Simplicial Cohomology of Totally Ordered Semigroup Algebras

David Peter Elliott

Thesis submitted for the degree of
Doctor of Philosophy

School of Mathematics and Statistics
Newcastle University
Newcastle upon Tyne
United Kingdom

June 2012

Acknowledgements

One of the most poignant pieces of advice given to me when considering my PhD was that choosing the right supervisor is just as important, if not more so, than the topic you choose to research. To that end I believe I was in the strongest possible position throughout my studies and started with the battle half won. Therefore, first and foremost, my heartfelt thanks must go to my supervisor, Michael White, whose mathematical insight, humour and patience cannot be (co)bounded. I am also grateful for his constant encouragement, guidance and supreme flexibility, especially with his time and my work turnaround.

Being able to do my PhD would not have been possible without the financial support provided to me by EPSRC, to whom I am extremely grateful. I would also like to thank the School of Mathematics and Statistics for their constant and uplifting support over the years.

My examiners, Joel Feinstein and Zina Lykova, also deserve a special thanks for their time and attention to detail, putting me through the wringer and really making me feel like I earned my pass.

Without my fellow PhD students I am entirely certain I would have retained far less of my sanity than I managed to; therefore I would like to thank them for their encouragement and comradeship over the years, which was and is so very valuable to me. In particular I am grateful to Kevin, my constant companion on this journey and others, Daniel, my gracious housemate, and the five-a-siders, helping me to blow off steam and putting up with me when my venting became intense.

In producing this thesis some technical help has been necessary for me to avoid writing everything in crayon. First, my thanks must go to Paul Taylor, whose diagrams.sty package I have used extensively throughout my work. Second, and very importantly, my eternal thanks to resident TEX-pert Anthony Youd, without whom this work would have been far shorter (and uglier), not least for the use of his superlative thesis template.

Being able to step outside of the academic bubble on a regular basis has gone a long way to helping me recharge and stay sharp. Thus to the Air Training Corps and its wonderful personnel, especially my friends and colleagues, I am, and always will be, extremely grateful.

My sincere thanks must go to my parents and family for their constant support and encouragement, not just for my PhD but for everything, helping me through the lows and accompanying me through the highs, despite never actually knowing just what it is that I do no matter how many times I try to explain.

And finally my fondest and heartfelt thanks go to Helene, my confidante and my rock, for always being there and offering me unwavering support, for helping me see the way forward and for so much more I cannot express here in mere words. Thank you for making this even more meaningful.

Abstract

It is known that some discrete semigroup algebras have trivial continuous simplicial cohomology, at least in high dimensions. The aim of this work is to investigate the situation for the locally compact case, which even for the important example of the positive real numbers is not clear.

The initial focus of this thesis is on the continuous simplicial cohomology groups for the algebra $L^1(\mathbb{R}_+, \vee)$. We then adapt our methods and progress to investigating the more general case of $L^1(X, \leq, \mu_c)$ where $(X, \leq)$ is a totally ordered semigroup with the binary operation max and which is locally compact in its order topology and μ_c is a continuous, σ-finite, positive, regular Borel measure on X.

The first continuous simplicial cohomology of $L^1(\mathbb{R}_+, \vee)$ was already known, but we offer a new method of deriving this result and then use this method to prove the triviality of higher dimensional continuous simplicial cohomology groups for this algebra. We then modify our method to derive analogous results for the algebra $L^1(X, \leq, \mu_c)$ by analysing the algebra $L^1(\mathbb{R}_+, \mu_c)$.

Contents

Chapter 1

Introduction

Cohomology arose as a natural part of algebraic topology, growing over the years to occupy a fundamental place within several areas of pure mathematics.

A large and increasing variety of cohomology theories exist, each offering insights into other areas of mathematics and providing tools with which we are able to explore even further.

The aim of this thesis is to consider an analytical version of Hochschild cohomology, particularly simplicial cohomology, for certain Banach algebras. In this chapter we will give some motivation and historical context for considering such problems.

1.1 Motivation

The dual notion to homology and considered by functional analysts the more natural course of study due to the appearance of dual spaces and functionals, cohomology was first glimpsed in Henri Poincaré's duality theorem in the 1890s (for more see §3.3 in [19]). The idea was made more formal over the next few decades and diverged into a wide class of cohomology theories such as de Rham cohomology, Čech cohomology and of course Hochschild cohomology.

All of these theories are algebraically based and mostly centre around groups or algebraic objects. Several mathematicians attempted to extend these methods to a Banach algebra setting in the 1960s and early 1970s.

One of the first attempts was made by Herbert Kamowitz who investigated continuous cohomology groups for commutative C^*-algebras with, as we would say, coefficients in commutative or finite Banach modules. Alain Guichardet followed this up by defining homology and cohomology groups and investigating their relationships before considering certain cases involving the group algebra $L^1(G)$ for some group G.

Alexander Helemskiĭ, who has since made even more progress in this field (see his book, [20]), introduced the concept of normed modules over Banach algebras at about the same time J.L.Taylor considered topological algebras.

The biggest step came in 1972 when Barry Johnson published his memoir *Cohomology in Banach Algebras* [23], placing the definitions of Hochschild cohomology in a Banach algebra setting on a firm footing, as well as adapting the notion of amenability. After this W.G.Bade, P.C.Curtis and H.G.Dales [1] were able to define the concept of weak amenability for commutative Banach algebras, particularly Beurling and Lipschitz algebras, which was naturally extended to all Banach algebras by Johnson himself, and N.Grønbæk looked at bounded derivations from $L^1(\mathbb{R}_+, w)$ into its dual for a weight function w.

I would like to thank Dr Zinaida Lykova and Dr Volker Runde for providing me with extra insight into the history behind simplicial cohomology through their illuminating seminar talks.

1.2 Recent work

This thesis builds upon some more recent work in this area. The research of T.D.Blackmore, the quartet Y.Choi, F.Gourdeau, Z.A.Lykova and M.C.White, who have worked independently and in various collaborations, Baker, Pym, Vasudeva and Helemskiĭ in particular has considered several problems similar to those in my work.

A lot of work has been done on simplicial cohomology for discrete algebras, that is those algebras with an underlying discrete measure. In [17] Gourdeau and White showed that the third simplicial cohomology group for the algebra $\ell^1(\mathbb{Z}_+)$, where the binary operation on the underlying semigroup is addition, is trivial.

Gourdeau and White with B.E.Johnson in [13] showed that for the algebra $\ell^1(\mathbb{N})$, again with addition serving as the binary operation on the underlying semigroup, the cyclic cohomology groups are trivial in odd degree and one dimensional in even degree above two which implies, via the Connes-Tzygan long exact sequence, that the simplicial cohomology groups vanish in degree two and higher.

Increased dimensionality of the underlying semigroup is the subject for Gourdeau, Lykova and White in [15] where, amongst other things, they look at the simplicial cohomology groups for $\ell^1(\mathbb{Z}_+^k)$. In this paper they show that the simplicial cohomology groups are Banach spaces, and the n^{th} groups are trivial when $n \geq k$ and nonzero otherwise for $n \in \mathbb{N}$.

Much of the ground work in extending the ideas of Johnson in [23] was done by Alexander Helemskiĭ. In particular his book [20] considers the homology of Banach and topological algebras and gives a lot of the required definitions, including general concepts, using tensor products and resolutions among others. His paper [21] is also of great importance as it considers the links between simplicial and cyclic cohomology through the Connes-Tzygan long exact sequence, which we make use of extensively throughout this thesis.

Yemon Choi, both in his PhD thesis and afterwards, has done a lot of work in this area. In particular his paper [8] showed that the ℓ^1-convolution algebra of a semilattice with symmetric coefficients has trivial simplicial cohomology in all degrees greater than or equal to one. In [9] he extends the work done by Gourdeau, Lykova and White in [15] by considering more general symmetric coefficients among other things. This work is extended in collaboration with Gourdeau and White in [10] where they investigate simplicial triviality for the ℓ^1-convolution algebra of a band semigroup.

Gourdeau and White also looked at simplicial cohomology for the bicyclic semigroup algebra via an interesting admissible resolution approach and the Connes-Tzygan long exact sequence in [18]. They obtained once again that the cyclic cohomology groups vanish in odd degree and are one-dimensional in even degree greater than two while the simplicial cohomology groups all vanish in degree greater than or equal to two.

T.D.Blackmore in his Ph.D thesis and subsequent papers concerned himself with weak amenability, that is the first simplicial cohomology groups, for certain Banach algebras. In [4] he, amongst other things, investigated the weak amenability of ℓ^1-convolution algebras of algebraic discrete semigroups with a weight function.

Although a lot of analysis has been conducted in the discrete case the level of consideration of what happens when the underlying measure on a semigroup is continuous, or a combination of continuous and discrete measures, is not nearly so expansive.

In their paper [2] Baker, Pym and Vasudeva considered compact totally ordered measure spaces and investigated their L^p algebras. They formalised important definitions and concepts, such as how to impose multiplication on such algebras. In addition to this they also analysed the links to Gelfand transforms and isomorphisms between L^p algebras of spaces with regular continuous measures on them. A very important theorem they put forward states that the algebra $L^p(\mu)$ for a regular continuous measure μ is commutative and semisimple for $1 \leq p \leq \infty$ and that for $p = 1$ this algebra has a bounded approximate identity. They also considered multipliers but that is of little concern to us in this thesis.

Gourdeau, Lykova and White have produced numerous results in investigating the simplicial cohomology of $L^1(\mathbb{R}^k_+, +)$. In [14] they showed that the n^{th} simplicial cohomology groups for $L^1(\mathbb{R}^k_+, +)$ are trivial when $n \geq k$ and nontrivial otherwise for $n \in \mathbb{N}$.

The other main work in this area, and the paper on which this thesis starts and builds, is Blackmore's paper on derivations from totally ordered semigroup algebras into their duals [3]. In this work Blackmore considered totally ordered semigroups X which are locally compact in their order topology and have an underlying σ-finite, regular, positive Borel measure μ. He stated that the measure can be uniquely decomposed into its discrete and continuous parts, μ_d and μ_c respectively, and then went on to investigate each part independently. He arrived at the conclusion that the first simplicial cohomology group for the L^1 algebra for the discrete part, $L^1(X, \mu_d)$, is trivial while the continuous part, $L^1(X, \mu_c)$, is isomorphic to its dual. He also went on to investigate the bounded derivations from $L^p([0,1], m)$ into its dual for $1 \leq p < \infty$ and m Lebesgue measure, but that is not followed up in the present work.

1.3 Section by section overview

The work in this thesis is concerned with analysing the higher simplicial cohomology groups for the continuous part of the measure in the case Blackmore considered when investigating bounded derivations. It furthers his work in that we consider the higher dimensional simplicial cohomology groups of certain semigroup algebras and consider cyclic cohomology groups as well. It is also in some respects a complement to the work of Gourdeau, Lykova and White in [14] in that we consider as part of this work the simplicial cohomology groups of $L^1(\mathbb{R}_+)$. However where they considered the binary operation on the underlying semigroup to be addition we use the max operation instead. Additionally, the methods used here to approach this problem are different from those used by Gourdeau, Lykova and White and allow the analysis of a more general class of Banach semigroup algebras.

The initial groundwork is laid out in Chapter 2, where the main definitions and core results we will need are given. We define Banach semigroup convolution algebras in the discrete and continuous case; for clarity we refer to these last as *locally compact*. We also give the necessary definitions of homological algebra and simplicial cohomology before looking at the Connes-Tzygan long exact sequence, a fundamental tool we will make repeated use of throughout this work.

Chapter 3 proves the same result given by Choi, Gourdeau and White in [10] and Johnson in [24] concerning the triviality of the first simplicial cohomology group of $\ell^1(\mathbb{Z}_+, \vee)$, but we use a different methodology for doing this. We then use this method to obtain the same result given by Blackmore in [3] regarding the isomorphism between the first simplicial cohomology group of $L^1(\mathbb{R}_+, \vee)$ and its dual space again in a different way.

The purpose of Chapter 3 is to create a method which we could use to analyse the higher dimensional cohomology groups; by obtaining the same results as before we have validated these methods, and in Chapter 4 we adapt and apply them to the second simplicial cohomology group of $L^1(\mathbb{R}_+, \vee)$ by once again first gaining insight via the case when the algebra under consideration is $\ell^1(\mathbb{Z}_+, \vee)$. We do this by first considering the cyclic cohomology groups and attempting to cobound a given 2-cocycle, investigate its required properties, show that the cyclic cohomology group is trivial and then show that simplicial cohomology groups are also trivial by using the Connes-Tzygan long exact sequence.

The closing stages of Chapter 4 give an example of cobounding a specific 2-cocycle and then an example of a semigroup algebra with non-vanishing second simplicial cohomology.

We once again modify our methods slightly in Chapter 5 where we show that the n^{th} simplicial cohomology group for $L^1(\mathbb{R}_+, \vee)$ vanishes. The substance of the chapter is concerned with generalising our method to n dimensions and then applying it to get the main result.

Chapter 6 considers the general case, that of the algebra $L^1(X, \leq, \mu_c)$ where (X, μ_c) is a totally ordered semigroup which is locally compact in its order topology and μ_c is a continuous, σ-finite, positive, regular Borel measure on X. Some necessary definitions are given before we once more adapt and extend our method to deal with this general case before giving the last main result, which is that the simplicial cohomology groups in this case are isomorphic to the dual space of the semigroup algebra in degree one and trivial in all higher degrees.

This thesis is then rounded off in Chapter 7 where the main points are summarised and some other questions naturally arising from this work are proposed for further research.

Chapter 2

Preliminaries

Many of the central concepts and common results that form much of the background for the work contained in this thesis can be found in many publications including [20] and [5]. The papers [2] and [3] also contain many of these definitions in a very similar form. We present some of the notation and terminology that is key to obtaining our results here for further clarity.

Throughout this section all vector spaces are defined using the underlying ground field of $\mathbb{C}$ as this is the convention we use. However they are also applicable for any other field $\mathbb{F}$.

The notions of chain maps and chain homotopies are used in later sections and more background on these can be found in [30]. Although not explicitly used an underlying topic here is that of category theory; [22] gives a nice introduction to categories and functors.

2.1 Banach semigroup convolution algebras

In this section we introduce the concepts of semigroup convolution algebras, giving some basic but useful definitions. We then focus particularly on the algebras the work in this thesis is centred around.

Definition. Let S be a set with an associative binary operation $\cdot : S \times S \to S$

mapping (s,t) to $s \cdot t$, i.e. such that

$$(s \cdot t) \cdot r = s \cdot (t \cdot r) \;\; \forall s,t,r \in S. \tag{2.1}$$

Then S is called a *semigroup* and its associated binary operation is referred to as *multiplication*. For convenience we often omit the symbol $\cdot$ and simply write st for $s \cdot t$.

In the case where it is necessary to identify the binary operation $\cdot$ associated with a semigroup S we will write this as $(S, \cdot)$.

Notice that the condition for being a semigroup is one of the three requirements of a group; therefore every group is also a semigroup. If a semigroup S has an identity $1 \in S$ such that $1 \cdot s = s \cdot 1 = s$ for all $s \in S$ then we have satisfied a second condition for being a group and we call S a *monoid*.

Proposition 2.1. *Let S be a monoid with identities 1 and e. Then $1 = e$.*

The proof is trivial, mimicking the proof for the analogous result for groups, and can be found in many books on algebra, for example [12]. Thus it is unambiguous to talk about *the* identity element of S.

Discrete semigroup algebras

Definition. Let $(S, \cdot)$ be a countable semigroup. Then we define

$$\ell^1(S,\cdot) = \left\{ f : S \to \mathbb{C} : f = \sum_{s \in S} a_s \delta_s \text{ with } \sum_{s \in S} |f(s)| = \sum_{s \in S} |a_s| < \infty \right\}, \tag{2.2}$$

where $a_s \in \mathbb{C}$ and for $t \in S$

$$\delta_s(t) = \begin{cases} 1, & s = t, \\ 0, & s \neq t. \end{cases} \tag{2.3}$$

Once again when the binary operation is unambiguous we will simply suppress the symbol and write $\ell^1(S)$. In fact a natural binary operation which we impose on this semigroup when constructed to be locally compact within its order topology, which we consider in a later case, is *maximum multiplication*, and we use this throughout the thesis.

We define addition and scalar multiplication on $\ell^1(S,\vee)$ to be pointwise and give it a multiplication $*$ defined by convolution,

$$(f * g)(s) = \sum_{tu=s} f(t)g(u), \quad s,t,u \in S. \tag{2.4}$$

We define this sum to be zero if $tu = s$ has no solutions. The usual norm for this space is given by

$$\|f\| = \sum_{s\in S} |f(s)| \tag{2.5}$$

for $f \in \ell^1(S,\vee)$. It is a well-known result that this is a Banach algebra and it is called the *discrete semigroup algebra of S*.

We now look at a specific example of a discrete semigroup algebra, one which we shall consider carefully in further chapters.

The discrete semigroup algebra of $(\mathbb{Z}_+,\vee)$

Consider the semigroup which is given by the set of nonnegative integers and the binary operation of $\vee : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$ given by

$$x \vee y = \max\{x,y\}, \quad x,y \in \mathbb{Z}_+. \tag{2.6}$$

We denote this semigroup as $(\mathbb{Z}_+,\vee)$. This is in fact a monoid, with identity element 0.

Since the underlying semigroup here is discrete the corresponding discrete semigroup algebra is in fact a space of sequences. Thus given a function $f \in \ell^1(\mathbb{Z}_+,\vee)$ we can write this as a series

$$f = \sum_{n=0}^{\infty} a_n \delta_n, \tag{2.7}$$

where $a_n \in \mathbb{C}$ and for $m \in \mathbb{Z}_+$

$$\delta_n(m) = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases} \tag{2.8}$$

The usual norm on this algebra is by definition given by $\|f\| = \sum_{n=0}^{\infty} |a_n|$, and the formula for multiplying two elements $f, g \in \ell^1(\mathbb{Z}_+, \vee)$ given by $f = \sum_{t=0}^{\infty} a_t \delta_t$ and $g = \sum_{u=0}^{\infty} b_u \delta_u$, for $a_t, b_u \in \mathbb{C}, t, u \in \mathbb{Z}_+$, can then be simplified for $N \in \mathbb{Z}_+$ as

$$\begin{aligned} (f * g)(N) &= \sum_{n \vee m = N} f(n) g(m) \\ &= \sum_{n \vee m = N} \left\{ \left(\sum_{t=0}^{\infty} a_t \delta_t(n) \right) \left(\sum_{u=0}^{\infty} b_u \delta_u(m) \right) \right\} \\ &= \sum_{n \vee m = N} a_n b_m \\ &= a_N \sum_{m=0}^{N} b_m + b_N \sum_{n=0}^{N-1} a_n. \end{aligned} \tag{2.9}$$

This is also consistent with simply multiplying f and g as defined above together in the usual way. The summation range of the second term does not include the endpoint N because that term has already appeared in the first sum. We could equally well have written the summation limits the other way around instead, but have chosen not to and this does not affect the overall meaning.

Notice that for fixed t, u we have

$$(\delta_t * \delta_u)(N) = \sum_{n \vee m = N} \delta_t(n) \delta_u(m). \tag{2.10}$$

If $t \vee u \neq N$ then every term in this sum is zero. On the other hand, if $t \vee u = N$ then exactly one term in this sum is nonzero, namely $\delta_t(t)\delta_u(u)$, which has value 1. Therefore we have that

$$(\delta_t * \delta_u)(N) = \begin{cases} 1, & t \vee u = N, \\ 0, & t \vee u \neq N, \end{cases} \tag{2.11}$$

which is the same as $\delta_{t\vee u}(N)$.

Definition. Let A and B be algebras over $\mathbb{C}$. A linear mapping ϕ from A into B such that

$$\phi(xy) = \phi(x)\phi(y), \quad x, y \in A, \tag{2.12}$$

is called a *homomorphism.* If ϕ is injective we call it a *monomorphism*, surjective an *epimorphism* and bijective an *isomorphism.*

Definition. Let A be an algebra over $\mathbb{C}$. A *character on A* is a nonzero linear homomorphism mapping A into the field $\mathbb{C}$ (i.e. a multiplicative functional). Where A is a Banach algebra it automatically follows that all characters are bounded and hence continuous; see §16 in [5]. The space of all characters on A is denoted by $M(A)$.

Proposition 2.2. *Let $f, g \in \ell^1(\mathbb{Z}_+, \vee)$ be given by $f = \sum_{n=0}^{\infty} a_n\delta_n$ and $g = \sum_{m=0}^{\infty} b_m\delta_m$ respectively and define $\chi_M : \ell^1(\mathbb{Z}_+, \vee) \to \mathbb{C}$ for $M \in \mathbb{N} \cup \{\infty\}$ by*

$$\chi_M(f) = \sum_{n<M} a_n. \tag{2.13}$$

Then χ_M is a character. Furthermore, all characters $\chi \in M(\ell^1(\mathbb{Z}_+, \vee))$ are of this form.

It should be noted that the calculations of such characters are relatively simple to perform. The result is Banach algebra folklore.

Proof. First, we show that χ_M is linear. Let $\lambda \in \mathbb{C}$. Then we have

$$\begin{aligned}
\chi_M(f + \lambda g) &= \chi_M\left(\sum_{n=0}^{\infty} a_n\delta_n + \sum_{m=0}^{\infty} \lambda b_m\delta_m\right)\\
&= \chi_M\left(\sum_{k=0}^{\infty}(a_k + \lambda b_k)\delta_k\right)\\
&= \sum_{k<M}(a_k + \lambda b_k)\\
&= \sum_{k<M} a_k + \lambda\sum_{k<M} b_k\\
&= \chi_M(f) + \lambda\chi_M(g).
\end{aligned} \tag{2.14}$$

Note that it is enough to prove the relation holds for the generators δ_n, δ_m. For if

$$\chi_M(\delta_n)\chi_M(\delta_m) = \chi_M(\delta_{n\vee m}), \tag{2.15}$$

then

$$\chi_M(a_n\delta_n)\chi_M(b_m\delta_m) = \chi_M(a_n b_m\delta_{n\vee m}). \tag{2.16}$$

By linearity and continuity this extends to give

$$\Rightarrow \chi_M\left(\sum_{n=0}^{\infty} a_n\delta_n\right)\chi_M\left(\sum_{m=0}^{\infty} b_m\delta_m\right) = \chi_M\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_m\delta_{n\vee m}\right), \tag{2.17}$$

which is simply $\chi_M(f)\chi_M(g) = \chi_M(f * g)$ since

$$\begin{aligned} f * g &= \left(\sum_{n=0}^{\infty} a_n\delta_n\right)\left(\sum_{m=0}^{\infty} b_m\delta_m\right) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_m \delta_n\delta_m \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_m\delta_{n\vee m}. \end{aligned} \tag{2.18}$$

So it remains to prove that $\chi_M(\delta_n)\chi_M(\delta_m) = \chi_M(\delta_{n\vee m})$, i.e.

$$\begin{cases} 0, & \text{if } n \geq M, \\ 1, & \text{if } n < M, \end{cases} \times \begin{cases} 0, & \text{if } m \geq M, \\ 1, & \text{if } m < M, \end{cases} = \begin{cases} 0, & \text{if } n\vee m \geq M, \\ 1, & \text{if } n\vee m < M. \end{cases} \tag{2.19}$$

It is clear that the LHS $\neq 0$ when both $n, m < M$ and occurs if, and only if, $n\vee m < M$, which is the condition for the RHS $\neq 0$.

Finally we prove that all characters $\chi \in M(\ell^1(\mathbb{Z}_+, \vee))$ are of the form χ_M. Note that $\chi(\delta_n * \delta_n) = \chi(\delta_n)$ and so it follows that $\chi(\delta_n)$ is equal to either 0 or 1.

It must be that if $\chi(\delta_N) = 0$ for some $N \in \mathbb{Z}_+$ that $\chi(\delta_n) = 0$ for all $n \geq N$. To show this let $N \in \mathbb{N}$ be such that $\chi(\delta_N) = 0$ and let $m \in \mathbb{N}$ be such that $m \geq N$.

It follows that

$$\begin{aligned}\chi(\delta_m) &= \chi(\delta_m * \delta_N) \\ &= \chi(\delta_m) \cdot \chi(\delta_N) \\ &= \chi(\delta_m) \cdot 0,\end{aligned} \tag{2.20}$$

and so $\chi(\delta_m) = 0$ for all $m \geq N$.

If $\chi(\delta_m) \neq 0$ for all $m \in \mathbb{Z}_+$, i.e. there is no instance of $\chi(\delta_m) = 0$, then

$$\begin{aligned}\chi(f) &= \chi\left(\sum_{n=0}^{\infty} a_n \delta_n\right) \\ &= \sum_{n=0}^{\infty} a_n \chi(\delta_n) \\ &= \sum_{n=0}^{\infty} a_n \cdot 1 \\ &= \chi_\infty(f).\end{aligned} \tag{2.21}$$

Now suppose that N is the first instance when $\chi(\delta_n) = 0$ for $n \in \mathbb{Z}_+$ and therefore

$$\begin{aligned}\chi(f) &= \chi\left(\sum_{n=0}^{\infty} a_n \delta_n\right) \\ &= \sum_{n=0}^{\infty} a_n \chi(\delta_n) \\ &= \sum_{n=0}^{N-1} a_n \cdot 1 \\ &= \chi_N(f).\end{aligned} \tag{2.22}$$

Together these prove that $\chi \equiv \chi_N$ for $N = 2, 3, \ldots$ as required. $\square$

Locally compact semigroup algebras

We now extend the notion of semigroup convolution algebra to the locally compact case. We will introduce the concept of measure theory here; an excellent text which covers the necessary theory of measures is [29] with further reading to be found in [7] and [31].

Given a semigroup convolution algebra on a semigroup X, it is important to consider the notion of size, or *measure*, on subsets of X. In a way analogous to changing the topology on a space, changing the nature of the measure alters the structure of the semigroup. Because this measure is a notion of size, changing it therefore also affects the norm we place on the semigroup convolution algebra we construct from the semigroup X.

Thus it follows that semigroup convolution algebras are equipped with an underlying measure with respect to which integrals, norms and such are defined. For example, in the discrete case just considered, the underlying measure is defined as *counting measure*,

$$m(A) = |A|, \quad A \subseteq S. \tag{2.23}$$

In other words the measure is simply the number of elements in A for each subset A of the discrete semigroup S. It then follows that the algebra we defined in the previous section is exactly as before, but the norm is given by

$$\|f\| = \sum_{n=0}^{\infty} |f(s)| m(\{s\}). \tag{2.24}$$

However $m(\{s\}) = 1$ for all $s \in S$, returning the definition we have already made.

In Section 4.4 we will consider the discrete case equipped with a different measure. In the locally compact case, where we have a locally compact semigroup equipped with a continuous measure, the measure will affect the definition of the integral.

We now present some important definitions.

Definition. Let $(X, \cdot)$ be a semigroup with binary operation $\cdot$. A *total order* $\leq$ on $(X, \cdot)$ is a relation where for $x, y \in X$

- $x \leq y$ or $y \leq x$;
- if both $x \leq y$ and $y \leq x$ hold then $x = y$;
- if $x \leq y$ and $y \leq z$ then $x \leq z$.

The *order topology on* $(X, \cdot)$ is the topology where the open sets are generated by the subbase of 'open rays'

$$\begin{aligned}(-\infty, a) &:= x \text{ such that } -\infty < x < a,\\ (b, \infty) &:= x \text{ such that } b < x < \infty.\end{aligned} \tag{2.25}$$

Then the open sets in X are arbitrary unions of finite intersections of these open rays. The open intervals

$$(a, b) := x \text{ such that } a < x < b \tag{2.26}$$

in $(X, \cdot)$ are obtained as $(a, b) = (-\infty, b) \cap (a, +\infty)$. For particular choices of a and b this also generates the empty set $\emptyset$; for instance set $b = a$ to obtain $(a, a) = (-\infty, a) \cap (a, \infty) = \emptyset$.

We now consider a locally compact semigroup as found in [3]. Throughout this thesis X will denote a totally ordered set which is locally compact in its order topology. A natural binary operation on this set to enable it to become a topological semigroup is once more that of *maximum multiplication*, denoted by $\vee$, which, similar to that of the discrete semigroup algebra, we will use throughout this thesis. We denote by $(X, \vee)$ a *locally compact semigroup.*

It should be noted that in other literature such objects are referred to as *totally ordered semigroups.*

As mentioned at the start of this subsection, we equip such semigroups with a measure. In defining the norm we naturally use integrals, looking at the area under the graph of the function. One of the most natural, and accepted, ways to define such integrals is to use *Lebesgue integration*, and for that we need to use *Lebesgue measure* and its associated theory. Again we could change the nature of the underlying measure, which would in turn alter the value of subsequent integrals and, hence, norms. This is illustrated below; see [29], [7] and [31] for more details.

Let $(X, \vee)$ be a locally compact semigroup with binary operation $\vee$. We equip $(X, \vee)$ with a measure, μ; we call functions which are measurable with respect to μ, the measure on our semigroup, *μ-measurable.* Then in an analogous way to that in the discrete case we define

$$L^1(X, \vee) = \left\{ f : X \to \mathbb{C} : f \text{ is } \mu\text{-measurable and } \int_X |f(x)| d\mu(x) < \infty \right\} \quad (2.27)$$

as the space of equivalence classes of μ-measurable functions from X into $\mathbb{C}$ (under the equivalence relation defined by equality almost everywhere). Also as before addition and scalar multiplication is defined pointwise and multiplication is that of order convolution, given in [2] as

$$(f * g)(x) = f(x) \int_{(-\infty,x]} g(u) d\mu(u) + g(x) \int_{(-\infty,x)} f(u) d\mu(u), \quad (2.28)$$

for $f, g \in L^1(X, \vee)$ with underlying measure μ. As with the discrete case the point x is only included in the first of the integrals. This is for a rather clear reason for if the point x has measure zero, that is $\mu(\{x\}) = 0$, then its inclusion or exclusion from one or both of the integrals will not affect the value of the convolution. If, on the other hand, the point x has nonzero measure then the term $f(x)g(x)\mu(\{x\})$ will be duplicated if x is included in both integrals or will not appear at all if x appears in neither.

The norm on this space is then

$$\|f\| = \int_X |f(x)| d\mu(x), \quad (2.29)$$

which makes $L^1(X, \vee)$ into a Banach algebra, called the *locally compact semigroup algebra of* X. Once again where ambiguity will not arise we omit the symbol $\vee$ in writing the semigroup.

It should be noted that we will often denote by f a function and also the equivalence class containing that function, where we use f as a representative of the class.

It should also be noted here that there are two possible types of Banach algebra. A *unital* Banach algebra A is one which contains an element 1 such that for all $a \in A$ we have $1*a = a*1 = a$. A *non-unital* Banach algebra is one devoid of such an element. However there are two solutions to this; one is to force a unitisation, the other is to use the concept of *bounded approximate identities (bai)*. We will make extensive use of these elements and an excellent introduction covering all of the necessary points can be found in §11 of [5].

We now look at a specific example of a locally compact semigroup algebra that is central to this work.

The locally compact semigroup algebra of $(\mathbb{R}_+, \vee)$

In the definition of the discrete semigroup algebra of $(\mathbb{Z}_+, \vee)$ we replace the discrete set $\mathbb{Z}_+$ with the continuous set $\mathbb{R}_+$, the set of all nonnegative *real* numbers. This is also a monoid with identity 0.

We set the underlying measure on this semigroup to be *Lebesgue measure* (see [29] for the definition and background), denoted by λ, where the usual norm is defined as

$$\|f\| = \int_{\mathbb{R}_+} |f(x)| d\lambda(x) = \int_0^\infty |f(x)| dx \tag{2.30}$$

and multiplication

$$(f * g)(x) = f(x) \int_0^x g(u)du + g(x) \int_0^x f(u)du, \tag{2.31}$$

for $f, g \in L^1(\mathbb{R}_+, \vee)$. The endpoint x is included in the limits of both integrals in the order convolution without confusion as the singleton set $\{x\}$ has Lebesgue measure zero and consequently does not affect the value of the integral and norm.

Notation. For convenience, we will make the definition

$$\widehat{f}(x) = \int_0^x f(u)du, \quad x \in (0, \infty] \tag{2.32}$$

and thus we are able to write the order convolution more simply as

$$(f * g)(x) = f(x)\widehat{g}(x) + g(x)\widehat{f}(x). \tag{2.33}$$

We now present several well-known and standard, yet useful, results regarding the multiplication on this algebra.

Lemma 2.1. *Let f and g be elements of $L^1(\mathbb{R}_+, \vee)$. Then for $x \in (0, \infty]$ it follows that $\widehat{(f * g)}(x) = \widehat{f}(x)\widehat{g}(x)$.*

Proof.

$$\begin{aligned}
\widehat{(f*g)}(x) &= \int_0^x (f*g)(t)dt = \int_0^x \left(\widehat{f}(t)g(t) + f(t)\widehat{g}(t)\right) dt \\
&= \int_0^x \left(\int_0^t f(u)du\right) g(t)dt + \int_0^x f(t) \left(\int_0^t g(u)du\right) dt \\
&= \int_{t=0}^x \int_{u=0}^t f(u)g(t)dudt + \int_{t=0}^x \int_{u=0}^t f(t)g(u)dudt. \qquad (2.34)
\end{aligned}$$

Now,

$$\int_{t=0}^x \int_{u=0}^t f(t)g(u)dudt = \int_{u=0}^x \int_{t=0}^u f(u)g(t)dtdu \qquad (2.35)$$

using the change of variables $\{u \mapsto t, t \mapsto u\}$, and hence

$$\begin{aligned}
\widehat{(f*g)}(x) &= \int_{t=0}^x \int_{u=0}^t f(u)g(t)dudt + \int_{u=0}^x \int_{t=0}^u f(u)g(t)dtdu \\
&= \int_{u=0}^x \int_{t=0}^x f(u)g(t)dudt \\
&= \int_0^x g(t) \int_0^x f(u)dudt \\
&= \left(\int_0^x f(u)du\right) \left(\int_0^x g(t)dt\right) = \widehat{f}(x)\widehat{g}(x), \qquad (2.36)
\end{aligned}$$

as required. □

Corollary 2.1. *Let $f, g, h \in L^1(\mathbb{R}_+, \vee)$. Then the order convolution product $*$ is associative, i.e. $(f*g)*h = f*(g*h)$.*

Proof. First, recall that $(f*g)(x) = \widehat{f}(x)g(x) + f(x)\widehat{g}(x)$. Then

$$\begin{aligned}
((f*g)*h)(x) &= \widehat{(f*g)}(x)h(x) + (f*g)(x)\widehat{h}(x) \\
&= \widehat{f}(x)\widehat{g}(x)h(x) + \left(\widehat{f}(x)g(x) + f(x)\widehat{g}(x)\right)\widehat{h}(x) \\
&= \widehat{f}(x)\widehat{g}(x)h(x) + \widehat{f}(x)g(x)\widehat{h}(x) + f(x)\widehat{g}(x)\widehat{h}(x) \\
&= \widehat{f}(x)\left(\widehat{g}(x)h(x) + g(x)\widehat{h}(x)\right) + f(x)\widehat{g}(x)\widehat{h}(x) \\
&= \widehat{f}(x)\left((g*h)(x)\right) + f(x)\left(\widehat{(g*h)}(x)\right) \\
&= (f*(g*h))(x), \qquad (2.37)
\end{aligned}$$

and therefore $(f*g)*h = f*(g*h)$. □

It is not difficult to see that the morphism $\chi : L^1(\mathbb{R}_+, \vee) \rightarrow \mathbb{C}$ defined by

$$\chi(f) = \int_0^x f(u)du = \widehat{f}(x) \tag{2.38}$$

is a character on $L^1(\mathbb{R}_+, \vee)$. For if $f, g \in L^1(\mathbb{R}_+, \vee)$ and $\lambda \in \mathbb{C}$ then χ is linear,

$$\begin{aligned} \chi(f + \lambda g) &= \int_0^x (f + \lambda g)(u)du \\ &= \int_0^x f(u)du + \lambda \int_0^x g(u)du \\ &= \chi(f) + \lambda\chi(g), \end{aligned} \tag{2.39}$$

and multiplicative,

$$\chi(f * g) = \widehat{f * g}(x) = \widehat{f}(x)\widehat{g}(x) = \chi(f)\chi(g), \tag{2.40}$$

by Lemma (2.1).

The next result shows that *all* characters on $L^1(\mathbb{R}_+, \vee)$ are of this form.

Proposition 2.3. *All characters on $L^1(\mathbb{R}_+, \vee)$ are of the form*

$$\chi(f) = \int_0^x f(u)du = \widehat{f}(x), \tag{2.41}$$

where $x \in (0, \infty]$.

Proof. First note that $\chi : L^1((\mathbb{R}_+, \vee)) \longrightarrow \mathbb{C}$ is a bounded multiplicative linear functional and so $\chi(f)$ can be written in the form

$$\chi(f) = \int_0^\infty e(y)f(y)dy, \tag{2.42}$$

where $e(y) \in L^\infty((\mathbb{R}_+, \vee))$. To prove the result we need to show that the only possibility here is to have $e(y) = 1$ for $y \leq t$ and zero otherwise, i.e. to have $e(y)$ being equal to 1 almost everywhere up to a point t and then zero afterwards.

From $\chi(f * g) = \chi(f)\chi(g)$ we have that

$$\int_0^\infty e(y)\left(\widehat{f}(y)g(y) + f(y)\widehat{g}(y)\right)dy = \int_0^\infty e(u)f(u)du \int_0^\infty e(y)g(y)dy, \tag{2.43}$$

or in other words

$$\int_0^\infty e(y)\widehat{f}(y)g(y)dy + \int_0^\infty e(y)f(y)\widehat{g}(y)dy = \int_0^\infty \int_0^\infty e(u)e(y)f(u)g(y)dudy. \tag{2.44}$$

The LHS of this equation gives us

$$\int_{y=0}^\infty \int_{u=0}^y e(y)f(u)g(y)dudy + \int_{y=0}^\infty \int_{u=0}^y e(y)f(y)g(u)dudy, \tag{2.45}$$

which, under the change of variables $\{y \mapsto u, u \mapsto y\}$, becomes

$$\int_{y=0}^\infty \int_{u=0}^y e(y)f(u)g(y)dudy + \int_{u=0}^\infty \int_{y=0}^u e(u)f(u)g(y)dydu, \tag{2.46}$$

and thus the LHS of (2.44) finally becomes

$$\int_0^\infty \int_0^\infty \left[e(y)\chi_{E_2}(y,u) + e(u)\chi_{E_1}(y,u)\right] f(u)g(y)dudy, \tag{2.47}$$

where

$$\chi_{E_1}(y,u) = \begin{cases} 1, & u \geq y, \\ 0, & \text{else,} \end{cases} \text{ and } \chi_{E_2}(y,u) = \begin{cases} 1, & u < y, \\ 0, & \text{else.} \end{cases} \tag{2.48}$$

Thus (2.44) becomes

$$\begin{aligned} &\int_0^\infty \int_0^\infty \left[e(y)\chi_{E_2}(y,u) + e(u)\chi_{E_1}(y,u)\right] f(u)g(y)dudy \\ =&\int_0^\infty \int_0^\infty e(u)e(y)f(u)g(y)dudy, \end{aligned} \tag{2.49}$$

which in turn implies that

$$\int_0^\infty \int_0^\infty \left[e(y)\chi_{E_2}(y,u) + e(u)\chi_{E_1}(y,u) - e(u)e(y)\right] f(u)g(y)dudy = 0, \tag{2.50}$$

and this holds for all $f, g \in L^1(\mathbb{R}_+, \vee)$.

Since $f, g \in L^1((\mathbb{R}_+, \vee))$ are arbitrary this now reduces to

$$e(y)\chi_{E_2}(y,u) + e(u)\chi_{E_1}(y,u) - e(u)e(y) = 0 \text{ a.e. } (y,u). \tag{2.51}$$

Without loss of generality suppose $u \geq y$, for one of $u \geq y$ or $u < y$ holds and choosing the other option will yield the same end result. Thus

$$e(u) = e(u)e(y) \Rightarrow e(u)\,(1 - e(y)) = 0 \text{ a.e. } (y \leq u). \tag{2.52}$$

Here we have that u is the biggest variable and so

$$e(u) \int_{\epsilon_1}^{\epsilon_2} (1 - e(y))\, dy = 0 \text{ a.e. } (u) \text{ for } \epsilon_1 < \epsilon_2 < u, \tag{2.53}$$

and hence for $0 \leq \epsilon_1 < \epsilon_2 \leq \delta_1 < \delta_2$

$$\int_{\delta_1}^{\delta_2} e(u)\left[(\epsilon_2 - \epsilon_1) - (\widehat{e}(\epsilon_2) - \widehat{e}(\epsilon_1))\right] du = 0, \tag{2.54}$$

or simply

$$\left[\widehat{e}(\delta_2) - \widehat{e}(\delta_1)\right]\left[(\epsilon_2 - \epsilon_1) - (\widehat{e}(\epsilon_2) - \widehat{e}(\epsilon_1))\right] = 0. \tag{2.55}$$

This last holds *everywhere* as it is continuous. The first thing to notice is that $\widehat{e}(0) = 0$. If we choose $\epsilon_1 = 0$, we are then left with

$$\left[\widehat{e}(\delta_2) - \widehat{e}(\delta_1)\right]\left[\epsilon_2 - \widehat{e}(\epsilon_2)\right] = 0, \tag{2.56}$$

for all $\epsilon_2 \leq \delta_1 < \delta_2$.

Since $\widehat{e}(0) = 0$, we assume that $\widehat{e}(y) = y$ up until some point t, $t \geq 0$. Let $E = \{y : \widehat{e}(y) \neq y\}$ and $t = \inf E$. There are two possibilities here, remembering that our function in (2.55) is continuous.

The first is that $t \in E$, in which case $\widehat{e}(t) \neq t$ while $\widehat{e}(y) = y$ for all $y < t$. Then by (2.56) we have that

$$\widehat{e}(\delta_2) - \widehat{e}(\delta_1) = 0 \tag{2.57}$$

for all $\delta_2 > \delta_1 \geq t$, or simply

$$\widehat{e}(\delta_2) = \widehat{e}(\delta_1). \tag{2.58}$$

Therefore,

$$\widehat{e}(\delta_2) = \widehat{e}(\delta_1) = \widehat{e}(t) = C, \tag{2.59}$$

where C is a constant.

The second possibility is that $t \notin E$, i.e. if E was not closed, in which case for all $\epsilon > 0$ there exists $t_0 > t$ such that $|t_0 - t| < \epsilon$ and $\widehat{e}(t_0) \neq t_0$. Then we apply the same argument as above to t_0 and obtain the same result. Hence it follows that

$$\int_0^t e(u)du = u \Rightarrow e(u) = 1 \text{ a.e. } (u < t) \tag{2.60}$$

and

$$\int_t^{\delta_2} e(u)du = 0 \ \ \forall \delta_2 \geq t \Rightarrow e(u) = 0 \text{ a.e. } (u \geq t). \tag{2.61}$$

Thus,

$$e(y) = \begin{cases} 1, & \text{a.e. } y < t, \\ 0, & \text{a.e. } y \geq t, \end{cases} \tag{2.62}$$

and so,

$$\chi(f) = \int_0^\infty e(y)f(y)dy = \int_0^t f(y)dy, \tag{2.63}$$

proving our result, as required. Note that the point t has measure zero, and so

$$\int_{(-\infty,t)} e(u)du = \int_{(-\infty,t]} e(u)du.$$

Also, as $\epsilon \to 0$ then

$$\int_{(-\infty,t]} e(u)du = \int_{(-\infty,t_0)} e(u)du.$$

This proves our result. □

2.2 Homological algebra and simplicial cohomology

Homology is an important part of algebraic topology. It is a procedure by which we associate a sequence of abelian groups or modules to a mathematical object that helps us to classify the space we are interested in. For a good introduction to general homological algebra we refer the reader to [22].

Cohomology is the dual notion to homology, and is often a more natural construct to deal with in functional analysis. Computing (co)homology groups is usually easier than calculating homotopy groups and other similar classification methods, giving us an immediate payoff for our work. Finally, (co)homology has useful implications and applications for areas such as geometry, abstract algebra, algebraic geometry, algebraic number theory, complex analysis and even mathematical physics amongst other things. Bonsall and Duncan [5] give a nice introduction to cohomology as applied to Banach algebras.

In short, cohomology is the study of sequences of abelian groups or modules defined using a *cochain complex*. In general, homological algebra is an abstract study, meaning that there is no reference to the underlying space but rather we assign algebraic invariants to the space.

The standard canonical reference we rely on here is Johnson's 1972 treatise [23]. We give the definitions and results that directly influence our results here.

2.2.1 Cochain complexes

Although we do not consider algebraic versions of cochain complexes we present them here for illustrative purposes. After this, in Section 2.2.2, we will present the Banach space version, which is applicable to this thesis.

Let $(A_i)_{i\in\mathbb{Z}}$ be a sequence of abelian groups or modules, and let $(d_n)_{n\in\mathbb{Z}}$ be a collection of homomorphisms $d_n : A_n \to A_{n-1}$ such that $d_n \circ d_{n+1} = 0$ for all n. We call such a collection of groups and maps a *chain complex* and write it as a diagram

$$\cdots \xleftarrow{d_{n-1}} A_{n-1} \xleftarrow{d_n} A_n \xleftarrow{d_{n+1}} A_{n+1} \xleftarrow{d_{n+2}} \cdots .$$

The homomorphisms connecting the objects in the chain complex are called *boundary operators.*

Dual to this is the definition of a *cochain complex*, a reversal of the above diagram

$$\cdots \xrightarrow{\delta^{n-2}} A'_{n-1} \xrightarrow{\delta^{n-1}} A'_n \xrightarrow{\delta^{n}} A'_{n+1} \xrightarrow{\delta^{n+1}} \cdots ,$$

but now the boundary operators are such that $\delta^n : A'_n \to A'_{n+1}$ and $\delta^{n+1} \circ \delta^n = 0$. Here A' is the algebraic dual space of A, but the objects contained in a general cochain complex need not be dual objects.

We call the index i in A_i the *degree* in the (co)chain complex. If the chain complex is finite and extended to the left and right by the zero objects, in other words if the complex is zero in almost all places, then we say it is *bounded*. If it is extended by zeroes to the right above some degree N then we say it is *bounded above*, and similarly for *bounded below*. Dual notions apply to cochains.

Definition. The elements of an individual group in the (co)chain complex, A_i for some i, are called *(co)chains*.

The (co)chains which are contained in the image of a boundary operator are called *(co)boundaries*, while those contained in the kernel of a boundary operator are called *(co)cycles*.

From these definitions, it is immediately clear that the (co)boundaries of a given group in the complex form a subgroup of the (co)cycles of that group.

Given an algebra A we need to know how to turn its dual A' into a dual bimodule.

Definition. Given an algebra A and an A-bimodule X we make the dual space of X, written as X', into a *dual bimodule* by defining the left and right module multiplications as

$$(a \cdot f)(x) = f(x * a) \tag{2.64}$$

and

$$(f \cdot a)(x) = f(a * x) \tag{2.65}$$

respectively, where $x \in X$, $a \in A$ and $f \in X'$.

In this way the dual space A' also becomes a dual bimodule.

Notation. For convenience we write the fundamental relation on the coboundary operators, $\delta^{n+1} \circ \delta^n = 0$, where $n \in \mathbb{Z}$ as simply $\delta^2 = 0$. Also, where ambiguity does not arise, we will often drop the superscripts on the δ operators. Similar conventions apply to the boundary operators d in chain complexes.

