Properties of Triangular Matrix and Gorenstein Differential Graded Algebras

Daniel Maycock

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School of Mathematics & Statistics
Newcastle University
Newcastle upon Tyne
United Kingdom

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Abstract

The main goal of this thesis is to investigate properties of two types of Differential Graded Algebras (or DGAs), namely upper triangular matrix DGAs and Gorenstein DGAs. In doing so we extend a number of corresponding ring theory results to the more general setting of DGAs and DG modules.

Chapters 2 and 3 contain background material. In chapter 2 we give a brief summary of some important aspects of homological algebra. Starting with the definition of an abelian category we give the construction of the derived category and the definition of derived functors. In chapter 3 we present the basics about Differential Graded Algebras and Differential Graded Modules in particular extending the definitions of the derived category and derived functors to the Differential Graded case before providing some results on Recollement of DGAs, Dualising DG-modules and Gorenstein DGAs.

Chapters 4 and 5 contain the bulk of the work for the Thesis. In chapter 4 we look at upper triangular matrix DGAs and in particular we generalise a result for upper triangular matrix rings to the situation of upper triangular matrix differential graded algebras. An upper triangular matrix DGA has the form

$$\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$$

where R and S are DGAs and M is a DG R-S^{op}-bimodule. We show that under certain conditions on the DG-module M, and given the existence of a DG R-module X from which we can build the derived category $D(R)$, that there exists a derived equivalence between the upper triangular matrix DGAs

$$\begin{bmatrix} R & M \\ 0 & S \end{bmatrix} \text{ and } \begin{bmatrix} S & M' \\ 0 & R' \end{bmatrix},$$

where the DG-bimodule M' is obtained from M and X, and R' is the endomorphism differential graded algebra of a K-projective resolution of X.

In chapter 5 we turn our attention to Gorenstein DGAs and generalise some results from Gorenstein rings to Gorenstein DGAs. We present a number of Gorenstein Theorems which state, for certain types of DGAs, that being Gorenstein is equivalent to the bounded and finite versions of the Auslander and Bass classes being maximal. We also provide a new definition of a Gorenstein morphism for DGAs by considering a DG bimodule as a generalised morphism of DGAs. We then show that some existing

results for Gorenstein morphism extend to these "Generalised Gorenstein Morphisms". We finally conclude with some examples of generalised Gorenstein morphisms for some well known DGAs.

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Contents

Chapter 1

Introduction

Homological algebra as a subject evolved out of algebraic topology and in particular the study of chain complexes associated with topological spaces. It was found that these chain complexes could provide some useful invariants associated with properties of the space. While the origins of homological algebra can be traced back to the beginning of the 20th century it was not until the 1940s and 50s that it started to develop into a subject in its own right. This early development culminated in the publication of Cartan and Eilenberg's seminal book [5] in 1956.

The principal objects of early homological algebra were chain complexes of modules over a ring. These chain complexes consist of a sequence of modules together with morphisms linking them together in a diagram

$$\cdots M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots$$

with the property that the composition of any two consecutive morphisms is the zero morphism. The tools of the classical homological algebra presented by Cartan and Eilenburg included the projective and injective resolutions; these are special complexes constructed of projective and injective modules respectively. One of the most important uses of the of these projective and injective resolutions is the construction of the classical derived functors $\mathrm{Ext}(-,-)$ and $\mathrm{Tor}(-,-)$ from the functors $\mathrm{Hom}(-,-)$ and $-\otimes-$. Since the publication of [5] the subject of homological algebra has blossomed and can now be found being employed in many areas of mathematics including representation theory, algebraic topology and algebraic geometry.

While the work of Cartan and Eilenberg codified much of classical homological algebra, and indeed their book remained the principal text on the subject for many years, other large advances in the field where being made. Perhaps the most important of these was the work of Grothendieck and in particular the 1957 publication of his paper "Sur quelques points d'algèbre homologique" [15]. Following this the notion of an abelian category became a central aspect of homological algebra. These are categories with the exactly those properties needed to perform the basic operations of classical homological algebra. From this point on homological algebra could be viewed from a generalised category theory outlook with complexes of objects of an abelian category replacing complexes of modules or abelian groups and the notions of projective and injective resolutions and derived functors became the central focus of the subject.

The work of Grothendieck was followed by the notions of triangulated categories and the derived category. Since their introduction in the thesis of Verdier [31], a student of Grothendieck, these have become increasingly important tools in the study of homological algebra throughout the second half of the twentieth century.

The structure of triangulated categories is weaker than that of possessed by abelian categories, despite this we are still able to carry out some operations of homological algebra with triangulated categories. The name triangulated category comes from the fact that the role played by short exact sequences in abelian categories is played by diagrams of the form

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X,$$

called distinguished triangles in triangulated categories. In fact, for many of the properties associated with short exact sequences in abelian categories there are similar corresponding properties for distinguished triangles in triangulated categories.

The derived category of an abelian category is an example of a triangulated category. For an abelian category $\mathscr{A}$ we construct the derived category $D(\mathscr{A})$ by the formal inversion of the class of morphisms of chain complexes which are quasi-isomorphisms, by a process of localisation. A quasi-isomorphism is a morphism of chain maps which induces an isomorphism at the level of homology. This means that the objects of $D(\mathscr{A})$ are the chain complexes of objects of $\mathscr{A}$ however any two such complexes are isomorphic in $D(\mathscr{A})$ if there is a quasi isomorphism between them. Thus the derived category of an abelian category preserves a number of the homological properties of the abelian category while allowing for simplifications of some processes that previously could only be performed by studying complicated spectral sequences, in particular the construction of the hyper-homological derived functors, generalisations of the classical derived functors, to functors between complexes rather than objects. Another application of the derived category is in studying when two objects are derived equivalent, i.e. their derived categories are equivalent. Since the derived categories preserve a number of the homological properties this can lead to a greater understanding of one or both of the objects in question.

Differential graded algebras are objects which can be viewed as straddling both ring theory and homological algebra. A differential graded algebra has the structure of a graded algebra $R = \{R_n\}$ together with a differential $d^R : R \to R$, an additive map of degree -1, such that $d_{n-1}d_n = 0$ and which satisfies the Leibniz rule. This gives differential graded algebras a natural complex structure and makes them ideal candidates for carrying out homological algebra. On the other hand differential graded algebras can be viewed as generalisations of rings, this is due to the fact that any ring may be viewed as a differential graded algebra concentrated in degree zero. Differential graded modules are the differential graded versions of the modules over a ring. A differential graded module consists of a graded module $M = \{M_n\}$ over a differential graded algebra together with a differential $d_n^M : M \to M$, a morphism of degree -1, such that $d_{n-1}d_n = 0$ and which satisfies a Leibniz rule. Thus differential graded modules have a natural complex structure. In fact the differential graded modules over a ring A, considered as a differential graded algebra concentrated in degree zero, are precisely the complexes of A-modules.

The natural complex structures of differential graded modules allow us to apply the methods of homological algebra to differential graded objects. In particular we can construct the derived category of a differential graded algebra R from the category of all differential graded R-modules by a method similar to the one used to construct the derived category of an abelian category from its complexes of objects. The construction of derived functors can also be applied to the differential graded case to obtain derived functors between the derived categories of differential graded algebras. Since differential graded algebras and their differential graded modules can be viewed as generalisations of rings and their modules it is also possible to extend some ring theoretical results to obtain differential graded versions. An example of this used in this thesis is that of dualising DG-modules which are generalisations of the dualising complexes over rings. Differential graded algebras occur naturally in mathematics such as in the cases of endomorphism differential graded algebras of complexes and the Koszul complex.

Chapters 2 and 3 of this thesis consist of the background material and provide a general overview and recap of some of the major topics in homological algebra. A reader familiar with the subject area can safely skip these chapters and use them purely as a reference.

In chapter 2 we give a brief background and summary of some of the main aspects of classical homological algebra. We approach this from the starting point of the abelian category and complexes of objects of such categories. We briefly introduce projective and injective resolutions before utilizing them to obtain the classical derived functors $\operatorname{Ext}(-,-)$ and $\operatorname{Tor}(-,-)$, important tools in the study of homological algebra. We also give the definition of a triangulated category and give a number of results showing the similarities of distinguished triangles to short exact exact sequences including a triangulated version of the 5-lemma. We follow this with the construction of the derived category by way of localisation at the quasi-isomorphisms of the homotopy category. We conclude the chapter with the definition of K-injective and K-projective resolutions and the construction of the “hyper-homological” derived functors and illustrate that they are generalisations of the classical derived functors.

Chapter 3 is concerned with providing an introduction to differential graded homological algebra. We begin with the definitions of differential graded algebras and differential graded modules and present their basic properties. In particular we define versions of the adjoint functors $- \underset{R}{\otimes} -$ and $\mathrm{Hom}_R(-,-)$ for differential graded modules over a differential graded algebra. Having laid out the basics of differential graded algebras we then adapt the constructions of the derived category and derived functors from chapter 2 to the differential graded setting. The second part of the chapter presents a number of special properties which differential graded algebras or differential graded modules may possess that will be required in the following chapters. This brings together a number of existing definitions and results which we shall need for the following chapters.

Chapters 4 and 5 contain the original work of the thesis.

In chapter 4 we consider upper triangular matrix differential graded algebras of the form

$$\Lambda = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix},$$

where R and S are DGAs and ${}_RM_S$ is a DG R-S^{op}-bimodule. In particular we look at when two such DGAs are derived equivalent, that is to say that their derived categories are equivalent.

The question of when two upper triangular matrix rings are derived equivalent was considered by Ladkani in [20]. By taking the approach developed by Rickard in [28], of using tilting modules, Ladkani was able to produce criteria under which two upper triangular matrix rings are derived equivalent. In this chapter we shall extend the result of Ladkani to the more generalised situation of upper triangular matrix differential graded algebras.

We begin by introducing the DG Λ-modules

$$B = \begin{bmatrix} R \\ 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} M \\ S \end{bmatrix}$$

which we use to construct a recollement diagram of the form

$$\begin{tikzcd} D(R) \arrow[r, "i_*"] & D(\Lambda) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D(S) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} ,$$

where $i_*({}_RR) \cong B$ and $j_!({}_SS) \cong C$. From this diagram we are then able to investigate the properties of the various objects involved.

Equipped with a better understanding of the objects we are working with we turn our attention to obtaining a generalisation of the main theorem of Ladkani for differential graded algebras. For this we employ a similar method to that used by Ladkani, by considering the DG-module $T = \Sigma i_*X \oplus j_*j^*\Lambda$ where X is compact and $\langle X \rangle = D(R)$. We can apply Keller's Theorem to obtain the following theorem.

Theorem. *Let ${}_RX$ be a compact DG R-module such that $\langle X \rangle = D(R)$. Let ${}_RM_S$ be a DG R-S^{op}-bimodule which is compact as a DG R-module. Then for the DG Λ-module $T = \Sigma i_*X \oplus j_*j^*\Lambda$ set $\mathscr{E} = \operatorname{End}_\Lambda(P)$, where P is a K-projective resolution of T. Then $\mathscr{E}$ is an DGA with $D(\Lambda) \simeq D(\mathscr{E}^{\text{op}})$.*

We then proceed to investigate the structure of P, the K-projective resolution of T, with the aim of constructing the differential graded algebra $\mathscr{E} = \operatorname{End}_\Lambda(P)$ explicitly. By doing this we are able to prove the following theorem, a differential graded version of Ladkani's main theorem, [20, Theorem 4.5].

Theorem. *Let X be a DG R-module such that ${}_RX$ is compact and $\langle {}_RX \rangle = D(R)$. Let ${}_RM_S$ be compact as a DG R-module and let U and V be K-projective resolutions of X and M respectively. Then for the upper triangular differential graded algebras*

$$\Lambda = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix} \text{ and } \tilde{\Lambda} = \begin{bmatrix} S & \operatorname{Hom}_R(V,U) \\ 0 & \operatorname{Hom}_R(U,U)^{\text{op}} \end{bmatrix}$$

we have that $D(\Lambda) \simeq D(\tilde{\Lambda})$.

The remainder of chapter 4 is concerned with looking at some special cases. We show that by restricting ourselves to the assumptions of Ladkani and by considering rings as differential graded algebras concentrated in degree 0 that our differential graded version of the theorem is in fact a generalisation of the main result of Ladkani. We also consider the special case where ${}_RX = {}_RR$. Our final example looks at the case where our base ring is a field and R is self dual in the sense that $\operatorname{Hom}_k(R,k) \cong R$ as DG R-bimodules. This gives us the following corollary.

Corollary. *Let R be a self dual finite dimensional DGA and S be a DGA, both over a field k. Let ${}_RM_S$ be compact as a DG-R-module. Then*

$$\Lambda = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix} \text{ and } \tilde{\Lambda} = \begin{bmatrix} S & \mathrm{Hom}_k(M,k) \\ 0 & R \end{bmatrix}$$

are derived equivalent.

In chapter 5 we consider properties of Gorenstein differential graded algebras and introduce generalised Gorenstein morphisms.

We begin by considering dualising equivalences. By taking the approach of Frankild and Jørgensen in [11], for ${}_{R,S}M$ a DG R-S-bimodule, we have that the adjoint pair of functors

$$D(R^{\mathrm{op}}) \underset{\mathrm{RHom}_S(M,-)}{\overset{-\overset{\mathrm{L}}{\underset{R}{\otimes}} M}{\rightleftarrows}} D(S)$$

restricts to quasi-inverse equivalences between the associated Auslander and Bass categories

$$\mathscr{A}_M(R^{\mathrm{op}}) = \{L \in D(R^{\mathrm{op}}) : \eta_L \text{ is an isomorphism}\}$$

and

$$\mathscr{B}_M(S) = \{N \in D(S) : \epsilon_N \text{ is an isomorphism}\}$$

where η_L and ϵ_N denote the unit morphism, $\eta_L : L \to \mathrm{RHom}_R(M, M \overset{\mathrm{L}}{\underset{R}{\otimes}} L)$, and counit morphism, $\epsilon_N : M \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathrm{RHom}_R(M,L) \to L$, of the adjunction respectively. We expand on this by defining the bounded and finite Auslander and Bass classes and show that the quasi-inverse equivalences between them restrict to quasi-inverse equivalences in both the bounded and finite cases.

We then turn our attention to obtaining a number of Gorenstein theorems for DGAs. The definition of a Gorenstein DGA is a generalisation of the definition of a Gorenstein ring. The definition of a Gorenstein DGA that we use is that of Frankild and Jørgensen in [12] and is reproduced in Definition 3.2.22. Gorenstein Theorems take the form of showing that a DGA R being Gorenstien is equivalent to R possessing certain other properties. In particular we generalise the results for Gorenstein rings given by Christensen in [6, Theorems 3.1.12 and 3.2.10]. These theorems show that for a local ring R which admits a dualising complex, R being Gorenstein is equivalent to the Auslander and Bass classes of R, with respect to the dualising complexes, being maximal. We show that for two special types of DGAs that being Gorenstein is equivalent to the bounded and finite versions of the Auslander and Bass classes being maximal.

In the third section of the chapter we look to extend the ring theory concept of a Gorenstein morphism to DGAs. A definition of a Gorenstein morphism for DGAs was given by Frankild and Jørgensen in [12]. However this definition was lacking in some respects and failed to allow for some results for Gorenstein morphisms in the classical ring case to be generalised to the DG case. The approach this new definition takes is to consider DG-bimodules as generalised morphisms of DGAs. This approach of viewing DG-bimodules as a generalisation of morphisms was used by Keller in [19] and Pauksztello in [24]. This gives us the following definition.

Definition. The bimodule ${}_RM_S$ is a *generalised Gorenstein morphism from S to R* if it satisfies the following conditions:

(i) M_S is compact in $D(S^{\mathrm{op}})$.

(ii) There exist dualising DG-modules ${}_RD_R$ and ${}_SE_S$ for R and S respectively such that there exist isomorphisms

$$\phi : {}_RD_R \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_RM_S \xrightarrow{\cong} {}_RM_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SE_S$$

and

$$\theta : {}_SZ_R \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_RD_R \xrightarrow{\cong} {}_SE_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SZ_R$$

where $Z = \mathrm{RHom}_{S^{\mathrm{op}}}({}_RM_S, {}_SS_S)$.

(iii) The isomorphisms ϕ and θ are compatible in the sense that the following diagram commutes.

$$\begin{tikzcd}
Z \overset{\mathrm{L}}{\underset{R}{\otimes}} D \overset{\mathrm{L}}{\underset{R}{\otimes}} M \arrow[r, "1\otimes\phi"] \arrow[d, "\theta\otimes 1"] & Z \overset{\mathrm{L}}{\underset{R}{\otimes}} M \overset{\mathrm{L}}{\underset{S}{\otimes}} E \arrow[r, "\tau\otimes 1"] & S \overset{\mathrm{L}}{\underset{S}{\otimes}} E \arrow[d, "\cong"] \\
E \overset{\mathrm{L}}{\underset{S}{\otimes}} Z \overset{\mathrm{L}}{\underset{R}{\otimes}} M \arrow[r, "1\otimes\tau"] & E \overset{\mathrm{L}}{\underset{S}{\otimes}} S \arrow[r, "\cong"] & E
\end{tikzcd}$$

where $\tau : Z \overset{\mathrm{L}}{\underset{R}{\otimes}} M \to S$ is the canonical morphism which corresponds to the map $\mathrm{Hom}_{S^{\mathrm{op}}}(U, S) \underset{R}{\otimes} U \to S$ given by $\mu \otimes u \mapsto \mu(u)$

Having defined when a bimodule is a generalised Gorenstein morphism we then apply the definition to obtain a DG version of the [3, Proposition 3.7(b)] a base change for the Auslander class.

