1, s^1_{in})$ and $\mathcal{N}_2 = (B_2, E_2, F_2, I_2, c^2_{in})$. From the definitions of the functors,

$$\begin{array}{rcccl} J(TS_1) & = & \mathcal{N}_{TS_1} & = & (R_{TS_1}, E_{TS_1}, F_{TS_1}, I_{TS_1}, R_{s^1_{in}}) \\ H(\mathcal{N}_2) & = & TS_{\mathcal{N}_2} & = & (C_{\mathcal{N}_2}, U_{\mathcal{N}_2}, \to_{\mathcal{N}_2}, c^2_{in}). \end{array}$$

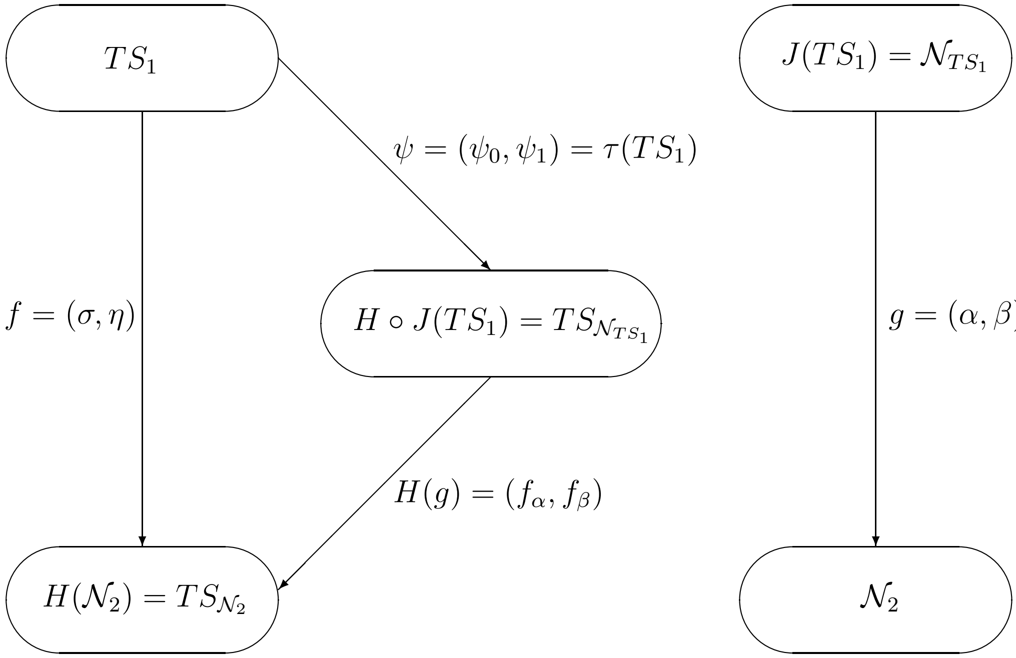


Figure 5.3: Illustration for theorem 5.4.1.

Let $\tau(TS_1) = \psi$. It follows from theorem 3.6.1 that ψ is an isomorphism and a well defined transition system morphism from TS_1 to $TS_{\mathcal{N}_{TS_1}}$. For $f = (\sigma, \eta)$, we define $g = (\alpha, \beta)$ in the following way. $\alpha : B_2 \to R_{TS_1}$ is a mapping such that, for $b \in B_2$,

$$\alpha(b) = \begin{cases} \sigma^{-1}(r_b) & \text{if } S_1 \neq \sigma^{-1}(r_b) \neq \emptyset \\ \textit{undefined} & \text{otherwise} \end{cases}$$

and $\beta : E_{TS_1} \to E_2$ is defined by $\beta(e) = \eta(e)$. Notice that $\sigma^{-1}(r_b) \neq S_1$ and $\sigma^{-1}(r_b) \neq \emptyset$ implies $r_b \neq C_{\mathcal{N}_2}$ and $r_b \neq \emptyset$.

We will prove that $g = (\alpha, \beta)$ is a net morphism from $J(TS_1)$ to $\mathcal{N}_2$. We observe that (MENI1) and (MENI2) hold since α is a partial function and β is a partial function (as $\eta : E_{TS_1} \to E_{TS_{\mathcal{N}_2}} \subseteq E_2$ is a partial function). To show (MENI3), for all $b \in dom(\alpha)$, we need to demonstrate that $\alpha(b) \in R_{s^1_{in}} \Leftrightarrow b \in c^2_{in}$. This holds, since

$$\begin{array}{lcl} \alpha(b) = \sigma^{-1}(r_b) \in R_{s^1_{in}} & \Leftrightarrow & s^1_{in} \in \sigma^{-1}(r_b) \;\; \Leftrightarrow \;\; \sigma(s^1_{in}) \in r_b \\ & \overset{(MTS1)}{\Leftrightarrow} & c^2_{in} \in r_b \qquad\quad\;\; \Leftrightarrow \;\; b \in c^2_{in}. \end{array}$$

To prove (MENI4), for every $e \in E_{TS_1} \setminus dom(\beta)$, we need to show that $\alpha^{-1}({}^\bullet e) = \emptyset = \alpha^{-1}(e^\bullet)$. Note that $\eta(e)$ is not defined. Assume that $\alpha^{-1}({}^\bullet e) \neq \emptyset$. Then there is $b \in B_2$ such that $b \in \alpha^{-1}({}^\bullet e)$ which means $\alpha(b) \in {}^\bullet e$. From the definition of α and the fact that (in $\mathcal{N}_{TS_1}$) ${}^\bullet e = {}^\circ e$, we have $\sigma^{-1}(r_b) \in {}^\circ e$. Hence, from proposition 4.1.2(1) we obtain that $\eta(e)$ is defined, a contradiction. The same way of reasoning applies to $\alpha^{-1}(e^\bullet)$.

Finally, to show (MENI5), we need to prove that for all $e \in dom(\beta) = dom(\eta)$,

$$\alpha^{-1}({}^\bullet e) = {}^\bullet\beta(e) \quad \text{and} \quad \alpha^{-1}(e^\bullet) = \beta(e)^\bullet \quad \text{and} \quad \beta(\overset{\blacksquare}{e}) \cap \mathcal{M}_{(\alpha,\beta)} \subseteq \alpha^{-1}(\overset{\blacksquare}{e}).$$

First we prove that $\alpha^{-1}({}^\bullet e) = {}^\bullet\beta(e)$.

$$\begin{array}{lcl}
b \in \alpha^{-1}({}^\bullet e) & \Leftrightarrow & \alpha(b) \in {}^\bullet e = {}^\circ e \text{ (in } \mathcal{N}_{TS_1}) \\
& \Leftrightarrow & \sigma^{-1}(r_b) \in {}^\circ e \text{ (a condition in } \mathcal{N}_{TS_1} \text{ is a region in } TS_1) \\
& \overset{(prop.\ 4.1.2(1))}{\Leftrightarrow} & r_b \in {}^\circ\eta(e) \text{ (in } H(\mathcal{N}_2)) \\
& \overset{(prop.\ 5.4.1(1))}{\Leftrightarrow} & b \in {}^\bullet\eta(e) \text{ (in } \mathcal{N}_2) \\
& \Leftrightarrow & b \in {}^\bullet\beta(e).
\end{array}$$

That $\alpha^{-1}(e^\bullet) = \beta(e)^\bullet$ can be proved in a similar way. We now prove that $\beta(\overset{\blacksquare}{e}) \cap \mathcal{M}_{(\alpha,\beta)} \subseteq \alpha^{-1}(\overset{\blacksquare}{e})$ holds (in $\mathcal{N}_2$). $b \in \beta(\overset{\blacksquare}{e}) \cap \mathcal{M}_{(\alpha,\beta)}$ implies $b \in \eta(\overset{\blacksquare}{e})$. From (A2) and corollary 3.1.1 it follows that there exist $c, c' \in C_{\mathcal{N}_2}$ such that $c \xrightarrow{\{\eta(e)\}}_{\mathcal{N}_2} c'$ and $c = \sigma(s)$, $c' = \sigma(s')$ for some $s, s' \in S_1$. From (3.3) we have $c \backslash c' = {}^\bullet\eta(e)$, $c' \backslash c = \eta(e)^\bullet$ and $\eta(\overset{\blacksquare}{e}) \cap c = \emptyset$. Since $b \in \eta(\overset{\blacksquare}{e})$, we have that $b \notin c$. By the definition of an inhibitor net, we have $\eta(e)^\bullet \cap \eta(\overset{\blacksquare}{e}) = \emptyset$. Since $b \in \eta(\overset{\blacksquare}{e})$, we have $b \notin \eta(e)^\bullet = c' \setminus c$, which together with the fact that $b \notin c$ means $b \notin c'$. Hence $b \notin c$ and $b \notin c'$, and so $c, c' \notin r_b$.

Recall that $b \in \mathcal{M}_{(\alpha,\beta)} = \{b \in B_2 \mid b \in c^2_{in} \ \vee \ \exists g \in dom(\beta) \ : \ b \in \beta(g)^\bullet\}$. If $b \in \beta(g)^\bullet$ for some $g \in dom(\beta)$, then from the already proved part of (MENI5) we have $b \in \alpha^{-1}(g^\bullet)$, so $b \in dom(\alpha)$. If $b \in \alpha^{-1}(R_{s^1_{in}})$ we have again $b \in dom(\alpha)$. If $b \in c^2_{in} \setminus \alpha^{-1}(R_{s^1_{in}})$, then b belongs to every case reachable in $\mathcal{N}_2$ when $\mathcal{N}_{TS_1}$ is simulated (this follows from proposition 4.2.1 and the fact that for $g \in dom(\beta)$, $\alpha^{-1}({}^\bullet g) = {}^\bullet\beta(g)$ and $\alpha^{-1}(g^\bullet) = \beta(g)^\bullet$ which was already proved). But this contradicts $b \notin c, c'$. So $b \in \beta(\overset{\blacksquare}{e}) \cap \mathcal{M}_{(\alpha,\beta)}$ implies $b \in dom(\alpha)$, and thus that $\sigma^{-1}(r_b)$ is not trivial. Consequently, r_b is non-trivial and hence $\mathcal{B}^{\eta(e)}_{C_{\mathcal{N}_2} \backslash r_b} \neq \emptyset$. Suppose now that $f \xrightarrow{\{\eta(e)\}}_{\mathcal{N}_2} f'$ belongs to $\mathcal{B}^{\eta(e)}_{r_b}$. Then $f, f' \in r_b$ and we have $b \in f$ and $b \in f'$. But this and (3.3) contradicts $b \in \eta(\overset{\blacksquare}{e})$. Hence $\mathcal{B}^{\eta(e)}_{r_b} = \emptyset$ and, as a result, $r_b \in \eta(\overset{\square}{e})$ (in $H(\mathcal{N}_2)$). From proposition 4.1.2(2) and $\sigma^{-1}(r_b) \neq \emptyset$, $\sigma^{-1}(r_b) \in \overset{\square}{e}$. Since $b \in dom(\alpha)$, $\alpha(b) \in \overset{\square}{e}$. Hence, in $\mathcal{N}_{TS_1}$, $\alpha(b) \in \overset{\blacksquare}{e}$ and so $b \in \alpha^{-1}(\overset{\blacksquare}{e})$ in $\mathcal{N}_2$. This means that the inclusion $\beta(\overset{\blacksquare}{e}) \cap \mathcal{M}_{(\alpha,\beta)} \subseteq \alpha^{-1}(\overset{\blacksquare}{e})$ holds. Thus we have shown that $g = (\alpha, \beta)$ is a net morphism from $J(TS_1)$ to $\mathcal{N}_2$.

We now want to show that $H(\alpha, \beta) \circ \tau(TS_1) = f$ where $f = (\sigma, \eta)$, $\tau(TS_1) = (\psi_0, \psi_1)$ and $H(\alpha, \beta) = (f_\alpha, f_\beta)$. What we need to show is that $(f_\alpha, f_\beta) \circ (\psi_0, \psi_1) = (\sigma, \eta)$, i.e.

$(f_\alpha \circ \psi_0, f_\beta \circ \psi_1) = (\sigma, \eta)$. It is enough to prove that $f_\beta \circ \psi_1 = \eta$, and then $f_\alpha \circ \psi_0 = \sigma$ follows from proposition 4.1.1.

It is easy to show that $f_\beta \circ \psi_1 = \eta$ holds. The first of the functions involved $\psi_1 : E_{TS_1} \to E_{TS_{\mathcal{N}_{TS_1}}}$ (see (5.3)) is a total identity function (notice that from proposition 3.6.1(2) we have $E_{TS_1} = E_{TS_{\mathcal{N}_{TS_1}}}$). The second function $f_\beta : E_{TS_{\mathcal{N}_{TS_1}}} \to E_{TS_{\mathcal{N}_2}}$ is defined as follows: $f_\beta = \beta = \eta$, where $\eta : E_{TS_1} \to E_{TS_{\mathcal{N}_2}}$. So $f_\beta \circ \psi_1(e) = f_\beta(\psi_1(e)) = f_\beta(e) = \eta(e)$ for all $e \in dom(\eta)$.

We now prove the uniqueness of $g = (\alpha, \beta)$. Assume that there is another net morphism $g' = (\alpha', \beta')$ satisfying (5.4). Then $H(\alpha, \beta) \circ (\psi_0, \psi_1) = (\sigma, \eta)$ and $H(\alpha', \beta') \circ (\psi_0, \psi_1) = (\sigma, \eta)$. From the above it follows that $(f_\alpha, f_\beta) \circ (\psi_0, \psi_1) = (f_{\alpha'}, f_{\beta'}) \circ (\psi_0, \psi_1)$ which means that:

$$f_\alpha \circ \psi_0 = f_{\alpha'} \circ \psi_0 \tag{5.5}$$

$$f_\beta \circ \psi_1 = f_{\beta'} \circ \psi_1. \tag{5.6}$$

From (5.6) we obtain $f_\beta = f_{\beta'}$ and consequently $\beta = \beta'$.[3] From (5.5) we have $f_\alpha(R_s) = f_{\alpha'}(R_s)$, for all $s \in S_1$, which means

$$\forall\, s \in S_1 \;:\; \alpha^{-1}(R_s) \cup (c^2_{in} \setminus \alpha^{-1}(R_{s^1_{in}})) = \alpha'^{-1}(R_s) \cup (c^2_{in} \setminus \alpha'^{-1}(R_{s^1_{in}})).$$

Observe that the sets $\alpha^{-1}(R_{s^1_{in}})$ and $\alpha'^{-1}(R_{s^1_{in}})$ are equal. From corollary 3.1.2 and (3.6) we know that every condition in $\mathcal{N}_{TS_1}$ is a pre- or post-condition of some event. This and (MENI3), (MENI4) and (MENI5) gives us

$$\begin{array}{rcl} \alpha^{-1}(R_{s^1_{in}}) & = & (\bigcup_{e \in dom(\beta)} {}^\bullet\beta(e) \;\;\cup\;\; \bigcup_{e \in dom(\beta)} \beta(e)^\bullet) \;\;\cap\;\; c^2_{in} \\ & \overset{(\beta=\beta')}{=} & (\bigcup_{e \in dom(\beta')} {}^\bullet\beta'(e) \;\;\cup\;\; \bigcup_{e \in dom(\beta')} \beta'(e)^\bullet) \;\;\cap\;\; c^2_{in} \\ & = & \alpha'^{-1}(R_{s^1_{in}}). \end{array}$$

Hence we obtain

$$\begin{array}{llrcl} \forall\, s \in S_1 \;: & & \alpha^{-1}(R_s) & = & \alpha'^{-1}(R_s) \\ \forall\, s \in S_1 & \forall\, b \in B_2 \;: & b \in \alpha^{-1}(R_s) & \Leftrightarrow & b \in \alpha'^{-1}(R_s) \\ \forall\, s \in S_1 & \forall\, b \in B_2 \;: & \alpha(b) \in R_s & \Leftrightarrow & \alpha'(b) \in R_s \\ \forall\, s \in S_1 & \forall\, b \in B_2 \;: & s \in \alpha(b) & \Leftrightarrow & s \in \alpha'(b) \\ \forall\, b \in B_2 & \forall\, s \in S_1 \;: & s \in \alpha(b) & \Leftrightarrow & s \in \alpha'(b) \\ \forall\, b \in B_2 \;: & & \alpha(b) & = & \alpha'(b) \end{array}$$

Thus $\alpha = \alpha'$, which gives $g = g'$, a contradiction. □

[3] It is essential to prove that $\alpha = \alpha'$ as well, as we do not have the net version of proposition 4.1.1. Notice that proving $\beta = \beta'$ first was important.

The adjunction between J and H is in fact a coreflection as every $\tau(TS)$ is an isomorphism (theorem 3.6.1).

Example 5.4.1 *We now illustrate the last theorem with the following example (see figure 5.4). Consider the TSENI transition system TS_1 shown in figure 5.4. It has four non-trivial regions: $r_1 = \{s^1_{in}, s_2\}$, $r_2 = \{s^1_{in}, s_1\}$, $r_3 = \{s_1, s_3\}$ and $r_4 = \{s_2, s_3\}$. Moreover, the pre-, post- and I-regions of a and b are: ${}^\circ a = \{r_1\}$, ${}^\circ b = \{r_2\}$, $a^\circ = \{r_3\}$, $b^\circ = \{r_4\}$, $\overset{\square}{a} = \{r_4\}$ and $\overset{\square}{b} = \{r_3\}$. We can build $\mathcal{N}_{TS_1}$ and then $TS_{\mathcal{N}_{TS_1}}$ using constructions from sections 3.4 and 3.5. Note that, according to theorem 3.6.1, $TS_{\mathcal{N}_{TS_1}}$ is isomorphic to TS_1. Consider now ENI-system $\mathcal{N}_2$ and the TSENI transition system generated by it, $TS_{\mathcal{N}_2}$. The reachable cases of $\mathcal{N}_2$ are:*

$$\begin{aligned} c^2_{in} &= \{b_1, b_2, b_5\} & c_1 &= \{b_3, b_2, b_5\} \\ c_2 &= \{b_1, b_4, b_5\} & c_3 &= \{b_3, b_4, b_5\}. \end{aligned}$$

We have the following regions of $TS_{\mathcal{N}_2}$ associated with every $b \in B_2$:

$$\begin{aligned} r_{b_1} &= \{c^2_{in}, c_2\} & r_{b_2} &= \{c^2_{in}, c_1\} & r_{b_3} &= \{c_1, c_3\} \\ r_{b_4} &= \{c_2, c_3\} & r_{b_5} &= \{c^2_{in}, c_1, c_2, c_3\} & r_{b_6} &= \emptyset. \end{aligned}$$

Observe that $r_{b_5} = C_{\mathcal{N}_2}$ and $r_{b_6} = \emptyset$ are trivial regions. Let us define a transition system morphism $f = (\sigma, \eta)$ from TS_1 to $TS_{\mathcal{N}_2}$ in a following way.

$$\begin{aligned} \sigma(s^1_{in}) &= c^2_{in} & \sigma(s_1) &= c_1 & \sigma(s_2) &= c^2_{in} \\ \sigma(s_3) &= c_1 & \eta(a) &= e & \eta(b) &- \textit{not defined.} \end{aligned}$$

According to the construction in theorem 5.4.1 the net morphism $g = (\alpha, \beta)$ from $\mathcal{N}_{TS_1}$ to $\mathcal{N}_2$ is defined by:

$$\begin{aligned} \alpha(b_1) &= \sigma^{-1}(r_{b_1}) = r_1 \\ \alpha(b_2) &- \textit{not defined, because } \sigma^{-1}(r_{b_2}) = S_1 \\ \alpha(b_3) &= \sigma^{-1}(r_{b_3}) = r_3 \\ \alpha(b_4) &- \textit{not defined, because } \sigma^{-1}(r_{b_4}) = \emptyset \\ \alpha(b_5) &- \textit{not defined, because } r_{b_5} = C_{\mathcal{N}_2} \\ \alpha(b_6) &- \textit{not defined, because } r_{b_6} = \emptyset \\ \beta(a) &= e \\ \beta(b) &- \textit{not defined.} \end{aligned}$$

Recall that $f_\alpha(\psi_0(s)) = f_\alpha(R_s) = \alpha^{-1}(R_s) \cup (c^2_{in} \setminus \alpha^{-1}(R_{s^1_{in}}))$. In our example, we have

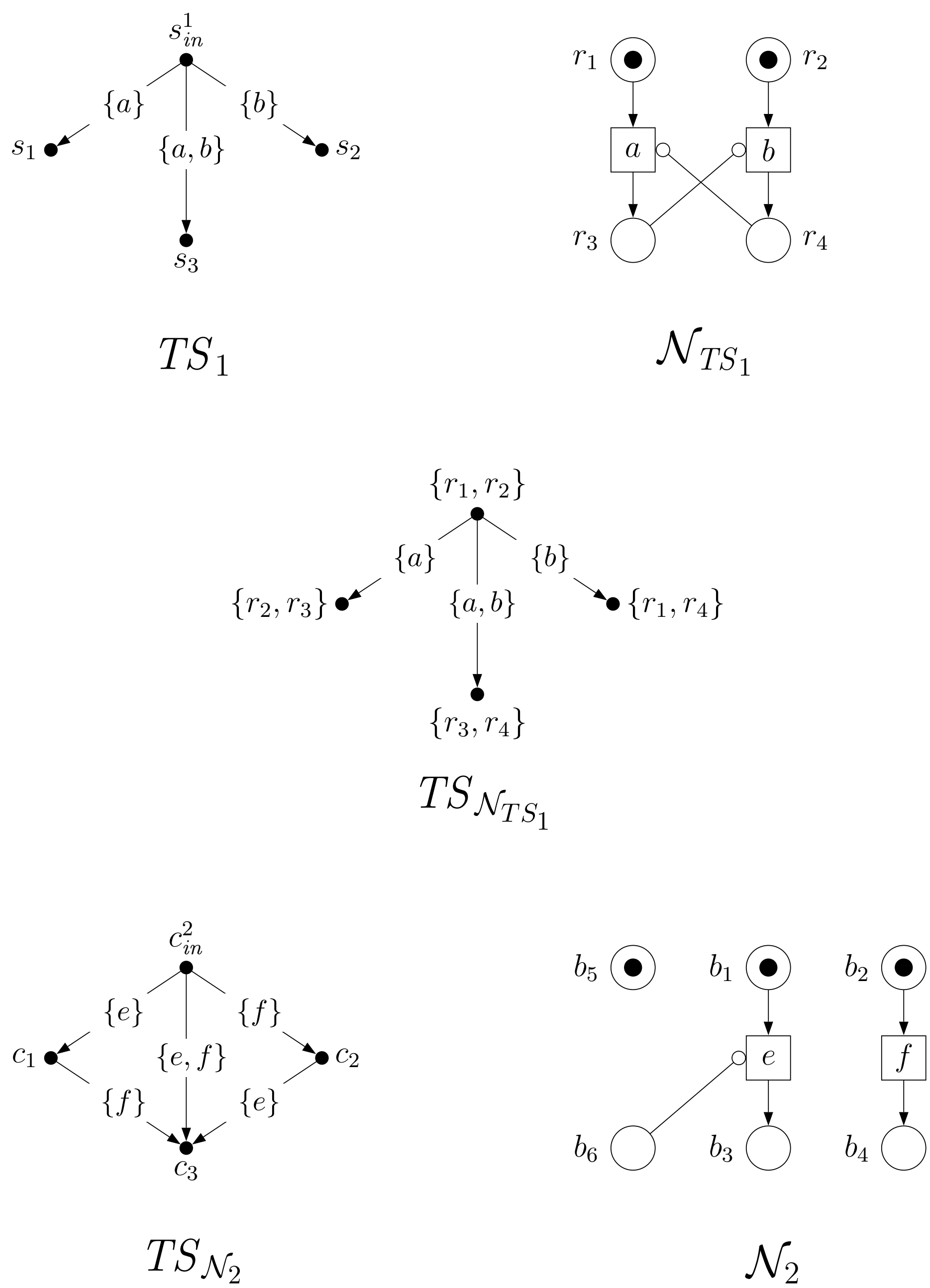


Figure 5.4: Example 5.4.1 (an illustration for theorem 5.4.1).