Consider a cochain complex $\mathfrak{X}$ given by

$$\mathfrak{X} : \cdots \xrightarrow{\delta^{n-2}} A_{n-1} \xrightarrow{\delta^{n-1}} A_n \xrightarrow{\delta^n} A_{n+1} \xrightarrow{\delta^{n+1}} \cdots .$$

It is immediately clear from above that the space of coboundaries $\mathrm{im}(\delta^{n-1})$, which we denote as $\mathcal{B}^n(\mathfrak{X})$, forms a normal subgroup of the space of cocycles $\ker(\delta^n)$, denoted by $\mathcal{Z}^n(\mathfrak{X})$. We can therefore form the quotient group

$$\mathcal{H}^n(\mathfrak{X}) := \frac{\mathcal{Z}^n(\mathfrak{X})}{\mathcal{B}^n(\mathfrak{X})}, \tag{2.66}$$

and we call this the n^{th} *cohomology group of* $\mathfrak{X}$.

All of these concepts are dualized from similar definitions for chain complexes, which consider *homology groups* instead.

Definition (Exactness)**.** A cochain complex $\mathfrak{X}$ is *exact in degree* n, or simply *exact at* A_n, if $\mathrm{im}(\delta^{n-1}) = \ker(\delta^n)$. We say that the cochain complex is simply *exact* if it is exact at A_n for all $n \in \mathbb{Z}$.

It is immediate that every exact sequence is a complex. Further to this, it is also immediate that a complex is also an exact sequence if, and only if, all of its (co)homology groups are 0. Thus, it is natural to think of the (co)homology groups of a (co)chain complex as being a measure of how far the sequence is from being exact at that degree.

2.2.2 Hochschild, simplicial and cyclic (co)homology

As expected, the central reference for this subsection is that of Johnson [23]. We will not define tensor products here as we mainly make use of them for convenience in notation, but [28] is a very good place to look for an introduction to the subject in a Banach algebra setting. Let us present some of the key definitions and notation on this topic for completeness.

Definition. Let X and Y be Banach algebras (or, more generally, Banach spaces). We denote the *projective tensor product* of X and Y as $X\widehat{\otimes}Y$, which is the completion of an algebraic tensor product $X \otimes Y$ with respect to the projective tensor norm; if $\sum_{i=1}^n x_i \otimes y_i$ is a representation of $u \in X \otimes Y$ then the projective tensor norm is given by

$$\|u\| = \inf\left\{\sum_{i=1}^n \|x_i\|\|y_i\|\right\}, \tag{2.67}$$

where the infimum is taken over all representations of u. This is the largest possible natural norm that can be placed on $X\widehat{\otimes}Y$ such that $\|x \otimes y\| \leq \|x\|\|y\|$.

When denoting elements of $X\widehat{\otimes}Y$ it is usual to drop the hat notation, i.e. $x \otimes y \in X\widehat{\otimes}Y$.

It should be noted that there are other natural norms which can be placed on $X\widehat{\otimes}Y$, such as the *injective norm*, but we will be focused purely on the projective setting.

Let A be a Banach algebra and X be a Banach A-bimodule. Now for $n \geq 0$ define

$$\mathcal{C}_n(A,X) = \underbrace{A\widehat{\otimes}\cdots\widehat{\otimes}A}_{n \text{ times}}\widehat{\otimes}X = A^{\widehat{\otimes}n}\widehat{\otimes}X \tag{2.68}$$

and form the complex

$$0 \xleftarrow{\ d_0\ } \mathcal{C}_0(A,X) \xleftarrow{d_1} \mathcal{C}_1(A,X) \xleftarrow{d_2} \mathcal{C}_2(A,X) \xleftarrow{\ d_3\ } \cdots, \tag{2.69}$$

where the boundary operators d_n are given by

$$d_n(a_1 \otimes \cdots \otimes a_n \otimes x) = \begin{cases} a_2 \otimes \cdots \otimes a_n \otimes xa_1 \\ +\displaystyle\sum_{j=1}^{n-1}(-1)^j a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n \otimes x \\ +(-1)^n a_1 \otimes \cdots \otimes a_n x \end{cases} \tag{2.70}$$

and naturally $\mathcal{C}_0(A,X) = X$. Note that when $n = 1$ this simply reduces to

$$d_1(a_1 \otimes x) = xa_1 - a_1 x. \tag{2.71}$$

It is straightforward to show that $d_n \circ d_{n+1} = 0$ for all $n \geq 0$, and we leave this for the reader. This is the *Hochschild chain complex of* (A,X).

Definition. For a Banach space X we define the *dual space of* X, X^*, to be the Banach space of all bounded functionals on X. Often in the literature this is described as the *continuous dual space* of X to differentiate it from the more general notion of the *algebraic dual space*; as we only encounter the continuous dual space in this thesis we will omit the word continuous without ambiguity.

The dual of $\mathcal{C}_n(A, X)$ is, by definition, the space of bounded linear functionals from $A^{\widehat{\otimes}n}\widehat{\otimes}X$ into $\mathbb{C}$. Let f be such a functional. Then for each choice of $a_1, \ldots, a_n \in A$ we obtain a functional $g \in X^*$, the dual space of X, i.e. we can identify $f(a_1 \otimes \ldots \otimes a_n \otimes x)$ with $g(a_1, \ldots, a_n)(x)$. Hence we can identify $\mathcal{C}_n(A, X)^*$ with $\mathcal{C}^n(A, X^*)$, which is the space of bounded n-linear mappings of $A \times \cdots \times A$ into X^*. See [23] for further information.

Remark. It should be noted that this identification can be made as we are dealing with the Banach case; this is not true, for instance, in the category of Fréchet spaces.

Thus the dual of (2.69) is then

$$0 \xrightarrow{\delta^0} \mathcal{C}^0(A, X^*) \xrightarrow{\delta^1} \mathcal{C}^1(A, X^*) \xrightarrow{\delta^2} \mathcal{C}^2(A, X^*) \xrightarrow{\delta^3} \cdots, \tag{2.72}$$

where $\delta^n = d_n^*$ and as such for $T \in \mathcal{C}^{n-1}(A, X^*)$ is defined as

$$(\delta^n T)(a_1, \ldots, a_n) = \begin{cases} a_1 T(a_2, \ldots, a_n) \\ +\sum_{j=1}^{n-1}(-1)^j T(a_1, \ldots, a_j a_{j+1}, \ldots, a_n) \\ +(-1)^n T(a_1, \ldots, a_{n-1})a_n, \end{cases} \tag{2.73}$$

for $n \geq 2$.

In the case when $n = 1$ we have that $\delta^1 : X^* \to \mathcal{C}^1(A, X^*)$. To define this map choose $f \in X^*$ and $a_0 \in A$, then set

$$(\delta^1 f)(a_0) = a_0 f - f a_0. \tag{2.74}$$

Again it is a simple exercise to show that $\delta^{n+1} \circ \delta^n = 0$. This is the *Hochschild cochain complex of* (A, X).

Now we denote the following

$$\begin{aligned} \mathcal{Z}_n(A, X) &= \ker(d_n) \\ \mathcal{B}_n(A, X) &= \operatorname{im}(d_{n+1}) \\ \mathcal{H}_n(A, X) &= \frac{\mathcal{Z}_n(A, X)}{\mathcal{B}_n(A, X)} \end{aligned} \tag{2.75}$$

as the space of n-cycles, space of n-boundaries and n^{th} homology group of A with coefficients in X respectively.

Analogously we also let

$$\begin{aligned} \mathcal{Z}^n(A, X^*) &= \ker(\delta^{n+1}) \\ \mathcal{B}^n(A, X^*) &= \operatorname{im}(\delta^n) \\ \mathcal{H}^n(A, X^*) &= \frac{\mathcal{Z}^n(A, X^*)}{\mathcal{B}^n(A, X^*)} \end{aligned} \tag{2.76}$$

as the space of n-cocycles, space of n-coboundaries and n^{th} cohomology group of A with coefficients in X^* respectively.

Remark. It can be seen in many texts that the spaces we have defined are called the *continuous Hochschild (co)homology groups.* We omit the term continuous throughout this thesis as we do not consider the purely algebraic setting, removing any possible ambiguity in our definitions.

The dual space X^* is also a dual bimodule as shown earlier. Thus the cochain complex (2.72) is not restricted to having coefficients in X^*; it can be defined for some other Banach A-bimodule in place of X^*. However this thesis is concerned primarily with the case as it is written here defined with X^*.

Throughout this work our interest is almost purely with regard to the cohomology groups in the case when the dual module X^* here is in fact the dual of the algebra itself A^*. Thus the cohomology groups we are interested in are $\mathcal{H}^n(A, A^*)$ for $n \geq 0$, and we call these the *simplicial cohomology groups of* A.

For convenience and ease of notation we will write the simplicial cohomology groups of A, $\mathcal{H}^n(A, A^*)$, as simply $\mathcal{HH}^n(A)$. We also define the spaces $\mathcal{Z}^n(A)$ for $\mathcal{Z}^n(A, A^*)$ and $\mathcal{B}^n(A)$ for $\mathcal{B}^n(A, A^*)$. However it should be noted that although it is true for Banach algebras with bai's that $\mathcal{H}^n(A, A^*) \cong \mathcal{HH}^n(A)$ it does not necessarily follow in general; see [6] for further details.

Simplicial cohomology is intricately connected to the notion of *cyclic cohomology* via the work of Connes and Tzygan, details of which can be found in [21]. This is discussed in greater detail in the next subsection, but here we define the cyclic cohomology groups.

Definition. Let $T \in \mathcal{C}^n(A, A^*)$ be an n-cochain. We say that T is *cyclic* if for

$a_i \in A, i = 0, \ldots, n,$

$$T(a_1, \ldots, a_n)(a_0) = (-1)^n T(a_0, a_1, \ldots, a_{n-1})(a_n). \tag{2.77}$$

The space of all cyclic n-cochains is written as $\mathcal{CC}^n(A)$. A well-known result by Connes and furthered in the Banach Algebra context by Helemskiĭ [21] shows that the the sequence of objects $\mathcal{CC}^n(A)$ form a subcomplex of the sequence of objects $\mathcal{C}^n(A, A^*)$ using the same boundary operators δ^n, this time defined from $\mathcal{CC}^n(A)$ into $\mathcal{CC}^{n+1}(A)$.

We are now also able to define cyclic versions of the spaces defined above (2.76) as $\mathcal{ZC}^n(A)$, $\mathcal{BC}^n(A)$ and $\mathcal{HC}^n(A)$ respectively.

The cyclic and simplicial cohomology groups are connected via the Connes-Tzygan long exact sequence, which we introduce next.

2.2.3 Connes-Tzygan long exact sequence

During the work in this thesis we make use of the Connes-Tzygan long exact sequence, given in a Banach Algebra setting in [21], where Theorem 16 states the sequence exists for every Banach algebra with left or right bai, and reproduced here using our notation.

For a given Banach algebra A the Connes-Tzygan long exact sequence is given by

$$0 \to \mathcal{HC}^1(A) \to \mathcal{HH}^1(A) \to \mathcal{HC}^0(A) \to \mathcal{HC}^2(A) \to \mathcal{HH}^2(A) \to \mathcal{HC}^1(A) \to \cdots . \tag{2.78}$$

For a general section in the middle we can see that the form this sequence takes is

$$\begin{aligned}\cdots \to \mathcal{HC}^{n-1}(A) \to \mathcal{HH}^{n-1}(A) \to \mathcal{HC}^{n-2}(A) \to \mathcal{HC}^n(A)& \\ \to \mathcal{HH}^n(A) \to \mathcal{HC}^{n-1}(A) \to \mathcal{HC}^{n+1}(A) \to \cdots .&\end{aligned} \tag{2.79}$$

Chapter 3

The first simplicial cohomology group of $\ell^1(\mathbb{Z}_+, \vee)$ and $L^1(\mathbb{R}_+, \vee)$

The aim of this chapter is to give two main results. Initially we show that the first simplicial cohomology group for the algebra $\mathfrak{A} = \ell^1(\mathbb{Z}_+, \vee)$ is trivial, that is there are no derivations from $\mathfrak{A}$ into its dual. This has already been done by Choi, Gourdeau and White in [10] and Blackmore in [3], and it is also covered in [8] and [9], but we present the proof using different methods.

A well-known result of B.E.Johnson in [24] states that a Banach algebra which is a closed linear span of idempotents is weakly amenable.

The strategy behind this is to make the existing proofs more accessible and then adapt our new method to look at higher dimensional simplicial cohomology groups.

The second result is similar. It is an alternative presentation of the method for identifying the first simplicial cohomology group for the algebra $\mathfrak{B} = L^1(\mathbb{R}_+, \vee)$ found in [3] in order that we might use this to analyse the higher cohomology groups here, too.

Throughout this chapter we denote by A a Banach algebra, A^* the dual of A, X a Banach A-bimodule and X^* a Banach dual module.

3.1 Derivations

As outlined in Chapter 2 we have the cochain complex (2.72) and its boundary operators (2.73).

Definition. Let $\mathcal{D}$ be a bounded linear map from A into A^* which satisfies the derivation equality

$$\mathcal{D}(a_1a_2) = \mathcal{D}(a_1)a_2 + a_1\mathcal{D}(a_2) \tag{3.1}$$

for all $a_1, a_2 \in A$. We call $\mathcal{D}$ a *simplicial derivation.*

In the more general case where $\mathcal{D}$ is a bounded linear map from A into a general Banach A-bimodule X we say that $\mathcal{D}$ is a *derivation.*

Setting $n = 2$ in (2.73) for $T \in \mathcal{C}^1(A, A^*)$ and $a_1, a_2 \in A$ gives us

$$(\delta^2 T)(a_1, a_2) = a_1T(a_2) - T(a_1a_2) + T(a_1)a_2. \tag{3.2}$$

Thus it immediately follows that $\mathcal{Z}^1(A)$ is simply the space of all bounded derivations from A into A^*.

Definition. Let $f \in A^*$ and $a_0 \in A$. Now let δ_f be a bounded linear mapping from A into A^* such that

$$\delta_f(a_0) = a_0f - fa_0. \tag{3.3}$$

It is trivial to show that δ_f is a derivation. Such a mapping is called an *inner simplicial derivation.*

As before in the more general case with X instead of A^* we say that δ_f is simply an *inner derivation.*

Comparing this to (2.74) immediately shows that there exists $f \in A^* = \mathcal{C}^0(A, A^*)$ such that

$$\delta^1(f) = \delta_f, \tag{3.4}$$

i.e. that $\delta_f \in \mathcal{B}^1(A)$. This means that the space $\mathcal{B}^1(A)$ is in fact simply the space of all inner derivations from A into A^*.

So the first simplicial cohomology group of A is now simply the space of all derivations from A into A^* quotiented out by the space of all inner derivations. If

A, (resp. in the general case the module X), is commutative (so that $f(a_0a_1) = f(a_1a_0)$ for $a_0, a_1 \in A, f \in A^*$), then the space $\mathcal{B}^1(A)$ (resp. $\mathcal{B}^1(A, X)$) is trivial and so the first simplicial cohomology group (resp. first cohomology group of A with coefficients in X) in this case is simply the space of all derivations.

Definition. Let A be a Banach algebra. Then A is *amenable* if $\mathcal{H}^1(A, X^*) = 0$ for every Banach A-bimodule X, that is every derivation from A into X^* is an inner derivation.

We say that A is *weakly amenable* if $\mathcal{HH}^1(A) = \mathcal{H}^1(A, A^*) = 0$. Clearly all amenable algebras A are also weakly amenable.

In the rest of this chapter we are concerned with identifying the first simplicial cohomology group for the algebras $\mathfrak{A} = \ell^1(\mathbb{Z}_+, \vee)$ and $\mathfrak{B} = L^1(\mathbb{R}_+, \vee)$. Both of these are commutative and so we are therefore simply trying to find all derivations from $\mathfrak{A}$ (resp. $\mathfrak{B}$) into $\mathfrak{A}^*$ (resp. $\mathfrak{B}^*$).

3.2 Weak amenability in the discrete case

In this section we are going to show that there are no nontrivial simplicial derivations on $\mathfrak{A} = \ell^1(\mathbb{Z}_+, \vee)$.

As with the proof of Proposition 2.2 it is enough for us to prove that this result holds on generators δ_n and δ_m. In other words we are going to show that $\mathcal{D}(\delta_n)(\delta_m) = 0$ for $\delta_n, \delta_m \in \mathfrak{A}$, where $n, m \in \mathbb{Z}_+$.

The result has been proved by H.G.Dales and again by B.E.Johnson. The proof is relatively short and holds for all Banach algebras spanned by idempotents. A version is presented here for illustration; the original can be found in [24].

Theorem 3.1. *The first simplicial cohomology group for $\mathfrak{A}$ is trivial.*

Proof. As this algebra is spanned by idempotents (the generators δ_n) and derivations are linear it is enough to prove the result holds on idempotents. Let $e \in \mathfrak{A}$

be an idempotent function. Then $e^2 = e * e = e$ in $\mathfrak{A}$ and for a derivation $\mathcal{D}$

$$\begin{aligned} \mathcal{D}(e) &= \mathcal{D}(e^2) \\ &= e\mathcal{D}(e) + \mathcal{D}(e)e \\ &= 2e\mathcal{D}(e) \end{aligned} \tag{3.5}$$

by commutativity. Then it follows that

$$e\mathcal{D}(e) = 2e^2\mathcal{D}(e) = 2e\mathcal{D}(e), \tag{3.6}$$

which implies that $e\mathcal{D}(e) = 0$. As (3.5) above implies $\mathcal{D}(e) = e\mathcal{D}(e)$ it must be that

$$\mathcal{D}(e) = 0. \tag{3.7}$$

□

Our proof is much longer and more technical, but this is a necessity for we are attempting to create a proof that does not rely on idempotent properties; when we migrate these methods to the locally compact case there is no longer a spanning set of idempotents, indeed there are no nontrivial idempotents at all, and the idea is to adapt these techniques.

In order to proceed with this we need to consider two cases: first when $n \leq m$ and second when $n \geq m$.

Case 1: $n \leq m$

In this case we have that $\delta_n * \delta_m = \delta_{n\vee m} = \delta_m$. In $\mathbb{Z}_+$ we can always find ℓ such that $n \leq \ell$ and so

$$\mathcal{D}(\delta_n * \delta_m)(\delta_\ell) = \mathcal{D}(\delta_n)(\delta_{m\vee\ell}) + \mathcal{D}(\delta_m)(\delta_{\ell\vee n}), \tag{3.8}$$

which is then simply

$$\mathcal{D}(\delta_m)(\delta_\ell) = \mathcal{D}(\delta_n)(\delta_{m\vee\ell}) + \mathcal{D}(\delta_m)(\delta_\ell). \tag{3.9}$$

Remark. It should be noted at this point that we could easily have taken $\ell = m$

here. However, just as we wish to avoid using idempotent properties in order to translate our methods to the locally compact case we will also wish to avoid simply duplicating the maximal element, making the calculations much more accessible.

This rearranges to finally give that $\mathcal{D}(\delta_n)(\delta_{m\vee\ell}) = 0$. Notice that $n \leq m\vee\ell$ and so we are able to conclude that

$$\mathcal{D}(\delta_n)(\delta_m) = 0 \text{ whenever } n \leq m. \tag{3.10}$$

Case 2: $n \geq m$

We now have $\delta_n * \delta_m = \delta_n$. Thus for $\ell \in \mathbb{Z}_+$ the derivation identity becomes

$$\mathcal{D}(\delta_n * \delta_m)(\delta_\ell) = \mathcal{D}(\delta_n)(\delta_m * \delta_\ell) + \mathcal{D}(\delta_m)(\delta_\ell * \delta_n), \tag{3.11}$$

which is then

$$\mathcal{D}(\delta_n)(\delta_\ell) = \mathcal{D}(\delta_n)(\delta_{m\vee\ell}) + \mathcal{D}(\delta_m)(\delta_{\ell\vee n}). \tag{3.12}$$

The final form of this identity is now dependent upon the place within the order structure of $\mathbb{Z}_+$ of ℓ in relation to n and m. Thus there are three possibilities to consider.

[A] *$n \geq \ell \geq m$:* Here the derivation identity becomes

$$\mathcal{D}(\delta_n)(\delta_\ell) = \mathcal{D}(\delta_n)(\delta_\ell) + \mathcal{D}(\delta_m)(\delta_n), \tag{3.13}$$

yielding the result that $\mathcal{D}(\delta_m)(\delta_n) = 0$ where $n \geq m$.

[B] *$\ell \geq n \geq m$:* In this situation the derivation identity reduces to

$$\mathcal{D}(\delta_n)(\delta_\ell) = \mathcal{D}(\delta_n)(\delta_\ell) + \mathcal{D}(\delta_m)(\delta_\ell), \tag{3.14}$$

giving us that $\mathcal{D}(\delta_m)(\delta_\ell) = 0$ where $\ell \geq m$.

Neither of these give us anything new; they both agree with Case 1 above.

[C] *$n \geq m \geq \ell$:* When n, m, ℓ occur in this order we have that

$$\mathcal{D}(\delta_n)(\delta_\ell) = \mathcal{D}(\delta_n)(\delta_m) + \mathcal{D}(\delta_m)(\delta_n). \tag{3.15}$$

By Case 1 above it follows that the final term here is trivial, i.e. $\mathcal{D}(\delta_m)(\delta_n) = 0$. Thus

$$\mathcal{D}(\delta_n)(\delta_\ell) = \mathcal{D}(\delta_n)(\delta_m). \tag{3.16}$$

Since $n \geq m$ and the above holds for all m, choose $m = n$. Then by Case 1 we have that $\mathcal{D}(\delta_n)(\delta_m) = \mathcal{D}(\delta_n)(\delta_n) = 0$. Thus we can conclude that

$$\mathcal{D}(\delta_n)(\delta_m) = 0 \text{ whenever } n \geq m. \tag{3.17}$$

Putting Case 1 and Case 2 together we have

$$\mathcal{D}(\delta_n)(\delta_m) = \begin{cases} 0, & n \leq m, \\ 0, & n \geq m. \end{cases} \tag{3.18}$$

In other words we are led to the conclusion that for all $n, m \in \mathbb{Z}_+$

$$\mathcal{D}(\delta_n)(\delta_m) = 0. \tag{3.19}$$

Since we showed that it was enough to prove that the result holds on generators, the result follows in general; given $f, g \in \mathfrak{A}$ it follows that

$$\mathcal{D}(f)(g) = 0. \tag{3.20}$$

Thus there are no nontrivial derivations on $\mathfrak{A}$ and hence by the arguments given in Section 3.1 we have proved the following.

Theorem 3.2. *Let $\mathfrak{A}$ be the semigroup convolution algebra $\ell^1(\mathbb{Z}_+,\vee)$. Then*

$$\mathcal{H}\mathcal{H}^1(\mathfrak{A}) = 0, \tag{3.21}$$

i.e. the first simplicial cohomology group of $\mathfrak{A}$ is trivial.

3.3 Weak amenability in the locally compact case

We now turn our attention to identifying the derivations on $\mathfrak{B} = L^1(\mathbb{R}_+, \vee)$. Essentially, we weill prove the following theorem:

Theorem 3.3. *Let $\mathfrak{B}$ be the Banach algebra $L^1(\mathbb{R}_+, \vee)$. Then all simplicial derivations on $\mathfrak{B}$ are of the form*

$$\mathcal{D}(f)(g) = \int_0^\infty \int_0^t H(t)f(t)g(s)dsdt, \tag{3.22}$$

for $f, g \in \mathfrak{B}$ and $H \in L^\infty(\mathbb{R}_+, \vee)$. This in turn implies that

$$\mathcal{H}\mathcal{H}^1(\mathfrak{B}) \cong \mathfrak{B}^* \left(\cong L^\infty(\mathbb{R}_+, \vee)\right), \tag{3.23}$$

i.e. there is a direct correspondence between the first simplicial cohomology group of $\mathfrak{B}$ and the dual space of $\mathfrak{B}$.

We approach this problem by first considering the situation for *indicator functions* and building on this to obtain the full solution to the problem. In order to proceed, we will need the following definitions.

Definition. Let X be a set and U a subset of X. Then the *characteristic function on U* is the function $\chi_U : X \to \{0, 1\}$ such that

$$\chi_U(x) = \begin{cases} 1, x \in U, \\ 0, x \notin U. \end{cases} \tag{3.24}$$

This is also known as the *indicator function on U*.

A function $f : X \to \mathbb{C}$ is a *step function* if it can be written as a linear combination of characteristic functions on intervals of X, i.e.

$$f(x) = \sum_{k=0}^{n} \alpha_k \chi_{U_i}(x), \tag{3.25}$$

where $\alpha_k \in \mathbb{C}$ and U_k are intervals in X for $k \geq 0$.

Definition. A *normalised indicator function*, written as $I_{[a_n,b_n]}$ or more simply as I_n where $a_n, b_n \in \mathbb{R}_+$ with $a_n < b_n$, is the function on the interval $[a_n, b_n]$ defined by,

$$I_n(x) = \begin{cases} \frac{1}{b_n - a_n}, & x \in [a_n, b_n], \\ 0, & \text{else.} \end{cases} \tag{3.26}$$

Notation. For convenience we will write $I_m \ll I_n$ to mean $a_m < b_m < a_n < b_n$, where $I_j = I_{[a_j,b_j]}$. Essentially this means that the intervals $[a_m, b_m]$ and $[a_n, b_n]$ are disjoint and that there exist points $c_{m,n}$ and $d_{m,n}$ such that $b_m < c_{m,n} < d_{m,n} < a_n$. Also we denote by $|I_n|$ the size, or *measure*, of the support of I_n, that is $\lambda([a_n, b_n]) = |[a_n, b_n]|$. Finally when we wish to talk about the intersection or union of the supports of normalised indicator functions I_n and I_m we will write $I_n \cap I_m$ and $I_n \cup I_m$ respectively.

We use the term *normalised* to say that I_n is a positive valued function which integrates over its whole domain with respect to the underlying measure to give 1, that is $\int_{\mathbb{R}_+} I_n(x)dx = 1$ in $\mathfrak{B}$. We make this particular definition for convenience as it ensures that the multiplication of two of these functions is straightforward,

$$I_1 * I_2 = \begin{cases} I_1, & I_2 \ll I_1, \\ I_2, & I_1 \ll I_2, \end{cases} \tag{3.27}$$

Now we are able to consider derivations on indicator functions with pairwise disjoint supports,

$$\mathcal{D}(I_1 * I_2)(I_3) = \mathcal{D}(I_1)(I_2 * I_3) + \mathcal{D}(I_2)(I_3 * I_1). \tag{3.28}$$

There are three main possibilities here for us to consider.

Case 1: $I_1 \ll I_2 \ll I_3$

$$\begin{aligned} \mathcal{D}(I_2)(I_3) &= \mathcal{D}(I_1)(I_3) + \mathcal{D}(I_2)(I_3) \\ \text{therefore } \mathcal{D}(I_1)(I_3) &= 0. \end{aligned} \tag{3.29}$$

Case 2: $I_3 \ll I_2 \ll I_1$

$$\begin{aligned} \mathcal{D}(I_1)(I_3) &= \mathcal{D}(I_1)(I_2) + \mathcal{D}(I_2)(I_1) \\ \mathcal{D}(I_1)(I_3) &= \mathcal{D}(I_1)(I_2), \text{ using (3.29)} \\ \text{therefore } \mathcal{D}(I_1)(I_3) &= \mathcal{D}(I_1)(I_2). \end{aligned} \tag{3.30}$$

This case illustrates that $\mathcal{D}(I_1)(I_2)$ is only dependent on the first argument (this is also true for non-normalised indicator functions barring a scaling factor).

Case 3: $I_1 \ll I_3 \ll I_2$

$$\begin{aligned} \mathcal{D}(I_2)(I_3) &= \mathcal{D}(I_1)(I_2) + \mathcal{D}(I_2)(I_3) \\ \text{therefore } \mathcal{D}(I_1)(I_2) &= 0. \end{aligned} \tag{3.31}$$

This case, and in fact any other ordering of $\{I_1, I_2, I_3\}$, gives us nothing new.

Our aim here is to identify all of the simplicial derivations on our algebra $\mathfrak{B}$ and hence completely identify the first cohomology group of $\mathfrak{B}$ with coefficients in the dual module $\mathfrak{B}^*$. We wish to show that there is a one-one correspondence between the simplicial derivations and the elements of the dual space $(L^1(\mathbb{R}_+,\vee))^* \cong L^\infty(\mathbb{R}_+,\vee)$. To do this, we will first find a derivation given an element of the dual space, $\mathcal{D}_F$, then find an element of the dual space given a derivation, $F_\mathcal{D}$, and then show that they are mutually inverse, i.e. $F_{(\mathcal{D}_F)} \equiv F$ and $\mathcal{D}_{(F_\mathcal{D})} \equiv \mathcal{D}$.

Once this has been established we will have found the derivations given an element of the dual space, and shown that there are no more, thus giving us all of the derivations on the space and achieving our aim. We are trying to show that there is an isomorphism between the dual space and $\mathcal{HH}^1(\mathfrak{B})$.

First, given an element $F \in \mathfrak{A}^* \cong L^\infty(\mathbb{R}_+,\vee)$ we need to find a derivation $\mathcal{D}_F$ satisfying the conditions given by the three cases above, i.e.

$$\begin{aligned} &\mathcal{D}(I_1)(I_2) = 0, \quad I_1 \ll I_2, \\ &\mathcal{D}(I_1)(I_2) = \mathcal{D}(I_1)(I_3), \quad I_2, I_3 \ll I_1. \end{aligned} \tag{3.32}$$

The second equation in (3.32) is only dependent on I_1, making this equivalent to a bounded linear functional F on $\mathfrak{A}$, that is

$$F(I_1) = \mathcal{D}(I_1)(I_2), \tag{3.33}$$

where $I_2 \ll I_1$, $F \in \mathfrak{A}^*$. Then using the Riesz Representation Theorem for L^p spaces found in [31] (Theorem 1, § 17.4), this can be written in an integral form, i.e.

$$\mathcal{D}(I_1)(I_2) = F(I_1) = \int_{\mathbb{R}_+} H(t)I_1(t)dt, \tag{3.34}$$

where $F \in \mathfrak{A}^*$ ($\cong L^\infty(\mathbb{R}_+, \vee)$) and $H \in L^\infty(\mathbb{R}_+, \vee)$.

Let I_0, I_1 be normalised indicator functions such that $I_0 \ll I_1$. Then our derivation becomes

$$\begin{aligned} \mathcal{D}(I_1)(I_0) &= F(I_1) \cdot 1 \\ &= \int_0^\infty H(t)I_1(t)dt \cdot 1 \\ &= \int_0^\infty H(t)I_1(t)dt \int_0^\infty I_0(s)ds \\ &= \int_0^\infty \int_0^\infty H(t)I_1(t)I_0(s)dsdt \\ &= \int_0^\infty \int_0^t H(t)I_1(t)I_0(s)dsdt, \end{aligned} \tag{3.35}$$

as $I_1 \gg I_0$.

Observe that

$$\begin{aligned} \int_{t=0}^\infty \int_{s=0}^t H(t)I_1(t)I_0(s)dsdt = &\int_{t=0}^{a_1} \int_{s=0}^t H(t)I_1(t)I_0(s)dsdt \\ &+ \int_{a_1}^{b_1} \int_{s=0}^t H(t)I_1(t)I_0(s)dsdt \\ &+ \int_{b_1}^\infty \int_{s=0}^t H(t)I_1(t)I_0(s)dsdt. \end{aligned} \tag{3.36}$$

The first and last terms are zero as the range of integration for the t variable is outside the support of I_1, while the middle term is simply

$$\int_{a_1}^{b_1} H(t)I_1(t)dt. \tag{3.37}$$

All of this is equivalent to

$$\mathcal{D}(I_1)(I_0) = \int_0^\infty H(t)I_1(t)dt, \tag{3.38}$$

and so we arrive back where we started in (3.34), showing that our reasoning is correct.

Thus we have that

$$\mathcal{D}(I_1)(I_0) = \int_{t=0}^\infty \int_{s=0}^t H(t)I_1(t)I_0(s)dsdt. \tag{3.39}$$

This is constructed to satisfy the second condition of (3.32). It also automatically satisfies the first, for if $I_1 \ll I_0$ then (3.39) becomes

$$\mathcal{D}(I_1)(I_0) = \int_0^\infty H(t)I_1(t)\widehat{I_0}(t)dt, \tag{3.40}$$

but notice that

$$\begin{cases} \widehat{I_0}(t) = 0 \text{ for all } t < a_0, \\ I_1(t) = 0 \text{ for all } t \geq a_0, \end{cases} \tag{3.41}$$

and so

$$\mathcal{D}(I_1)(I_0) = 0, \tag{3.42}$$

as required.

Since the derivation we have been investigating here is defined by an element $F \in \mathfrak{A}^* \cong L^\infty(\mathbb{R}_+, \vee)$ we shall relabel it to read $\mathcal{D}_F$.

In the following analysis we will need the following two results.

Theorem 3.4 (Lusin's Theorem [27]). *For an interval $[a, b]$, let $f : [a, b] \longrightarrow \mathbb{C}$ be a measurable function. Then for all $\epsilon > 0$, there exists a compact set $E \subset [a, b]$ such that f restricted to E, $f|_E$, is continuous and $\mu(E^c) < \epsilon$.*

Essentially, this says that every measurable function is continuous on nearly all of its support. Here, E^c is the complement of E in $[a, b]$ and the continuity of f is with respect to the subspace topology that E inherits from $[a, b]$.

This means that it is enough for us to use indicator functions defined on intervals rather than the more general situation of characteristic functions defined on measurable sets.

Theorem 3.5 ([29]). *The step/characteristic functions are dense in $L^1(\mathbb{R}_+, \vee)$.*

These two results allow us to approximate any function in $L^1(\mathbb{R}_+, \vee)$ by linear combinations of disjoint normalised indicator functions as closely as we like.

Let $f, g \in \mathfrak{B}$ be L^1 functions approximated by

$$f_1 = \sum_{i=1}^{N} \alpha_i I_i \tag{3.43}$$

and

$$g_1 = \sum_{j=1}^{M} \beta_j J_j \tag{3.44}$$

respectively. Let $\epsilon > 0$; now define

$$\epsilon_f = \min \left\{ \frac{\epsilon}{2 \|\mathcal{D}_H\| \|g_1\| + 1}, \frac{\epsilon}{2 \|H\| \|g_1\| + 1} \right\}, \tag{3.45}$$

and

$$\epsilon_g = \min \left\{ \frac{\epsilon}{2 \|\mathcal{D}_H\| \|f\| + 1}, \frac{\epsilon}{2 \|H\| \|f\| + 1} \right\}. \tag{3.46}$$

Then we can refine f_1 and g_1 so that for all $\epsilon > 0$ (making $\epsilon_f, \epsilon_g > 0$), we have $\|f - f_1\| < \epsilon_f$ and $\|g - g_1\| < \epsilon_g$.

Then it follows that

$$\begin{aligned} |\mathcal{D}_F(f)(g) - \mathcal{D}_F(f_1)(g_1)| &= |\mathcal{D}_F(f)(g) - \mathcal{D}_F(f)(g_1) + \mathcal{D}_F(f)(g_1) - \mathcal{D}_F(f_1)(g_1)| \\ &\leq |\mathcal{D}_F(f)(g - g_1)| + |\mathcal{D}_F(f - f_1)(g_1)| \\ &< \|\mathcal{D}_F\| \|f\| \epsilon_g + \|\mathcal{D}_F\| \epsilon_f \|g_1\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned} \tag{3.47}$$

by (3.45) and (3.46). Hence

$$|\mathcal{D}_F(f)(g) - \mathcal{D}_F(f_1)(g_1)| < \epsilon. \tag{3.48}$$

In the next part of the analysis we will consider the case where given two functions $f, g \in \mathfrak{B}$ we can approximate them using linear combinations of normalised indicator functions which *do not overlap.* The overlapping case will be considered separately later.

Given f, g, f_1, g_1 as above, assume that the normalised indicator functions used to define f_1 and g_1 never overlap, that is $I_i \cap J_j = \emptyset$ for all $i, j \in \mathbb{N}$. We can see therefore that

$$\begin{aligned}
\mathcal{D}_F(f_1)(g_1) &= \mathcal{D}_F\left(\sum_{i=1}^{N} \alpha_i I_i\right)\left(\sum_{j=1}^{M} \beta_j J_j\right) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{M} \alpha_i \beta_j \mathcal{D}_F(I_i)(J_j) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{M} \alpha_i \beta_j \int_0^\infty \int_0^t H(t) I_i(t) J_j(s) ds dt \\
&= \int_0^\infty \int_0^t H(t)\left(\sum_{i=1}^{N} \alpha_i I_i(t)\right)\left(\sum_{j=1}^{M} \beta_j J_j(s)\right) ds dt \\
&= \int_0^\infty \int_0^t H(t) f_1(t) g_1(s) ds dt,
\end{aligned} \tag{3.49}$$

and hence for all $\epsilon > 0$

$$\left| \mathcal{D}_F(f)(g) - \int_0^\infty \int_0^t H(t) f_1(t) g_1(s) ds dt \right| < \epsilon. \tag{3.50}$$

Now consider the accuracy of approximating

$$\int_0^\infty \int_0^t H(t) f(t) g(s) ds dt \tag{3.51}$$

by

$$\int_0^\infty \int_0^t H(t) f_1(t) g_1(s) ds dt. \tag{3.52}$$

We will now show that this approximation can be as accurate as we desire by looking at the modulus of the difference between these terms and showing it is less than ϵ for any $\epsilon > 0$, similar to our earlier strategy.

The modulus of this difference can be written as

$$\left|\int_0^\infty \int_0^t H(t)[f_1(t)g_1(s) - f(t)g(s)]dsdt\right|, \tag{3.53}$$

which is equal to

$$\left|\int_0^\infty \int_0^t H(t)\left[(f_1(t) - f(t))g_1(s) + f(t)(g_1(s) - g(s))\right]dsdt\right|. \tag{3.54}$$

This in turn is less than or equal to

$$\|H\|\,\|f_1 - f\|\,\|g_1\| + \|H\|\,\|f\|\,\|g_1 - g\|. \tag{3.55}$$

Treating this similarly to (3.47) as well as using (3.45) and (3.46) transforms this last equation into

$$\begin{aligned}\|H\|\,\|f_1 - f\|\,\|g_1\| + \|H\|\,\|f\|\,\|g_1 - g\| &< \|H\|\,\epsilon_f\,\|g_1\| + \|H\|\,\|f\|\,\epsilon_g \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned} \tag{3.56}$$

and thus

$$\left|\int_0^\infty \int_0^t H(t)f_1(t)g_1(s)dsdt - \int_0^\infty \int_0^t H(t)f(t)g(s)dsdt\right| < \epsilon, \tag{3.57}$$

as required.

The final step here is to consider

$$\mathcal{D}_F(f)(g) - \int_0^\infty \int_0^t H(t)f(t)g(t)dsdt. \tag{3.58}$$

Note that this can be rewritten as

$$\begin{aligned}&\mathcal{D}_F(f)(g) - \int_0^\infty \int_0^t H(t)f_1(t)g_1(t)dsdt \\ &\qquad + \int_0^\infty \int_0^t H(t)f_1(t)g_1(t)dsdt - \int_0^\infty \int_0^t H(t)f(t)g(t)dsdt,\end{aligned} \tag{3.59}$$

which in modulus is less than or equal to

$$\begin{aligned}\left|\mathcal{D}_F(f)(g) - \int_0^\infty \int_0^t H(t)f_1(t)g_1(t)dsdt\right| \\ + \left|\int_0^\infty \int_0^t H(t)f_1(t)g_1(t)dsdt - \int_0^\infty \int_0^t H(t)f(t)g(t)dsdt\right| \\ < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad (3.60)\end{aligned}$$

by (3.50) and (3.57), replacing ϵ with $\frac{\epsilon}{2}$.

Hence we have that

$$\left|\mathcal{D}_F(f)(g) - \int_0^\infty \int_0^t H(t)f(t)g(s)dsdt\right| < \epsilon, \quad \text{for all } \epsilon > 0, \tag{3.61}$$

and since neither of these terms depend on our approximations, this means we can conclude that they are actually equal, that is

$$\mathcal{D}_F(f)(g) = \int_0^\infty \int_0^t H(t)f(t)g(s)dsdt. \tag{3.62}$$

We must now deal with the situation where there *can exist overlap* between the supports of the normalised indicator functions used to define f_1 and g_1, that is $I_i \cap J_j$ does not necessarily equate to the empty set $\emptyset$.

In order to do this we will consider the case for normalised indicator functions first.

Let I and J be normalised indicator functions such that $I \cap J \neq \emptyset$, that is the interval supports of these functions overlap. We can divide each of these intervals into M equal subintervals. In the worst case, each of these subintervals for one function overlap at most three subintervals from the other function, as shown by the following argument.

To illustrate this point more clearly, we draw these two intervals on a pair of axes. The interval defining I is placed on the x-axis, while the corresponding interval for J is drawn onto the y-axis. The division of these intervals into M equal subintervals results in an $M \times M$ grid being formed on our graph, as outlined in Figure(3.1), below.

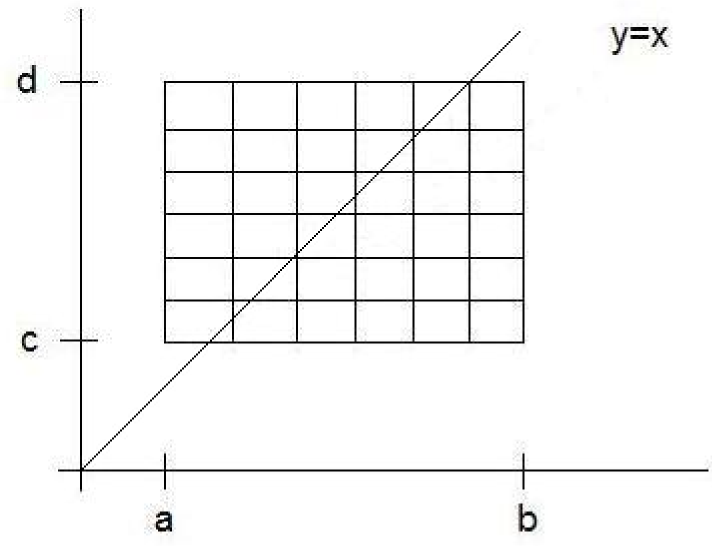


Figure 3.1: An illustration of our $M \times M$ grid

If a subinterval of the interval support for I overlaps a subinterval of the interval support for J then it must mean they have a point in common. Obviously, all points of overlap are shared points and so all of the points contained in all of the overlaps can be represented by the line $y = x$ on our graph. This overlap line has gradient 1, and if it touches a box in our grid then that box represents a subinterval of the interval support for I and a subinterval of the interval support for J overlapping. Our aim, then, is to show that the line $y = x$ only touches *at most* $3M$ of our M^2 boxes.

In the first instance we consider the case when the interval supports on which I and J are defined are equal in length. This results in all of the boxes in our grid being square. For our overlap line to touch more than three boxes in a particular row, it must have a gradient no steeper than that of the line connecting the bottom left corner of one box to the top right corner of the box immediately right adjacent to it. A little thought shows that this is indeed the highest gradient a line touching more than 3 boxes in a given row can have.

Since the gradient of such a line is then at most $\frac{1}{2}$ and our overlap line is steeper than this with gradient 1, it is immediately obvious that our line can pass through at most 3 boxes in a given row. Passing from corner to corner through one box is a line of gradient 1, and so it is indeed possible for our line to touch three boxes in any given row.

The next case we consider is when the interval supports are not equal in length to each other and so the boxes of our grid become rectangles. This is slightly more tricky. We deal with the case of when the rectangles are longer than they are tall, as the opposite case follows via a very similar argument.

If the boxes are long then say they have a height x and a length y such that $y > x$. The steepest gradient for a line to touch three boxes is from corner to corner across one box, as before. Such a line, therefore, must have a gradient less than or equal to $\frac{x}{y}$, and since $x < y$ this gradient is strictly less than 1. As our overlap line has a gradient of precisely 1, this line cannot touch three, or indeed more, boxes. Our overlap line, it follows, must touch at most two boxes in any given row.

A similar argument follows for when the boxes are taller than they are long, except that in this particular case the line can only touch a maximum of two boxes in any given *column*. Thus, we are now able to conclude that in any given row (or column in the last case) our line touches at most three boxes and since there are M rows (and columns) we must have a maximum number of $3M$ overlaps.

Thus it follows that we have normalised indicator functions I and J and if the interval supports on which they are defined overlap then we can subdivide these intervals up into M equal subintervals. Then given a derivation $\mathcal{D}_F$ with arguments I and J we can rewrite it to give

$$\mathcal{D}_F(I)(J) = \sum_{i=1}^{M} \sum_{j=1}^{M} \mathcal{D}_F(I_i)(J_j). \tag{3.63}$$

From this we have that

$$\begin{aligned}\sum_{i=1}^{M}\sum_{j=1}^{M}\mathcal{D}_F(I_i)(J_j) &= \sum_{I_i\cap J_j=\emptyset}\mathcal{D}_F(I_i)(J_j) + \sum_{I_i\cap J_j\neq\emptyset}\mathcal{D}_F(I_i)(J_j),\\ &= \sum_{I_i\cap J_j=\emptyset}\int_0^\infty\int_0^t H(t)I_i(t)J_j(s)dsdt + \sum_{I_i\cap J_j\neq\emptyset}\mathcal{D}_F(I_i)(J_j),\\ &= \sum_{i=1}^{M}\sum_{j=1}^{M}\int_0^\infty\int_0^t H(t)I_i(t)J_j(s)dsdt\\ &+ \sum_{I_i\cap J_j\neq\emptyset}\left(\mathcal{D}_F(I_i)(J_j) - \int_0^\infty\int_0^t H(t)I_i(t)J_j(s)dsdt\right),\end{aligned} \tag{3.64}$$

as

$$\begin{aligned}\sum_{i=1}^{M}\sum_{j=1}^{M}\int_0^\infty\int_0^t H(t)I_i(t)J_j(s)dsdt &= \sum_{I_i\cap J_j=\emptyset}\int_0^\infty\int_0^t H(t)I_i(t)J_j(s)dsdt\\ &+ \sum_{I_i\cap J_j\neq\emptyset}\int_0^\infty\int_0^t H(t)I_i(t)J_j(s)dsdt.\end{aligned} \tag{3.65}$$

Notice that

$$|\mathcal{D}_F(I_i)(J_j)| \leq \|\mathcal{D}_F\|\,\|I_i\|\,\|J_j\| = \|\mathcal{D}_F\|\frac{|I|}{M}\frac{|J|}{M} = \frac{\|\mathcal{D}_F\|}{M^2} \tag{3.66}$$

and

$$\left|\int_0^\infty\int_0^t H(t)I_i(t)J_j(s)dsdt\right| \leq \|H\|\frac{|I|}{M}\frac{|J|}{M} = \frac{\|H\|}{M^2}, \tag{3.67}$$

as $\|I_i\| = \frac{|I|}{M} = \frac{1}{M}$ and $\|J_i\| = \frac{|I|}{M} = \frac{1}{M}$, with I, J normalised.

As we have already shown earlier, the maximum number of places of overlap here is $3M$. this implies that the maximum number of terms we have in the summation over $I_i \cap J_j \neq \emptyset$ is $3M$. Thus

$$\left| \sum_{I_i \cap J_j \neq \emptyset} \left(\mathcal{D}_F(I_i)(J_j) - \int_0^\infty \int_0^t H(t) I_i(t) J_j(s) ds dt \right) \right| \\ \leq 3M \cdot \left(\frac{\|\mathcal{D}_F\|}{M^2} - \frac{\|H\|}{M^2} \right), \quad (3.68)$$

which clearly tends to 0 as M tends to infinity.