Theorem. *Let the DG R-S^{op}-bimodule ${}_RM_S$ be a generalised Gorenstein morphism from R to S. Then*

$${}_SN \in \mathscr{A}_E(S) \Rightarrow {}_RM_S \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} {}_SN \in \mathscr{A}_D(R),$$

and

$$N'_S \in \mathscr{A}_E(S^{\mathrm{op}}) \Rightarrow N'_S \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} {}_SZ_R \in \mathscr{A}_D(R^{\mathrm{op}}),$$

where ${}_RD_R$ and ${}_SE_S$ are the dualising modules which satisfy conditions (ii) and (iii) of the definition of a generalised Gorenstein morphism.
Furthermore if the functors $M \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} -$ or $- \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} Z$ reflect isomorphisms then the corresponding reverse implications also hold.

We also obtain a result corresponding to the main result for Gorenstein morphisms of DGAs from [12], the ascent theorem of Gorenstein DGAs.

Theorem. *Let R and S be DGAs. Suppose that there exists a DG R-S^{op}-bimodule ${}_RM_S$ satisfying the following properties:*

(i) ${}_RM_S$ *is a generalised Gorenstein morphism from R to S,*

(ii) *The functors $M \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} -$ and $- \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} Z$ reflect isomorphisms,*

(iii) ${}_SN \in D^f(S) \Leftrightarrow M \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} N \in D^f(R)$ *and* $N'_S \in D^f(S^{\mathrm{op}}) \Leftrightarrow N' \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} Z \in D^f(R^{\mathrm{op}})$
where $D^f(R) = \{M \in D(R) \mid M$ *is a finite DG R-module*$\}$

Then R is a Gorenstein DGA $\Rightarrow$ S is a Gorenstein DGA.

We conclude the chapter with some examples of generalised Gorenstein morphisms for well known DGAs including endomorphism DGAs and the Koszul complex.

Chapter 2

Homological Algebra and the Derived Category

The aim of this chapter is to give a brief background and summary of some of the main aspects of classical homological algebra as developed originally by Cartan and Eilenberg and later expanded on by the work of Grothendieck and Verdier. In particular the construction of the derived category and the definition of derived functors.

The approach in this chapter is from the generalised category theory viewpoint of Grothendieck with the starting point being the definition of an abelian category. This is followed by the basic definitions of complexes of objects of an abelian category and the homology of such complexes. We then give a brief account of projective and injective objects and how they are used in the classical definition of the derived functors and in particular the functors $\mathrm{Ext}(-,-)$ and $\mathrm{Tor}(-,-)$, the derived functors of $\mathrm{Hom}(-,-)$ and $- \otimes -$. More detailed accounts of classical homological algebra can be found in [17], [32] and [5].

Our attention then turns to the theory of triangulated categories and in particular the derived category of an abelian category. Triangulated categories are categories which, while having a weaker structure than abelian categories, do display a number of similar properties to those of abelian categories. Perhaps the most useful example of a triangulated category is the derived category associated with an abelian category. We give the definition of a triangulated category and some results obtained from the definition before describing the construction of the derived category via the construction of the homotopy category, another example of a triangulated category. The construction of the derived category from the homotopy category is due to a process of localisation at a multiplicative system of morphisms which we describe in detail. In the case of the derived category this multiplicative set is the class of all quasi-isomorphisms between complexes. These are the morphisms of complexes which are isomorphisms at the level of homology. The result of this is a category in which the objects are the complexes of an abelian category while any two complexes with a quasi-isomorphism between will be isomorphic. The material covered in this section is mostly taken from [16].

One of the principal benefits of working with derived categories is that they provide a suitable framework for extending the concept of classical derived functors to derived functors between complexes rather than just between individual objects. For that we need a tool which performs a similar role for complexes as projective and injective resolutions do for individual objects. Such a concept is that of K-projective and K-injective resolutions. Having defined these, we conclude the chapter by presenting the construction of the derived functors.

2.1 Classical Theory of Homological Algebra

2.1.1 Abelian Categories

An abelian category is a category in which the operations of homological algebra can be performed. They appear throughout mathematics with the motivating example being the category of abelian groups. We shall give a brief summary of some basic definitions and results regarding abelian categories. The main reference for this section, as well as source for further information on category theory, is [21], from which most of the definitions in this section are taken. Alternatively, [17, Chapter II] also gives a good background to the aspects of category theory important to homological algebra.

We begin by recalling the definition of a category and the category theory definitions of a coproduct and a zero object.

Definition 2.1.1. A category, $\mathscr{C}$, has of the following items (i)-(iii) such that the axioms (a) and (b) below hold

(i) A class of objects.

(ii) A class of morphisms (or arrows) between the objects, where such a morphism f has an unique source object A and an unique target object B. We denote such a morphism by $f : A \to B$ and denote the class of all morphisms from A to B by $\mathrm{Hom}_{\mathscr{C}}(A, B)$.

(iii) A composition of morphisms, i.e. for objects A, B and C in $\mathscr{C}$ there exists an operation $\mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(B, C) \to \mathrm{Hom}_{\mathscr{C}}(A, C)$. We denote the composition of two morphisms $f : A \to B$ and $g : B \to C$ by $g \circ f$.

The axioms are the following.

(a) For each object A, there exists an identity morphism $\mathrm{id}_A : A \to A$ with the property that for any morphism $f : A \to B$, $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$.

(b) Associativity of composition of morphisms, that is, for morphisms $f : A \to B$, $g : B \to C$ and $h : C \to D$, $h \circ (g \circ f) = (h \circ g) \circ f$.

Definition 2.1.2. Let $\{X_i\}$ be a family of objects of a category $\mathscr{C}$. Then the *coproduct* of the X_i's is an object, denoted by $\coprod X_i$, together with morphisms, $\iota_i : X_i \to \coprod X_i$, called *injections*, with the universal property: Given any object, Y, and a family of morphisms, $f_i : X_i \to Y$, then there exists a unique morphism $f : \coprod X_i \to Y$ with $f\iota_i = f_i$.

Definition 2.1.3. For a category $\mathscr{C}$, an object T is *terminal* in $\mathscr{C}$ if, for each object A in C, there is exactly one morphism $A \to T$. Similarly an object S is *initial* in $\mathscr{C}$ if for each object A in $\mathscr{C}$, there is exactly one morphism $S \to A$.

A *zero object* in $\mathscr{C}$ is an object which is both terminal and initial, such an object is unique up to isomorphism and we denote it by 0.

Definition 2.1.4. For a category $\mathscr{C}$, a morphism $f : X \to Y$ in $\mathscr{C}$ is called a *constant morphism* if for any object W in $\mathscr{C}$ and morphisms $g, h : W \to X$ we have that $fg = fh$. Similarly f is called a *coconstant morphism* if for any object Z in $\mathscr{C}$ and morphisms $g, h : Y \to Z$ we have that $gf = hf$.

A *zero morphism* in $\mathscr{C}$ is a morphism which is both constant and conconstant.

We now define the additive categories. This is a more general definition than that of abelian categories and serves as a precursor to the definition of abelian categories.

Definition 2.1.5. A category $\mathscr{A}$ is called *additive* if it has the following properties:

(i) $\mathscr{A}$ has a zero object.

(ii) For any pair of objects X and Y of $\mathscr{A}$ the set of morphisms $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ forms an abelian group.

(iii) For any objects X, Y and Z of $\mathscr{A}$ the composition

$$\mathrm{Hom}_{\mathscr{A}}(X, Y) \times \mathrm{Hom}_{\mathscr{A}}(Y, Z) \to \mathrm{Hom}_{\mathscr{A}}(X, Z)$$

is bilinear.

(iv) For any pair of objects X and Y of $\mathscr{A}$, the coproduct $X \coprod Y$ exists.

Remark 2.1.6. In an additive category $\mathscr{A}$ every set of morphisms $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ is an abelian group and therefore contains a zero element. These zero elements give us a family of zero morphisms for $\mathscr{A}$ making it a category with zero morphism.

We can look to expand the definition of an additive category to that of an abelian category. Before we do so we recall the category theory definitions of the kernel, cokernel and image of a morphism.

Definition 2.1.7. Let $\mathscr{C}$ be a category. A morphism $f : X \to Y$ in $\mathscr{C}$ is called a *monomorphism* if for any object Z in $\mathscr{C}$ and pair of morphisms $g, h : Z \to X$ we have that $fg = fh \Rightarrow g = h$. Similarly f is called an *epimorphism* if for any object W in $\mathscr{C}$ and pair of of morphisms $g, h : Y \to W$ we have that $gf = hf \Rightarrow g = h$.

Definition 2.1.8. Let $\mathscr{C}$ be a category with a zero object and let $f : X \to Y$ be a morphism. The *kernel* of f, denoted $\mathrm{Ker}\, f$, is a morphism $k : K \to X$ where $fk = 0$ and for every $h : U \to X$ with $fh = 0$ there exists a unique morphism $h' : U \to K$ such that $h = kh'$.

The *cokernel* of f, denoted $\mathrm{Coker}\, f$, is a morphism $u : Y \to C$ where $uf = 0$ and for every $v : Y \to Z$ with $vf = 0$ there exists a unique morphism $v' : C \to Z$ such that $v = v'u$.

The *image* of f, denoted by $\mathrm{Im}\, f$, is defined as $\mathrm{Ker}(\mathrm{Coker}\, f)$.

The *coimage* of f, denoted by $\mathrm{Coim}\, f$ is defined as $\mathrm{Coker}(\mathrm{Ker}\, f)$.

For a given category $\mathscr{C}$ and a morphism $f : X \to Y$ in that category the kernel and cokernel of f do not necessarily exist. The existence of kernels and cokernels is one of the defining aspects of abelian categories as we see below.

Definition 2.1.9. An additive category $\mathscr{A}$ is called *abelian* if it satisfies the following additional properties:

(i) Every morphism f of $\mathscr{A}$ has both a kernel, $\mathrm{Ker}\, f$, and a cokernel, $\mathrm{Coker}\, f$.

(ii) Every monomorphism in $\mathscr{A}$ is a kernel and every epimorphism in $\mathscr{A}$ is a cokernel.

Proposition 2.1.10. *Let $\mathscr{A}$ be an abelian category. Then every morphism f has a factorisation $f = me$ where $m = \mathrm{Ker}(\mathrm{Coker}\, f)$ is a monomorphism and $e = \mathrm{Coker}(\mathrm{Ker}\, f)$ is an epimorphism.*

Proof. See [21, Proposition VIII.3.1] □

Definition 2.1.11. A functor $F : \mathscr{A} \to \mathscr{B}$ between additive categories is called *additive* if for a pair of morphisms $f, f' : A \to A'$ in $\mathscr{A}$ we have that $F(f + f') = F(f) + F(f')$.

Definition 2.1.12. Let $\mathscr{C}$ be a category and let A be an object of $\mathscr{C}$.

Two monomorphisms u and v with codomain A are equivalent if $u = v\theta$ for some invertible morphism θ. This gives us an equivalence relation for the monomorphisms of $\mathscr{C}$ with codomain A. We define the *subobjects* of A to be the equivalence classes of such monomorphisms.

Dually we have that two epimorphisms r and s with domain A are equivalent if $r = \phi s$ for some invertible morphism ϕ. This gives us an equivalence relation for the epimorphisms of $\mathscr{C}$ with domain A. We define the *quotient objects* of A to be the equivalence classes of such epimorphisms.

We now give two well known examples of abelian categories.

Examples 2.1.13. The category of abelian groups, which we denote Ab, whose objects consist of all abelian groups and whose morphisms are group homomorphisms, is an abelian category.

For a ring R. The category of left R-modules, $\mathrm{Mod}(R)$, whose objects are all left R-modules and whose morphisms are module homomorphisms, is an abelian category.

Remarks 2.1.14. When working in more concrete settings such as that of the category of abelian groups we usually consider a kernel $\mathrm{Ker}\, f$ as being the object K rather than a morphism $k : K \to X$. This is possible since we can view K as being a subobject of X and the morphism as an inclusion. By taking this outlook the categorical definition of a kernel coincides with the more familiar algebraic definition.

Similarly, in such settings, we also have that the catgorical definitions of cokernels, images and coimages can also be though of as taking the form of an object rather than a morphism and thus also agree with the more familar algebraic definitons.

Likewise, for subobjects and quotient objects we can in most concrete settings, including all those considered throughout this document, consider these to be the objects rather than morphisms. The monomorphisms in the case of subobjects are inclusions while the epimorphisms in the case of quotient objects are projections. Thus for the concrete objects we have that the category theory definition of subobjects and quotient objects also agree with the familiar algebraic definitions.

For every monomorphism in the equivalence class of a subobject we can take the cokernel. These cokernels are all, by definition, epimorphisms and satisfy the equivalence relation for quotient objects and therefore give us a quotient object associated with the subobject. When working in more concrete setting, where we can consider subobjects and quotient objects as objects rather than equivalence classes of morphisms, we donote the quotient object associated with a subobject B of an object A by $\dfrac{A}{B}$. In these cases this notation agrees with the familiar algebraic notation.

We now conclude this section by considering sequences of objects and morphisms of an abelian category and in particular when such sequences can be said to be exact.

Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of objects and morphisms of an abelian category $\mathscr{A}$ such that $g \circ f = 0$. Then we have the kernel and image morphisms $\operatorname{Ker} g : K \to B$ and $\operatorname{Im} f : I \to B$ which are both monomorphisms with $g \circ \operatorname{Ker} g = 0$ and $g \circ \operatorname{Im} f = 0$. So, by the definition of $\operatorname{Ker} g$, we have a canonical map $i : I \to K$ such that $\operatorname{Ker} g \circ i = \operatorname{Im} f$ and thus i is a monomorphism. The converse also holds, that is, if there exists a monomorphism $i : I \to K$ such that $\operatorname{Ker} g \circ i = \operatorname{Im} f$ then $g \circ f = 0$.

Additionally there exists a morphism $i' : K \to I$ if and only if every morphism $a : X \to B$ with $ga = 0$ factors through I, that is to say that there exists a morphism $a' : X \to I$ such that $a = \operatorname{Im} f \circ a'$.

These canonical morphisms allows us to make the following definition of an exact sequence.

Definitions 2.1.15. A sequence is a diagram

$$\cdots \to X^{-2} \xrightarrow{f^{-2}} X^{-1} \xrightarrow{f^{-1}} X^{0} \xrightarrow{f^{0}} X^{1} \xrightarrow{f^{1}} X^{2} \to \cdots$$

which consists of objects and morphisms in an abelian category, $\mathscr{A}$, such that $f^n \circ f^{n-1} = 0$.

We say that such a sequence is *exact* at X^n if the canonical morphism $\operatorname{Im} f^{n-1} \to \operatorname{Ker} f^n$ is an isomorphism.
It is an *exact sequence* if it is exact at X^n for all n.

A *short exact sequence* is an exact sequence of the form:

$$0 \to K \overset{m}{\hookrightarrow} X \overset{e}{\twoheadrightarrow} C \to 0$$

where m is a monomorphism and the kernel of e, while e is an epimorphism and the cokernel of m.

2.1.2 Complexes and Homology

Complexes of objects are the basic objects of interest in Homological Algebra. These complexes consist of a sequence of objects and morphisms (called differentials) of an abelian category such that the composition of two consecutive morphisms is zero. Complexes originated in algebraic topology where they provided an algebraic representation of certain properties of spaces. Homological algebra is concerned with the study of various invariants obtained from these complexes of which one of the most important is homology.

The following definitions are taken from [17]; other good references for the material in this section include [32].

Definition 2.1.16. A *cochain complex* C of objects of an abelian category $\mathscr{A}$ is a family of objects $\{C^n | n \in \mathbb{Z}\}$ of $\mathscr{A}$ and a family of morphisms $\{d^n : C^n \to C^{n+1} | n \in \mathbb{Z}\}$, called *differentials*, such that $d^{n+1}d^n = 0$ for all $n \in \mathbb{Z}$.

$$C : \cdots \to C^{n-1} \overset{d^{n-1}}{\to} C^n \overset{d^n}{\to} C^{n+1} \to \cdots .$$

Similarly a *chain complex* C of objects of an abelian category $\mathscr{A}$ is a family of objects $\{C_n | n \in \mathbb{Z}\}$ of $\mathscr{A}$ and a family of morphisms $\{d_n : C_n \to C_{n-1} | n \in \mathbb{Z}\}$ such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$.

$$C : \cdots \to C_{n+1} \overset{d_{n+1}}{\to} C_n \overset{d_n}{\to} C_{n_1} \to \cdots .$$

Example 2.1.17. Note that for an abelian category $\mathscr{A}$ we can consider any object A in $\mathscr{A}$ as a complex concentrated in degree 0 i.e. as a complex of the form:

$$\cdots \to 0 \to 0 \to A \to 0 \to 0 \to \cdots .$$

Now that we have defined (co)chain complexes it is natural to define morphisms between them.