$c_{in}^2 \setminus \alpha^{-1}(R_{s_{in}^1}) = \{b_2, b_5\}$. *We can now verify that* $f_\alpha(\psi_0(s)) = \sigma(s)$ *for all* $s \in S_1$.

$$\begin{array}{lllllllll}
R_{s_{in}^1} & = & \{r_1, r_2\} & \alpha^{-1}(R_{s_{in}^1}) & = & \{b_1\} & f_\alpha(R_{s_{in}^1}) & = & \{b_1\} \cup \{b_2, b_5\} = c_{in}^2 \\
R_{s_1} & = & \{r_2, r_3\} & \alpha^{-1}(R_{s_1}) & = & \{b_3\} & f_\alpha(R_{s_1}) & = & \{b_3\} \cup \{b_2, b_5\} = c_1 \\
R_{s_2} & = & \{r_1, r_4\} & \alpha^{-1}(R_{s_2}) & = & \{b_1\} & f_\alpha(R_{s_2}) & = & \{b_1\} \cup \{b_2, b_5\} = c_{in}^2 \\
R_{s_3} & = & \{r_3, r_4\} & \alpha^{-1}(R_{s_3}) & = & \{b_3\} & f_\alpha(R_{s_3}) & = & \{b_3\} \cup \{b_2, b_5\} = c_1
\end{array}$$

We observe that $f_\alpha(C_{\mathcal{N}_{TS_1}}) = \{c_{in}^2, c_1\}$. □

Chapter 6

Minimisation of ENI-systems

In this chapter we consider the synthesis of ENI-systems using *minimal regions* (i.e. minimal w.r.t. set inclusion). We show that minimal regions are sufficient to solve the synthesis problem for ENI-systems. We show as well how to reduce the number of inhibitor arcs without changing the behaviour of the resulting net. It turns out that the redundancy in the number of regions and in the number of inhibitor arcs are linked, and both can be tackled at the same time. The synthesis problem for the Elementary Nets Systems with Inhibitor Arcs was studied in [17], but only for sequential behaviours. We compare the method of eliminating inhibitor arcs presented in this chapter with the one developed in [17]. As it turns out, the two methods delete the same inhibitor arcs.

6.1 Properties of (Minimal) Regions

Let $TS = (S, U, T, s_{in})$ be a TSENI transition system fixed for the rest of this chapter. The results in this section were formulated for transition systems describing sequential behaviour: Elementary Transition Systems in [12, 17, 22], and Condition Event Transition Systems in [12]. Here we show that they hold for the TSENI Transition Systems, where non-sequential behaviour is represented explicitly.

Proposition 6.1.1 *If r' and r are regions in R_{TS} such that $r' \subset r$ then $r_{\mathit{diff}} = r \setminus r' \in R_{TS}$.*

Proof: First we prove that definition 3.1.1(1) holds for r_{diff}. Let $s \xrightarrow{u} s'$, $s \in r_{\mathit{diff}} = r \setminus r'$ and $s' \notin r_{\mathit{diff}}$. We need to consider two cases (see figure 6.1).
Case 1: $s' \in r'$. Since r' is a region, there is $e \in u$ such that:

(i) If $u' \subseteq u \setminus \{e\}$ and $s \xrightarrow{u'} s''$ then $s'' \notin r'$.

(ii) If $q \xrightarrow{v} q'$ and $e \in v$ then $q \notin r'$ and $q' \in r'$.

To show definition 3.1.1(1) for r_{diff} it suffices to prove that $s'', q \in r$ in the formulae above.

Suppose that $s \xrightarrow{u'} s''$, $u' \subseteq u \setminus \{e\}$ and $s'' \in S \setminus r$ in (i). Then we have $s \in r$ (by $s \in r_{diff}$) and $s'' \notin r$ (by $s'' \in S \setminus r$). Since r is a region, there is $e' \in u'$ such that:

(iii) If $w \xrightarrow{h} w'$ and $e' \in h$ then $w \in r$ and $w' \notin r$.

From (iii) with $w = s$, $w' = s'$ and $h = u$ (notice that $e' \in u$) we obtain $s' \notin r$, which produces a contradiction with $s' \in r' \subset r$. Hence $s'' \in r$ in (i).

Suppose now that $q \xrightarrow{v} q'$, $e \in v$ and $q \in S \setminus r$ in (ii). Then we have $q \notin r$ and $q' \in r' \subset r$. Since r is a region, there exists $e'' \in v$ such that:

(iv) If $u'' \subseteq v \setminus \{e''\}$ and $q \xrightarrow{u''} s'''$ then $s''' \notin r$.

(v) If $p \xrightarrow{v'} p'$ and $e'' \in v'$ then $p \notin r$ and $p' \in r$.

From (A4) and $q \xrightarrow{v} q'$ it follows that there exists q'' such that $q \xrightarrow{\{e\}} q''$. By (ii), $q'' \in r' \subset r$. If $e \neq e''$ then $q'' \notin r$, by (iv) with $u'' = \{e\}$ and $s''' = q''$, producing a contradiction. Suppose $e = e''$. Then (v) is satisfied with $p = s$, $p' = s'$ and $v' = u$. This implies $s \notin r$, contradicting $s \in r_{diff} \subset r$. Hence $q \in r$ in (ii).

Case 2: $s' \notin r$. Since r is a region, there is $e \in u$ such that:

(vi) If $u' \subseteq u \setminus \{e\}$ and $s \xrightarrow{u'} s''$ then $s'' \in r$.

(vii) If $q \xrightarrow{v} q'$ and $e \in v$ then $q \in r$ and $q' \notin r$.

Now, to show definition 3.1.1(1) for r_{diff} it suffices to prove that $s'', q \notin r'$ in the formulae above.

Suppose that $s \xrightarrow{u'} s''$, $u' \subseteq u \setminus \{e\}$ and $s'' \in r'$ in (vi). Since r' is a region and $s \notin r'$, there exists $e' \in u'$ such that:

(viii) If $w \xrightarrow{h} w'$ and $e' \in h$ then $w \notin r'$ and $w' \in r'$.

From (viii) with $w = s$, $w' = s'$ and $h = u$ (notice that $e' \in u$) we obtain $s' \in r'$, which contradicts $s' \notin r$ (because $r' \subset r$). Hence $s'' \notin r'$ in (vi).

Suppose now that $q \xrightarrow{v} q'$, $e \in v$ and $q \in r'$ in (vii). Then $q \in r'$ and $q' \notin r'$ (because $q' \notin r$). Since r' is a region, there exists $e'' \in v$ such that:

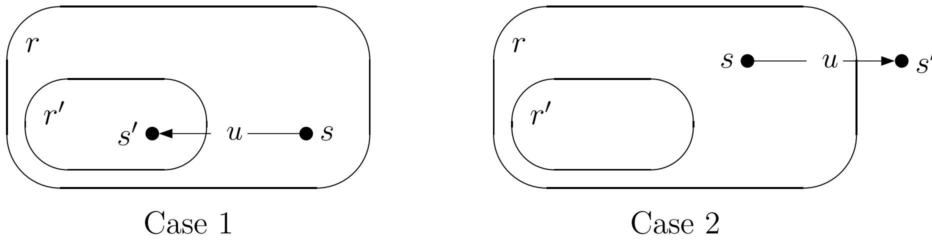


Figure 6.1: An illustration for proposition 6.1.1.

(ix) If $u'' \subseteq v \setminus \{e''\}$ and $q \xrightarrow{u''} s'''$ then $s''' \in r'$.

(x) If $p \xrightarrow{v'} p'$ and $e'' \in v'$ then $p \in r'$ and $p' \notin r'$.

From (A4) and $q \xrightarrow{v} q'$ it follows that there exists q'' such that $q \xrightarrow{\{e\}} q''$. By (vii), $q'' \notin r$ and hence $q'' \notin r'$. If $e \neq e''$ then $q'' \in r'$, by (ix) with $u'' = \{e\}$ and $s''' = q''$, producing a contradiction. Suppose $e = e''$. Then (x) is satisfied with $p = s$, $p' = s'$ and $v' = u$. This implies $s \in r'$, contradicting $s \in r_{\mathit{diff}} = r \setminus r'$. Hence $q \notin r'$ in (vii).

That definition 3.1.1(2) holds for r_{diff} can be proved in a similar way. Hence r_{diff} is a region. Moreover, as $r_{\mathit{diff}} \neq \emptyset$, $r_{\mathit{diff}} \in R_{TS}$. □

Proposition 6.1.2 *If r' and r'' are disjoint regions in R_{TS} then $r' \cup r''$ is a (possibly trivial) region.*

Proof: Define $r = r' \cup r''$. If $r = S$ then r is a trivial region in TS. Suppose $r \neq S$. From $r' \in R_{TS}$ it follows that $S \setminus r' \in R_{TS}$. Moreover, $r'' \subset S \setminus r'$ (because $r' \cap r'' = \emptyset$ and $r \neq S$). Hence, by proposition 6.1.1, $(S \setminus r') \setminus r'' = S \setminus (r' \cup r'') \in R_{TS}$ which in turn gives $r' \cup r'' \in R_{TS}$. □

Definition 6.1.1 *A region $r \in R_{TS}$ is* minimal *if $r' \not\subset r$ for every $r' \in R_{TS}$.* □

The proof of the next result is similar to that of property 3.3 in [22].

Theorem 6.1.1 *Every $r \in R_{TS}$ can be represented as a disjoint union of minimal regions.*

Proof: If r is minimal then the result holds. If r is non-minimal then there exists a minimal region $r' \subset r$. From proposition 6.1.1 it follows that $r'' = r \setminus r'$ is a region in R_{TS}. If r'' is minimal we have $r = r' \cup r''$. Otherwise, we continue in the same way with

r'' instead of r. In this way we will build a sequence of mutually disjoint, minimal regions which will be finite as S is finite[1], and whose union is equal to r. $\square$

Proposition 6.1.3 *Let r be a non-minimal region in R_{TS}, $u \in U$, $e \in E_{TS}$ and $s \in S$.*

1. *If $r \in {}^{\circ}u$ then there exists a minimal region $r' \subset r$ such that $r' \in {}^{\circ}u$.*

2. *If $r \in u^{\circ}$ then there exists a minimal region $r' \subset r$ such that $r' \in u^{\circ}$.*

3. *If $r \in \overset{\square}{e}$ then for every minimal region $r' \subset r$, $r' \in \overset{\square}{e}$.*

4. *If $r \in R_s$ then there exists a minimal region $r' \subset r$ such that $r' \in R_s$.*

Proof: (1) There exists $s \xrightarrow{u} s'$ such that $s \in r$ and $s' \notin r$. From theorem 6.1.1 it follows that r can be represented as a disjoint union of a set R of minimal regions. Let r' be a minimal region in R such that $s \in r'$. Since $s' \notin r$, $s' \notin r'$. Hence $r' \in {}^{\circ}u$.
(2) Can be proved similarly as (1).
(3) From the definition of an inhibitor region of e, it follows that for every non-trivial region $r' \subset r$, $r' \in \overset{\square}{e}$.
(4) Follows directly from theorem 6.1.1. $\square$

6.2 Minimal ENI-systems

Let $\mathcal{N}_{TS} = (R_{TS}, E_{TS}, F_{TS}, I_{TS}, R_{s_{in}})$ be an ENI-system associated with TS (see (3.5)). $\mathcal{N}_{TS}$ will be called *saturated* because it uses all the non-trivial regions as conditions; we will denote it by $\mathcal{N}_{Sat}$.

Let $\mathcal{R} \in 2^{R_{TS}}$ be a set of non-trivial regions of TS. Then

$$Min(\mathcal{R}) = \{r \in \mathcal{R} \mid r \text{ is minimal}\}$$

will denote the set of minimal regions in $\mathcal{R}$.

We now define a net system $\mathcal{N}_{Min}$ (called *minimal*), which is obtained from $\mathcal{N}_{Sat}$ by deleting all the conditions associated with non-minimal regions and adjacent arcs:

$$\mathcal{N}_{Min} = (Min(R_{TS}), E_{TS}, \widehat{F}_{TS}, \widehat{I}_{TS}, Min(R_{s_{in}}))$$

[1] It is only here that we use the assumption that S is finite.

where $\widehat{F}_{TS}$ and $\widehat{I}_{TS}$ are defined thus:

$$\begin{aligned} \widehat{F}_{TS} &= \{(r,e) \in R_{TS} \times E_{TS} \mid r \in Min({}^\circ e)\} \\ &\cup \{(e,r) \in E_{TS} \times R_{TS} \mid r \in Min(e^\circ)\} \\ \widehat{I}_{TS} &= \{(r,e) \in R_{TS} \times E_{TS} \mid r \in Min(\overset{\square}{e})\}. \end{aligned} \tag{6.1}$$

Directly from the definition of $\mathcal{N}_{Min}$ we have that, for every $e \in E_{TS}$,

$${}^\bullet e = Min({}^\circ e) \quad \text{and} \quad e^\bullet = Min(e^\circ) \quad \text{and} \quad \overset{\blacksquare}{e} = Min(\overset{\square}{e}). \tag{6.2}$$

Proposition 6.2.1 *$\mathcal{N}_{Min}$ is an ENI-system.*

Proof: Since $\mathcal{N}_{Sat}$ is an ENI-system, it suffices to show that, for every $e \in E_{TS}$, $e^\bullet$ and ${}^\bullet e$ are both non-empty sets in $\mathcal{N}_{Min}$. Thus, by (6.2), it suffices to show that for all $e \in E_{TS}$, $Min(e^\circ) \neq \emptyset \neq Min({}^\circ e)$. From proposition 3.2.1 it follows that $e^\circ \neq \emptyset \neq {}^\circ e$, for all $e \in E_{TS}$. And the former follows directly from corollary 3.1.1 and proposition 6.1.3(1,2). $\square$

The following proposition shows that any active step of events from $\mathcal{N}_{Min}$ is a valid step in $\mathcal{N}_{Sat}$, although in the latter there are more conditions.

Proposition 6.2.2 $U_{\mathcal{N}_{Min}} \subseteq V_{\mathcal{N}_{Sat}}$.

Proof: Let $u \in U_{\mathcal{N}_{Min}} \subseteq V_{\mathcal{N}_{Min}}$. We need to show that $u \in V_{\mathcal{N}_{Sat}}$. From the definition of a valid step in ENI-system, (3.2), (3.6) and (6.2) we have:

$$\begin{aligned} V_{\mathcal{N}_{Sat}} &= \{u \subseteq E_{TS} \mid u \neq \emptyset \ \wedge \ \forall\, e,f \in u \ : \ (e \neq f \Rightarrow ({}^\circ e \cup e^\circ) \cap ({}^\circ f \cup f^\circ) = \emptyset)\} \\ V_{\mathcal{N}_{Min}} &= \{u \subseteq E_{TS} \mid u \neq \emptyset \ \wedge \ \forall\, e,f \in u \ : \\ &\qquad (e \neq f \Rightarrow (Min({}^\circ e) \cup Min(e^\circ)) \cap (Min({}^\circ f) \cup Min(f^\circ)) = \emptyset)\}. \end{aligned}$$

Let $e, f \in u$ and $e \neq f$.

We will prove that ${}^\circ e \cap {}^\circ f = \emptyset$. Suppose there is $r \in {}^\circ e \cap {}^\circ f$. Then $r \in R_{TS}$ is non-minimal due to the definition of $V_{\mathcal{N}_{Min}}$ and $U_{\mathcal{N}_{Min}} \subseteq V_{\mathcal{N}_{Min}}$. From corollary 3.1.1 and proposition 6.1.3(1) it follows that there is a minimal region $r' \subset r$ such that $r' \in {}^\circ e$. We consider two cases.

Case 1: $r' \in {}^\circ f$. Then $r' \in Min({}^\circ e) \cap Min({}^\circ f)$. Since $u \in U_{\mathcal{N}_{Min}} \subseteq V_{\mathcal{N}_{Min}}$, we obtain a contradiction.

Case 2: $r' \notin {}^\circ f$. Then $r \setminus r' \in {}^\circ f$ (see figure 6.2(a)). (Notice that proposition 6.1.1 guarantees that $r \setminus r' \in R_{TS}$.) We observe that $r' \in \overset{\square}{f}$. From $u \in U_{\mathcal{N}_{Min}}$ we have that

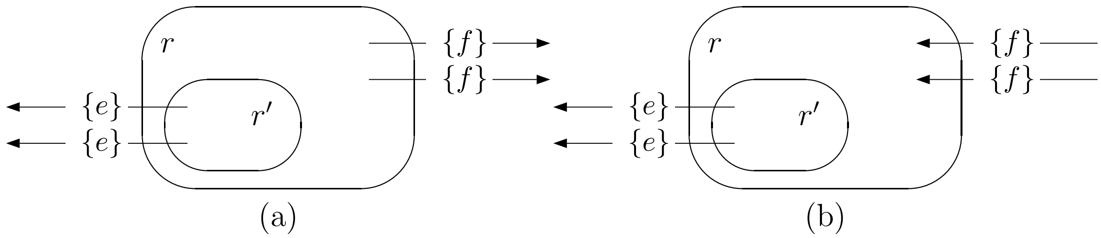


Figure 6.2: An illustration for proposition 6.2.2.

there exist $c, c' \in C_{\mathcal{N}_{Min}}$ such that $c \xrightarrow{u}_{\mathcal{N}_{Min}} c'$. From proposition 3.3.1(1) it follows that ${}^\bullet u \subseteq c$ and $\overset{\blacksquare}{u} \cap c = \emptyset$ (in $\mathcal{N}_{Min}$). Hence, ${}^\bullet e \subseteq c$ and $\overset{\blacksquare}{f} \cap c = \emptyset$ which, after applying (6.2), means that $Min({}^\circ e) \subseteq c$ and $Min(\overset{\square}{f}) \cap c = \emptyset$. But $r' \in {}^\circ e$, $r' \in \overset{\square}{f}$ and the fact that r' is minimal imply $r' \in Min({}^\circ e)$ and $r' \in Min(\overset{\square}{f})$, a contradiction. Hence ${}^\circ e \cap {}^\circ f = \emptyset$.

To prove $e^\circ \cap f^\circ = \emptyset$, suppose that there exists r in $e^\circ \cap f^\circ$. From proposition 3.1.5(2) it follows that $S \setminus r \in {}^\circ e \cap {}^\circ f$, which contradicts the previously proven fact.

What remains to be shown is ${}^\circ e \cap f^\circ = \emptyset$ (the case $e^\circ \cap {}^\circ f = \emptyset$ is symmetric). Suppose that there is a non-minimal region $r \in {}^\circ e \cap f^\circ$. From corollary 3.1.1 and proposition 6.1.3(1) it follows that there is a minimal region $r' \subset r$ such that $r' \in {}^\circ e$. We again consider two cases.

Case 1: $r' \in f^\circ$. Then $r' \in Min({}^\circ e) \cap Min(f^\circ)$. Since $u \in U_{\mathcal{N}_{Min}} \subseteq V_{\mathcal{N}_{Min}}$, we obtain a contradiction.

Case 2: $r' \notin f^\circ$. Then $r \setminus r' \in f^\circ$ (see figure 6.2(b)). We observe that $r' \in \overset{\square}{f}$. From $u \in U_{\mathcal{N}_{Min}}$ we have that there exist $c, c' \in C_{\mathcal{N}_{Min}}$ such that $c \xrightarrow{u}_{\mathcal{N}_{Min}} c'$. From proposition 3.3.1(1) it follows that ${}^\bullet u \subseteq c$ and $\overset{\blacksquare}{u} \cap c = \emptyset$ (in $\mathcal{N}_{Min}$). Hence, ${}^\bullet e \subseteq c$ and $\overset{\blacksquare}{f} \cap c = \emptyset$ which, after applying (6.2), means that $Min({}^\circ e) \subseteq c$ and $Min(\overset{\square}{f}) \cap c = \emptyset$. But $r' \in {}^\circ e$, $r' \in \overset{\square}{f}$ and the fact that r' is minimal imply $r' \in Min({}^\circ e)$ and $r' \in Min(\overset{\square}{f})$, a contradiction. Hence ${}^\circ e \cap f^\circ = \emptyset$. $\square$

6.3 $TS_{\mathcal{N}_{Sat}}$ and $TS_{\mathcal{N}_{Min}}$ are Isomorphic

In this section we examine the relationship between the behaviour of the saturated and minimal net constructed for a TSENI transition system $TS = (S, U, T, s_{in})$. First we define a mapping between ENI-systems $\mathcal{N}_{Sat}$ and $\mathcal{N}_{Min}$ as follows: $(\widetilde{\alpha}, \widetilde{\beta}) : \mathcal{N}_{Sat} \to \mathcal{N}_{Min}$, where $\widetilde{\alpha} : Min(R_{TS}) \to R_{TS}$ and $\widetilde{\beta} : E_{TS} \to E_{TS}$ are both total identity functions. Notice that,

$$\forall\, X \subseteq R_{TS} \;:\; \widetilde{\alpha}^{-1}(X) = Min(X). \tag{6.3}$$

Proposition 6.3.1 $(\widetilde{\alpha}, \widetilde{\beta})$ *is a net morphism from* $\mathcal{N}_{Sat}$ *to* $\mathcal{N}_{Min}$.