Hence, as M tends to ∞ by (3.68)

$$\begin{aligned} \mathcal{D}_F(I)(J) &= \sum_{i=1}^M \sum_{j=1}^M \mathcal{D}_F(I_i)(J_j) \\ &\longrightarrow \sum_{i=1}^M \sum_{j=1}^M \int_0^\infty \int_0^t H(t) I_i(t) J_j(s) ds dt \\ &= \int_0^\infty \int_0^t H(t) \left(\sum_{i=1}^M I_i(t) \right) \left(\sum_{j=1}^M J_j(s) \right) ds dt \\ &= \int_0^\infty \int_0^t H(t) I(t) J(s) ds dt. \end{aligned} \quad (3.69)$$

The error term here, given in (3.68), is therefore actually the difference between two constants, as by linearity

$$\sum_{i=1}^M \sum_{j=1}^M \mathcal{D}_F(I_i)(J_j) = \mathcal{D}_F(I)(J) \quad (3.70)$$

and

$$\int_0^\infty \int_0^t H(t) \left(\sum_{i=1}^M I_i(t) \right) \left(\sum_{j=1}^M J_j(s) \right) ds dt = \int_0^\infty \int_0^t H(t) I(t) J(s) ds dt. \quad (3.71)$$

This then allows us to draw the conclusion that actually

$$\mathcal{D}_F(I)(J) = \int_{t=0}^\infty \int_{s=0}^t H(t) I(t) J(s) ds dt \quad (3.72)$$

instead of this just holding in the limit as M tends to ∞.

Finally, it follows from this and equation (3.50) that we end up with

$$\mathcal{D}_F(f)(g) = \int_{t=0}^{\infty} \int_{s=0}^{t} H(t)f(t)g(t)dsdt. \tag{3.73}$$

We now show that this result is indeed a derivation.

Lemma 3.1. *The map $\mathcal{D}_F$ induced by the element $F \in \mathfrak{A}^*$ is a derivation.*

Proof. Given an element $H \in L^\infty(\mathbb{R}_+, \vee)$ we obtain an element $F \in \mathfrak{A}^*$ and then a map

$$\mathcal{D}_F(f)(g) = \int_{t=0}^{\infty} \int_{s=0}^{t} H(t)f(t)g(s)dsdt = \int_{t=0}^{\infty} H(t)f(t)\widehat{g}(t)dt, \tag{3.74}$$

where $f, g \in \mathfrak{B}$ and so we have that

$$\begin{aligned}
\mathcal{D}_F(f * g)(h) &= \int_{t=0}^{\infty} \int_{s=0}^{t} H(t)(f * g)(t)h(s)dsdt \\
&= \int_{t=0}^{\infty} H(t)f(t)\widehat{g}(t)\widehat{h}(t)dt + \int_{t=0}^{\infty} H(t)g(t)\widehat{f}(t)\widehat{h}(t)dt \\
&= \int_{t=0}^{\infty} H(t)f(t)\widehat{(g * h)}(t)dt + \int_{t=0}^{\infty} H(t)g(t)\widehat{(h * f)}(t)dt \\
&= \int_{t=0}^{\infty} \int_{s=0}^{t} H(t)f(t)(g * h)(s)dsdt + \int_{t=0}^{\infty} \int_{s=0}^{t} H(t)g(t)(h * f)(s)dsdt \\
&= \mathcal{D}_F(f)(g * h) + \mathcal{D}_F(g)(h * f),
\end{aligned} \tag{3.75}$$

as required. ∎

As we have already shown that given a functional $F \in \mathfrak{A}^*$, which is dependant on a unique element $H \in L^\infty(\mathbb{R}_+, \vee)$, we can find a derivation $\mathcal{D}_H$ it is now natural to ask if the reverse is also possible; that is given a derivation $\mathcal{D} \in \mathcal{ZH}^1(\mathbb{R}_+, \vee)$ can we find a functional $F_D \in \mathfrak{A}^*$, which is dependent on a unique element $H_\mathcal{D} \in L^\infty(\mathbb{R}_+, \vee)$? Looking at the second condition in (3.32), it follows that for $f \in \mathfrak{B}$ we should define

$$F_\mathcal{D}(f) = \lim_{n\to\infty} \mathcal{D}(f)(nI_{[0,\frac{1}{n})}) = \lim_{n\to\infty} \int_{t=0}^{\infty} \int_{s=0}^{t} H_\mathcal{D}(t)f(t)nI_{[0,\frac{1}{n})}(s)dsdt \tag{3.76}$$

to be the answer we are seeking.

Before we can continue, however, we will need the following result, which is well-known in Banach algebra folklore.

Lemma 3.2. *Let A be a Banach algebra, and let $(e_\alpha)_{\alpha \in I}$ be a bounded approximate identity for A. Also, let $\mathcal{D}$ be a derivation from A into A^*. Then $\mathcal{D}(a)(e_\alpha)$ tends to a limit for all $a \in A$.*

Proof. By Cohen's Factorisation Theorem [5] (Theorem 10 and Corollary 11 in § 11) we are able to write $a = bc$ for some $b, c \in A$. Then, invoking the derivation equality, we have that

$$\begin{aligned} \lim_\alpha \mathcal{D}(a)(e_\alpha) &= \lim_\alpha \mathcal{D}(bc)(e_\alpha) \\ &= \lim_\alpha \mathcal{D}(c)(e_\alpha b) + \mathcal{D}(b)(ce_\alpha) \\ &= \mathcal{D}(c)(b) + \mathcal{D}(b)(c). \end{aligned} \tag{3.77}$$

Note that $\mathcal{D}(c)(b) + \mathcal{D}(b)(c) \in \mathbb{C}$ and so we have found the limit of $\mathcal{D}(a)(e_\alpha)$ as required. □

Using this result we now show that the element $H_\mathcal{D}$ is indeed induced by a derivation $\mathcal{D}$ given in (3.76).

Lemma 3.3. *Given a simplicial derivation $\mathcal{D}$ on $\mathfrak{A}$ we can find an element $F_\mathcal{D} \in \mathfrak{A}^* \cong L^\infty(\mathbb{R}_+, \vee)$.*

Proof. In order to prove this result we need to show that $F_\mathcal{D}$ is indeed an element of $A^* \cong L^\infty(\mathbb{R}_+, \vee)$.

First of all it is clear that $F_\mathcal{D}$ is bounded since $\mathcal{D}$ is bounded by definition. Secondly, the linearity of $F_\mathcal{D}$ also follows quite naturally from the fact that $\mathcal{D}$ is bilinear, as for $f, g \in \mathfrak{B}$

$$\begin{aligned} F_\mathcal{D}(f + \lambda g) &= \lim_{n\to\infty} \mathcal{D}(f + \lambda g)(nI_{[0,\frac{1}{n})}) \\ &= \lim_{n\to\infty} \mathcal{D}(f)(nI_{[0,\frac{1}{n})}) + \lambda \lim_{\to\infty} \mathcal{D}(g)(nI_{[0,\frac{1}{n})}) \\ &= F_\mathcal{D}(f) + \lambda F_\mathcal{D}(g). \end{aligned} \tag{3.78}$$

So all that remains for us to prove is to show that this limit actually exists. We will show that the function $nI_{[0,\frac{1}{n})}(x)$ is a bounded approximate identity for our Banach algebra and invoke the results of the previous lemma.

We see that

$$\left(f * nI_{[0,\frac{1}{n})}\right)(x) = f(x)\left(n\widehat{I_{[0,\frac{1}{n})}}(x)\right) + nI_{[0,\frac{1}{n})}(x)\widehat{f}(x). \tag{3.79}$$

If we take the limit as n tends to infinity, then we can see that $nI_{[0,\frac{1}{n})}(x)$ is defined on the empty set and so equates to the zero function while

$$\lim_{n\to\infty} n\widehat{I_{[0,\frac{1}{n})}}(x) = \lim_{n\to\infty} n\int_0^x I_{[0,\frac{1}{n})}(t)dt = 1, \tag{3.80}$$

and so $\left(f * nI_{[0,\frac{1}{n})}\right)(x) \to f(x)$ as $n \to \infty$.

Also, a similar result holds for $\left(nI_{[0,\frac{1}{n})} * f\right)(x)$.

So we have a bounded approximate identity and thus, by the results of the previous lemma, this limit exists, as required. □

We now make a fundamental observation concerning the maps $F_{\mathcal{D}}$ and $\mathcal{D}_F$, which is related to the canonical map from $\mathcal{HH}^1 \to \mathcal{HC}^0$ in the Connes-Tzygan long exact sequence taking the derivation $\mathcal{D}$ to the trace $\tau_{\mathcal{D}}$ where $\tau_{\mathcal{D}}(f) = \mathcal{D}(f)(1)$. This of course relies on the existence of a unit element 1 in our algebra. In our case we do not have a 1 but we do have a bounded approximate identity. See [21] for more details.

Lemma 3.4. *Given elements $\mathcal{D}$ and F as above to induce maps $F_{\mathcal{D}}$ and $\mathcal{D}_F$ respectively and $f, g \in \mathfrak{B}$, we get the following:*

1. $\mathcal{D}_{(F_{\mathcal{D}})} \equiv \mathcal{D}$ *and*
2. $F_{(\mathcal{D}_F)} \equiv F$.

Proof. $F_{(\mathcal{D}_F)} \equiv F$**:** Here we have that

$$\begin{aligned} F_{(\mathcal{D}_F)}(f) &= \lim_{n\to\infty} \mathcal{D}_F(f)(nI_{[0,\frac{1}{n})}), \text{ by the definition in (3.76)}, \\ &= \lim_{n\to\infty} F(f), \text{ as } nI_{[0,\frac{1}{n})} \text{ is a bai}, \\ &= F(f), \end{aligned} \tag{3.81}$$

as required.

$D_{(F_\mathcal{D})} \equiv D$**:**

Note that $\mathcal{D}$ and $\mathcal{D}_{(F_\mathcal{D})}$ are the same derivation if and only if they induce the same element F. Suppose, without loss of generality, that $F_\mathcal{D} \equiv 0$. Then it follows that the unique element $H_\mathcal{D} \in L^\infty(\mathbb{R}_+, \vee)$ on which $F_\mathcal{D}$ depends is trivially equal to zero almost everywhere. In other words, $F_\mathcal{D}$ is a bounded linear functional on $\mathfrak{A}$ and so can be written in the form

$$F_\mathcal{D}(f) = \int_{\mathbb{R}_+} H_\mathcal{D}(t)f(t)dt \tag{3.82}$$

for $f \in \mathfrak{A}$ and $H_\mathcal{D} \in L^\infty(\mathbb{R}_+, \vee)$. Since this holds for arbitrary f it follows that $H_\mathcal{D} = 0$ almost everywhere. From this we can define $D_{F_\mathcal{D}}$ using $H_\mathcal{D}$ and hence it follows that

$$\mathcal{D}_{(F_\mathcal{D})}(f)(g) = \int_0^\infty \int_0^t H_\mathcal{D}(t)f(t)g(s)dsdt = 0. \tag{3.83}$$

So all that remains is to prove that the derivation we started with, $\mathcal{D}$, is in fact 0 itself. The element $F_\mathcal{D}$ is induced by the derivation $\mathcal{D}$, so for $f \in \mathfrak{A}$

$$F_\mathcal{D}(f) = \lim_{n\to\infty} \mathcal{D}(f)(nI_{[0,\frac{1}{n})}) = 0. \tag{3.84}$$

Since this has to hold for arbitrary f we can conclude that $\mathcal{D}(f) \equiv 0$ and hence $\mathcal{D} \equiv 0$, as required. $\square$

What we have shown here is that there is indeed a correspondence between the spaces $\mathfrak{A}^* \cong L^\infty(\mathbb{R}_+, \vee)$ and $\mathcal{HH}^1(\mathbb{R}_+, \vee)$, and so we conclude with the fact that all of the derivations from $L^1(\mathbb{R}_+, \vee)$ into $\mathfrak{A}^* \cong L^\infty(\mathbb{R}_+, \vee)$ are of the form

$$\mathcal{D}(f)(g) = \int_0^\infty \int_0^t H(t)f(t)g(s)dsdt, \tag{3.85}$$

where $H \in L^\infty(\mathbb{R}_+, \vee)$. This completely identifies the first cohomology group of $L^1(\mathbb{R}_+, \vee)$ with coefficients in the dual module $(L^1(\mathbb{R}_+, \vee))^* \cong L^\infty(\mathbb{R}_+, \vee)$, as desired.

Chapter 4

The second simplicial cohomology group of $\ell^1(\mathbb{Z}_+, \vee)$ and $L^1(\mathbb{R}_+, \vee)$

In the last chapter we successfully developed a new approach to calculating the first simplicial cohomology group for the Banach semigroup algebra $\mathfrak{A} = \ell^1(\mathbb{Z}_+, \vee)$ and then adapted this new method to the case where the algebra was replaced by $\mathfrak{B} = L^1(\mathbb{R}_+, \vee)$. The methods we employed confirmed the well-known results that $\mathcal{HH}^1(\mathfrak{A}) \cong 0$ and $\mathcal{HH}^1(\mathfrak{B}) \cong \mathfrak{B}^*$.

The aim of this chapter is to further the work we have done in Chapter 3 and calculate the previously unknown higher dimensional simplicial cohomology groups for the algebra $\mathfrak{B}$. Initially we consider the adaptation of our methods to the second simplicial cohomology group for the algebra $\mathfrak{A}$ before extending this to the locally compact case in $\mathfrak{B}$. This result is already known from the work of Choi, Gourdeau and White in [10] but we are presenting an alternative *method* which will be adaptable to the locally compact case.

First we calculate this similarly to the way we did it in Chapter 3, that is to say by hand, and then adapt this method into a more general mechanism for attacking the general case.

Let A be a Banach algebra. Then the *second simplicial cohomology group of A*, as defined in Section 2.2.2 and particularly equation (2.76), is simply

$$\mathcal{HH}^2(A) = \frac{\mathcal{Z}^2(A)}{\mathcal{B}^2(A)}. \tag{4.1}$$

Instead of computing the second simplicial cohomology group of $\mathfrak{A}$ directly we first look at the *cyclic* cohomology group. We then provide the full solution for the simplicial case via the Connes-Tzygan long exact sequence, the details of which are given in Section 2.2.3.

4.1 The discrete case

Throughout this section we are considering the algebra $\mathfrak{A} = \ell^1(\mathbb{Z}_+, \vee)$ and we aim to show, via the second cyclic cohomology group and the Connes-Tzygan long exact sequence, that the second simplicial cohomology group of $\mathfrak{A}$ is trivial, that is

$$\mathcal{HH}^1(\mathfrak{A}) = \{0\}. \tag{4.2}$$

To begin with we consider the elements of $\mathcal{Z}^2(\mathfrak{A})$. As we did before it is enough to work with generators rather than more general elements of $\mathfrak{A}$.

Thus for a 2-cocycle $\varphi \in \mathcal{Z}^2(\mathfrak{A})$ and generators δ_{a_i} for $i = 0, 1, 2, 3$ defined in Section 2.1 we have that

$$\begin{aligned}(\delta^3\varphi)(\delta_{a_1}, \delta_{a_2}, \delta_{a_3})(\delta_{a_0}) &= \varphi(\delta_{a_2}, \delta_{a_3})(\delta_{a_0} * \delta_{a_1}) - \varphi(\delta_{a_1} * \delta_{a_2}, \delta_{a_3})(\delta_{a_0}) \\ &\quad + \varphi(\delta_{a_1}, \delta_{a_2} * \delta_{a_3})(\delta_{a_0}) - \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_3} * \delta_{a_0}) = 0. \end{aligned} \tag{4.3}$$

Each of the generators $\delta_{a_i}, i = 0, 1, 2, 3$ takes the value 1 at a particular point a_i in $\mathbb{Z}_+$ and is zero for all other points. When we consider a strict ordering these points occur in some order and there are 24 possibilities. In each of these order structures we allow equality to occur between adjacent pairs as this does not change the overall result. Each ordering gives information via the cocycle identity it yields. However it is possible to considerably reduce the number of those we have to consider in order to gain useful information.

For any given $\varphi \in \mathcal{ZC}^2(\mathfrak{A})$, the cyclicity condition given in (2.77) states that any cyclic permutation of the ordering of the three arguments will not change the value of the cyclic 2-cocycle, that is

$$\varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}) = \varphi(\delta_{a_2}, \delta_{a_0})(\delta_{a_1}) = \varphi(\delta_{a_0}, \delta_{a_1})(\delta_{a_2}). \tag{4.4}$$

The effect of this is such that cycling the four arguments around in the element $(\delta^3\varphi)(\delta_{a_1}, \delta_{a_2}, \delta_{a_3})(\delta_{a_0})$ will not result in any new information; we will be able to transform the resulting cocycle identity into the cocycle identity given by the effect of cycling the elements around in $\delta^3\varphi$. Thus we now have only six different identities to consider.

For convenience we also assume a left-right symmetry is present in our identities, that is

$$\varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}) = \varphi(\delta_{a_0}, \delta_{a_2})(\delta_{a_1}). \tag{4.5}$$

It will be shown in the following analysis that this is a sufficient condition for considering all cyclic 2-cocycles. Again the effect of this is that swapping the left and right elements in $(\delta^3\varphi)(\delta_{a_1}, \delta_{a_2}, \delta_{a_3})(\delta_{a_0})$ will not give us any new information; we will once more be able to transform the resulting cocycle identity into the original one.

This halves the remaining number of permutations we actually need to consider. These three orderings must be such that they cannot be transformed into each other via the operations of left-right symmetry or cycling. After a little careful consideration we arrive at the following cases:

1. $a_1 \leq a_2 \leq a_3 \leq a_0$;
2. $a_1 \leq a_2 \leq a_0 \leq a_3$;
3. $a_1 \leq a_3 \leq a_0 \leq a_2$.

So for $\varphi \in \mathcal{ZC}^2(\mathfrak{A})$ we now investigate these three cases.

Case 1: $a_1 \leq a_2 \leq a_3 \leq a_0$

When the arguments are ordered in this way the cocycle identity becomes

$$\begin{aligned}(\delta^3\varphi)(\delta_{a_1}, \delta_{a_2}, \delta_{a_3})(\delta_{a_0}) &= \varphi(\delta_{a_2}, \delta_{a_3})(\delta_{a_0} * \delta_{a_1}) - \varphi(\delta_{a_1} * \delta_{a_2}, \delta_{a_3})(\delta_{a_0}) \\ &\qquad + \varphi(\delta_{a_1}, \delta_{a_2} * \delta_{a_3})(\delta_{a_0}) - \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_3} * \delta_{a_0}) \\ &= \varphi(\delta_{a_2}, \delta_{a_3})(\delta_{a_0}) - \varphi(\delta_{a_2}, \delta_{a_3})(\delta_{a_0}) \\ &\qquad + \varphi(\delta_{a_1}, \delta_{a_3})(\delta_{a_0}) - \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}) \\ &= 0.\end{aligned} \tag{4.6}$$

So from this we deduce that

$$\varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}) = \varphi(\delta_{a_1}, \delta_{a_3})(\delta_{a_0}). \tag{4.7}$$

This implies that if the second argument is not the largest or the smallest then φ is independent of this term, depending entirely on the others. In other words if the smallest generator occurs first and the largest last then we can replace the second term with any other generator falling between the smallest and the largest without affecting the final result.

Case 2: $a_1 \leq a_2 \leq a_0 \leq a_3$

In this instance the cocycle identity becomes

$$\begin{aligned}(\delta^3\varphi)(\delta_{a_1}, \delta_{a_2}, \delta_{a_3})(\delta_{a_0}) &= \varphi(\delta_{a_2}, \delta_{a_3})(\delta_{a_0} * \delta_{a_1}) - \varphi(\delta_{a_1} * \delta_{a_2}, \delta_{a_3})(\delta_{a_0}) \\ &\qquad + \varphi(\delta_{a_1}, \delta_{a_2} * \delta_{a_3})(\delta_{a_0}) - \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_3} * \delta_{a_0}) \\ &= \varphi(\delta_{a_2}, \delta_{a_3})(\delta_{a_0}) - \varphi(\delta_{a_2}, \delta_{a_3})(\delta_{a_0}) \\ &\qquad + \varphi(\delta_{a_1}, \delta_{a_3})(\delta_{a_0}) - \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_3}) \\ &= 0,\end{aligned} \tag{4.8}$$

which gives

$$\varphi(\delta_{a_1}, \delta_{a_3})(\delta_{a_0}) = \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_3}). \tag{4.9}$$

By Case 1 above, we have that $\varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_3}) = \varphi(\delta_{a_1}, \delta_{a_0})(\delta_{a_3})$, because of the independence of φ on the middle term here, and so

$$\varphi(\delta_{a_1}, \delta_{a_3})(\delta_{a_0}) = \varphi(\delta_{a_1}, \delta_{a_0})(\delta_{a_3}). \tag{4.10}$$

This essentially means that when we cycle the terms to put the smallest first we can swap the positions of the two remaining terms to put the biggest on the right hand side. A consequence of this is that the middle term by definition of order is also the middle term as listed in the arguments.

Case 3: $a_1 \leq a_3 \leq a_0 \leq a_2$

The final case gives us that

$$\begin{aligned}(\delta^3\varphi)(\delta_{a_1}, \delta_{a_2}, \delta_{a_3})(\delta_{a_0}) &= \varphi(\delta_{a_2}, \delta_{a_3})(\delta_{a_0} * \delta_{a_1}) - \varphi(\delta_{a_1} * \delta_{a_2}, \delta_{a_3})(\delta_{a_0}) \\ &\qquad + \varphi(\delta_{a_1}, \delta_{a_2} * \delta_{a_3})(\delta_{a_0}) - \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_3} * \delta_{a_0}) \\ &= \varphi(\delta_{a_2}, \delta_{a_3})(\delta_{a_0}) - \varphi(\delta_{a_2}, \delta_{a_3})(\delta_{a_0}) \\ &\qquad + \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}) - \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}) \\ &= 0,\end{aligned} \tag{4.11}$$

which implies that $0 = 0$. This gives us nothing new.

We have shown that we can cycle the arguments of φ in order to place the smallest element first. We can then swap the argument listed second with the right hand argument to put the biggest as the rightmost of the three terms. Finally we can then replace the second (also now the middle) term by anything else that falls in between the biggest and the smallest in the order structure, making φ independent of its middle argument in this manner.

These identities confirm that our assumption of a left-right symmetry being present is correct for all cyclic 2-cocycles, which we will now demonstrate, meaning that we need only consider such elements. For assume that we have generators δ_{a_i} for $i = 0, 1, 2$. Since we have just shown that we can cycle the elements around to make the argument listed first the smallest it is without loss of generality that we assume δ_{a_1} to be the smallest element here. As we can also then swap the positions of the remaining elements around there is no need to specify the order relation between δ_{a_0} and δ_{a_2}.

Now using cyclicity and the cocycle identities obtained above in Cases 1 and 2 we have that

$$\varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}) = \varphi(\delta_{a_1}, \delta_{a_0})(\delta_{a_2}) = \varphi(\delta_{a_0}, \delta_{a_2})(\delta_{a_1}), \tag{4.12}$$

as required.

In order to prove the triviality of the second cyclic cohomology group of $\mathfrak{A}$ we must show that for every cyclic cocycle $\varphi \in \mathcal{ZC}^2(\mathfrak{A})$ there exists a $\psi \in \mathcal{CC}^1(\mathfrak{A}, \mathfrak{A}^*)$ such that

$$\varphi = \delta^2\psi. \tag{4.13}$$

In other words we must be able to cobound every cyclic cocycle φ. Recall by the definition of cyclicity that for ψ to be cyclic here this means that

$$\psi(\delta_{a_1})(\delta_{a_0}) = -\psi(\delta_{a_0})(\delta_{a_1}). \tag{4.14}$$

We now analyse the cyclic coboundary ψ and try to ascertain what exactly it would have to be for a given cyclic cocycle φ.

First we have that

$$\begin{aligned}(\delta^2\psi)(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}) &= \psi(\delta_{a_2})(\delta_{a_0} * \delta_{a_1}) - \psi(\delta_{a_1} * \delta_{a_2})(\delta_{a_0}) + \psi(\delta_{a_1})(\delta_{a_2} * \delta_{a_0}) \\ &= \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}).\end{aligned} \tag{4.15}$$

If the order structure of the three generators here is such that $a_1 < a_2 < a_0$ then it follows that

$$\psi(\delta_{a_1})(\delta_{a_0}) = \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}). \tag{4.16}$$

This assertion makes sense given that φ is independent of δ_{a_2} here; we can simply remove the generator that lies between δ_{a_1} and δ_{a_2} in our given φ to define our ψ.

This rearranges to give

$$(\varphi - \delta^2\psi)(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}) = 0, \tag{4.17}$$

and if the identified ψ does indeed cobound our given φ then (4.17) must hold for any other order structure placed on the arguments.

Now assume the order structure is given by $a_1 < a_0 < a_2$. Then (4.17) becomes

$$\begin{aligned}(\varphi - \delta^2\psi)(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}) &= \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}) - (\delta^2\psi)(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}) \\ &= \varphi(\delta_{a_1}, \delta_{a_0})(\delta_{a_2}) - (\psi(\delta_{a_2})(\delta_{a_0}) - \psi(\delta_{a_2})(\delta_{a_0}) \\ &\qquad +\psi(\delta_{a_1})(\delta_{a_2})) \\ &= \varphi(\delta_{a_1}, \delta_{a_0})(\delta_{a_2}) - \psi(\delta_{a_1})(\delta_{a_2}) \\ &= \varphi(\delta_{a_1}, \delta_{a_0})(\delta_{a_2}) - \varphi(\delta_{a_1}, \delta_{a_0})(\delta_{a_2}) = 0.\end{aligned} \tag{4.18}$$

Because of cyclicity we can now conclude that (4.17) holds for any distinct generators in $\mathfrak{A}$ here. We now consider what happens if the terms are *not* all distinct.

From our assertion in (4.16) it seems to follow that

$$\psi(\delta_{a_1})(\delta_{a_0}) = \varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_0}). \tag{4.19}$$

This is acceptable given the cocycle identities from earlier. In fact we can show that $\varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_0})$ is equivalent to $\varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_0})$ and also $\varphi(\delta_{a_1}, \delta_{a_0})(\delta_{a_0})$.

Begin with the order structure on the generators defined by $a_1 \le a_2 \le a_0$. Then we have that

$$\begin{aligned}0 &= (\delta^3\varphi)(\delta_{a_1}, \delta_{a_1}, \delta_{a_2})(\delta_{a_2}) \\ &= \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_2}) - \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_2}) + \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_2}) - \varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_2}),\end{aligned} \tag{4.20}$$

and so $\varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_2}) = \varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_2})$.

We also have that

$$\begin{aligned}0 &= (\delta^3\varphi)(\delta_{a_1}, \delta_{a_1}, \delta_{a_2})(\delta_{a_0}) \\ &= \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}) - \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}) + \varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}) - \varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_0}),\end{aligned} \tag{4.21}$$

and hence $\varphi(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}) = \varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_0}) = \varphi(\delta_{a_1}, \delta_{a_0})(\delta_{a_0})$ by (4.20).

Now we are in a position to analyse the case when we have equality within the order structure of our generators.

If we have that $a_1 = a_2 < a_0$ then (4.17) becomes

$$\begin{aligned}(\varphi - \delta^2\psi)(\delta_{a_1}, \delta_{a_1})(\delta_{a_0}) &= \varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_0}) - (\delta^2\psi)(\delta_{a_1}, \delta_{a_1})(\delta_{a_0}) \\ &= \varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_0}) - (\psi(\delta_{a_1})(\delta_{a_0}) - \psi(\delta_{a_1})(\delta_{a_0}) \\ &\qquad +\psi(\delta_{a_1})(\delta_{a_0})) \\ &= \varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_0}) - \psi(\delta_{a_1})(\delta_{a_0}) \\ &= \varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_0}) - \varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_0}) = 0.\end{aligned} \tag{4.22}$$

On the other hand if we have the order structure defined as $a_1 < a_2 = a_0$ then (4.17) becomes

$$\begin{aligned}(\varphi - \delta^2\psi)(\delta_{a_1}, \delta_{a_0})(\delta_{a_0}) &= \varphi(\delta_{a_1}, \delta_{a_0})(\delta_{a_0}) - (\delta^2\psi)(\delta_{a_1}, \delta_{a_0})(\delta_{a_0}) \\ &= \varphi(\delta_{a_1}, \delta_{a_0})(\delta_{a_0}) - (\psi(\delta_{a_0})(\delta_{a_0}) - \psi(\delta_{a_0})(\delta_{a_0}) \\ &\qquad +\psi(\delta_{a_1})(\delta_{a_0})) \\ &= \varphi(\delta_{a_1}, \delta_{a_0})(\delta_{a_0}) - \psi(\delta_{a_1})(\delta_{a_0}) \\ &= \varphi(\delta_{a_1}, \delta_{a_0})(\delta_{a_0}) - \varphi(\delta_{a_1}, \delta_{a_0})(\delta_{a_0}) = 0,\end{aligned} \tag{4.23}$$

following the same logic.

Everything works so far, but difficulties arise in the case when all three generators are equal, i.e. the order structure is defined by $a_1 = a_2 = a_0$. In this case (4.17) becomes

$$\begin{aligned}(\varphi - \delta^2\psi)(\delta_{a_1}, \delta_{a_1})(\delta_{a_1}) &= \varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_1}) - (\delta^2\psi)(\delta_{a_1}, \delta_{a_1})(\delta_{a_1}) \\ &= \varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_1}) - (\psi(\delta_{a_1})(\delta_{a_1}) - \psi(\delta_{a_1})(\delta_{a_1}) \\ &\qquad +\psi(\delta_{a_1})(\delta_{a_1})) \\ &= \varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_1}) - \psi(\delta_{a_1})(\delta_{a_1}).\end{aligned} \tag{4.24}$$

Since ψ is cyclic, we must have that $\psi(\delta_{a_1})(\delta_{a_1}) = -\psi(\delta_{a_1})(\delta_{a_1})$ and so this leads us to conclude that $\psi(\delta_{a_1})(\delta_{a_1}) = 0$. Thus for our identified ψ to cobound our given φ it follows that $\varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_1}) = 0$. It is not always the case that φ is zero on such arguments since the only real condition this must satisfy is

$$\begin{aligned}(\delta^3\varphi)(\delta_{a_1},\delta_{a_1},\delta_{a_1})(\delta_{a_1}) &= \varphi(\delta_{a_1},\delta_{a_1})(\delta_{a_1}) - \varphi(\delta_{a_1},\delta_{a_1})(\delta_{a_1}) \\ &\quad + \varphi(\delta_{a_1},\delta_{a_1})(\delta_{a_1}) - \varphi(\delta_{a_1},\delta_{a_1})(\delta_{a_1}) = 0, \end{aligned} \tag{4.25}$$

which it clearly does so for any choice of value for $\varphi(\delta_{a_1},\delta_{a_1})(\delta_{a_1})$. Indeed, one perfectly viable choice for φ is

$$\varphi(\delta_{a_1},\delta_{a_2})(\delta_{a_0}) = \begin{cases} 1, & \delta_{a_1} = \delta_{a_2} = \delta_{a_0}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.26}$$

This φ is clearly an element of $\mathcal{ZC}^2(l^1((\mathbb{Z}_+,\vee)))$, and so the ψ we have identified does *not* cobound this given φ, as $(\varphi - \delta^2\psi)(\delta_{a_1},\delta_{a_1})(\delta_{a_1}) \neq 0$ necessarily.

We now turn to the Connes-Tzygan long exact sequence, given in Section 2.2.3, in order to identify a way around this problem. We are going to show that the triviality of the second simplicial cohomology group of $\mathfrak{A}$ follows from the surjectivity of the map from $\mathcal{HC}^0(\mathfrak{A})$ into $\mathcal{HC}^2(\mathfrak{A})$ in this sequence.

Recall that for our algebra $\mathfrak{A}$ the start of the Connes-Tzygan long exact sequence is

$$0 \to \mathcal{HC}^1(\mathfrak{A}) \to \mathcal{HH}^1(\mathfrak{A}) \to \mathcal{HC}^0(\mathfrak{A}) \to \mathcal{HC}^2(\mathfrak{A}) \to \mathcal{HH}^2(\mathfrak{A}) \to \mathcal{HC}^1(\mathfrak{A}) \to \cdots . \tag{4.27}$$

We know that $\mathcal{HH}^1(\mathfrak{A}) = 0$, and hence $\mathcal{HC}^1(\mathfrak{A}) = 0$ also. This is because $\mathcal{B}^1(\mathfrak{A})$ and $\mathcal{BC}^1(\mathfrak{A})$ are equal to 0, forcing $\mathcal{HH}^1(\mathfrak{A}) = \mathcal{Z}^1(\mathfrak{A})$ and $\mathcal{HC}^1(\mathfrak{A}) = \mathcal{ZC}^1(\mathfrak{A})$, with $\mathcal{ZC}^1(\mathfrak{A}) \subseteq \mathcal{Z}^1(\mathfrak{A})$. Thus we have that $\mathcal{ZC}^1(\mathfrak{A}) = 0$ and so the Connes-Tzygan long exact sequence becomes

$$0 \to 0 \to 0 \to \mathcal{HC}^0(\mathfrak{A}) \to \mathcal{HC}^2(\mathfrak{A}) \to \mathcal{HH}^2(\mathfrak{A}) \to 0 \to \cdots . \tag{4.28}$$

Notice that

$$\begin{aligned}\mathcal{HH}^2(\mathfrak{A}) &= \ker\left(\mathcal{HH}^2(\mathfrak{A}) \to 0\right) \\ &= \operatorname{im}\left(\mathcal{HC}^2(\mathfrak{A}) \to \mathcal{HH}^2(\mathfrak{A})\right) \text{ by exactness,}\end{aligned} \tag{4.29}$$

and so in order to show that the second simplicial cohomology group for $\mathfrak{A}$ is trivial we need to show that

$$\operatorname{im}\big(\mathcal{HC}^2(\mathfrak{A}) \to \mathcal{HH}^2(\mathfrak{A})\big) = 0. \tag{4.30}$$

This is true if and only if

$$\begin{aligned} \mathcal{HC}^2(\mathfrak{A}) &= \ker\big(\mathcal{HC}^2(\mathfrak{A}) \to \mathcal{HH}^2(\mathfrak{A})\big) \\ &= \operatorname{im}\big(\mathcal{HC}^0(\mathfrak{A}) \to \mathcal{HC}^2(\mathfrak{A})\big) \text{ by exactness,} \end{aligned} \tag{4.31}$$

i.e. that the map from $\mathcal{HC}^0(\mathfrak{A})$ into $\mathcal{HC}^2(\mathfrak{A})$ is a surjection.

In the Connes-Tzygan long exact sequence, the map from $\mathcal{HC}^0(\mathfrak{A})$ to $\mathcal{HC}^2(\mathfrak{A})$ is given by $[\tau] \to [\widetilde{\tau}]$, where, for generators defined as above,

$$\tau(\delta_{a_1} * \delta_{a_2} * \delta_{a_0}) = \widetilde{\tau}(\delta_{a_1}, \delta_{a_2})(\delta_{a_0}). \tag{4.32}$$

Notice that because of the max multiplication imposed on the underlying semigroup of our algebra, we have that $\widetilde{\tau}(\delta_{a_1}, \delta_{a_1})(\delta_{a_1}) = \tau(\delta_{a_1} * \delta_{a_1} * \delta_{a_1}) = \tau(\delta_{a_1})$. The function $[\tau] \to [\widetilde{\tau}]$ is a well-defined map between equivalence classes of the quotient spaces $\mathcal{HC}^0(\mathfrak{A})$ and $\mathcal{HC}^2(\mathfrak{A})$.

Thus we have that given an element $\varphi \in \mathcal{ZC}^2(\mathfrak{A})$ we are now able to obtain another element in $\mathcal{ZC}^2(\mathfrak{A})$ via the Connes-Tzygan long exact sequence which we label as $\widetilde{\tau}_\varphi$. Since this space is closed under addition we have that $\varphi - \widetilde{\tau}_\varphi \in \mathcal{ZC}^2(\mathfrak{A})$ and

$$\begin{aligned} (\varphi - \widetilde{\tau}_\varphi)(\delta_{a_1}, \delta_{a_1})(\delta_{a_1}) &= \varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_1}) - \widetilde{\tau}_\varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_1}) \\ &= \varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_1}) - \tau_\varphi(\delta_{a_1}). \end{aligned} \tag{4.33}$$

We are free to set $\tau_\varphi(\delta_{a_1}) = \varphi(\delta_{a_1}, \delta_{a_1})(\delta_{a_1})$ as this is still a trace function, and so we have that

$$(\varphi - \widetilde{\tau}_\varphi)(\delta_{a_1}, \delta_{a_1})(\delta_{a_1}) = 0. \tag{4.34}$$

We can cobound $\varphi - \widetilde{\tau}_\varphi$ in $\mathcal{ZC}^2(\mathfrak{A})$ by our identified coboundary ψ as the problematical case of what happens when all three arguments are equal is easily satisfied here, i.e.

$$(\varphi - \widetilde{\tau}_\varphi - \delta^2\psi)(\delta_{a_1}, \delta_{a_1})(\delta_{a_1}) = 0. \tag{4.35}$$

Thus we can conclude that

$$\varphi - \widetilde{\tau}_\varphi = \delta^2\psi, \tag{4.36}$$

that is φ and $\widetilde{\tau}_\varphi$ differ by an element of $\mathcal{BC}^2(\mathfrak{A})$. Since we have that

$$\mathcal{HC}^2(\mathfrak{A}) = \frac{\mathcal{ZC}^2(\mathfrak{A})}{\mathcal{BC}^2(\mathfrak{A})} \tag{4.37}$$

it must be that φ and $\widetilde{\tau}_\varphi$ are in the same equivalence class in $\mathcal{HC}^2(\mathfrak{A})$.

However we know that $\widetilde{\tau}_\varphi$ is in the equivalence class $[\widetilde{\tau}_\varphi]$, which has the preimage $[\tau_\varphi]$ in $\mathcal{HC}^0(\mathfrak{A})$. We chose φ arbitrarily and every φ belongs to an equivalence class of $\mathcal{HC}^2(\mathfrak{A})$, so it follows that the equivalence class containing our given φ is exactly $[\widetilde{\tau}_\varphi]$, which does have a preimage in $\mathcal{HC}^0(\mathfrak{A})$.

Hence we have shown that the map from $\mathcal{HC}^0(\mathfrak{A})$ into $\mathcal{HC}^2(\mathfrak{A})$ is indeed a surjection and therefore we have, finally, that

$$\mathcal{HH}^2(\mathfrak{A}) = 0, \tag{4.38}$$

i.e. that the second simplicial cohomology group for the algebra $\mathfrak{A} = \ell^1(\mathbb{Z}_+, \vee)$ is indeed trivial.

Note that a consequence of our analysis is that the second cyclic cohomology group of $\mathfrak{A}$ has been shown to be *not* trivial. It can be seen that the Connes-Tzygan long exact sequence in this case reduces to give the section

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{HC}^0(\mathfrak{A}) \longrightarrow \mathcal{HC}^2(\mathfrak{A}) \longrightarrow 0 \longrightarrow \cdots .$$

Following a similar logic to that found in [22] we have that

$$\begin{aligned}\mathcal{HC}^2(\mathfrak{A}) &= \ker\left(\mathcal{HC}^2(\mathfrak{A}) \to 0\right) \\ &= \operatorname{im}\left(\mathcal{HC}^0(\mathfrak{A}) \to \mathcal{HC}^2(\mathfrak{A})\right), \text{ by exactness,}\end{aligned} \tag{4.39}$$

implying that the map from $\mathcal{HC}^0(\mathfrak{A})$ into $\mathcal{HC}^2(\mathfrak{A})$ is surjective, and

$$\begin{aligned} 0 &= \operatorname{im}\left(0 \to \mathcal{HC}^0(\mathfrak{A})\right) \\ &= \ker\left(\mathcal{HC}^0(\mathfrak{A}) \to \mathcal{HC}^2(\mathfrak{A})\right), \text{ by exactness,} \end{aligned} \tag{4.40}$$

implying that the map from $\mathcal{HC}^0(\mathfrak{A})$ into $\mathcal{HC}^2(\mathfrak{A})$ is injective. Thus this map is an isomorphism and hence the second cyclic cohomology group on $\mathfrak{A}$ is isomorphic to the space of traces on $\mathfrak{A}$. This is what we expect given the problem we encountered in trying to cobound a cyclic cocycle.

We now turn our attentions to the locally compact case and attempt to adapt these methods in solving this problem.

4.2 The locally compact case

In considering the locally compact case, where the algebra under consideration is now $\mathfrak{B} = L^1(\mathbb{R}_+, \vee)$, we use the method from the discrete case described in Section 4.1 to demonstrate the triviality of the second simplicial cohomology group of $\mathfrak{B}$. Therefore, as before, we first consider the second *cyclic* cohomology group of $\mathfrak{B}$ and then use the Connes-Tzygan long exact sequence to obtain the final result.

First we prove a vital result regarding the first cyclic cohomology group of $\mathfrak{B}$.

Lemma 4.1. *The first cyclic cohomology group of $\mathfrak{B}$ is trivial.*

Proof. Recall from Section 3.3 that all simplicial derivations, which are the elements of $\mathcal{Z}^1(\mathfrak{B})$, are of the form

$$\mathcal{D}_F(f)(g) = \int_0^\infty \int_0^t H(t)f(t)g(s)dsdt. \tag{4.41}$$

Recall also that $\mathcal{B}^1(\mathfrak{B}) \equiv 0$, and so $\mathcal{HH}^1(\mathfrak{B})$ is identically $\mathcal{Z}^1(\mathfrak{B})$. The elements of $\mathcal{ZC}^1(\mathfrak{B})$ here then are those derivations $\mathcal{D}_F$ that are also cyclic and so satisfy the equation

$$\mathcal{D}_F(f)(g) = -\mathcal{D}_F(g)(f) \tag{4.42}$$

which occurs if and only if for all $f, g, \in \mathfrak{B}$

$$\mathcal{D}_F(f)(g) + \mathcal{D}_F(g)(f) = 0. \tag{4.43}$$

Now it follows that

$$\begin{aligned}
0 &= \mathcal{D}_F(f)(g) + \mathcal{D}_F(g)(f) \\
&= \int_0^\infty \int_0^t H(t)f(t)g(s)dsdt + \int_0^\infty \int_0^t H(t)g(t)f(s)dsdt \\
&= \int_0^\infty H(t)f(t)\widehat{g}(t)dt + \int_0^\infty H(t)g(t)\widehat{f}(t)dt \\
&= \int_0^\infty H(t)\left[f(t)\widehat{g}(t) + g(t)\widehat{f}(t)\right]dt \\
&= \int_0^\infty H(t)\left(f * g\right)(t)dt.
\end{aligned} \tag{4.44}$$

Since f and g are arbitrary this must hold also for when g is a bounded approximate identity in $\mathfrak{B}$, i.e. when $g = e_\alpha$. Then we have that

$$\begin{aligned}
0 &= \int_0^\infty H(t)\left(f * e_\alpha\right)(t)dt \\
&= \int_0^\infty H(t)f(t)dt = 0,
\end{aligned} \tag{4.45}$$

and this holds if and only if $H(t)f(t) = 0$ almost everywhere. Since f is arbitrary, it forces $H(t) = 0$ almost everywhere. Then $\mathcal{D}_F \equiv \mathcal{D}_0$, which is the trivial zero derivation. Thus it follows that $\mathcal{D}_F$ is cyclic if and only if $\mathcal{D}_F \equiv \mathcal{D}_0$, making the first cyclic cohomology group of $\mathfrak{B}$, $\mathcal{HC}^1(\mathfrak{B}) \equiv \mathcal{ZC}^1(\mathfrak{B})$ trivially zero as required. □

We now proceed to show that the second cyclic cohomology group of $\mathfrak{B}$ is also trivial.

Let $\varphi \in \mathcal{ZC}^2(\mathfrak{B})$ be a 2-cocycle. As in Section 3.3 it is enough for us to consider cocycles and coboundaries acting on normalised indicator functions.

Notation. Throughout the rest of this section we will continue to denote by I_i

in $\mathfrak{B}$, for $i = 0, 1, 2, 3$, the normalised indicator function

$$I_i(x) = \begin{cases} \frac{1}{b_i - a_i}, & x \in [a_i, b_i], \\ 0, & \text{else}, \end{cases} \tag{4.46}$$

where $a_i < b_i$ and $a_i, b_i \in \mathbb{R}_+$. We say normalised because $\|I_i\| = 1$ for $i = 0, 1, 2, 3$.

However it will become necessary for convenience to rewrite this slightly as

$$I_i(x) = \frac{\chi_{[a_i,b_i]}(x)}{b_i - a_i}, \text{ where } x \in \mathbb{R}_+, \tag{4.47}$$

where $a_i, b_i \in \mathbb{R}_+$ such that $a_i < b_i$ and $\chi_{[a_i,b_i]}$ denotes the characteristic function defined by

$$\chi_{[a_i,b_i]}(x) = \begin{cases} 1, x \in [a_i, b_i], \\ 0, \text{ else}. \end{cases} \tag{4.48}$$

Using *normalised* indicator functions is useful as it ensures that the multiplication of two such elements is straightforward, namely

$$(I_1 * I_2)(x) = \begin{cases} I_1(x), & I_2 \ll I_1, \\ I_2(x), & I_1 \ll I_2, \end{cases} \tag{4.49}$$

where the notation $\ll$ is as defined in Section 3.3. Note we are still able to write functions as sums of indicator functions despite the scaling involved.

As in Section 3.3 we will first consider the case when the intervals on which our indicator functions are defined *do not* overlap and then progress to the case where they do.

Exactly as with the discrete case there are 24 possible orderings of the four arguments in $\delta^3\varphi$ and using the same logic found there we are once again able to reduce these to the same three cases that we have to consider.

The cocycle identity on normalised indicator functions which φ has to satisfy here is

$$\begin{aligned} 0 &= (\delta^3\varphi)(I_1, I_2, I_3)(I_0) \\ &= \varphi(I_2, I_3)(I_0 * I_1) - \varphi(I_1 * I_2, I_3)(I_0) + \varphi(I_1, I_2 * I_3)(I_0) - \varphi(I_1, I_2)(I_3 * I_0). \end{aligned} \tag{4.50}$$

Case 1: $I_1 \ll I_2 \ll I_3 \ll I_0$

Under this order structure the cocycle identity becomes

$$\begin{aligned} 0 &= (\delta^3\varphi)(I_1, I_2, I_3)(I_0) \\ &= \varphi(I_2, I_3)(I_0) - \varphi(I_2, I_3)(I_0) + \varphi(I_1, I_3)(I_0) - \varphi(I_1, I_2)(I_0) \\ &= \varphi(I_1, I_3)(I_0) - \varphi(I_1, I_2)(I_0) \end{aligned} \tag{4.51}$$

which then leads to

$$\varphi(I_1, I_2)(I_0) = \varphi(I_1, I_3)(I_0). \tag{4.52}$$

Once again this implies that when the second term is not the smallest or the largest in the order structure φ is independent of this term and so we can replace it with any other middle term.

Case 2: $I_1 \ll I_2 \ll I_0 \ll I_3$

The cocycle identity in this case becomes

$$\begin{aligned} 0 &= (\delta^3\varphi)(I_1, I_2, I_3)(I_0) \\ &= \varphi(I_2, I_3)(I_0) - \varphi(I_2, I_3)(I_0) + \varphi(I_1, I_3)(I_0) - \varphi(I_1, I_2)(I_3) \\ &= \varphi(I_1, I_3)(I_0) - \varphi(I_1, I_2)(I_3), \end{aligned} \tag{4.53}$$

but $\varphi(I_1, I_2)(I_3) = \varphi(I_1, I_0)(I_3)$ by Case 1 due to the ordering of the terms here, which implies that

$$\varphi(I_1, I_3)(I_0) = \varphi(I_1, I_0)(I_3). \tag{4.54}$$

Thus, as in the discrete case, once we have cycled the arguments to make the first term listed the smallest we are allowed to exchange the remaining two terms without affecting the value of φ.