Definitions 2.1.18. Let $C = \{C, d_C\}$ and $D = \{D, d_D\}$ be cochain complexes of objects of an abelian category $\mathscr{A}$.
A map $\phi : C \to D$ is called a *homomorphism of degree p*, if it consists of a family of morphisms, $\{\phi^n : C^n \to D^{n+p}\}_{n\in\mathbb{Z}}$, in $\mathscr{A}$.

A *morphism of complexes* $\phi : C \to D$, is a homomorphism of degree 0 such that $d_D^n \circ \phi^n = \phi^{n+1} \circ d_C^n$ for all $n \in \mathbb{Z}$. That is that the following diagram commutes:

$$\begin{tikzcd}
C: \arrow[d, "\phi"] & \cdots \arrow[r] & C^{n-1} \arrow[r, "d_C^{n-1}"] \arrow[d, "\phi^{n-1}"] & C^n \arrow[r, "d_C^n"] \arrow[d, "\phi^n"] & C^{n+1} \arrow[r] \arrow[d, "\phi^{n+1}"] & \cdots \\
D: & \cdots \arrow[r] & D^{n-1} \arrow[r, "d_D^{n-1}"'] & D^n \arrow[r, "d_D^n"'] & D^{n+1} \arrow[r] & \cdots .
\end{tikzcd}$$

We can obtain corresponding definitions for chain complexes.

The (co)chain complexes of objects of an abelian category $\mathscr{A}$, together with the morphisms of complexes defined above form a new category, that of the complexes of $\mathscr{A}$, which we denote by $C(\mathscr{A})$. This new category is itself an abelian category.

Since the canonical morphism $\operatorname{Im} d^{n-1} \to \operatorname{Ker} d^n$ is a monomorphism we can view $\operatorname{Im} d^{n-1}$ as a subobject of $\operatorname{Ker} d^n$. This allows us to make the following important definitions of homology and cohomology of a complex.

Definition 2.1.19. Let C be a cochain complex of objects of an abelian category $\mathscr{A}$. We define the *cohomology* of C to be the graded object

$$\mathrm{H}^* C = \{\mathrm{H}^n C\}, \text{ where } \mathrm{H}^n C = \frac{\operatorname{Ker} d^n}{\operatorname{Im} d^{n-1}}.$$

Then $\mathrm{H}^n C$ is called the n^{th}-*cohomology object* of C.

Similarly, let C be a chain complex of objects of an abelian category $\mathscr{A}$. We define the *homology* of C to be the graded object

$$\mathrm{H}_* C = \{\mathrm{H}_n C\}, \text{ where } \mathrm{H}_n C = \frac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}}.$$

Then $\mathrm{H}_n C$ is called the n^{th}-*homology object* of C.

Definition 2.1.20. A complex for which the (co)homology is 0 is said to be *exact*.

Remark 2.1.21. Let $f : C \to D$ be a morphism of chain complexes of objects of an abelian category $\mathscr{A}$. This induces a family of morphisms $\mathrm{H} f = \{H^n f\}$ where each $H^n f : \mathrm{H}^n C \to \mathrm{H}^n D$ is given by $H^i f(\overline{k}) = \overline{f(k)}$ for $k \in \operatorname{Ker} d^n$.

2.1.3 Projective and Injective Objects

Projective objects are objects of an abelian category with the property that any morphism from such an object can be "lifted through a surjection" while Injective objects are those objects with the dual property that any morphism to such an object can be "extended through any monomorphism". These two types of objects perform important roles in Homological algebra, not least for the construction of the special complexes of Projective and Injective resolutions of objects of the abelian category in question. These complexes are required to calculate the derived functors and thus the existence of such complexes is a desirable property for an abelian category. As before the definitions and results in this section are taken from [17] where further information and results regarding these objects can be found.

Definition 2.1.22. (i) An object P of an abelian category $\mathscr{A}$ is a *projective object* if for any epimorphism $\epsilon : A \to B$ and any morphism $\phi : P \to B$ there exists a morphism $\mu : P \to A$ such that $\epsilon \circ \mu = \phi$. That is to say that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\mu"'] \arrow[d, "\phi"] \\ A \arrow[r, two heads, "\epsilon"'] & B. \end{tikzcd}$$

An abelian category, $\mathscr{A}$, is said to have *enough projectives* if, for every object $X \in \mathscr{A}$, there is an epimorphism $P \to X$, where P is a projective object of $\mathscr{A}$.

(ii) An object I of an abelian category $\mathscr{A}$ is an *injective object* if for any monomorphism $\iota : C \to D$ and any morphism $\theta : C \to I$ there exists a morphism $\nu : D \to I$ such that $\nu \circ \iota = \theta$. That is to say that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, hook, "\iota"] \arrow[d, "\theta"'] & D . \arrow[dl, dashed, "\nu"] \\ I & \end{tikzcd}$$

An abelian category, $\mathscr{A}$, has *enough injectives* if, for every object $X \in \mathscr{A}$, there is a monomorphism $X \to I$, where I is an injective object of $\mathscr{A}$.

Our main interest in projective and injective objects is to construct special kinds of complexes, namely the projective and injective resolutions associated with a given object in an abelian category.

Definition 2.1.23. Let $\mathscr{A}$ be an abelian category.

(i) A chain complex

$$C : \cdots \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$$

of objects in $\mathscr{A}$, is called *acyclic* if $\mathrm{H}_i(C) = 0$ for all $i \geq 1$. It is called *projective* if each C_i is a projective object in $\mathscr{A}$.

A *projective resolution* of an object X in $\mathscr{A}$ consists of a projective and acyclic complex

$$P : \cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0$$

and an isomorphism $\mathrm{H}_0(P) \xrightarrow{\cong} X$.

(ii) A cochain complex

$$D : 0 \to D^0 \to D^1 \to \cdots \to D^{n-1} \to D^n \to \cdots$$

of objects in $\mathscr{A}$, is called *acyclic* if $\mathrm{H}^i(D) = 0$ for all $i \geq 1$ and *injective* if each D^i is an injective object of $\mathscr{A}$.

An *injective resolution* of an object X in $\mathscr{A}$ consists of an injective and acyclic complex

$$I : 0 \to I^0 \to I^1 \to \cdots \to I^{n-1} \to I^n \to \cdots$$

and an isomorphism $X \xrightarrow{\cong} \mathrm{H}^0(I)$.

The following proposition shows that we can always construct projective (injective) resolutions for objects of an abelian category with enough projectives (injectives).

Proposition 2.1.24. *Let $\mathscr{A}$ be an abelian category and let X be an object of $\mathscr{A}$.*

(i) If $\mathscr{A}$ has enough projectives then X has a projective resolution.

(ii) If $\mathscr{A}$ has enough injectives then X has an injective resolution.

Proof. (i) Let $\mathscr{A}$ have enough projectives. Then there exists a short exact sequence

$$0 \to K_0 \to P_0 \to X \to 0$$

where P_0 is a projective object. Now for K_0 there exists a short exact sequence

$$0 \to K_1 \to P_1 \to K_0 \to 0$$

where P_1 is projective. Hence we have an exact complex

$$0 \to K_1 \to P_1 \to P_0 \to X \to 0.$$

By continuing in this way for K_1 and beyond we obtain a, possibly infinite, projective resolution for X.

$$\cdots \to P_2 \to P_1 \to P_0 \to X \to 0.$$

The proof for (ii) is similar. $\square$

We end with the following simple example of a projective resolution.

Example 2.1.25. Consider the $\mathbb{Z}$-module $\frac{\mathbb{Z}}{a\mathbb{Z}}$. There is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{a} \mathbb{Z} \longrightarrow \frac{\mathbb{Z}}{a\mathbb{Z}} \longrightarrow 0$$

where the morphism a denotes multiplication by a. Since $\mathbb{Z}$ is a projective $\mathbb{Z}$-module this gives us the following projective resolution of $\frac{\mathbb{Z}}{a\mathbb{Z}}$:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{a} \mathbb{Z} \to 0.$$

2.1.4 Classical Derived Functors

In this section we build upon the previous definitions to briefly give a summary of the classical construction of derived functors and in particular the functors Ext and Tor. For a more in depth account of this construction see [17, Chapter IV Sections 5-11].

Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories such that $\mathscr{A}$ has enough projectives and suppose that $F : \mathscr{A} \to \mathscr{B}$ is an additive covariant functor. Our aim is to define a family of functors $\mathrm{L}_n F : \mathscr{A} \to \mathscr{B}$. Define a functor $P : \mathscr{A} \to C(\mathscr{A})$ which sends each object A in $\mathscr{A}$ to an arbitrarily chosen projective resolution of A

$$P(A) = \cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0.$$

We can apply the functor F to the complex P to obtain the complex

$$FP(A) = \cdots \to FP_n \to FP_{n-1} \to \cdots \to FP_1 \to FP_0 \to 0.$$

We define $\mathrm{L}_n^P F(A) = \mathrm{H}_n(FP(A))$.

Let $P' : \mathscr{A} \to C(\mathscr{A})$ be a functor which sends each object A in $\mathscr{A}$ to an alternative arbitrary choice of projective resolution

$$P'(A) = \cdots \to P'_n \to P'_{n-1} \to \cdots \to P'_1 \to P'_0 \to 0.$$

Then $\mathrm{L}_n^P F(A) \cong \mathrm{L}_n^{P'} F(A)$, for details see [17, Chapter IV Proposition 5.1]. Thus we can define $\mathrm{L}_n F(A) = \mathrm{L}_n^P F(A)$ where P is any functor which sends objects of $\mathscr{A}$ to one of their projective resolutions. Furthermore, a morphism $\alpha : A \to A'$ induces a morphism of complexes $P(A) \to P(A')$ which is unique up to homotopy, as detailed in [17, Chapter IV Section 5 Theorem 4.1]. By applying the functor $\mathrm{H}_n(F-)$ to this morphism of complexes we obtain a morphism $\alpha_* : \mathrm{L}_n F(A) \to \mathrm{L}_n F(A')$. Thus $\mathrm{L}_n F(-)$ is a functor.

Definition 2.1.26. Let $F : \mathscr{A} \to \mathscr{B}$ be an additive covariant functor. If $\mathscr{A}$ has enough projectives, then the family of functors $\mathrm{L}_n F(-)$ defined by $\mathrm{L}_n F(A) = \mathrm{H}_n(FP(A))$, where $P : \mathscr{A} \to C(\mathscr{A})$ is a functor which sends each object $A \in \mathscr{A}$ to a projective resolution of A, are called the *(classical) left derived functors* of F. If $\mathscr{A}$ has enough injectives, then the family of functors $\mathrm{R}^n F(-)$ defined by $\mathrm{R}^n F(A) = \mathrm{H}^n(FI(A))$, where $I : \mathscr{A} \to C(\mathscr{A})$ is a functor which sends each object $A \in \mathscr{A}$ to an injective resolution of A,, are called the *(classical) right derived functors* of F.

Similarly, for an additive contravariant functor $G : \mathscr{A} \to \mathscr{B}$. If $\mathscr{A}$ has enough projectives, then the family of functors $\mathrm{R}_n G(-)$ defined by $\mathrm{R}_n G(A) = \mathrm{H}_n(GP(A))$, where $P : \mathscr{A} \to C(\mathscr{A})$ is a functor which sends each object $A \in \mathscr{A}$ to a projective resolution of A, are called the *(classical) right derived functors* of G. If $\mathscr{A}$ has enough injectives, then the family of functors $\mathrm{L}^n F(-)$ defined by $\mathrm{L}^n G(A) = \mathrm{H}^n(GI(A))$, where $I : \mathscr{A} \to C(\mathscr{A})$ is a functor which sends each object $A \in \mathscr{A}$ to an injective resolution of A, are called the *(classical) left derived functors* of G.

Theorem 2.1.27. *Let $F : \mathscr{A} \to \mathscr{B}$ be an additive covariant functor between abelian categories where $\mathscr{A}$ has enough projectives. Suppose $0 \to A' \overset{a'}{\to} A \overset{a}{\to} A'' \to 0$ is a right exact sequence in $\mathscr{A}$. Then there exist connecting morphisms $\omega_n : \mathrm{L}_n F(A'') \to \mathrm{L}_{n-1} F(A')$ for all $n = 1, 2, ...$ such that we have the long exact sequence:*

$$\cdots \to \mathrm{L}_n F(A') \overset{a'_*}{\to} \mathrm{L}_n F(A) \overset{a_*}{\to} \mathrm{L}_n F(A'') \overset{\omega_n}{\to} \mathrm{L}_{n-1} F(A') \to \cdots$$

$$\cdots \to \mathrm{L}_1 F(A'') \overset{\omega_1}{\to} \mathrm{L}_0 F(A') \overset{a'_*}{\to} \mathrm{L}_0 F(A) \overset{a_*}{\to} \mathrm{L}_0 F(A'') \to 0.$$

Proof. See [17, Theorem IV.6.1]. □

We can obtain corresponding theorems for the left derived functor of a contravariant functor as well as for right derived functors of both covariant and contravariant functors.

Examples 2.1.28. Let R be a ring. Let A be a right R-module and B be a left R-module. Then the tensor products give the additive covariant functors

$$A \underset{R}{\otimes} - : \mathrm{Mod}(R) \to \mathrm{Ab} \ \text{ and } \ - \underset{R}{\otimes} B : \mathrm{Mod}(R^{\mathrm{op}}) \to \mathrm{Ab}\,.$$

We can therefore define the left derived functors

$$\operatorname{Tor}_n^R(A,-) = \mathrm{L}_n(A \underset{R}{\otimes} -) \text{ and } \overline{\operatorname{Tor}}_n^R(-,B) = \mathrm{L}_n(- \underset{R}{\otimes} B).$$

Since the tensor product $- \underset{R}{\otimes} -$ is a bifunctor we have, for any morphism $A \to A'$ in $\mathrm{Mod}(R^{\mathrm{op}})$, a natural transformation $A \underset{R}{\otimes} - \to A' \underset{R}{\otimes} -$. This in turn gives us a natural transformation $L_n(A \underset{R}{\otimes} -) \to L_n(A' \underset{R}{\otimes} -)$. So for all $B \in \mathrm{Mod}(R)$ there is a natural transformation $\operatorname{Tor}_n^R(A,B) \to \operatorname{Tor}_n^R(A',B)$ and thus $\operatorname{Tor}_n^R(-,-)$ is a bifunctor.

By a similar argument we also have that $\overline{\operatorname{Tor}}_n^R(-,-)$ is a bifunctor. Furthermore these bifunctors can be shown to be equivalent, see [17, Proposition 11.1]. This gives us a bifunctor $\operatorname{Tor}_n^R(-,-)$ which can be computed by taking a projective resolution of either variable.

Similarly for the additive functors $\operatorname{Hom}_R(-,B) : \mathrm{Mod}(R) \to \mathrm{Ab}$, which is contravariant, and $\operatorname{Hom}_R(A,-) : \mathrm{Mod}(R) \to \mathrm{Ab}$, which is covariant, we can define the right derived functors $\operatorname{Ext}_R^n(-,B) = \mathrm{R}^n \operatorname{Hom}_R(-,B)$ and $\overline{\operatorname{Ext}}_R^n(A,-) = \mathrm{R}_n \operatorname{Hom}_R(A,-)$. By a similar method to that of the example of $\operatorname{Tor}_n^R(A,-)$ above, $\operatorname{Ext}_R^n(-,-)$ and $\overline{\operatorname{Ext}}_R^n(-,-)$ can be shown to be bifunctors which, by [17, Proposition 8.1], are equivalent.

This gives us a bifunctor $\operatorname{Ext}_R^n(-,-)$ which can be computed by taking a projective resolution of the first variable or an injective resolution in the second variable.

2.2 Triangulated Categories and the Derived Category

2.2.1 Triangulated Categories

Triangular categories are a type of category which, while having a weaker structure than abelian categories, share some similar properties. The key aspect of triangulated categories are diagrams of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

called distinguished triangles. These perform a similar function in triangulated categories to that performed by short exact sequences in abelian categories, and a number of familiar results for short exact sequences have corresponding versions for distinguished triangles. The axioms for triangulated categories given below, together with the following results, are mostly taken from [16, Chapter I]. Another good reference for the theory of triangulated categories is in [14].

Definition 2.2.1. A *triangulated category* is an additive category $\mathscr{T}$, together with the following.

- An isomorphism of categories $\Sigma : \mathscr{T} \to \mathscr{T}$ called the *suspension functor*.
- A class of diagrams, called *distinguished triangles* of the form

  $$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

  where X, Y and Z are objects of $\mathscr{T}$ and $f : X \to Y$, $g : Y \to Z$ and $h : Z \to \Sigma X$ are morphisms in $\mathscr{T}$.

  A morphism of distinguished triangles is a commutative diagram,

  $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] \arrow[d, "\gamma"] & \Sigma X \arrow[d, "\Sigma\alpha"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z \arrow[r, "h'"] & \Sigma X' \end{tikzcd}$$

  where each row is a distinguished triangle. If α, β and γ are isomorphisms then the distinguished triangles are called isomorphic.