Proof: (MENI1) and (MENI2) are clearly satisfied. For (MENI3) we need to show that for every $r \in dom(\widetilde{\alpha})$, $\widetilde{\alpha}(r) \in R_{s_{in}} \Leftrightarrow r \in Min(R_{s_{in}})$. It follows easily from the fact that $Min(R_{s_{in}}) = \widetilde{\alpha}^{-1}(R_{s_{in}})$ (see (6.3)). (MENI4) holds since for all $e \in E_{TS}$, $e \in dom(\widetilde{\beta})$. Finally, we show that (MENI5) holds as follows. For every $e \in E_{TS}$,

$$\begin{array}{llllll}
r \in {}^{\bullet}\widetilde{\beta}(e) & (\text{in } \mathcal{N}_{Min}) & \Leftrightarrow \quad \widetilde{\alpha}(r) \in {}^{\bullet}e & (\text{in } \mathcal{N}_{Sat}) & \Leftrightarrow \quad r \in \widetilde{\alpha}^{-1}({}^{\bullet}e) & (\text{in } \mathcal{N}_{Min}) \\
r \in \widetilde{\beta}(e)^{\bullet} & (\text{in } \mathcal{N}_{Min}) & \Leftrightarrow \quad \widetilde{\alpha}(r) \in e^{\bullet} & (\text{in } \mathcal{N}_{Sat}) & \Leftrightarrow \quad r \in \widetilde{\alpha}^{-1}(e^{\bullet}) & (\text{in } \mathcal{N}_{Min}) \\
r \in \overset{\blacksquare}{\widetilde{\beta}(e)} & (\text{in } \mathcal{N}_{Min}) & \Leftrightarrow \quad \widetilde{\alpha}(r) \in \overset{\blacksquare}{e} & (\text{in } \mathcal{N}_{Sat}) & \Leftrightarrow \quad r \in \widetilde{\alpha}^{-1}(\overset{\blacksquare}{e}) & (\text{in } \mathcal{N}_{Min})
\end{array}$$

Hence $(\widetilde{\alpha}, \widetilde{\beta})$ is a well defined net morphism from $\mathcal{N}_{Sat}$ to $\mathcal{N}_{Min}$. $\square$

Consider the mappings f_{α} and f_{β} defined in proposition 5.2.1 for a net morphism (α, β) between two ENI-systems $\mathcal{N}_1$ and $\mathcal{N}_2$. According to proposition 5.2.1, $(f_{\alpha}, f_{\beta}) : TS_{\mathcal{N}_1} \to TS_{\mathcal{N}_2}$ is a transition system morphism. We will show that for the specific $(\widetilde{\alpha}, \widetilde{\beta})$ defined above, $(f_{\widetilde{\alpha}}, f_{\widetilde{\beta}})$ is in fact an isomorphism. Before proving this we have the following result.

Proposition 6.3.2 *Let* $e \in E_{TS}$ *and* $s \in S$ *in* TS.

1. *If* $Min({}^{\circ}e) \subseteq Min(R_s)$ *then* ${}^{\circ}e \subseteq R_s$ *and* $e^{\circ} \cap R_s = \emptyset$.

2. *If* $Min(\overset{\square}{e}) \cap Min(R_s) = \emptyset$ *then* $\overset{\square}{e} \cap R_s = \emptyset$.

Proof: (1) Suppose that $r \in {}^{\circ}e$ is a non-minimal region such that $r \notin R_s$. From proposition 6.1.3(1) and corollary 3.1.1 it follows that there exists a minimal region $r' \subset r$ such that $r' \in {}^{\circ}e$. Clearly, $r \notin R_s$ implies $r' \notin R_s$, a contradiction with $Min({}^{\circ}e) \subseteq Min(R_s)$. Hence ${}^{\circ}e \subseteq R_s$ holds.

Suppose now that there exists $r \in e^{\circ} \cap R_s$. Then $S \setminus r \in {}^{\circ}e$ and $S \setminus r \notin R_s$, and we proceed as before, obtaining a contradiction with $Min({}^{\circ}e) \subseteq Min(R_s)$. Hence $e^{\circ} \cap R_s = \emptyset$ is satisfied.

(2) Suppose that $r \in \overset{\square}{e} \cap R_s$ is a non-minimal region. From proposition 6.1.3(4) we have that there exists a minimal region $r' \subset r$ such that $r' \in R_s$. From proposition 6.1.3(3) we have $r' \in \overset{\square}{e}$, which contradicts $Min(\overset{\square}{e}) \cap Min(R_s) = \emptyset$. Hence $\overset{\square}{e} \cap R_s = \emptyset$. $\square$

Proposition 6.3.3 $(f_{\widetilde{\alpha}}, f_{\widetilde{\beta}})$ *is an isomorphism from* $TS_{\mathcal{N}_{Sat}}$ *to* $TS_{\mathcal{N}_{Min}}$.

Proof: From proposition 3.6.1(1) it follows that for $\mathcal{N}_{Sat} = \mathcal{N}_{TS}$, $C_{\mathcal{N}_{Sat}} = \{R_s \mid s \in S\}$. As a result, $f_{\widetilde{\alpha}} : \{R_s \mid s \in S\} \to C_{\mathcal{N}_{Min}}$ and, for all $s \in S$,

$$\begin{array}{rcl} f_{\widetilde{\alpha}}(R_s) & \overset{(prop.5.2.1)}{=} & \widetilde{\alpha}^{-1}(R_s) \cup (Min(R_{s_{in}}) \setminus \widetilde{\alpha}^{-1}(R_{s_{in}})) \\ & \overset{(6.3)}{=} & \widetilde{\alpha}^{-1}(R_s) \cup (\widetilde{\alpha}^{-1}(R_{s_{in}}) \setminus \widetilde{\alpha}^{-1}(R_{s_{in}})) \\ & = & \widetilde{\alpha}^{-1}(R_s). \end{array}$$

Hence, for all $s \in S$, $f_{\widetilde{\alpha}}(R_s) = \widetilde{\alpha}^{-1}(R_s) \overset{(6.3)}{=} Min(R_s) \in C_{\mathcal{N}_{Min}}$. Thus $f_{\widetilde{\alpha}}$ maps the set of regions containing a specific state into its subset of minimal regions. We will prove that $f_{\widetilde{\alpha}}$ is a bijection.

First we show that $f_{\widetilde{\alpha}}$ is injective. Suppose $R_{s_1} \neq R_{s_2}$ and $Min(R_{s_1}) = Min(R_{s_2})$. Then, there exists a non-minimal region $r \in R_{s_1} \setminus R_{s_2}$ (the case $r \in R_{s_2} \setminus R_{s_1}$ is symmetric). From proposition 6.1.3(4) it follows that there exists a minimal region $r' \subset r$ such that $r' \in R_{s_1}$. Since $Min(R_{s_1}) = Min(R_{s_2})$ and r' is a minimal region, we obtain $r' \in R_{s_2}$. This implies that $s_2 \in r' \subset r$ and, as a result, that $r \in R_{s_2}$. Hence we obtained a contradiction, and so $f_{\widetilde{\alpha}}$ is injective.

We now show that $f_{\widetilde{\alpha}}$ is onto. For all $s \in S$, $f_{\widetilde{\alpha}}(R_s) \in C_{\mathcal{N}_{Min}}$. We need to prove that for every $c \in C_{\mathcal{N}_{Min}}$, there exists $s \in S$ such that $Min(R_s) = c$. To the contrary, suppose that this is not the case. We observe that $f_{\widetilde{\alpha}}(R_{s_{in}}) = Min(R_{s_{in}})$. Thus there exists a step sequence $\varrho = \varrho' u$ of sets of $U_{\mathcal{N}_{Min}}$ such that $Min(R_{s_{in}})[\varrho\rangle c'$ and $c' \neq Min(R_s)$, for all $s \in S$, and there exists $s' \in S$ such that $Min(R_{s_{in}})[\varrho'\rangle Min(R_{s'}) \xrightarrow{u}_{\mathcal{N}_{Min}} c'$. We will show that u is enabled at $R_{s'}$ in $\mathcal{N}_{Sat}$, i.e.

$$Min(R_{s'}) \xrightarrow{u}_{\mathcal{N}_{Min}} \quad \Rightarrow \quad R_{s'} \xrightarrow{u}_{\mathcal{N}_{Sat}} . \tag{6.4}$$

From proposition 3.3.1(1) we have ${}^\bullet u \subseteq Min(R_{s'})$, $u^\bullet \cap Min(R_{s'}) = \emptyset$ and $\overset{\blacksquare}{u} \cap Min(R_{s'}) = \emptyset$ (in $\mathcal{N}_{Min}$). Hence, ${}^\bullet e \subseteq Min(R_{s'})$ and $\overset{\blacksquare}{e} \cap Min(R_{s'}) = \emptyset$, for all $e \in u \subseteq E_{TS}$. By (6.2) we have $Min({}^\circ e) \subseteq Min(R_{s'})$ and $Min(\overset{\square}{e}) \cap Min(R_{s'}) = \emptyset$, for all $e \in u$. From this and proposition 6.3.2(1,2) it follows that ${}^\circ e \subseteq R_{s'}$, $e^\circ \cap R_{s'} = \emptyset$ and $\overset{\square}{e} \cap R_{s'} = \emptyset$, for all $e \in u$ which, after applying (3.6), means that ${}^\bullet e \subseteq R_{s'}$, $e^\bullet \cap R_{s'} = \emptyset$ and $\overset{\blacksquare}{e} \cap R_{s'} = \emptyset$, for all $e \in u$ (in $\mathcal{N}_{Sat}$). We recall that from proposition 6.2.2 we have $u \in U_{\mathcal{N}_{Min}} \subseteq V_{\mathcal{N}_{Sat}}$, and $R_{s'} \in C_{\mathcal{N}_{Sat}}$ is satisfied as well. So, we can apply proposition 3.3.1(1) to obtain $R_{s'} \xrightarrow{u}_{\mathcal{N}_{Sat}}$ which proves (6.4). This implies that there exists $s'' \in S$ such that $R_{s'} \xrightarrow{u}_{\mathcal{N}_{Sat}} R_{s''}$ and then from proposition 3.3.1(2) and (3.6) we get $R_{s''} = (R_{s'} \setminus {}^\circ u) \cup u^\circ$. Notice that u is a step in TS as $u \in U_{\mathcal{N}_{Sat}} = U$ (see proposition 3.6.1(2)). From $Min(R_{s'}) \xrightarrow{u}_{\mathcal{N}_{Min}} c'$ and proposition 3.3.1(2) we have the following:

$$\begin{array}{rcl} c' & = & (Min(R_{s'}) \setminus {}^\bullet u) \cup u^\bullet \\ & \overset{(6.2)}{=} & (Min(R_{s'}) \setminus Min({}^\circ u)) \cup Min(u^\circ) \\ & \overset{(6.3)}{=} & (\widetilde{\alpha}^{-1}(R_{s'}) \setminus \widetilde{\alpha}^{-1}({}^\circ u)) \cup \widetilde{\alpha}^{-1}(u^\circ) \\ & = & \widetilde{\alpha}^{-1}((R_{s'} \setminus {}^\circ u) \cup u^\circ) \\ & = & \widetilde{\alpha}^{-1}(R_{s''}) \\ & \overset{(6.3)}{=} & Min(R_{s''}). \end{array}$$

Hence we obtained a contradiction, and thus proved that $f_{\widetilde{\alpha}}$ is onto. Thus $f_{\widetilde{\alpha}}$ is a bijection from $\{R_s \mid s \in S\}$ to $\{Min(R_s) \mid s \in S\}$, and $f_{\widetilde{\alpha}}(R_{s_{in}}) = Min(R_{s_{in}})$.

The second mapping, $f_{\widetilde{\beta}} : E_{TS_{\mathcal{N}_{Sat}}} \to E_{TS_{\mathcal{N}_{Min}}}$, defined in proposition 5.2.1 by $f_{\widetilde{\beta}} = \widetilde{\beta}$ is a bijection as well, as $\widetilde{\beta}$ is a total identity function from E_{TS} to E_{TS}, $E_{TS_{\mathcal{N}_{Sat}}} = E_{TS}$ (by proposition 3.6.1(2)) and $E_{TS_{\mathcal{N}_{Min}}} = E_{TS_{\mathcal{N}_{Sat}}}$ (by proposition 4.2.2(3) and (6.4)).

Finally, we need to prove that

$$R_s \xrightarrow{u}_{\mathcal{N}_{Sat}} R_{s'} \quad \Leftrightarrow \quad Min(R_s) \xrightarrow{u}_{\mathcal{N}_{Min}} Min(R_{s'}).$$

The "$\Rightarrow$" implication follows from proposition 4.2.2(3). We need to show that the reverse implication holds as well. Let $Min(R_s) \xrightarrow{u}_{\mathcal{N}_{Min}} Min(R_{s'})$. From the already proved (6.4) we have that $R_s \xrightarrow{u}_{\mathcal{N}_{Sat}}$. This implies that there exists $s'' \in S$ such that $R_s \xrightarrow{u}_{\mathcal{N}_{Sat}} R_{s''}$ and then from proposition 3.3.1(2) and (3.6) we get $R_{s''} = (R_s \setminus {}^\circ u) \cup u^\circ$. From this and (6.3) we obtain

$$\begin{array}{rcl} Min(R_{s''}) & = & \widetilde{\alpha}^{-1}((R_s \setminus {}^\circ u) \cup u^\circ) \\ & = & (\widetilde{\alpha}^{-1}(R_s) \setminus \widetilde{\alpha}^{-1}({}^\circ u)) \cup \widetilde{\alpha}^{-1}(u^\circ) \\ & = & (Min(R_s) \setminus Min({}^\circ u)) \cup Min(u^\circ) \\ & \overset{(prop.3.3.1(2),(6.2))}{=} & Min(R_{s'}). \end{array}$$

Hence, $Min(R_{s''}) = Min(R_{s'})$. Since $f_{\widetilde{\alpha}}$ is an injective function, $R_{s''} = R_{s'}$. Consequently, we have $R_s \xrightarrow{u}_{\mathcal{N}_{Sat}} R_{s'}$. □

Theorem 6.3.1 *TS is isomorphic to $TS_{\mathcal{N}_{Min}}$.*

Proof: From theorem 3.6.1 we have that TS is isomorphic to $TS_{\mathcal{N}_{Sat}}$. Proposition 6.3.3 states, on the other hand, that $TS_{\mathcal{N}_{Sat}}$ is isomorphic to $TS_{\mathcal{N}_{Min}}$. Hence TS is isomorphic to $TS_{\mathcal{N}_{Min}}$. □

An alternative way of proving that $TS_{\mathcal{N}_{Sat}}$ and $TS_{\mathcal{N}_{Min}}$ are isomorphic (proposition 6.3.3) would be to employ the theory presented in [26], which is based on admissible sets of regions in a transition system TS. An *admissible set of regions* is $m \subseteq R_{TS}$ such that the saturated net, and the net obtained from TS by the process of synthesis which uses only regions from m as conditions, generate isomorphic transition systems. For the Elementary Net Systems, it was proved that m is admissible if it contains witnesses for every instance of the state separation and event/state separation axioms. This theory can easily be extended to the ENI-systems. As a consequence, to prove that $TS_{\mathcal{N}_{Sat}}$ and $TS_{\mathcal{N}_{Min}}$ are isomorphic, it is enough to show that the set of minimal regions of a TSENI transition system TS is admissible. That is, we need to prove that $Min(R_{TS})$ contains witnesses for every instance of the separation axioms (A5) and (A6). We first show this for (A5).

Let $s, s' \in S$. We need to prove that if $Min(R_s) = Min(R_{s'})$ then $s = s'$. Suppose that $Min(R_s) = Min(R_{s'})$ and $s \neq s'$. Since TS satisfies (A5) as a TSENI transition system, we have $R_s \neq R_{s'}$. Hence, without loss of generality, there is $r \in R_s \setminus R_{s'}$ and r is non-minimal (the case $r \in R_{s'} \setminus R_s$ is symmetric). Note that $s \in r$ and $s' \notin r$. From proposition 6.1.3(4) we obtain that there exists minimal region $r' \subset r$ such that $r' \in R_s$. We also have $s' \notin r'$ as $s' \notin r$. Hence $r' \in R_s \setminus R_{s'}$, which contradicts $Min(R_s) = Min(R_{s'})$.

We now show that there are witnesses among minimal regions for the satisfaction of every instance of (A6). Let $s \in S$, $u \in \{u \subseteq E_{TS} \mid u \neq \emptyset \ \wedge \ \forall\, e, f \in u \ : \ (e \neq f \Rightarrow (Min({}^\circ e) \cup Min(e^\circ)) \cap (Min({}^\circ f) \cup Min(f^\circ)) = \emptyset)\}$ and, for every $e \in u$, $Min({}^\circ e) \subseteq Min(R_s)$ and $Min(\overset{\square}{e}) \cap Min(R_s) = \emptyset$. We need to show that $s \xrightarrow{u}$. From $u \in \{u \subseteq E_{TS} \mid u \neq \emptyset \ \wedge \ \forall\, e, f \in u \ : \ (e \neq f \Rightarrow (Min({}^\circ e) \cup Min(e^\circ)) \cap (Min({}^\circ f) \cup Min(f^\circ)) = \emptyset)\}$ and, $Min({}^\circ e) \subseteq Min(R_s)$ and $Min(\overset{\square}{e}) \cap Min(R_s) = \emptyset$, for every $e \in u$, we can deduce that $u \in V_{TS}$, applying reasoning similar to that in proposition 6.2.2. From proposition 6.3.2 we have ${}^\circ e \subseteq R_s$ and $\overset{\square}{e} \cap R_s = \emptyset$, for every $e \in u$. And TS is a TSENI transition system which satisfies (A6). Hence $s \xrightarrow{u}$.

6.4 Reduced ENI-systems

In this section we will further reduce $\mathcal{N}_{Min}$ without changing its behaviour, by removing some inhibitor arcs. Below we denote the disjoint union of sets by $\uplus$.

Proposition 6.4.1 *Let $r' \subseteq r$ be regions in R_{TS} and $u \in U$.*

1. *If* $r \in {}^{\circ}u$ *then* $r' \in {}^{\circ}u \cup \overset{\Box}{u}$.

2. *If* $r \in u^{\circ}$ *then* $r' \in u^{\circ} \cup \overset{\Box}{u}$.

Proof: (1) There exists $s \xrightarrow{u} s'$ such that $s \in r$ and $s' \notin r$. If $s \in r'$ then, because $s' \notin r'$ (by $s' \notin r$), we have $r' \in {}^{\circ}u$. Suppose that $s \notin r'$. From the definition of a region and (A4) it follows that there exist $e \in u$ and $s'' \in S$ such that $s \xrightarrow{\{e\}} s''$ and $s'' \notin r$. From proposition 3.1.1(1) and $r \in {}^{\circ}e$ we obtain that for all $p \xrightarrow{\{e\}} p'$, $p \in r$ and $p' \notin r$. This means $p' \notin r'$, and therefore there is no arc labelled with e inside r' or coming into r'. There are no arcs labelled with e coming out of r' as well, because, by proposition 3.1.1(1), this would mean that all such arcs would be coming out of r', contradicting $s \notin r'$. So, in this case $r' \in \overset{\Box}{e} \subseteq \overset{\Box}{u}$.
(2) The proof of this part is similar to that of (1). $\Box$

Proposition 6.4.2 *Let* r *be a non-minimal region of* R_{TS} *and* $u \in U$.

1. *If* $r \in {}^{\circ}u$ *then there exist minimal regions* r' *and* r_i *(*$i = 1, \ldots, n$*) such that* $r' \in {}^{\circ}u$, $r_i \in \overset{\Box}{u}$ *(for* $i = 1, \ldots, n$*) and* $r = r' \uplus \biguplus_{i=1}^{n} r_i$.

2. *If* $r \in u^{\circ}$ *then there exist minimal regions* r' *and* r_i *(*$i = 1, \ldots, n$*) such that* $r' \in u^{\circ}$, $r_i \in \overset{\Box}{u}$ *(for* $i = 1, \ldots, n$*) and* $r = r' \uplus \biguplus_{i=1}^{n} r_i$.

Proof: (1) From proposition 6.1.3(1) it follows that there exists a minimal region $r' \subset r$ such that $r' \in {}^{\circ}u$. Then $r'' = r \setminus r'$, which according to proposition 6.1.1 is a region in R_{TS}, does not belong to ${}^{\circ}u$ (see proposition 3.1.1(1)). Hence from proposition 6.4.1 it follows that $r'' \in \overset{\Box}{u}$. Thus there is $e \in u$ such that $r'' \in \overset{\Box}{e}$. If r'' is minimal then $n = 1$ and $r_1 = r''$. If r'' is non-minimal, theorem 6.1.1 says that it can be represented as a disjoint union of minimal regions $r_1, \ldots, r_n$ ($n \geq 2$), and from proposition 6.1.3(3) it follows that for all $i = 1, \ldots, n$, $r_i \in \overset{\Box}{e}$. Consequently, in both cases, $r_i \in \overset{\Box}{u}$ (for $i = 1, \ldots, n$).
(2) The proof of this part is similar to that of (1). $\Box$

Note that the representation of a non-minimal region r, given in proposition 6.4.2, does not need to be unique (see the last paragraph of example 6.4.1).

Proposition 6.4.3 *Let* $e \in E_{TS}$ *and* r *be a non-minimal region in* R_{TS} *such that* $r \in {}^{\circ}e$. *Then there are minimal regions* $r' \in {}^{\circ}e$ *and* $r_i \in \overset{\Box}{e}$ *(*$i = 1, \ldots, n;\ n \geq 1$*) such that*

$r = r' \uplus \biguplus_{i=1}^{n} r_i$. *Moreover, if one deletes the set of inhibitor arcs* $\mathcal{I} = \{(r_1, e), \ldots, (r_n, e)\}$ *from* $\mathcal{N}_{Sat}$ *or* $\mathcal{N}_{Min}$ *then the transition system of the resulting net remains the same (up to isomorphism).*

Proof: From proposition 6.4.2(1) and corollary 3.1.1 it follows that the above representation of r is possible. Recall that $C_{\mathcal{N}_{Sat}} = \{R_s \mid s \in S\}$ and $C_{\mathcal{N}_{Min}} = \{Min(R_s) \mid s \in S\}$. Suppose a condition corresponding to the region r' is marked at R_s. This means $r' \in R_s$ and so $s \in r'$. Consequently $s \notin r_i$ $(i = 1, \ldots, n)$ as the minimal regions in the representation are mutually disjoint. Hence $r_i \notin R_s$ $(i = 1, \ldots, n)$ which means they are not marked. In this case the inhibitor arcs (r_i, e) are not needed. If r' is not marked at R_s then e is not enabled and it does not matter whether the r_i's are marked or not. Thus in both cases the marking of the r_i's does not change the enabledness of e at any case R_s. Hence the inhibitor arcs in $\mathcal{I}$ can be removed without changing the transition system generated by the net. □

We will denote by $\mathcal{I}_{TS}$ the union of all the sets $\mathcal{I}$ in proposition 6.4.3, after taking into account every $e \in E_{TS}$, every non-minimal pre-region r of e, and every possible representation of r described there. The net obtained from $\mathcal{N}_{Min}$ by deleting all the inhibitor arcs in $\mathcal{I}_{TS}$, will be called *reduced* and denoted by

$$\mathcal{N}_{Rcd} = (Min(R_{TS}), E_{TS}, \widehat{F}_{TS}, \widehat{I}_{TS} \setminus \mathcal{I}_{TS}, Min(R_{s_{in}})).$$

Clearly $\mathcal{N}_{Rcd}$ is an ENI-system and, directly from proposition 6.4.3, we obtain:

Theorem 6.4.1 $TS_{\mathcal{N}_{Min}}$ *is isomorphic to* $TS_{\mathcal{N}_{Rcd}}$.