This means we can always manufacture the order in which the indicator functions are listed as the arguments of φ to ensure that the smallest is first and the largest last. Then we are free to exchange the term in the middle with any other normalised indicator function provided it remains between the smallest and largest in the order structure.

Case 3: $I_0 \ll I_3 \ll I_0 \ll I_2$

As before we glean no new information from the final case, as the cocycle identity becomes

$$0 = (\delta^3\varphi)(I_1, I_2, I_3)(I_0) = \varphi(I_2, I_3)(I_0) - \varphi(I_2, I_3)(I_0) + \varphi(I_1, I_2)(I_0) + \varphi(I_1, I_2)(I_0), \tag{4.55}$$

which implies that $0 = 0$.

So far this analysis follows the same lines as that of the discrete case. Thus, like in the discrete case, we are going to investigate cobounding our given φ by the element $\psi \in \mathcal{C}^1(\mathfrak{B}, \mathfrak{B}^*)$ given by

$$\psi(I_1)(I_0) = \varphi(I_1, I_2)(I_0), \tag{4.56}$$

where $I_1 \ll I_2 \ll I_0$.

Our aim in the following investigation of ψ is to discover exactly what properties ψ would need to have *if* it cobounded our given φ. We are unsure at this stage whether or not ψ does cobound φ.

Now we have to investigate ψ and show that it does indeed cobound our given 2-cocycle φ.

The notation we have used up to this point for the normalised indicator functions has been useful, but it will present some issues in the following arguments. Thus we will now introduce a small rewriting where I_1 is now simply denoted by I, I_2 by J and I_0 by K.

Following the example found in Equation (3.34), given our φ and ψ we can use the Riesz Representation Theorem for L^p spaces once again to write them as

$$\varphi(I,J)(K) = \int_{\mathbb{R}_+^3} \Phi(s,t,u)I(s)J(t)K(u)dsdtdu, \tag{4.57}$$

$$\psi(I)(K) = \int_{\mathbb{R}_+^2} \Psi(s,u)I(s)K(u)dsdu, \tag{4.58}$$

for the indicator functions $I \equiv I_1$, $J \equiv I_2$, $K \equiv I_0$ and the bounded functions $\Phi \in L^\infty(\mathbb{R}_+^3, \vee)$, $\Psi \in L^\infty(\mathbb{R}_+^2, \vee)$. For consider φ and define the multilinear functional $\widetilde{\varphi}$ on $L^1(\mathbb{R}_+, \vee) \times L^1(\mathbb{R}_+, \vee) \times L^1(\mathbb{R}_+, \vee) \cong L^1(\mathbb{R}_+^3, \vee)$ as

$$\widetilde{\varphi}((I,J,K)) = \varphi(I,J)(K). \tag{4.59}$$

As $\widetilde{\varphi}$ is now a linear functional on $L^1(\mathbb{R}_+^3, \vee)$ the Reisz Representation Theorem for L^p spaces now allows us to write this as

$$\widetilde{\varphi}((I,J,K)) = \int_{\mathbb{R}_+} \Phi(s,t,u)(I,J,K)(s,t,u)dsdtdu \tag{4.60}$$

for the triple $(s,t,u) \in \mathbb{R}_+^3$, with $\Phi \in L^\infty(\mathbb{R}_+^3, \vee)$. Putting these last two equations together and noting that $(I,J,K)(s,t,u) = I(s)J(t)K(u)$ leads us to the desired result. An identical argument applies for the integral representation of ψ.

Then we have

$$\int_{\mathbb{R}_+^2} \Psi(s,u)I(s)K(u)dsdu = \int_{\mathbb{R}_+^3} \Phi(s,t,u)I(s)J(t)K(u)dsdtdu. \tag{4.61}$$

The right-hand-side of the above equation becomes

$$\int_{\mathbb{R}_+^2} I(s)K(u)\left(\int_0^\infty \Phi(s,t,u)J(t)dt\right)dsdu, \tag{4.62}$$

but

$$\int_0^\infty \Phi(s,t,u)J(t)dt = \int_0^\infty \frac{\chi_{[a_2,b_2]}(t)}{(b_2-a_2)}\Phi(s,t,u) = \int_{a_2}^{b_2} \frac{\Phi(s,t,u)}{(b_2-a_2)}dt, \tag{4.63}$$

and so

$$\int_{\mathbb{R}^2_+} \Psi(s,u)I(s)K(u)dsdu = \int_{\mathbb{R}^2_+} \left(\int_{a_2}^{b_2} \frac{\Phi(s,t,u)}{(b_2 - a_2)} dt \right) I(s)K(u)dsdu. \tag{4.64}$$

From this and the dependence of Ψ on s and u we claim that

$$\Psi(s,u) = \int_s^u \frac{\Phi(s,t,u)}{(u-s)} dt, \tag{4.65}$$

where $s < u$. We then also define $\Psi(u,s) = -\Psi(s,u)$ in order to satisfy the cyclicity required of ψ (which incidentally means that $\Phi(s,t,u) = \Phi(u,t,s)$, preserving the cyclicity of φ). This would then make (4.61)

$$\int_{\mathbb{R}^2_+} \int_s^u \frac{\Phi(s,t,u)}{(u-s)} dt I(s)K(u)dsdu = \int_{\mathbb{R}^3_+} \Phi(s,t,u)I(s)J(t)K(u)dsdtdu. \tag{4.66}$$

We are now left with the task of showing that this is indeed true.

We appeal to the method used in Section 3.3 to calculate the first simplicial cohomology group of $\mathfrak{B}$ and divide up the intervals on which I and K are defined into n equal subintervals:

$$I = \frac{\chi_{[a_1,b_1]}}{(b_1 - a_1)} = \frac{1}{n}\sum_{i=1}^n I_i, \text{ where } I_i = \frac{\chi_{[\alpha_i^I, \beta_i^I]}}{(\beta_i^I - \alpha_i^I)}, \tag{4.67}$$

$$K = \frac{\chi_{[a_3,b_3]}}{(b_3 - a_3)} = \frac{1}{n}\sum_{k=1}^n K_k, \text{ where } K_k = \frac{\chi_{[\alpha_k^K, \beta_k^K]}}{(\beta_k^K - \alpha_k^K)}, \tag{4.68}$$

where $[\alpha_i^I, \beta_i^I]$ denotes the i^{th} subinterval of $[a_1, b_1]$ and $[\alpha_k^K, \beta_k^K]$ denotes the k^{th} subinterval of $[a_0, b_0]$. In other words,

$$\alpha_i^I = a_1 + (i-1)\left(\frac{b_1 - a_1}{n}\right), \quad \beta_i^I = a_1 + i\left(\frac{b_1 - a_1}{n}\right), \tag{4.69}$$

$$\alpha_k^K = a_0 + (k-1)\left(\frac{b_0 - a_0}{n}\right), \quad \beta_k^K = a_0 + i\left(\frac{b_0 - a_0}{n}\right). \tag{4.70}$$

As a check, notice that

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} I_i &= \frac{1}{n}\sum_{i=1}^{n} \frac{\chi_{[\alpha_i^I, \beta_i^I]}}{(\beta_i^I - \alpha_i^I)} \\
&= \frac{1}{n}\left[\frac{\chi_{[\alpha_1^I, \beta_1^I]}}{(\beta_1^I - \alpha_1^I)} + \frac{\chi_{[\alpha_2^I, \beta_2^I]}}{(\beta_2^I - \alpha_2^I)} + \cdots + \frac{\chi_{[\alpha_n^I, \beta_n^I]}}{(\beta_n^I - \alpha_n^I)}\right] \\
&= \frac{1}{n}\left[\frac{n}{(b_1 - a_1)}\left\{\chi_{[\alpha_1^I, \beta_1^I]} + \chi_{[\alpha_2^I, \beta_2^I]} + \cdots + \chi_{[\alpha_n^I, \beta_n^I]}\right\}\right], \qquad (4.71) \\
&\qquad \text{as } \beta_i^I - \alpha_i^I = \frac{b_1 - a_1}{n} \quad \forall i = 1, \ldots, n \\
&= \frac{1}{b_1 - a_1}\left\{\chi_{[a_1, b_1]}\right\} \\
&= I.
\end{aligned}$$

Consider the left-hand-side of (4.66). By dividing the intervals up as described, we can rewrite this as

$$\begin{aligned}
\frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}_+^2}\Bigg[\int_s^{\gamma_i^I} \frac{\varPhi(s,t,u)}{(u-s)}dt + \int_{\gamma_i^I}^{\gamma_k^K} \frac{\varPhi(s,t,u)}{(u-s)}dt& \\
+ \int_{\gamma_k^K}^{u} \frac{\varPhi(s,t,u)}{(u-s)}dt\Bigg] I_i(s)K_k(u)dsdu,& \qquad (4.72)
\end{aligned}$$

where

$$\gamma_i^I = \min\left\{\frac{a_2 + \beta_i^I}{2}, \beta_i^I + \frac{b_1 - a_1}{n}\right\}, \qquad (4.73)$$

$$\gamma_k^K = \min\left\{\frac{b_2 + \alpha_k^K}{2}, \alpha_k^K - \frac{b_0 - a_0}{n}\right\}. \qquad (4.74)$$

Looking at the first term here, which is

$$\frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}_+^2}\int_s^{\gamma_i^I} \frac{\varPhi(s,t,u)}{(u-s)}dt I_i(s)K_k(u)dsdu, \qquad (4.75)$$

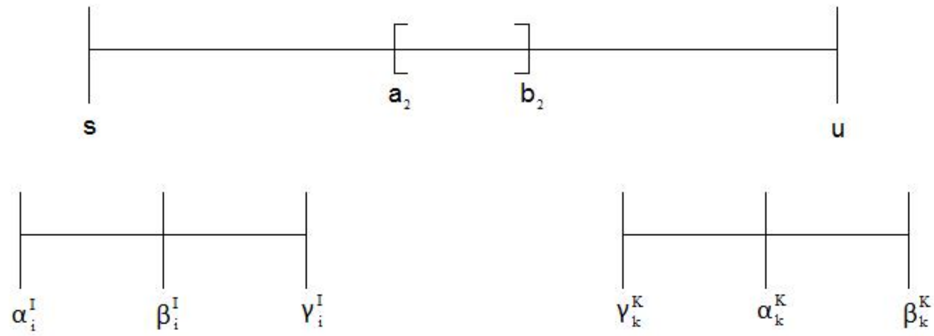


Figure 4.1: An illustration of our intervals and where the end points lie in relation to each other

we have that

$$\begin{aligned}
&\left|\frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}_+^2}\int_{s}^{\gamma_i^I}\frac{\Phi(s,t,u)}{(u-s)}dtI_i(s)K_k(u)dsdu\right| \\
&\leq \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}_+^2}\frac{\|\Phi\|\,(\gamma_i^I-s)}{(u-s)}I_i(s)K_k(u)dsdu \\
&\leq \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}_+^2}\frac{\|\Phi\|\,2(b_1-a_1)}{(b_2-a_2)n}I_i(s)K_k(u)dsdu \\
&= \frac{2\,\|\Phi\|\,(b_1-a_1)}{n(b_2-a_2)} \to 0 \text{ as } n\to\infty.
\end{aligned} \tag{4.76}$$

A similar argument follows for the third term, meaning that

$$\frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}_+^2}\int_{\gamma_k^K}^{u}\frac{\Phi(s,t,u)}{(u-s)}dtI_i(s)K_k(u)dsdu \to 0 \text{ as } n\to\infty. \tag{4.77}$$

This leaves us with only the second term to consider, which is

$$\frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}_+^2}\int_{\gamma_i^I}^{\gamma_k^K}\frac{\Phi(s,t,u)}{(u-s)}dtI_i(s)K_k(u)dsdu. \tag{4.78}$$

We rewrite this slightly as

$$\frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}_+^2}\frac{(\gamma_k^K-\gamma_i^I)}{(u-s)}\int_{\gamma_i^I}^{\gamma_k^K}\frac{\Phi(s,t,u)}{(\gamma_k^K-\gamma_i^I)}dtI_i(s)K_k(u)dsdu, \tag{4.79}$$

and observe that

$$\begin{aligned}\frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}_+^2}\frac{(\gamma_k^K-\gamma_i^I)}{(u-s)}\int_{\gamma_i^I}^{\gamma_k^K}\frac{\Phi(s,t,u)}{(\gamma_k^K-\gamma_i^I)}dtI_i(s)K_k(u)dsdu \\ -\frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}_+^2}\int_{\gamma_i^I}^{\gamma_k^K}\frac{\Phi(s,t,u)}{(\gamma_k^K-\gamma_i^I)}dtI_i(s)K_k(u)dsdu \\ =\frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}_+^2}\left(\frac{(\gamma_k^K-\gamma_i^I)}{(u-s)}-1\right)\int_{\gamma_i^I}^{\gamma_k^K}\frac{\Phi(s,t,u)}{(\gamma_k^K-\gamma_i^I)}dtI_i(s)K_k(u)dsdu.\end{aligned} \tag{4.80}$$

It then follows that

$$\begin{aligned}&\left|\frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}_+^2}\left(\frac{(\gamma_k^K-\gamma_i^I)}{(u-s)}-1\right)\int_{\gamma_i^I}^{\gamma_k^K}\frac{\Phi(s,t,u)}{(\gamma_k^K-\gamma_i^I)}dtI_i(s)K_k(u)dsdu\right| \\ &\le\frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\left|\int_{\mathbb{R}_+^2}\left(\frac{(\gamma_k^K-\gamma_i^I)}{(u-s)}-1\right)\int_{\gamma_i^I}^{\gamma_k^K}\frac{\Phi(s,t,u)}{(\gamma_k^K-\gamma_i^I)}dtI_i(s)K_k(u)dsdu\right| \\ &\le\frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}_+^2}\int_{\gamma_i^I}^{\gamma_k^K}\left|\frac{(\gamma_k^K-\gamma_i^I)}{(u-s)}-1\right|\left|\frac{\Phi(s,t,u)}{(\gamma_k^K-\gamma_i^I)}I_i(s)K_k(u)\right|dtdsdu.\end{aligned} \tag{4.81}$$

Now notice that

$$\begin{aligned}\left|\frac{(\gamma_k^K-\gamma_i^I)}{(u-s)}-1\right| &= \left|\frac{\gamma_k^K-\gamma_i^I-u+s}{u-s}\right| \\ &\le \frac{\left|\gamma_k^K-u\right|+\left|s-\gamma_i^I\right|}{|u-s|}.\end{aligned} \tag{4.82}$$

It follows that $\left|\gamma_k^K-u\right|$ and $\left|s-\gamma_i^I\right|$ both tend to zero as n increases to infinity. For consider $\left|s-\gamma_i^I\right|$. By choosing $n \ge N$ such that $\gamma_i^I=\beta_i^I+\frac{b_1-a_1}{n}$, which we can do as this becomes the smaller of the two possibilities for γ_i^I when n is large enough, we have

$$\begin{aligned}\left|s-\gamma_i^I\right| &= \left|\gamma_i^I - s\right| = \gamma_i^I - s \\ &= \beta_i^I + \frac{b_1-a_1}{n} - s \\ &\leq \beta_i^I + \frac{b_1-a_1}{n} - \alpha_i^I, \text{ as } \alpha_i^I \leq s, \\ &= \beta_i^I - \alpha_i^I + \frac{b_1-a_1}{n} \\ &= \frac{b_1-a_1}{n} + \frac{b_1-a_1}{n} \to 0 \text{ as } n\to\infty.\end{aligned} \tag{4.83}$$

A similar argument follows for $\left|\gamma_k^K - u\right|$. Thus

$$\left|\frac{\gamma_k^K-\gamma_i^I}{u-s} - 1\right| \leq \frac{\left|\gamma_k^K - u\right| + \left|s-\gamma_i^I\right|}{u-s} \to 0 \text{ as } n\to\infty, \tag{4.84}$$

and hence as $n\to\infty$

$$\frac{1}{n^2}\sum_{i=1}^n\sum_{k=1}^n\int_{\mathbb{R}_+^2}\int_{\gamma_i^I}^{\gamma_k^K}\left|\frac{(\gamma_k^K-\gamma_i^I)}{(u-s)}-1\right|\left|\frac{\Phi(s,t,u)}{(\gamma_k^K-\gamma_i^I)}I_i(s)K_k(u)\right|dtdsdu \to 0. \tag{4.85}$$

In turn this implies that

$$\begin{aligned}&\lim_{n\to\infty}\left\{\frac{1}{n^2}\sum_{i=1}^n\sum_{k=1}^n\int_{\mathbb{R}_+^2}\frac{(\gamma_k^K-\gamma_i^I)}{(u-s)}\int_{\gamma_i^I}^{\gamma_k^K}\frac{\Phi(s,t,u)}{(\gamma_k^K-\gamma_i^I)}dtI_i(s)K_k(u)dsdu\right\} \\ &\qquad = \lim_{n\to\infty}\left\{\frac{1}{n^2}\sum_{i=1}^n\sum_{k=1}^n\int_{\mathbb{R}_+^2}\int_{\gamma_i^I}^{\gamma_k^K}\frac{\Phi(s,t,u)}{(\gamma_k^K-\gamma_i^I)}dtI_i(s)K_k(u)dsdu\right\}.\end{aligned} \tag{4.86}$$

Note that

$$\frac{1}{n^2}\sum_{i=1}^n\sum_{k=1}^n\int_{\mathbb{R}_+^2}\int_{\gamma_i^I}^{\gamma_k^K}\frac{\Phi(s,t,u)}{(\gamma_k^K-\gamma_i^I)}dtI_i(s)K_k(u)dsdu = \frac{1}{n^2}\sum_{i=1}^n\sum_{k=1}^n\varphi(I_i,J_{i,k})(K_k), \tag{4.87}$$

where

$$J_{i,k} = \frac{\chi_{[\gamma_i^I,\gamma_k^K]}}{(\gamma_k^K-\gamma_i^I)}. \tag{4.88}$$

Since $\beta_i^I \leq \gamma_i^I \leq a_2$, $b_2 \leq \gamma_k^K \leq \alpha_k^K$ and φ is independent of its middle function so long as the interval on which it is defined falls between those defining the functions on either end without any overlap, we have that

$$\varphi(I_i, J_{i,k})(K_k) = \varphi(I_i, J)(K_k), \tag{4.89}$$

where $J = \dfrac{\chi_{[a_2,b_2]}}{(b_2 - a_2)}$ as before. So

$$\begin{aligned}\frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}_+^2}\int_{\gamma_i^I}^{\gamma_k^K}\frac{\Phi(s,t,u)}{(\gamma_k^K - \gamma_i^I)}dtI_i(s)K_k(u)dsdu&\\ = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\varphi(I_i, J)(K_k) = \varphi(I,J)(K)&\end{aligned} \tag{4.90}$$

since the functions I and K are regained due to the linearity of φ.

Putting all of this together gives us that

$$\begin{aligned}&\int_{\mathbb{R}_+^2}\int_s^u \frac{\Phi(s,t,u)}{(u-s)}dtI(s)K(u)dsdu\\ &\quad= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_{\mathbb{R}_+^2}\left[\int_s^{\gamma_i^I}\frac{\Phi(s,t,u)}{(u-s)}dt + \int_{\gamma_i^I}^{\gamma_k^K}\frac{\Phi(s,t,u)}{(u-s)}dt\right.\\ &\qquad\left. + \int_{\gamma_k^K}^{u}\frac{\Phi(s,t,u)}{(u-s)}dt\right] I_i(s)K_k(u)dsdu,\end{aligned} \tag{4.91}$$

and from this it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\left\{\int_{\mathbb{R}^2_+}\int_s^u \frac{\Phi(s,t,u)}{(u-s)}dtI(s)K(u)dsdu\right\}\\
&=\lim_{n\to\infty}\left\{\frac{1}{n^2}\sum_{i=1}^n\sum_{k=1}^n\int_{\mathbb{R}^2_+}\left[\int_s^{\gamma_i^I}\frac{\Phi(s,t,u)}{(u-s)}dt+\int_{\gamma_i^I}^{\gamma_k^K}\frac{\Phi(s,t,u)}{(u-s)}dt\right.\right.\\
&\qquad\left.\left.+\int_{\gamma_k^K}^u\frac{\Phi(s,t,u)}{(u-s)}dt\right]I_i(s)K_k(u)dsdu\right\}\\
&=\lim_{n\to\infty}\left\{\frac{1}{n^2}\sum_{i=1}^n\sum_{k=1}^n\int_{\mathbb{R}^2_+}\left[\int_{\gamma_i^I}^{\gamma_k^K}\frac{\Phi(s,t,u)}{(u-s)}dt\right]I_i(s)K_k(u)dsdu\right\}\\
&=\lim_{n\to\infty}\left\{\frac{1}{n^2}\sum_{i=1}^n\sum_{k=1}^n\int_{\mathbb{R}^2_+}\left[\int_{\gamma_i^I}^{\gamma_k^K}\frac{\Phi(s,t,u)}{(\gamma_k^K-\gamma_i^I)}dt\right]I_i(s)K_k(u)dsdu\right\}\\
&=\lim_{n\to\infty}\left\{\varphi(I,J)(K)\right\},
\end{aligned}\tag{4.92}$$

i.e.

$$\lim_{n\to\infty}\left\{\int_{\mathbb{R}^2_+}\int_s^u \frac{\Phi(s,t,u)}{(u-s)}dtI(s)K(u)dsdu\right\}=\lim_{n\to\infty}\left\{\varphi(I,J)(K)\right\}.\tag{4.93}$$

Since neither of these terms depend in anyway upon n we simply have that

$$\int_{\mathbb{R}^2_+}\int_s^u \frac{\Phi(s,t,u)}{(u-s)}dtI(s)K(u)dsdu=\varphi(I,J)(K),\tag{4.94}$$

which shows that (4.66) is true, as required.

For characteristic functions defined on non-overlapping intervals this means we can cobound our given φ by

$$\psi(I)(K)=\int_{\mathbb{R}^2_+}\Psi(s,u)I(s)K(u)dsdu,\tag{4.95}$$

with

$$\Psi(s,u)=\int_s^u\frac{\Phi(s,t,u)}{(u-s)}dt,\tag{4.96}$$

where $u > s$ and $\Psi(u,s) = -\Psi(s,u)$.

Next we consider the situation where the intervals on which the normalised indicator functions are defined *do* overlap.

First we consider the case when the intervals on which two of the indicator functions are defined overlap. Without loss of generality, suppose that the intervals on which J and K are defined overlap. Using the same idea we employed in Section 3.3 we divide the intervals on which J and K are defined into M equal subintervals. Then

$$\varphi(I, J)(K) = \sum_{j=1}^{M} \sum_{k=1}^{M} \varphi(I, J_j)(K_k). \tag{4.97}$$

Then we have that

$$\begin{aligned}
\varphi(I, J)(K) &= \sum_{J_j \cap K_k = \emptyset} \varphi(I, J_j)(K_k) + \sum_{J_j \cap K_k \neq \emptyset} \varphi(I, J_j)(K_k) \\
&= \sum_{J_j \cap K_k = \emptyset} (\delta^2 \psi)(I, J_j)(K_k) + \sum_{J_j \cap K_k \neq \emptyset} \varphi(I, J_j)(K_k) \\
&\quad \text{(as we can cobound } \varphi \text{ by } \psi \text{ when the intervals do not overlap)} \\
&= \sum_{j=1}^{M} \sum_{k=1}^{M} (\delta^2 \psi)(I, J_j)(K_k) \\
&\qquad\qquad + \sum_{J_j \cap K_k \neq \emptyset} \big(\varphi(I, J_j)(K_k) - (\delta^2 \psi)(I, J_j)(K_k)\big) \\
&= (\delta^2 \psi)(I, J)(K) + \sum_{J_j \cap K_k \neq \emptyset} \big(\varphi(I, J_j)(K_k) - (\delta^2 \psi)(I, J_j)(K_k)\big).
\end{aligned} \tag{4.98}$$

The main piece of information that we need here is the maximum *number* of places of overlap, i.e. we need to find out how many *at most* of the subintervals of the interval on which one characteristic function is defined overlaps with a subinterval of the interval on which another characteristic function is defined.

Once again as in Section 3.3 the maximum number of places of overlap in this case is simply $3M$, and thus

$$\begin{aligned}
&\left| \sum_{J_j \cap K_k \neq \emptyset} \big(\varphi(I, J_j)(K_k) - (\delta^2\psi)(I, J_j)(K_k)\big) \right| \\
&\leq \sum_{J_j \cap K_k \neq \emptyset} \big|\varphi(I, J_j)(K_k) - (\delta^2\psi)(I, J_j)(K_k)\big| \\
&\leq 3M \cdot \big(\|\varphi\| \, \|I\| \, \|J_j\| \, \|K_k\| + \left\|\delta^2\psi\right\| \, \|I\| \, \|J_j\| \, \|K_k\|\big) \\
&= 3M \cdot \left(\|\varphi\| \, |I| \frac{|J|}{M} \frac{|K|}{M} + \left\|\delta^2\psi\right\| \, |I| \frac{|J|}{M} \frac{K}{M} \right) \longrightarrow 0,
\end{aligned} \tag{4.99}$$

as M tends to ∞. Hence

$$\begin{aligned}
&\varphi(I, J)(K) \\
&= \lim_{M \to \infty} (\varphi(I, J)(K)) \\
&= \lim_{M \to \infty} \left((\delta^2\psi)(I, J)(K) + \sum_{J_j \cap K_k \neq \emptyset} \big(\varphi(I, J_j)(K_k) - (\delta^2\psi)(I, J_j)(K_k)\big) \right) \\
&= (\delta^2\psi)(I, J)(K) + \lim_{M \to \infty} \left(\sum_{J_j \cap K_k \neq \emptyset} \big(\varphi(I, J_j)(K_k) - (\delta^2\psi)(I, J_j)(K_k)\big) \right) \\
&= (\delta^2\psi)(I, J)(K),
\end{aligned} \tag{4.100}$$

or, in other words,

$$\varphi(I, J)(K) = (\delta^2\psi)(I, J)(K). \tag{4.101}$$

An identical argument follows for the other instances of overlapping intervals that occur in pairs.

We now consider the situation when all three of the intervals on which our indicator functions are defined overlap; the main piece of information that we need here is again the maximum number of places of overlap, which occurs when the subintervals are all of the same size and, when the intervals are arranged on a three-dimensional set of axes (cf. first simplicial cohomology group where it was two-dimensional), the *planes* (cf. line in two-dimensions) of overlap $x = y$ and $y = z$ runs from bottom corner to the opposite top corner in the resulting cube.

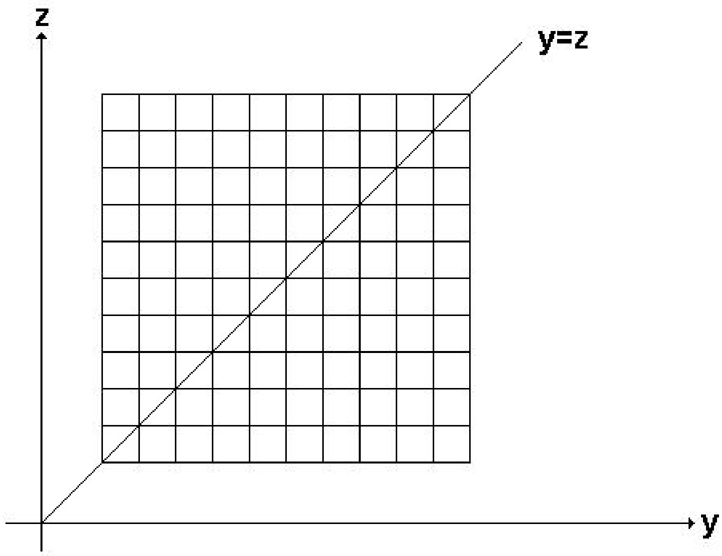


Figure 4.2: A side view of our $M \times M \times M$ cube and $y = z$ plane

We have *two* planes of overlap; we are interested in the scenario when I and J overlap, corresponding to the plane $x = y$, and the case when J and K overlap, corresponding to the plane $y = z$, occuring simultaneously. Notice that the line of intersection of these two planes is $x = y = z$, which is precisely the instance when all three of I, J, K overlap together.

First consider an individual face, or vertical layer, of boxes and one of the planes, in this case $y = z$. Looking on that layer from the side we have an identical view to that of Figure 4.2. From this it is straightforward to see that the plane $y = z$, running corner to corner across all of the cubes, touches at most three boxes per horizontal row in a way analagous to the line in the two-dimensional case. Since there are M rows this translates to $3M$ boxes per layer.

It is also easy to see that a plane touches a box in a particular row if and only if it also touches the box on the same level immediately left and right of it; in other words it cuts through the same boxes in each layer. Therefore as there are M layers this means the maximum places of overlap is $3M^2$.

This holds for each plane, so the maximum number of places of overlap is $3M^2$. In the worst possible scenario, not removing any double-counted boxes, both of these planes will contribute the maximum number of places, giving us an upper bound of $6M^2$ for the total number of places of overlap.

So, proceeding as we did above, we divide the intervals on which I, J, K are defined into M equal subintervals, obtaining

$$\varphi(I,J)(K) = \sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{k=1}^{M}\varphi(I_i, J_j)(K_k), \tag{4.102}$$

and this becomes

$$\begin{aligned}
\varphi(I,J)(K) &= \sum_{I_i\cap J_j\cap K_k=\emptyset}\varphi(I_i,J_j)(K_k) + \sum_{I_i\cap J_j\cap K_k\neq\emptyset}\varphi(I_i,J_j)(K_k)\\
&= \sum_{I_i\cap J_j\cap K_k=\emptyset}(\delta^2\psi)(I_i,J_j)(K_k) + \sum_{I_i\cap J_j\cap K_k\neq\emptyset}\varphi(I_i,J_j)(K_k)\\
&\quad\text{(as we can cobound } \varphi \text{ by } \psi \text{ when the intervals do not overlap)}\\
&= \sum_{i,j,k=1}^{M}(\delta^2\psi)(I_i,J_j)(K_k)\\
&\qquad + \sum_{I_i\cap J_j\cap K_k\neq\emptyset}\big(\varphi(I_i,J_j)(K_k) - (\delta^2\psi)(I_i,J_j)(K_k)\big)\\
&= (\delta^2\psi)(I,J)(K) + \sum_{I_i\cap J_j\cap K_k\neq\emptyset}\big(\varphi(I_i,J_j)(K_k) - (\delta^2\psi)(I_i,J_j)(K_k)\big)\,.
\end{aligned} \tag{4.103}$$

The maximum number of places of overlap is $6M^2$ and so

$$\begin{aligned}
&\left|\sum_{I_i\cap J_j\cap K_k\neq\emptyset}\big(\varphi(I_i,J_j)(K_k) - (\delta^2\psi)(I_i,J_j)(K_k)\big)\right|\\
&\le \sum_{I_i\cap J_j\cap K_k\neq\emptyset}\big|\varphi(I_i,J_j)(K_k) - (\delta^2\psi)(I_i,J_j)(K_k)\big|\\
&\le 6M^2\cdot\big(\|\varphi\|\,\|I_i\|\,\|J_j\|\,\|K_k\| + \|\delta^2\psi\|\,\|I_i\|\,\|J_j\|\,\|K_k\|\big)\\
&= 6M^2\cdot\left(\|\varphi\|\frac{|I|}{M}\frac{|J|}{M}\frac{|K|}{M} + \|\delta^2\psi\|\frac{|I|}{M}\frac{|J|}{M}\frac{|K|}{M}\right)\longrightarrow 0,
\end{aligned} \tag{4.104}$$

as M tends to ∞. Therefore we have that

$$\begin{aligned}
&\varphi(I,J)(K)\\
&= \lim_{M\to\infty} (\varphi(I,J)(K))\\
&= \lim_{M\to\infty} \left((\delta^2\psi)(I,J)(K) + \sum_{I_i\cap J_j\cap K_k\neq\emptyset} \big(\varphi(I_i,J_j)(K_k) - (\delta^2\psi)(I_i,J_j)(K_k)\big) \right)\\
&= (\delta^2\psi)(I,J)(K) + \lim_{M\to\infty} \left(\sum_{I_i\cap J_j\cap K_k\neq\emptyset} \big(\varphi(I_i,J_j)(K_k) - (\delta^2\psi)(I_i,J_j)(K_k)\big) \right)\\
&= (\delta^2\psi)(I,J)(K),
\end{aligned} \tag{4.105}$$

i.e.

$$\varphi(I,J)(K) = (\delta^2\psi)(I,J)(K). \tag{4.106}$$

We are thus able to conclude that whether the intervals overlap or not we can cobound our given φ with our chosen ψ. Even if the intervals are not defined in any order, with arbitrary ways of overlapping, we can divide them up into particular smaller subintervals and the result follows from the linearity and properties of both φ and $\delta^2\psi$.

It is necessary to check that ψ does indeed have all the desired properties, that is it is bounded, linear and measurable. It is straightforward to do all of this from first principles, but we can more easily just use the fact that ψ is defined using φ, and so the result for ψ automatically follows from the same facts that are true for φ.

We have shown that given a cyclic 2-cocycle $\varphi \in \mathcal{ZC}^2(\mathfrak{B})$ we can cobound it by $\psi \in \mathcal{C}^1(\mathfrak{B}, \mathfrak{B}^*)$, demonstrating that

$$\mathcal{HC}^2(\mathfrak{B}) = 0, \tag{4.107}$$

the second cyclic cohomology group of $\mathfrak{B} = L^1(\mathbb{R}_+, \vee)$ is trivial.

Consider the section of the Connes-Tzygan long exact sequence given in Section

2.2.3 consisting of

$$\cdots \longrightarrow \mathcal{HC}^2(\mathfrak{B}) \longrightarrow \mathcal{HH}^2(\mathfrak{B}) \longrightarrow \mathcal{HC}^1(\mathfrak{B}) \longrightarrow \cdots.$$

We know that $\mathcal{HC}^1(\mathfrak{B})$ and $\mathcal{HC}^2(\mathfrak{B})$ are trivial and that the sequence is exact at $\mathcal{HH}^2(\mathfrak{B})$ and so

$$\mathcal{HH}^2(\mathfrak{B}) = \ker(\mathcal{HH}^2(\mathfrak{B}) \to 0) = \mathrm{im}(0 \to \mathcal{HH}^2(\mathfrak{B})) = 0, \tag{4.108}$$

i.e.

$$\mathcal{HH}^2(L^1(\mathbb{R}_+, \vee)) = 0, \tag{4.109}$$

and so we have shown that the second simplicial cohomology group for the algebra $\mathfrak{B} = L^1(\mathbb{R}_+, \vee)$ is trivial.

4.3 An example of cobounding a particular cyclic 2-cocycle

We may ask how a specific 2-cocycle cobounds. Let $\varphi(f,g)(h) = \tau(fgh)$, where

$$\tau(k) = \int_0^\infty k(t)dt. \tag{4.110}$$

We can show this is a 2-cocycle by direct calculation, or we could simply observe that in the Connes-Tzygan long exact sequence the map between $\mathcal{HC}^0(\mathfrak{B})$ into $\mathcal{HC}^2(\mathfrak{B})$ takes the trace τ to the cyclic 2-cocycle φ_τ which is exactly our φ here. From our analysis recall that ψ was chosen based on indicator functions I, J, K as

$$\psi(I)(K) = \varphi(I,J)(K), \tag{4.111}$$

where φ is independent of the middle term. Thus let

$$\psi(f)(g) = \tau(fg). \tag{4.112}$$

Then

$$\begin{aligned}(\delta^2\psi)(f,g)(h) &= \psi(g)(h*f) - \psi(f*g)(h) + \psi(f)(g*h)\\ &= \tau(gfh) - \tau(fgh) + \tau(fgh)\\ &= \tau(fgh)\\ &= \varphi(f,g)(h),\end{aligned} \tag{4.113}$$

and so ψ cobounds φ. This ψ is not cyclic, and since $\mathcal{HC}^2(L^1(\mathbb{R}_+, \vee)) = 0$ it must be possible to *cyclically* cobound φ also where the requirement is for $\psi(f)(g) = -\psi(g)(f)$. On interval functions I and K we wrote ψ as

$$\psi(I)(K) = \int_0^\infty \int_0^\infty \Psi(s,u) I(s) K(u) ds du. \tag{4.114}$$

For indicator functions I, J, K with order structure $I \ll J \ll K$ it follows from our analysis that

$$\begin{aligned}\varphi(I,J)(K) &= \tau(IJK)\\ &= \int_0^\infty (I * J * K)(t) dt\\ &= \int_0^\infty K(t) dt = 1,\end{aligned} \tag{4.115}$$

and so

$$\varphi(I,J)(K) = \psi(I)(K) = \int_0^\infty \int_0^\infty \Psi(s,u) I(s) K(u) ds du = 1. \tag{4.116}$$

This rearranges to give

$$\frac{1}{|I|}\frac{1}{|K|}\int_{a_0}^{b_0}\int_{a_1}^{b_1} \Psi(s,u) ds du = 1, \tag{4.117}$$

recalling that I is defined on $[a_1, b_1]$ and K is defined on $[a_0, b_0]$. If we set

$$\Psi(s,u) = \begin{cases} 1, & s < t, \\ -1, & s > t, \end{cases} \tag{4.118}$$

we will recover what we want, i.e. that ψ is cyclic and it does indeed cobound φ for indicator functions. What happens when $s = u$ is of little interest as this occurs on a line, an area of the plane which has Lebesgue measure zero; incidentally this is precisely why the second cyclic cohomology group in the discrete case is not trivial. This extends by linearity to general $L^1(\mathbb{R}_+, \vee)$ functions.

Thus we have proven that the second simplicial cohomology groups for the algebras $\mathfrak{A} = \ell^1(\mathbb{Z}_+, \vee)$ and $\mathfrak{B} = L^1(\mathbb{R}_+, \vee)$ are both trivial.

4.4 An example of an algebra with nontrivial second simplicial cohomology

One may be forgiven for thinking that our methods are applicable to any Banach algebra spanned by idempotents and that therefore there must not be such a Banach algebra with nontrivial second simplicial cohomology. The aim of this section is to provide a counterexample to dispel this notion.

We consider the algebra $A_\infty = \ell^1(\mathbb{N}, \vee, \mu)$ with measure on the underlying semigroup given by $\mu(\{n\}) = n$. The key point of note here is that the measure is unbounded on atoms, and is essentially a weighted algebra with weight $w(n) = n$.

The idea is to uniquely cobound a 2-cocycle $\varphi \in \mathcal{Z}^2(A_\infty)$ by an unbounded ψ; because ψ is unbounded it cannot be in $C^1(A_\infty, (A_\infty)^*)$ and hence is not an element of $\mathcal{B}^2(A_\infty)$, therefore making the second simplicial cohomology group of A_∞ nontrivial as required.

Let $f \in A_\infty$. Then f is a sequence, which in a way analogous to that for the algebra $\ell^1(\mathbb{Z}_+, \vee)$ can be written as

$$f = \sum_{i=1}^{\infty} f_i \delta_i, \tag{4.119}$$

where $f_i \in \mathbb{C}$ and δ_i is the indicator function

$$\delta_i(n) = \begin{cases} 1, & n = i, \\ 0, & \text{else}, \end{cases} \tag{4.120}$$

which are the *generators* of this algebra.

It also follows that the usual norm on this algebra is given by

$$\|f\| = \sum_{i=1}^{\infty} |f_i|\mu_c(i) = \sum_{i=1}^{\infty} |f_i| \cdot i < \infty. \tag{4.121}$$

As we have done throughout this chapter let us assume that we are working with $f, g, h \in A_\infty$ which are all normalised, that is have norm equal to 1.

Notation. For convenience let $\psi_{i,j} = \psi(\delta_i)(\delta_j)$ and $\varphi_{i,j,k} = \varphi(\delta_i, \delta_j)(\delta_k)$.

It is once again sufficient by linearity arguments to define ψ and φ on the generators for A_∞. We now define $\psi : A_\infty \to (A_\infty)^*$ to be

$$\psi(\delta_i)(\delta_j) = \begin{cases} 0, & i = j, \\ i^2 j, & i < j, \\ -j^2 i, & i > j. \end{cases} \tag{4.122}$$

It follows that $\psi_{i,j} = -\psi_{j,i}$ and it is well-defined.

Then we have that

$$\begin{aligned} \varphi_{i,j,k} &= (\delta^2\psi)(\delta_i, \delta_j)(\delta_k) \\ &= \psi_{j,k\vee i} - \psi_{i\vee j,k} + \psi_{i,j\vee k} \end{aligned} \tag{4.123}$$

Because of the fact that $\psi_{i,j} = -\psi_{j,i}$ and the indices i, j, k must occur in some relative order in $\mathbb{N}$ we are able to arrive at the same conclusion regardless of the actual ordering. Thus without loss of generality let us say that $i \leq j \leq k$. Then

$$\varphi_{i,j,k} = \psi_{i,k}. \tag{4.124}$$

Going through the other orderings in turn will show us that in fact

$$\varphi_{i,j,k} = \psi_{\min(i,j,k),\max(i,j,k)}. \tag{4.125}$$

It is straightforward to show that $\delta^3\varphi = 0$. It also follows that

$$\begin{aligned}
\|\varphi\| &= \sup_{\|f\|,\|g\|,\|h\|\leq 1} |\varphi(f,g)(h)| \\
&= \sup_{\|f\|,\|g\|,\|h\|\leq 1} \left| \sum_{i,j,k=1}^{\infty} f_i \cdot g_j \cdot h_k \cdot (\delta^2\psi)(\delta_i, \delta_j)(\delta_k) \right| \\
&= \sup_{\|f\|,\|g\|,\|h\|\leq 1} \left| \sum_{i,j,k=1}^{\infty} f_i \cdot i \cdot g_j \cdot j \cdot h_k \cdot k \cdot \frac{(\delta^2\psi)(\delta_i, \delta_j)(\delta_k)}{i \cdot j \cdot k} \right| \\
&\leq \sup_{\|f\|,\|g\|,\|h\|\leq 1} \sum_{i,j,k=1}^{\infty} |f_i| \cdot i \cdot |g_j| \cdot j |h_k| \cdot k \cdot \frac{|(\delta^2\psi)(\delta_i, \delta_j)(\delta_k)|}{i \cdot j \cdot k}
\end{aligned} \tag{4.126}$$

Assume without loss of generality that $i \leq j \leq k$; of course any ordering will lead to the same conclusion. Then

$$|(\delta^2\psi)(\delta_i, \delta_j)(\delta_k)| = |\psi_{ik}| = i^2 k. \tag{4.127}$$

Also we have that

$$\frac{i^2 k}{i \cdot j \cdot k} \leq \sup_{i,j,k\in\mathbb{N}} \left(\frac{i^2 k}{i \cdot j \cdot k} \right) < 1 \tag{4.128}$$

as $i \leq j \leq k$. Hence

$$\begin{aligned}
\|\varphi\| &\leq \sup_{\|f\|,\|g\|,\|h\|\leq 1} \sum_{i,j,k=1}^{\infty} |f_i| \cdot i \cdot |g_j| \cdot j |h_k| \cdot k \cdot 1 \\
&= \sup_{\|f\|,\|g\|,\|h\|\leq 1} \left(\sum_{i=1}^{\infty} |f_i| \cdot i \right) \left(\sum_{j=1}^{\infty} |g_j| \cdot j \right) \left(\sum_{k=1}^{\infty} |h_k| \cdot k \right) \\
&= \sup_{\|f\|,\|g\|,\|h\|\leq 1} \|f\| \, \|g\| \, \|h\| \\
&= 1,
\end{aligned} \tag{4.129}$$

i.e. $\|\varphi\| \leq 1$ and so φ is bounded. Therefore we can conclude that $\varphi = \delta^2\psi \in \mathcal{Z}^2(A_\infty)$; i.e. φ is a 2-cocycle cobounded by ψ.

However, notice that following the same steps as before

$$\begin{aligned}\|\psi\| &= \sup_{\|f\|,\|g\|\leq 1} |\psi(f)(g)| \\ &\leq \sup_{\|f\|,\|g\|\leq 1} \sum_{i,j=1}^{\infty} |f_i| \cdot i \cdot |g_j| \cdot j \cdot \frac{|\psi_{i,j}|}{i \cdot j}.\end{aligned} \tag{4.130}$$

As we are still assuming that $i \leq j$ then it follows that $|\psi_{i,j}| = i^2 j$ and thus

$$\frac{|\psi_{i,j}|}{i \cdot j} = \frac{i^2 j}{i \cdot j} \leq \sup_{i,j\in\mathbb{N}} (i) = \infty, \tag{4.131}$$

and hence

$$\|\psi\| \leq \sup_{\|f\|,\|g\|\leq 1} \sum_{i,j=1}^{\infty} |f_i| \cdot i \cdot |g_j| \cdot j \cdot \infty, \tag{4.132}$$

and finally then $\|\psi\| \leq \infty$.

It is not difficult to manufacture arguments to make $\|\psi\| = \infty$. For consider the functions $f_1 = \frac{1}{x}\delta_x$ and $g_1 = \frac{1}{y}\delta_y$ with $x, y \in \mathbb{N}$ such that $x < y$. Then we have that $\|f_1\| = \|g_1\| = 1$ and

$$\begin{aligned}\|\psi\| &= \sup_{\|f\|,\|g\|\leq 1} |\psi(f)(g)| \\ &\geq |\psi(f_1)(g_1)| \\ &= \left|\psi(\frac{1}{x}\delta_x)(\frac{1}{y}\delta_y)\right| \\ &= \frac{1}{xy}|\psi_{x,y}| \\ &= \frac{1}{xy} \cdot x^2 y \\ &= x.\end{aligned} \tag{4.133}$$

Since $x \in \mathbb{N}$ was an arbitrary choice it must be that $\|\psi\| \geq x$ for all $x \in \mathbb{N}$, and therefore we conclude that ψ is unbounded. Thus $\psi \notin \mathcal{C}^1(A_\infty, (A_\infty)^*)$ and therefore $\psi \notin \mathcal{B}^2(A_\infty)$, as required.

The final point to note here is that given $\varphi \in \mathcal{Z}^2(A_\infty)$ it must cobound uniquely.

For if ψ_1 and ψ_2 are two coboundings of φ then it follows that

$$\varphi = \delta^2\psi_1 = \delta^2\psi_2 \tag{4.134}$$

and thus $\delta^2(\psi_1 - \psi_2) = 0$, making $\psi_1 - \psi_2$ a derivation. As A_∞ is spanned by idempotents, namely the generators δ_i for $i \in \mathbb{N}$, it follows from Theorem 3.1 that there are no nontrivial derivations on this algebra. Therefore $\psi_1 - \psi_2 = 0$ and hence

$$\psi_1 = \psi_2 \tag{4.135}$$

and the cobounding is unique.

Hence we have proven the result we set out to show at the beginning of this section, allowing us to finally say that $\mathcal{HH}^2(A_\infty)$ is nontrivial, as required.

Chapter 5

The general simplicial cohomology group of $L^1(\mathbb{R}_+, \vee)$

The main aim of this thesis is to obtain results about the n^{th} simplicial cohomology group for the semigroup algebra $\mathfrak{B} = L^1(\mathbb{R}_+, \vee)$. Having considered the first and second simplicial cohomology groups for $\mathfrak{B}$ we now generalise the methods used in Chapters 3 and 4 in order to show that, for all $n \in \mathbb{N}$, $n \geq 2$,

$$\mathcal{H}\mathcal{H}^n(\mathfrak{B}) = 0. \tag{5.1}$$

When we try to adapt the methods found in Chapters 3 and 4 directly to higher cohomology groups we run into problems. For instance the cocycle identities no longer yield the same important information and it becomes very difficult to perform any of the calculations required. These difficulties are compounded further by the fact that the increasing dimensions vastly increases the number of these calculations, their complexity and their length making such attempts nearly impossible to compute by hand as we have been doing so far.

In order to overcome the above difficulties we sought inspiration from the discrete case. The results are already known and we attempted to arrive at the same conclusions via our methods.