These data are subject to the following axioms.

(TR1) Any diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

isomorphic to a distinguished triangle is a distinguished triangle.

For any object $X \in \mathscr{T}$ the diagram

$$X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow \Sigma X$$

is a distinguished triangle.

(TR2) For any morphism $f : X \to Y$ in $\mathscr{T}$ there exists a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X.$$

(TR3) The diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is a distinguished triangle if and only if the diagram

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is a distinguished triangle.

(TR4) Given two distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and

$$X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X',$$

and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] & \Sigma X \arrow[d, "\Sigma\alpha"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X' \end{tikzcd}$$

then there exists a morphism $\gamma : Z \to Z'$ which completes the morphism of distinguished triangles, that is to say that it makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] \arrow[d, dashed, "\gamma"] & \Sigma X \arrow[d, "\Sigma\alpha"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'. \end{tikzcd}$$

(TR5) (The octahedral axiom) Consider three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{i} Z' \xrightarrow{i'} \Sigma X,$$

$$Y \xrightarrow{g} Z \xrightarrow{j} X' \xrightarrow{j'} \Sigma Y$$

and

$$X \xrightarrow{g \circ f} Z \xrightarrow{k} Y' \xrightarrow{k'} \Sigma X.$$

Then there exist morphisms $u : Z' \to Y'$ and $v : Y' \to X'$ such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"] & X \arrow[r] \arrow[d, "g\circ f"] & 0 \arrow[r] \arrow[d] & \Sigma X \arrow[d, "\Sigma f"] \\
Y \arrow[r, "g"'] \arrow[d, "i"] & Z \arrow[r, "j"'] \arrow[d, "k"] & X' \arrow[r, "j'"'] \arrow[d, equal] & \Sigma Y \arrow[d, "\Sigma i"] \\
Z' \arrow[r, "u"'] \arrow[d, "i'"] & Y' \arrow[r, "v"'] \arrow[d, "k'"] & X' \arrow[r, "\Sigma i\circ j'"'] \arrow[d] & \Sigma Z' \arrow[d, "\Sigma i'"] \\
\Sigma X \arrow[r, equal] & \Sigma X \arrow[r] & 0 \arrow[r] & \Sigma^2 X
\end{tikzcd}$$

and each row and column is a distinguished triangle.

Definition 2.2.2. Let $\mathscr{T}$ and $\mathscr{T}'$ be triangulated categories. A *triangulated functor* consists of a pair (F, α) where $F : \mathscr{T} \to \mathscr{T}'$ is an additive functor and $\alpha : F \circ \Sigma_{\mathscr{T}} \to \Sigma_{\mathscr{T}'} \circ F$ is a natural equivalence such that if $X \stackrel{x}{\to} Y \stackrel{y}{\to} Z \stackrel{z}{\to} \Sigma_{\mathscr{T}} X$ is a distinguished triangle in $\mathscr{T}$, then $FX \stackrel{Fx}{\to} FY \stackrel{Fy}{\to} FZ \stackrel{\alpha\circ Fz}{\to} \Sigma_{\mathscr{T}'} FX$ is a distinguished triangle in $\mathscr{T}'$.

Thus a triangulated functor is one which preserves distinguished triangles.

The distinguished triangles perform a similar role for triangulated categories as short exact sequences do for abelian categories. These similarities are demonstrated by the following results.

Proposition 2.2.3. *The composition of any two consecutive morphisms in a distinguished triangle is zero.*

Proof. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be a distinguished triangle. By (TR3) it is sufficient to show that $gf = 0$. By (TR1) and (TR2) we have that $Z \xrightarrow{id_Z} Z \longrightarrow 0 \longrightarrow \Sigma Z$ and $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$ are distinguished triangles. From the diagram

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "g"] & Z \arrow[r, "h"] \arrow[d, "\mathrm{id}_Z"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \arrow[d, "\Sigma g"] \\
Z \arrow[r, "\mathrm{id}_Z"'] & Z \arrow[r] & 0 \arrow[r] & \Sigma Z
\end{tikzcd}$$

and (TR4) we can conclude that there exists a map $\gamma : \Sigma X \to 0$ so that we have a morphism of distinguished triangles. Therefore we have that $\Sigma g \Sigma f = 0$, or, since Σ is an automorphism, that $gf = 0$. $\square$

Definition 2.2.4. Let $\mathscr{T}$ be a triangulated category and $\mathscr{A}$ be an abelian category. An additive functor $G : \mathscr{T} \to \mathscr{A}$ is called a *covariant cohomological functor* if for a distinguished triangle,

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

G induces a long exact sequence

$$\cdots \to G(\Sigma^i X) \to G(\Sigma^i Y) \to G(\Sigma^i Z) \to G(\Sigma^{i+1} X) \to \cdots .$$

Similarly we call G a *contravariant cohomological functor* if it induces a long exact sequence

$$\cdots \to G(\Sigma^i Z) \to G(\Sigma^i Y) \to G(\Sigma^i X) \to G(\Sigma^{i-1} Z) \to \cdots .$$

If G is a covariant or contravariant cohomological functor, we write $G^i(X) = G(\Sigma^i X)$.

Theorem 2.2.5. *Let $\mathscr{T}$ be a triangulated category and suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$, is a distinguished triangle. Then for M an arbitrary object of $\mathscr{T}$ we have that*

(i) $\mathrm{Hom}_{\mathscr{T}}(M, -)$ *is a covariant cohomological functor, that is to say that there is a long exact sequence*

$$\cdots \to \mathrm{Hom}_{\mathscr{T}}(M, \Sigma^i X) \to \mathrm{Hom}_{\mathscr{T}}(M, \Sigma^i Y) \to \mathrm{Hom}_{\mathscr{T}}(M, \Sigma^i Z)$$
$$\to \mathrm{Hom}_{\mathscr{T}}(M, \Sigma^{i+1} X) \to \cdots .$$

(ii) $\mathrm{Hom}_{\mathscr{T}}(-, M)$ *is a contravariant cohomological functor, that is to say that there is a long exact sequence*

$$\cdots \to \mathrm{Hom}_{\mathscr{T}}(\Sigma^i Z, M) \to \mathrm{Hom}_{\mathscr{T}}(\Sigma^i Y, M) \to \mathrm{Hom}_{\mathscr{T}}(\Sigma^i X, M)$$
$$\to \mathrm{Hom}_{\mathscr{T}}(\Sigma^{i-1} Z, M) \to \cdots .$$

Proof. In order to show that $\mathrm{Hom}_{\mathscr{T}}(M, -)$ is a covariant cohomological functor it is sufficient, by (TR3), to show that the sequence

$$\mathrm{Hom}_{\mathscr{T}}(M, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathscr{T}}(M, Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathscr{T}}(M, Z)$$

is exact.

From Proposition 2.2.3 we know that the composition of f and g is zero, so it follows that $\mathrm{Im}\, f_* \subseteq \mathrm{Ker}\, g_*$. Now let $u \in \mathrm{Hom}_{\mathscr{T}}(M, Y)$ such that $gu = 0$ i.e $u \in \mathrm{Ker}\, g_*$. By considering the diagram

$$\begin{tikzcd}
\Sigma^{-1}M \arrow[r] \arrow[d, "\Sigma^{-1}u"] & 0 \arrow[r] \arrow[d] & M \arrow[r, "\mathrm{id}_M"] & M \arrow[d, "u"] \\
\Sigma^{-1}Y \arrow[r, "-\Sigma^{-1}g"'] & \Sigma^{-1}Z \arrow[r, "-\Sigma^{-1}h"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

we have from (TR4) that there exists $v : M \to X$ such that $fv = u$, i.e $u \in \mathrm{Im}\, f_*$. So $\mathrm{Ker}\, g_* = \mathrm{Im}\, f_*$ and the sequence above is exact.

The proof that $\mathrm{Hom}_{\mathscr{T}}(-, M)$ is a contravariant cohomological functor is similar. □

Finally the following theorem is a version of the 5-lemma for distinguished triangles.

Theorem 2.2.6. *Let $\mathscr{T}$ be a triangulated category. Given a commutative diagram of distinguished triangles:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] \arrow[d, "\gamma"] & \Sigma X \arrow[d, "\Sigma\alpha"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z \arrow[r, "h'"'] & \Sigma X'.
\end{tikzcd}$$

Then if α and β are isomorphisms we also have that γ is also an isomorphism.

Proof. Begin by applying the functor $\mathrm{Hom}_{\mathscr{T}}(Z', -)$ to the diagram to obtain the commutative diagram of exact sequences:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{T}}(Z',X) \arrow[r] \arrow[d, "\alpha_*"] & \mathrm{Hom}_{\mathscr{T}}(Z',Y) \arrow[r] \arrow[d, "\beta_*"] & \mathrm{Hom}_{\mathscr{T}}(Z',Z) \arrow[r] \arrow[d, "\gamma_*"] & \mathrm{Hom}_{\mathscr{T}}(Z',\Sigma X) \arrow[r] \arrow[d, "(\Sigma\alpha)_*"] & \mathrm{Hom}_{\mathscr{T}}(Z',\Sigma Y) \arrow[d, "(\Sigma\beta)_*"] \\
\mathrm{Hom}_{\mathscr{T}}(Z',X') \arrow[r] & \mathrm{Hom}_{\mathscr{T}}(Z',Y') \arrow[r] & \mathrm{Hom}_{\mathscr{T}}(Z',Z') \arrow[r] & \mathrm{Hom}_{\mathscr{T}}(Z',\Sigma X') \arrow[r] & \mathrm{Hom}_{\mathscr{T}}(Z',\Sigma Y').
\end{tikzcd}$$

Now, since α and β are isomorphisms in $\mathscr{T}$, it follows that α_*, β_*, $(\Sigma\alpha)_*$ and $(\Sigma\beta)_*$ are isomorphisms of abelian groups. Hence by the five lemma, we have that γ_* is also an isomorphism of abelian groups and therefore there exists $\phi \in \mathrm{Hom}_{\mathscr{T}}(Z', Z)$ such that $\gamma_*(\phi) = \gamma \circ \phi = id_{Z'}$.

Similarly by using the contravariant cohomological functor $\mathrm{Hom}_{\mathscr{T}}(-, Z')$ we can conclude that there exists $\psi \in \mathrm{Hom}_{\mathscr{T}}(Z', Z)$ such that $\psi \circ \gamma = \mathrm{id}_Z$. Hence γ is an isomorphism. □

2.2.2 The Homotopy Category

The homotopy category serves as both an example of a triangulated category and as stepping stone towards the derived category. In this section we shall describe how to construct the the homotopy category from the category of complexes of objects of an abelian category.

Throughout this section we shall let $\mathscr{A}$ denote an abelian category.

Before we can give the definition of the homotopy category we first need to define when two morphisms of complexes are homotopic.

Definition 2.2.7. Let $f, g : X \to Y$ be morphisms of complexes of objects of $\mathscr{A}$. Then f and g are said to be *homotopic* if there is a collection of maps $h = (h^n)$, where $h^n : X^n \to Y^{n-1}$, such that

$$f^n - g^n = d_Y^{n-1} h^n + h^{n+1} d_X^n$$

for all $n \in \mathbb{Z}$. Such a collection of maps h is called a *homotopy.*

A morphism of complexes f is called *null homotopic* if it is homotopic to the zero map. We denote all the null homotopic maps from X to Y by $\mathrm{Null}(X, Y)$.

Definition 2.2.8. The *homotopy category* of $\mathscr{A}$, denoted by $K(\mathscr{A})$, is defined as consisting of objects which are complexes of objects of $\mathscr{A}$ and morphisms of the form

$$\mathrm{Hom}_{K(\mathscr{A})}(X, Y) = \frac{\mathrm{Hom}_{C(\mathscr{A})}(X, Y)}{\mathrm{Null}(X, Y)}.$$

The homotopy category $K(\mathscr{A})$ consists of the same objects as $C(\mathscr{A})$. The morphisms are the equivalence classes of the form $f + \mathrm{Null}(X, Y)$, where $f \in \mathrm{Hom}_{C(\mathscr{A})}(X, Y)$. We shall denote these equivalence classes by $\overline{f}$.

We now want to show that $K(\mathscr{A})$ is a triangulated category. For this we need to define what the distinguished triangles in the homotopy category are. To this end we need the following definition of the mapping cone of a morphism.

Definition 2.2.9. Let $f : X \to Y$ be a morphism of complexes of objects of $\mathscr{A}$. The *mapping cone* of f, which we shall denote by $\mathrm{C}(f)$, is the complex:

$$\cdots \to \mathrm{C}^{n-1} \stackrel{d^{n-1}}{\to} \mathrm{C}^n \stackrel{d^n}{\to} \mathrm{C}^{n+1} \to \cdots$$

where $(\mathrm{C}(f))^n = X^{n+1} \oplus Y^n$ and $d^n_{\mathrm{C}(f)} = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix}$.

Proposition 2.2.10. *The mapping cone of the identity map $X \overset{\mathrm{id}_X}{\to} X$ is homotopic to 0.*

Proof. First note that $(\mathrm{C}(\mathrm{id}_X))^n = X^{n+1} \oplus X^n$ and $d^n_{\mathrm{C}(\mathrm{id}_X)} = \begin{pmatrix} -d^{n+1}_X & 0 \\ \mathrm{id}_X & d^n_X \end{pmatrix}$. Now define the collection of maps $h^n : X^{n+1} \oplus X^n \to X^n \oplus X^{n-1}$ by $h^n = \begin{pmatrix} 0 & \mathrm{id}_X \\ 0 & 0 \end{pmatrix}$. Then

$$(d^{n-1}_{\mathrm{C}(\mathrm{id}_X)} h^n + h^{n+1} d^n_{\mathrm{C}(\mathrm{id}_X)}) = \begin{pmatrix} -d^n_X & 0 \\ \mathrm{id}_X & d^{n-1}_X \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id}_X \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{id}_X \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -d^{n+1}_X & 0 \\ \mathrm{id}_X & d^n_X \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -d^n_X \\ 0 & \mathrm{id}_X \end{pmatrix} + \begin{pmatrix} \mathrm{id}_X & d^n_X \\ 0 & 0 \end{pmatrix} = \mathrm{id}_{\mathrm{C}(\mathrm{id}_X)}\,.$$

Therefore $\mathrm{id}_{\mathrm{C}(\mathrm{id}_X)}$ is nullhomotopic. □

Theorem 2.2.11. *The homotopy category $K(\mathscr{A})$ is a triangulated category.*

Proof. For $K(\mathscr{A})$ to be a triangulated category we require that there exists a suspension functor and distinguished triangles which satisfy the axioms (TR1-5).

We define the suspension functor, Σ, on $K(\mathscr{A})$ as the operation of shifting one place to the left and changing the sign on the differential, thus $\Sigma(X)^n = X^{n+1}$ and $d^n_{\Sigma X} = -d^{n+1}_X$ while for a morphism $f : X \to Y$ we have that $\Sigma(f) : \Sigma X \to \Sigma Y$ is given by $(\Sigma f)^n = f^{n+1}$.

We define distinguished triangles of $K(\mathscr{A})$ to be the triangles isomorphic in $K(\mathscr{A})$ to those of the form

$$X \xrightarrow{f} Y \xrightarrow{i} \mathrm{C}(f) \xrightarrow{p} \Sigma X$$

where X and Y are objects of $\mathscr{A}$ and $f \in \mathrm{Hom}_{K(\mathscr{A})}(X, Y)$ while $i : Y \to C(f)$ and $p : C(f) \to \Sigma X$ denote the obvious injection and projection morphisms.

It is now routine to verify that the axioms hold, for the details see [14, Chapter IV, Theorem 1.9]. □

Remark 2.2.12. Let $F : \mathscr{A} \to \mathscr{B}$ be a additive covarient functor between abelian categories. Then this extends naturally to a functor

$$F_C : C(\mathscr{A}) \to C(\mathscr{B})$$

between the categories of complexes of objects in $\mathscr{A}$ and $\mathscr{B}$. This functor is defined on objects by sending a complex

$$A = \cdots A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} \cdots$$

in $C(\mathscr{A})$ to the complex

$$F_C(A) = \cdots F(A^{n-1}) \xrightarrow{F(d^{n-1})} F(A^n) \xrightarrow{F(d^n)} F(A^{n+1}) \xrightarrow{F(d^{n+1})} \cdots$$

in $C(\mathscr{B})$ and on morphisms by sending a morphism of complexes $\phi = \phi^n : A \to A'$ in $C(\mathscr{A})$ to the morphism of complexes $F(\phi) = F(\phi^n) : F(A) \to F(A')$. We can then extend this further to a functor

$$F_K : K(\mathscr{A}) \to K(\mathscr{B})$$

between the homotopy categories of $\mathscr{A}$ and $\mathscr{B}$. We define F_K to act on objects in the same way as F_C and to act on morphisms by sending a morphism $\overline{f}$ in $K(\mathscr{A})$ to the morphism $\overline{F_C(f)}$. It is easy to see that $F_K(-)$ defined in this way is well defined.