Definition 6.4.1 *We introduce the following notions.*

1. *An ENI-system* $\mathcal{N}'$ *is a* state machine *if its initial case is a singleton set and every event has exactly one pre-condition and one post-condition.*

2. *A* state machine component *of an ENI-system* $\mathcal{N} = (B, E, F, I, c_{in})$ *is a state machine* $\mathcal{N}' = (B', E', F', I', c'_{in})$ *such that* $B' \subseteq B$, $E' = \{e \in E \mid (e^\bullet \cup {}^\bullet e) \cap B' \neq \emptyset\}$, $F' = F \cap (B' \times E' \cup E' \times B')$, $I' = I \cap (B' \times E')$ *and* $c'_{in} = c_{in} \cap B'$.

3. *A* state machine decomposition *of* $\mathcal{N}$ *is a set of state machine components,* $\mathcal{N}_i = (B_i, E_i, F_i, I_i, c^i_{in})$ $(i = 1, \ldots, n)$ *such that* $B = \bigcup_{i=1}^{n} B_i$, $E = \bigcup_{i=1}^{n} E_i$ *and* $F = \bigcup_{i=1}^{n} F_i$. □

In [12] it was shown that the states of an elementary transition system can be decomposed into disjoint minimal regions; moreover any such decomposition induces a state machine component. The set of all possible decompositions determines a set of state machine components which cover the minimal net associated with this elementary transition system. In this chapter we have proved, in theorem 6.1.1, that any non-trivial region of a TSENI transition system can be represented as a disjoint union of minimal regions. The decomposability of minimal ENI-systems into state machines can then be proved in a similar way as it was done in [12] for Elementary Net Systems. For example $\mathcal{N}_{Min}$ ($\mathcal{N}_{Rcd}$) considered in example 6.4.1 has two state machine components: one induced by the decomposition $S = r_2 \uplus r_3 \uplus r_5$ and the other by $S = r_1 \uplus r_4 \uplus r_5$.

The ability of decomposing a net into state machine components can be useful for finding those inhibitor arcs which can be removed from the net without changing its behaviour. In [17], where the sequential behaviour of Elementary Net Systems with Inhibitor Arcs was investigated, it was shown that inhibitor arcs which are present within a state machine component are superfluous. We will show that the method of eliminating inhibitor arcs introduced in this section for ENI-systems is similar in effect to the method described in [17].

Theorem 6.4.2 *Let $SM_i = (B_i, E_i, F_i, I_i, c^i_{in})$ $(i = 1, \ldots, l)$ be the state machine components of $\mathcal{N}_{Min}$. Then $(r_{inh}, e) \in \mathcal{I}_{TS}$ if and only if there exists SM_k $(1 \le k \le l)$ such that $(r_{inh}, e) \in I_k$.*

Proof: Let $(r_{inh}, e) \in \mathcal{I}_{TS}$. Then there exists a non-minimal region $r \in R_{TS}$ such that $r \in {}^\circ e$ and r can be represented as $r = r' \uplus \biguplus_{i=1}^{n} r_i$ $(n \ge 1)$, where $r' \in {}^\circ e$ and $r_i \in \overset{\Box}{e}$ (for $i = 1, \ldots, n$) are minimal regions. Let $1 \le i_k \le n$ be such that $r_{i_k} = r_{inh}$. We have $S \setminus r \in e^\circ$. Define r'' as $S \setminus r$, if it is minimal; otherwise define r'' as a minimal post-region of e appearing in the representation of $S \setminus r$ in proposition 6.4.2(2). Then $S = r' \uplus r'' \uplus \biguplus_{i=1}^{n} r_i \uplus \biguplus_{j=1}^{m} \overline{r}_j$, where $m \ge 0$ and $\overline{r}_j \in \overset{\Box}{e}$ $(j = 1, \ldots, m)$ are minimal regions. Define SM_k as a state machine component of $\mathcal{N}_{Min}$ induced by the decomposition of S given above. Clearly, $(r_{inh}, e) \in I_k$.

To prove the reverse implication we assume that $(r_{inh}, e) \in I_k$ for some $1 \le k \le l$. Then there are $r_{pred},\ r_{succ} \in B_k$ such that $(r_{pred}, e),\ (e, r_{succ}) \in F_k$ and r_{pred}, r_{succ} and r_{inh} are mutually disjoint non-empty sets (they are minimal regions from the decomposition associated with SM_k). Hence, by proposition 6.1.2, $r = r_{pred} \cup r_{inh}$ is a non-trivial region in TS and $r \in {}^\bullet e = {}^\circ e$ in $\mathcal{N}_{Sat}$. By proposition 6.4.3, we finally have $(r_{inh}, e) \in \mathcal{I}_{TS}$. $\square$

Example 6.4.1 *Figure 6.3 shows the saturated ENI-system $\mathcal{N}_{Sat} = \mathcal{N}_{TS}$ associated with a TSENI transition system TS, and two stages of minimisation of $\mathcal{N}_{Sat}$. The regions in TS are:*

$$\begin{array}{lcllcllcl} r_1 & = & \{s_{in}, s_1\} & r_2 & = & \{s_{in}, s_2\} & r_3 & = & \{s_1, s_3\} \\ r_4 & = & \{s_2, s_3\} & r_5 & = & \{s_4\} & r_6 & = & \{s_{in}, s_1, s_2, s_3\} \\ r_7 & = & \{s_{in}, s_1, s_4\} & r_8 & = & \{s_{in}, s_2, s_4\} & r_9 & = & \{s_1, s_3, s_4\} \\ r_{10} & = & \{s_2, s_3, s_4\} & & & & & & \end{array}$$

and the pre-regions, post-regions and I-regions of events are:

$$\begin{array}{lcllcllcl} {}^\circ a & = & \{r_2, r_8\} & a^\circ & = & \{r_3, r_9\} & \overset{\square}{a} & = & \{r_4, r_5, r_{10}\} \\ {}^\circ b & = & \{r_1, r_7\} & b^\circ & = & \{r_4, r_{10}\} & \overset{\square}{b} & = & \{r_3, r_5, r_9\} \\ {}^\circ c & = & \{r_3, r_4, r_6\} & c^\circ & = & \{r_5, r_7, r_8\} & \overset{\square}{c} & = & \{r_1, r_2\}. \end{array}$$

The minimal regions of TS are: r_1, r_2, r_3, r_4 and r_5. To obtain $\mathcal{N}_{Min}$, we minimise $\mathcal{N}_{Sat}$ by removing conditions associated with non-minimal regions and the adjacent arcs. At this stage two inhibitor arcs are deleted: (r_{10}, a) and (r_9, b). The resulting $\mathcal{N}_{Min}$ has still redundant inhibitor arcs which can be identified by looking at non-minimal pre-regions of events in $\mathcal{N}_{Sat}$, and representing them as disjoint unions of minimal pre-regions and I-regions, as described in proposition 6.4.2. For event a we have: $r_8 = r_2 \uplus r_5$, for b: $r_7 = r_1 \uplus r_5$, for c: $r_6 = r_3 \uplus r_2$ and $r_6 = r_4 \uplus r_1$. Thus, by proposition 6.4.3, the following inhibitor arcs are redundant: (r_5, a), (r_5, b), (r_2, c) and (r_1, c). Notice that the representation of a non-minimal pre-region, given in proposition 6.4.2, does not need to be unique; for example, as in the case of r_6. In such a situation we can eliminate more inhibitor arcs. At the end of this process we obtain $\mathcal{N}_{Rcd}$. $\square$

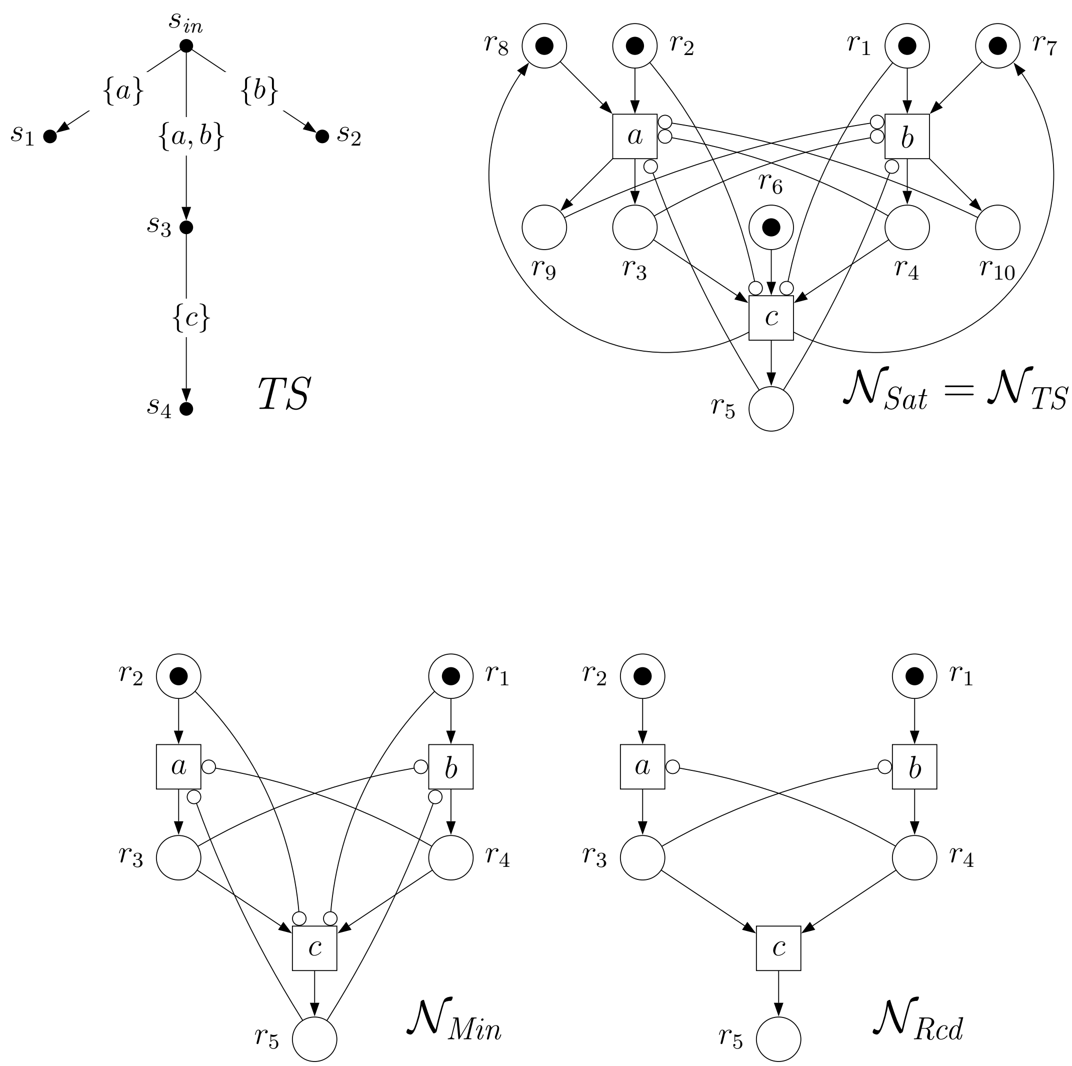
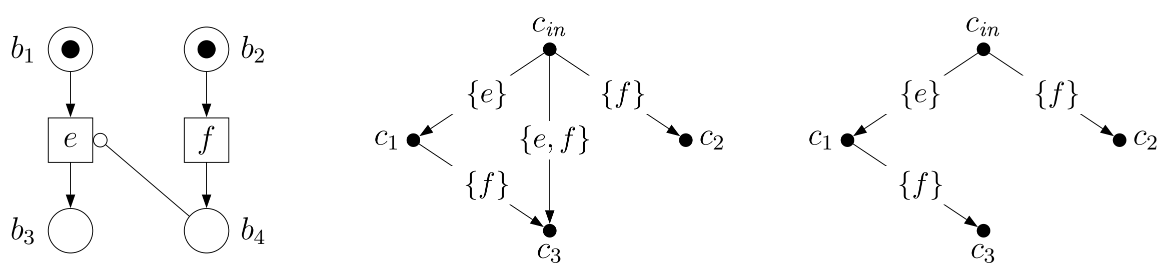


Figure 6.3: Minimisation of the ENI-system for a given TSENI transition system.

Chapter 7

A-posteriori Semantics

In this chapter, we will be interested in Elementary Net Systems with Inhibitor Arcs executed under the a-posteriori semantics (*ENI$_{apost}$-systems*). We provide here a complete characterisation of the class of transition systems generated by ENI$_{apost}$-systems which we call *Transition Systems Modelling Elementary Nets with Inhibitor Arcs under the a-posteriori semantics (TSENI$_{apost}$)*. For the elementary net system with inhibitor arcs in figure 7.1(a), $\mathcal{N}$, the corresponding TSENI transition system is shown in figure 7.1(b) and the TSENI$_{apost}$ transition system in figure 7.1(c).



(a) $\mathcal{N}$ (b) $TS_{\mathcal{N}}$ (a-priori) (c) $TS_{\mathcal{N}}$ (a-posteriori)

Figure 7.1: Elementary net system with inhibitor arcs $\mathcal{N}$ and the transition systems it generates.

In section 7.2, we formulate some important properties of the TSENI$_{apost}$ Transition Systems. In particular, like other classes of transition systems (see [37]), TSENI$_{apost}$ Transition Systems enjoy the 'intermediate state' property. This property, as we recall, does not hold for the TSENI Transition Systems.

7.1 TSENI$_{apost}$ Transition Systems

In this section, we introduce TSENI$_{apost}$ Transition Systems which will later be shown to be the class of transition systems generated by ENI$_{apost}$-systems. We approach the final definition gradually, by introducing the seven axioms characterising TSENI$_{apost}$ Transition Systems. We prove the properties of TSENI$_{apost}$ Transition Systems if they differ from the ones introduced and proved for TSENI Transition Systems. Otherwise, we state them without proofs. The definition of a region and the definitions of pre-, post- and I-regions remain the same as for TSENI Transition Systems.

Let $TS = (S, U, T, s_{in})$ be a transition system defined as in chapter 3 by TS1-TS4 (fixed throughout the rest of this section). We assume that TS satisfies the following four axioms:

A1* For every $(s, u, s') \in T$, $s \neq s'$.

A2* For every $u \in U$, there are $s, s' \in S$ such that $(s, u, s') \in T$.

A3* For every $s \in S \setminus \{s_{in}\}$, there are $(s_0, u_0, s_1), (s_1, u_1, s_2), \ldots, (s_{n-1}, u_{n-1}, s_n) \in T$ such that $s_0 = s_{in}$ and $s_n = s$.

A4* If $s \xrightarrow{u}$ and $e \in u$ then $s \xrightarrow{\{e\}}$.

The above axioms are shared by the TSENI$_{apost}$ and TSENI Transition Systems. As a consequence, the following properties which were true for the TSENI Transition Systems hold for the TSENI$_{apost}$ Transition Systems as well. The proofs were given for the corresponding properties of the TSENI Transition Systems in chapter 3. We quote their numbers in square brackets.

Proposition 7.1.1 *[proposition 3.1.1] If* $s \xrightarrow{u} s'$ *then*

1. $r \in {}^\circ u$ *implies* $s \in r$ *and* $s' \notin r$,

2. $r \in u^\circ$ *implies* $s \notin r$ *and* $s' \in r$. □

Corollary 7.1.1 *[corollary 3.1.1] For every* $e \in E_{TS}$, $\{e\} \in U$. □

Proposition 7.1.2 *[proposition 3.1.2] If* $u \in U$ *then* ${}^\circ u = \bigcup_{e \in u} {}^\circ e$ *and* $u^\circ = \bigcup_{e \in u} e^\circ$. □

Proposition 7.1.3 *[proposition 3.1.3] There exists exactly one event $e \in u$ which satisfies definition 3.1.1(1) (or 3.1.1(2)).* $\square$

Proposition 7.1.4 *[proposition 3.1.5] The following hold:*

1. *$r \subseteq S$ is a region if and only if $S \setminus r$ is a region.*

2. *If $u \in U$ then $u^\circ = \{S \setminus r \mid r \in {}^\circ u\}$.*

3. *If $s \xrightarrow{u} s'$ then $R_s \setminus R_{s'} = {}^\circ u$ and $R_{s'} \setminus R_s = u^\circ$.*
   Moreover, ${}^\circ u \subseteq R_s$ and $u^\circ \cap R_s = \emptyset$ and $R_{s'} = (R_s \setminus {}^\circ u) \cup u^\circ$. $\square$

Proposition 7.1.5 *[proposition 3.1.6] Let $s \in S$ and $e \in E_{TS}$ be such that ${}^\circ e \subseteq R_s$. Then $e^\circ \cap R_s = \emptyset$.* $\square$

Proposition 7.1.6 *[proposition 3.1.8] If $s \xrightarrow{\{e\}} s'$ then $r \in \overset{\scriptscriptstyle\square}{e}$ implies $s, s' \notin r$.* $\square$

To characterise fully TSENI$_{apost}$ Transition Systems we will need a new notion of a potential step in TS. The set of all *potential steps* SV_{TS} is defined as follows:

$$SV_{TS} = V_{TS} \cap \{u \subseteq E_{TS} \mid u \neq \emptyset \ \wedge \ \forall e, f \in u : (e \neq f \Rightarrow e^\circ \cap \overset{\scriptscriptstyle\square}{f} = \emptyset \ \wedge \ f^\circ \cap \overset{\scriptscriptstyle\square}{e} = \emptyset)\}.$$ [1]

SV_{TS} comprises sets of events which share neither pre- nor post-regions. Moreover, a post-region of an event from $u \in SV_{TS}$ cannot be an I-region of some other event from u. The above definition of the set of potential steps in TS is more restrictive than the one used for TSENI Transition Systems. There the conditions involving I-regions were not needed and the set of all potential steps of a transition system TS was defined as V_{TS}.

We will assume from now on that the transition system TS satisfies an additional axiom which was not used for TSENI Transition Systems.

A5* If $\xrightarrow{u} s$ and $e \in u$ then $\xrightarrow{\{e\}} s$.

The new axiom (A5*) will be necessary to prove that the definition of the set of potential steps of TS is consistent with the definition of U.

Proposition 7.1.7 *$U \subseteq SV_{TS}$.*

[1]Recall that $V_{TS} = \{u \subseteq E_{TS} \mid u \neq \emptyset \ \wedge \ \forall e, f \in u : (e \neq f \Rightarrow ({}^\circ e \cup e^\circ) \cap ({}^\circ f \cup f^\circ) = \emptyset)\}$.

Proof: Let $u \in U$ and $e \neq f \in u$. By (A2*), there is $s \xrightarrow{u} s'$.

Suppose that $r \in {}^\circ e \cap {}^\circ f$. This and (A4*) and proposition 7.1.1(1) implies that there are $s^e, s^f \notin r$ such that $s \xrightarrow{\{e\}} s^e$ and $s \xrightarrow{\{f\}} s^f$. By proposition 7.1.2, we have $r \in {}^\circ u$, so, by proposition 7.1.1(1), $s \in r$ and $s' \notin r$. Hence, by proposition 7.1.3, there is a unique $g \in u$ such that $s \xrightarrow{\{g\}} s''$ and $s'' \notin r$, for some s''. But this produces a contradiction with the already established properties of e and f. That $e^\circ \cap f^\circ = \emptyset$ can be proved similarly.

Now, we prove that $e^\circ \cap {}^\circ f = \emptyset$ (the case $f^\circ \cap {}^\circ e = \emptyset$ is symmetric). Let $r \in e^\circ \cap {}^\circ f$. From (A4*) it follows that $s \xrightarrow{\{e\}} s^e$ and $s \xrightarrow{\{f\}} s^f$, for some $s^e, s^f \in S$. On the one hand, by $r \in e^\circ$ and proposition 7.1.1(2), $s \notin r$. On the other hand, by $r \in {}^\circ f$ and proposition 7.1.1(1), $s \in r$. We obtained a contradiction.

Finally, we prove that $e^\circ \cap \overset{\square}{f} = \emptyset$ (the case $f^\circ \cap \overset{\square}{e} = \emptyset$ is symmetric). Let $r \in e^\circ \cap \overset{\square}{f}$. From (A5*) it follows that $s^e \xrightarrow{\{e\}} s'$ and $s^f \xrightarrow{\{f\}} s'$, for some $s^e, s^f \in S$. On the one hand, by $r \in e^\circ$ and proposition 7.1.1(2), $s' \in r$. On the other hand, by $r \in \overset{\square}{f}$ and proposition 7.1.6, $s' \notin r$. We obtained a contradiction. □

It is straightforward to show that a step can be executed at a state only if the I-regions of the former do not comprise the latter. Due to the new axiom (A5*) we can also prove that a step can only lead to a state which is not contained by its I-regions.

Proposition 7.1.8 *If* $s \xrightarrow{u} s'$ *then* $\overset{\square}{u} \cap R_s = \emptyset$ *and* $\overset{\square}{u} \cap R_{s'} = \emptyset$.

Proof: Suppose that $r \in \overset{\square}{u} \cap R_s \neq \emptyset$. Then there is $e \in u$ such that $r \in \overset{\square}{e}$. Hence, by proposition 7.1.6, if $p \xrightarrow{\{e\}} p'$ then $p, p' \notin r$. In particular, by (A4*) and $s \xrightarrow{u} s'$ and $e \in u$, we have $s \notin r$. On the other hand, by $r \in R_s$, we have $s \in r$, a contradiction.

Suppose now that $r \in \overset{\square}{u} \cap R_{s'} \neq \emptyset$. Then there is $e \in u$ such that $r \in \overset{\square}{e}$. Hence, by proposition 7.1.6, if $p \xrightarrow{\{e\}} p'$ then $p, p' \notin r$. By axiom (A5*) and $s \xrightarrow{u} s'$ and $e \in u$, we have $s' \notin r$. On the other hand, by $r \in R_{s'}$, we have $s' \in r$, a contradiction. □

We now can define the desired class of transition systems. A transition system *TS* is a *TSENI*$_{apost}$ *transition system* if it satisfies, in addition to (A1*)-(A5*), the following two axioms:

A6* For all $s, s' \in S$, if $R_s = R_{s'}$ then $s = s'$.

A7* Let $s \in S$ and $u \in SV_{TS}$ be such that, for every $e \in u$, ${}^{\circ}e \subseteq R_s$ and $\overset{\Box}{e} \cap R_s = \emptyset$. Then $s \xrightarrow{u}$.

The first of the last two axioms was used for the TSENI Transition Systems as well. It excludes non-deterministic transition systems like TS_1 shown in figure 3.3. The second axiom is a variation of the forward closure property [38] (or the event/state separation property [9]). It was used for the TSENI Transition Systems, but there u was a set of events from V_{TS}. Axiom (A7*) excludes transition systems like TS_2 in figure 3.3. As a consequence the transition systems from figure 3.3 are neither TSENI nor TSENI$_{apost}$ transition systems.

7.2 Properties of TSENI$_{apost}$ Transition Systems

We now formulate some properties of a TSENI$_{apost}$ transition system $TS = (S, U, T, s_{in})$. The properties shared with TSENI Transition Systems are given without proofs.