However the main problem we will encounter is due to the inability to cobound cocycles *uniquely*. For let φ be a cyclic 2-cocycle that is cobounded by both ψ_1

and ψ_2 in $\mathcal{CC}^1(\mathcal{B})$. Then

$$\varphi = \delta^2\psi_1 = \delta^2\psi_2, \tag{5.2}$$

and so we have $\delta^2(\psi_1 - \psi_2) = 0$, i.e. $\psi_1 - \psi_2 \in \ker(\delta^2)$. Since $\mathcal{HC}^1(\mathfrak{B})$ is trivial we know $\ker(\delta^2) = \operatorname{im}(\delta^1)$ and thus there exists $\chi \in \mathfrak{B}^*$ such that

$$\psi_1 - \psi_2 = \delta^1\chi. \tag{5.3}$$

However

$$\delta^1\chi(f_1)(f_0) = \chi(f_0 * f_1) - \chi(f_1 * f_0) \tag{5.4}$$

for $f_1, f_0 \in \mathfrak{B}$, and by the commutativity of $\mathfrak{B}$ this reduces to give $\delta^1\chi \equiv 0$. This holds for all $\chi \in \mathfrak{B}^*$ and so

$$\psi_1 - \psi_2 = 0 \Rightarrow \psi_1 = \psi_2. \tag{5.5}$$

This uniqueness of coboundary does not continue into higher dimensions. For example consider φ to be a cyclic 3-cocycle. If φ is cobounded by ψ then it is also cobounded by $\psi + \delta^2\chi$ for any $\chi \in \mathcal{CC}^1(\mathfrak{B})$. Then $\delta^2\chi \in \operatorname{im}(\delta^2)$, which is $\mathcal{BC}^2(\mathfrak{B})$. This space is not trivial and so it does not necessarily follow that $\delta^2\chi = 0$, meaning that our cyclic 3-cocycle could indeed be cobounded by several candidates.

This makes searching for coboundings directly much more problematic in higher dimensions as there is no route to *the* solution. Previously we stated that if we could cobound a cocycle φ by ψ then ψ had to exhibit certain properties which helped us to identify it precisely. This method no longer yields fruitful results when the solution is not unique; as there is almost no unique cobounding then it is harder to pick a particular choice of coboundary.

Therefore it has become necessary to adopt a more abstract method for proving the triviality of the n^{th} simplicial cohomology group of $\mathfrak{B}$.

At the start of this chapter we use the method of Section 4.2 to guide us in formulating a more general and abstract method in order to obtain the same result for $\mathcal{HH}^2(\mathfrak{B})$. The rest of the chapter is dedicated to applying this method to the n^{th} case.

5.1 Re-evaluating the second simplicial cohomology group

In this section we will re-evaluate the second simplicial cohomology group case from Section 4.2 and produce a more direct method for proving its triviality.

The general strategy that we shall adopt to do this will begin with a result about the cobounding of a 2-cocycle that is defined on arguments which are L^1 functions where the supports of these functions are ordered and disjoint such that the first term has support to the left of the second term which in turn has support to the left of the final term. Then we extend this cobounding via several interesting methods, first to other distinct orderings and then to the general case using some results concerned with the rewriting and approximating of the general case as a linear combination of the various ordered cases. Finally, having shown that these coboundings occur and the general case can be approximated by this, we will have shown that the second simplicial cohomology group for this semigroup algebra is trivial.

5.1.1 Cobounding a 2-cocycle on a particular ordering of arguments

Let $\varphi \in \mathcal{Z}^2(\mathfrak{B})$ be a 2-cocycle. This section proves the following theorem:

Theorem 5.1. *Let I_1, I_2, I_0 be normalised indicator functions in $\mathfrak{B}$ such that $I_1 \ll I_2 \ll I_0$. Then there exists $\psi \in \mathcal{C}^1(\mathfrak{B})$ such that*

$$\varphi(I_1, I_2)(I_0) = (\delta^2\psi)(I_1, I_2)(I_0). \tag{5.6}$$

In Section 4.2 the idea was to cobound a general $\varphi \in \mathcal{ZC}^2(\mathfrak{B})$ with a $\psi \in \mathcal{CC}^1(\mathfrak{B})$ defined using the cyclic 2-cocyle we started with. We showed that it was sufficient to prove the result only for normalised indicator functions and that these functions could always be ordered as arguments of φ such that the first one listed was the smallest, the last the largest and the second term between these in order.

Section 4.2 then went on to show that φ was independent of this second term and so given $\varphi(I_1, I_2)(I_0)$ where the I_i are normalised indicator functions for

$i = 0, 1, 2$, we replaced I_2 with a normalised indicator function defined on the *largest possible support* that lay between the supports of I_1 and I_0. We then used this to define our ψ and showed that this cobounded $\varphi(I_1, I_2)(I_0)$.

A version of this theorem has been proven in Section 4.2 in the *cyclic* case. We now prove it for the general case.

Recall that it is possible to write $\varphi \in \mathcal{Z}^2(\mathfrak{B})$ and $\psi \in \mathcal{C}^1(\mathfrak{B})$ as

$$\begin{aligned} \varphi(I_1, I_2)(I_0) &= \int_{\mathbb{R}_+^3} \Phi(x_1, x_2, x_0) I_1(x_1) I_2(x_2) I_0(x_0) dx_1 dx_2 dx_0, \\ \psi(I_1)(I_0) &= \int_{\mathbb{R}_+^2} \Psi(x_1, x_0) I_1(x_1) I_0(x_0) dx_1 dx_0, \end{aligned} \tag{5.7}$$

respectively, where $\Phi \in L^\infty(\mathbb{R}_+^3)$ and $\Psi \in L^\infty(\mathbb{R}_+^2)$.

Before giving the proof of Theorem (5.1) we need the following two results.

Lemma 5.1. *Let φ be as above with $I_1 \ll I_2 \ll I_3 \ll I_0$ normalised indicator functions in $\mathfrak{B}$. Then*

$$\begin{aligned} &\int_{\mathbb{R}_+^4} \Phi(x_1, x_3, x_0) I_1(x_1) I_2(x_2) I_3(x_3) I_0(x_0) dx_1 dx_2 dx_3 dx_0 \\ &\qquad = \int_{\mathbb{R}_+^4} \Phi(x_1, x_2, x_0) I_1(x_1) I_2(x_2) I_3(x_3) I_0(x_0) dx_1 dx_2 dx_3 dx_0. \end{aligned} \tag{5.8}$$

Proof. The cocycle identity gives us that

$$\begin{aligned} 0 &= (\delta^3 \varphi)(I_1, I_2, I_3)(I_0) \\ &= \varphi(I_2, I_3)(I_0 * I_1) - \varphi(I_1 * I_2, I_3)(I_0) + \varphi(I_1, I_2 * I_3)(I_0) - \varphi(I_1, I_2)(I_3 * I_0) \\ &= \varphi(I_2, I_3)(I_0) - \varphi(I_2, I_3)(I_0) + \varphi(I_1, I_3)(I_0) - \varphi(I_1, I_2)(I_0), \end{aligned} \tag{5.9}$$

which rearranges to give

$$\varphi(I_1, I_3)(I_0) = \varphi(I_1, I_2)(I_0). \tag{5.10}$$

Using the rewritten versions above this becomes

$$\begin{aligned}\int_{\mathbb{R}_+^3} &\Phi(x_1, x_3, x_0) I_1(x_1) I_3(x_3) I_0(x_0) dx_1 dx_3 dx_0 \\ &= \int_{\mathbb{R}_+^3} \Phi(x_1, x_2, x_0) I_1(x_1) I_2(x_2) I_0(x_0) dx_1 dx_2 dx_0. \end{aligned} \tag{5.11}$$

Now notice that

$$\int_{\mathbb{R}_+} I_i(x) dx_i = 1, \quad i = 0, 1, 2, 3 \tag{5.12}$$

and so the above equation becomes

$$\begin{aligned} &1 \times \int_{\mathbb{R}_+^3} \Phi(x_1, x_3, x_0) I_1(x_1) I_3(x_3) I_0(x_0) dx_1 dx_3 dx_0 \\ &= 1 \times \int_{\mathbb{R}_+^3} \Phi(x_1, x_2, x_0) I_1(x_1) I_2(x_2) I_0(x_0) dx_1 dx_2 dx_0 \\ &\Rightarrow \int_{\mathbb{R}_+^4} \Phi(x_1, x_3, x_0) I_1(x_1) I_2(x_2) I_3(x_3) I_0(x_0) dx_1 dx_2 dx_3 dx_0 \\ &= \int_{\mathbb{R}_+^4} \Phi(x_1, x_2, x_0) I_1(x_1) I_2(x_2) I_3(x_3) I_0(x_0) dx_1 dx_2 dx_3 dx_0, \end{aligned} \tag{5.13}$$

as required. □

Lemma 5.2. *Let I_0, I_1, I_2 be normalised indicator functions in $\mathfrak{B}$ such that $I_1 \ll I_2 \ll I_0$. Then*

$$\varphi(I_1, I_2)(I_0) = \varphi(I_1, I_0)(I_2). \tag{5.14}$$

Proof. Choose $I_3 \in \mathfrak{B}$ such that $I_1 \ll I_3 \ll I_2$; such a choice can always be made given the disjoint nature of the supports of I_1 and I_2. Using these the cocycle identity gives us

$$\begin{aligned} 0 &= (\delta^3 \varphi)(I_1, I_3, I_0)(I_2) \\ &= \varphi(I_3, I_0)(I_2) - \varphi(I_3, I_0)(I_2) + \varphi(I_1, I_0)(I_2) - \varphi(I_1, I_3)(I_0), \end{aligned} \tag{5.15}$$

which becomes

$$\varphi(I_1, I_0)(I_2) = \varphi(I_1, I_3)(I_0). \tag{5.16}$$

By Lemma (5.1) it follows that $\varphi(I_1, I_3)(I_0) = \varphi(I_1, I_2)(I_0)$, and so

$$\varphi(I_1, I_0)(I_2) = \varphi(I_1, I_2)(I_0), \tag{5.17}$$

as required. $\square$

This result can be extended to the case with a non-normalised function f instead of I_2 such that the support of f lies entirely in the order structure between the supports of I_1 and I_0 by linearity. For approximate f by a linear combination of disjoint normalised indicator functions J_j. Note that each J_j for $j = 0, 1, \ldots$ has support between those of I_1 and I_0 also. Then we have, where $\alpha_j \in \mathbb{C}$ for $j \in \mathbb{Z}_+$,

$$\begin{aligned} \varphi(I_1, f)(I_0) &= \sum_{j=0}^{\infty} \alpha_j \varphi(I_1, J_j)(I_0) \\ &= \sum_{j=0}^{\infty} \alpha_j \varphi(I_1, I_0)(J_j) \\ &= \varphi(I_1, I_0)(f). \end{aligned} \tag{5.18}$$

We can also extend the result of Lemma (5.1) to such an f. For the cocycle identity on normalised indicator functions $I_1 \ll I_2 \ll I_0$ with f fitting within the order structure such that its support is between the supports of I_2 and I_0 gives us

$$\begin{aligned} 0 = (\delta^3\varphi)(I_1, I_2, f)(I_0) = \varphi(I_2, f)(I_0) - \varphi(I_2, f)(I_0) + \varphi(I_1, f)(I_0) \\ - \varphi(I_1, I_2)(\vartheta(f)I_0) \end{aligned} \tag{5.19}$$

as it can easily be shown that $(I_2 * f)(x) = f(x)$ and $(f * I_0)(x) = \vartheta(f)I_0$, where ϑ is the *augmentation character* defined by

$$\vartheta(f) = \int_0^{\infty} f(x)dx = \widehat{f}(\infty) \tag{5.20}$$

for $f \in L^1(\mathfrak{B})$. Thus

$$\varphi(I_1, f)(I_0) = \vartheta(f)\varphi(I_1, I_2)(I_0). \tag{5.21}$$

If f is positive and normalised then this result is even simpler as $\vartheta(f) = 1$.

Remark. The result from Lemma (5.1) states that for the ordering $I_1 \ll I_2 \ll I_3 \ll I_0$ we have

$$\varphi(I_1, I_2)(I_0) = \varphi(I_1, I_3)(I_0). \tag{5.22}$$

However this is not as restrictive as it seems; we can replace the middle term by *any* normalised indicator function with support disjoint from and between the supports of I_1 and I_0. For consider I such that $I_1 \ll I \ll I_0$. Note that we make no restriction on the relation between I and I_2; it may be that $I \cap I_2 \neq \emptyset$. Begin with $\varphi(I_1, I_2)(I_0)$ and note that we can always find a normalised indicator function I_3 such that both $I_2 \ll I_3 \ll I_0$, $I \ll I_3 \ll I_0$ hold. Then by Lemma (5.1) it follows that

$$\varphi(I_1, I_2)(I_0) = \varphi(I_1, I_3)(I_0) = \varphi(I_1, I)(I_0). \tag{5.23}$$

We are now in a position to give the proof of Theorem (5.1).

Proof. [Theorem (5.1)] As in Chapter 4 we set

$$\Psi(x_1, x_0) = \begin{cases} \displaystyle\int_{x_1}^{x_0} \frac{\Phi(x_1, t, x_0)}{x_0 - x_1} dt, & x_1 < x_0, \\ 0, & \text{else}. \end{cases} \tag{5.24}$$

Then if ψ is to cobound φ it must be that

$$\begin{aligned} \varphi(I_1, I_2)(I_0) &= (\delta^2 \psi)(I_1, I_2)(I_0) \\ &= \psi(I_2)(I_0 * I_1) - \psi(I_1 * I_2)(I_0) + \psi(I_1)(I_2 * I_0) \\ &= \psi(I_2)(I_0) - \psi(I_2)(I_0) + \psi(I_1)(I_0) \\ &= \psi(I_1)(I_0), \end{aligned} \tag{5.25}$$

i.e. that $\psi(I_1)(I_0) = \varphi(I_1, I_2)(I_0)$. Therefore we are required to show that

$$\int_{\mathbb{R}_+^2} \Psi(x_1, x_0) I_1(x_1) I_0(x_0) dx_1 dx_0 = \int_{\mathbb{R}_+^3} \Phi(x_1, x_2, x_0) I_1(x_1) I_2(x_2) I_0(x_0) dx_1 dx_2 dx_0, \tag{5.26}$$

which after making the substitution for Ψ becomes

$$\int_{\mathbb{R}_+^2}\int_{x_1}^{x_0}\frac{\Phi(x_1,t,x_0)}{x_0-x_1}dtI_1(x_1)I_0(x_0)dx_1dx_0 \\ = \int_{\mathbb{R}_+^3}\Phi(x_1,x_2,x_0)I_1(x_1)I_2(x_2)I_0(x_0)dx_1dx_2dx_0. \quad (5.27)$$

Looking at the left hand side this becomes

$$\begin{aligned}
&\int_{\mathbb{R}_+^2}\int_{\mathbb{R}_+}\Phi(x_1,t,x_0)\frac{I_{[x_1,x_0)(t)}}{x_0-x_1}dtI_1(x_1)I_0(x_0)dx_1dx_0 \\
&=\int_{\mathbb{R}_+^3}\Phi(x_1,t,x_0)\left[\frac{I_{[x_1,x_0)(t)}}{x_0-x_1}\left(\chi_{[x_2,\infty)}(t)+\chi_{[0,x_2)}(t)\right)\right]dtI_1(x_1)I_0(x_0)dx_1dx_0 \\
&=\int_{\mathbb{R}_+^3}\Phi(x_1,t,x_0)\left[\frac{I_{[x_1,x_0)(t)}}{x_0-x_1}\times\chi_{[x_2,\infty)}(t)\right]dtI_1(x_1)I_0(x_0)dx_1dx_0 \\
&\quad+\int_{\mathbb{R}_+^3}\Phi(x_1,t,x_0)\left[\frac{I_{[x_1,x_0)(t)}}{x_0-x_1}\times\chi_{[0,x_2)}(t)\right]dtI_1(x_1)I_0(x_0)dx_1dx_0,
\end{aligned} \quad (5.28)$$

where $\chi_{[x_2,\infty)}(t)$ and $\chi_{[0,x_2)}(t)$ are the characteristic functions (not necessarily normalised) defined on the supports $[x_2,\infty)$ and $[0,x_2)$ respectively.

A little thought makes it clear that

$$\begin{aligned}
\frac{I_{[x_1,x_0)(t)}}{x_0-x_1}\times\chi_{[x_2,\infty)}(t)&=\frac{I_{[x_2,x_0)}(t)}{x_0-x_1}, \\
\frac{I_{[x_1,x_0)(t)}}{x_0-x_1}\times\chi_{[0,x_2)}(t)&=\frac{I_{[x_1,x_2)}(t)}{x_0-x_1},
\end{aligned} \quad (5.29)$$

and so the equation in (5.28) then becomes

$$\int_{\mathbb{R}_+^3}\Phi(x_1,t,x_0)\frac{I_{[x_2,x_0)(t)}}{x_0-x_1}dtI_1(x_1)I_0(x_0)dx_1dx_0 \\ +\int_{\mathbb{R}_+^3}\Phi(x_1,t,x_0)\frac{I_{[x_1,x_2)(t)}}{x_0-x_1}dtI_1(x_1)I_0(x_0)dx_1dx_0. \quad (5.30)$$

Note that

$$\int_{\mathbb{R}_+}I_2(x_2)dx_2=1 \quad (5.31)$$

by construction, and so it follows that equation (5.30) becomes

$$\int_{\mathbb{R}^4_+} \Phi(x_1,t,x_0)\frac{I_{[x_2,x_0)(t)}}{x_0-x_1}dtI_1(x_1)I_2(x_2)I_0(x_0)dx_1dx_2dx_0 \\ + \int_{\mathbb{R}^4_+} \Phi(x_1,t,x_0)\frac{I_{[x_1,x_2)(t)}}{x_0-x_1}dtI_1(x_1)I_2(x_2)I_0(x_0)dx_1dx_2dx_0. \quad (5.32)$$

We then appeal to Lemma (5.1) to yield

$$\int_{\mathbb{R}^4_+} \Phi(x_1,x_2,x_0)\frac{I_{[x_2,x_0)(t)}}{x_0-x_1}dtI_1(x_1)I_2(x_2)I_0(x_0)dx_1dx_2dx_0 \\ + \int_{\mathbb{R}^4_+} \Phi(x_1,x_2,x_0)\frac{I_{[x_1,x_2)(t)}}{x_0-x_1}dtI_1(x_1)I_2(x_2)I_0(x_0)dx_1dx_2dx_0. \quad (5.33)$$

This in turn reduces to

$$\int_{\mathbb{R}^3_+} \Phi(x_1,x_2,x_0)\left[\frac{x_0-x_2+x_2-x_1}{x_0-x_1}\right]I_1(x_1)I_2(x_2)I_0(x_0)dx_1dx_2dx_0 \quad (5.34)$$

which is simply

$$\int_{\mathbb{R}^3_+} \Phi(x_1,x_2,x_0)I_1(x_1)I_2(x_2)I_0(x_0)dx_1dx_2dx_0 = \varphi(I_1,I_2)(I_0), \quad (5.35)$$

as required. □

This shows that for the order structure $I_1 \ll I_2 \ll I_0$ we can approximate these functions by normalised indicator functions and by setting

$$\psi(I_1)(I_0) = \varphi(I_1, I_{[a,b]})(I_0) \quad (5.36)$$

for $I_{[a,b]}$ with the largest possible support between those of I_1 and I_0 and still disjoint will yield a cobounding of $\varphi(I_1,I_2)(I_0)$ in this case. However we require this result to be extended to other possible orderings of the disjoint supports of the normalised functions I_i; doing so by this method is possible but very difficult.

The difficulty lies in proving that the choice of a similar ψ for our given 2-cocycle φ defined on the case where the supports of I_1, I_2, I_0 are *decreasing disjoint* is well-defined. It requires an analogue of Lemma (5.1) (where the orders are all reversed), which is difficult if not impossible to prove.

Thus we are forced at this point to amend the method we have created so far. The difficulty lies primarily in the notion of using the normalised indicator function of *largest* support between those of I_1 and I_0 to define ψ using φ as it is not always clear to see if this is well-defined. This suggests that the logical way forward is to somehow invoke the use of the normalised indicator function of *smallest* support between those of I_1 and I_0 instead, in a sense to be explained later.

An alternative approach

It is convenient at this point to introduce a change in approach to the definitions of φ and ψ also. First define

$$\widetilde{\varphi}\left((I_1, I_2, I_0)\right) = \varphi(I_1, I_2)(I_0), \tag{5.37}$$

which is a bounded multilinear map from $L^1(\mathbb{R}_+, \vee) \times L^1(\mathbb{R}_+, \vee) \times L^1(\mathbb{R}_+, \vee)$ into $\mathbb{C}$. Then this multilinear map on $L^1(\mathbb{R}_+, \vee) \times L^1(\mathbb{R}_+, \vee) \times L^1(\mathbb{R}_+, \vee)$ defines a linear map on the tensor product space $L^1(\mathbb{R}_+, \vee)\widehat{\otimes}L^1(\mathbb{R}_+, \vee)\widehat{\otimes}L^1(\mathbb{R}_+, \vee)$ in the usual manner by the fact this is isometrically isomorphic to $L^1(\mathbb{R}^3_+)$; we denote such a linear map by $\widetilde{\varphi}$. So

$$\varphi(I_1, I_2)(I_0) = \widetilde{\varphi}\left((I_1, I_2, I_0)\right) = \widetilde{\varphi}(I_1 \otimes I_2 \otimes I_0). \tag{5.38}$$

Similarly,

$$\psi(I_1)(I_0) = \widetilde{\psi}\left((I_1, I_0)\right) = \widetilde{\psi}(I_1 \otimes I_0). \tag{5.39}$$

It is now more convenient to work with $\widetilde{\varphi}$ and $\widetilde{\psi}$.

Definition. Let the function $\epsilon_k : \mathfrak{B} \to \mathfrak{B}\widehat{\otimes}\mathfrak{B} \cong L^1(\mathbb{R}^2_+, \vee)$ be defined for $I_i \in \mathfrak{B}$ with $i \in \mathbb{Z}_+$ by the formula

$$\epsilon_k(I_i)(x, y) = \epsilon_{k,x}(I_i)(y) = I_i(y)\frac{\chi_{[0,\frac{1}{k})}(y - x)}{\left(\frac{1}{k}\right)} = k \cdot I_i(y) \cdot \chi_{[x,x+\frac{1}{k})}(y). \tag{5.40}$$

We call ϵ_k the *left spreading out function.*

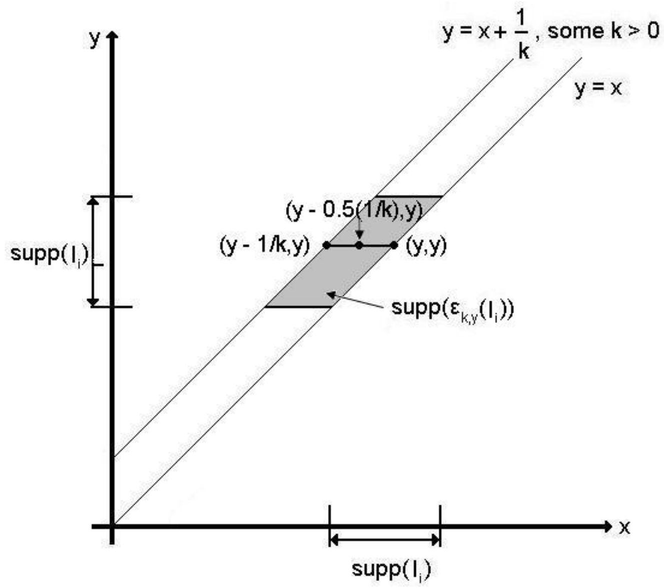


Figure 5.1: An illustration showing the support of $\epsilon_k(I_i)$

The *right spreading out function* $\gamma_k : \mathfrak{B} \to \mathfrak{B}\widehat{\otimes}\mathfrak{B}$ is defined by

$$\gamma_k(I_i)(x,y) = \gamma_{k,y}(I_i)(x) = I_i(x)\frac{\chi_{[0,\frac{1}{k})}(x-y)}{\left(\frac{1}{k}\right)} = k \cdot I_i(x) \cdot \chi_{[y,y+\frac{1}{k})}(x). \tag{5.41}$$

Note that $\epsilon_k(I_i)(x,y)$ has two-dimensional support but cannot be written as an elementary two-dimensional tensor, i.e. in the form $I_i \otimes I_j$. It is a function which takes the one-dimensional support of a function I_i in $\mathfrak{B}$ and turns it into a two-dimensional support by 'stretching it out' to the left. The function $\gamma_k(I_i)(x,y)$ is similar to this but creates the two-dimensional support by 'stretching it out' to the right.

Notation. In the following analysis we will apply an order structure between a one-dimensional function in $\mathfrak{B}$ and the two-dimensional spreading out functions. The notation $\ll$ we have used so far needs extending to include this new case. Therefore for $I_0, I_1 \in \mathfrak{B}$ we use the notation $I_1 \ll_k \epsilon_k(I_0)$ to denote the following

- $I_1 \ll I_0$;
- $k \in \mathbb{N}$ is large enough to ensure $a_0 - \frac{1}{k} > b_1$, i.e. the extended support of I_0 under the action of ϵ_k is still disjoint and to the right of the support of I_1.

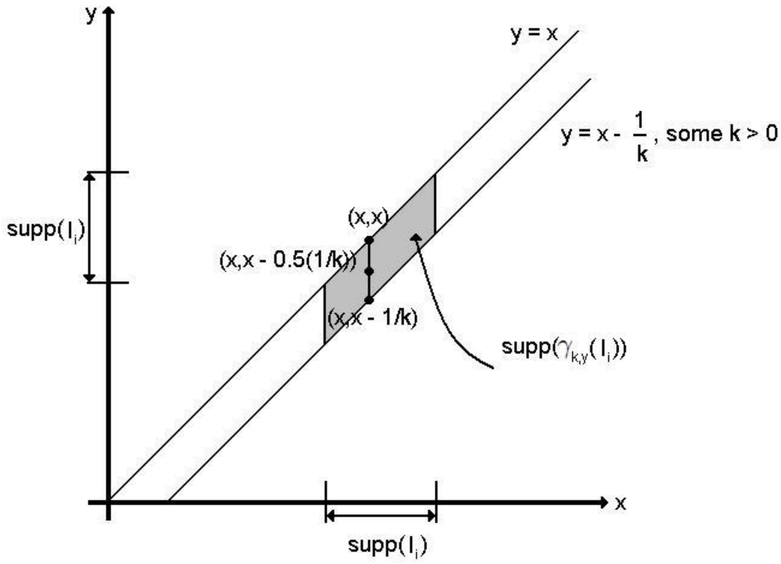


Figure 5.2: An illustration showing the support of $\gamma_k(I_i)$

This last also implies that $\epsilon_k(I_0) \cap I_1 = \emptyset$. This set up merely says that the supports of I_0 and I_1 are ordered with I_1 being the smallest, and the left spreading out function does not spread the support of I_0 as far to the left so as to intercept the support of I_1.

A very similar definition applies to $I_1 \ll_k \gamma_k(I_0)$.

Remark. Note that in order for a similar situation to be applied to the expression $\epsilon_k(I_1) \ll_k I_0$ (sim. $\gamma_k(I_0) \ll_k I_1$) it is only necessary to impose the condition that $I_0 \ll I_1$; there is no dependence on k in order to preserve disjointed order between the two-dimensional support of $\epsilon_k(I_0)$ (sim. $\gamma_k(I_0)$) and the one-dimensional support of I_1. Hence we drop the subscript k on the order relation and simply say that $\epsilon_k(I_0) \ll I_1$ (sim. $\gamma_k(I_0) \ll I_1$) here.

This is illustrated in Figures (5.3) and (5.4), which show the necessity of demanding k to be large enough in order to ensure a disjoint separation of supports.

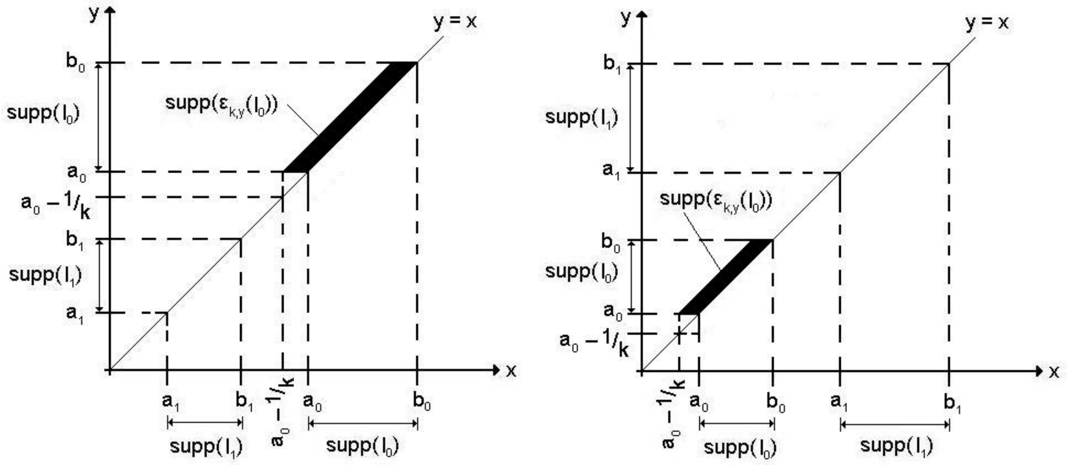


Figure 5.3: An illustration of the relations $I_1 \ll_k \epsilon_k(I_0)$ (left) and $\epsilon_k(I_0) \ll I_1$ (right).

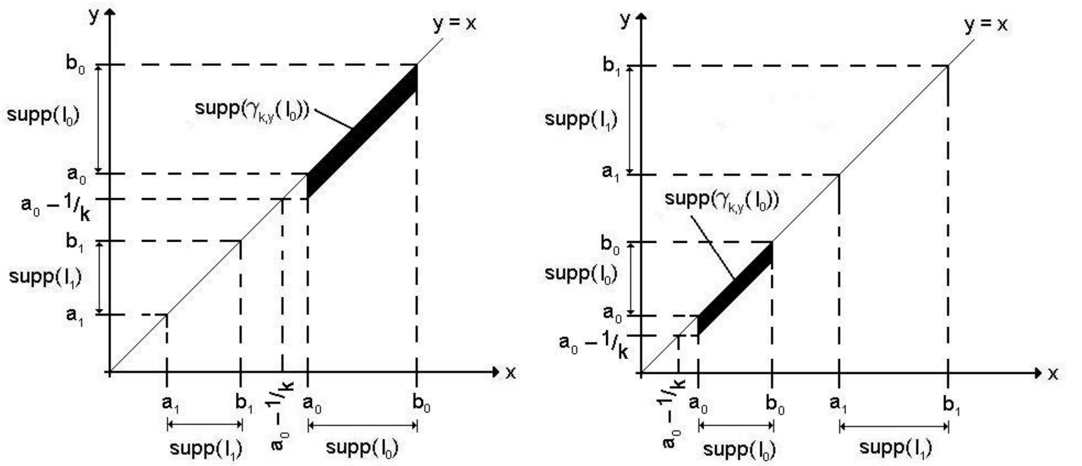


Figure 5.4: An illustration of the relations $I_1 \ll_k \gamma_k(I_0)$ (left) and $\gamma_k(I_0) \ll I_1$ (right).

Approximations by elementary tensor products of normalised functions

Before we continue it is necessary for us to prove a particularly useful result. In order to get to this however we will need the following lemma.

Lemma 5.3. *Given $F \in L^1(\mathbb{R}_+^2)$ there exists $f_i, g_i \in L^1(\mathbb{R}_+)$ with $\alpha_i \in \mathbb{C}$ for $i \in \mathbb{N}$ such that*

$$F(x, y) = \sum_{i=1}^{\infty} \alpha_i f_i \otimes g_i, \tag{5.42}$$

with $\sum_{i=1}^{\infty}|\alpha_i| < \infty$, $\|f_i\| = \|g_i\| = 1$.

The proof of this Lemma is a mixture of Example (14) and Proposition (12) in §42 of [5]. First of all there is the existence of a linear isometric isomorphism of $L^1(\mathbb{R}_+)\widehat{\otimes}L^1(\mathbb{R}_+)$ onto $L^1(\mathbb{R}_+^2)$. For by the (universal) properties of tensor products there exists a unique linear mapping $T : L^1(\mathbb{R}_+)\widehat{\otimes}L^1(\mathbb{R}_+) \to L^1(\mathbb{R}_+^2)$ such that

$$T(f \otimes g)(x, y) = f(x)g(y), \quad x, y \in \mathbb{R}_+, \quad f, g \in L^1(\mathbb{R}_+). \tag{5.43}$$

Then it is observed that $L^1(\mathbb{R}_+)\widehat{\otimes}L^1(\mathbb{R}_+)$ can be represented as the linear subspace of $BL^2(L^1(\mathbb{R}_+)^*; \mathbb{C})$ consisting of all elements of the form

$$u = \sum_{n=1}^{\infty} f_n \otimes g_n, \tag{5.44}$$

where $\sum_{n=1}^{\infty} \|f_n\|\,\|g_n\| < \infty$.

The result we are aiming at in this section is an extension of this lemma. Before giving the result we need to define our objects of interest.

Definition. Recall that $L^1(\mathbb{R}_+)\widehat{\otimes}L^1(\mathbb{R}_+)\widehat{\otimes}L^1(\mathbb{R}_+)$ is isometrically isomorphic to $L^1(\mathbb{R}_+^3)$. Let W be the permutation group of the three elements x_0, x_1, x_2. Then for $w \in W$ we define the *wedge*

$$Q_w = \left\{(x_1, x_2, x_0) \in \mathbb{R}_+^3 : x_1, x_2, x_0 \text{ satisfy the order structure given by } w\right\}. \tag{5.45}$$

For example, if $w_1 = (x_1, x_2, x_0)$ then

$$Q_{w_1} = \left\{(x_1, x_2, x_0) \in \mathbb{R}_+^3 : 0 < x_1 < x_2 < x_0\right\}. \tag{5.46}$$

Then for $w \in W$ we are able to define the *quadrant* $L^1(Q_w)$ as a subspace of $L^1(\mathbb{R}_+^3)$ such that $F \in L^1(Q_w)$ has support on the wedge Q_w.

We are now ready to give the following result, which will allow us to write

$F \in L^1(Q_w)$ as a linear combination of elementary tensor products of disjoint normalised functions.

Lemma 5.4. *Given $F \in L^1(Q_w)$, where $w \in W$, F can be written as*

$$\sum_{i=1}^{\infty} \alpha_i f_1^{(i)} \otimes f_2^{(i)} \otimes f_0^{(i)}, \tag{5.47}$$

where $\alpha_i \in \mathbb{C}$, $\sum_{i=1}^{\infty} |\alpha_i| < \infty$, $f_1^{(i)}, f_2^{(i)}, f_0^{(i)}$ are normalised $L^1(\mathfrak{B})$ functions for all $i \in \mathbb{N}$, and for each i the supports of these functions occur in the disjoint order defined by the permutation w.

Proof. Without loss of generality we simply consider the two-dimensional case and let $L^1(Q)$ to be the quadrant where functions are defined on the wedge $\{(x_1, x_0) \in \mathbb{R}_+^2 : x_0 < x_1\}$.

Define the following:

$$\begin{aligned} R_m &= \{(x_1, x_0) : x_1 > x_0 \text{ and } m-1 \leq x_1 < m\}, \\ I_{n,m}^k &= \left[(m-1) + \frac{2k-1}{2^{n+1}}, (m-1) + \frac{2k}{2^{n+1}}\right), \\ J_{n,m}^k &= \left[(m-1) + \frac{2k-2}{2^{n+1}}, (m-1) + \frac{2k-1}{2^{n+1}}\right), \end{aligned} \tag{5.48}$$

where $k = 1, \ldots, 2^n$, $m \in \mathbb{N}$ and $n = 0, 1, \ldots$. This is represented pictorially in Figures 5.5 and 5.6.

Then it follows that

$$R_1 = \bigcup_{n=0}^{\infty} \left(\bigcup_{k=1}^{2^n} I_{n,1}^k \times J_{n,1}^k \right), \tag{5.49}$$

and for $m \geq 2$

$$R_m = \left[\bigcup_{n=0}^{\infty} \left(\bigcup_{k=1}^{2^n} I_{n,m}^k \times J_{n,m}^k \right) \right] \cup \left[[m-1, m) \times [0, m) \right]. \tag{5.50}$$

Thus each R_m is a countable union of countable boxes. Since $Q = \bigcup_{m=1}^{\infty} R_m$, it follows that Q is countable union of the segments R_m and so is a countable union

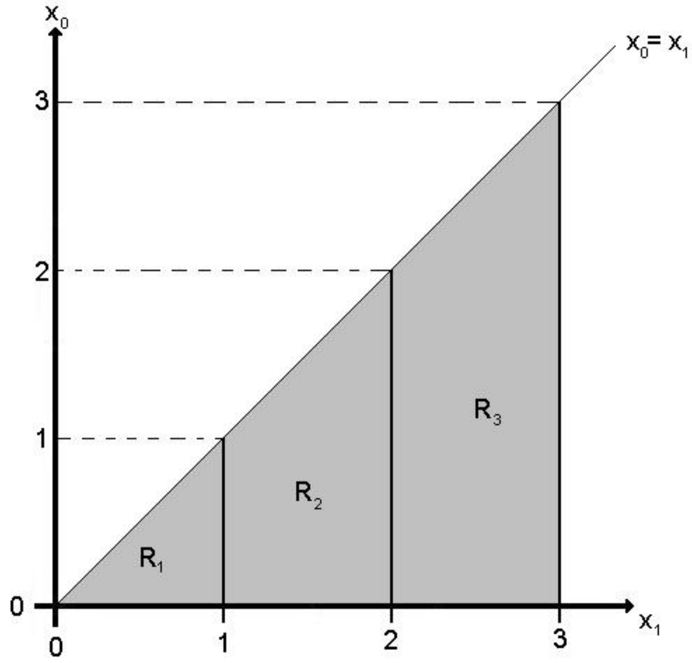


Figure 5.5: An illustration of the regions R_m for $m = 1, 2, \ldots$

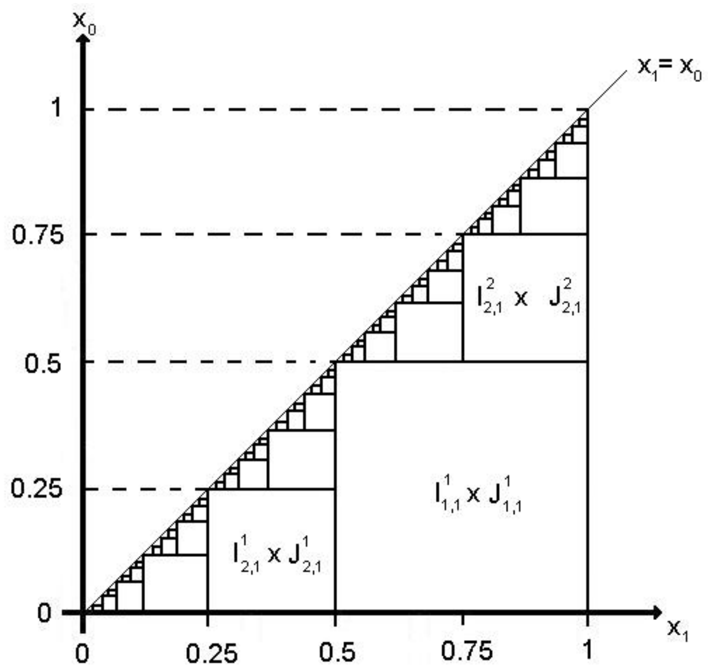


Figure 5.6: An illustration of splitting R_1 into boxes $I^k_{n,1}$ for $n = 0, 1, \ldots$ and $k = 1, \ldots, 2^n$

of a countable union of countable boxes. Hence the region $\{(x_1, x_0) : x_0 < x_1\}$ is itself also a countable union of boxes. Label each of these boxes in turn as $X_1, X_2, \ldots,$, so that

$$\{(x_1, x_0) : x_0 < x_1\} = \bigcup_{x=1}^{\infty} X_x. \tag{5.51}$$

Given $F \in L^1(Q)$ it follows that $F(x_1, x_0)$ is equivalent to $\sum_{x=1}^{\infty} F \cdot \chi_{X_x}$. For notational convenience, relabel $F \cdot \chi_{X_x}$ as F_x. Each F_x is defined on a box, and so is an element of $L^1(\mathbb{R}_+^2)$, and it follows that F_x is zero outside of the box which defines its support. Thus by Lemma (5.3) F_x can be written as

$$F_x = \sum_{y=1}^{\infty} \alpha_y f_1^{(y)} \otimes f_0^{(y)}, \tag{5.52}$$

with $\alpha_y \in \mathbb{C}$, $\sum_{y=1}^{\infty} |\alpha_y| < \infty$ and $\left\| f_i^{(y)} \right\| = 1$ for $i = 0, 1$ and $y \in \mathbb{N}$. Notice that by the nature of the wedge we are in it must follow that $f_0 \ll f_1$. Then

$$F(x_1, x_0) = \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} \alpha_y f_1^{(y)} \otimes f_0^{(y)}. \tag{5.53}$$

This is a countable sum of a countable sum and so is itself (trivially) a countable sum. As such, relabel the terms in turn and index them by i to yield

$$F(x_1, x_0) = \sum_{i=1}^{\infty} \alpha_i f_1^{(i)} \otimes f_0^{(i)}, \tag{5.54}$$

and hence proving the result as required. $\square$

It is clear after a little thought that this result extends to n dimensions, making this a useful tool for the higher-order cohomology groups we shall consider later.

The left and right spreading out functions

We now give some results regarding the left and right spreading-out functions. First we explore an inverse to both ϵ_k and γ_k before investigating the results of convolving I_1 in one variable with the 2-dimensional tensors $\epsilon_k(I_0)$ and $\gamma_k(I_0)$, where I_1, I_0 are normalised indicator functions.

Lemma 5.5. *Let $F \in \mathfrak{B}\widehat{\otimes}\mathfrak{B}$ and $I_i \in \mathfrak{B}$. Let $\pi_k : \mathfrak{B}\widehat{\otimes}\mathfrak{B} \to \mathfrak{B}$ be the function defined by*

$$\pi_k(F)(x) = \int_0^x F(t,x)dt + \int_0^x F(x,t)dt. \tag{5.55}$$

Then π_k is a left inverse to both ϵ_k and γ_k, that is to say $\pi_k(\epsilon_k(I_i))(x) = I_i(x)$ and $\pi_k(\gamma_k(I_i))(x) = I_i(x)$.

Proof. We only offer the proof for ϵ_k here; the proof for γ_k is very similar and can be deduced from the following method.

Calculating the value of $\pi_k \circ \epsilon_k$ on I_i it follows that

$$\begin{aligned}\pi_k(\epsilon_k(I_i))(x) &= \int_0^x \epsilon_k(I_i)(t,x)dt + \int_0^x \epsilon_k(I_i)(x,t)dt \\ &= \int_0^x I_i(x)\cdot k\cdot \chi_{[0,\frac{1}{k})}(x-t)dt + \int_0^x I_i(t)\cdot k \cdot \chi_{[0,\frac{1}{k})}(t-x)dt.\end{aligned} \tag{5.56}$$

The second term here reduces to zero as the range of the integral does not include the support of $\chi_{[0,\frac{1}{k})}(t-x)$, so

$$\begin{aligned}\pi_k(\epsilon_k(I_i))(x) &= I_i(x)\cdot k\int_0^x \chi_{[0,\frac{1}{k})}(x-t)dt \\ &= I_i(x)\cdot k\cdot\frac{1}{k} = I_i(x).\end{aligned} \tag{5.57}$$

□

A natural question which arises now concerns what happens when a function in one variable $I_i \in \mathfrak{B}$ is convolved with a two-dimensional function $F \in \mathfrak{B}\widehat{\otimes}\mathfrak{B}$. The natural module action of $\mathfrak{B}$ on $\mathfrak{B}\widehat{\otimes}\mathfrak{B}$ defines left and right multiplication as

$$(I_i * F)(x,y) = \int_0^x I_i(t)dt \cdot F(x,y) + I_i(x)\cdot\int_0^x F(t,y)dt \tag{5.58}$$

and

$$(F * I_i)(x,y) = \int_0^y F(x,t)dt\cdot I_i(y) + F(x,y)\cdot\int_0^y I_i(t)dt \tag{5.59}$$

respectively. This ensures that for $F = I_j \otimes I_k$ we have $I_i * (I_j \otimes I_k) = (I_i * I_j) \otimes I_k$ and $(I_j \otimes I_k) * I_i = I_j \otimes (I_k * I_i)$ as expected.

Lemma 5.6. *Let $I_0, I_1 \in \mathfrak{B}$ and let $k \in \mathbb{N}$ be large enough to ensure that that $I_1 \ll_k \epsilon_k(I_0)$. Then*

$$(I_1 * \epsilon_k(I_0))(x, y) = \epsilon_k(I_0)(x, y) = (\epsilon_k(I_0) * I_1)(x, y). \tag{5.60}$$

The necessity of proving both cases is due to the lack of commutativity regarding the product of a 1-dimensional tensor with a 2-dimensional tensor.

Proof. Again simply calculating the value of $I_1 * \epsilon_k(I_0)$ on x, y yields

$$\begin{aligned}(I_1 * \epsilon_k(I_0))(x, y) &= \int_0^x I_1(t)dt \cdot \epsilon_k(I_0)(x, y) + I_1(x) \cdot \int_0^x \epsilon_k(I_0)(t, y)dt \\ &= \int_0^x I_1(t)dt \cdot I_0(y) \cdot k \cdot \chi_{[0,\frac{1}{k})}(y - x) \\ &\qquad + I_1(x) \cdot k \int_0^x I_0(y)\chi_{[0,\frac{1}{k})}(y - t)dt.\end{aligned} \tag{5.61}$$

If $x \notin (y - \frac{1}{k}, y]$ then this whole expression reduces to zero. This is because $\chi_{[0,\frac{1}{k})}(y - x) = 0$ in the first term. If furthermore x is not in the support of I_1 then the second term is zero. If x *does* fall into the support of I_1 then $y - t > \frac{1}{k}$ for all $t \in [0, x]$ as $I_1 \ll I_0$ and y is in the support of I_0.

If on the other hand $x \in (y - \frac{1}{k}, y]$ then we have that x is not in the support of I_1 and so the second term is zero, leaving us with

$$\begin{aligned}(I_1 * \epsilon_k(I_0))(x, y) &= \int_0^x I_1(t)dt \cdot I_0(y) \cdot k \cdot \chi_{[0,\frac{1}{k})}(y - x) \\ &= 1 \cdot I_0(y) \cdot k \cdot \chi_{[0,\frac{1}{k})}(y - x) = \epsilon_k(I_0)(x, y).\end{aligned} \tag{5.62}$$

Thus we are able to conclude that

$$(I_1 * \epsilon_k(I_0))(x, y) = \epsilon_k(I_0)(x, y) \tag{5.63}$$

as $\epsilon_k(I_0)(x, y) = 0$ if $x \notin (y - \frac{1}{k}, y]$ as required.

For the other case the multiplication yields

$$
\begin{aligned}
(\epsilon_k(I_0) * I_1)(x,y) &= \int_0^y \epsilon_k(I_0)(x,t)dt \cdot I_1(y) + \epsilon_k(I_0)(x,y) \cdot \int_0^y I_1(t)dt \\
&= kI_1(y)\int_0^y I_0(t)\chi_{[x,x+\frac{1}{k})}(t)dt + kI_0(y)\chi_{[x,x+\frac{1}{k})}(y)\int_0^y I_1(t)dt.
\end{aligned}
\tag{5.64}
$$

For this to be nontrivial we require that y is in the support of either I_1 or I_0; obviously it cannot be in both as they are ordered and disjoint.