Similarly, $G : \mathscr{A} \to \mathscr{B}$ an additive contravarient functor between abelian categories can be extended to the functors

$$F_C : C(\mathscr{B}) \to C(\mathscr{A}) \text{ and } F_K : K(\mathscr{B}) \to K(\mathscr{A}).$$

2.2.3 Localisation at a Multiplicative Set

We shall construct the derived category of an abelian category $\mathscr{A}$ from the homotopy category $K(\mathscr{A})$ through a localisation process where we formally invert a class of morphisms. In this section we describe the localisation process in an abstract setting.

We begin by setting out the properties which the class of morphisms we wish to invert must satisfy.

Definition 2.2.13. For any category $\mathscr{C}$ a *multiplicative system* S is a collection of morphisms of $\mathscr{C}$ which satisfy the following axioms:

(MS1) For $f, g \in S$ such that fg exists, then $fg \in S$.

For any $X \in \mathscr{C}$ we have that $\mathrm{id}_X \in S$.

(MS2) Given a diagram

$$\begin{tikzcd} & Z \arrow[d, "s"] \\ X \arrow[r, "u"] & Y \end{tikzcd}$$

with $s \in S$, then there is a commutative diagram

$$\begin{tikzcd} W \arrow[r, "v"] \arrow[d, "t"] & Z \arrow[d, "s"] \\ X \arrow[r, "u"] & Y \end{tikzcd}$$

with $t \in S$.

The dual statement is also satisfied.

(MS3) If $f, g : X \to Y$ are morphisms in $\mathscr{C}$, then the following statements are equivalent:

(i) $\exists\ s : Y \to Y'$ in S such that $sf = sg$,

(ii) $\exists\ t : X' \to X$ in S such that $ft = gt$.

In the case where we are dealing with a triangulated category, such as $K(\mathscr{A})$, it is desirable that the multiplicative set also satisfies the following properties. These additional properties ensure that the resulting category is also triangulated, as we shall see later.

Definition 2.2.14. Let $\mathscr{T}$ be a triangulated category and S a multiplicative system of morphisms of $\mathscr{T}$. Then S is *compatible with the triangulation* if the following axioms hold:

(MS4) $s \in S$ if and only if $\Sigma s \in S$, where Σ is the suspension functor of $\mathscr{T}$.

(MS5) Given two distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and

$$X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'$$

in $\mathscr{T}$, together with morphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$ in S which form the commutative diagram below,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] & \Sigma X \arrow[d, "\Sigma\alpha"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z \arrow[r, "h'"] & \Sigma X' \end{tikzcd}$$

there exists a morphism $\gamma : Z \to Z'$ in S such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] \arrow[d, dashed, "\gamma"] & \Sigma X \arrow[d, "\Sigma\alpha"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z \arrow[r, "h'"] & \Sigma X'.
\end{tikzcd}$$

We now define the process by which we formally invert a multiplicative system of morphisms of a category in order to obtain a new category.

Definition 2.2.15. Let $\mathscr{C}$ be a category with a multiplicative system S. Then the *localisation of $\mathscr{C}$ with respect to S* is a category, denoted $S^{-1}\mathscr{C}$, where the objects are the objects of $\mathscr{C}$ whilst the morphisms are defined as follows.

For $X, Y \in S^{-1}\mathscr{C}$, $\mathrm{Hom}_{S^{-1}\mathscr{C}}(X, Y)$ is the class of equivalence classes of diagrams of the form

$$\begin{tikzcd}[column sep=small]
& Z \arrow[dl, "s"'] \arrow[dr, "a"] & \\
X & & Y
\end{tikzcd}$$

where $s \in S$. These diagrams are often referred to as *roofs*.

Two roofs, $\begin{tikzcd}[column sep=small] & Z \arrow[dl, "s"'] \arrow[dr, "a"] & \\ X & & Y \end{tikzcd}$ and $\begin{tikzcd}[column sep=small] & Z' \arrow[dl, "t"'] \arrow[dr, "b"] & \\ X & & Y \end{tikzcd}$ with $s, t \in S$, are equivalent if and only if there exists a diagram

$$\begin{tikzcd}
& & Z'' \arrow[dl, "f"'] \arrow[dr, "g"] & & \\
& Z \arrow[dl, "s"'] \arrow[drrr, "a"' near end] & & Z' \arrow[dlll, "t" near end] \arrow[dr, "b"] & \\
X & & & & Y
\end{tikzcd}$$

which commutes and $sf = tg \in S$.

We denote the set of roofs equivalent to $\begin{tikzcd}[column sep=small] & Z \arrow[dl, "s"'] \arrow[dr, "a"] & \\ X & & Y \end{tikzcd}$ by $\left[\begin{tikzcd}[column sep=small] & Z \arrow[dl, "s"'] \arrow[dr, "a"] & \\ X & & Y \end{tikzcd}\right]$.

The composition of two morphisms in $S^{-1}\mathscr{C}$ is given by

$$\left[\begin{tikzcd}[column sep=small] & W' \arrow[dl, "t"'] \arrow[dr, "b"] & \\ Y & & Z \end{tikzcd}\right] \cdot \left[\begin{tikzcd}[column sep=small] & W \arrow[dl, "s"'] \arrow[dr, "a"] & \\ X & & Y \end{tikzcd}\right] = \left[\begin{tikzcd}[column sep=small] & W'' \arrow[dl, "su"'] \arrow[dr, "bv"] & \\ X & & Z \end{tikzcd}\right]$$

where

$$\begin{tikzcd}[column sep=small]
& & W'' \arrow[dl, "u"'] \arrow[dr, "v"] & & \\
& W \arrow[dl, "s"] \arrow[dr, "a"] & & W' \arrow[dl, "t"'] \arrow[dr, "b"] & \\
X & & Y & & Z
\end{tikzcd}$$

is a commutative diagram with $u \in S$. The existence of this is due to (MS2). It is straightforward to check that this operation is well defined.

Notation 2.2.16. We shall, in some situations, denote a morphism $\left[\begin{tikzcd}[cramped, sep=small] & W' \arrow[dl, "s"'] \arrow[dr, "a"] & \\ Y & & Z \end{tikzcd}\right]$ in $S^{-1}\mathscr{C}$ by the more concise $a \circ s^{-1}$.

Remark 2.2.17. There are some potential set theoretical considerations regarding the existence of the category $S^{-1}\mathscr{C}$. However in the case we are considering here, that of the derived category of an abelian category we can safely ignore these. For a further explanation of these issues as well as an explanation as to why in our case we can ignore them can be found in [32, Remark 10.3.3].

Definition 2.2.18. Let $\mathscr{C}$ be a category with a multiplicative system S. Then we define the *quotient functor* $Q : \mathscr{C} \to S^{-1}\mathscr{C}$ to be the following functor.

For any object $X \in \mathscr{C}$, $Q(X) = X \in S^{-1}\mathscr{C}$ and for a morphism $f : X \to Y$, in $\mathscr{C}$, $Q(f)$, in $S^{-1}\mathscr{C}$ is the morphism

$$\left[\begin{tikzcd}[cramped, sep=small] & X \arrow[dl, equal] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}\right].$$

The aim of the localisation process we have described is to formally invert the morphisms of a multiplicative set. The following proposition shows that morphisms of the this multiplicative set become isomorphisms in the new category we have constructed.

Proposition 2.2.19. *Let $\mathscr{C}$ be a category with a multiplicative system S. Then, for a morphism $s : X \to Y$ in S we have that $Q(s)$ is an isomorphism in $S^{-1}\mathscr{C}$ with inverse* $\left[\begin{tikzcd}[cramped, sep=small] & X \arrow[dl, "s"'] \arrow[dr, equal] & \\ Y & & X \end{tikzcd}\right]$.

Proof. This is simply a case of checking that $\left[\begin{tikzcd}[cramped, sep=small] & X \arrow[dl, "s"'] \arrow[dr, equal] & \\ Y & & X \end{tikzcd}\right]$ is in fact the inverse of $Q(s) = \left[\begin{tikzcd}[cramped, sep=small] & X \arrow[dl, equal] \arrow[dr, "s"] & \\ X & & Y \end{tikzcd}\right]$.

Firstly from the commutative diagram

$$\begin{tikzcd}[sep=small] & & X \arrow[dl, equal] \arrow[dr, equal] & & \\ & X \arrow[dl, "s"'] \arrow[dr, equal] & & X \arrow[dl, equal] \arrow[dr, "s"] & \\ Y & & X & & Y. \end{tikzcd}$$

we have that

$$\left[\begin{tikzcd}[sep=small] & X \arrow[dl, "s"'] \arrow[dr, equal] & \\ Y & & X \end{tikzcd}\right] \cdot \left[\begin{tikzcd}[sep=small] & X \arrow[dl, equal] \arrow[dr, "s"] & \\ X & & Y \end{tikzcd}\right] = \left[\begin{tikzcd}[sep=small] & X \arrow[dl, "s"'] \arrow[dr, "s"] & \\ Y & & Y \end{tikzcd}\right].$$

Furthermore $\left[\begin{tikzcd}[sep=small] & X \arrow[dl, "s"'] \arrow[dr, "s"] & \\ Y & & Y \end{tikzcd}\right] = \left[\begin{tikzcd}[sep=small] & Y \arrow[dl, equal] \arrow[dr, equal] & \\ Y & & Y \end{tikzcd}\right] = Q(\mathrm{id}_Y)$ as we can construct the commutative diagram

$$\begin{tikzcd} & & X \arrow[dl, equal] \arrow[dr, "s"] & & \\ & X \arrow[dl, "s"'] \arrow[drrr, "s"] & & Y \arrow[dlll, equal] \arrow[dr, equal] & \\ Y & & & & Y. \end{tikzcd}$$

Similarly, since we have the commutative diagram

$$\begin{tikzcd}[sep=small] & & X \arrow[dl, equal] \arrow[dr, equal] & & \\ & X \arrow[dl, equal] \arrow[dr, "s"] & & X \arrow[dl, "s"'] \arrow[dr, equal] & \\ X & & Y & & X. \end{tikzcd}$$

we have that

$$\left[\begin{tikzcd}[sep=small] & X \arrow[dl, equal] \arrow[dr, "s"] & \\ X & & Y \end{tikzcd}\right] \cdot \left[\begin{tikzcd}[sep=small] & X \arrow[dl, "s"'] \arrow[dr, equal] & \\ Y & & X \end{tikzcd}\right] = \left[\begin{tikzcd}[sep=small] & X \arrow[dl, equal] \arrow[dr, equal] & \\ X & & X \end{tikzcd}\right] = Q(id_X).$$

Thus $\left[\begin{tikzcd}[sep=small] & X \arrow[dl, "s"'] \arrow[dr, equal] & \\ Y & & X \end{tikzcd}\right]$ and $Q(s)$ are inverses of each other in $S^{-1}\mathscr{C}$ and so they are isomorphisms. $\square$

The category, $S^{-1}\mathscr{C}$ together with the quotient functor $Q : \mathscr{C} \to S^{-1}\mathscr{C}$ satisfy the following universal property.

Proposition 2.2.20. *Let $\mathscr{C}$ be a category with a multiplicative system S and let $Q : \mathscr{C} \to S^{-1}\mathscr{C}$ be the quotient functor defined in Definition 2.2.18 above. Then for any functor $F : \mathscr{C} \to \mathscr{D}$ such that $F(s)$ is an isomorphism for all $s \in S$ there exists a unique functor $G : S^{-1}\mathscr{C} \to \mathscr{D}$ such that $GQ = F$.*

Proof. Clearly since the objects of $S^{-1}\mathscr{C}$ are the same as the objects of $\mathscr{C}$ we must have that $G(X) = F(X)$ for all objects $X \in \mathscr{C}$.

Now consider the morphism $\left[\begin{tikzcd}[sep=small] & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y\end{tikzcd}\right]$ in $S^{-1}\mathscr{C}$.

$$
\begin{aligned}
G\left(\left[\begin{tikzcd}[sep=small] & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y\end{tikzcd}\right]\right) &= G\left(\left[\begin{tikzcd}[sep=small] & Z \arrow[dl, equal] \arrow[dr, "f"] & \\ Z & & Y\end{tikzcd}\right] \circ \left[\begin{tikzcd}[sep=small] & Z \arrow[dl, "s"'] \arrow[dr, equal] & \\ X & & Z\end{tikzcd}\right]\right) \\
&= G\left(\left[\begin{tikzcd}[sep=small] & Z \arrow[dl, equal] \arrow[dr, "f"] & \\ Z & & Y\end{tikzcd}\right]\right) \circ G\left(\left[\begin{tikzcd}[sep=small] & Z \arrow[dl, "s"'] \arrow[dr, equal] & \\ X & & Z\end{tikzcd}\right]\right) \\
&= GQ(f) \circ G(Q(s)^{-1}).
\end{aligned}
$$

However

$$\mathrm{id}_F(X) = F(\mathrm{id}_X) = G(\mathrm{id}_X) = G(Q(s) \circ Q(s)^{-1}) = GQ(s) \circ G(Q(s)^{-1}) = F(s) \circ G(Q(s)^{-1})$$

and

$$\mathrm{id}_F(Z) = F(\mathrm{id}_Z) = G(\mathrm{id}_Z) = G(Q(s)^{-1} \circ Q(s)) = G(Q(s)^{-1}) \circ GQ(s) = G(Q(s)^{-1}) \circ F(s).$$

This forces $G(Q(s)^{-1}) = F(s)^{-1}$ and so $G\left(\left[\begin{tikzcd}[sep=small] & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y\end{tikzcd}\right]\right)$ must equal $F(f) \circ F(s)^{-1}$. This proves the uniqueness of G.

For existence we need to show that G is well defined. To do this let

$$\left[\begin{tikzcd}[sep=small] & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y\end{tikzcd}\right] = \left[\begin{tikzcd}[sep=small] & Z' \arrow[dl, "s'"'] \arrow[dr, "f'"] & \\ X & & Y\end{tikzcd}\right].$$

Then there exists a commutative diagram

$$
\begin{tikzcd}
 & & Z'' \arrow[dl, "t"'] \arrow[dr, "u"] & & \\
 & Z \arrow[dl, "s"'] \arrow[drrr, "f"' near end] & & Z' \arrow[dlll, "s'" near end] \arrow[dr, "f'"] & \\
X & & & & Y
\end{tikzcd}
$$

with $st = s'u \in S$.

Now consider $F(ft) \circ F(st)^{-1}$.

We have that

$$F(ft) \circ F(st)^{-1} = F(f) \circ F(t) \circ F(t)^{-1} \circ F(s)^{-1} = F(f) \circ F(s)^{-1}$$

and

$$F(ft)\circ F(st)^{-1} = F(f'u)\circ F(s'u)^{-1} = F(f')\circ F(u)\circ F(u)^{-1}\circ F(s')^{-1} = F(f')\circ F(s')^{-1}.$$

Hence $G\left(\left[\begin{tikzcd}[column sep=small, row sep=small] & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y\end{tikzcd}\right]\right) = F(f)\circ F(s)^{-1} = F(f')\circ F(s')^{-1} = G\left(\left[\begin{tikzcd}[column sep=small, row sep=small] & Z' \arrow[dl, "s'"'] \arrow[dr, "f'"] & \\ X & & Y\end{tikzcd}\right]\right).$
Thus G is well defined. □

Remark 2.2.21. A dual construction, in which the roofs take the form

$$\begin{tikzcd}[column sep=small, row sep=small] & Z & \\ X \arrow[ur, "u"] & & Y \arrow[ul, "s"']\end{tikzcd}$$

with $s \in S$, is also possible.

This dual construction also satisfies the universal property given in Proposition 2.2.20 and hence results in an equivalent category. It is therefore possible to construct the category $S^{-1}\mathscr{C}$ by using roofs of either form.

We shall mostly, for the purpose of this document, restrict ourselves to using roofs of the form

$$\begin{tikzcd}[column sep=small, row sep=small] & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y.\end{tikzcd}$$

There are some situations in which it is preferable to consider roofs in the form of this dual construction.

We now complete this section with the following Theorem showing that for a triangulated category with a multiplicative system which is compatible with the triangulation then the the localisation process preserves the triangulated structure.

Theorem 2.2.22. *Let $\mathscr{T}$ be a triangulated category with a multiplicative system S which is compatible with the triangulation. Then $S^{-1}\mathscr{T}$ is also a triangulated category.*

Proof. Let Σ' be the suspension functor on $\mathscr{T}$. Then we define the suspension functor for $S^{-1}\mathscr{T}$, Σ, to be the functor such that $\Sigma(X) = \Sigma'(X)$ for all objects $X \in \mathscr{T}$ and for any morphism $\left[\begin{tikzcd}[column sep=small, row sep=small] & Z \arrow[dl, "s"'] \arrow[dr, "u"] & \\ X & & Y\end{tikzcd}\right] \in \mathscr{T}$,

$$\Sigma\left(\left[\begin{tikzcd}[column sep=small, row sep=small] & Z \arrow[dl, "s"'] \arrow[dr, "u"] & \\ X & & Y\end{tikzcd}\right]\right) = \left[\begin{tikzcd}[column sep=small, row sep=small] & \Sigma Z \arrow[dl, "\Sigma' s"'] \arrow[dr, "\Sigma' u"] & \\ \Sigma X & & \Sigma Y\end{tikzcd}\right]$$

where $\Sigma' s \in S$ by (MS4). It is straightforward to see that this is well defined.