Proposition 7.2.1 *[proposition 3.2.1] For every $e \in E_{TS}$, ${}^{\circ}e$ and e° are non-empty sets and ${}^{\circ}e$, e° and $\overset{\Box}{e}$ are mutually disjoint sets.* $\Box$

Proposition 7.2.2 *[proposition 3.2.2] For every $u \in U$, ${}^{\circ}u$ and u° are non-empty disjoint sets.* $\Box$

Proposition 7.2.3 *[proposition 3.2.3] If $s \xrightarrow{u}$ and $\emptyset \neq v \subset u$ then $s \xrightarrow{v}$.* $\Box$

Proposition 7.2.4 *[proposition 3.1.7] If $s \xrightarrow{u} s'$ and $s \xrightarrow{u} s''$ then $s' = s''$.* $\Box$

It is worth noting that, unlike TSENI Transition Systems, TSENI$_{apost}$ enjoy the 'intermediate state' property which is true of other classes of transition systems considered in the literature [37].

Proposition 7.2.5 *If $s \xrightarrow{u} s'$ then for every non-empty $v \subset u$ there exists $s'' \in S$ such that $s \xrightarrow{v} s''$ and $s'' \xrightarrow{u \setminus v} s'$.*

Proof: From proposition 7.2.3 it follows that $v, u \setminus v \in U$ and $s \xrightarrow{v} s''$ for some $s'' \in S$. By proposition 7.1.7, we have $U \subseteq SV_{TS}$. Hence, $u \setminus v \in SV_{TS}$. To prove that $s'' \xrightarrow{u \setminus v}$ we need to show that the conditions in the axiom (A7*) hold. First we show that for every

$e \in u \setminus v$, ${}^{\circ}e \subseteq R_{s''}$. From proposition 7.1.2 it follows that ${}^{\circ}u = \bigcup\limits_{e \in u} {}^{\circ}e$, for every $u \in U$. Hence, for every $e \in u \setminus v$,

$$ {}^{\circ}e \subseteq {}^{\circ}(u \setminus v) \overset{u \in SV_{TS}}{=} {}^{\circ}u \setminus {}^{\circ}v \overset{prop.\ 7.1.4(3)}{=} (R_s \setminus R_{s'}) \setminus (R_s \setminus R_{s''}) \subseteq R_{s''}. $$

Next we need to prove that for every $e \in u \setminus v$, $\overset{\square}{e} \cap R_{s''} = \emptyset$. To the contrary, suppose there is $r \in \overset{\square}{e} \cap R_{s''} \neq \emptyset$ for some $e \in u \setminus v$. Since $r \in \overset{\square}{e}$ there exist $p, p' \in S$ such that $p \xrightarrow{\{e\}} p'$ and $p, p' \notin r$. From $s \xrightarrow{u} s'$ and (A4*) we have $s \xrightarrow{\{e\}}$ which, together with proposition 7.1.6, gives $s \notin r$. Since $s \xrightarrow{v} s''$, $s \notin r$ and $s'' \in r$ (by $r \in R_{s''}$) we can apply definition 3.1.1(2) and obtain that there is $f \in v$ such that if $q \xrightarrow{\{f\}} q'$ then $q \notin r$ and $q' \in r$. From (A4*) we have $s \xrightarrow{\{f\}} s^f$ for some $s^f \in S$. Hence, $s \notin r$ and $s^f \in r$. As a result, $r \in f^{\circ}$. Since $r \in \overset{\square}{e}$, we have $r \in \overset{\square}{e} \cap f^{\circ} \neq \emptyset$. But this produces a contradiction with $u \in SV_{TS}$, as $e, f \in u$ and $e \neq f$ ($e \in u \setminus v$ and $f \in v$). Hence $\overset{\square}{e} \cap R_{s''} = \emptyset$, for every $e \in u \setminus v$. Thus all the conditions in axiom (A7*) are satisfied for s'' and $u \setminus v$. Hence $s'' \xrightarrow{u \setminus v} s'''$, for some $s''' \in S$.

We finally need to prove that $s' = s'''$. From proposition 7.1.4(3), $s \xrightarrow{u} s'$, $s \xrightarrow{v} s''$ and $s'' \xrightarrow{u \setminus v} s'''$ we have:

$$
\begin{aligned}
R_{s'} &= (R_s \setminus {}^{\circ}u) \cup u^{\circ}, \\
R_{s''} &= (R_s \setminus {}^{\circ}v) \cup v^{\circ}, \\
R_{s'''} &= \big(R_{s''} \setminus {}^{\circ}(u \setminus v)\big) \cup (u \setminus v)^{\circ}.
\end{aligned}
$$

It is then easy to verify, using $v \subset u \in SV_{TS}$ and proposition 7.2.2, that:

$$
\begin{aligned}
R_{s'''} &= \Big(\big((R_s \setminus {}^{\circ}v) \cup v^{\circ}\big) \setminus {}^{\circ}(u \setminus v)\Big) \cup (u \setminus v)^{\circ} \\
&= \Big(\big((R_s \setminus {}^{\circ}v) \cup v^{\circ}\big) \setminus ({}^{\circ}u \setminus {}^{\circ}v)\Big) \cup (u^{\circ} \setminus v^{\circ}) \\
&= (R_s \setminus {}^{\circ}u) \cup u^{\circ} \\
&= R_{s'}.
\end{aligned}
$$

Hence $R_{s'''} = R_{s'}$ and, by (A6*), we obtain $s''' = s'$. □

Corollary 7.2.1 *If* $\xrightarrow{u} s$ *and* $\emptyset \neq v \subset u$ *then* $\xrightarrow{v} s$.

Proof: Follows directly from proposition 7.2.5. □

Corollary 7.2.2 *Let* $u \in U$ *and* $|u| = n$. *If* $s \xrightarrow{u} s'$ *then for every enumeration of the events from* u, $(e_{i_1}, e_{i_2}, \ldots, e_{i_n})$, *there exist transitions*

$$ (s_0, \{e_{i_1}\}, s_1), (s_1, \{e_{i_2}\}, s_2), \ldots, (s_{n-1}, \{e_{i_n}\}, s_n) $$

in T *such that* $s_0 = s$ *and* $s_n = s'$.

Proof: Follows easily from proposition 7.2.5. $\square$

An *event sequence* of TS is a sequence $\sigma = e_1 e_2 \dots e_n$ of events from E_{TS} for which there are states $s_0, s_1, \dots, s_n$ satisfying $(s_0, \{e_1\}, s_1), (s_1, \{e_2\}, s_2), \dots, (s_{n-1}, \{e_n\}, s_n) \in T$. We will denote it by $s_0 \overset{\sigma}{\leadsto} s_n$, and call s_0 the *source* and s_n the *target* of σ. We will say that an event sequence σ is *enabled* at a state $s \in S$ if there is $s' \in S$ such that $s \overset{\sigma}{\leadsto} s'$. We will denote this by $s \overset{\sigma}{\leadsto}$.

Corollary 7.2.3 *Let $u \in V_{TS}$ (where $|u| = n$), and $(e_{i_1}, e_{i_2}, \dots, e_{i_n})$ and $(e_{j_1}, e_{j_2}, \dots, e_{j_n})$ be enumerations of the events from u. Let $\sigma_1 = e_{i_1} e_{i_2} \dots e_{i_n}$ and $\sigma_2 = e_{j_1} e_{j_2} \dots e_{j_n}$ be event sequences enabled at s, $s \overset{\sigma_1}{\leadsto} s_1$ and $s \overset{\sigma_2}{\leadsto} s_2$. Then $s_1 = s_2$.*

Proof: Follows from the fact that $u \in V_{TS}$, proposition 7.1.4(3) and axiom (A6*). $\square$

7.3 ENI$_{apost}$-systems

An *elementary net system with inhibitor arcs* (ENI$_{apost}$-system) is a tuple

$$\mathcal{N} = (B, E, F, I, c_{in})$$

such that $N_{\mathcal{N}} = (B, E, F, I)$ is the (underlying) net with inhibitor arcs, defined as for ENI-systems, and $c_{in} \subseteq B$ is the *initial case*. We will assume that $\mathcal{N}$ is fixed until the end of this section.

The difference between ENI$_{apost}$-systems and ENI-systems lies in the definitions of their concurrency semantics, precisely in the definition of the valid steps. The concurrency semantics of ENI$_{apost}$-systems will be based, as before, on steps of simultaneously executed events. We first introduce a new definition of a valid step. A non-empty set of events $u \subseteq E$ is a *valid step*, denoted $u \in SV_{\mathcal{N}}$, if for all $e \neq f \in u$,

$$({}^\bullet e \cup e^\bullet) \cap ({}^\bullet f \cup f^\bullet) = \emptyset \text{ and } e^\bullet \cap \overset{\blacksquare}{f} = \emptyset \text{ and } f^\bullet \cap \overset{\blacksquare}{e} = \emptyset. \tag{7.1}$$

We recall that for ENI-systems the set of valid steps $V_{\mathcal{N}}$ was defined[2] using only the first out of the three constraints of (7.1).

The transition relation of $N_{\mathcal{N}}$, denoted by $\rightarrow_{N_{\mathcal{N}}}$, is given by:

$$\rightarrow_{N_{\mathcal{N}}} = \{(c, u, c') \in 2^B \times SV_{\mathcal{N}} \times 2^B \mid c \setminus c' = {}^\bullet u \ \wedge \ c' \setminus c = u^\bullet \wedge \ \overset{\blacksquare}{u} \cap c = \emptyset\}. \tag{7.2}$$

[2] $V_{\mathcal{N}} = \{u \subseteq E \mid u \neq \emptyset \ \wedge \ \forall e, f \in u : \ (e \neq f \Rightarrow ({}^\bullet e \cup e^\bullet) \cap ({}^\bullet f \cup f^\bullet) = \emptyset)\}$.

Notice that the above definition differs from a similar definition for ENI-system, (3.3), only in the fact that now $u \in SV_{\mathcal{N}}$.

The *state space* of $\mathcal{N}$, denoted by $C_{\mathcal{N}}$, is the least subset of 2^B containing c_{in} such that if $c \in C_{\mathcal{N}}$ and $(c, u, c') \in \rightarrow_{N_{\mathcal{N}}}$ then $c' \in C_{\mathcal{N}}$. The *transition relation* of $\mathcal{N}$, denoted by $\rightarrow_{\mathcal{N}}$, is then defined as $\rightarrow_{N_{\mathcal{N}}}$ restricted to $C_{\mathcal{N}} \times SV_{\mathcal{N}} \times C_{\mathcal{N}}$. The set of *active steps* of $\mathcal{N}$ is given by

$$U_{\mathcal{N}} = \{u \in SV_{\mathcal{N}} \mid \exists c, c' : (c, u, c') \in \rightarrow_{\mathcal{N}}\}.$$

The above definition of the operational semantics of $\mathcal{N}$ is what is referred to as the *a-posteriori* semantics in [19].

Proposition 7.3.1 *The following hold:*

1. *Let $c \in C_{\mathcal{N}}$ and $u \in SV_{\mathcal{N}}$. Then $c \xrightarrow{u}_{\mathcal{N}}$ if and only if ${}^{\bullet}u \subseteq c$ and $(u^{\bullet} \cup \overset{\blacksquare}{u}) \cap c = \emptyset$.*

2. *If $c \xrightarrow{u}_{\mathcal{N}} c'$ then $c' = (c \setminus {}^{\bullet}u) \cup u^{\bullet}$ and $\overset{\blacksquare}{u} \cap c' = \emptyset$.*

Proof: (1) Suppose $c \xrightarrow{u}_{\mathcal{N}}$. Then there is $c' \in C_{\mathcal{N}}$ such that $c \xrightarrow{u}_{\mathcal{N}} c'$. From (7.2), ${}^{\bullet}u \subseteq c$ and $u^{\bullet} \cap c = \emptyset$ and $\overset{\blacksquare}{u} \cap c = \emptyset$.
Suppose now that ${}^{\bullet}u \subseteq c$ and $(u^{\bullet} \cup \overset{\blacksquare}{u}) \cap c = \emptyset$. Define $c' = (c \setminus {}^{\bullet}u) \cup u^{\bullet}$. It is easy to show that $c \setminus c' = {}^{\bullet}u$ and $c' \setminus c = u^{\bullet}$. Hence, by (7.2), $c \xrightarrow{u}_{\mathcal{N}} c'$ and thus $c \xrightarrow{u}_{\mathcal{N}}$.
(2) The first part follows easily from (7.2). We need to prove that $\overset{\blacksquare}{u} \cap c' = \emptyset$. Suppose there is $b \in \overset{\blacksquare}{u} \cap c'$. Then either $b \in c' \setminus c$ or $b \in c' \cap c$. In the first case $b \in u^{\bullet}$, and since $b \in \overset{\blacksquare}{u}$ there exist $e, f \in u$ such that $b \in e^{\bullet}$ and $b \in \overset{\blacksquare}{f}$, and we obtain a contradiction with $u \in SV_{\mathcal{N}}$ (if $e \neq f$) or with (3.1) (if $e = f$). In the second case $b \in c$, and we obtain a contradiction with $\overset{\blacksquare}{u} \cap c = \emptyset$. $\square$

Notice that by using stronger definition for a valid step, $c \xrightarrow{u}_{\mathcal{N}} c'$ means not only that $\overset{\blacksquare}{u} \cap c = \emptyset$ (which was true for ENI-systems), but that $\overset{\blacksquare}{u} \cap c' = \emptyset$ is satisfied as well.

To compare solutions to the synthesis problem (in chapter 8), we will need net isomorphism up to the names of conditions. Let $\mathcal{N}_i = (B_i, E, F_i, I_i, c^i_{in})$ $(i = 1, 2)$ be net systems with inhibitor arcs (ENI$_{apost}$-systems or ENI-systems) with the same sets of events. $\mathcal{N}_1$ and $\mathcal{N}_2$ are *isomorphic* if there exists a bijection $f : B_1 \to B_2$ satisfying, for every $b \in B_1$

and $e \in E$, the following conditions:

$$\begin{array}{llcl} 1. & (b,e) \in F_1 & \Leftrightarrow & (f(b),e) \in F_2, \\ 2. & (e,b) \in F_1 & \Leftrightarrow & (e,f(b)) \in F_2, \\ 3. & (b,e) \in I_1 & \Leftrightarrow & (f(b),e) \in I_2, \\ 4. & b \in c^1_{in} & \Leftrightarrow & f(b) \in c^2_{in}. \end{array}$$

We will denote this by $\mathcal{N}_1 \cong \mathcal{N}_2$.

7.4 Transition Systems of ENI$_{apost}$-systems

The construction of a transition system for a given ENI$_{apost}$-system is straightforward.

Let $\mathcal{N} = (B, E, F, I, c_{in})$ be an ENI$_{apost}$-system. Then

$$TS_{\mathcal{N}} = (C_{\mathcal{N}}, U_{\mathcal{N}}, \rightarrow_{\mathcal{N}}, c_{in})$$

is the *transition system generated* by $\mathcal{N}$.

Theorem 7.4.1 *$TS_{\mathcal{N}}$ is a TSENI$_{apost}$ transition system.*

Proof: Clearly, $TS_{\mathcal{N}}$ is a transition system. We need to prove that it satisfies (A1*)-(A7*).

(A1*) Suppose $c \xrightarrow{u}_{\mathcal{N}} c'$ and $c = c'$. Then, by (7.2), $u^\bullet = {}^\bullet u = \emptyset$, contradicting (3.1).

(A2*) and **(A3*)** follow directly from the definition of $C_{\mathcal{N}}$ and $U_{\mathcal{N}}$.

(A4*) Suppose $c \xrightarrow{u}_{\mathcal{N}}$ and $e \in u$. By proposition 7.3.1(1), ${}^\bullet u \subseteq c$ and $(u^\bullet \cup \overset{\blacksquare}{u}) \cap c = \emptyset$. We also have ${}^\bullet e \subseteq {}^\bullet u$, $e^\bullet \subseteq u^\bullet$ and $\overset{\blacksquare}{e} \subseteq \overset{\blacksquare}{u}$, so ${}^\bullet e \subseteq c$ and $(e^\bullet \cup \overset{\blacksquare}{e}) \cap c = \emptyset$. Thus, from proposition 7.3.1(1) it follows that $c \xrightarrow{\{e\}}_{\mathcal{N}}$.

(A5*) Suppose $\xrightarrow{u}_{\mathcal{N}} c$ and $e \in u$. Then there is $c' \in C_{\mathcal{N}}$ such that $c' \xrightarrow{u}_{\mathcal{N}} c$. From proposition 7.3.1(2) we have $c = (c' \setminus {}^\bullet u) \cup u^\bullet$. From proposition 7.3.1(1) we have ${}^\bullet u \subseteq c'$ and $(u^\bullet \cup \overset{\blacksquare}{u}) \cap c' = \emptyset$, and as a result ${}^\bullet(u \setminus \{e\}) \subseteq c'$, $(u \setminus \{e\})^\bullet \cap c' = \emptyset$ and $\overset{\blacksquare}{(u \setminus \{e\})} \cap c' = \emptyset$. Since $u \setminus \{e\} \in SV_{\mathcal{N}}$ we can apply proposition 7.3.1(1) to obtain $c' \xrightarrow{u\setminus\{e\}}_{\mathcal{N}}$. Let $c'' \in C_{\mathcal{N}}$ be such that $c' \xrightarrow{u\setminus\{e\}}_{\mathcal{N}} c''$. From proposition 7.3.1(2), $c'' = (c' \setminus {}^\bullet(u \setminus \{e\})) \cup (u \setminus \{e\})^\bullet$. It can be easily verified that ${}^\bullet e \subseteq c''$, $e^\bullet \cap c'' = \emptyset$ and $\overset{\blacksquare}{e} \cap c'' = \emptyset$ (by $\overset{\blacksquare}{e} \cap c' = \emptyset$ and $\overset{\blacksquare}{e} \cap (u \setminus \{e\})^\bullet \overset{u \in SV_{\mathcal{N}}}{=} \emptyset$). Hence $c'' \xrightarrow{\{e\}}_{\mathcal{N}} c^e$, for some $c^e \in C_{\mathcal{N}}$. From proposition 7.3.1(2) we have $c^e = (c'' \setminus {}^\bullet e) \cup e^\bullet$. It is then easy to verify that $c^e = c$. Hence we have proved that $\xrightarrow{\{e\}}_{\mathcal{N}} c$, for every $e \in u$.

Before proving (A6*) and (A7*) we show that, for every $b \in B$, $r_b = \{c \in C_{\mathcal{N}} \mid b \in c\}$ is (possibly trivial) region in $TS_{\mathcal{N}}$. Moreover,

$$\emptyset \neq r_b \neq C_{\mathcal{N}} \Rightarrow r_b \in R_{TS_{\mathcal{N}}}. \tag{7.3}$$

Suppose $c \xrightarrow{u}_{\mathcal{N}} c'$, where $c \in r_b$ and $c' \notin r_b$. Then $b \in c$ and $b \notin c'$. By (7.2), $c \setminus c' = {}^\bullet u$ and $c' \setminus c = u^\bullet$. Hence $b \in {}^\bullet u$ and $b \notin u^\bullet$, and we can choose $e \in u$ such that $b \in {}^\bullet e$. We now observe that if $d \xrightarrow{v}_{\mathcal{N}} d'$ and $e \in v$ then $d \in r_b$ and $d' \notin r_b$ (since, by (7.2), $b \in d$ and $b \notin d'$). Moreover, if $v \subseteq u \setminus \{e\}$ and $c \xrightarrow{v}_{\mathcal{N}} c''$ then $c'' \in r_b$, since by (7.1), $b \notin v^\bullet \cup {}^\bullet v$. Thus the first part of definition 3.1.1 is satisfied; the second part can be shown in a similar way. Hence r_b is a region in $TS_{\mathcal{N}}$. Clearly, if $\emptyset \neq r_b \neq C_{\mathcal{N}}$ then r_b is a non-trivial region and (7.3) holds.

(A6*) Suppose that $c \neq c' \in C_{\mathcal{N}}$. Without loss of generality, we may assume that there is $b \in c \setminus c'$. Hence $c \in r_b$ and $c' \notin r_b$. Thus, by (7.3) and $r_b \in R_c \setminus R_{c'}$, (A6*) holds.

(A7*) Suppose that $c \in C_{\mathcal{N}}$ and $u \in SV_{TS_{\mathcal{N}}}$ are such that, for every $e \in u$, ${}^\circ e \subseteq R_c$ and $\overset{\square}{e} \cap R_c = \emptyset$. We first show that $c \xrightarrow{\{e\}}_{\mathcal{N}}$, for every $e \in u$.

Let $e \in u$. Since $e \in E_{TS_{\mathcal{N}}}$ and (A4*) and (A2*) hold, there are $d, d' \in C_{\mathcal{N}}$ such that $d \xrightarrow{\{e\}}_{\mathcal{N}} d'$.

Consider any $b \in {}^\bullet e$. Then $b \in d$ and $b \notin d'$, and so $d \in r_b$ and $d' \notin r_b$. Hence, by (7.3), $r_b \in R_{TS_{\mathcal{N}}}$ and $r_b \in {}^\circ e$. From ${}^\circ e \subseteq R_c$ we have $r_b \in R_c$ which means $b \in c$. As a result, ${}^\bullet e \subseteq c$.

Consider now any $b \in e^\bullet$. Then $b \notin d$ and $b \in d'$, and so $d \notin r_b$ and $d' \in r_b$. Hence, by (7.3), $r_b \in e^\circ$. This and $e^\circ \cap R_c = \emptyset$ (follows from ${}^\circ e \subseteq R_c$ and proposition 7.1.5) means that $r_b \notin R_c$, and so $b \notin c$. Hence $e^\bullet \cap c = \emptyset$.

Suppose that $b \in \overset{\blacksquare}{e} \cap c \neq \emptyset$. Then $c \in r_b$. By (7.2) and $\overset{\blacksquare}{e} \cap e^\bullet = \emptyset$, $b \notin d$ and $b \notin d'$. Thus $d \notin r_b$ and $d' \notin r_b$. As a result, by (7.3), $r_b \in R_{TS_{\mathcal{N}}}$ and $d, d' \in C_{\mathcal{N}} \setminus r_b$. Hence $\mathcal{B}^e_{C_{\mathcal{N}} \setminus r_b} \neq \emptyset$. Suppose now that $f \xrightarrow{\{e\}}_{\mathcal{N}} f'$ belongs to $\mathcal{B}^e_{r_b}$. This means $f, f' \in r_b$ and we have $b \in f$ and $b \in f'$. But this and (7.2) contradict $b \in \overset{\blacksquare}{e}$. Hence $\mathcal{B}^e_{r_b} = \emptyset$ and, as a result, $r_b \in \overset{\square}{e}$. Since $\overset{\square}{e} \cap R_c = \emptyset$, $r_b \notin R_c$ which means $b \notin c$, a contradiction with $b \in \overset{\blacksquare}{e} \cap c$. Hence $\overset{\blacksquare}{e} \cap c = \emptyset$ which, together with ${}^\bullet e \subseteq c$ and $e^\bullet \cap c = \emptyset$, yields $c \xrightarrow{\{e\}}_{\mathcal{N}}$.