If y is in the support of I_1 then the integral in the first term is zero while $I_0(y) = 0$ in the second term, returning an overall trivial solution.

Thus if y is in the support of I_0 the first term is zero while the second term becomes

$$
kI_0(y)\chi_{[x,x+\frac{1}{k})}(y) \cdot 1 = \epsilon_k(I_0)(x,y), \tag{5.65}
$$

proving the second case as required. $\square$

The results of these multiplications are quite different when the order structure of I_0 and I_1 are reversed.

Proposition 5.1. *Let $I_0, I_1 \in \mathfrak{B}$ be normalised indicator functions such that $I_0 \ll I_1$. Then*

$$
(I_1 * \epsilon_k(I_0))(x,y) = I_1(x)I_0(y) \tag{5.66}
$$

and

$$
(\epsilon_k(I_0) * I_1)(x,y) = I_1(y)f(x), \tag{5.67}
$$

where

$$
f(x) = \begin{cases} 0, & x \notin (a_0 - \frac{1}{k}, b_0), \\ \frac{(x-a_0)k+1}{b_0-a_0}, & x \in (a_0 - \frac{1}{k}, a_0], \\ \frac{1}{b_0-a_0}, & x \in (a_0, b_0 - \frac{1}{k}], \\ \frac{k(b_0-x)}{b_0-a_0}, & (b_0 - \frac{1}{k}, b_0]. \end{cases} \tag{5.68}
$$

Proof. **Case 1:** $I_1 * \epsilon_k(I_0)$

Calculating the value of $I_1 * \epsilon_k$ on I_0 we have that

$$\begin{aligned}(I_1 * \epsilon_k(I_0))(x, y) &= \int_0^x I_1(t)dt \cdot \epsilon_k(I_0)(x, y) + I_1(x) \cdot \int_0^x \epsilon_k(I_0)(t, y)dt \\ &= k \cdot I_0(y)\chi_{[x,x+\frac{1}{k})}(y) \int_0^x I_1(t)dt \\ &\quad + k \cdot I_1(x) \cdot I_0(y) \int_0^x \chi_{[0,\frac{1}{k})}(y-t)dt.\end{aligned} \tag{5.69}$$

In order for this to be nontrivial we need y to be in the support of I_0.

If $x \notin (y - \frac{1}{k}, y]$ then the first term is zero as $\chi_{[x,x+\frac{1}{k})}(y) = 0$. There are now two possibilities for where x can be. If x is not in the support of I_1 then the second term is also trivial. On the other hand if x *is* in the support of I_1 then the range of the integral in the second term covers the whole of the support of $\chi_{[0,\frac{1}{k})}(y-t)$, and thus this term becomes

$$k \cdot I_1(x) \cdot I_0(y) \cdot \frac{1}{k} = I_1(x)I_0(y). \tag{5.70}$$

Alternatively if $x \in (y - \frac{1}{k}, y]$ then it follows that $x \leq y$. Thus as y is in the support of I_0 and $I_0 \ll I_1$ it follows that x is not in the support of I_1. This makes the integral in the first term and $I_1(x)$ in the second term both equal to 0, making the entire expression trivial.

The conditions and definitions of this expression allow us to conclude that

$$(I_1 * \epsilon_k(I_0))(x, y) = I_1(x)I_0(y) \tag{5.71}$$

as required.

Case 2: $\epsilon_k(I_0) * I_1$

This case is a lot trickier. In calculating $\epsilon_k(I_0) * I_1$ on I_0 we simply get

$$\begin{aligned}(\epsilon_k(I_0) * I_1)(x, y) &= \int_0^y \epsilon_k(I_0)(x, t)dt \cdot I_1(y) + \epsilon_k(I_0)(x, y) \cdot \int y_0 I_1(t)dt \\ &= k \cdot I_1(y) \int_0^y I_0(t)\chi_{[0,\frac{1}{k})}(t - x)dt \\ &\qquad + k \cdot I_0(y) \cdot \chi_{[0,\frac{1}{k})}(y - x) \int_0^y I_1(t)dt.\end{aligned} \tag{5.72}$$

In order for this to be nontrivial we again require that y is in the support of either I_0 or I_1; obviously it cannot be in both as their supports are ordered and disjoint.

If y is in the support of I_0 then $I_1(y)$ in the first term and the integral in the second term are both zero, making the overall result trivial; the integral is zero because the range of the integral is not large enough to reach the support of I_1.

On the other hand if y is in the support of I_1 then the second term is zero as $I_0(y) = 0$. The first term however simply becomes

$$k \cdot I_1(y) \int_0^y I_0(t)\chi_{[0,\frac{1}{k})}(t - x)dt = k \cdot I_1(y) \cdot \int_{a_0}^{b_0} \frac{1}{b_0 - a_0} \cdot \chi_{[x,x+\frac{1}{k})}(t)dt \tag{5.73}$$

as $[a_0, b_0] \subset [0, y]$ and $I_0(t) = 0$ for all $t \notin [a_0, b_0]$ and using the definition of I_0 on $[a_0, b_0]$.

It is immediately clear after a little thought that $\chi_{[x,x+\frac{1}{k})}(t) = 0$ for all $x \notin (a_0 - \frac{1}{k}, b_0]$.

When $x \in (a_0 - \frac{1}{k}, a_0]$ then the overlap between the support of the function $\chi_{[x,x+\frac{1}{k})}(t)$ and the range of the integral $[a_0, b_0]$ is $x + \frac{1}{k} - a_0$. This makes the integral simply the area of the box with height one and width $(x + \frac{1}{k} - a_0)$ and hence the expression becomes

$$k \cdot I_1(y) \cdot \frac{1}{b_0 - a_0} \cdot (x + \frac{1}{k} - a_0) = I_1(y) \cdot \frac{(x - a_0)k + 1}{b_0 - a_0}. \tag{5.74}$$

When $x \in (a_0, b_0 - \frac{1}{k}]$ the whole support of the function $\chi_{[x,x+\frac{1}{k})}(t)$ is contained within the range of the integral. Thus the integral is the area under the whole of

$\chi_{[x,x+\frac{1}{k})}(t)$, which is $\frac{1}{k}$ and the expression becomes

$$k \cdot I_1(y) \cdot \frac{1}{b_0 - a_0} \cdot \frac{1}{k} = I_1(y) \cdot \frac{1}{b_0 - a_0}. \tag{5.75}$$

Finally if $x \in (b_0 - \frac{1}{k}, b_0]$ then the amount of overlap between the range of the integral $[a_0, b_0]$ and the support of the function $\chi_{[x,x+\frac{1}{k})}(t)$ is $(b_0 - x)$. Then the integral is simply the area of the box of height one and width $(b_0 - x)$, making the expression become

$$k \cdot I_1(y) \cdot \frac{1}{b_0 - a_0} \cdot (b_0 - x) = I_1(y) \cdot \frac{k(b_0 - x)}{b_0 - a_0}. \tag{5.76}$$

Putting all of these together yields the second case of the proposition, as required.

□

Proposition (5.1) essentially shows that both $(I_1 * \epsilon_k(I_0))(x, y) = I_1(x)I_0(y)$ and $(\epsilon_k(I_0) * I_1)(x, y) = I_1(y)f(x)$ are 2-dimensional elementary tensor products.

In the first case this is simply $I_1 \otimes I_0$ with order structure $I_0 \ll I_1$.

The second one is the one which will appear in our future calculations. Here the tensor product is $I_1 \otimes f$, but f is not a normalised indicator function. Notice though that the support of f is disjoint from and entirely below the support of I_1. Also f is defined on an interval $(a_0 - \frac{1}{k}, b_0]$ and it is normalised; we leave this last as an exercise. We are also able to approximate this by a linear combination of normalised indicator functions

$$f(x) = \sum_{j=0}^{\infty} \alpha_j J_j(x), \tag{5.77}$$

such that $J_j(x) \ll I_1$ for all j such that $0 \leq j < \infty$, with $\alpha_j \in \mathbb{C}$. Then for $I_2 \ll I_0 \ll I_1$ and $I_2 \ll_k \epsilon_k(I_0)$ we have that

$$\begin{aligned}\widetilde{\varphi}(I_2 \otimes \epsilon_k(I_0) * I_1) &= \sum_{j=0}^{\infty} \alpha_j \widetilde{\varphi}(I_2 \otimes I_1 \otimes J_j) \\ &= \sum_{j=0}^{\infty} \alpha_j \widetilde{\varphi}(I_2 \otimes J_j \otimes I_1),\end{aligned} \tag{5.78}$$

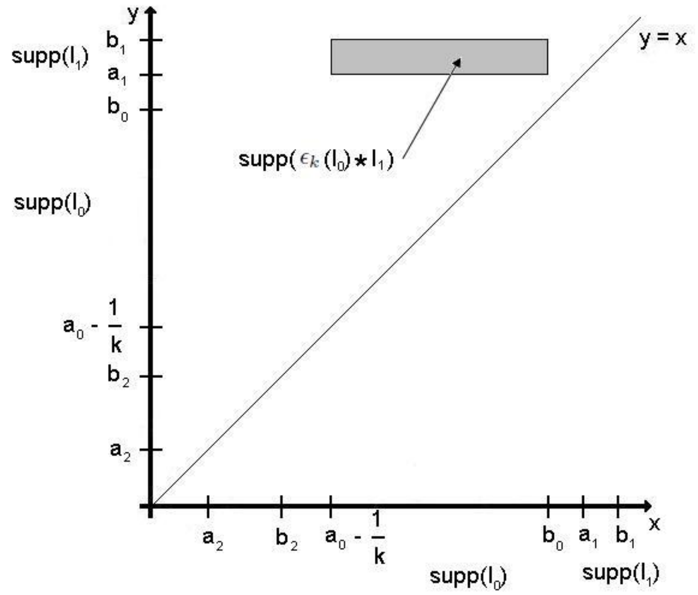


Figure 5.7: An illustration of the support of $\epsilon_k(I_0) * I_1$ for $I_0 \ll I_1$

by Lemma (5.2) and here we have that $I_2 \ll J_j \ll I_1$ such that all are normalised for $j = 0, 1, \ldots$. This will be very useful in the next subsection.

Similar results hold for the right spreading-out function γ_k.

Proposition 5.2. *Let I_1, I_0 be normalised indicator functions. If $I_1 \ll \gamma_k(I_0)$ then*

$$(I_1 * \gamma_k(I_0))(x, y) = (\gamma_k(I_0) * I_1)(x, y) = \gamma_k(I_0)(x, y). \tag{5.79}$$

If on the other hand $I_0 \ll I_1$ then

$$(\gamma_k(I_0) * I_1)(x, y) = I_0(x)I_1(y) \tag{5.80}$$

and

$$(I_1 * \gamma_k(I_0))(x, y) = I_1(x)f(y), \tag{5.81}$$

where

$$f(y) = \begin{cases} \frac{k(y-a_0)+1}{b_0-a_0}, & y \in (a_0 - \frac{q}{k}, a_0], \\ \frac{1}{b_0-a_0}, & y \in (a_0, b_0 - \frac{1}{k}], \\ \frac{k(b_0-y)}{b_0-a_0}, & y \in (b_0 - \frac{1}{k}, b_0], \\ 0, & \textit{else}. \end{cases} \tag{5.82}$$

The proof of this proposition is similar to the combined proofs of Lemma (5.6) and Proposition(5.1).

Remark. We may worry about spreading out a function so that its domain is extended beyond zero and into a negative value. In our approach this is never a problem as we only apply ϵ_k and γ_k to indicator functions which in the imposed order structure are defined to be the largest. Thus there is always another indicator function present which has support lying between 0 and the support of the function we wish to spread out.

An alternative solution to this problem is to invoke the argument used in Section 6.4 which is a density argument for a separation from zero.

The alternative proof of Theorem (5.1)

We now present the alternative proof of Theorem (5.1). The reason for doing this is explained at the start of this chapter and is due to our need for a more direct approach to calculating the higher level simplicial cohomology groups of $\mathfrak{B}$; the method we use here is the method that we have sought from the start.

Proof. [Alternative for Theorem (5.1)] With $I_1 \ll I_0$, replacing I_0 with $\epsilon_k(I_0)$ has a similar effect to choosing an I_2 such that $I_1 \ll I_2 \ll I_0$ with $|I_2| = \frac{1}{k}$. To ensure the support of $\epsilon_k(I_0)$ remains disjoint from that of I_1 and that the the spreading out is as minimal as possible it is necessary to take the limit as k tends to ∞.

Then in a manner similar to the first proof we set

$$\widetilde{\psi}(I_1 \otimes I_0) = \lim_{k\to\infty} \widetilde{\varphi}(I_1 \otimes \epsilon_k(I_0)). \tag{5.83}$$

This can clearly always be done, and as long as the limit exists this function is always well-defined. In this case we will see in (5.89) that the limit *does* exist for cocycles as the value of $\widetilde{\varphi}(I_1 \otimes \epsilon_k(I_0))$ is eventually constant for large enough k. Notice that this only holds for cocycles and not cochains in general due to fact this follows from the cocycle identity. It is also clearly bounded and linear given its definition.

Recall that $I_1 \ll I_2 \ll I_0$. Also choose $k \in \mathbb{N}$ such that $I_2 \ll_k \epsilon_k(I_0)$. Then using the cocycle identity we get that

$$\begin{aligned}
0 &= (\delta^3\widetilde{\varphi})(I_1 \otimes I_2 \otimes \epsilon_k(I_0)) \\
&= \widetilde{\varphi}(I_2 \otimes \epsilon_k(I_0) * I_1) - \widetilde{\varphi}(I_1 * I_2 \otimes \epsilon_k(I_0)) \\
&\qquad + \widetilde{\varphi}(I_1 \otimes I_2 * \epsilon_k(I_0)) - \widetilde{\varphi}(I_1 \otimes I_2 \otimes \pi_k(\epsilon_k(I_0))).
\end{aligned} \tag{5.84}$$

This equation is verified if you write $\epsilon_k(I_0)$ as a linear combination of elementary tensor products of normalised functions of disjoint support which in turn can, via density arguments, be approximated by normalised indicator functions of disjoint support. So let $\epsilon_k(I_0) = \sum_{i=1}^{\infty} \alpha_i f_i \otimes g_i$ where $\alpha_i \in \mathbb{C}$, $f_i, g_i \in L^1(\mathfrak{B})$ and $\|f_i\| = \|g_i\| = 1$ as in Lemma (5.4). We are also able to assure the order structure $I_2 \ll f_i \ll g_i$ for $i = 1, 2, \ldots$. Thus we have that

$$\begin{aligned}
0 &= (\delta^3\widetilde{\varphi})(I_1 \otimes I_2 \otimes \epsilon_k(I_0)) \\
&= \sum_{i=1}^{\infty} \alpha_i(\delta^3\widetilde{\varphi})(I_1 \otimes I_2 \otimes f_i \otimes g_i) \\
&= \sum_{i=1}^{\infty} \alpha_i \left[\varphi(I_2 \otimes f_i \otimes g_i * I_1) - \varphi(I_1 * I_2 \otimes f_i \otimes g_i) + \varphi(I_1 \otimes I_2 * f_i \otimes g_i)\right. \\
&\qquad \left. -\varphi(I_1 \otimes I_2 \otimes f_i * g_i)\right].
\end{aligned} \tag{5.85}$$

The first two terms cancel each other out, leaving us with

$$\begin{aligned}
\sum_{i=1}^{\infty} \alpha_i \varphi(I_1 \otimes f_i \otimes g_i) &= \sum_{i=1}^{\infty} \alpha_i \varphi(I_1 \otimes I_2 \otimes f_i * g_i) \\
\text{i.e. } \varphi(I_1 \otimes \epsilon_k(I_0)) &= \varphi(I_1 \otimes I_2 \otimes \sum_{i=1}^{\infty} \alpha_i f_i * g_i).
\end{aligned} \tag{5.86}$$

Now note that by definition

$$\begin{aligned}\pi_k(f_i \otimes g_i)(x) &= \int_0^x (f_i \otimes g_i)(x,t)dt + \int_0^x (f_i \otimes g_i)(t,x) \\ &= f_i(x)\int_0^x g_i(t)dt + g_i(x)\int_0^x f_i(t)dt \\ &= (f_i * g_i)(x)\end{aligned} \tag{5.87}$$

and so

$$\begin{aligned}I_0 &= \pi_k(\epsilon_k(I_0)) \\ &= \sum_{i=1}^{\infty} \alpha_i \pi_k(f_i \otimes g_i) \\ &= \sum_{i=1}^{\infty} \alpha_i f_i * g_i.\end{aligned} \tag{5.88}$$

Therefore Equation (5.84) reduces to give

$$\widetilde{\varphi}(I_1 \otimes I_2 \otimes I_0) = \widetilde{\varphi}(I_1 \otimes \epsilon_k(I_0)), \tag{5.89}$$

and since the left hand side is a constant for given I_0, I_1, I_2 it follows that for large enough k the right hand side is as well. Therefore taking the limit as k tends to ∞ *does* exist and so

$$\widetilde{\varphi}(I_1 \otimes I_2 \otimes I_0) = \lim_{k\to\infty} \widetilde{\varphi}(I_1 \otimes \epsilon_k(I_0)) = \widetilde{\psi}(I_1 \otimes I_0). \tag{5.90}$$

Finally we have that

$$\begin{aligned}(\delta^2\widetilde{\psi})(I_1 \otimes I_2 \otimes I_0) &= \widetilde{\psi}(I_2 \otimes I_0 * I_1) - \widetilde{\psi}(I_1 * I_2 \otimes I_0) + \widetilde{\psi}(I_1 \otimes I_2 * I_0) \\ &= \widetilde{\psi}(I_1 \otimes I_0),\end{aligned} \tag{5.91}$$

and hence

$$\widetilde{\varphi}(I_1 \otimes I_2 \otimes I_0) = (\delta^2\widetilde{\psi})(I_1 \otimes I_2 \otimes I_0), \tag{5.92}$$

i.e.

$$\varphi(I_1, I_2)(I_0) = (\delta^2\psi)(I_1, I_2)(I_0) \tag{5.93}$$

for our given φ and our chosen ψ as required. □

We now need to extend this result to include other order structures of the functions I_0, I_1, I_2 in $\mathfrak{B}$.

5.1.2 The remaining cases

Case 1: $I_2 \ll I_0$ and $I_2 \ll I_1$

All that has changed in this case is that we now allow I_1 to fit anywhere disjointly within the order structure such that it is not the smallest any more, i.e. either of $I_1 \ll I_0$ or $I_0 \ll I_1$ could hold here. This ensures that the first case is not repeated.

Switching to tensor notation and taking k to be large enough so that $I_2 \ll_k \epsilon_k(I_0)$ we have

$$\begin{aligned} 0 &= (\delta^3\widetilde{\varphi})(I_1 \otimes I_2 \otimes \epsilon_k(I_0)) \\ &= \widetilde{\varphi}(I_2 \otimes \epsilon_k(I_0) * I_1) - \widetilde{\varphi}(I_1 \otimes \epsilon_k(I_0)) + \widetilde{\varphi}(I_1 \otimes \epsilon_k(I_0)) \\ &\quad - \widetilde{\varphi}(I_1 \otimes I_2 \otimes \pi(\epsilon_k(I_0))), \end{aligned} \tag{5.94}$$

and thus

$$\widetilde{\varphi}(I_1 \otimes I_2 \otimes I_0) = \widetilde{\varphi}(I_2 \otimes \epsilon_k(I_0) * I_1). \tag{5.95}$$

If $I_1 \ll I_0$ then this is simply the previous case. On the other hand if $I_0 \ll I_1$ then this becomes

$$\widetilde{\varphi}(I_1 \otimes I_2 \otimes I_0) = \widetilde{\varphi}(I_2 \otimes I_1 \otimes f) \tag{5.96}$$

for f as defined in Proposition (5.1). Writing f as a linear combination of normalised indicator functions, $\sum_{j=0}^{\infty} a_j J_j$, where $a_i \in \mathbb{C}$, we have that

$$\begin{aligned}
\widetilde{\varphi}(I_2 \otimes I_1 \otimes f) &= \widetilde{\varphi}(I_2 \otimes I_1 \otimes \sum_{j=0}^{\infty} a_j J_j) \\
&= \sum_{j=0}^{\infty} a_j \widetilde{\varphi}(I_2 \otimes I_1 \otimes J_j) \\
&= \sum_{j=0}^{\infty} a_j \widetilde{\varphi}(I_2 \otimes J_j \otimes I_1) \text{ (by Lemma 5.2)} \\
&= \sum_{j=0}^{\infty} a_j (\delta^2 \widetilde{\psi})(I_2 \otimes J_j \otimes I_1) \text{ (by Section 5.1.1)} \\
&= (\delta^2 \widetilde{\psi})(I_2 \otimes \sum_{j=0}^{\infty} a_j J_j \otimes I_1) \\
&= (\delta^2 \widetilde{\psi})(I_2 \otimes f \otimes I_1),
\end{aligned} \tag{5.97}$$

where $\widetilde{\psi}$ is defined as in Section 5.1.1. By the extension to Lemma (5.1) this is equivalent to $(\delta^2 \widetilde{\psi})(I_2 \otimes I_0 \otimes I_1)$ in tensor notation and thus

$$\widetilde{\varphi}(I_2 \otimes I_0 \otimes I_1) = (\delta^2 \widetilde{\psi})(I_2 \otimes I_0 \otimes I_1) \tag{5.98}$$

for $\widetilde{\psi}$ as defined in Section 5.1.1. This shows that the second case is also cobounded by our chosen ψ.

Case 2: $I_0 \ll I_2 \ll I_1$

This is the very difficult case to deal with using our original method, but here it becomes a lot more straightforward.

In the original method we attempt to cobound $\varphi(I_1, I_2)(I_0)$ and we would define $\psi'(I_1)(I_0)$ by finding the function of largest support between and disjoint from the supports of I_1 and I_0, say I, then writing

$$\psi'(I_1)(I_0) = \varphi(I_1, I)(I_0). \tag{5.99}$$

Then we would show that this ψ' cobounds φ.

In this method we are looking at the largest normalised indicator function in our order structure and stretching its support out into a 2-dimensional tensor product. We have to be careful that this stretching out is in the right direction in order to preserve order relations.

We have been given $\varphi \in \mathcal{Z}^2(\mathfrak{B})$. With the order case defined here cobounding this directly is difficult. Therefore we switch to tensor notation and make the definition

$$\widetilde{\varphi}' = \widetilde{\varphi} - \delta^2 \widetilde{\psi} \tag{5.100}$$

where $\widetilde{\psi}$ is defined as in the previous cases. This ensures that if both $I_2 \ll I_0$, $I_2 \ll I_1$ and one of $I_1 \ll I_0$ or $I_0 \ll I_0$ hold then $\widetilde{\varphi}'(I_1 \otimes I_2 \otimes I_0) = 0$. In other words the order structure of the previous cases makes this a trivial result.

Thus following this and the logic of calculations done in the previous cases and switching to tensor notation we make the definition

$$\widetilde{\psi}'(I_1 \otimes I_0) = \begin{cases} -\lim\limits_{k\to\infty} \widetilde{\varphi}'(\gamma_k(I_1) \otimes I_0), & I_0 \ll I_1, \\ 0, & \text{else}, \end{cases} \tag{5.101}$$

in order to attempt to cobound $\widetilde{\varphi}'$.

This is well-defined as it can always be done without ambiguity. The reason for the minus sign will become apparent.

Writing $\gamma_k(I_1)$ as a linear combination of normalised elementary tensor products, the cocycle identity then says

$$\begin{aligned} 0 &= (\delta^3 \widetilde{\varphi}')(\gamma_k(I_1) \otimes I_2 \otimes I_0) \\ &= \sum_{i=1}^{\infty} \alpha_i (\delta^3 \widetilde{\varphi}')(f_i \otimes g_i \otimes I_2 \otimes I_0). \end{aligned} \tag{5.102}$$

Note that for each $i \in \mathbb{N}$ we have that $I_2 \ll g_i \ll f_i$ and so this becomes

$$\begin{aligned}
0 &= \sum_{i=1}^{\infty} \alpha_i \Big[\widetilde{\varphi}'(g_i \otimes I_2 \otimes I_0 * f_i) - \widetilde{\varphi}'(f_i * g_i \otimes I_2 \otimes I_0) + \widetilde{\varphi}'(f_i \otimes g_i * I_2 \otimes I_0) \\
&\qquad - \widetilde{\varphi}'(f_i \otimes g_i \otimes I_2 * I_0) \Big] \\
&= \sum_{i=1}^{\infty} \alpha_i \widetilde{\varphi}'(g_i \otimes I_2 \otimes f_i) - \widetilde{\varphi}'(I_1 \otimes I_2 \otimes I_0) + \widetilde{\varphi}'(\gamma_k(I_1) \otimes I_0) \\
&\qquad - \widetilde{\varphi}'(\gamma_k(I_1) \otimes I_2),
\end{aligned} \tag{5.103}$$

where the second term arises because

$$I_1 = \pi_k(\gamma_k(I_1)) = \sum_{i=1}^{\infty} \alpha_i \pi_k(f_i \otimes g_i) = \sum_{i=1}^{\infty} \alpha_i f_i * g_i \tag{5.104}$$

as earlier.

Each of the terms in the sum in the first term here has terms with order structure satisfying the condition for triviality given above, i.e. it is an order structure from the previous cases and so

$$\sum_{i=1}^{\infty} \alpha_i \widetilde{\varphi}'(g_i \otimes I_2 \otimes f_i) = \sum_{i=1}^{\infty} \alpha_i \cdot 0 = 0. \tag{5.105}$$

Thus (5.103) becomes

$$\widetilde{\varphi}'(I_1 \otimes I_2 \otimes I_0) = 0 + \widetilde{\varphi}'(\gamma_k(I_1) \otimes I_0) - \widetilde{\varphi}'(\gamma_k(I_1) \otimes I_2). \tag{5.106}$$

Taking the limit as k tends to ∞ then yields

$$\begin{aligned}
\widetilde{\varphi}'(I_1 \otimes I_2 \otimes I_0) &= 0 + \lim_{k \to \infty} \widetilde{\varphi}'(\gamma_k(I_1) \otimes I_0) - \lim_{k \to \infty} \widetilde{\varphi}'(\gamma_k(I_1) \otimes I_2) \\
&= \widetilde{\psi}'(I_2 \otimes I_1) - \widetilde{\psi}'(I_1 \otimes I_0) + \widetilde{\psi}'(I_1 \otimes I_2) \\
&= (\delta^2 \widetilde{\psi}')(I_1 \otimes I_2 \otimes I_0),
\end{aligned} \tag{5.107}$$

i.e.

$$\widetilde{\varphi}'(I_1 \otimes I_2 \otimes I_0) = (\delta^2 \widetilde{\psi}')(I_1 \otimes I_2 \otimes I_0). \tag{5.108}$$

Now in $\mathcal{HH}^2(B)$ since $\widetilde{\varphi}' = \delta^2\widetilde{\psi}'$ it follows that $\widetilde{\varphi}' \in [0]$, the equivalence class of zero. Also $\widetilde{\varphi} - \widetilde{\varphi}' = \delta^2\widetilde{\psi}$ and so $\widetilde{\varphi}$ and $\widetilde{\varphi}'$ are equivalent, meaning that we have cobounded $\widetilde{\varphi}$ in this order case.

Case 3: $I_0 \ll I_2$ **and** $I_1 \ll I_2$

This case is similar to Case 1, but with the orders reversed. It is the final case we need to consider here.

For convenience we define

$$\widetilde{\varphi''} = \widetilde{\varphi}' - \delta^2\widetilde{\psi}'. \tag{5.109}$$

Thus $\widetilde{\varphi''}$ is in the same equivalence class in $\mathcal{HH}^2(\mathfrak{B})$ as $\widetilde{\varphi}'$ and hence $\widetilde{\varphi}$, and $\widetilde{\varphi''}(I_1 \otimes I_2 \otimes I_0) = 0$ for the order structures considered so far.

This time writing $\gamma_k(I_2)$ as a linear combination of normalised elementary tensor products the cocycle identity gives us that

$$\begin{aligned} 0 &= (\delta^3\widetilde{\varphi''})(I_1 \otimes \gamma_k(I_2) \otimes I_0) \\ &= \sum_{i=1}^{\infty} \alpha_i(\delta^3\widetilde{\varphi''})(I_1 \otimes f_i \otimes g_i \otimes I_0). \end{aligned} \tag{5.110}$$

Again note that $I_1 \ll g_i \ll f_i$ and $I_0 \ll g_i \ll f_i$ here for all $i \in \mathbb{N}$. Thus

$$\begin{aligned} 0 = \sum_{i=1}^{\infty} \alpha_i \Big[&\widetilde{\varphi''}(f_i \otimes g_i \otimes I_0 * I_1) - \widetilde{\varphi''}(I_1 * f_i \otimes g_i \otimes I_0) + \widetilde{\varphi''}(I_1 \otimes f_i * g_i \otimes I_0) \\ &- \widetilde{\varphi''}(I_1 \otimes f_i \otimes g_i * I_0) \Big] \\ = \sum_{i=1}^{\infty} \alpha_i \Big[&\widetilde{\varphi''}(f_i \otimes g_i \otimes I_0 * I_1) - \widetilde{\varphi''}(f_i \otimes g_i \otimes I_0) + \widetilde{\varphi''}(I_1 \otimes f_i * g_i \otimes I_0) \\ &- \widetilde{\varphi''}(I_1 \otimes f_i \otimes g_i) \Big]. \end{aligned} \tag{5.111}$$

In the first and second terms we have that $I_0 * I_2 \ll g_i \ll f_i$, as $I_0 * I_1$ is either I_0 or I_1 depending on their relative ordering, and $I_0 \ll g_i \ll f_i$. These are both exactly the order structure from Case 2 making these terms trivial.

The third term, as before, is simply $\widetilde{\varphi''}(I_1 \otimes I_2 \otimes I_0)$ in precisely the same way as in the previous cases.

The final term becomes $\widetilde{\varphi''}(I_1 \otimes g_i \otimes f_i)$ by Lemma (5.2). Here we have that $I_1 \ll g_i \ll f_i$, which is the order structure from the initial case, making this term also trivial.

Hence we conclude that

$$\widetilde{\varphi''}(I_1 \otimes I_2 \otimes I_0) = 0, \tag{5.112}$$

which is trivially cobounded by $\widetilde{\psi''} \equiv 0$.

The final case

Reverting back to our standard notation we have so far shown that given the normalised indicator functions I_1, I_2, I_0 in $\mathfrak{B}$ such that their supports occur in some disjoint order structure it is possible to cobound $\varphi(I_1, I_2)(I_0)$. We now extend this to general $\mathfrak{B}$ functions f_1, f_2, f_0 in place of I_1, I_2, I_0.

Since the indicator functions are dense in $\mathfrak{B}$ we need only consider the result on normalised indicator functions; the final outcome follows by linearity. The difference here is that our normalised indicator functions no longer necessarily have disjoint ordered supports.

However, following the examples found at the ends of both Chapter 3 and Chapter 4 by subdividing the intervals on which each normalised indicator function is defined further and further we can show that the eventual contribution from the overlapping supports is negligible, enabling us to once again simply focus on those normalised indicator functions of ordered disjoint support.

The fundamental concept to this idea is to note that subdividing the supports of each of these indicator functions into M equal subintervals we have that

$$\left| \sum_{I_i \cap J_j \cap K_k \neq \emptyset} \big(\varphi(I_i, J_j)(K_k) - (\delta^2\psi)(I_i, J_j)(K_k)\big) \right| \tag{5.113}$$

is less than or equal to

$$6M^2 \cdot \left(\|\varphi\| \frac{|I|}{M}\frac{|J|}{M}\frac{|K|}{M} + \left\|\delta^2\psi\right\| \frac{|I|}{M}\frac{|J|}{M}\frac{|K|}{M} \right) \tag{5.114}$$

which tends to zero as M increases to infinity. Following the logic in Equation (4.105) this shows that the contribution from places of overlap is reduced to nothing, or simply that the volume, or Lebesgue measure, of the supports in places of overlap tends to zero.

So let $F \in \mathfrak{B}\widehat{\otimes}\mathfrak{B}\widehat{\otimes}\mathfrak{B} \cong L^1(\mathbb{R}_+^3)$. We define

$$F_w(x_1, x_2, x_0) = \begin{cases} F(x_1, x_2, x_0), x_1, x_2, x_0 \text{ satisfy the order structure defined by } w, \\ 0, \text{ else.} \end{cases} \tag{5.115}$$

Since $\bigsqcup_{w \in W} Q_w$ is dense in $\mathbb{R}_+^3$ and $\lambda\left(\mathbb{R}_+^3 \setminus \bigsqcup_{w \in W} Q_w\right) = 0$, where we are able to use the disjoint union symbol $\bigsqcup$ (as the Q_w are pairwise disjoint), then by the above argument on subdividing supports further and further we have that $\bigoplus_{w \in W} L^1(Q_w) = L^1(\mathbb{R}_+^3)$. Therefore it follows that we can now write F as the ℓ^1-sum

$$F(x_1, x_2, x_0) = \bigoplus_{w \in W} F_w(x_1, x_2, x_0), \tag{5.116}$$

where $\|F\| = \sum_{w \in W} \|F_w\|$.

As $F_w \in L^1(Q_w)$ then by Lemma (5.4) we are able to write this as a linear combination of elementary tensor products of disjoint normalised functions. Thus we are able to write F as

$$F = \bigoplus_{w \in W} \sum_{i=1}^{\infty} \alpha_i f_1^{(i)} \otimes f_2^{(i)} \otimes f_3^{(i)}. \tag{5.117}$$

Hence it is enough by linearity to only consider our given 2-cocycle to be defined on such elementary tensor products of disjoint normalised functions. Finally, by using density arguments, we can approximate these functions by linear combinations of normalised *indicator* functions of disjoint supports.

So we have that $\varphi(f_1, f_2)(f_0)$ is equivalent to $\sum_{i=1}^{\infty} \alpha_i \varphi(I_1^i, I_2^i)(I_0^i)$ for disjoint normalised indicator functions I_1^i, I_2^i, I_0^i. Switching to tensor notation this is $\sum_{i=1}^{\infty} \alpha_i \widetilde{\varphi}(I_1^i \otimes I_2^i \otimes I_0^i)$.

For each $i \in \mathbb{N}$ we have that $I_1^i \otimes I_2^i \otimes I_0^i \in L^1(Q_w)$ for some $w \in W$, the permutation group of three elements. Thus by defining $\widetilde{\psi}$ by quadrants, which is clearly possible, as given by each of the order structure cases throughout this section we can cobound each of the terms in this sum individually. Thus we have that

$$\begin{aligned}
\widetilde{\varphi}(f_1 \otimes f_2 \otimes f_0) &= \widetilde{\varphi}\left(\sum_{i=1}^{\infty} \alpha_i I_1^i \otimes I_2^i \otimes I_0^i\right) \\
&= \sum_{i=1}^{\infty} \alpha_i \widetilde{\varphi}(I_1^i \otimes I_2^i \otimes I_0^i) \\
&= \sum_{i=1}^{\infty} \alpha_i (\delta^2 \widetilde{\psi})(I_1^i \otimes I_2^i \otimes I_0^i) \\
&= (\delta^2 \widetilde{\psi})\left(\sum_{i=1}^{\infty} \alpha_i I_1^i \otimes I_2^i \otimes I_0^i\right) \\
&= (\delta^2 \widetilde{\psi})(f_1 \otimes f_2 \otimes f_0),
\end{aligned} \tag{5.118}$$

by linearity.

In standard notation we thus have

$$\varphi(f_1, f_2)(f_0) = (\delta^2 \psi)(f_1, f_2)(f_0). \tag{5.119}$$

Therefore given a 2-cocycle $\varphi \in \mathcal{Z}^2(\mathfrak{B})$ we are able to cobound it, showing that the second simplicial cohomology group of $\mathfrak{B} = L^1(\mathbb{R}_+, \vee)$, $\mathcal{HH}^2(L^1(\mathbb{R}_+, \vee))$, is trivial, as required.

We now use these methods to consider the n^{th} simplicial cohomology group of $\mathfrak{B}$.

5.2 Investigating the n^{th} simplicial cohomology group of $L^1(\mathbb{R}_+, \vee)$

In this section we show that the n^{th} simplicial cohomology group of the algebra $\mathfrak{B} = L^1(\mathbb{R}_+, \vee)$ is trivial. To do this it will be necessary to combine all of what we have accomplished so far.

The strategy is to first introduce the notion of *contracting homotopy* and use this to investigate the n^{th} cyclic cohomology group of $\mathfrak{B}$; we will use the methods of

Chapter 5 thus far to define our contracting homotopy and hence show that this group is trivial for all n. Finally we will invoke the Connes-Tzygan long exact sequence to obtain the required result regarding the n^{th} simplicial cohomology group of $\mathfrak{B}$ when $n \geq 2$.

Due to the density of the linear span of n-dimensional tensor products of normalised indicator functions in $\overbrace{\mathfrak{B}\widehat{\otimes}\cdots\widehat{\otimes}\mathfrak{B}}^{n \text{ times}} = \widehat{\bigotimes}^n \mathfrak{B}$ it is enough to consider n-cocycles acting on normalised indicator functions. In fact, analogous to the arguments we have completed before, it is enough to consider normalised indicator functions with ordered and disjoint support.

5.2.1 Contracting homotopy

In order to define a contracting homotopy we first need a (co)chain complex. Let $(A_i)_{i\in\mathbb{Z}_+}$ be a sequence of Banach spaces (one can also define versions for more general abelian groups or modules) and define the chain complex (as in Section 2.2.1) to be

$$0 \longleftarrow A_0 \xleftarrow{d_1} \cdots \xleftarrow{d_{n-1}} A_{n-1} \xleftarrow{d_n} A_n \xleftarrow{d_{n+1}} A_{n+1} \xleftarrow{d_{n+2}} \cdots .$$

Definition. A *contracting homotopy* for our complex is a family of bounded linear operators $\{s_i\}_{i\in\mathbb{N}}$ where $s_i : A_{i-1} \to A_i$ such that

$$s_i \circ d_i + d_{i+1} \circ s_{i+1} = 1_{A_i} \text{ for } i > 1 \tag{5.120}$$

and

$$d_0 \circ s_0 = 1_{A_0}, \tag{5.121}$$

where 1_{A_i} is the identity map from A_i into itself. In other literature it may also be said that the complex *splits.*

If this only holds for $i = n$ for some $n \in \mathbb{N}$ then we call this a *contracting homotopy in degree n.* Similar, dual definitions apply to cochain complexes.

The existence of a contracting homotopy makes the complex exact. In this definition we only specify that s_i is simply a function; for our complexes in a Banach algebra context we also require that s_i is both continuous and linear for all $i \in \mathbb{N}$.

As we know that the first simplicial cohomology group of $\mathfrak{B}$ here is not trivial and the existence of a contracting homotopy would show that our cochain complex is exact, and hence making every simplicial cohomology group of $\mathfrak{B}$ trivial, we cannot produce it. As we are trying to show that all but the first simplicial cohomology group of $\mathfrak{B}$ are trivial we are therefore only interested in obtaining a contracting homotopy in degree n for nearly all n.

5.2.2 Constructing our contracting homotopy in degree n

Recall the complexes we are working with from Section 2.2.2. We repeat them here for clarity, recalling that the Banach algebra under consideration is $\mathfrak{B} = L^1(\mathbb{R}_+, \vee)$.

The simplicial cochain complex is given by

$$0 \xrightarrow{\delta^0} \mathfrak{B}^* \xrightarrow{\delta^1} \mathcal{C}^1(\mathfrak{B}, \mathfrak{B}^*) \xrightarrow{\delta^2} \mathcal{C}^2(\mathfrak{B}, \mathfrak{B}^*) \xrightarrow{\delta^3} \cdots, \tag{5.122}$$

where $\mathcal{C}^n(\mathfrak{B}, \mathfrak{B}^*)$ for $n \in \mathbb{N}$ is the space of bounded n-linear mappings of $\mathfrak{B} \times \cdots \times \mathfrak{B}$ into $\mathfrak{B}^*$ and for $T \in \mathcal{C}^{n-1}(\mathfrak{B}, \mathfrak{B}^*)$ the boundary operator δ^n is defined as

$$(\delta^n T)(a_1, \ldots, a_n) = \begin{cases} a_1 T(a_2, \ldots, a_n) \\ + \sum_{j=1}^{n-1} (-1)^j T(a_1, \ldots, a_j a_{j+1}, \ldots, a_n) \\ +(-1)^n T(a_1, \ldots, a_{n-1}) a_n, \end{cases} \tag{5.123}$$

for $n \geq 2$.

In the case when $n = 1$ we have that $\delta^1 : \mathfrak{B}^* \to \mathcal{C}^1(\mathfrak{B}, \mathfrak{B}^*)$. To define this map choose $f \in \mathfrak{B}^*$ and $a_0 \in \mathfrak{B}$, then set

$$(\delta^1 f)(a_0) = a_0 f - f a_0. \tag{5.124}$$

This is the dual of the simplicial chain complex

$$0 \xleftarrow{d_0} \mathfrak{B} \xleftarrow{d_1} \mathfrak{B} \widehat{\otimes} \mathfrak{B} \xleftarrow{d_2} \mathfrak{B} \widehat{\otimes} \mathfrak{B} \widehat{\otimes} \mathfrak{B} \xleftarrow{d_3} \cdots, \tag{5.125}$$

where the boundary operators d_n are given by

$$d_n(a_1 \otimes \cdots \otimes a_n \otimes a_0) = \begin{cases} a_2 \otimes \cdots \otimes a_n \otimes a_0 a_1 \\ + \sum_{j=1}^{n-1} (-1)^j a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n \otimes a_0 \\ + (-1)^n a_1 \otimes \cdots \otimes a_n a_0. \end{cases} \tag{5.126}$$

However, in this section we are concerned with *cyclic* cohomology and so need the cyclic versions of these complexes. We denote by $\mathcal{CC}^n(\mathfrak{B}, \mathfrak{B}^*)$ the subspace of $\mathcal{C}^n(\mathfrak{B}, \mathfrak{B}^*)$ consisting of all cyclic bounded n-linear mappings of $\mathfrak{B} \times \ldots \times \mathfrak{B}$ into $\mathfrak{B}^*$. Then we consider the subcomplex of (5.122)

$$0 \xrightarrow{\delta^0} \mathcal{CC}^0(\mathfrak{B}, \mathfrak{B}^*) \xrightarrow{\delta^1} \mathcal{CC}^1(\mathfrak{B}, \mathfrak{B}^*) \xrightarrow{\delta^2} \mathcal{CC}^2(\mathfrak{B}, \mathfrak{B}^*) \xrightarrow{\delta^3} \cdots, \tag{5.127}$$

where δ^n for $n \geq 0$ is defined exactly as before but restricted to the subspace; it is straightforward to show that the δ-maps respect cyclicity.

We now require the following definition.

Definition. Let $\tau_n : \widehat{\bigotimes}^{n+1} \mathfrak{B} \to \widehat{\bigotimes}^{n+1} \mathfrak{B}$ be the *cyclic shift operator*, that is for $f_1 \otimes \cdots \otimes f_{n+1} \in \widehat{\bigotimes}^{n+1} \mathfrak{B}$

$$\tau_n(f_1 \otimes \cdots \otimes f_{n+1}) = (-1)^n (f_2 \otimes \cdots \otimes f_{n+1} \otimes f_1). \tag{5.128}$$

It now becomes necessary to consider the predual to this, i.e. the cyclic version of (5.125). It is well-known (see [21]) that the predual of $\mathcal{CC}^n(\mathfrak{B}, \mathfrak{B}^*)$ is the space $\widehat{\bigotimes}^{n+1} \mathfrak{B}$, which we will denote by V, quotiented by the space of cyclic elements, namely

$$\frac{\widehat{\bigotimes}^{n+1} \mathfrak{B}}{(1 - \tau_n) \widehat{\bigotimes}^{n+1} \mathfrak{B}}, \tag{5.129}$$

where 1 is the identity map on V. For convenience we shall denote this space as $\mathcal{CC}_n(\mathfrak{B}, \mathfrak{B}^*)$ and $\widehat{\bigotimes}^{n+1} \mathfrak{B}$ as V. Note that for $w \in V$ we have that $w - \tau_n w \in (1 - \tau_n)V$, $\tau_n w - \tau_n^2 w \in (1 - \tau_n)V$ as $\tau_n w \in V$ and so on; hence $[\tau_n^i w] = [w]$.

One can easily show that the subspace $(1 - \tau_n)V$ is complemented in V and so is isomorphic to a subspace, but this has no bearing on the current work.

We then give the predual of (5.127) as

$$0 \xleftarrow{d_0} \mathcal{CC}_0(\mathfrak{B}, \mathfrak{B}^*) \xleftarrow{d_1} \mathcal{CC}_1(\mathfrak{B}, \mathfrak{B}^*) \xleftarrow{d_2} \mathcal{CC}_2(\mathfrak{B}, \mathfrak{B}^*) \xleftarrow{d_3} \cdots, \tag{5.130}$$

where the d-maps are restricted to the quotient spaces $\mathcal{CC}_n(\mathfrak{B}, \mathfrak{B}^*)$ for all $n \geq 0$; again it is straightforward to see that the d-maps respect the quotient.

We are now in a position to analyse a possible contracting homotopy here in the cyclic case.

The following definition is a necessary adjustment to that found in Section 5.1.1.

Definition. We are able to define $Q_i^n \subset \mathbb{R}_+^n$ to be the *wedge*

$$Q_i^n = \{(x_1, \ldots, x_n) : x_1, \ldots x_n \text{ are strictly ordered with } x_i > x_j, j \neq i\}. \tag{5.131}$$

In other words we have that all the x_k occur in some order structure for $1 \leq k \leq n$ and the only stipulation we require of this order is for the i^{th} variable to be larger than all of the others.

Let $I_1 \otimes \cdots \otimes I_n$ be a n-dimensional elementary tensor product of normalised indicator functions. We are then able to define the *quadrant* $L^1(Q_i^n)$ as a subspace of $L^1(\mathbb{R}_+^n)$ such that $F \in L^1(Q_i^n)$ has support on the wedge Q_i^n.

Notice that $\bigoplus_{i=1}^n L^1(Q_i^n) = L^1(\mathbb{R}_+^n, \vee)$ (as the boundaries between quadrants have measure zero and the Q_i^n are disjoint, measurable sets whose union has measure zero complement in $\mathbb{R}_+^n$). We may ask what happens on areas of overlap between the supports of the normalised indicator functions. These areas are subspaces of $\mathbb{R}_+^n$ and using better and better approximations we can eventually reduce these subspaces to hyperplanes of Lebesgue measure zero.

For example, if I_1 and I_2 have overlapping supports we create better approximations to reduce this area of overlap to a hyperplane contained in a collection of boxes with smaller and smaller measure. The maximum error caused by these places of overlap, as laid out at the end of both Chapter 3 and Chapter 4, is then at most $C \cdot \dfrac{M}{M^2}$ where C is a constant. Combined with Fubini's Theorem (§6 in [7]) we can see that this trivially extends to more general cases of overlapping supports for multiple functions, yielding an error of $C' \cdot \dfrac{M^{n-1}}{M^n}$ for constant C'.

Both of these tend to zero as the subdivision is refined further, i.e. as M tends to ∞.

Therefore consider the elementary tensor F in the space $\widehat{\bigotimes}^n L^1(\mathbb{R}_+, \vee)$ given as $F(\mathbf{x}) = (f_1 \otimes \cdots \otimes f_n)(\mathbf{x})$. Then if $F(\mathbf{x}) \in Q_i^n$ we have that the functions f_k occur in some order structure for $1 \leq k \leq n$ and that $f_j \ll f_i$ for all $j \neq i$ where $1 \leq i, j \leq n$.