We define the distinguished triangles of $S^{-1}\mathscr{T}$ to be those triangles isomorphic to triangles of the form

$$X \xrightarrow{Qf} Y \xrightarrow{Qg} Z \xrightarrow{Qh} \Sigma X,$$

where

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is a distinguished triangle in $\mathscr{T}$ and $Q(-)$ is the quotient functor introduced in Definition 2.2.18.

It remains to show that the axioms (TR1)-(TR5) hold.

The axioms (TR1) and (TR3) follow from our definition of a distinguished triangle in $S^{-1}\mathscr{T}$.

To show that (TR2) holds, let $u \circ s^{-1}$ denote the morphism $\left[\begin{tikzcd}[sep=small] & Z \arrow[dl, "s"'] \arrow[dr, "u"] & \\ X & & Y\end{tikzcd}\right]$ from X to Y in $S^{-1}\mathscr{T}$ and let $Z \overset{u}{\to} Y \overset{v}{\to} W \overset{w}{\to} \Sigma Z$ be a distinguished triangle in $\mathscr{T}$ containing u. Then $Z \overset{Qu}{\to} Y \overset{Qv}{\to} W \overset{Qw}{\to} \Sigma Z$ is a distinguished triangle in $S^{-1}\mathscr{T}$ and we have the following commutative diagram,

$$\begin{tikzcd}
Z \arrow[r, "Qu"] \arrow[d, "Qs"'] & Y \arrow[r, "Qv"] \arrow[d, equal] & W \arrow[r, "Qw"] \arrow[d, equal] & \Sigma Z \arrow[d, "\Sigma Qs"] \\
X \arrow[r, "u \circ s^{-1}"'] & Y \arrow[r, "Qv"'] & W \arrow[r, "\Sigma Qs \circ Qw"'] & \Sigma X.
\end{tikzcd}$$

Since Qs is an isomorphism in $S^{-1}\mathscr{T}$ this gives us that the triangle

$$X \xrightarrow{u \circ s^{-1}} Y \xrightarrow{Qv} W \xrightarrow{\Sigma Qs \circ Qw} \Sigma X$$

is isomorphic to the distinguished triangle $Z \overset{Qu}{\to} Y \overset{Qv}{\to} W \overset{Qw}{\to} \Sigma Z$ and so is itself a distinguished triangle. Hence any morphism in $S^{-1}\mathscr{T}$ can be embedded in a distinguished triangle.

For (TR4), consider the following diagram of in $S^{-1}\mathscr{T}$

$$\begin{tikzcd}
X \arrow[r, "Qf"] \arrow[d, "\alpha"'] & Y \arrow[r, "Qg"] \arrow[d, "\beta"'] & Z \arrow[r, "Qh"] & \Sigma X \arrow[d, "\Sigma\alpha"'] \\
X' \arrow[r, "Qf'"'] & Y' \arrow[r, "Qg'"'] & Z' \arrow[r, "Qh'"'] & \Sigma X'
\end{tikzcd}$$

where both rows are distinguished triangles while

$$\alpha = \left[\begin{tikzcd}[column sep=small] & A \arrow[dl, "s"'] \arrow[dr, "u"] & \\ X & & X' \end{tikzcd}\right] \text{ and } \beta = \left[\begin{tikzcd}[column sep=small] & B \arrow[dl, "t"'] \arrow[dr, "v"] & \\ Y & & Y' \end{tikzcd}\right].$$

We can now construct, in $\mathscr{T}$, a diagram of the form

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \\ A \arrow[u, "s"] \arrow[d, "u"'] & B \arrow[u, "t"] \arrow[d, "v"'] & & \Sigma A \arrow[u, "\Sigma s"'] \arrow[d, "\Sigma u"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X', \end{tikzcd}$$

where the top and bottom rows are distinguished triangles in $\mathscr{T}$ and $\Sigma s \in S$, by (MS4).

Furthermore, by (MS2) we can choose A in such a way that there exists a morphism $h : A \to B$ such that the diagram commutes. This morphism can in turn be embedded into a distinguished triangle $A \xrightarrow{h} B \xrightarrow{k} C \xrightarrow{l} \Sigma A$.

By (TR4) and (MS5) there exists a morphism $r : C \to Z$ in S and a morphism $w : C \to Z'$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \\ A \arrow[u, "s"] \arrow[d, "u"'] \arrow[r, "h"] & B \arrow[u, "t"] \arrow[d, "v"'] \arrow[r, "k"] & C \arrow[u, "r"] \arrow[d, "w"'] \arrow[r, "l"] & \Sigma A \arrow[u, "\Sigma s"'] \arrow[d, "\Sigma u"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X' \end{tikzcd}.$$

Hence we have a morphism $\gamma = \left[\begin{tikzcd}[column sep=small] & C \arrow[dl, "r"'] \arrow[dr, "w"] & \\ Z & & Z' \end{tikzcd}\right]$ in $S^{-1}\mathscr{T}$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "Qf"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & \Sigma X \arrow[d, "\Sigma\alpha"'] \\ X' \arrow[r, "Qf'"'] & Y' \arrow[r, "Qg'"'] & Z' \arrow[r, "Qh'"'] & \Sigma X', \end{tikzcd}$$

in $S^{-1}\mathscr{T}$, commutes.

For (TR5), let $f \circ s^{-1} = \left[\begin{tikzcd}[column sep=small, row sep=small] & U \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y\end{tikzcd}\right]$ and $g \circ t^{-1} = \left[\begin{tikzcd}[column sep=small, row sep=small] & V \arrow[dl, "t"'] \arrow[dr, "g"] & \\ Y & & Z\end{tikzcd}\right]$ be morphisms in $S^{-1}\mathscr{T}$. Then $\left[\begin{tikzcd}[column sep=small, row sep=small] & V \arrow[dl, "t"'] \arrow[dr, "g"] & \\ Y & & Z\end{tikzcd}\right] \circ \left[\begin{tikzcd}[column sep=small, row sep=small] & U \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y\end{tikzcd}\right] = \left[\begin{tikzcd}[column sep=small, row sep=small] & W \arrow[dl, "s\circ t'"'] \arrow[dr, "g\circ f'"] & \\ X & & Z\end{tikzcd}\right]$ where the diagram

$$\begin{tikzcd}[column sep=small, row sep=small] & & W \arrow[dl, "t'"'] \arrow[dr, "f'"] & & \\ & U \arrow[dl, "s"'] \arrow[dr, "f"] & & V \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

commutes and $t' \in S$.

A consequence of this diagram commuting is that $\left[\begin{tikzcd}[column sep=small, row sep=small] & U \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y\end{tikzcd}\right] = \left[\begin{tikzcd}[column sep=small, row sep=small] & W \arrow[dl, "st'"'] \arrow[dr, "tf'"] & \\ X & & Y\end{tikzcd}\right]$.

Now suppose that we have following distinguished triangle in $S^{-1}\mathscr{T}$ containing $f \circ s^{-1}$,

$$X \xrightarrow{f\circ s^{-1}} Y \longrightarrow Z'' \longrightarrow \Sigma X$$

and that the mapping cone of f' in $\mathscr{T}$ is

$$W \xrightarrow{f'} V \xrightarrow{i} Z' \xrightarrow{i'} \Sigma W.$$

Then we can construct the following commutative diagram $S^{-1}\mathscr{T}$

$$\begin{tikzcd} W \arrow[r, "Q(f')"] \arrow[d, "Q(s\circ t')"'] & V \arrow[r, "Q(i)"] \arrow[d, "Q(t)"'] & Z' \arrow[r, "Q(i')"] & \Sigma W \arrow[d, "Q(\Sigma s\circ t')"'] \\ X \arrow[r, "f\circ s^{-1}"'] & Y \arrow[r] & Z'' \arrow[r] & \Sigma X. \end{tikzcd}$$

where both rows are distinguished triangles, hence by (TR4) and Theorem 2.2.6 the distinguished triangles are isomorphic.

By similar arguments, the distinguished triangles in $S^{-1}\mathscr{T}$ containing $g \circ t^{-1}$ and the composition $(g \circ t^{-1}) \circ (f \circ s^{-1})$ are isomorphic to the distinguished triangles

$$V \xrightarrow{Qg} Z \xrightarrow{Qj} X' \xrightarrow{Qj'} \Sigma V$$

and

$$W \xrightarrow{Q(g\circ f')} Z \xrightarrow{Qk} Y' \xrightarrow{Qk'} \Sigma W$$

respectively. Here $V \xrightarrow{g} Z \xrightarrow{j} X' \xrightarrow{j'} \Sigma V$ and $W \xrightarrow{g\circ f'} Z \xrightarrow{k} Y' \xrightarrow{k'} \Sigma W$ are distinguished triangles in $\mathscr{T}$ containing the morphisms g and $g \circ f'$ respectively.

Thus, by relabeling we need only to consider three distinguished triangles in $S^{-1}\mathscr{T}$ of the form:

$$X \xrightarrow{Qf} Y \xrightarrow{Qi} Z' \xrightarrow{Qi'} \Sigma X,$$

$$Y \xrightarrow{Qg} Z \xrightarrow{Qj} X' \xrightarrow{Qj'} \Sigma Y$$

and

$$X \xrightarrow{Q(g\circ f)} Z \xrightarrow{Qk} Y' \xrightarrow{Qk'} \Sigma X$$

where $X \xrightarrow{f} Y \xrightarrow{i} Z' \xrightarrow{i'} \Sigma X$, $Y \xrightarrow{g} Z \xrightarrow{j} X' \xrightarrow{j'} \Sigma Y$ and $X \xrightarrow{g\circ f} Z \xrightarrow{k} Y' \xrightarrow{k'} \Sigma X$ are distinguished triangles in $\mathscr{T}$.

By the octahedral axiom for $\mathscr{T}$, there exists morphism u and v in $\mathscr{T}$ such that we can we can construct a commutative diagram

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"] & X \arrow[r] \arrow[d, "g\circ f"] & 0 \arrow[r] \arrow[d] & \Sigma X \arrow[d, "\Sigma f"] \\
Y \arrow[r, "g"'] \arrow[d, "i"] & Z \arrow[r, "j"'] \arrow[d, "k"] & X' \arrow[r, "j'"'] \arrow[d, equal] & \Sigma Y \arrow[d, "\Sigma i"] \\
Z' \arrow[r, "u"'] \arrow[d, "i'"] & Y' \arrow[r, "v"'] \arrow[d, "k'"] & X' \arrow[r, "\Sigma i\circ j'"'] \arrow[d] & \Sigma Z' \arrow[d, "\Sigma i'"] \\
\Sigma X \arrow[r, equal] & \Sigma X \arrow[r] & 0 \arrow[r] & \Sigma^2 X
\end{tikzcd}$$

in $\mathscr{T}$, where each row and column is a distinguished triangle.

Hence we have morphisms Qu and Qv in $S^{-1}\mathscr{T}$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "Qf"] & X \arrow[r] \arrow[d, "Qg\circ Qf"] & 0 \arrow[r] \arrow[d] & \Sigma X \arrow[d, "\Sigma Qf"] \\
Y \arrow[r, "Qg"'] \arrow[d, "Qi"] & Z \arrow[r, "j"'] \arrow[d, "Qk"] & X' \arrow[r, "Qj'"'] \arrow[d, equal] & \Sigma Y \arrow[d, "\Sigma Qi"] \\
Z' \arrow[r, "Qu"'] \arrow[d, "Qi'"] & Y' \arrow[r, "Qv"'] \arrow[d, "Qk'"] & X' \arrow[r, "\Sigma Qi\circ Qj'"'] \arrow[d] & \Sigma Z' \arrow[d, "\Sigma Qi'"] \\
\Sigma X \arrow[r, equal] & \Sigma X \arrow[r] & 0 \arrow[r] & \Sigma^2 X
\end{tikzcd}$$

in $S^{-1}\mathscr{T}$ is commutative and each row and column is a distinguished triangle. $\square$

Remark 2.2.23. A consequence of how the distinguished triangles in the localisation of a triangulated category are defined, in the above theorem, is that the quotient functor $Q(-)$ in this case is a triangulated functor.

2.2.4 Constructing the Derived Category

We now use the localisation construction from the previous section to obtain the derived category for an abelian category, $\mathscr{A}$. We do this by formally inverting a multiplicative set of morphisms of the homotopy category $K(\mathscr{A})$. The multiplicative set used is the set of all quasi-isomorphisms. This results in a triangulated category in which the objects are complexes of objects of $\mathscr{A}$ with the property that any two objects with the same homology are isomorphic.

Definition 2.2.24. A morphism $f : X \to Y$ in $K(\mathscr{A})$ is called a *quasi-isomorphism* if it induces an isomorphism on the cohomology. That is if $\mathrm{H}(f) : \mathrm{H}(X) \to \mathrm{H}(Y)$ is an isomorphism.

Proposition 2.2.25. *The collection of all quasi-isomorphisms in $K(\mathscr{A})$ forms a multiplicative set which is compatible with triangulation.*

Proof. To do this we must show that the axioms (MS1)-(MS5) hold for the collection of all quasi-isomorphisms.

The axioms (MS1) and (MS4) hold trivially.

For (MS2), consider the diagram

$$\begin{tikzcd} & Z \arrow[d, "s"] \\ X \arrow[r, "u"] & Y \end{tikzcd}$$

where s is a quasi-isomorphism. Then s can be embedded into a distinguished triangle $Z \xrightarrow{s} Y \xrightarrow{f} N \xrightarrow{g} \Sigma Z$. Similarly $fu : X \to N$ can be embedded into a distinguished triangle $W \xrightarrow{t} X \xrightarrow{fu} N \xrightarrow{h} \Sigma W$. By (TR4) there exists a morphism $v : W \to Z$ which gives the morphism of distinguished triangles

$$\begin{tikzcd} W \arrow[r, "t"] \arrow[d, "v"] & X \arrow[r, "fu"] \arrow[d, "u"] & N \arrow[r, "h"] \arrow[d, "\mathrm{id}_N"] & \Sigma W \arrow[d, "\Sigma v"] \\ Z \arrow[r, "s"'] & Y \arrow[r, "f"'] & N \arrow[r, "g"] & \Sigma Z. \end{tikzcd}$$

Since $sv = ut$, it remains to prove that t is a quasi-isomorphism. Applying the covariant cohomological functor $\mathrm{H}(-)$ to the the distinguished triangles gives us the long exact sequences

$$\cdots \to \mathrm{H}^{i-1}(N) \to \mathrm{H}^i(Z) \xrightarrow{s_*} \mathrm{H}^i(Y) \to \mathrm{H}^i(N) \to \cdots$$

and

$$\cdots \to \mathrm{H}^{i-1}(N) \to \mathrm{H}^i(W) \xrightarrow{t_*} \mathrm{H}^i(X) \to \mathrm{H}^i(N) \to \cdots .$$

Since s is a quasi-isomorphism we have that s_* is an isomorphism and so from the first exact sequence $H^i(N) = 0$ for all i. This in turn gives us from the second exact sequence that t_* is an isomorphism and so t is a quasi-isomorphism as required.

The proof for the dual condition of (MS2) is similar.

To prove (MS3) let $f, g : X \to Y$ be morphisms and set $h = f - g$. Then it is sufficient to show that the following are equivalent:

(i) $\exists$ a quasi-isomorphism $s : Y \to Y'$ such that $sh = 0$,

(ii) $\exists$ a quasi-isomorphism $t : X' \to X$ such that $ht = 0$.

Suppose (i) holds, then there exists a distinguished triangle $Z \xrightarrow{v} Y \xrightarrow{s} Y' \xrightarrow{u} \Sigma Z$. The fact that $sh = 0$ gives us that the following diagram commutes,

$$\begin{tikzcd}
X \arrow[r, "0"] \arrow[d, "h"] & 0 \arrow[r, "0"] \arrow[d, "0"] & \Sigma X \arrow[r, "-\Sigma\,\mathrm{id}_X"] & \Sigma X \arrow[d, "\Sigma h"] \\
Y \arrow[r, "s"'] & Y' \arrow[r, "u"'] & \Sigma Z \arrow[r, "-\Sigma v"'] & \Sigma Y.
\end{tikzcd}$$

Hence by (TR4) there exists a morphism $k : X \to Z$ such that $h = vk$. We can now embed k into a distinguished triangle $X \xrightarrow{k} Z \xrightarrow{w} \Sigma X' \xrightarrow{-\Sigma t} \Sigma X$ and by (TR2) together with the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, "h"] & Z \arrow[r, "w"] \arrow[d, "v"] & \Sigma X' \arrow[r, "-\Sigma t"] & \Sigma X \arrow[d, "\Sigma h"] \\
Y \arrow[r, "\mathrm{id}_Y"'] & Y \arrow[r, "0"'] & 0 \arrow[r, "0"'] & \Sigma Y
\end{tikzcd},$$

we have that $ht = 0$.

We now need to show that t is a quasi-isomorphism. However since s is a quasi-isomorphism we have, from the long exact sequence of cohomology for the distinguished triangle $Z \xrightarrow{v} Y \xrightarrow{s} Y' \xrightarrow{u} \Sigma Z$, that $\mathrm{H}^i(Z) = 0$ for all i. This in turn implies, from the long exact sequence of cohomology for the distinguished triangle $X \xrightarrow{g} Z \xrightarrow{w} X' \xrightarrow{t} \Sigma X$, that t is a quasi-isomorphism.