We proved that $c \xrightarrow{\{e\}}_{\mathcal{N}}$, for every $e \in u$. Moreover, we have already shown that $b \in {}^\bullet e$ implies $r_b \in {}^\circ e$, $b \in e^\bullet$ implies $r_b \in e^\circ$, and $b \in \overset{\blacksquare}{e}$ together with $r_b \neq \emptyset$ implies $r_b \in \overset{\square}{e}$, for all $e \in u$. This and $u \in SV_{TS_{\mathcal{N}}}$ means that $u \in SV_{\mathcal{N}}$. Hence $c \xrightarrow{u}_{\mathcal{N}}$. $\square$

7.5 ENI$_{apost}$-systems of TSENI$_{apost}$ Systems

The reverse translation, from TSENI$_{apost}$ Transition Systems to ENI$_{apost}$-systems, is based on the pre- post- and I-regions of events appearing in a transition system.

Let $TS = (S, U, T, s_{in})$ be a TSENI$_{apost}$ transition system. The net system *associated* with TS is defined as

$$\mathcal{N}_{TS} = (R_{TS}, E_{TS}, F_{TS}, I_{TS}, R_{s_{in}})$$

where F_{TS} and I_{TS} are defined thus:

$$\begin{array}{lcl} F_{TS} & = & \{(r,e) \in R_{TS} \times E_{TS} \mid r \in {}^{\circ}e\} \;\cup\; \{(e,r) \in E_{TS} \times R_{TS} \mid r \in e^{\circ}\}, \\ I_{TS} & = & \{(r,e) \in R_{TS} \times E_{TS} \mid r \in \overset{\square}{e}\}. \end{array} \tag{7.4}$$

Directly from the definition of $\mathcal{N}_{TS}$ we obtain that, for every $e \in E_{TS}$,

$${}^{\circ}e = {}^{\bullet}e \text{ and } e^{\circ} = e^{\bullet} \text{ and } \overset{\square}{e} = \overset{\blacksquare}{e}\,. \tag{7.5}$$

The proof of the next theorem is omitted as it is similar to the proof of the corresponding property of ENI-systems, theorem 3.5.1.

Theorem 7.5.1 *$\mathcal{N}_{TS}$ is an ENI$_{apost}$-system.* □

The above construction produces a net which is saturated both with conditions and inhibitor arcs.

7.6 Consistency of the Two Translations

In this section, we show that the ENI$_{apost}$-system associated with a TSENI$_{apost}$ transition system TS generates a transition system which is isomorphic to TS.

Proposition 7.6.1 *Let $TS = (S, U, T, s_{in})$ be a TSENI$_{apost}$ transition system and $\mathcal{N} = \mathcal{N}_{TS}$ be the ENI$_{apost}$-system associated with it.*

1. $C_{\mathcal{N}} = \{R_s \mid s \in S\}$.

2. $\rightarrow_{\mathcal{N}} = \{(R_s, u, R_{s'}) \mid (s, u, s') \in T\}$.

Proof: Note that from the definition of $C_{\mathcal{N}}$, every $c \in C_{\mathcal{N}}$ is reachable from c_{in} in $\mathcal{N}$; and that from axiom (A3*), every $s \in S$ is reachable from s_{in} in TS.

We first show that if $c \xrightarrow{u}_{\mathcal{N}} c'$ and $c = R_s$, for some $s \in S$, then there is $s' \in S$ such that $s \xrightarrow{u} s'$ and $c' = R_{s'}$. We have that $c \setminus c' = {}^\bullet u$ and $c' \setminus c = u^\bullet$ and $\overset{\blacksquare}{u} \cap c = \emptyset$. This means ${}^\bullet e \subseteq c$ and $\overset{\blacksquare}{e} \cap c = \emptyset$, for all $e \in u$. This and (7.5) implies that ${}^\circ e \subseteq c$ and $\overset{\square}{e} \cap c = \emptyset$, for all $e \in u$. Hence ${}^\circ e \subseteq R_s$ and $\overset{\square}{e} \cap R_s = \emptyset$, for all $e \in u$. Moreover, by $u \in SV_{\mathcal{N}}$ and (7.5), we have $u \in SV_{TS}$. Hence from (A7*) it follows that $s \xrightarrow{u} s'$, for some $s' \in S$. Then, by proposition 7.1.4(3), $R_{s'} = (R_s \setminus {}^\circ u) \cup u^\circ$. At the same time, from proposition 7.3.1(2), $c' = (c \setminus {}^\bullet u) \cup u^\bullet$. Hence, by (7.5) and proposition 7.1.2 and $c = R_s$, $c' = R_{s'}$.

As a result, we have shown (note that $c_{in} = R_{s_{in}} \in \{R_s \mid s \in S\}$) that $C_{\mathcal{N}} \subseteq \{R_s \mid s \in S\}$ and $\rightarrow_{\mathcal{N}} \subseteq \{(R_s, u, R_{s'}) \mid (s, u, s') \in T\}$.

We now will prove that $\{R_s \mid s \in S\} \subseteq C_{\mathcal{N}}$. By definition, $R_{s_{in}} \in C_{\mathcal{N}}$. What needs to be shown is that if $s \xrightarrow{u} s'$ and $R_s \in C_{\mathcal{N}}$ then $R_{s'} \in C_{\mathcal{N}}$. By propositions 7.1.4(3) and 7.1.8, we have ${}^\circ u \subseteq R_s$ and $(u^\circ \cup \overset{\square}{u}) \cap R_s = \emptyset$. So, using (7.5) and proposition 7.1.2, ${}^\bullet u \subseteq R_s$ and $(u^\bullet \cup \overset{\blacksquare}{u}) \cap R_s = \emptyset$. Moreover, from proposition 7.1.7 and (7.5) we obtain that u is a valid step in $\mathcal{N}$. Hence, by proposition 7.3.1(1), we have $R_s \xrightarrow{u}_{\mathcal{N}}$. This implies $(R_s \setminus {}^\bullet u) \cup u^\bullet \in C_{\mathcal{N}}$. On the other hand, by proposition 7.1.4(3) and $s \xrightarrow{u} s'$, we have $R_{s'} = (R_s \setminus {}^\circ u) \cup u^\circ$. Hence, by (7.5) and proposition 7.1.2, $R_{s'} \in C_{\mathcal{N}}$.

What remains to be shown is that $\{(R_s, u, R_{s'}) \mid (s, u, s') \in T\} \subseteq \rightarrow_{\mathcal{N}}$. Suppose $s \xrightarrow{u} s'$. From propositions 7.1.4(3) and 7.1.8 it follows that $R_s \setminus R_{s'} = {}^\circ u$, $R_{s'} \setminus R_s = u^\circ$ and $\overset{\square}{u} \cap R_s = \emptyset$. We have already proved that $C_{\mathcal{N}} = \{R_s \mid s \in S\}$. So there are $c, c' \in C_{\mathcal{N}}$ such that $c = R_s$ and $c' = R_{s'}$. From (7.5) and proposition 7.1.2 it follows that $c \setminus c' = {}^\bullet u$ and $c' \setminus c = u^\bullet$ and $\overset{\blacksquare}{u} \cap c = \emptyset$. Since $s \xrightarrow{u} s'$, from proposition 7.1.7 and (7.5), it follows that u is a valid step. Hence, by (7.2), $c \xrightarrow{u}_{\mathcal{N}} c'$. $\square$

The proof of the following theorem is omitted as it is similar to the proof of the corresponding property of ENI-systems, theorem 3.6.1.

Theorem 7.6.1 *Let $TS = (S, U, T, s_{in})$ be a $TSENI_{apost}$ transition system and $\mathcal{N} = \mathcal{N}_{TS}$ be the ENI_{apost}-system associated with it. Then $TS_{\mathcal{N}}$ is isomorphic to TS.* $\square$

Chapter 8

Comparing A-priori and A-posteriori Semantics

In this chapter we will compare the TSENI_{apost} and TSENI Transition Systems. We will give (in section 8.1) sufficient conditions for building, for any $TS \in \mathrm{TSENI}_{apost} \setminus \mathrm{TSENI}$, a transition system called $sat(TS)$ such that $sat(TS) \in \mathrm{TSENI} \setminus \mathrm{TSENI}_{apost}$ and the nets associated with them by the process of synthesis are isomorphic ($\mathcal{N}_{TS} \cong \mathcal{N}_{sat(TS)}$). Similarly, we will formulate (in section 8.2) sufficient conditions to create, for any $TS \in \mathrm{TSENI} \setminus \mathrm{TSENI}_{apost}$, a transition system called $prun(TS)$ such that $prun(TS) \in \mathrm{TSENI}_{apost} \setminus \mathrm{TSENI}$ and $\mathcal{N}_{TS} \cong \mathcal{N}_{prun(TS)}$. In both cases, we discuss the possibility of weakening the present conditions (see section 8.3).

To compare TSENI and TSENI_{apost} Transition Systems we observe that neither class is a proper subset of the other, and that there are transition systems which satisfy the axioms of both TSENI and TSENI_{apost} class. This is illustrated in figure 8.1, where $TS_1 \in \mathrm{TSENI} \setminus \mathrm{TSENI}_{apost}$, $TS_2 \in \mathrm{TSENI} \cap \mathrm{TSENI}_{apost}$ and $TS_3 \in \mathrm{TSENI}_{apost} \setminus \mathrm{TSENI}$.

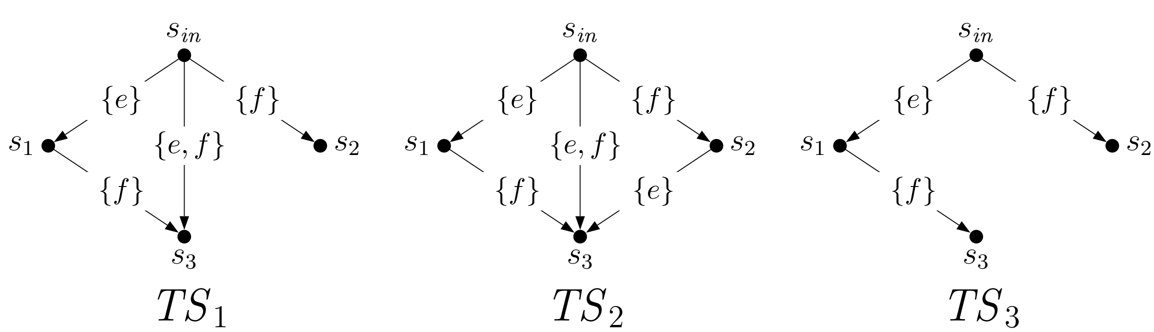


Figure 8.1: Comparison between TSENI and TSENI_{apost} Transition Systems.

8.1 Saturating TSENI$_{apost}$ Transition Systems

In this section, we consider a transition system $TS = (S, U, T, s_{in}) \in$ TSENI$_{apost} \setminus$ TSENI and investigate whether it is possible to find a TSENI transition system whose associated net would be isomorphic to that of TS.

Proposition 8.1.1 *If $TS \in TSENI_{apost} \setminus TSENI$ then there exists $u \in V_{TS} \setminus SV_{TS}$ and $s \in S$ such that for every $e \in u$, ${}^{\circ}e \subseteq R_s$ and $\overset{\Box}{e} \cap R_s = \emptyset$.*

Proof: Since $TS \in$ TSENI$_{apost} \setminus$ TSENI we have that all axioms (A1*)-(A7*) are satisfied for TS and as a consequence (A1)-(A5) are satisfied as well. The only axiom which makes TS fail to be a TSENI transition system is (A6). Hence, (A7*) is satisfied and (A6) is not satisfied for TS. We introduce some symbols for the subformulae appearing in (A7*) and (A6), where $u \subseteq E_{TS}$ and $s \in S$ are such that (A6) fails to hold:

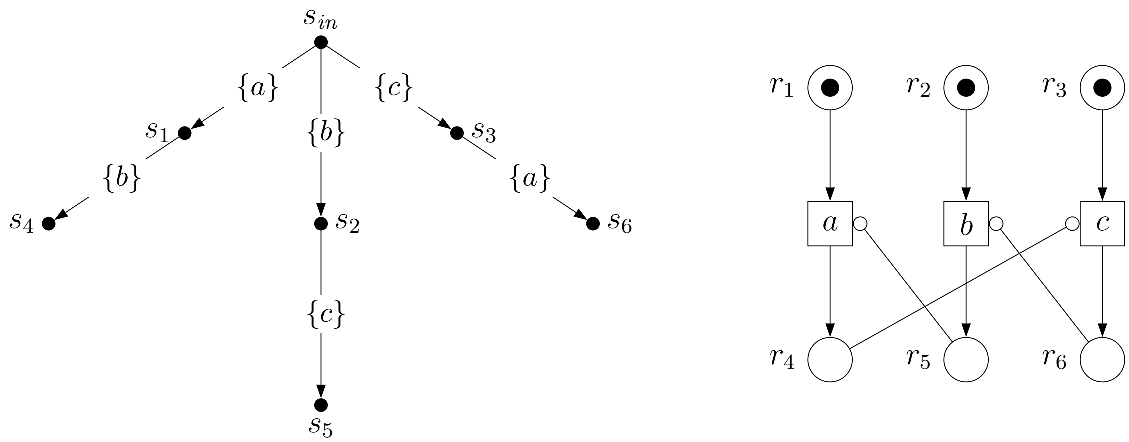
$$\begin{array}{ll} \alpha & u \in SV_{TS} \\ \beta & u \in V_{TS} \\ \gamma & \forall e \in u: \ {}^{\circ}e \subseteq R_s \wedge \overset{\Box}{e} \cap R_s = \emptyset \\ \delta & s \xrightarrow{u} \end{array}$$

(A6) is false, so β is true, γ is true and δ is false. (A7*) is true, so $\alpha \wedge \gamma \Rightarrow \delta$ is true, which means $\alpha \wedge \gamma$ is false. Since γ is true, α is false. So $\beta \wedge \neg\alpha \wedge \gamma$ is true. $\Box$

From proposition 8.1.1 it follows that in $TS \in$ TSENI$_{apost} \setminus$ TSENI there is a set of events $u \subseteq E_{TS}$ and a state $s \in S$ such that u is not enabled as a step at s according to the a-posteriori axioms (A1*)-(A7*), but it would be enabled at s under the a-priori axioms (A1)-(A6). This suggests that by adding to TS an appropriate transition, for every $s \in S$ and $u \subseteq E_{TS}$ which satisfy the conditions of proposition 8.1.1, we could obtain a TSENI transition system whose associated net is isomorphic to that of TS. Before proving this hypothesis, we need to define the targets of transitions added in that way. A good candidate for the target of the transition associated with certain $s \in S$ and $u \subseteq E_{TS}$ would be s' such that there exists an event sequence

$$\rho_u = e_{i_1} e_{i_2} \ldots e_{i_n}, \text{ where } (e_{i_1}, e_{i_2}, \ldots, e_{i_n}) \text{ is an enumeration of events from } u, \tag{8.1}$$

and $s \overset{\rho_u}{\rightsquigarrow} s'$. Notice that corollary 7.2.2 guarantees that for $u \in U$ such an event sequence always exists, but for $u \in V_{TS}$ we can only say, following corollary 7.2.3, that if it exists then the state s' is well defined as it does not depend on the chosen enumeration. Unfortunately, for some $u \in V_{TS}$ and $s \in S$ such a sequence does not exist, as shown in figure 8.2.



(a) $TS \in \mathrm{TSENI}_{apost} \setminus \mathrm{TSENI}$ (b) $\mathcal{N}_{TS}$

Figure 8.2: $TS \in \mathrm{TSENI}_{apost} \setminus$ TSENI and the associated net, $\mathcal{N}_{TS}$.

The regions of TS depicted in figure 8.2 are:

$$
\begin{array}{lllllllll}
r_1 & = & \{s_{in}, s_2, s_3, s_5\} & r_2 & = & \{s_{in}, s_1, s_3, s_6\} & r_3 & = & \{s_{in}, s_1, s_2, s_4\} \\
r_4 & = & \{s_1, s_4, s_6\} & r_5 & = & \{s_2, s_4, s_5\} & r_6 & = & \{s_3, s_5, s_6\}
\end{array}
$$

and the pre-regions, post-regions and I-regions of events are given by:

$$
\begin{array}{lllllllll}
{}^\circ a & = & \{r_1\} & a^\circ & = & \{r_4\} & \overset{\scriptscriptstyle\square}{a} & = & \{r_5\} \\
{}^\circ b & = & \{r_2\} & b^\circ & = & \{r_5\} & \overset{\scriptscriptstyle\square}{b} & = & \{r_6\} \\
{}^\circ c & = & \{r_3\} & c^\circ & = & \{r_6\} & \overset{\scriptscriptstyle\square}{c} & = & \{r_4\}.
\end{array}
$$

Notice that $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\} \in V_{TS}$, but $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\} \notin SV_{TS}$, because $b^\circ \cap \overset{\scriptscriptstyle\square}{a} = \{r_5\} \neq \emptyset$, $a^\circ \cap \overset{\scriptscriptstyle\square}{c} = \{r_4\} \neq \emptyset$ and $c^\circ \cap \overset{\scriptscriptstyle\square}{b} = \{r_6\} \neq \emptyset$. The transition system TS satisfies axioms (A1*)-(A7*) and (A1)-(A5), but does not satisfy (A6). Hence $TS \in \mathrm{TSENI}_{apost} \setminus$ TSENI. The set $u = \{a,b,c\} \in V_{TS} \setminus SV_{TS}$ cannot be enumerated in any way to constitute an event sequence of three events which is enabled at $s = s_{in}$. In such a case, it is difficult to tell whether the target s' for the transition associated with $s \in S$ and $u \subseteq E_{TS}$ should be sought among the existing states of TS or a new state should be added. Foreseeing many complications if adding new states was necessary, we will only be interested in the situation when for every $s \in S$ and $u \subseteq E_{TS}$ satisfying the conditions stated in proposition 8.1.1, there is an event sequence ρ_u as in (8.1) with a source at s.

Let $TS = (S, U, T, s_{in})$ be a transition system in $\text{TSENI}_{apost} \setminus \text{TSENI}$ that satisfies the following condition.

$$
\begin{array}{l}
\text{If } s \in S \ \wedge \ u \in V_{TS} \setminus SV_{TS} \ \wedge \ \forall e \in u: \ {}^{\circ}e \subseteq R_s \wedge \ \overset{\square}{e} \cap R_s = \emptyset \text{ then there is} \\
\text{an event sequence } \rho_u \text{ (as in (8.1)) such that } s \overset{\rho_u}{\leadsto} s', \text{ for some } s' \in S. \\
\text{The target of the event sequence } \rho_u, \ s', \text{ will be denoted by } fin(s, u).
\end{array}
\tag{8.2}
$$

We then define the *saturation* of TS as the quadruple $sat(TS) = (S', U', T', s'_{in})$ given by:

$$
\begin{array}{lcl}
T' & = & T \cup \{(s, u, fin(s, u)) \mid s \in S \ \wedge \ u \in V_{TS} \setminus SV_{TS} \wedge \\
 & & \qquad\qquad\qquad\qquad \forall e \in u: \ {}^{\circ}e \subseteq R_s \wedge \ \overset{\square}{e} \cap R_s = \emptyset\}, \\
U' & = & U \cup \{u \subseteq E_{TS} \mid \exists s \in S: \ (s, u, fin(s, u)) \in T' \setminus T\}, \\
S' & = & S, \\
s'_{in} & = & s_{in}.
\end{array}
$$

It is immediate to see that $sat(TS)$ is a transition system, i.e. it satisfies (TS1)-(TS4). Before showing that $sat(TS)$ is a TSENI transition system, we need to prove some properties which relate the regions of TS with the regions of $sat(TS)$.

Proposition 8.1.2 *If $r \in R_{TS}$ then $r \in R_{sat(TS)}$.*

Proof: We prove the first part of definition 3.1.1. Let $s \xrightarrow{u} s'$ and $s \in r$ and $s' \notin r$ in $sat(TS)$.

Case 1: $u \in U$.

Hence $s \xrightarrow{u} s'$ and $s \in r$ and $s' \notin r$ in TS. Since $r \in R_{TS}$ there exists an r-crossing event e in u, in TS. We will show that e is the r-crossing event in u in $sat(TS)$ as well.

Let $u' \subseteq u \setminus \{e\}$ and $s \xrightarrow{u'} s''$ in $sat(TS)$. Notice that $u' \in U$ since $u' \neq \emptyset$ and u' is a subset of u (proposition 7.2.3). Since r is a region in TS we have $s'' \in r$. Let $q \xrightarrow{v} q'$ and $e \in v$ in $sat(TS)$ (note that e is the r-crossing event in u, in TS). We need to consider two cases.

1. If $v \in U$ then from definition 3.1.1, for r in TS, we have $q \in r$ and $q' \notin r$.

2. If $v \in U' \setminus U$ then, from the definition of U', for every $f \in v$, ${}^{\circ}f \subseteq R_q$ and $\overset{\square}{f} \cap R_q = \emptyset$ (in TS). Hence from axiom (A7*) for TS we have, for every $f \in v$, $q \xrightarrow{\{f\}} q^f$ for some $q^f \in S$. In particular, $q \xrightarrow{\{e\}} q^e$, where $q \in r$ and $q^e \notin r$ as e is the r-crossing event in u in TS. Since $v \in U' \setminus U$, we have $v \in V_{TS}$, which together with $r \in {}^{\circ}e$

and $q \in r$ gives $q^f \in r$, for all $f \neq e,\ f \in v$ (in TS). From (8.2) we have that q' is the target of some event sequence ρ_v (as in (8.1)), such that $q \overset{\rho_v}{\rightsquigarrow} q'$ in TS. Since none of the transitions associated with ρ_v except the one labelled with e crosses the border of r, we have $q' \notin r$.

Case 2: $u \in U' \setminus U$.

From the definition of U', we have that for every $f \in u$, ${}^\circ f \subseteq R_s$ and $\overset{\square}{f} \cap R_s = \emptyset$ (in TS). Hence from axiom (A7*) for TS we have, for every $f \in u$, $s \xrightarrow{\{f\}} s^f$ for some $s^f \in S$. From (8.2) we have that s' is the target of some event sequence ρ_u (as in (8.1)), such that $s \overset{\rho_u}{\rightsquigarrow} s'$ in TS. Hence there exists $e \in u$ such that the transitions labelled with it leave r. So, for $s \xrightarrow{\{e\}} s^e$, we have $s^e \notin r$. Since $u \in V_{TS}$ and $s \in r$ and $r \in {}^\circ e$ in TS we have $s^f \in r$, for all $f \neq e$, $f \in u$. We will prove that e is the r-crossing event in u, in $sat(TS)$. Let $u' \subseteq u \setminus \{e\}$ and $s \xrightarrow{u'} s''$ in $sat(TS)$. Since s'' is the target of some event sequence $\rho_{u'}$ (as in (8.1)), such that $s \overset{\rho_{u'}}{\rightsquigarrow} s''$ in TS, and all the events from u' are enabled at s and none of the transitions labelled with them crosses the border of r, we have $s'' \in r$. Let $q \xrightarrow{v} q'$ and $e \in v$ in $sat(TS)$ (note that e is the event from u for whom $s \xrightarrow{\{e\}} s^e$ and $s \in r$ and $s^e \notin r$). We consider two cases.