For $i = 1, \ldots, n$ we can define

$$F_i(\mathbf{x}) = \begin{cases} F(\mathbf{x}), \mathbf{x} \in Q_i^n, \\ 0, \text{ else}, \end{cases} \tag{5.132}$$

and it follows by density arguments that we can write F as $\bigoplus_{i=1}^n F_i$; since these two functions differ on sets of measure zero, as explained above, they are in the same equivalence class in $\mathfrak{B}$. Hence for our purposes it is enough to consider $\bigoplus_{i=1}^n L^1(Q_i^n)$ here.

The idea here is to split the function F into n quadrants, where the i^{th} quadrant has the i^{th} L^1 function as the largest in the order structure.

Since F_i is an element of $L^1(Q_i^n)$ is follows from an extension of Lemma (5.4) into n that it can be written as a linear combination of n-dimensional elementary tensors of normalised $\mathfrak{B}$ functions of disjoint support with order structure given by the wedge Q_i^n, i.e.

$$F_i(\mathbf{x}) = \sum_{l=1}^{\infty} \alpha_l f_l^{(1)} \otimes \cdots \otimes f_l^{(n)}(\mathbf{x}), \tag{5.133}$$

where $\alpha_l \in \mathbb{C}$ for $l \in \mathbb{N}$ such that $\sum_{l=1}^{\infty} |\alpha_l| < \infty$ and

$$\left\| f_l^{(m)} \right\|_1 = \int_{\mathbb{R}_+} \left| f_l^{(m)}(x) \right| dx = 1, \quad m = 1, \ldots, n. \tag{5.134}$$

We are now in a position to construct our contracting homotopy in degree n. First we compare our cyclic cochain complex with

$$\cdots \xleftarrow{s_*^{n-1}} \mathcal{CC}^{n-1}(\mathfrak{B}; \mathfrak{B}^*) \xleftarrow{s_*^n} \mathcal{CC}^n(\mathfrak{B}; \mathfrak{B}^*) \xleftarrow{s_*^{n+1}} \mathcal{CC}^{n+1}(\mathfrak{B}; \mathfrak{B}^*) \xleftarrow{s_*^{n+2}} \cdots .$$

Our cyclic cochain complex is the dual of our cyclic chain complex which in turn should be compared with

$$\cdots \xrightarrow{s_k^{n-1}} \mathcal{CC}_{n-1}(\mathfrak{B}, \mathfrak{B}^*) \xrightarrow{s_k^n} \mathcal{CC}_n(\mathfrak{B}, \mathfrak{B}^*) \xrightarrow{s_k^{n+1}} \mathcal{CC}_{n+1}(\mathfrak{B}, \mathfrak{B}^*) \xrightarrow{s_k^{n+2}} \cdots,$$

where the map s_*^n is related to the dual of s_k^n and both require defining; the inclusion of the k subscript will become apparent. We will first define the map s_k^n on $F \in \widehat{\bigotimes}^n \mathfrak{B}$ and then show this respects the quotient and is therefore well-defined on $\dfrac{\widehat{\bigotimes}^n \mathfrak{B}}{(1-\tau_{n-1})\widehat{\bigotimes}^n \mathfrak{B}}$. Thus for $F \in \widehat{\bigotimes}^n \mathfrak{B}$ we have that

$$s_k^n(F) = s_k^n\left(\bigoplus_i F_i\right) = \bigoplus_i s_k^n(F_i), \tag{5.135}$$

where

$$s_k^n(F_i)(\mathbf{x}, y) = \sum_{j=1}^m \alpha_j s_k^n(f_1^{(j)} \otimes \cdots \otimes f_n^{(j)})(\mathbf{x}, y), \tag{5.136}$$

for $\mathbf{x} = (x_1, \ldots, x_n)$ and $F_i \in L^1(Q_i^n)$ written as a linear combination of elementary tensor products as given in Lemma (5.4), where $\alpha_j \in \mathbb{C}$.

If F_i is an elementary tensor, say $f_1 \otimes \cdots \otimes f_n$ in $L^1(Q_i^n)$ we have

$$\begin{aligned} s_k^n(f_1 \otimes \cdots \otimes f_n)(\mathbf{x}, y) &= (-1)^i (f_1 \otimes \cdots \otimes f_n)(\mathbf{x}) \cdot k \cdot \chi_{[0,\frac{1}{k})}(x_i - y) \\ &= (-1)^i f_1(x_1) \cdots f_i(x_i) \cdots f_n(x_n) \cdot k \cdot \chi_{[0,\frac{1}{k})}(x_i - y) \\ &= (-1)^i f_1(x_1) \cdots \gamma_{k,y}(f_i)(x_i) \cdots f_n(x_n) \\ &= (-1)^i (f_1 \otimes \cdots \otimes \gamma_{k,y}(f_i) \otimes \cdots \otimes f_n)(\mathbf{x}), \end{aligned} \tag{5.137}$$

where $\gamma_{k,y}$ acts on the function f_i and the i in $(-1)^i$ is given by the the fact that f_i is in the i^{th} position (acting on x_i) and so

$$s_k^n(f_1 \otimes \cdots \otimes f_n) = (-1)^i (f_1 \otimes \cdots \otimes \gamma_{k,y}(f_i) \otimes \cdots \otimes f_n) \tag{5.138}$$

for elementary tensors. The factor $(-1)^i$ appears instead of simply (-1) due to the principle of cyclic equivalency which we introduce later.

It is a trivial calculation to show that s_k^n is well-defined on $\mathcal{CC}_{n-1}(\mathfrak{B}, \mathfrak{B}^*)$, i.e. it respects the quotient. For observe that applying s_k^n to $\tau(f_1 \otimes \cdots \otimes f_n)$ yields

$$\begin{aligned} s_k^n \left((-1)^{n-1} f_2 \otimes \cdots \otimes f_n \otimes f_1\right)(\mathbf{x}, y) \\ = (-1)^{n-1}(-1)^{i-1}(f_2 \otimes \cdots \gamma_{k,y}(f_i) \otimes \cdots \otimes f_n \otimes f_1)(\mathbf{x}) \end{aligned} \tag{5.139}$$

as f_i, the largest function here, is in position $i-1$ now (that is, it acts on x_{i-1}). Cycling this around (writing $\gamma_{k,y}(f_i)$ as a linear combination of elementary tensors if neccessary) gives us

$$(-1)^{n-1}(-1)^{i-1}(-1)^n(f_1 \otimes \cdots \otimes \gamma_{k,y}(f_i) \otimes \cdots \otimes f_n)(\mathbf{x}). \tag{5.140}$$

It should be noted that if f_1 is the largest function in our elementary tensor here then the calculation still works, but more care has to be taken with the sign changes; in fact the coefficient in front of the tensor in the last equation changes to $(-1)^{n^2+2n-1}$, which is always odd regardless if n is odd or even.

The coefficient $(-1)^{n-1}(-1)^{i-1}(-1)^n$ can be written as $(-1)^{2(n-1)}(-1)^i$ and

$$(-1)^{2(n-1)} = (-1)^{2^{n-1}} = 1^{n-1} = 1. \tag{5.141}$$

Hence we can see it follows that $s_k^n\left((-1)^{n-1} f_2 \otimes \cdots \otimes f_n \otimes f_1\right)(\mathbf{x}, y)$ is equivalent to $s_k^n\left(f_1 \otimes \cdots \otimes f_n\right)(\mathbf{x}, y)$, proving that the maps s_k^n do indeed respect the quotient.

Since the elementary tensors of normalised functions have dense linear span it follows that

$$\begin{aligned} s_k^n(F_i) &= s_k^n\left(\sum_{l=1}^{\infty} \alpha_l f_l^{(1)} \otimes \cdots \otimes f_l^{(n)}\right) \\ &= \sum_{l=1}^{\infty} \alpha_l s_k^n\left(f_l^{(1)} \otimes \cdots \otimes f_l^{(n)}\right) \\ &= (-1)^i \sum_{l=1}^{\infty} \alpha_l \left(f_l^{(1)} \otimes \cdots \otimes \gamma_{k,y}(f_l^{(i)}) \otimes \cdots \otimes f_l^{(n)}\right). \end{aligned} \tag{5.142}$$

Thus it is enough for us to check our results only on such elementary tensors; it is enough that our formulae and identities are defined on these elementary tensors of normalised functions. Our results will then extend by linearity and density arguments to the whole space $\mathcal{CC}_{n-1}(\mathfrak{B}, \mathfrak{B}^*)$.

In order to perform cohomology calculations here we need that all of the functions in our elementary tensor belong to some wedge Q_i^n, i.e. are of disjoint ordered support. To make this happen it is necessary that k be sufficiently large, large enough to ensure that the two-dimensional support of $\gamma_{k,y}(f_i)$ remains disjoint and larger than f_j for all $j \neq i$ with $1 \leq i, j \leq n$. In other words so that $f_j \ll_k \gamma_{k,y}(f_i)$ for fixed f_i and f_j.

The ideal way to ensure this is to take the limit as k tends to infinity; however there are several inherent problems which arise as a result. Since we work on a dense subset it follows that $\lim\limits_{k\to\infty} \gamma_{k,y}$ exists and is well-behaved on this dense subset, but it is difficult to show that this holds anywhere else in $\mathfrak{B}$. Additionally our contracting homotopy maps need t be independent of k; a single map rather than a family.

To eliminate these problems we move to the dual space, making taking the limit a well-defined operation; additionally, once k is large enough to ensure the functions f_i are well-separated taking larger k does not change the overall result (compare with Lemma (5.2)), meaning that taking the limit is a viable thing to do. This is because $\varphi(F) \in \mathbb{C}$ and so we can, if necessary, select an appropriate ultrafilter to force the limit to exist as this will be a bounded net of linear functionals; however, using normal limits is sufficient in our case.

Thus for $\varphi \in \mathcal{CC}^n(\mathfrak{B}, \mathfrak{B}^*)$ and $F \in L^1(\mathbb{R}_+^n)$ we have that

$$s_*^n \circ \varphi(F) = \lim_{k\to\infty} (s_k^n)^* \circ \varphi(F) = \lim_{k\to\infty} \varphi \circ s_k^n(F). \tag{5.143}$$

By switching to tensor notation and writing F as a linear combination of elementary tensors of disjoint support we can see that it is enough to define this map on such tensor products. In other words, switching between regular and tensor notation as appropriate, and looking at the case with support in wedge Q_i^n, we have that

$$\begin{aligned}(s_*^n\varphi)(f_1, \ldots, f_{n-1})(f_n) &= (s_*^n\varphi)(f_1 \otimes \cdots \otimes f_n)\\ &= (-1)^i\varphi(f_1 \otimes \cdots \otimes \gamma_{k,y}(f_i) \otimes \cdots \otimes f_n)\\ &= (-1)^i\varphi(f_1, \ldots, \gamma_{k,y}(f_i), \ldots, f_{n-1})(f_n).\end{aligned} \tag{5.144}$$

In other words, moving to the dual makes verifying this operation a much more straightforward task, and means we now have a contracting homotopy in the dual case which is independent of k.

By density arguments for a fixed f_i the error between the expression

$$\varphi(f_1, \ldots, \gamma_{k,y}(f_i), \ldots, f_{n-1})(f_n) \tag{5.145}$$

and

$$\varphi(f_1, \ldots, f_i^{'}, f_i, \ldots, f_{n-1}))(f_n), \tag{5.146}$$

where $f_i^{'}$ is a function such that $f_j \ll f_i^{'} \ll_k \gamma_{k,y}(f_i)$ with $j \in \{1, \ldots, n\}\backslash\{i\}$ tends to zero, as illustrated by Equation (5.90). This means that not only does the limit exist but we are led to expect what that limit should be.

Thus there are natural equalities (via duality)

$$\delta^n \circ \varphi = \varphi \circ d^n \text{ and } s_*^n \circ \varphi = \lim_{k\to\infty} \varphi \circ s_k^n. \tag{5.147}$$

It must also be noted that the maps s_k^n and s_*^n are compatible, i.e. they can be applied in sequence, for $n \in \mathbb{N}$, which can be seen through simple direct calculation; for clarity we give a specific example here but do not demonstrate the general case as it becomes notationally convoluted.

Example. Consider $F \in \widehat{\bigotimes}^2 \mathfrak{B} \cong L^1(\mathbb{R}_+^2, \vee)$. Then using Lemma (5.4) and the fact that $F = F_1 \oplus F_2$,

$$\begin{aligned}s_k^2(F) &= s_k^2(F_1) \oplus s_k^2(F_2)\\ &= s_k^2\left(\sum_{n=1}^{\infty} \alpha_n f_n^{(1)} \otimes f_n^{(2)}\right) \oplus s_k^2\left(\sum_{m=1}^{\infty} \beta_m g_m^{(1)} \otimes g_m^{(2)}\right)\\ &= \left(-\sum_{n=1}^{\infty} \alpha_n \gamma_{k,y}(f_n^{(1)}) \otimes f_n^{(2)}\right) \oplus \left(\sum_{m=1}^{\infty} \beta_m g_m^{(1)} \otimes \gamma_{k,y}(g_m^{(2)})\right).\end{aligned} \tag{5.148}$$

Then consider

$$s_k^3(s_k^2(F)) = s_k^3\left(-\sum_{n=1}^{\infty}\alpha_n\gamma_{k,y}(f_n^{(1)})\otimes f_n^{(2)}\right)\oplus s_k^3\left(\sum_{m=1}^{\infty}\beta_m g_m^{(1)}\otimes\gamma_{k,y}(g_m^{(2)})\right). \tag{5.149}$$

In the first term we can write $\gamma_{k,y}(f_n^{(1)})$ as $\sum_{p=1}^{\infty}\delta_p h_p^{(1)}\otimes h_p^{(2)}$, where $f_n^{(2)} \ll h_p^{(2)} \ll h_p^{(1)}$ for all $n, p \in \mathbb{N}$; this is because $f_n^{(2)} \ll_k \gamma_{k,y}(f_n^{(1)})$. Then the first term becomes

$$-\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\alpha_n\delta_p s_k^3(h_p^{(1)}\otimes h_p^{(2)}\otimes f_n^{(2)}) \tag{5.150}$$

and then

$$\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\alpha_n\delta_p\gamma_{k,y}(h_p^{(1)})\otimes h_p^{(2)}\otimes f_n^{(2)}. \tag{5.151}$$

Writing the expression $\gamma_{k,y}(h_p^{(1)}) = \sum_{q=1}^{\infty}\zeta_q v_q^{(1)}\otimes v_q^{(2)}$ we have that this finally becomes

$$\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\alpha_n\delta_p\zeta_q v_q^{(1)}\otimes v_q^{(2)}\otimes h_p^{(2)}\otimes f_n^{(2)}, \tag{5.152}$$

where $f_n^{(2)} \ll h_p^{(2)} \ll v_q^{(2)} \ll v_q^{(1)}$ and $\alpha_n, \delta_p, \zeta_q \in \mathbb{C}$ by construction for all $n, p, q \in \mathbb{N}$. Then by definition this is simply a function $G_1 \in L^1(Q_1^4)$.

By a similar method we conclude that the second term in (5.149) is equivalent to a function $G_2 \in L^1(Q_2^4)$ and therefore

$$s_k^3(s_k^2(F)) = G_1 \oplus G_2 \oplus 0 \oplus 0 = G \in L^1(\mathbb{R}_+^4, \vee), \tag{5.153}$$

i.e. we have a function $G \in \widehat{\bigotimes}^4 \mathfrak{B} \cong L^1(\mathbb{R}_+^4, \vee)$ as required.

5.2.3 A locally compact case

In the discrete case, the method simply involves considering an elementary tensor product $x_1 \otimes \cdots \otimes x_n$, where the x_i satisfy some order structure, and extending it from n-dimensions to $(n+1)$-dimensions by inserting a duplicate of the maximal element. There are problems which arise when the maximal element is not unique, as is possible in the discrete case. The method we use here ensures that such an element *is* unique and allows us to perform a similar insertion technique using our spreading-out functions.

In this section we will consider an elementary tensor product in $\widehat{\bigotimes}^{n+1}\mathfrak{B}$ under particular order conditions and use this to show that our s and d maps form a contracting homotopy in degree $n+1$ for our chain complex. We will then apply φ to η and, via our ability to switch between tensor and non-tensor notation, use dualisation to show that we can cobound $\varphi(\eta)$. This result will extend naturally to all other ordered cases.

Let $\eta = f_1 \otimes f_2 \otimes \cdots \otimes f_n \otimes f_0 \in \widehat{\bigotimes}^{n+1}\mathfrak{B}$ be the $(n+1)$-dimensional tensor product of normalised functions of disjoint support with the functions $f_i \in \mathfrak{B}$ satisfying some order structure for $i = 0, \ldots, n$. We do not make any assumptions about the relative orders of the functions but note that this construction immediately gives us that there is a (unique) maximal element, i.e. for some $0 \leq i \leq n$ we have $f_j \ll f_i$ for all $j \neq i$. As we are looking at *cyclic* cohomology here, meaning we are able to cycle the elements around without affecting the result, we can assume without loss of generality that this element is f_1.

For $\varphi \in \mathcal{ZC}^n(\mathfrak{B})$ and, in tensor notation, $f_1 \otimes \cdots \otimes f_n \otimes f_0 \in \widehat{\bigotimes}^{n+1}\mathfrak{B}$ recall that the definition of cyclicity is

$$\varphi(f_1 \otimes \cdots \otimes f_n \otimes f_0) = (-1)^n \varphi(f_0 \otimes f_1 \otimes \cdots \otimes f_n), \tag{5.154}$$

for $f_i \in \mathfrak{B}$.

Also recall that the d maps for our chain complex are given in Section 2.2.2. We make a similar definition for the map $d_1^n : \widehat{\bigotimes}^{n}\mathfrak{B} \to \widehat{\bigotimes}^{n-1}\mathfrak{B}$ given by

$$d_1^n(f_1 \otimes \cdots f_n) = f_1 f_2 \otimes \cdots \otimes f_n - f_1 \otimes f_2 f_3 \otimes \cdots \otimes f_n + \cdots + (-1)^n f_1 \otimes \cdots \otimes f_{n-1} f_n. \tag{5.155}$$

This is entirely for notational convenience.

Once again for notational convenience we will also require the following definitions. We define $\omega_1 \in \widehat{\otimes}^n\mathfrak{B}$ and $\omega_2 \in \widehat{\bigotimes}^{n-1}\mathfrak{B}$ as

$$\omega_1 = f_2 \otimes \cdots \otimes f_n \otimes f_0 \tag{5.156}$$

and

$$\omega_2 = f_3 \otimes \cdots \otimes f_n \otimes f_0. \tag{5.157}$$

respectively. Note that the f_i are placeholders from our chosen η. Then $\eta = f_1 \otimes \omega_1 = f_1 \otimes f_2 \otimes \omega_2$. Note also that for $f \in \mathfrak{B}$

$$f \cdot \omega_1 = f * f_2 \otimes \cdots \otimes f_n \otimes f_0, \tag{5.158}$$

and

$$\omega_1 \cdot f = f_2 \otimes \cdots \otimes f_n \otimes f_0 * f, \tag{5.159}$$

and similarly for ω_2.

Finally we will need to make a new definition.

Definition. Let $F, G \in \widehat{\bigotimes}^n \mathfrak{B}$ be n-dimensional elementary tensor products. For $\varphi \in \mathcal{ZC}^{n-1}(\mathfrak{B})$ we write $F \approx G$ if $\varphi(F) = \pm\varphi(G)$. We say that F and G are *cyclically equivalent*.

This last definition is useful since we only ultimately consider what the value of φ is on elementary tensors and as φ is cyclic we use this relation to make connections.

Calculating $s_k^n \circ d^n(f_1 \otimes f_2 \otimes \omega_2)$ gives us

$$\begin{aligned} s_k^n \circ d^n(f_1 \otimes f_2 \otimes \omega_2) &= s_k^n(f_2 \otimes \omega_2 \cdot f_1) - s_k^n(f_1 * f_2 \otimes \omega_2) + s_k^n(f_1 \otimes f_2 \cdot \omega_2) \\ &\qquad - s_k^n(f_1 \otimes f_2 \otimes d_1^{n-1}(\omega_2)) \\ &= s_k^n(f_2 \otimes \cdots \otimes f_n \otimes f_1) - s_k^n(f_1 \otimes \omega_2) \\ &\qquad + s_k^n(f_1 \otimes f_2 \cdot \omega_2) - s_k^n(f_1 \otimes f_2 \otimes d_1^{n-1}(\omega_2)) \\ &= (-1)^n(f_2 \otimes \cdots \otimes f_n \otimes \gamma_{k,y}(f_1)) - (-1)(\gamma_{k,y}(f_1) \otimes \omega_2) \\ &\qquad + (-1)(\gamma_{k,y}(f_1) \otimes f_2 \cdot \omega_2) \\ &\qquad - (-1)(\gamma_{k,y}(f_1) \otimes f_2 \otimes d_1^{n-1}(\omega_2)). \end{aligned} \tag{5.160}$$

In the first term we can approximate $\gamma_{k,y}(f_1)$ by a linear combination of elementary tensor products of normalised functions of disjoint support, $\sum_{i=1}^{\infty} \alpha_i x_i \otimes y_i$ where $x_i, y_i \in \mathfrak{B}$. Note once more that $f_k \ll y_i \ll x_i$ for all $i = 1, 2, \ldots$ and $k \neq 1$. This means that we can express the first term as

$$(-1)^n(f_2 \otimes \cdots \otimes f_n \otimes \gamma_{k,y}(f_1)) = (-1)^n \sum_{i=1}^{\infty} \alpha_i f_2 \otimes \cdots \otimes f_n \otimes x_i \otimes y_i. \tag{5.161}$$

Under cyclicity this is cyclically equivalent to

$$(-1)^n(-1)^n \sum_{i=1}^{\infty} \alpha_i y_i \otimes f_2 \otimes \cdots \otimes f_n \otimes x_i, \tag{5.162}$$

and since $(-1)^n(-1)^n = (-1)^{2n} = 1$ for all $n \in \mathbb{N}$ these cancel out.

Thus we conclude that

$$\begin{aligned} s_k^n \circ d^n(f_1 \otimes f_2 \otimes \omega_2) = \sum_{i=1}^{\infty} \alpha_i(y_i \otimes f_2 \otimes \cdots \otimes f_n \otimes x_i) + \gamma_{k,y}(f_1) \otimes \omega_2 \\ - \gamma_{k,y}(f_1) \otimes f_2 \cdot \omega_2 + \gamma_{k,y}(f_1) \otimes f_2 \otimes d_1^{n-1}(\omega_2). \end{aligned} \tag{5.163}$$

Now computing $d^{n+1} \circ s_k^{n+1}(f_1 \otimes f_2 \otimes \omega_2)$ we get that

$$d^{n+1} \circ s_k^{n+1}(\eta) = d^{n+1}(-\gamma_{k,y}(f_1) \otimes f_2 \otimes \cdots f_n \otimes f_0). \tag{5.164}$$

Again writing $\gamma_{k,y}(f_1)$ as a linear combination of elementary tensor products of normalised functions of disjoint support we have that this is

$$-\sum_{i=1}^{\infty} \alpha_i d^{n+1}(x_i \otimes y_i \otimes f_2 \otimes \cdots f_n \otimes f_0), \tag{5.165}$$

which becomes

$$\begin{aligned} -\sum_{i=1}^{\infty} \alpha_i \left[y_i \otimes f_2 \otimes \cdots \otimes f_n \otimes f_0 * x_i - x_i * y_i \otimes f_2 \otimes \omega_2 \right. \\ \left. + x_i \otimes y_i * f_2 \otimes \omega_2 - x_i \otimes y_i \otimes f_2 \cdot \omega_2 + x_i \otimes y_i \otimes f_2 \otimes d_1^{n-1}\omega_2 \right] \end{aligned} \tag{5.166}$$

or more simply

$$\begin{aligned} -\sum_{i=1}^{\infty} \alpha_i(y_i \otimes f_2 \otimes \cdots \otimes f_n \otimes x_i) + f_1 \otimes f_2 \otimes \omega_2 \\ - \gamma_{k,y}(f_1) \otimes \omega_2 + \gamma_{k,y}(f_1) \otimes f_2 \cdot \omega_2 - \gamma_{k,y}(f_1) \otimes f_2 \otimes d_1^{n-1}\omega_2 \end{aligned} \tag{5.167}$$

as $y_i * f_2 = y_i$, recalling that $\sum_{i=1}^{\infty} \alpha_i x_i \otimes y_i = \gamma_{k,y}(f_1)$ and $\sum_{i=1}^{\infty} \alpha_i x_i * y_i = \pi_k(\gamma_{k,y}(f_1)) = f_1$ analogously from (5.104) earlier.

Notice that the second term above is simply η and that we are therefore able to see that

$$\eta = s_k^n \circ d^n(\eta) + d^{n+1} \circ s_k^{n+1}(\eta). \tag{5.168}$$

Thus it follows for $\varphi \in \mathcal{ZC}^n(\mathfrak{B})$ that

$$\varphi(\eta) = \varphi(s_k^n \circ d^n(\eta)) + \varphi(d^{n+1} \circ s_k^{n+1}(\eta)), \tag{5.169}$$

where the operation of taking limits ensures our disjointedness criteria are met. Taking the limit of both sides then yields

$$\varphi(\eta) = \lim_{k\to\infty} \varphi(s_k^n \circ d^n(\eta)) + \lim_{k\to\infty} \varphi(d^{n+1} \circ s_k^{n+1}(\eta)), \tag{5.170}$$

as $\varphi(\eta)$ is independent of k.

It is immediate, by duality, that

$$\lim_{k\to\infty} \varphi(d^{n+1} \circ s_k^{n+1}(\eta)) = \lim_{k\to\infty} (\delta^{n+1}\varphi)(s_k^{n+1}(\eta)) = \lim_{k\to\infty} 0 = 0 \tag{5.171}$$

as $\varphi \in \ker(\delta^{n+1})$. It also follows that

$$\begin{aligned} \lim_{k\to\infty} \varphi(s_k^n \circ d^n(\eta)) &= \lim_{k\to\infty} ((s_k^n)^* \circ \varphi)(d^n(\eta)) \\ &= (\delta^n(s_*^n\varphi))(\eta). \end{aligned} \tag{5.172}$$

Since $s_*^n\varphi \in \mathcal{CC}^{n-1}(\mathfrak{B})$ relabel this to be ψ and hence we have that

$$\varphi(\eta) = (\delta^n\psi)(\eta) \tag{5.173}$$

i.e. given a cyclic n-cocycle $\varphi \in \mathcal{ZC}^n(\mathfrak{B})$ we are able to cobound it by $\psi = s_*^n\varphi \in \mathcal{CC}^{n-1}(\mathfrak{B})$. As our choice of φ was arbitrary this holds for all $\varphi \in \mathcal{ZC}^n(\mathfrak{B})$.

It should be remarked that this holds for all $n \in \mathbb{N}$ by construction.

5.2.4 The locally compact case in degree n

In the last subsection, we used cyclicity to allow us without loss of generality to assume that the maximal element in the n-dimensional elementary tensor product η was f_1. Therefore by cyclic equivalence the result also holds when f_i is the maximal element, where $i = 0, \ldots, n$. Since we can split our general function F into a sum of functions F_i defined on wedges where we know which normalised function in the tensor product is the maximal element, the general result follows by linearity: we can cobound each of the terms, and the coboundary is multilinear.

As has been mentioned earlier, the linear span of such elementary tensors are dense in $\mathfrak{B}$ and hence it follows that

$$\mathcal{HC}^n(\mathfrak{B}) \cong 0, \tag{5.174}$$

for all $n = 1, 2, \ldots$; our contracting homotopy holds for all $n \in \mathbb{N}$.

Looking at the general locally compact case in the n^{th} degree for the algebra $\mathfrak{B}$ we once more appeal to the Connes-Tzygan long exact sequence in Section 2.2.3. In a way similar to that at the end of Section 4.2 we look at the section of the Connes-Tzygan long exact sequence containing

$$\cdots \longrightarrow \mathcal{HC}^n(\mathfrak{B}) \longrightarrow \mathcal{HH}^n(\mathfrak{B}) \longrightarrow \mathcal{HC}^{n-1}(\mathfrak{B}) \longrightarrow \cdots .$$

We know that $\mathcal{HC}^{n-1}(\mathfrak{B})$ and $\mathcal{HC}^n(\mathfrak{B})$ are trivial for $n \geq 2$ and that the sequence is exact at $\mathcal{HH}^n(\mathfrak{B})$ and so

$$\mathcal{HH}^n(\mathfrak{B}) = \ker(\mathcal{HH}^n(\mathfrak{B}) \to 0) = \operatorname{im}(0 \to \mathcal{HH}^n(\mathfrak{B})) = 0, \tag{5.175}$$

for $n \geq 2$. In other words we have shown that

$$\mathcal{HH}^n(\mathfrak{B}) \cong 0 \tag{5.176}$$

for the algebra $\mathfrak{B} = L^1(\mathbb{R}_+, \vee)$ and $n \geq 2$, that is the n^{th} simplicial cohomology group for the algebra $\mathfrak{B}$ is trivial for all $n \in \mathbb{N}$ such that $n \geq 2$.

Originally this was the ultimate aim of this thesis, but we are able to extend our results to more general locally compact cases. This is the aim of the rest of the thesis and will be explained more in the next chapter.

Chapter 6

A more general class of locally compact semigroup algebras

In the previous chapters we have analysed simplicial cohomology groups of the algebras $\mathfrak{A} = \ell^1(\mathbb{Z}_+, \vee)$ and $\mathfrak{B} = L^1(\mathbb{R}_+, \vee)$ with the focus being on using results from the former to obtain results for the latter.

It is known ([15], Theorem 7.5 with $k = 1$) that $\mathcal{HH}^n(\mathfrak{A}) \cong 0$ for all $n \in \mathbb{N}$ while we have shown that $\mathcal{HH}^1(\mathfrak{B}) \cong \mathfrak{B}^*$ and $\mathcal{HH}^n(\mathfrak{B}) \cong 0$ for all $n \geq 2$.

The aim now is to take our methods further and ultimately consider the algebra $L^1(X, \leq, \mu)$ where

- X is a semigroup with the binary operation $\vee$,
- $(X, \leq)$ is a general totally ordered space with the order topology, in which it is locally compact and
- μ is a σ-finite, positive, regular Borel measure on X.

The definition of a totally ordered semigroup which is locally compact in its order topology can be found in Section 2.1. The measure theory definitions are given in the next section.

If we set $X = \mathbb{R}_+$ and $\mu = \lambda$, which is Lebesgue measure, then we simply have the algebra $\mathfrak{B}$. Similarly setting $X = \mathbb{Z}_+$ and $\mu = \delta$, which is counting measure, we have the algebra $\mathfrak{A}$.

As outlined in [3] the measure μ can be uniquely decomposed into a linear combination of its *continuous* and *discrete* parts. In this chapter we consider only the continuous part of the decomposition in order to generalise the results of Chapter 5.

Throughout this chapter we will denote by $\mathfrak{C}$ the algebra $L^1(\mathbb{R}_+, \leq, \mu_c)$ where μ_c is a *continuous*, σ-finite, positive, regular Borel measure on $\mathbb{R}_+$.

6.1 Some definitions

Throughout this section we will make some important definitions regarding measures; for the background and a more comprehensive account see [7, 29].

First, let X be a set, Σ be a σ-algebra over X and μ a measure. It follows that (X, Σ) is called a *measurable space*, (X, Σ, μ) a *measure space* and the members of Σ are termed *measurable sets*.

A measure μ, acting on the members of Σ, is *continuous* if the measure of any singleton set, and hence any discrete or countable set, is zero. Alternatively μ is a *discrete* measure if its support is at most a countable set.

When we say that μ is *σ-finite* we mean that the space X can be written as a countable union of measurable sets each of which has finite measure with respect to μ.

We say that μ is *positive* if it only takes nonnegative values.

A measure μ is *inner regular* if every measurable set in X can be approximated from within by compact sets, that is for a measurable set M

$$\mu(M) = \sup\{\mu(K) : K \subseteq M, K \text{ compact}\}. \tag{6.1}$$

It should be noted that $\mu(K)$ is finite for all compact sets K if the measure is *locally finite*, that is each point in X has a neighbourhood in Σ which has finite measure; this is not always automatic given the definition.

On the other hand μ is *outer regular* if any measurable set can be approximated from without by open sets, that is for a measurable set M

$$\mu(M) = \inf\{\mu(U) : M \subseteq U, U \text{ open}\}. \tag{6.2}$$

We call μ *regular* if it is both inner regular and outer regular.

The notions of *continuity from above* and *continuity from below* are applicable to all measures. A measure μ is *continuous from below* if for measurable sets $E_1, E_2, \ldots$ such that $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$ then $\bigcup_{i\in\mathbb{N}} E_i$ is measurable and

$$\mu\left(\bigcup_{i\in\mathbb{N}} E_i\right) = \lim_{i\to\infty} \mu(E_i). \tag{6.3}$$

A measure μ is *continuous from above* if for measurable sets $E_1, E_2, \ldots$ such that $E_n \supseteq E_{n+1}$ for all $n \in \mathbb{N}$ then $\bigcap_{i\in\mathbb{N}} E_i$ is measurable and

$$\mu\left(\bigcap_{i\in\mathbb{N}} E_i\right) = \lim_{i\to\infty} \mu(E_i). \tag{6.4}$$

A small note: on σ-algebras, continuity from below is automatic, while continuity from above is automatic if any of the sets in the nested sequence have finite measure.

We now consider what it means for a measure to be a Borel measure.

6.2 Borel measures

An excellent text introducing Borel measures, along with other concepts of measure theory, is the book [31].

Consider a topological space X. Then the smallest possible σ-algebra containing all open (equivalently closed) sets in X is called the *Borel algebra on X*. Quite clearly such an algebra exists. Another way to form such a σ-algebra is to generate it by the open (equivalently closed) sets in X, i.e. start with the open (equivalently closed) sets and add to this collection via the operations of countable unions, countable intersections and relative complements, then take the intersection of all such σ-algebras.

Sets which are contained within the Borel algebra are called *Borel sets*.

A measure μ on the topological space X is a *Borel measure* if the space X is a locally compact Hausdorff space and μ is defined on the σ-algebra of Borel sets.

In the case where $X = \mathbb{R}$ these definitions mean that the Borel algebra is the smallest σ-algebra which contains the intervals. A particular Borel measure, sometimes referred to as 'the' Borel measure on $\mathbb{R}$, is the measure on this algebra giving the intervals, i.e. $[a, b]$, the measure $b-a$: in other words Lebesgue measure restricted to intervals I.

Unfortunately Borel measures are not generally complete, which is why in most cases Lebesgue measure is usually preferred. A *complete measure space* is a measure space where the subsets of all null sets are measurable. Naturally, such subsets must therefore have measure zero.

6.3 The diagonal in $\mathbb{R}_+$

Before conducting any further analysis an investigation of the diagonal is required. Let $\mathcal{D}$ denote the diagonal $\{(x, x) : x \in \mathbb{R}_+\}$ in $\mathbb{R}_+^2$. The necessary result which is being sought here is that the continuous measure of $\mathcal{D}$ in $\mathbb{R}_+^2$ is zero. Some of the techniques from this proof will also be useful later on.

Lemma 6.1. *Let μ_c be a continuous, σ-finite, positive, regular Borel measure on $\mathbb{R}_+$, and let $\mathcal{D}$ be the diagonal in $\mathbb{R}_+^2$. Then $\mu_c^{(2)}(\mathcal{D}) = 0$, where $\mu_c^{(2)}$ is the product measure $\mu_c \otimes \mu_c$ on $\mathbb{R}_+^2$.*

Proof. Let $\mathcal{D}_i$ denote the section of the diagonal $\mathcal{D}$ contained in the box $[i, i+1)^2$ for $i = 0, 1, 2, \ldots$. Then

$$\mathcal{D} = \bigcup_{i=0}^{\infty} \mathcal{D}_i, \tag{6.5}$$

with $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$ whenever $i \neq j$.

So by countable additivity

$$\mu_c^{(2)}(\mathcal{D}) = \mu_c^{(2)}\left(\bigcup_{i=0}^{\infty}\mathcal{D}_i\right) = \sum_{i=0}^{\infty}\mu_c^{(2)}(\mathcal{D}_i). \tag{6.6}$$

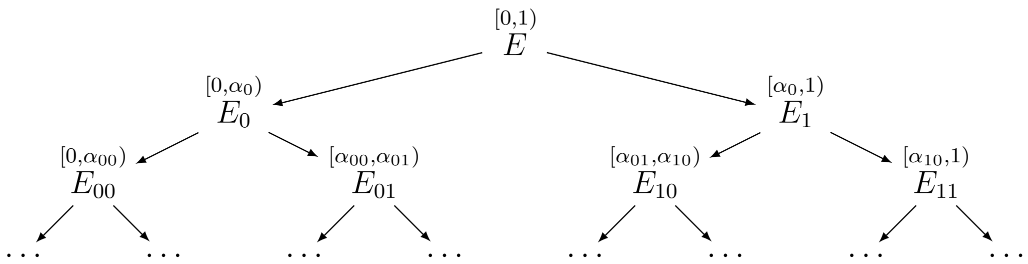
Without loss of generality consider $\mathcal{D}_0 \subset [0,1)^2$. The method is to dyadically split the interval $[0,1)$ into subintervals of equal measure.

Now we define $E_{a_1\cdots a_n}$ as

$$E_{a_1\cdots a_n} = \begin{cases} [0,\alpha_{0\cdots 0}), & a_1\cdots a_n = 0\cdots 0, \\ [\alpha_{a_1\cdots a_n - 0\cdots 01}, \alpha_{a_1\cdots a_n}), & \text{else,} \end{cases} \tag{6.7}$$

with $a_i = 0$ or 1 for $i = 1,\ldots,n$ and $\alpha_{1\cdots 1} = 1$.

Pictorially this is:



Notice that the top level is where $n = 0$, the second level (containing E_0 and E_1) is where $n = 1$ and so on.

Consider $[0,1)$, and define

$$\alpha_0 = \sup\left\{t \in [0,1) : \mu_c([0,t)) \le \frac{1}{2}\mu_c([0,1))\right\}. \tag{6.8}$$

Then $\mu_c([0,\alpha_o)) = \frac{1}{2}\mu_c([0,1))$. For if $t < \alpha_0$ then $\mu_c([0,t)) \le \frac{1}{2}\mu_c([0,1))$ by definition. Also $\mu_c([0,\alpha_0 - \frac{1}{n})) \le \mu_c([0,\alpha_0)) \le \frac{1}{2}\mu_c([0,1))$, again by definition for all $n \in \mathbb{N}$. Finally $\mu_c([0,\alpha_0 + \frac{1}{n})) \ge \frac{1}{2}\mu_c([0,1))$ for all $n \in \mathbb{N}$, else α_0 would not be the supremum.

Thus by continuity from above and below

$$\mu_c([0,\alpha_0)) = \frac{1}{2}\mu_c([0,1)). \tag{6.9}$$

Then in turn

$$\mu_c([\alpha_0, 1)) = \frac{1}{2}\mu_c([0, 1)) \tag{6.10}$$

by countable additivity.

Similar arguments hold for every splitting. Thus

$$\mu_c(E_{a_1 \cdots a_n 0}) = \mu_c(E_{a_1 \cdots a_n 1}) = \frac{1}{2}\mu_c(E_{a_1 \cdots a_n}), \tag{6.11}$$

or in other words every *child* receives exactly half the measure of its *parent*. The reason for this construction will become clear later on.

Now make the inductive definitions

$$M_n = \bigcup_{a_i = 0 \text{ or } 1} E^2_{a_1 \cdots a_n}, \tag{6.12}$$

such that

$$\begin{aligned} M_1 &= E_0^2 \cup E_1^2, \\ M_2 &= E_{00}^2 \cup E_{01}^2 \cup E_{10}^2 \cup E_{11}^2, \\ &\vdots \end{aligned} \tag{6.13}$$

with $M_0 = E^2 = [0, 1)^2$.

Then $\mathcal{D}_0 \subseteq M_n$ for all $n \in \mathbb{Z}_+$ and $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$. In fact

$$\lim_{n \to \infty} M_n = \mathcal{D}_0 \text{ i.e. } \bigcap_{n=0}^{\infty} M_n = \mathcal{D}_0. \tag{6.14}$$

Thus the box $[0, 1)^2$ has been split up into smaller and smaller disjoint boxes which collectively contain the diagonal.

It is now necessary to prove that $\mu_c^{(2)}(M_n) \to 0$ as $n \to \infty$.

It follows that

$$\mu_c^{(2)}(M_n) = \sum_{a_i = 0 \text{ or } 1} \mu_c^{(2)}(E^2_{a_1 \cdots a_n}) = \sum_{a_i = 0 \text{ or } 1} \mu_c^2(E_{a_1 \cdots a_n}), \tag{6.15}$$

as $M_n = \bigcup_{a_i = 0 \text{ or } 1} E^2_{a_1 \cdots a_n}$ and these boxes are pairwise disjoint.

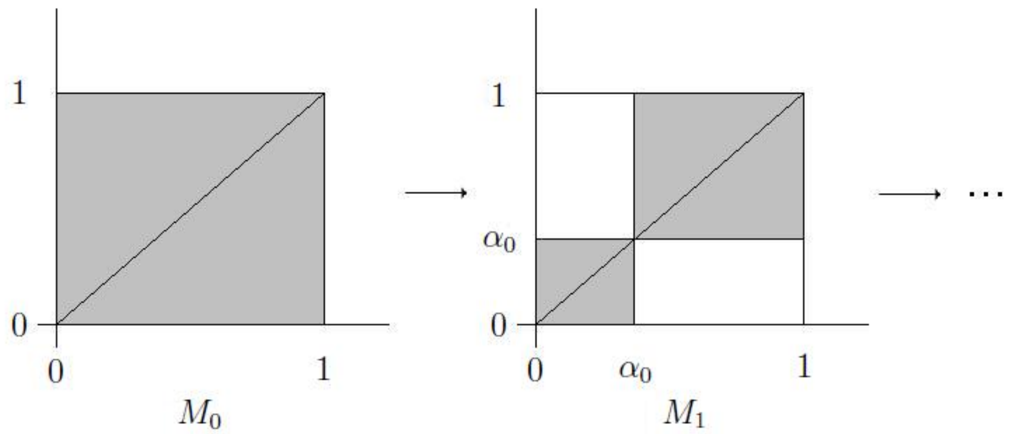


Figure 6.1: Smaller and smaller disjoint boxes collectively containing $\mathcal{D}_0$

Now note that

$$\sum_{a_i=0 \text{ or } 1} \mu_c(E_{a_1\cdots a_n}) = 1. \tag{6.16}$$

The next part is analogous to König's Lemma (Lemma 8.1.2 in [11]; also §49, Lemma 10 in [25]).

Let $\epsilon > 0$. Eventually there exists a level n where $\mu_c(E_{a_1\cdots a_n}) < \epsilon$ for all $a_1 \cdots a_n$ where $a_i = 0$ or 1 and $i = 1, \dots, n$.

For suppose this is not the case. Then at every level n there exists an interval $E_{a_1\cdots a_n}$ with $\mu_c(E_{a_1\cdots a_n}) > \epsilon$. It follows by monotonicity of the measure that the parent of such an interval also has measure greater than ϵ, and so it is possible to construct an infinite path in the tree in Figure (6.1) starting at E such that the intervals in this path form a nested sequence and that each has measure greater than ϵ. In other words a nested sequence

$$E \supseteq E_{a_1} \supseteq E_{a_1 a_2} \supseteq \cdots, \quad a_i = 0 \text{ or } 1, \tag{6.17}$$

is created such that $\mu_c(E_{a_1\cdots a_n}) > \epsilon$ for all $n \in \mathbb{Z}_+$.

Note that $\lim\limits_{n\to\infty} E_{a_1\cdots a_n}$ is a singleton point, of measure zero here (by virtue of the fact that the measure is continuous in this case). This contradicts the fact that $\mu_c(E_{a_1\cdots a_n}) > \epsilon$ for every $n \in \mathbb{Z}_+$.

Hence for large enough n it follows that

$$0 \leq \mu_c(E_{a_1\cdots{}_n}) < \epsilon. \tag{6.18}$$

Thus, again assuming n is large enough,

$$\begin{aligned}
\mu_c^{(2)}(M_n) &= \sum_{a_i=0 \text{ or } 1} \mu_c^{(2)}(E^2_{a_1\cdots a_n}) \\
&= \sum_{a_i=0 \text{ or } 1} (\mu_c(E_{a_1\cdots a_n}))^2, \text{ (product measure)}, \\
&< \sum_{a_i=0 \text{ or } 1} \epsilon \cdot \mu_c(E_{a_1\cdots a_n}), \text{ (by above argument)}, \\
&= \epsilon \cdot \sum_{a_i=0 \text{ or } 1} \mu_c(E_{a_1\cdots a_n}) \\
&= \epsilon \cdot 1, \text{ (by (6.16))} \\
&= \epsilon,
\end{aligned} \tag{6.19}$$

and thus

$$\mu_c^{(2)}(M_n) < \epsilon \text{ for all } n > N \in \mathbb{N} \text{ and } \epsilon > 0. \tag{6.20}$$

Hence by definition $\mu_c^{(2)}(M_n) \to 0$ as $n \to \infty$.

Using this it now follows that

$$\mu_c^{(2)}(\mathcal{D}_0) = \mu_c^{(2)}\left(\bigcap_{n=0}^{\infty} M_n\right) = \lim_{n\to\infty} \mu_c^{(2)}(M_n) = 0 \tag{6.21}$$

by continuity from above and below, i.e. $\mu_c^{(2)}(\mathcal{D}_0) = 0$.

A similar argument holds for all $i \in \mathbb{Z}_+$ to obtain the result

$$\mu_c^{(2)}(\mathcal{D}_i) = 0. \tag{6.22}$$

Hence, finally, we are able to conclude that

$$\mu_c^{(2)}(\mathcal{D}) = \sum_{i=0}^{\infty} \mu_c^{(2)}(\mathcal{D}_i) = \sum_{i=0}^{\infty} 0 = 0, \tag{6.23}$$

or in other words

$$\mu_c^{(2)}(\mathcal{D}) = 0, \tag{6.24}$$

as required. □

6.4 Adjusting the spreading out function

In Chapters 4 and 5 we necessarily created and used the *spreading-out* functions ϵ_k and γ_k. Such functions are once again necessary here but they need adjusting to accommodate the changes in the measure; this will become more apparent as the proceeding analysis continues.

Consider the division of $\mathbb{R}_+$ into intervals of length one, where $I_i = [i, i+1)$. As in the previous section divide each of these intervals dyadically so that each division imparts half of the currently available measure, and let $E^{(i)}_{a_1\cdots a_n}$ denote the dyadic intervals at the n^{th} level of interval I_i.

Now let $f_{E^{(i)}_{a_1\cdots a_n}} \in \mathfrak{C}$, where $f_{E^{(i)}_{a_1\cdots a_n}}$ is a function in $\mathfrak{C}$ defined on the interval $E^{(i)}_{a_1\cdots a_n}$.

Definition. The *spreading-out* function $\gamma_k : \mathfrak{C} \to \mathfrak{C}\widehat{\otimes}\mathfrak{C}$ is defined for $f_{E^{(i)}_{a_1\cdots a_n}}$ by the formula

$$\gamma_k(f_{E^{(i)}_{a_1\cdots a_n}})(x,y)=\gamma_{k,y}(f_{E^{(i)}_{a_1\cdots a_n}})(x)=\begin{cases} f_{E^{(i)}_{0\cdots 0}}(x)\dfrac{\chi_{E^{(i-1)}_{1\cdots 1}}(y)}{\mu_c(E^{(i-1)}_{1\cdots 1})}, & a_0\cdots a_n = 0\cdots 0, i \geq 1, \\ \\ f_{E^{(i)}_{a_1\cdots a_n}}(x)\dfrac{\chi_{E^{(i)}_{a_1\cdots a_n}}(y)}{\mu_c(E^{(i)}_{(a_1\cdots a_n)'})}, & \text{else}, \end{cases} \tag{6.25}$$

where $(a_1\cdots a_n)' = a_1\cdots a_n - 0\cdots 01$; in other words $(a_1\cdots a_n)'$ is the binary number $(a_1\cdots a_n)$ minus 1.