The proof that (ii)$\Rightarrow$(i) is similar.

Finally to show (MS5) let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'$ be distinguished triangles and let $\alpha : X \to X'$ and $\beta : Y \to Y'$ be quasi-isomorphisms such that we have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] & \Sigma X \arrow[d, "\Sigma\beta"] \\ X' \arrow[r, "f'"'] & Y \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X' \end{tikzcd} .$$

By applying the covariant cohomological functor, $H^*(-)$, we obtain the following diagram of long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{H}^i X \arrow[r] \arrow[d, "\alpha_*"] & \mathrm{H}^i Y \arrow[r] \arrow[d, "\beta_*"] & \mathrm{H}^i Z \arrow[r] & \mathrm{H}^{i+1} X \arrow[r] \arrow[d, "\alpha_*"] & \mathrm{H}^{i+1} Y \arrow[r] \arrow[d, "\beta_*"] & \cdots \\ \cdots \arrow[r] & \mathrm{H}^i X' \arrow[r] & \mathrm{H}^i Y' \arrow[r] & \mathrm{H}^i Z' \arrow[r] & \mathrm{H}^{i+1} X' \arrow[r] & \mathrm{H}^{i+1} Y' \arrow[r] & \cdots \end{tikzcd}$$

where α_* and β_* are isomorphisms. Hence by the five lemma there exists a morphism $\gamma : Z \to Z'$ which is a quasi-isomorphism. □

Now that we have that the collection of all quasi-isomorphisms forms a multiplicative system we can formally invert the quasi-isomorphisms of the homotopy category. Furthermore, since the multiplicative system of quasi-isomorphisms is compatible with triangulation we also have by Theorem 2.2.22 that the category obtained is a triangulated category.

Definition 2.2.26. Let $\mathscr{A}$ be an abelian category and let S be the multiplicative system, compatible with triangulation, which consists of all quasi-isomorphisms in $K(\mathscr{A})$. Then the *derived category* of $\mathscr{A}$, denoted $D(\mathscr{A})$, is the triangulated category $S^{-1}K(\mathscr{A})$.

2.3 Derived Functors

2.3.1 K-projective and K-injective Objects

The rest of this chapter is concerned with constructing the (hyper-homological) derived functors. Before we can do so we need to introduce the concepts of K-projective and K-injective objects. These objects perform a similar function in calculating the hyper-homological derived functors to that of projective and injective resolutions in the classical theory.

Definition 2.3.1. Let $\mathscr{C}$ be a category with a multiplicative system S.

(i) An object P of $\mathscr{C}$ is called *K-projective* if, for any morphism $s : X \to Y$ in S, the induced map $\mathrm{Hom}_{\mathscr{C}}(P, s) : \mathrm{Hom}_{\mathscr{C}}(P, X) \to \mathrm{Hom}_{\mathscr{C}}(P, Y)$ is a bijection.

Denote by $\mathscr{P}_{\mathscr{C}}$ the full subcategory of $\mathscr{C}$ consisting of the K-projective objects on $\mathscr{C}$.

(ii) An object I of $\mathscr{C}$ is called *K-injective* if, for any morphism $s : X \to Y$ in S, the induced map $\mathrm{Hom}_{\mathscr{C}}(s, I) : \mathrm{Hom}_{\mathscr{C}}(Y, I) \to \mathrm{Hom}_{\mathscr{C}}(X, I)$ is a bijection.

Denote by $\mathscr{I}_{\mathscr{C}}$ the full subcategory of $\mathscr{C}$ consisting of the K-injective objects of $\mathscr{C}$.

The following results give a number of useful properties for K-projective objects.

Lemma 2.3.2. *Let $\mathscr{T}$ be a triangulated category with a multiplicative system S which is compatible with triangulation. Then for a K-projective object P we have that*

$$Q : \mathrm{Hom}_{\mathscr{T}}(P, X) \to \mathrm{Hom}_{S^{-1}\mathscr{T}}(P, X),$$

given by $f \mapsto \left[\begin{array}{ccc} & P & \\ \diagup\!\!\diagup & & \searrow^{f} \\ P & & X \end{array}\right]$*, is a bijection.*

Proof. We simply have to check that Q is both surjective and injective.

To see that it is surjective let $\left[\begin{array}{ccc} & Z & \\ {}^{s}\swarrow & & \searrow^{u} \\ P & & X \end{array}\right] \in \mathrm{Hom}_{S^{-1}\mathscr{T}}(P, X)$. Then, since P is K-projective, we have that there exists $t \in \mathrm{Hom}_{\mathscr{T}}(P, Z)$ such that $s \circ t = id_P$. Hence we can construct a diagram of the form

$$\begin{tikzcd}
 & & P \arrow[dl, equal] \arrow[dr, "t"] & & \\
 & P \arrow[dl, equal] \arrow[drrr, "ut" near end] & & Z \arrow[dlll, "s"' near start] \arrow[dr, "u"] & \\
P & & & & X
\end{tikzcd}$$

and so $\left[\begin{array}{ccc} & Z & \\ {}^{s}\swarrow & & \searrow^{u} \\ P & & X \end{array}\right] = \left[\begin{array}{ccc} & P & \\ \diagup\!\!\diagup & & \searrow^{ut} \\ P & & X \end{array}\right] = Q(ut)$ and thus Q is surjective.

For injectivity, let $Qf = 0$, i.e. $\left[\begin{array}{ccc} & P & \\ \diagup\!\!\diagup & & \searrow^{f} \\ P & & X \end{array}\right] = \left[\begin{array}{ccc} & P & \\ \diagup\!\!\diagup & & \searrow^{0} \\ P & & X \end{array}\right]$. Hence we have a commutative diagram

$$\begin{tikzcd}
 & & P \arrow[dl, equal] \arrow[dr, equal] & & \\
 & P \arrow[dl, equal] \arrow[drrr, "f" near end] & & P \arrow[dlll, equal] \arrow[dr, "0"] & \\
P & & & & X
\end{tikzcd}$$

Since the diagram is commutative we have that $f = 0$, and thus $Qf = 0$ iff $f = 0$ and so Q is injective. $\square$

Lemma 2.3.3. *Let $\mathscr{T}$ be a triangulated category with a multiplictive system S compatible with the triangulation. Then for $f : X \to Y$ a morphism of K-projective objects of a $\mathscr{T}$ we have that the mapping cone of f is also K-projective.*

Proof. Let $J \xrightarrow{s} K$ be in S. By applying the functors $\mathrm{Hom}_R(-, J)$ and $\mathrm{Hom}_R(-, K)$ to the distinguished triangle $X \xrightarrow{f} Y \to Z \to \Sigma X$, where Z is the mapping cone of f, we get the pair of exact sequences

$$\begin{tikzcd}
... & \mathrm{Hom}_R(X,J) \arrow[l] \arrow[d, "\cong"] & \mathrm{Hom}_R(Y,J) \arrow[l] \arrow[d, "\cong"] & \mathrm{Hom}_R(Z,J) \arrow[l] \arrow[d, dotted] & ... . \arrow[l] \\
... & \mathrm{Hom}_R(X,K) \arrow[l] & \mathrm{Hom}_R(Y,K) \arrow[l] & \mathrm{Hom}_R(Z,K) \arrow[l] & ... \arrow[l]
\end{tikzcd}$$

Since X and Y are K-projective we have that both $\mathrm{Hom}_R(X, J) \cong \mathrm{Hom}_R(X, K)$ and $\mathrm{Hom}_R(Y, J) \cong \mathrm{Hom}_R(Y, K)$ and so $\mathrm{Hom}_R(Z, J) \cong \mathrm{Hom}_R(Z, K)$ and Z is K-projective.

$\square$

Proposition 2.3.4. *Let $\mathscr{T}$ be a triangulated category with a multiplicative system S which is compatible with triangulation. Then for a K-projective object P and a morphism $P \xrightarrow{s} X$ in S we have that* $\mathrm{Hom}_{S^{-1}\mathscr{T}}(X, Y) \cong \mathrm{Hom}_{\mathscr{T}}(P, Y)$

Proof. This follows from the diagram

$$\mathrm{Hom}_{\mathscr{T}}(P, Y) \xrightarrow{Q} \mathrm{Hom}_{S^{-1}\mathscr{T}}(P, Y) \xleftarrow{\cong} \mathrm{Hom}_{S^{-1}\mathscr{T}}(X, Y).$$

By Lemma 2.3.2 we have that Q is an isomorphism and the second isomorphism is due to P and X being isomorphic in $S^{-1}\mathscr{T}$. $\square$

Definition 2.3.5. Let $\mathscr{C}$ be a category with a multiplicative system S and let X be an object of $\mathscr{C}$. Then a *K-projective resolution* of X consists of a K-projective object P together with a morphism $\pi : P \to X$ in S.
If every object in $\mathscr{C}$ has a K-projective resolution then we say that $\mathscr{C}$ has *enough K-projectives*.

Similarly a *K-injective resolution* of X consists of a K-injective object I together with a morphism $\iota : X \to I$ in S.

If every object in $\mathscr{C}$ has a K-injective resolution then we say that $\mathscr{C}$ has *enough K-injectives.*

Example 2.3.6. Consider the case where $\mathscr{C}$ is the category of complexes of modules over some ring, R, and let S be the multiplicative system consisting of all quasi-isomorphisms. Consider the R-module M as the complex

$$\cdots \to 0 \to 0 \to M \to 0 \to 0 \to \cdots$$

then a projective resolution of M

$$P = \cdots \to P_2 \to P_1 \to P_0 \to 0 \to 0 \to \cdots$$

of M. It is easy to see that there exists a morphism $\pi : P \to M$ which is a quasi-isomorphism and thus in S. Furthermore the a projective resolution is a K-projective object in $\mathscr{C}$ and thus P is a K-projective resolution of M. For further details see [30, Example 3.2]

Similarly we also have that a injective resolution of M is also a K-injective resolution.

We therefore have that the category of complexes of modules has enough K-projectives and K-injectives.

The following theorem, concerning the existence of K-projective and K-injective resolutions in the homotopy category of complexes of modules over a ring, is due to Spaltenstein.

Theorem 2.3.7. *Let R be a ring. Then every complex in the homotopy category $K(R)$ has a K-projective and K-injective resolution.*

Proof. See [30, Corollary 3.5 and Proposition 3.11] □

Theorem 2.3.8. *Let $\mathscr{C}$ be a category with a multiplicative system S. Then any morphism $s \in S$ between two K-projective objects of $\mathscr{C}$ is invertible.*

Proof. Let $s : P \to Q$ be a morphism in S between two K-projective objects in $\mathscr{C}$. Then from the definition of K-projective we have the following two bijections

$$\mathrm{Hom}_{\mathscr{C}}(P, P) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(P,s)} \mathrm{Hom}_{\mathscr{C}}(P, Q)$$

and

$$\mathrm{Hom}_{\mathscr{C}}(Q,P) \overset{\mathrm{Hom}_{\mathscr{C}}(Q,s)}{\longrightarrow} \mathrm{Hom}_{\mathscr{C}}(Q,Q).$$

From the second bijection, there exists $t \in \mathrm{Hom}_{\mathscr{C}}(Q,P)$ such that $s \circ t = \mathrm{id}_Q$.

Also $t \circ s \in \mathrm{Hom}_{\mathscr{C}}(P,P)$ and

$$\mathrm{Hom}_{\mathscr{C}}(P,s)(t\circ s) = s\circ(t\circ s) = (s\circ t)\circ s = \mathrm{id}_Q \circ s = s = s\circ \mathrm{id}_P = \mathrm{Hom}_{\mathscr{C}}(P,s)(\mathrm{id}_P).$$

However $\mathrm{Hom}_{\mathscr{C}}(P,s)$ is a bijection and so $t\circ s = \mathrm{id}_P$ and hence t is the inverse of s. □

2.3.2 Derived Functors

We presented earlier in this chapter the classical definition of the derived functors. We now make use of the derived category in order to define the (hyper homological) derived functors. These extend the previous theory of derived functors which only allowed for derived functors between individual objects of an abelian category to allow for derived functors between complexes of objects. Further information regarding derived functors can be found in [16, Chapter I Section 5].

Proposition 2.3.9. *Let $\mathscr{C}$ be a category with a multiplicative system S. Let $\mathscr{D}$ be a full subcategory of $\mathscr{C}$, such that $\tilde{S} = S \cap \mathscr{D}$ is a multiplicative system in $\mathscr{D}$. Assume one of the following conditions holds.*

(i) For any morphism $s : X' \to X$ in S, with $X \in \mathscr{D}$, there exists a morphism $f : X'' \to X'$ such that $X'' \in \mathscr{D}$ and $sf \in S$.

(ii) For any morphism $s : X \to X'$ in S, with $X \in \mathscr{D}$, there exists a morphism $g : X' \to X''$ such that $X'' \in \mathscr{D}$ and $gs \in S$.

Then the natural functor $\tilde{S}^{-1}\mathscr{D} \to S^{-1}\mathscr{C}$ is fully faithful.

Proof. Assume that condition (i) is satisfied.

Let A and B be objects in $\mathscr{D}$ with $\begin{bmatrix} & Z & \\ {}^{s}\swarrow & & \searrow^{u} \\ A & & B \end{bmatrix} \in \mathrm{Hom}_{S^{-1}\mathscr{C}}(A,B)$. Then, since $s : Z \to A$ is in S and $A \in \mathscr{D}$, there exists $f : X \to Z$, where $X \in \mathscr{D}$ and $sf \in S$. Thus there exists a morphism $\begin{bmatrix} & X & \\ {}^{sf}\swarrow & & \searrow^{uf} \\ A & & B \end{bmatrix} \in \mathrm{Hom}_{\tilde{S}^{-1}\mathscr{D}}(A,B)$ such that $\begin{bmatrix} & X & \\ {}^{sf}\swarrow & & \searrow^{uf} \\ A & & B \end{bmatrix} =$ $\begin{bmatrix} & Z & \\ {}^{s}\swarrow & & \searrow^{u} \\ A & & B \end{bmatrix}$ in $S^{-1}\mathscr{C}$ and hence the natural functor is full.

Now let $\left[\begin{tikzcd}[sep=small] & Z \arrow[dl, "s"'] \arrow[dr, "u"] & \\ A & & B\end{tikzcd}\right]$ and $\left[\begin{tikzcd}[sep=small] & Z' \arrow[dl, "t"'] \arrow[dr, "v"] & \\ A & & B\end{tikzcd}\right]$ be morphisms in $\mathrm{Hom}_{\tilde{S}^{-1}\mathscr{D}}(A, B)$ such that $\left[\begin{tikzcd}[sep=small] & Z \arrow[dl, "s"'] \arrow[dr, "u"] & \\ A & & B\end{tikzcd}\right] = \left[\begin{tikzcd}[sep=small] & Z' \arrow[dl, "t"'] \arrow[dr, "v"] & \\ A & & B\end{tikzcd}\right]$ in $S^{-1}\mathscr{C}$. Then there exists the commutative diagram

$$\begin{tikzcd} & & Z'' \arrow[dl, "f"'] \arrow[dr, "g"] & & \\ & Z \arrow[dl, "s"'] \arrow[drrr, "u" near end] & & Z' \arrow[dlll, "t"' near end] \arrow[dr, "v"] & \\ A & & & & B \end{tikzcd}$$

in $\mathscr{C}$ such that $sf = tg \in S$.

However there exists $h : X \to Z''$ such that $X \in \mathscr{D}$ and $sfh \in S$ and so we have a commutative diagram

$$\begin{tikzcd} & & X \arrow[dl, "fh"'] \arrow[dr, "gh"] & & \\ & Z \arrow[dl, "s"'] \arrow[drrr, "u" near end] & & Z' \arrow[dlll, "t"' near end] \arrow[dr, "v"] & \\ A & & & & B \end{tikzcd}$$

in $\mathscr{D}$ such that $sfh = tgh \in S$ and so $\left[\begin{tikzcd}[sep=small] & Z \arrow[dl, "s"'] \arrow[dr, "u"] & \\ A & & B\end{tikzcd}\right] = \left[\begin{tikzcd}[sep=small] & Z' \arrow[dl, "t"'] \arrow[dr, "v"] & \\ A & & B\end{tikzcd}\right]$ in $\tilde{S}^{-1}\mathscr{D}$. Hence the natural functor is faithful.

The proof when condition (ii) holds is similar, the key difference being that it uses roofs of the form $\begin{tikzcd}[sep=small] & Z & \\ A \arrow[ur, "f"] & & B \arrow[ul, "s"']\end{tikzcd}$ instead. $\square$

Corollary 2.3.10. *Let $\mathscr{C}$ be a category with a multiplicative system S.*

(i) Suppose that $\mathscr{C}$ has enough K-projective objects. Then $\tilde{S}^{-1}\mathscr{P}_{\mathscr{C}}$ and $S^{-1}\mathscr{C}$ are equivalent categories.

(ii) Suppose that $\mathscr{C}$ has enough K-injective objects. Then $\hat{S}^{-1}\mathscr{I}_{\mathscr{C}}$ and $S^{-1}\mathscr{C}$ are equivalent categories.