1. If $v \in U$ then from the fact that r is a region in TS and $r \in {}^\circ e$ we have $q \in r$ and $q' \notin r$.

2. If $v \in U' \setminus U$ then from the definition of U' we have, ${}^\circ f \subseteq R_q$ and $\overset{\square}{f} \cap R_q = \emptyset$, for every $f \in v$ (in TS). Hence from axiom (A7*) for TS we have, for every $f \in v$, $q \xrightarrow{\{f\}} q^f$ for some $q^f \in S$. From the fact that $r \in {}^\circ e$ in TS we have $q \in r$ and $q^e \notin r$. Since $v \in V_{TS}$ we obtain $q^f \in r$, for all $f \neq e$, $f \in v$. From (8.2) we have that q' is the target of some event sequence ρ_v (as in (8.1)), such that $q \overset{\rho_v}{\rightsquigarrow} q'$ in TS. Since none of the transitions associated with ρ_v except the one labelled with e crosses the border of r, we have $q' \notin r$.

The second part of definition 3.1.1 for r in $sat(TS)$ can be shown in a similar way. Hence r is a region in $sat(TS)$. Moreover, it is non-trivial since $r \in R_{TS}$ and $S = S'$. □

Proposition 8.1.3 *If* $r \in R_{sat(TS)}$ *then* $r \in R_{TS}$.

Proof: Follows easily from the construction of $sat(TS)$. Specifically, from the fact that $S = S'$ and $T \subset T'$. □

Corollary 8.1.1 *Let TS be a transition system in $TSENI_{apost} \setminus TSENI$ that satisfies (8.2). Then*

1. $E_{TS} = E_{sat(TS)}$.
2. *For every* $e \in E_{TS}$: $\quad r \in {}^{\circ}e$ *(in TS)* $\quad\Leftrightarrow\quad r \in {}^{\circ}e$ *(in $sat(TS)$).*
3. *For every* $e \in E_{TS}$: $\quad r \in e^{\circ}$ *(in TS)* $\quad\Leftrightarrow\quad r \in e^{\circ}$ *(in $sat(TS)$).*
4. *For every* $e \in E_{TS}$: $\quad r \in \overset{\square}{e}$ *(in TS)* $\quad\Leftrightarrow\quad r \in \overset{\square}{e}$ *(in $sat(TS)$).*
5. *For every* $s \in S$: $\quad r \in R_s$ *(in TS)* $\quad\Leftrightarrow\quad r \in R_s$ *(in $sat(TS)$).*
6. $V_{TS} = V_{sat(TS)}$.

Proof: Follows directly from propositions 8.1.2 and 8.1.3, and the construction of the transition system $sat(TS)$. $\square$

Proposition 8.1.4 *$sat(TS)$ is a TSENI transition system.*

Proof: (A1) Let $(s, u, s') \in T'$. If $u \in U$ then $s \neq s'$ follows from (A1*) which is satisfied for $TS \in \mathrm{TSENI}_{apost}$. Suppose now that $u \in U' \setminus U$ and $s = s'$. From (8.2) we have that there exists an enumeration of the events from u, $(e_{i_1}, e_{i_2}, \ldots, e_{i_n})$, and an event sequence $\sigma = e_{i_1} e_{i_2} \ldots e_{i_n}$ such that $s \overset{\sigma}{\rightsquigarrow} s'$. Since $u \in U' \setminus U$, we have for every $e \in u$, ${}^{\circ}e \subseteq R_s$ in TS. In particular, ${}^{\circ}e_{i_n} \subseteq R_s$. Let $r \in {}^{\circ}e_{i_n}$ (${}^{\circ}e_{i_n} \neq \emptyset$, by proposition 7.2.1). From $\xrightarrow{\{e_{i_n}\}} s$ and proposition 7.1.1 we obtain $s \notin r$. Hence $r \notin R_s$, a contradiction.
(A2) and (A3) Follow directly from the construction of $sat(TS)$ and the fact that $TS \in \mathrm{TSENI}_{apost}$.
(A4) If $u \in U$ then this axiom is satisfied since (A4*) is satisfied for TS. Let $u \in U' \setminus U$ and $s \xrightarrow{u}$. From the definition of U' we have that for all $e \in u$, ${}^{\circ}e \subseteq R_s$ and $\overset{\square}{e} \cap R_s = \emptyset$ in TS. Since $TS \in \mathrm{TSENI}_{apost}$ and (A7*) is satisfied we obtain that $s \xrightarrow{\{e\}}$ in TS, and so in $sat(TS)$.
(A5) Follows from corollary 8.1.1(5) and axiom (A6*) for TS.
(A6) From corollary 8.1.1(6) and 8.1.1(5), we have that $V_{TS} = V_{sat(TS)}$ and that the sets of regions containing some $s \in S = S'$ are the same for TS and $sat(TS)$. Hence, in the antecedent of the implication of (A6) we have that: $s \in S$, $u \in V_{TS}$, and for every $e \in u$, ${}^{\circ}e \subseteq R_s$ and $\overset{\square}{e} \cap R_s = \emptyset$ in TS. We need to show that $s \xrightarrow{u}$ in $sat(TS)$. If $u \in SV_{TS}$ then, since (A7*) is satisfied for TS, we have $s \xrightarrow{u}$ in TS and thus $s \xrightarrow{u}$ in $sat(TS)$. If $u \notin SV_{TS}$ then $u \in V_{TS} \setminus SV_{TS}$. Since ${}^{\circ}e \subseteq R_s$ and $\overset{\square}{e} \cap R_s = \emptyset$, for every $e \in u$, we have from (8.2) and the construction of $sat(TS)$ that $(s, u, fin(s, u)) \in T' \setminus T$. So in this case u is enabled at s in $sat(TS)$ as well. $\square$

Theorem 8.1.1 *Let TS be a transition system in $TSENI_{apost} \setminus TSENI$ which satisfies (8.2). Then there is a transition system $sat(TS) \in TSENI$ such that $\mathcal{N}_{TS} \cong \mathcal{N}_{sat(TS)}$.*

Proof: Follows from propositions 8.1.2, 8.1.3, 8.1.4 and corollary 8.1.1. □

Proposition 8.1.5 *Let TS be a transition system in $TSENI_{apost} \setminus TSENI$ which satisfies (8.2). Then $sat(TS) \in TSENI \setminus TSENI_{apost}$.*

Proof: We show that $sat(TS)$ does not satisfy (A5*). From proposition 8.1.1 we have that there exists $u \in V_{TS} \setminus SV_{TS}$ and $s \in S$ such that for every $e \in u$, ${}^\circ e \subseteq R_s$ and $\overset{\square}{e} \cap R_s = \emptyset$. Since $u \notin SV_{TS}$ there are $f_1, f_2 \in u$ such that $f_1 \neq f_2$ and $f_1{}^\circ \cap \overset{\square}{f_2} \neq \emptyset$. From ${}^\circ e \subseteq R_s$ and $\overset{\square}{e} \cap R_s = \emptyset$, for $e \in \{f_1, f_2\}$, and (A7*) we have $s \xrightarrow{\{f_1\}}$ and $s \xrightarrow{\{f_2\}}$. These transitions are in $sat(TS)$ as well, together with $(s, \{f_1, f_2\}, s')$, where $s' = fin(s, \{f_1, f_2\})$. If $\overset{\{f_1\}}{\not\rightarrow} s'$ then $sat(TS)$ does not satisfy (A5*). Let $\xrightarrow{\{f_1\}} s'$. Hence $f_1{}^\circ \subseteq R_{s'}$. Suppose $\xrightarrow{\{f_2\}} s'$. Since $f_1{}^\circ \cap \overset{\square}{f_2} \neq \emptyset$ there exists $r \in f_1{}^\circ \cap \overset{\square}{f_2}$ and $s' \in r$. From $\xrightarrow{\{f_2\}} s'$ and $r \in \overset{\square}{f_2}$ and proposition 7.1.6 (or 3.1.8), we have $s' \notin r$, a contradiction. Thus, $\overset{\{f_2\}}{\not\rightarrow} s'$ and, as a consequence, $sat(TS)$ does not satisfy (A5*). □

We now give sufficient and necessary conditions for (8.2) to be satisfied. First we introduce the idea of a 'blocking' relationship for the events of TS. Let $\{e, f\} \in V_{TS}$. We will say that e *blocks* f if $e^\circ \cap \overset{\square}{f} \neq \emptyset$, and denote this by $e \dashv f$. Let $u \in V_{TS}$. A directed graph of the relation $\dashv$ on the events u will be called the *blocking graph* of u, i.e. it is defined as follows:

$$BG(u) = (u, \{(e, f) \in u \times u \mid e \dashv f\}).$$

The vertices of the graph are labelled with the events from u and an arc from $e \in u$ to $f \in u$ means that e blocks f. If TS is not clear from the context, we will use $BG_{TS}(u)$ to denote $BG(u)$.

Let $G = (V, A)$ be a directed graph. A *directed circuit* is a sequence $v_1, v_2, \ldots, v_n$ $(n \geq 1)$ of distinct vertices of G such that $(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1) \in A$. A directed graph that has no directed circuit is called *acyclic.*

The *adjacency matrix* $X = [x_{ij}]$ of G is a $|V| \times |V|$ binary matrix whose element

$$x_{ij} = \begin{cases} 1 & \text{if there is an arc from } i\text{th vertex to } j\text{th vertex,} \\ 0 & \text{otherwise.} \end{cases}$$

An adjacency matrix X is called *lower triangular* if $x_{ij} = 0$, for $i \leq j$.

We will need the following theorem from [25].

Theorem 8.1.2 *[25] A directed graph G is acyclic if and only if its vertices can be ordered such that the adjacency matrix X is a lower triangular matrix.* $\Box$

Proposition 8.1.6 *Let $TS \in TSENI_{apost}$. Suppose that there are $u \in V_{TS}$ and $s \in S$ such that $s \xrightarrow{\{f\}}$, for every $f \in u$. Then there is no enumeration of events from u which can be executed in a sequence from s if and only if $BG(u)$ contains a directed circuit.*

Proof: ($\Rightarrow$) To prove this implication we assume that $BG(u)$ contains no directed circuit and show how to order events from $u = \{f_1, \ldots, f_n\}$ to build an event sequence which is enabled at s. Since $s \xrightarrow{\{f_i\}}$, we have ${}^\circ f_i \subseteq R_s$, $f_i^\circ \cap R_s = \emptyset$ and $\overset{\Box}{f_i} \cap R_s = \emptyset$, for $i = 1, \ldots, n$ (see propositions 7.1.4(3) and 7.1.8). Suppose an event sequence $\sigma_i = f_1 f_2 \ldots f_i$, where $1 \leq i < n$, is enabled at s. Hence there is a sequence of transitions $(s_0, \{f_1\}, s_1), \ldots, (s_{i-1}, \{f_i\}, s_i)$ where $s_0 = s$ and $s_k \in S$ $(k = 1, \ldots, i)$. From proposition 7.1.4(3) we have $R_{s_k} = (R_{s_{k-1}} \setminus {}^\circ f_k) \cup f_k^\circ$ for $k = 1, \ldots, i$. Since $u \in V_{TS}$,

$$R_{s_i} = \Big(R_s \setminus ({}^\circ f_1 \cup \ldots \cup {}^\circ f_i)\Big) \cup (f_1^\circ \cup \ldots \cup f_i^\circ).$$

For f_{i+1} to be enabled at s_i we need to ensure that two conditions of (A7*) are satisfied. The first one, ${}^\circ f_{i+1} \subseteq R_{s_i}$, is satisfied as $u \in V_{TS}$ and ${}^\circ f_{i+1} \subseteq R_s$. The second one, $\overset{\Box}{f_{i+1}} \cap R_{s_i} = \emptyset$, can only be violated if $f_k^\circ \cap \overset{\Box}{f_{i+1}} \neq \emptyset$ for some $1 \leq k \leq i$. Hence an event sequence $\sigma = f_1 f_2 \ldots f_n$, for an enumeration $(f_1, f_2, \ldots, f_n)$ of events of u, would be enabled at s if $f_i^\circ \cap \overset{\Box}{f_j} = \emptyset$ $(f_i \not\dashv f_j)$ for every $i < j$, where $i, j = 1, \ldots, n$. The following shows it is possible. Since $BG(u)$ contains no directed circuit we have, by theorem 8.1.2, that its vertices can be ordered such that the adjacency matrix X is a lower triangular matrix. Let an enumeration $(f_{i_1}, \ldots, f_{i_n})$ be ordered in this way. Hence, in matrix X, we have $x_{f_{i_k}, f_{i_l}} = 0$, for $k \leq l$. This guarantees that in the event sequence $\sigma_X = f_{i_1} \ldots f_{i_n}$, $f_{i_k} \not\dashv f_{i_l}$ if $k < l$, where $k, l = 1, \ldots, n$. Hence, $s \overset{\sigma_X}{\rightsquigarrow}$.
($\Leftarrow$) Suppose there is an enumeration of events from u, $(f_1, f_2, \ldots, f_n)$, such that an event sequence $\sigma = f_1 f_2 \ldots f_n$ is enabled at s. We will write $f_i \overset{\sigma}{\prec} f_j$ if f_i precedes f_j, directly or indirectly, in the event sequence σ. We now show that the following holds for σ.

$$\text{For all } 1 \leq i, j \leq n,\ i \neq j: \quad \text{if } f_i \dashv f_j \text{ then } f_j \overset{\sigma}{\prec} f_i. \tag{8.3}$$

Let $f_i \dashv f_j$ and $f_i \overset{\sigma}{\prec} f_j$ for some $1 \leq i, j \leq n$, $i \neq j$. Then we have $f_i^\circ \cap \overset{\Box}{f_j} \neq \emptyset$ and a sequence of transitions in TS,

$$(s_0, \{f_1\}, s_1), \ldots, (s_{i-1}, \{f_i\}, s_i), \ldots, (s_{j-1}, \{f_j\}, s_j), \ldots, (s_{n-1}, \{f_n\}, s_n),$$

where $s_0 = s$. From proposition 7.1.4(3) we have for every transition $(s_{k-1}, \{f_k\}, s_k)$ $(k = 1, \ldots, n)$, $R_{s_k} = (R_{s_{k-1}} \setminus {}^\circ f_k) \cup f_k{}^\circ$. Since $u \in V_{TS}$, ${}^\circ f_k \cap f_i{}^\circ = \emptyset$, for every $k \geq i+1$. Hence $f_i{}^\circ \subseteq R_{s_{j-1}}$. From $(s_{j-1}, \{f_j\}, s_j) \in T$ and proposition 7.1.8 we have $\overset{\square}{f_j} \cap R_{s_{j-1}} = \emptyset$. But $f_i{}^\circ \cap \overset{\square}{f_j} \neq \emptyset$, a contradiction. Thus (8.3) holds.
Since $BG(u)$ contains a directed circuit, there are events $f_{i_1}, \ldots, f_{i_k} \in u$ $(2 \leq k \leq n)$ such that $f_{i_1} \dashv f_{i_2} \dashv \ldots \dashv f_{i_k} \dashv f_{i_1}$. From (8.3) we have $f_{i_1} \overset{\sigma}{\prec} f_{i_k} \overset{\sigma}{\prec} \ldots \overset{\sigma}{\prec} f_{i_2} \overset{\sigma}{\prec} f_{i_1}$. Notice that while $\dashv$ relation is not a transitive relation, $\overset{\sigma}{\prec}$ is. So, we obtain $f_{i_1} \overset{\sigma}{\prec} f_{i_1}$, a contradiction. $\square$

The blocking graph $BG(u)$ for $u = \{a, b, c\}$, for transition system TS in figure 8.2, is depicted in figure 8.3. We can observe that, since $BG(u)$ contains a directed circuit, this TS does not satisfy condition (8.2).

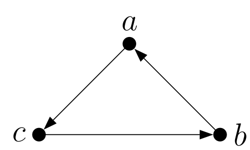


Figure 8.3: $BG(\{a, b, c\})$ for the transition system TS in figure 8.2.

8.2 Pruning TSENI Transition Systems

In this section, we consider a transition system $TS = (S, U, T, s_{in}) \in \text{TSENI} \setminus \text{TSENI}_{apost}$ and try to determine whether it is possible to find a TSENI_{apost} transition system whose associated net would be isomorphic to that of TS.

Proposition 8.2.1 *Let $TS \in$ TSENI $\setminus$ TSENI$_{apost}$. Then TS does not satisfy (A5*).*

Proof: Since $TS \in \text{TSENI} \setminus \text{TSENI}_{apost}$, it satisfies axioms (A1)-(A6) and, as a consequence, axioms (A1*)-(A4*) and (A6*). The only axioms which might not be satisfied by TS are (A5*) or (A7*). Suppose (A7*) is not satisfied. We introduce some symbols for the subformulae appearing in (A7*) and (A6), where $u \subseteq E_{TS}$ and $s \in S$ are such that (A7*) fails to hold:

$$\begin{array}{ll} \alpha & u \in SV_{TS} \\ \beta & u \in V_{TS} \\ \gamma & \forall e \in u: \ {}^\circ e \subseteq R_s \wedge \overset{\square}{e} \cap R_s = \emptyset \\ \delta & s \xrightarrow{u} \end{array}$$

(A7*) is not satisfied, so α is true (and so is β), γ is true and δ is false. Then $\beta \wedge \gamma$ is true and δ is false contradicting (A6). That means (A7*) is satisfied and the only axiom which can fail for TS is (A5*). $\square$

We now need a couple of results concerning TSENI Transition Systems.

Proposition 8.2.2 *Let $TS \in$ TSENI and $u, v, w \in U$ be steps such that $u = v \cup w$ and $v \cap w = \emptyset$. If $s \xrightarrow{u} s'$ and $s \xrightarrow{v} s''$ and $s'' \xrightarrow{w} s'''$ are transitions in TS, then $s' = s'''$.*

Proof: From $s \xrightarrow{u} s'$, $s \xrightarrow{v} s''$, $s'' \xrightarrow{w} s'''$ and proposition 3.1.5(3) we have:

$$\begin{aligned} R_{s'} &= (R_s \setminus {}^\circ u) \cup u^\circ, \\ R_{s''} &= (R_s \setminus {}^\circ v) \cup v^\circ, \\ R_{s'''} &= (R_{s''} \setminus {}^\circ w) \cup w^\circ. \end{aligned}$$

Hence,

$$R_{s'''} = \Big(\big((R_s \setminus {}^\circ v) \cup v^\circ\big) \setminus {}^\circ w\Big) \cup w^\circ.$$

Since $u \in V_{TS}$ (proposition 3.1.4), ${}^\circ u \subseteq R_s$ and $u^\circ \cap R_s = \emptyset$ (proposition 3.1.5), and $u = v \cup w$ we obtain

$$R_{s'''} = \big(R_s \setminus ({}^\circ v \cup {}^\circ w)\big) \cup (v^\circ \cup w^\circ).$$

Proposition 3.1.2 for u, v and w implies ${}^\circ u = {}^\circ v \cup {}^\circ w$ and $u^\circ = v^\circ \cup w^\circ$. Hence $R_{s'''} = R_{s'}$. Then, since TS satisfies (A5) as a TSENI transition system, we obtain $s' = s'''$. $\square$

Proposition 8.2.3 *Let $TS \in$ TSENI and there exists a transition $s_0 \xrightarrow{u} s$ such that $\overset{\{e\}}{\not\to} s$ for some $e \in u$. Then there is $f \in u$ such that $f \neq e$ and $f^\circ \cap \overset{\Box}{e} \neq \emptyset$.*

Proof: From axiom (A4) we have $s_0 \xrightarrow{\{e\}}$. Hence ${}^\circ e \subseteq R_{s_0}$ and $\overset{\Box}{e} \cap R_{s_0} = \emptyset$ (see propositions 3.1.5 and 3.1.9). From proposition 3.2.3 we have that $s_0 \xrightarrow{u \setminus \{e\}} s'$, for some $s' \in S$. So,

$$R_{s'} \overset{prop.\ 3.1.5}{=} \big(R_{s_0} \setminus {}^\circ(u \setminus \{e\})\big) \cup (u \setminus \{e\})^\circ.$$

Since $u \in V_{TS}$ (see proposition 3.1.4), ${}^\circ e \subseteq R_{s'}$. Suppose $\overset{\Box}{e} \cap R_{s'} = \emptyset$. Then, by (A6), we have $s' \xrightarrow{\{e\}}$, which by proposition 8.2.2 implies $s' \xrightarrow{\{e\}} s$. But $\overset{\{e\}}{\not\to} s$, a contradiction. Hence, $\overset{\Box}{e} \cap R_{s'} \neq \emptyset$. This and $\overset{\Box}{e} \cap R_{s_0} = \emptyset$ imply that there is $f \in u$ such that $f \neq e$ and $f^\circ \cap \overset{\Box}{e} \neq \emptyset$. $\square$

Corollary 8.2.1 *Let $TS \in TSENI$ and there exist $s \in S$ and $u \in U$ such that $\xrightarrow{u} s$ and $\overset{\{e\}}{\not\to} s$, for some $e \in u$. Then for every $s' \in S$, if $\xrightarrow{u} s'$ then there exists $e' \in u$ such that $\overset{\{e'\}}{\not\to} s'$.*

Proof: Let $s' \in S$ be such that $\xrightarrow{u} s'$ and $\xrightarrow{\{e'\}} s'$, for every $e' \in u$. From $\xrightarrow{u} s$ and $\overset{\{e\}}{\not\to} s$ and proposition 8.2.3 we have that there is $f \in u$ such that $f \neq e$ and $f^\circ \cap \overset{\Box}{e} \neq \emptyset$. Hence there is $r \in R_{TS}$ such that $r \in f^\circ \cap \overset{\Box}{e}$. Since $\xrightarrow{\{f\}} s'$ ($e' = f$) and $r \in f^\circ$ and proposition 3.1.1, we have $s' \in r$. But, $\xrightarrow{\{e\}} s'$ ($e' = e$) and $r \in \overset{\Box}{e}$ and proposition 3.1.8 imply $s' \notin r$, a contradiction. □

Observe that, according to the above corollary, if (A5*) is satisfied for a step $u \in U$ at some $s \in S$ then it will be satisfied for u at any state $s \in S$. So, we can say that 'a step u satisfies (A5*)' without mentioning the state at which it is satisfied.