The spreading-out function then applies to a general $f \in \mathfrak{C}$ by linearity; for given such a function f write it as

$$\begin{aligned} f &= \sum_{i=0}^{\infty} f_{I_i} \\ &= \sum_{i=1}^{\infty} \sum_{a_i=0 \text{ or } 1} f_{E^{(i)}_{a_1\cdots a_n}}. \end{aligned} \tag{6.26}$$

Then it follows by linearity that

$$\gamma_k(f) = \sum_{i=1}^{\infty} \sum_{a_i=0 \text{ or } 1} \gamma_k\left(f_{E^{(i)}_{a_1\cdots a_n}}\right). \tag{6.27}$$

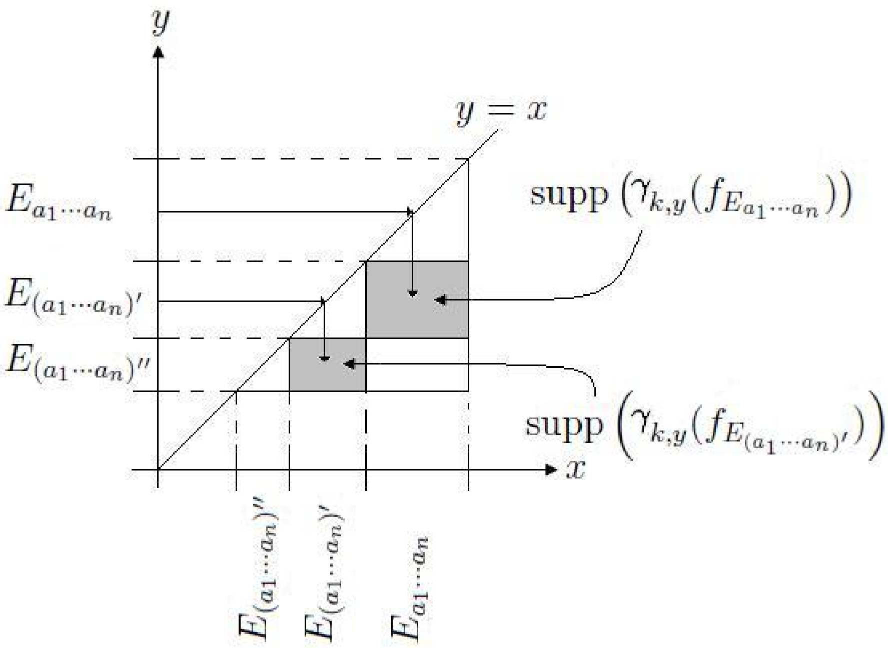


Figure 6.2: An illustration of how the spreading out function works

Note that $\mu_c(E^{(i)}_{a_1\cdots a_n}) \neq 0$ by construction for all $n \in \mathbb{Z}_+$.

The inherent problem with this method is what to do for $f_{E^{(0)}_{0\cdots 0}}$, as there is clearly no *predecessor* $E_{(0\cdots 0)'}$. One way around this problem is to further utilise the concept employed throughout this thesis, that of *disjointly supported functions*. This is a good idea since the only functions truly considered are those of well-separated supports.

Thus given a function $f \in \mathfrak{C}$ assume that $\chi_{\{(-1,0]\}} \ll f$, i.e. that there is a clear separation between the point 0 and the support of f. So when given such a function f there exists $N \in \mathbb{N}$ such that for all $n \geq N$ it follows that $E^{(0)}_{0\cdots 0} \nsubseteq \text{supp}(f)$.

The reason that this is possible is due to the fact that the step/characteristic function are dense in L^1-spaces (Chapter 3, Theorem 3.5). Thus the assumption that $\chi_{\{0\}} \ll f$ is representative of this fact.

An alternative is to note that, as in Section 5.1.1, we are only applying the spreading out function to indicator functions that are not only well-separated from zero but are also order related to another indicator function which has support between zero and the support of our function.

6.4.1 The inverse to the spreading out function

As before we will require a left inverse to the spreading-out function γ_k. Following the same premise as before we define this function $\pi_k : \mathfrak{C}\widehat{\otimes}\mathfrak{C} \to \mathfrak{C}$ as

$$\pi_k(F_{J\times J_-})(x) = \int_{[0,x)} F_{J\times J_-}(x,t)d\mu_c(t) + \int_{[0,x)} F_{J\times J_-}(t,x)d\mu_c(t), \tag{6.28}$$

where $F_{J\times J_-} \in \mathfrak{C}\widehat{\otimes}\mathfrak{C}$ is supported on the domain $J \times J_-$ where

$$\begin{aligned} J = E^{(i)}_{a_1\cdots a_n} &\Rightarrow J_- = E^{(i)}_{(a_1\cdots a_n)'}, \quad a_1\cdots a_n \neq 0\cdots 0, \\ \text{or } J = E^{(i)}_{0\cdots 0} &\Rightarrow J_- = E^{(i-1)}_{1\cdots 1}. \end{aligned} \tag{6.29}$$

The crucial assumption here is that n is taken large enough to ensure the clear separation from $\{0\}$ argument holds as above, making

$$f_{E^{(0)}_{0\cdots 0}} \equiv 0. \tag{6.30}$$

Then it follows for J and J_- defined above that

$$\begin{aligned}\pi_k(\gamma_k(f_J))(x) &= \int_0^x \gamma_k(f_J)(x,t)dt + \int_0^x \gamma_k(f_J)(t,x)dt \\ &= \int_0^x f_J(x)\frac{\chi_{J_-}(t)}{\mu_c(J_-)}dt + \int_0^x f_J(t)\frac{\chi_{J_-}(x)}{\mu_c(J_-)}dt \\ &= f_J(x)\frac{\mu_c(J_-)}{\mu_c(J_-)} + 0 \\ &= f_J(x).\end{aligned} \tag{6.31}$$

The second term vanishes because we require that $x \in J$ in order to avoid the trivial outcome where everything reduces simply to zero and so the characteristic function over J_- vanishes, while the first term sees the whole of the characteristic function in t integrated out leaving behind the measure of its domain, namely $\mu_c(J_-)$.

Finally observe that if $J = E^{(0)}_{0\cdots 0}$ then $f_J \equiv 0$ and it then follows that $\pi_k(\gamma_k(f_J)) \equiv 0$, too.

Hence it follows that π_k is a left inverse for γ_k.

6.4.2 The approximate module property

Consider f_J and g_K with K being defined similarly to J, which is given above, such that $g_K \ll f_J$ and f_J, g_K are normalised as always. It is now necessary to see what happens when $\gamma_k(f_J)$ and g_K are convolved together.

To maintain consistency in notation it is imperative to define, where $a \in \mathfrak{C}$, $G \in \mathfrak{C}\widehat{\otimes}\mathfrak{C}$,

$$(a * G)(x,y) = \int_0^x a(t)dt \cdot G(x,y) + a(x) \cdot \int_0^x G(t,y)dt, \tag{6.32}$$

and

$$(G * a)(x,y) = \int_0^y G(x,t)dt \cdot a(y) + G(x,y) \cdot \int_0^y a(t)dt, \tag{6.33}$$

for then it follows that for $G = f \otimes g$

$$(f \otimes g) * a = f \otimes (g * a) \ \ \& \ \ a * (f \otimes g) = (a * f) \otimes g. \tag{6.34}$$

Using this setup, and noting that n is sufficiently large enough to ensure that the intervals are well separated and ordered such that K is below J_- which is in turn below J, yields that

$$\begin{aligned}(\gamma_k(f_J) * g_K)(x,y) &= \int_0^y \gamma_k(f_J)(x,t)dt \cdot g_K(y) + \gamma_k(f_J)(x,y) \cdot \int_0^y g_K(t)dt \\ &= f_J(x) \cdot \int_0^y \frac{\chi_{J_-}(t)}{\mu_c(J_-)}dt \cdot g_K(y) + f_J(x) \cdot \frac{\chi_{J_-}(y)}{\mu_c(J_-)} \cdot \int_0^y g_K(t)dt.\end{aligned} \tag{6.35}$$

For this entire expression to be nontrivial it is necessary that $y \in J_-$, for if, in particular, $y \in K$ then everything reduces to zero. Then the first term reduces to zero regardless as the limits of the integral do not reach the domain of χ_{J_-} while the integral in the second term is simply 1 as the limits cover the whole of the domain of the normalised function g_K.

Hence this whole expression reduces simply to

$$(\gamma_k(f_J) * g_K)(x,y) = 0 + f_J(x) \cdot \frac{\chi_{J_-}(y)}{\mu_c(J_-)} \cdot 1 = \gamma_k(f_J)(x,y). \tag{6.36}$$

An almost identical calculation reveals that

$$(g_K * \gamma_k(f_J))(x,y) = \gamma_k(f_J)(x,y). \tag{6.37}$$

Consideration of what happens for these calculations when $f_J \ll g_k$, as in Chapter 5, is possible and follows in a similar way. As these calculations do not appear in our analysis we will not include them here and leave them for the reader.

6.5 The simplicial cohomology groups of $L^1(\mathbb{R}_+, \leq, \mu_c)$

It is already known from [3] that $\mathcal{HH}^1(\mathfrak{C}) \cong \mathfrak{C}^*$ which, as has already been stated, is isomorphic to $L^\infty(\mathbb{R}_+, \leq, \mu_c)$. We now show that for $n \geq 2$ we have that

$$\mathcal{HH}^n(\mathfrak{C}) \cong 0. \tag{6.38}$$

The analysis here follows identically to that in Sections 5.2.2 and 5.2.3.

First we consider the $(n+1)$-dimensional tensor product $\eta = f_1 \otimes \cdots \otimes f_n \otimes f_0$ in $\widehat{\bigotimes}^{n+1} \mathfrak{C}$ of normalised functions of disjoint support with the functions $f_i \in \mathfrak{C}$ satisfying some order structure for $i = 0, \ldots, n$ in the cyclic context. Because of cyclicity we are once again able to consider the case when $f_i \ll f_1$ for $i \neq 1$.

Because of the work done earlier in the current section we are in a position to define our contracting homotopy of our cyclic chain and cochain complexes here exactly as found previously, namely that in (5.130) and (5.127) but with $\mathbb{C}$ in place of $\mathfrak{B}$, defining s_k^n, s_*^n, δ^n and d^n in the same way. We are again able to approximate $\gamma_k(f_1)$ by a linear combination of elementary tensor products of normalised functions of disjoint support, $\sum_{i=1}^{\infty} x_i \otimes y_i$, with $f_j \ll x_i \ll y_i$ for all $i = 1, 2, \ldots$, $j \neq 1$ and large enough k; this is analogous to the approximation of $\gamma_{k,y}(f_1)$.

We follow the calculations found in Section 5.2.3 through and arrive at the conclusion

$$\eta = s_{k,y}^n \circ d^n(\eta) + d^{n+1} \circ s_{k,y}^{n+1}(\eta) \tag{6.39}$$

and thus for $\varphi \in \mathcal{ZC}^n(\mathfrak{B})$

$$\varphi(\eta) = \varphi(s_{k,y}^n \circ d^n(\eta)) + \varphi(d^{n+1} \circ s_{k,y}^{n+1}(\eta)), \tag{6.40}$$

where the operation of taking limits ensures our disjointedness criteria are met once more.

We are immediately able to conclude that the second term is trivial as before, and by duality arguments and a relabeling we therefore see that

$$\varphi(\eta) = (\delta^n \psi)(\eta) \tag{6.41}$$

and hence

$$\mathcal{HC}^n(\mathfrak{C}) \cong 0 \tag{6.42}$$

for all $n \in \mathbb{N}$.

An appeal to the Connes-Tzygan long exact sequence as in Section 5.2.4 yields that

$$\mathcal{HH}^n(\mathfrak{C}) \cong 0 \tag{6.43}$$

for all $n \geq 2$.

Considering the first part of the sequence as given in Section 2.2.3 we see that the sequence

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{HH}^1(\mathfrak{C}) \longrightarrow \mathcal{HC}^0(\mathfrak{C}) \longrightarrow 0 \longrightarrow \cdots$$

is exact.

Note that the space $\mathcal{HC}^0(\mathfrak{C})$ is exactly the space of traces on the algebra $\mathfrak{C}$ which is isomorphic to $\mathfrak{C}^*$; this is a well-known identification. Then we have that

$$0 = \mathrm{im}(0 \to \mathcal{HH}^1(\mathfrak{C})) = \ker(\mathcal{HH}^1(\mathfrak{C}) \to \mathfrak{C}^*) \tag{6.44}$$

and so the map from $\mathcal{HH}^1(\mathfrak{C})$ into $\mathfrak{C}^*$ is injective, while

$$\mathrm{im}(\mathcal{HH}^1(\mathfrak{C}) \to \mathfrak{C}^*) = \ker(\mathfrak{C}^* \to 0) = \mathfrak{C}^* \tag{6.45}$$

shows that the map is also surjective. Thus we have an isomorphism, showing us that

$$\mathcal{HH}^1(\mathfrak{C}) \cong \mathfrak{C}^* \tag{6.46}$$

exactly as before.

6.6 Extending to a general totally ordered, locally compact semigroup

We now extend our results to the most general case for a continuous, σ-finite, regular, positive Borel measure μ_c; changing $(\mathbb{R}_+, \vee)$ in our algebra to $(X, \vee)$, a general totally ordered semigroup which is locally compact its order topology, as defined in Section 2.1. Because there will be no ambiguity, we will drop the $\vee$ notation.

First we investigate the structure of the semigroup $(X, \leq, \mu_c)$. The method is similar to that found in [2].

Definition. Define the equivalence relation $\approx$ such that for $x, y \in X$ and $x \leq y$ we have $x \approx y$ if $\mu_c([x, y)) < \infty$. It is straightforward to see that this is indeed an equivalence relation, which we leave to the reader to show.

Definition. A *transversal* is a set containing exactly one element from each member of a collection.

We now prove the following necessary result about X.

Lemma 6.2. *The equivalence classes of X under the relation $\approx$ are disjoint and totally ordered intervals.*

Note. In our case, given the nature of later calculations, we are dealing with countably many equivalence classes here (therefore making the transveral here possess countably many elements). A problem arises if $\mu_c((a,b)) = 0$ for $a, b \in X$ such that $a \neq b$. One way to counter such a problem is to insist on the measure being *strictly positive*, that is the support of μ_c is the whole space X; this ensures that every nonempty open interval in X has strictly positive measure and is hence infinite. Lebesgue measure is strictly positive.

Proof. Let $x \approx x'$ in $[x]$ and $y \approx y'$ in $[y]$, where $[\cdot]$ denotes the equivalence class of $\cdot$, such that $x \lneqq y$ and therefore $x' \lneqq y'$.

As x and y are in different equivalence classes it must be that $\mu_c([x,y)) = \infty$. We also have that $\mu_c([x,x')) < \infty$ and $\mu_c([y,y')) < \infty$. Thus

$$\infty = \mu_c([x,y)) \leq \mu_c([x,x')) + \mu_c([x',y')) + \mu_c([y,y')) \tag{6.47}$$

by the triangle inequality. Since $\mu_c([x,x'))$ and $\mu_c([y,y'))$ are both finite it must be that $\mu_c([x',y')) = \infty$ and hence x' and y' are in different equivalence classes. Therefore $[x]$ and $[y]$ are disjoint.

To prove that the equivalence classes are totally ordered we use representatives.

For transitivity, if $[x] \leq [y]$ and $[y] \leq [z]$ then $x \leq y$ and $y \leq z$ for all $x \in [x], y \in [y]$ and $z \in [z]$. As $x, y, z \in X$ and X is totally ordered it trivially follows that $x \leq z$ for all $x \in [x]$ and $z \in [z]$, and thus $[x] \leq [z]$.

Now if $[x] \leq [y]$ then it must be that $x \leq y$ for all $x \in [x], y \in [y]$. As X is totally ordered we cannot have $y \leq x$ for any $x \in [x], y \in [y]$ and so we cannot have $[y] \leq [x]$.

The hardest part is proving antisymmetry. Let $[x] \leq [y]$ and $[y] \leq [x]$. Then we can choose representatives $x \in [x], y \in [y]$ such that $x \leq y$. We are also able to choose representatives $x' \in [x], y' \in [y]$ such that $y' \leq x'$.

Within the equivalence class $[x]$ one of $x \leq x'$ or $x' \leq x$ must hold. If $x \leq x'$ then we can conclude that $y' \leq x' \leq x \leq y$, and so by inclusion and non-negativity of measures $\mu_c([x, y)) \leq \mu_c([y', y)) < \infty$ as y and y' are in the same equivalence class.

On the other hand, if $x \leq x'$ then either $y' \leq x$ or $x \leq y'$. If $y' \leq x$ then we have that $y' \leq x \leq y$, and thus $\mu_c([x, y)) \leq \mu_c([y', y)) < \infty$. If $x \leq y'$ then we have that $x \leq y' \leq x'$ and hence $\mu_c([y', x')) \leq \mu_c([x, x')) < \infty$ as x and x' are in the same equivalence class.

This covers every possibility and in all of the cases we have for $x \in [x], y \in [y]$ that $\mu_c([x, y)) < \infty$ and so $x \approx y$ or $[x] = [y]$ as required.

To see that the equivalence classes are intervals in X we choose $a \in [a]$ and define the function $\gamma : [a] \to \mathbb{R}$ by the formula

$$\begin{aligned} \gamma(x) &= \mu_c([a, x)), && (6.48) \\ \gamma(x') &= \mu_c([x', a)), && (6.49) \end{aligned}$$

for $x, x' \in [a]$ such that $x' \leq a \leq x$. Then it is not hard to see, given the condition of strict positivity, that $\mathrm{im}\gamma = J$, an interval in $\mathbb{R}$ such that $0 \in J$. It follows that γ is a bijection from $[a]$ onto J and so the two are isomorphic. $\square$

The multiplication in our semigroup X is simply an extension of the multiplication in each of the individual semigroups that go into making its sum. Let $[a_i], [a_j]$ be equivalence classes in X under $\approx$ such that $i \leq j$, say. If $i < j$ then $a_i < a_j$ and $[a_i] \cap [a_j] = \emptyset$, i.e. the intervals are ordered and disjoint. If on the other hand $i = j$ then $[a_i] = [a_j]$.

Then for $s \in [a_i]$ and $t \in [a_j]$ we have that

$$s \vee t = \begin{cases} t, & i < j, \\ s \vee t, & i = j. \end{cases} \tag{6.50}$$

It should be noted at this point that equivalence classes under the relation $\approx$ in X are open. For if $x \in [a]$ then local compactness implies that there exists a compact neighbourhood of x. Using this neighbourhood we are able to find an open interval of finite length I (take the maximum and minimum elements of the totally ordered neighbourhood) which contains x such that $I \cup [a] = I$. So $I \subseteq [a]$.

As this holds for every point of $[a]$ we are able to write $[a]$ as an arbitrary union of open intervals, making it open.

It should also be noted that using the same logic and I^c, the complement of I, instead we can also write $[a]$ as an arbitrary intersection of closed intervals, making $[a]$ closed as well.

We are now able to see that X can be written as a union of disjoint, totally ordered intervals;

$$X = \bigsqcup_{i \in T} [x_i], \tag{6.51}$$

where T is a countable transversal of $X/\approx$.

We can therefore consider $L^1(X, \leq, \mu_c)$ to be equivalent to the ℓ^1-direct sum

$$\bigoplus_{i \in T} L^1([a_i], \leq, \mu_c). \tag{6.52}$$

The fundamental result of this section is now presented as:

Theorem 6.1. *Let $[a]$ be an equivalence class in X under the relation $\approx$. Then*

$$L^1([a], \mu_c) \cong L^1((b, c), \lambda), \tag{6.53}$$

where $b, c \in \mathbb{R} \cup \{\pm\infty\}$ and λ is Lebesgue measure.

Note that b, c could be $\pm\infty$ or simply finite numbers in $\mathbb{R}$. As we are dealing with continuous measures it follows that the inclusion or exclusion of the endpoints b, c is a null point, especially as $[a]$ is both open and closed.

Proof. We have shown that $[a]$ is an interval, say (x, y); this could be unbounded in either or both directions. Let $f \in L^1([a], \mu_c) \equiv L^1((x, y), \mu_c)$ and fix the point $a \in [a] = (x, y) \subseteq X$. It should be noted that a cannot be an unattainable endpoint (although it can be a *finite* endpoint) due to the definition of the equivalence relation $\approx$, i.e. there exist $x_1, x_2 \in [a]$ such that $x_1 < a < x_2$.

We now define the transformation $\tau : (x, y) \to (b, c)$ by the formula

$$\tau(t) = \begin{cases} -\mu_c(t, a), & t \leq a, \\ +\mu_c(a, t), & a < t. \end{cases} \tag{6.54}$$

Then we have that $\tau(a) = 0$. Thus $0 \in (b, c)$, where $b, c \in \mathbb{R} \cup \{\pm\infty\}$. Note that we have no control over what b and c are.

Now given $f(t) = \widehat{g}(t) \in L^1([a], \mu_c) \equiv L^1((x, y), \mu_c)$ such that

$$\widehat{g}(t) = \begin{cases} 1, t \in [x_1, x_2], \\ 0, \text{ else}, \end{cases} \tag{6.55}$$

we define the map $\widehat{g} \mapsto g$ such that

$$\widehat{g}(t) = g(\tau(t)) \tag{6.56}$$

for $t \in [x_1, x_2] = \mathrm{supp}(\widehat{g})$ where $x \leq x_1 < x_2 \leq y$ and $g \in L^1((b, c), \lambda)$. For these step functions, which we have scaled to have value 1 on their supports, this is clearly injective and onto. Additionally, we have that

$$\|\widehat{g}\| = \int_x^y |\widehat{g}(t)| d\mu_c = \mu_c((x_1, x_2)). \tag{6.57}$$

Also

$$\|g\| = \int_b^c |g(t)| dt = \int_{\tau(x_1)}^{\tau(x_2)} |g(t)| dt, \tag{6.58}$$

and since for all $t \in (\tau(x_1), \tau(x_2)) = \mathrm{supp}(g)$ there exists $t' \in (x_1, x_2) = \mathrm{supp}(\widehat{g})$ such that $\tau(t') = t$ it follows that $g(t) = g(\tau(t')) = \widehat{g}(t') = 1$ here and thus

$$\int_{\tau(x_1)}^{\tau(x_2)} |g(t)| dt = \int_{\tau(x_1)}^{\tau(x_2)} dt = \tau(x_2) - \tau(x_1). \tag{6.59}$$

Then

$$\tau(x_2) - \tau(x_1) = \begin{cases} \mu_c((a, x_2)) - \mu_c((a, x_1)), & a \leq x_1 \leq x_2, \\ -\mu_c((x_2, a)) + \mu_c((x_1, a)), & x_1 \leq x_2 \leq a, \\ \mu_c((a, x_2)) + \mu_c((x_1, a)), & x_1 \leq a \leq x_2, \end{cases} \tag{6.60}$$

which, by countable additivity of μ_c, is exactly $\mu_c((x_1, x_2))$ as required.

Since this holds for step functions, which are dense in L^1 algebras, it follows that the map $\widehat{g} \mapsto g$ is an isometric isomorphism.

Now consider step functions $\widehat{f}, \widehat{g} \in L^1([a], \mu_c)$. We need to show that our isometric isomorphism preserves the product structure.

Under the isomorphism $\widehat{f}$ is mapped to f such that $f(\tau(t)) = \widehat{f}(t)$ while $\widehat{g}$ is mapped to g such that $g(\tau(t)) = \widehat{g}(t)$ for all $t \in \operatorname{supp}(\widehat{g})$. The convolution of these images is then $f * g \in L^1((b, c), \lambda)$. But for all $x \in (b, c)$ there exists $t \in [a]$ such that $\tau(t) = x$ and so

$$(f * g)(y) = (f * g)(\tau(t)). \tag{6.61}$$

On the other hand, the product of $\widehat{f}$ and $\widehat{g}$ is a function $\widehat{h}$ in $L^1([a], \mu_c)$. Under our isometric isomorphism this is mapped to the function h such that $h(\tau(t)) = \widehat{h}(t)$ for $t \in [a]$. But

$$\begin{aligned} \widehat{h}(t) &= (\widehat{f} * \widehat{g})(t) \\ &= \int_{u<t} \widehat{f}(u) d\mu_c(u) \cdot \widehat{g}(t) + \widehat{f}(t) \cdot \int_{u<t} \widehat{g}(u) d\mu_c(u). \end{aligned} \tag{6.62}$$

Compare this to

$$(f * g)(\tau(t)) = (f * g)(x) = \int_{v<x} f(v) dv \cdot g(x) + f(x) \cdot \int_{v<x} g(v) dv. \tag{6.63}$$

First we can see that as τ is an order-preserving isomorphism it follows that for the step function with value 1 and $\operatorname{supp}(f) = (x_1, x_2)$

$$\int_{v<x} f(v) dv = x - x_1. \tag{6.64}$$

But there exist t, t_1 such that $\tau(t_1) = x_1$ and $\tau(t) = x$, and so this becomes

$$\tau(t) - \tau(t_1) = \begin{cases} \mu_c((a,t)) - \mu_c((a,t_1)), & a \leq t_1 \leq t, \\ \mu_c((a,t)) - (-\mu_c((t_1,a))), & t_1 \leq a \leq t, \\ -\mu_c((t,a)) - (-\mu_c((t_1,a))), & t_1 \leq t \leq a, \end{cases} \tag{6.65}$$

all of which equate to $\mu_c((t_1,t))$ via countable additivity of μ_c.

Compare this to

$$\int_{u<t} \widehat{f}(u)du = \mu_c((t_1,t)). \tag{6.66}$$

as $\mathrm{supp}(f) = (x_1, x_2)$ implies that $\mathrm{supp}(\widehat{f}) = (t_1, t_2)$ where $\tau(t_2) = x_2$.

Since for all x there exists t such that $\tau(t) = x$ it follows that $f(x) = f(\tau(t)) = \widehat{f}(t)$. As these hold for g as well we have that

$$\widehat{h}(t) = (f * g)(\tau(t)) \tag{6.67}$$

and since both $(f*g)(\tau(t))$ and $h(\tau(t))$ equal $\widehat{h}(t)$ it follows that $h \equiv f * g$, i.e. that $\widehat{f} * \widehat{g}$ is mapped to $f * g$ such that $(f*g)(\tau(t)) = (\widehat{f} * \widehat{g})(t)$; as f is obtained from $\widehat{f}$, and similarly for g, this therefore shows that our isomorphism preserves the product structure of our algebras on step functions.

As the step functions are dense in L^1-algebras it follows by density arguments that this holds for all functions; therefore we have an algebra isometric isomorphism from $L^1([a], \mu_c)$ to $L^1((b,c), \lambda)$ as required. $\square$

Essentially this allows us to consider the algebra $L^1(X, \leq, \mu_c)$ to be a sum of algebras which are in effect copies (or subalgebras) of $L^1(\mathbb{R}_+, \leq, \lambda)$, $L^1(\mathbb{R}_-, \leq, \lambda)$ or $L^1(\mathbb{R}, \leq, \lambda)$.

We are now in a position to make our conclusion. We are able to write

$$L^1(X, \leq, \mu_c) \equiv \bigoplus_{i \in T}^{\ell^1} L^1((b_i, c_i), i, \leq, \lambda), \tag{6.68}$$

where $b_i, c_i \in \mathbb{R} \cup \{\pm\infty\}$ and i denotes the transversal element indexing the summand.

The product on this algebra is a subtle one, but it must be such that it preserves the isomorphism of Banach algebras, not just Banach spaces.

Consider the case when X can be decomposed into two intervals; the general case for n intervals follows from this analysis. Thus we have

$$L^1(X, \leq, \mu_c) = L^1([x_{i_1}], \leq, \mu_c) \oplus L^1([x_{i_2}], \leq, \mu_c). \tag{6.69}$$

As the equivalence classes of X are ordered we may assume, without loss of generality, that $[x_{i_1}] < [x_{i_2}]$. Then $f, g \in L^1(X, \leq, \mu_c)$ are such that $f = f^{i_1} \oplus f^{i_2}$ and $g = g^{i_1} \oplus g^{i_2}$, where $f^i, g^i \in L^1([x_i], \leq, \mu_c)$ for $i = i_1, i_2$. Then since the multiplication on $L^1(X, , \leq, \mu_c)$ is that of convolution we have that

$$\begin{aligned}
(f * g)(x) &= \int_{u<x} f(u) d\mu_c(u) \cdot g(x) + f(x) \cdot \int_{u<x} g(u) d\mu_c(u) \\
&= \int_{u<x} (f^{i_1} \oplus f^{i_2})(u) d\mu_c(u) \cdot (g^{i_1} \oplus g^{i_2})(x) \\
&\qquad + (f^{i_1} \oplus f^{i_2})(x) \cdot \int_{u<x} (g^{i_1} \oplus g^{i_2})(u) d\mu_c(u) \\
&= \int_{u<x} f^{i_1}(u) d\mu_c(u) \cdot g^{i_1}(x) + \int_{u<x} f^{i_1}(u) d\mu_c(u) \cdot g^{i_2}(x) \\
&\qquad + \int_{u<x} f^{i_2}(u) d\mu_c(u) \cdot g^{i_1}(x) + \int_{u<x} f^{i_2}(u) d\mu_c(u) \cdot g^{i_2}(x) \\
&\qquad + f^{i_1}(x) \cdot \int_{u<x} g^{i_1}(u) d\mu_c(u) + f^{i_1}(x) \cdot \int_{u<x} g^{i_2}(u) d\mu_c(u) \\
&\qquad + f^{i_2}(x) \cdot \int_{u<x} g^{i_1}(u) d\mu_c(u) + f^{i_2}(x) \cdot \int_{u<x} g^{i_2}(u) d\mu_c(u).
\end{aligned} \tag{6.70}$$

After collecting terms it is easy to see that

$$(f * g)(x) = (f^{i_1} * g^{i_1})(x) + (f^{i_1} * g^{i_2})(x) + (f^{i_2} * g^{i_1})(x) + (f^{i_2} * g^{i_2})(x). \tag{6.71}$$

Hence it must be that the product on $\bigoplus_{i \in T} L^1([a_i], \leq, \lambda)$ is

$$\begin{aligned}
(f * g)(x) &= (f^{i_1} \oplus \cdots \oplus f^{i_n}) * (g^{i_1} \oplus \cdots \oplus g^{i_n}) \\
&= \sum_{\mathbf{t} \in T^2} (f^{t_1} * g^{t_2})(x),
\end{aligned} \tag{6.72}$$

where $T = \{i_1, \ldots, i_n\}$ and $\mathbf{t} = (t_1, t_2)$ here. It is easily seen that our isomorphism respects this product by linearity.

Remark. The order structure on this algebra is initially decided via transversals. In decomposing X into its collection of intervals indexed by a transversal, these intervals are ordered. Thus we can order the transversal elements to preserve this ordering, i.e. if $[x_i] < [x_j]$ in $X/\approx$ then $i < j$ in T.

Thus if $f_i \in L^1([x_i], \leq, \mu_c) \cong L^1((b_i, c_i), i, \leq, \lambda)$ for $i = i_1, i_2$ then $f_{i_1} \ll f_{i_2}$ regardless of the relative positioning of the intervals (b_{i_1}, c_{i_1}) and (b_{i_2}, c_{i_2}); essentially the transversal is the component used to determine order initially. If the two functions are indexed by the same transversal element then relative order is arrived at in the usual way, identical to that used for the case when $X = \mathbb{R}_+$.

We are thus able to consider $L^1(X, \leq, \mu_c)$ as an ℓ^1 direct sum of L^1 algebras defined on intervals (b_i, c_i) for $i \in T$, a transversal set for $X/\approx$.

We are now interested in showing the triviality of the cyclic cohomology groups for this algebra $\mathfrak{D} = L^1(X, \leq, \mu_c)$. To do this we will again show that a contracting homotopy exists here, again exactly as in (5.130) and (5.127) (but with $\mathfrak{D}$ instead of $\mathfrak{B}$), with s_k^n, s_*^n, δ^n and d_n defined as found there, by proving the identity

$$s_k^n \circ d^n + d^{n+1} \circ s_k^{n+1} = 1 \tag{6.73}$$

exactly as in Section 5.2.

Consider the elementary tensor product $\eta = f_1 \otimes \cdots \otimes f_n \otimes f_0 \in \bigotimes^{n+1} \mathfrak{D}$; by the density and overlapping arguments prevalent throughout this thesis it is enough to consider the functions f_j for $j = 0, \ldots, n$ to have ordered and disjoint supports in X. It should be noted that, as before, s_k^n is also well-defined for $\eta \in \mathcal{CC}_n(\mathfrak{D})$ too, but as we are following the method in Section 5.2.3 it is necessary to consider a general tensor product rather than a cyclic one; for in the analysis in Section 5.2.3 we then, implicitly, make the connection with $\mathcal{CC}_n(\mathfrak{D})$ enabling us to introduce $\varphi \in \mathcal{ZC}^n(\mathfrak{D})$, which is where the cyclicity comes in, and move to the dual and thus our ultimate conclusion: that $\mathcal{HC}^n(\mathfrak{D})$ is trivial for all $n \in \mathbb{N}$.

As X is a disjoint union of its equivalence classes under the relation $\approx$, we may write each f_j as the direct sum $\bigoplus_{i\in T} f_j^i$. It then follows by linearity that

$$(s_k^n \circ d^n + d^{n+1} \circ s_k^{n+1})(\eta) = \bigoplus_{\mathbf{t}\in T^{n+1}} (s_k^n \circ d^n + d^{n+1} \circ s_k^{n+1})(f_{j1}^{t_1} \otimes \cdots \otimes f_{jn}^{t_n} \otimes f_{j0}^{t_0}). \quad (6.74)$$

Calculating this directly would be very computationally intensive as we would have to consider all of the possible order cases within each interval defined by a transversal element, of which there are many. Therefore we try to emulate what we have done in the previous chapter; we will show that each of the arguments of the summands can be embedded into $L^1(\mathbb{R}_+, \leq, \lambda)$ and so conclude that the identity holds as before.

Let $f, g \in \mathfrak{D}$ and write X as a disjoint union of its equivalence classes,

$$X = \bigsqcup_{i\in T} [a_i]. \quad (6.75)$$

Using our map τ from $[a_i] \equiv (x, y) \subseteq X$ to $(b_i, c_i) \subseteq \mathbb{R}$, we define a family of submaps, τ_i such that

$$[a_i]_k = \tau_i^{-1}\left((b_i, c_i) \cap [k, k+1)\right). \quad (6.76)$$

Then we can see that $\mu_c([a_i]_k) \leq 1$ due to the isometric nature of the map τ, from which this family is derived, and

$$X = \bigsqcup_{(i,k)\in T\times\mathbb{Z}} [a_i]_k, \quad (6.77)$$

which induces the associated map $\tau_i^* : L^1([a_i]) \to L^1(b_i, c_i)$ such that $\tau_i^* \circ f = f \circ \tau_i$. Without loss of generality suppose that $f, g \in \mathfrak{D}$ are supported on $\bigsqcup_{(i,k)\in F} [a_i]_k$, where $F \subseteq T \times \mathbb{Z}$ such that $|F| < \infty$; in other words assume f, g are supported on finite (or compact) intervals. We obtain the general case by expanding through density arguments once again.

We then define $\tau_F : \bigsqcup_{(i,k)\in F} [a_i]_k \to \mathbb{R}_+$ with the formula

$$\tau_F([a_i]_k) = [2\ell_{(i,k)}, 2\ell_{(i,k)} + 1), \quad (6.78)$$

where $\ell_{(i,k)} = |\{(i',k') : (i',k') < (i,k)\}|$. For instance, the minimum in F (which exists due to the total ordering of $T \times \mathbb{Z}$), say, (i_0, k_0), results in the unit interval section $[a_{i_0}]_{k_0}$ being mapped to the unit interval $[0,1)$ in $\mathbb{R}_+$, with its immediate successor in F being mapped to $[2,3)$ and so on.

The map induced by τ_F, τ_F^*, is then a map from $L^1([a_i], \mu_c)$ to $L^1(\mathbb{R}_+, \mu_c)$, the latter of which is isometrically isomorphic to $L^1(\mathbb{R}_+, \lambda)$, such that

$$\tau_F^*(f * g) = \tau_F^*(f) * \tau_F^*(g), \tag{6.79}$$

by analogy to the isomorphism in Theorem (6.1), meaning that we have successfully embedded the element $f \in L^1(\bigsqcup_{(i,k)\in F}[a_i]_k, \mu_c)$ into $L^1(\mathbb{R}_+, \lambda)$.

This means that we may consider $f_{j_1}^{t_1} \otimes \cdots \otimes f_{j_n}^{t_n} \otimes f_{j_0}^{t_0}$ as an element of $L^1(\mathbb{R}_+^n, \lambda)$ for which we already know that

$$(s_k^n \circ d^n + d^{n+1} \circ s_k^{n+1})(f_{j_1}^{t_1} \otimes \cdots \otimes f_{j_n}^{t_n} \otimes f_{j_0}^{t_0}) = 1, \tag{6.80}$$

and hence it follows that

$$\bigoplus_{\mathbf{t}\in T^{n+1}} (s_k^n \circ d^n + d^{n+1} \circ s_k^{n+1})(f_{j_1}^{t_1} \otimes \cdots \otimes f_{j_n}^{t_n} \otimes f_{j_0}^{t_0}) = \bigoplus_{\mathbf{t}\in T^{n+1}} 1_{\bigoplus_{i\in T} L^1([a_i], \leq, \mu_c)}, \tag{6.81}$$

which is the identity on $\bigoplus_{i\in T} L^1([a_i], \leq, \mu_c)$ as required.

Remark. We may be worried that applying the s-maps to a function may spread the support of that function beyond the interval in which it is defined, but as we only apply these maps to the function with the biggest support in the order structure on the supports it means that we only ever apply it to functions that are well-separated from the lower edge of each interval. Therefore there will always be scope to apply these maps.

Alternatively we could also invoke the separation from zero argument found in Section 6.4.

From our analysis, identical to that in Section 5.2.3, it then follows that the cyclic cohomology groups of $\mathfrak{D} = L^1(X, \leq, \mu_c)$ are also trivial, that is

$$\mathcal{HC}^n(\mathfrak{D}) \cong 0, \text{ for all } n \in N. \tag{6.82}$$

The Connes-Tzygan long exact sequence used exactly as before then yields to us the final result of this thesis:

Theorem 6.2. *Let $(X, \leq)$ be a totally ordered semigroup which is locally compact in its order topology and let μ_c be a continuous, σ-finite, regular, positive Borel measure. Then we have for $n \in \mathbb{N}$,*

$$\mathcal{HH}^n(L^1(X, \leq, \mu_c)) \cong \begin{cases} (L^1(X, \leq, \mu_c))^* \cong L^\infty(X, \leq, \mu_c), & \text{for } n = 1, \\ 0, & \text{else.} \end{cases} \tag{6.83}$$

Chapter 7

Conclusions and further work

Throughout this thesis we have seen that calculating the simplicial cohomology groups for semigroup algebras in the locally compact case is not an easy task, despite appearances. Even proving the triviality of higher degrees of these groups can become very involved, technical and subtle, relying on a multitude of techniques and results.

In Chapter 3 we created an alternative method for calculating the first simplicial cohomology group with coefficients in $L^1(\mathbb{R}_+, \vee)$, reinforcing the result proven by Blackmore in [3]. This allowed us to move on to investigating higher degrees, with the second simplicial cohomology group for $L^1(\mathbb{R}_+, \vee)$ being the focus of Chapter 4. Our method allowed us to show that this group is trivial.

In considering the n^{th} simplicial cohomology group for this algebra our method became restrictive and so we were forced to make modifications in order to proceed, namely using the notion of a contracting homotopy. Chapter 5 dealt with these changes and then proved the results of Chapter 4 in order to demonstrate that these modifications do indeed work. The remainder of Chapter 5 took this modified method and showed that the n^{th} simplicial cohomology group with coefficients in $L^1(\mathbb{R}_+, \vee)$ was trivial for $n \geq 2$.

We were able to take these methods further in Chapter 6, analysing initially the case where the underlying semigroup for our algebra is $(\mathbb{R}_+, \leq, \mu_c)$, where μ_c is a more general continuous Borel measure than the Lebesgue measure we had before, and finally the most general case where $(\mathbb{R}_+, \vee)$ is replaced by a more general totally ordered, locally compact semigroup $(X, \leq, \vee)$; in both cases we were able to show that the same results apply as those for the algebra $L^1(\mathbb{R}_+, \vee)$.

Despite what we were able to do there are still a number of questions that could be the source of some further work on this subject. Some possible avenues of future research are given here.

Question. *What would be the effect of changing the underlying total order to a partial order? For instance one could consider the algebra $L^1(\mathbb{R}_+^2, \vee)$ and attempt to ascertain if this algebra has trivial higher order simplicial cohomology also.*

This question has already been answered in some cases by Gourdeau, Lykova and White in [14] who investigated the higher dimensional simplicial cohomology groups of $L^1(\mathbb{R}_+^k, +)$.

We might also consider what changes would be wrought to our results if we change the binary operation within the semigroup, but there appear to be no other natural operations left to consider; the min operation is intimately connected to max, which we have considered here, and $\times$ to $+$, which has been considered elsewhere.

Question. *What are the corresponding results for the more general algebras $L^p(X, \leq, \mu_c)$ when $1 < p < \infty$?*

This question has been partially answered by Blackmore in [3] where he looked at the first simplicial cohomology group of $L^p([0,1], m)$ where $1 \leq p < \infty$ and m is Lebesgue measure.

Question. *Would it be possible to change the continuous, σ-finite, positive, regular Borel measure on X to a more general Radon measure?*

Question. *Could we generate Künneth formulae for our work, for example to cover the algebras $L^1(\mathbb{R}_+^2, \vee)$ and $\ell^1(\mathbb{Z}_+^2, \vee)$?*

Note that for the algebra $\ell^1(\mathbb{Z}_+^2, \vee)$ the underlying semigroup is a unital semilattice (via pointwise operations) and so has already been covered by Choi in his paper [8].

The final point of consideration is to look at what happens if we have a general measure, not just a continuous measure. Blackmore in [3] states that a σ-finite, positive, regular Borel measure on X, μ, can be uniquely decomposed into its discrete and continuous parts, $\mu = \mu_d + \mu_c$. Based on the result from his paper and from the work we have conducted in Chapter 6 it appears that we are able to apply the same logic and hence make the following observation:

Conjecture. *Let $(X, \leq, \mu)$ be a totally ordered semigroup which is locally compact in its order topology, and let μ be a σ-finite, positive, regular Borel measure on X with discrete part μ_d and continuous part μ_c. Then for $n \in \mathbb{N}$ we have that*

$$\mathcal{HH}^n(L^1(X, \leq, \mu)) \cong \mathcal{HH}^n(L^1(X, \leq, \mu_d)) \oplus \mathcal{HH}^n(L^1(X, \leq, \mu_c)). \tag{7.1}$$

We would then be able to consider the discrete and continuous parts separately, into which a lot of work has gone already. That concludes this thesis.

Bibliography

[1] W. G. Bade, P. C. Curtis, and H. G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras*, Proc. London Math. Soc. (3), 55 (1987), pp. 359–377.

[2] J. W. Baker, J. S. Pym, and H. L. Vasudeva, *Totally ordered measure spaces and their L^p algebras*, Mathematika, (1982).

[3] T. D. Blackmore, *Derivations from totally ordered semigroup algebras into their duals*, Canad. Math. Bull., 40(2) (1997), pp. 133–142.

[4] ——, *Weak amenability of discrete semigroup algebras*, Semigroup Forum, 55 (1997), pp. 196–205.

[5] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 80, Springer-Verlag, Berlin, 1973.

[6] J. Brodzki and Z. A. Lykova, *Excision in cyclic type homology of Fréchet algebras*, Bulletin of the London Math. Soc., 33 (2001), pp. 283–291.

[7] M. Capiński and E. Kopp, *Measure, integral and probability*, Springer undergraduate mathematics series, Springer, London, 2004.

[8] Y. Choi, *Simplicial homology and Hochschild cohomology of Banach semilattice algebras*, Glasgow Mathematical Journal, 48 (2006), pp. 231–245.

[9] ——, *Hochschild homology and cohomology of $\ell^1(\mathbb{Z}_+^k)$*, The Quarterly Journal of Mathematics, 61 (2010), pp. 1–28.

[10] Y. Choi, F. Gourdeau, and M. C. White, *Simplicial cohomology of band semigroup alegras*, Preprint, arXiv:1004.2301v1 (2010).

[11] R. DIESTEL, *Graph Theory*, vol. 173 of Graduate Texts in Mathematics, Springer, Berlin, 2005.

[12] J. B. FRALEIGH, *A first course in abstract algebra*, Addison-Wesley Publishing Company, 1980.

[13] F. GOURDEAU, B. E. JOHNSON, AND M. C. WHITE, *The cyclic and simplicial cohomology of $\ell^1(\mathbb{N})$*, Transactions of the American Mathematical Society, 357 (2005), pp. 5097–5113.

[14] F. GOURDEAU, Z. A. LYKOVA, AND M. C. WHITE, *The simplicial cohomology of $L^1(\mathbb{R}_+^k)$*, 363 (2004), pp. 95–109.

[15] ——, *The simplicial cohomology of $\ell^1(\mathbb{Z}_+^k)$ and a Künneth formula*, Studia Mathematica, 166 (2005), pp. 29–54.

[16] F. GOURDEAU, A. POURABBAS, AND M. C. WHITE, *Simplicial cohomology of some semigroup algebras*, Canadian Mathematical Bulletin. Bulletin Canadien de Mathématiques, 50 (2007), pp. 56–70.

[17] F. GOURDEAU AND M. C. WHITE, *Vanishing of the third simplicial cohomology group of $\ell^1(\mathbb{Z}_+)$*, Transactions of the American Mathematical Society, 353 (2001), pp. 2003–2017 (electronic).

[18] ——, *The cyclic and simplicial cohomology of the bicyclic semigroup algebra*, to appear in Quarterly Journal of Mathematics, (2010).

[19] A. HATCHER, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

[20] A. Y. HELEMSKIĬ, *The homology of Banach and topological algebras*, Kluwer Academic Publishers, 1986.

[21] ——, *Banach cyclic (co)homology and the Connes-Tzygan exact sequence*, J. London Math. Soc. (2), 46 (1992), pp. 449–462.

[22] P. J. HILTON AND U. STAMMBACH, *A Course in Homological Algebra*, Springer-Verlag, New York, 1997.

[23] B. E. JOHNSON, *Cohomology in Banach algebras*, Memoirs of the American Mathematical Society, 127 (1972).

[24] ——, *Derivations from $L^1(G)$ into $L^1(G)$ and $L^\infty(G)$*, Harmonic analysis (Luxembourg, 1987), 1359 (1988), pp. 191–198.

[25] S. C. KLEENE, *Mathematical Logic*, John Wiley & Sons, inc., New York, 1967.

[26] J. R. MUNKRES, *Topology*, Pearson Education International, 2000.

[27] W. RUDIN, *Real and complex analysis*, 1987.

[28] R. A. RYAN, *Introduction to tensor products of Banach spaces*, Springer monographs in mathematics, Springer, London, 2002.

[29] K. SAXE, *Beginning functional analysis*, Springer, New York, 2002.

[30] C. A. WEIBEL, *An introduction to homological algebra*, Cambridge University Press, 1994.

[31] A. J. WEIR, *General integration and measure*, Cambridge University Press, 1974.