Proposition 8.2.4 *Let $TS \in TSENI$ and there is $u \in U$ which satisfies (A5*). Then for every $\emptyset \neq u' \subset u$, u' satisfies (A5*).*

Proof: From (A2) we have $s \xrightarrow{u} s'$, for some $s, s' \in S$. From proposition 3.2.3, $u' \in U$. Suppose u' does not satisfy (A5*). Then from proposition 8.2.3 there are $e, f \in u'$ such that $f \neq e$ and $f^\circ \cap \overset{\Box}{e} \neq \emptyset$. Hence, there is $r \in R_{TS}$ such that $r \in f^\circ \cap \overset{\Box}{e}$. Since $TS \in$ TSENI we have from proposition 3.1.2 that $u^\circ = \bigcup_{e \in u} e^\circ$. So, $r \in u^\circ$ and hence $s' \in r$. But this and $r \in \overset{\Box}{e}$ implies $\overset{\{e\}}{\not\to} s'$, contradicting the fact that u satisfies (A5*). □

Let $TS = (S, U, T, s_{in})$ be a transition system in TSENI $\setminus$ TSENI$_{apost}$ which satisfies the following condition.

$$\begin{array}{l} \text{If } (s, u, s') \in T \text{ and } u \text{ does not satisfy (A5*) then there is} \\ \text{an event sequence } \rho_u \text{ (as in (8.1)) such that } s \overset{\rho_u}{\rightsquigarrow} s'. \end{array} \tag{8.4}$$

We then define the *pruning* of TS as the quadruple $prun(TS) = (S', U', T', s'_{in})$ given by:

$$\begin{array}{lcl} T' & = & T \setminus \{(s, u, s') \in T \mid (s, u, s') \text{ does not satisfy (A5*)}\}, \\ U' & = & U \setminus \{u \in U \mid \exists (s, u, s') \in T \setminus T'\}, \\ S' & = & S, \\ s'_{in} & = & s_{in}. \end{array}$$

Notice that the condition (8.4) allows safe removal of transitions from TS without creating isolated (or non-reachable) states in $prun(TS)$. Corollary 8.2.1 guarantees, on the other hand, that U' is well defined. It is immediate to see that $prun(TS)$ is a transition system, i.e. it satisfies (TS1)-(TS4).

Before we show that $prun(TS)$ is a TSENI$_{apost}$ transition system, we need to prove some properties which relate the regions of TS with those of $prun(TS)$.

Proposition 8.2.5 *If* $r \in R_{TS}$ *then* $r \in R_{prun(TS)}$.

Proof: Follows easily from the construction of $prun(TS)$. Specifically, from the fact that $S = S'$ and $T' \subset T$. □

Proposition 8.2.6 *If* $r \in R_{prun(TS)}$ *then* $r \in R_{TS}$.

Proof: Let r be a region in $prun(TS)$. We need to show that it is a region in TS. Suppose $s \xrightarrow{u} s'$ and $s \in r$ and $s' \notin r$ in TS. We consider two cases.

Case 1: $u \in U'$.

Since $r \in R_{prun(TS)}$ there exists $e \in u$ such that the following are satisfied in $prun(TS)$:

(a) if $u' \subseteq u \setminus \{e\}$ and $s \xrightarrow{u'} s''$ then $s'' \in r$,

(b) if $q \xrightarrow{v} q'$ and $e \in v$ then $q \in r$ and $q' \notin r$.

We need to show that the above is true in TS as well. We will show that e is the r-crossing event in u in TS. Let $u' \subseteq u \setminus \{e\}$ and $s \xrightarrow{u'} s''$ in TS. Since u satisfies (A5*), u' satisfies (A5*) as well (see proposition 8.2.4). So $s'' \in r$, as $u' \in U'$ and (a) is satisfied in $prun(TS)$. Let $q \xrightarrow{v} q'$ and $e \in v$ in TS. If $v \in U'$ then $q \in r$ and $q' \notin r$ follow from the fact that (b) is satisfied in $prun(TS)$. If $v \in U \setminus U'$ then by (8.4) there exists in TS an event sequence $\rho_v = e_1 e_2 \dots e_n$, where $(e_1, e_2, \dots, e_n)$ is an enumeration of the events from v, such that $q \overset{\rho_v}{\rightsquigarrow} q'$ and $e_k = e$ for some $1 \leq k \leq n$. This event sequence is in $prun(TS)$ as well. For every event in ρ_v there is a transition $t_i = (q_{i-1}, \{e_i\}, q_i)$, where $i = 1, \dots, n$ and $q_0 = q$, $q_n = q'$. Since $TS \in$ TSENI and satisfies (A4) we have $q \xrightarrow{\{e_i\}}$ for $i = 1, \dots, n$. Moreover, since $r \in {}^\circ e_k$ (in $prun(TS)$) we have $q_{k-1} \in r$ and $q_k \notin r$, and $q \in r$. Since $r \in R_q$ in $prun(TS)$ and $q \xrightarrow{\{e_i\}}$ $(i = 1, \dots, n)$, we deduce that none of the transitions t_i $(i = 1, \dots, n)$ enters into region r. Hence, since $q_k \notin r$, we have that $q_i \notin r$ for $i = k+1, \dots, n$, as otherwise some t_i would need to enter into r. Thus $q' \notin r$.

Case 2: $u \in U \setminus U'$.

Then, $u \in U$ and u does not satisfy (A5*). By (8.4), there exists in TS an event sequence

$\rho_u = e_1 e_2 \dots e_n$, where $(e_1, e_2, \dots, e_n)$ is an enumeration of the events from u, such that $s \overset{\rho_u}{\leadsto} s'$. This event sequence is in $prun(TS)$ as well. For every event in ρ_u, there is a transition $t_i = (s_{i-1}, \{e_i\}, s_i)$, where $i = 1, \dots, n$ and $s_0 = s$ and $s_n = s'$. Since $s \in r$ and $s' \notin r$, there is $1 \leq k \leq n$ such that $s_{k-1} \in r$ and $s_k \notin r$. Since $TS \in$ TSENI and satisfies (A4) we have $s \xrightarrow{\{e_i\}}$ for $i = 1, \dots, n$. From the fact that $r \in R_s$ in $prun(TS)$ and $s \xrightarrow{\{e_i\}}$ $(i = 1, \dots, n)$ we deduce that none of the transitions t_i $(i = 1, \dots, n)$ enters into r. Hence, since $s, s_{k-1} \in r$ and $s_k \notin r$, t_k is the only transition among the t_i's which crosses the border of r. We need to prove that e_k is the r-crossing event in u in TS.

Let $u' \subseteq u \setminus \{e_k\}$ and $s \xrightarrow{u'} s''$ in TS. We need to show that $s'' \in r$. If $u' \in U'$ then $s'' \in r$ follows from $r \in R_{prun(TS)}$ and the fact that transitions labelled with the events from u' do not cross the border of r. If $u' \in U \setminus U'$ then by (8.4) there is an event sequence in TS, $\rho_{u'}$ (as in (8.1)), such that $s \overset{\rho_{u'}}{\leadsto} s''$. Since $s \in r$ and none of the transitions associated with the events in $\rho_{u'}$ crosses the border of r, we have $s'' \in r$. Suppose now that $q \xrightarrow{v} q'$ and $e_k \in v$ in TS. We need to show that $q \in r$ and $q' \notin r$. If $v \in U'$ then this follows from $r \in R_{prun(TS)}$ and the fact that $s_{k-1} \xrightarrow{\{e_k\}} s_k$, $s_{k-1} \in r$ and $s_k \notin r$. If $v \in U \setminus U'$ then we can apply similar reasoning as the one used in Case 1.

The second part of definition 3.1.1 for r in TS can be shown in a similar way. Hence r is a region in TS. Moreover, it is non-trivial since $r \in R_{prun(TS)}$ and $S = S'$. □

Corollary 8.2.2 *Let TS be a transition system in TSENI $\setminus$ TSENI$_{apost}$ which satisfies (8.4). Then*

1. $E_{TS} = E_{prun(TS)}$.
2. *For every* $e \in E_{TS}$: $r \in {}^{\circ}e$ *(in* TS*)* $\Leftrightarrow$ $r \in {}^{\circ}e$ *(in* $prun(TS)$*).*
3. *For every* $e \in E_{TS}$: $r \in e^{\circ}$ *(in* TS*)* $\Leftrightarrow$ $r \in e^{\circ}$ *(in* $prun(TS)$*).*
4. *For every* $e \in E_{TS}$: $r \in \overset{\square}{e}$ *(in* TS*)* $\Leftrightarrow$ $r \in \overset{\square}{e}$ *(in* $prun(TS)$*).*
5. *For every* $s \in S$: $r \in R_s$ *(in* TS*)* $\Leftrightarrow$ $r \in R_s$ *(in* $prun(TS)$*).*
6. $V_{TS} = V_{prun(TS)}$.
7. $SV_{TS} = SV_{prun(TS)}$.

Proof: Follows directly from propositions 8.2.5 and 8.2.6, and the construction of the transition system $prun(TS)$. □

Proposition 8.2.7 *$prun(TS)$ is a TSENI$_{apost}$ transition system.*

Proof: (A1*),(A2*) follow from $TS \in$ TSENI and the construction of $prun(TS)$. (A3*) follows from (A3) for TS, the construction of $prun(TS)$ and (8.4).

(A4*) holds due to the construction of $prun(TS)$ and the fact that TS satisfies (A4).

(A5*) follows from proposition 8.2.1 and the fact that the construction of $prun(TS)$ removes all the steps u which violate this axiom.

(A6*) follows from corollary 8.2.2(5) and axiom (A5) for TS.

(A7*) is satisfied for TS as it is shown in the proof of proposition 8.2.1. The construction of $prun(TS)$ removes steps which do not satisfy (A5*) in TS. From proposition 8.2.3 we have that such steps of TS are not potential steps in $prun(TS)$, $u \notin SV_{TS} \stackrel{coro.\ 8.2.2(7)}{=} SV_{prun(TS)}$. Hence the implication in the axiom (A7*) holds for $prun(TS)$ as well. $\square$

Theorem 8.2.1 *Let TS be a transition system in TSENI$\setminus$ TSENI$_{apost}$ which satisfies (8.4). Then there is a transition system $prun(TS) \in$ TSENI$_{apost}$ such that $\mathcal{N}_{TS} \cong \mathcal{N}_{prun(TS)}$.*

Proof: Follows from propositions 8.2.5, 8.2.6, 8.2.7 and corollary 8.2.2. $\square$

Proposition 8.2.8 *Let TS be a transition system in TSENI$\setminus$ TSENI$_{apost}$ which satisfies (8.4). Then $prun(TS) \in$ TSENI$_{apost}\setminus$ TSENI.*

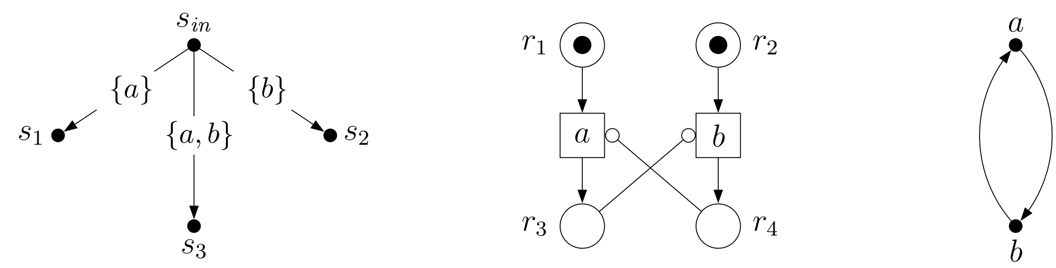
Proof: We need to show that $prun(TS) \notin$ TSENI. From proposition 8.2.1 we have that TS does not satisfy (A5*). Therefore, there is a transition $(s, u, s') \in T$ for which (A5*) does not hold and, according to the construction of $prun(TS)$, it is removed from TS ($(s, u, s') \notin T'$). But, from (A4) we have $s \xrightarrow{\{e\}}$ for every $e \in u$, in TS, and consequently in $prun(TS)$. By $u \in V_{TS}$ and corollary 8.2.2(6), $u \in V_{prun(TS)}$. So u and s satisfy all the conditions in (A6), but $(s, u, s') \notin T'$. Thus, $prun(TS)$ fails to satisfy (A6), and so $prun(TS) \notin$ TSENI. $\square$

Sufficient and necessary conditions for (8.4) to be satisfied are expressed using a blocking graph of a step appearing in condition (8.4).

Proposition 8.2.9 *Let $TS \in$ TSENI and $s \xrightarrow{u} s'$ be a transition in TS. Then there is no enumeration of events from u which can be executed in a sequence from s if and only if $BG(u)$ contains a directed circuit.*

Proof: Since $u \in U$ and $TS \in$ TSENI, we have from proposition 3.1.4 that $u \in V_{TS}$, and from (A4) that $s \xrightarrow{\{f\}}$, for every $f \in u$. The rest of the proof is similar to that of proposition 8.1.6, as it uses the common properties of TSENI and TSENI$_{apost}$ Transition Systems. $\square$

We observe that TS shown in figure 8.4 does not satisfy condition (8.4), since there is a step $\{a, b\} \in U$ such that $BG(\{a, b\})$ contains a directed circuit.

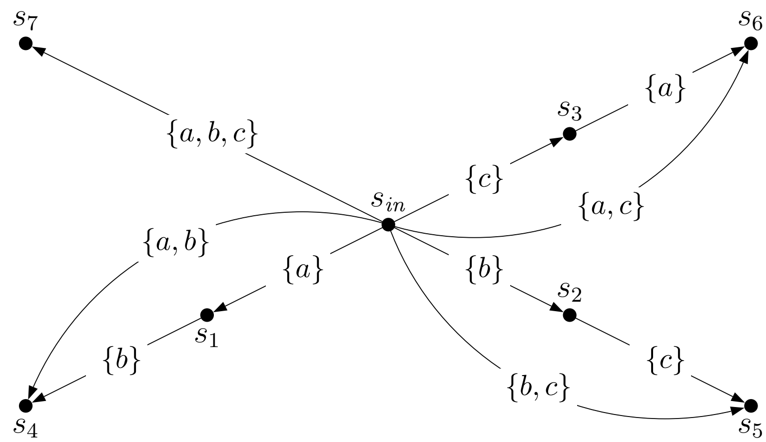


(a) $TS \in$ TSENI $\setminus$ TSENI$_{apost}$ (b) $\mathcal{N}_{TS}$ (c) $BG(\{a, b\})$

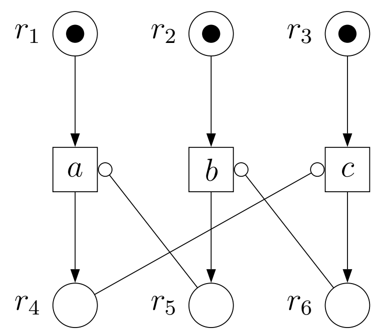
Figure 8.4: TSENI transition system which does not satisfy condition (8.4).

8.3 Discussion

In this chapter, we compared the TSENI$_{apost}$ and TSENI Transition Systems. It was shown that for any $TS \in$ TSENI$_{apost} \setminus$TSENI satisfying the condition (8.2), there is a transition system $sat(TS) \in$ TSENI $\setminus$ TSENI$_{apost}$, such that $\mathcal{N}_{TS} \cong \mathcal{N}_{sat(TS)}$. We mentioned that when $TS \in$ TSENI$_{apost} \setminus$TSENI does not satisfy the condition (8.2), the problem is much more complicated. In particular, some additional states might be required to build a TSENI transition system whose associated net is isomorphic to $\mathcal{N}_{TS}$. For the TS from figure 8.2(a), the procedure of 'saturation' leads to the TSENI transition system depicted in figure 8.5(a). We can see that one extra state, s_7, was added. The number of regions of the new 'saturated' transition system will be the same as number of regions of TS, and we only need to add s_7 to the post-regions of every event. The nets associated with TS, in figure 8.2(b), and its 'saturated' version, in figure 8.5(b), are isomorphic. Notice that the transition system in figure 8.5(a) is not a TSENI$_{apost}$ transition system. So, by adding extra transitions, we are loosing the ability to fulfill (A5*), exactly like when the process of 'saturation' is applied to the TSENI$_{apost}$ transition system satisfying the condition (8.2). The generalisation of the process of 'saturation' for TSENI$_{apost}$ (but not TSENI) transition systems which are not satisfying the condition (8.2) looks promising. One only needs to ensure that by adding extra states, we do not violate the state separation property, (A5), of the TSENI transition system we create.



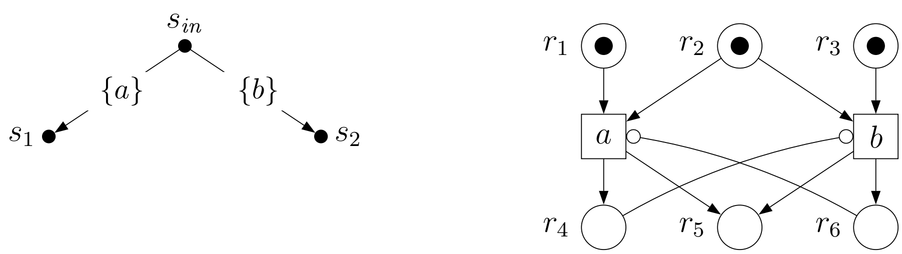
(a) TS



(b) $\mathcal{N}_{TS}$

Figure 8.5: $TS \in \text{TSENI} \setminus \text{TSENI}_{apost}$ and the net associated with it, $\mathcal{N}_{TS}$.

The generalisation of the 'pruning' procedure for a transition system $TS \in \mathrm{TSENI} \setminus \mathrm{TSENI}_{apost}$, which does not satisfy the condition (8.4), to obtain a TSENI_{apost} transition system with isomorphic net, will certainly fail. Take, for example, the TSENI transition system in figure 8.4(a). After deleting transition $(s_{in}, \{a, b\}, s_3)$, for which (A5*) is not satisfied, we obtain a transition system which is both TSENI and TSENI_{apost} transition system (see figure 8.6(a)), and the net associated with it (see figure 8.6(b)) is not isomorphic to that of the transition system in figure 8.4.



(a) TS (b) $\mathcal{N}_{TS}$

Figure 8.6: $TS \in \mathrm{TSENI} \cap \mathrm{TSENI}_{apost}$ and the net associated with it, $\mathcal{N}_{TS}$.

Chapter 9

Conclusions

This Thesis has tackled the synthesis problem for various extensions of Elementary Net Systems under the assumption that the resulting nets remain safe. These extensions are, in particular, relevant from the point of view of the modelling of the behaviour of asynchronous circuits, but one can also envisage their usefulness in other application areas (such as distributed systems).

From the technical point of view, the approach proposed by the Thesis is based on the notion of a region of a transition system which represents the global behaviour of the synthesised net. In the first class of transition systems considered here, introduced in chapter 2 and corresponding to the Semi-elementary Net Systems, it was sufficient to change only the set of axioms defining them, whereas the definition of a region was carried forward from the standard treatment of the Elementary Transition Systems. However, this extension was not entirely satisfactory from the application point of view, as the potential independence relation between the events in a transition system is not properly represented by the Semi-elementary Transition Systems. Such a realization motivated new line of research for the synthesis problem, characterised by explicit structural and semantical constructs for transition systems and nets based on inhibitor arcs and the derived step sequence semantics. The resulting class of transition systems, introduced in chapter 3, needs a completely new definition of a region as well as a new set of axioms (both regional axioms and step axioms). On the net level, self-loops used by the Semi-elementary Net Systems had been replaced by inhibitor arcs, while the semantics expressed in terms of interleaving sequences had been changed to a step semantics (a-priori and a-posteriori) directly capturing the concurrent behaviour of a system being modelled. Unlike any other model of concurrent behaviour, the a-priori step sequence semantics is capable of faithfully expressing the behaviour of asynchronous circuits with critical races. Consequently, Elementary Net Systems with Inhibitor Arcs equipped with the a-priori semantics became a focal point of the research presented in this Thesis.

A central result of the research presented in this Thesis is the novel definition of a region for the TSENI Transition Systems that enabled the development of an algorithm for the synthesis of ENI-systems. Moreover, it turned out that ENI-systems may be synthesised on the basis of the minimal regions only; such a result is important from the point of view of automatic tools, such as Petrify, which employ minimal regions for efficient synthesis. It is therefore expected that the work presented here would be of direct relevance for further development and extending the applicability of such tools.

The Thesis proposes a particularly efficient way of removing redundant inhibitor arcs from a synthesised ENI-system, and it is likely that such a method can be generalised to other classes of Petri nets, such as Place/Transition Nets with Inhibitor Arcs. It is worth noting that removing redundant inhibitor arcs is an important way of reducing the size of resulting implementations.

9.1 Directions for Further Research

There are several directions for extending the results presented in this Thesis both as far as the theory and implementation of the synthesis technique are concerned.

The work presented in Chapter 2 addressed problems related to structural transformations of transition systems that do not lead outside the class of Semi-elementary Transition Systems. To our knowledge, such a problem has not received sufficient attention in the literature, and the results presented here are the only ones currently available. There are at least two ways in which the work on structural transformations could be extended. First, it would be important to base the approach on a wider set of substructures in transition systems than the ladders of transition systems. In particular, it would be relevant to investigate lattice-like structures that are often identifiable in the behavioural patterns generated by asynchronous circuits. Secondly, it would be interesting to investigate what kinds of such substructures can still be used in the framework of step transition systems, such as TSENI Transition Systems. Moreover, future research on transition system transformations should be extended to the algorithmic aspects and practical implementations.

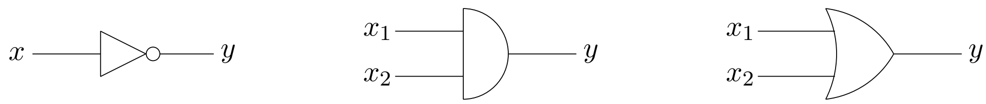
As already mentioned, the theory presented in this Thesis can provide a theoretical foundation for further development of automatic synthesis tools, such as Petrify, in order to cover also the class of nets with inhibitor arcs. On the other hand, the theory of synthesis of ENI-systems can be extended to the general class of Place/Transition Nets with Inhibitor Arcs by applying the idea of treating regions as morphisms. The set of axioms defining the initial transition system could be revised as well, so that the regional axioms would be replaced with ones which describe properties of the transition systems and not the nets which are synthesised.

In chapter 8, the comparison between the a-priori and a-posteriori semantics of nets with inhibitor arcs was carried out under certain restrictive conditions. In future, the processes of saturating the TSENI$_{apost}$ Transition Systems and the pruning of TSENI Transition Systems could be investigated after allowing the sets of states of transition systems to change, and thus generalise the results obtained here.

Finally, the Thesis has not addressed issues related to the semantics of the ENI-systems based on event structures which to our knowledge has not yet been discussed in the literature. Such a treatment would provide an additional insight into the fundamental relationships between the events in the system represented by a net with inhibitor arcs. We regard the problem of defining an event structure semantics of ENI-systems as a challenging one. The main reason is that the mutual dependencies between events in an ENI-system cannot be based on the usual independence relation [29]. Moreover, the standard causality semantics based on partial orders is also insufficient. A satisfactory treatment can be provided by replacing the independence relation with two new relations, and also adding the so-called weak causality to complement that used by the standard approach [29]. It is not yet clear how to define event structures taking a proper account of the differences between the way causality relationships are represented in ENI-systems and, e.g., in Elementary Net Systems, and we consider the problem of bridging this gap as a significant topic for future research.

Appendix

The diagram below gives the symbols of commonly used logic gates.



$y = \neg x$ $\quad$ $y = x_1 \wedge x_2$ $\quad$ $y = x_1 \vee x_2$

One- and two-input gates: NOT, AND, OR.

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