Proof. Follows from Proposition 2.3.9 and [21, Theorem 4.4.1]. $\square$

Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories, such that $K(\mathscr{A})$ has enough K-projectives and let

$F : \mathscr{A} \to \mathscr{B}$ be a covariant functor. Then, from Remark 2.2.12, we have a functor

$$F_K : K(\mathscr{A}) \to K(\mathscr{B}).$$

Let S be a multiplicative system for $\mathscr{A}$, then $\tilde{S} = S \cap \mathscr{P}_{\mathscr{A}}$, is a multiplicative system of $\mathscr{P}_{\mathscr{A}}$. From Corollary 2.3.10 we have an equivalence of categories

$$U : \tilde{S}^{-1}\mathscr{P}_{\mathscr{A}} \to D(\mathscr{A}).$$

Furthermore, from Theorem 2.3.8 we have that the multiplicative system $\tilde{S}$ consists entirely of isomorphisms and thus the quotient functor

$$Q_{\mathscr{P}_{\mathscr{A}}} : \mathscr{P}_{\mathscr{A}} \to \tilde{S}^{-1}\mathscr{P}_{\mathscr{A}}$$

becomes an equivalence of categories.

We can define a functor

$$P : D(\mathscr{A}) \to \mathscr{P}_{\mathscr{A}}$$

by $P = Q^{-1}_{\mathscr{P}_{\mathscr{A}}} \circ U^{-1}$, which sends an object, X, of $D(\mathscr{A})$ to a K-projective object, $P(X)$, in $K(\mathscr{A})$ which is quasi-isomorphic to X.

We can now define a functor

$$LF : D(\mathscr{A}) \to D(\mathscr{B})$$

by $LF(X) = Q_{\mathscr{B}} \circ F_K P(X)$.

$$\begin{tikzcd}
 & K(\mathscr{B}) \arrow[rr, "Q_{\mathscr{B}}"] & & D(\mathscr{B}) \\
K(\mathscr{A}) \arrow[ur, "F_K"] \arrow[rr, "Q_{\mathscr{A}}"] & & D(\mathscr{A}) \arrow[ur, "LF"'] \arrow[dll, "P"'] & \\
\mathscr{P}_{\mathscr{A}} \arrow[u, "\subseteq"] \arrow[rr, "Q_{\mathscr{P}_{\mathscr{A}}}"'] & & \tilde{S}^{-1}\mathscr{P}_{\mathscr{A}} \arrow[u, "U"'] &
\end{tikzcd}$$

Likewise for a contravariant functor $G : \mathscr{A} \to \mathscr{B}$ we can obtain a functor

$$RG : D(\mathscr{A}) \to D(\mathscr{B})$$

where $RG(X) = Q_{\mathscr{B}} \circ G_K P(X)$.

Similarly, if $K(\mathscr{A})$ has enough K-injectives, we can obtain a functor $I : D(\mathscr{A}) \to K(\mathscr{A})$

which sends an object, X, of $D(\mathscr{A})$ to a K-injective object, $I(X)$, in $K(\mathscr{A})$ which is quasi-isomorphic to X. We can therefore define the functors

$$RF : D(\mathscr{A}) \to D(\mathscr{B})$$

and

$$LG : D(\mathscr{A}) \to D(\mathscr{B})$$

where $RF(X) = Q_{\mathscr{B}} \circ F_K I(X)$ and $LG(X) = Q_{\mathscr{B}} \circ G_K I(X)$.

Definition 2.3.11. For a covariant functor $F : \mathscr{A} \to \mathscr{B}$ the associated functors

$$LF : D(\mathscr{A}) \to D(\mathscr{B})$$

and

$$RF : D(\mathscr{A}) \to D(\mathscr{B})$$

defined above are known as the *left and right derived functors* of F respectively.

Similarly for a contravariant functor $G : \mathscr{A} \to \mathscr{B}$ the associated functors

$$LF : D(\mathscr{A}) \to D(\mathscr{B})$$

and

$$RF : D(\mathscr{A}) \to D(\mathscr{B})$$

are also known as the *left and right derived functors* of G respectively.

We shall now show the connection between the classical derived functors and the derived functors defined above.

Let $F : \mathscr{A} \to \mathscr{B}$ be a covariant functor where $\mathscr{A}$ and $\mathscr{B}$ are abelian categories and $\mathscr{A}$ has enough projectives. Then for an object $A \in \mathscr{A}$ the classical left derived functor $L_i F$ is given by $L_i F(A) = \mathrm{H}_i\, F(Q)$ where

$$Q = \cdots \to Q_2 \to Q_1 \to Q_0 \to 0$$

is a projective resolution of A.

We can consider A as a complex in $K(\mathscr{A})$ concentrated in degree 0,

$$A = \cdots \to 0 \to 0 \to A \to 0 \to \cdots .$$

Then since Q is a projective resolution of the object A we have that it is a K-projective resolution of the complex A in $K(\mathscr{A})$ and we can set $P(A) = Q$. Hence the left derived functor $LF(A) = Q_{\mathscr{B}} \circ F_K \circ P(A) = Q_{\mathscr{B}} \circ F_K(Q)$.

However for any object $X \in K(\mathscr{B})$ we have that $H_i \circ Q_B(X) = \mathrm{H}_i(X)$, similarly for a morphism $f : X \to Y$ in $\mathscr{B}$ we have that

$$\mathrm{H}_i \circ Q_{\mathscr{B}}(f) = \mathrm{H}_i \left(\left[\begin{tikzcd}[column sep=small] & X \arrow[dl, equal] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \right] \right) = \mathrm{H}_i(f) \circ \mathrm{H}_i(\mathrm{id}_X) = \mathrm{H}_i(f).$$

Thus $\mathrm{H}_i \circ Q_{\mathscr{B}} = \mathrm{H}_i$. Hence taking the ith-homology of the left derived functor $LF(A)$ gives us $\mathrm{H}_i\, LF(A) = H_i \circ Q_{\mathscr{B}} \circ F_K(Q) = \mathrm{H}_i\, F_K(Q) = L_i F(A)$. By a similar arguement we also have that $\mathrm{H}^i\, RF(A) = R^i F(A)$.

Similarly for a contavarient functor $G : \mathscr{A} \to \mathscr{B}$ we can show that $\mathrm{H}^i\, LG(A) = L^i G(A)$ and $\mathrm{H}_i\, RG(A) = R_i G(A)$.

Thus when considering an object as a complex concentrated in degree 0 we can obtain the classical derived functors by simply taking the homology of the hyper-homological derived functors.

Chapter 3

Differential Graded Algebras

The aim of this chapter is to give an introduction to differential graded homological algebra. In this setting we study differential graded algebras and their differential graded modules rather than the abstract complexes of objects of abelian categorys of the previous chapter. Differential graded algebras and differential graded modules can be viewed, in certain situations, as generalisations of rings and modules. They also have a natural complex structure. It is this natural complex structure which makes differential graded objects an ideal setting for homological algebra and also allows a number of the results and techniques of the previous chapter to be easily extended to the differential graded setting.

In the first part of this chapter we define differential graded algebras and differential graded modules along with some of their basic properties. In particular we define versions of the adjoint functors $- \underset{R}{\otimes} -$ and $\mathrm{Hom}_R(-,-)$ for differential graded modules over a differential graded algebra. We also give two important examples of differential graded algebras, namely the endomorphism differential graded algebra of a perfect complex and the Koszul complex. We proceed to show that the construction of the derived category in the previous chapter can be adapted to the differential graded setting to obtain the derived category of a differential graded algebra. Similarly we also show that the construction of the derived functors can be extended to the differential graded setting and in particular we construct the derived functors $- \overset{\mathrm{L}}{\underset{R}{\otimes}} -$ and $\mathrm{RHom}_R(-,-)$ from $- \underset{R}{\otimes} -$ and $\mathrm{Hom}_R(-,-)$ respectively. Alternative introductions to differential graded algebras can be found in a number of places including in [4, Section 10], [9, Part I Chapter 3] and [1].

In the second part of this chapter we give a selection of special properties which differential graded algebras or differential graded modules may possess and some results associated with these properties. The properties covered are compact differential graded modules, recollement of differential graded algebras, dualising differential graded modules and Gorenstein differential graded algebras. All these properties and their associated results are used extensively in chapters 4 and 5.

3.1 Differential Graded Algebras and Differential Graded Modules

3.1.1 Differential Graded Algebras

Differential graded algebras (or DGAs) are objects with the structure of a graded algebra with the addition of a complex structure. This means that DGAs are a natural generalisation of rings, any ring can be considered as a DGA concentrated in degree zero, while also providing an environment which is ideally suited to carrying out the operations of homological algebra.

Definition 3.1.1. A *Differential Graded Algebra (DGA)* over a commutative base ring, k, is a $\mathbb{Z}$ graded algebra $R = \underset{i\in\mathbb{Z}}{\oplus} R_i$, over k, together with a differential ∂^R, a collection of k-linear maps $\partial_i : R_i \to R_{i-1}$ such that $\partial_i \circ \partial_{i+1} = 0$ and which also satisfy the Leibniz rule, that is, for $r \in R_i$ and $s \in R_j$

$$\partial_{i+j}(rs) = \partial_i(r)s + (-1)^i r\partial_j(s).$$

It is now natural to define morphisms between DGAs.

Definition 3.1.2. Let R and S be DGAs. We define a *morphism of DGAs* $\theta : R \to S$ as a morphism of graded algebras, of degree zero, which is compatible with the differential. Thus θ is a collection of morphisms of k-modules $\theta_i : R_i \to S_i$ such that:

(i) $\theta_{i+j}(r_i r_j) = \theta_i(r_i)\theta_j(r_j)$ for $r_i \in R_i$ and $r_j \in R_j$,

(ii) $\theta(1_R) = 1_S$,

(iii) $\partial_i^S \circ \theta_i = \theta_{i-1} \circ \partial_i^R$ for all i.

This now allows us to consider the category of DGAs which consists of all DGAs and the morphism between them.

Definition 3.1.3. We define *the forgetful functor on DGAs*, $(-)^\natural$, to be the functor which sends a DGA, R, to the graded algebra $R^\natural = \underset{i\in\mathbb{Z}}{\oplus} R_i$.

Thus the natural functor sends any DGA to its underlying graded algebra, i.e. it simply "forgets" about the differential.

A DGA R can be thought of as a complex of k-modules.

$$R = \cdots \to R_1 \overset{\partial_1^R}{\to} R_0 \overset{\partial_0^R}{\to} R_{-1} \to \cdots .$$

A consequence of this is that we can calculate the homology of a DGA as follows.

For a DGA R, set $Z_i(R) = \operatorname{Ker} \partial_i^R$ and $B_i(R) = \operatorname{Im} \partial_{i+1}^R$. We can define the homology groups of R in the normal way by $\mathrm{H}_i(R) = \frac{Z_i(R)}{B_i(R)}$.

For $z_i \in Z_i(R)$ and $z_j \in Z_j(R)$ we have by the Leibniz rule, that $\partial_{i+j}^R(z_i z_j) = 0$ and $\partial(1_R) = 0$. Therefore $z_i z_j \in Z_{i+j}(R)$ and $1_R \in Z_0(R)$.

Thus we have that $Z(R) = \underset{i\in\mathbb{Z}}{\oplus} Z_i(R)$ is a graded algebra. In fact it is a subalgebra of $R^\natural$.

Let $z_i \in Z_i(R)$ and $r_j \in R_j$. Then we have, by the Leibniz rule, that $z_i \partial_j^R(r_j) = (-1)^i \partial_{i+j}^R(z_i r_j)$ and $\partial_j^R(r_j) z_i = \partial_{i+j}^R(r_j z_i)$, so $z_i B_{j-1}(R)$ and $B_{j-1}(R) z_i$ are contained in $B_{i+j-1}(R)$. So $B(R) = \underset{i\in\mathbb{Z}}{\oplus} B_i(R)$ is a graded ideal of $Z(R)$. Thus the homology groups of R form a graded algebra $H(R) = \frac{Z(R)}{B(R)} = \underset{i\in\mathbb{Z}}{\oplus} H_i(R)$.

Definition 3.1.4. For a DGA R we can define the *opposite DGA*, denoted R^{op}. This consists of the same elements as R. However the product is reversed, thus $r \cdot s = (-1)^{|r||s|} sr$, where $\cdot$ denotes multiplication in R^{op}.

We now define some special types of DGAs which we shall work with.

Definition 3.1.5. A DGA R is said to be *commutative* if $rs = (-1)^{|r||s|} sr$, *connective* if $\mathrm{H}_i(R) = 0$ for $i < 0$ and *coconnective* if $\mathrm{H}_i(R) = 0$ for $i > 0$.

The following simple examples show that we can consider some familiar classical algebraic objects as DGAs.

Examples 3.1.6. (i) Let A be an algebra over a commutative base ring, k, then we can consider A as a DGA concentrated in degree 0:

$$\cdots \to 0 \to 0 \to A \to 0 \to 0 \to \cdots .$$

Furthermore a morphism $f : A \to A'$ between two algebras over k gives a morphism of DGAs

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & 0 \arrow[r] \arrow[d, "0"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, "0"] & 0 \arrow[r] \arrow[d, "0"] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & A' \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots . \end{tikzcd}$$

Thus we can think of the category of algebras over k as a subcategory of the category of DGAs over k.

(ii) Any graded algebra $B = \oplus B_i$ can be turned into a DGA with the addition of a trivial differential:

$$B = \cdots \to B_1 \xrightarrow{0} B_0 \xrightarrow{0} B_{-1} \to \cdots .$$

A morphism, $g = \{g_n\}$, between two graded algebras B and B' can be thought of as a morphism of DGAs:

$$\begin{tikzcd} \cdots \arrow[r] & B_{-2} \arrow[r] \arrow[d, "g_{-2}"] & B_{-1} \arrow[r] \arrow[d, "g_{-1}"] & B_0 \arrow[r] \arrow[d, "g_0"] & B_1 \arrow[r] \arrow[d, "g_1"] & B_2 \arrow[r] \arrow[d, "g_2"] & \cdots \\ \cdots \arrow[r] & B'_{-2} \arrow[r] & B'_{-1} \arrow[r] & B'_0 \arrow[r] & B'_1 \arrow[r] & B'_2 \arrow[r] & \cdots . \end{tikzcd}$$

Before we move on it is worth setting out the following terminology and conventions which we shall adopt throughout this document.

- For any element r belonging to a graded object we shall denote the degree of r by $|r|$.
- We shall adhere to the Koszul sign convention; that is to say that whenever two graded elements of degrees m and n are interchanged we introduce a sign $(-1)^{mn}$.
- We shall use homological notation, that is, lower indices and differential of degree -1.

3.1.2 Differential Graded Modules

We now look to give a brief introduction to Differential Graded Modules (or DG-Modules). These are, unsurprisingly, a generalisation of modules to the differential graded setting. As with DGAs DG-modules have an underlying complex structure which makes them ideal for carrying out homological algebra.

Throughout this section let R and S denote DGAs.

Definition 3.1.7. A *Differential Graded left R-module (DG left R-module)* is a graded left $R^\natural$-module $M = \oplus M_i$, so $R_i.M_j \subseteq M_{i+j}$, together with a a differential ∂^M a collection of maps $\partial_i : M_i \to M_{i-1}$ which satisfy $\partial_i^M \circ \partial_{i+1}^M = 0$ and the Leibniz rule, that is, for $r \in R_i$ and $m \in M_j$

$$\partial_{i+j}^M(rm) = \partial_i^R(r)m + (-1)^i r\partial_j^M(m).$$

A *DG right R-module* is similarly defined from a graded right R-module, N, together with a differential ∂^N a collection of maps $\partial_i : N_i \to N_{i-1}$ with $\partial_i^N \circ \partial_{i+1}^N = 0$ and the Leibniz rule that is, for $r \in R_i$ and $n \in N_j$

$$\partial_{i+j}^N(nr) = \partial_j^N(n)r + (-1)^j n\partial_i^N(r).$$

We will often denote a DG left R-module, M, by ${}_RM$ and a DG right R-module, N, by N_R.

Definition 3.1.8. Let M and N be DG left R-modules. A *morphism of DG-modules* $\theta : M \to N$ is a morphism of graded R-modules compatible with the differential. Thus, θ is a collection of morphisms of abelian groups $\theta_i : M_i \to N_i$ such that:

(i) $\theta_{i+j}(r_i m_j) = r_i\theta_j(m_j)$ for $r_i \in R_i$ and $m_j \in M_j$,

(ii) $\partial_i^N \circ \theta_i = \theta_{i-1} \circ \partial_i^M$ for all i.

Observation 3.1.9. Let N_R be a DG right R-module, then N is also a DG left R^{op}-modules via $r \cdot n = (-1)^{|r||n|}nr$. Thus we can identify the DG right R-modules with the DG left R^{op}-modules.

From now on when we refer to DG R-modules we shall mean the DG left R-modules and similarly when we refer to the DG R^{op}-modules we mean the DG right R-modules.

Definition 3.1.10. We define the *category of DG R-modules*, denoted $\mathrm{DMod}(R)$, as the category whose objects are DG left R-modules and whose morphisms are the morphisms of DG left R-modules.

Similarly, we denote the *category of DG R^{op}-modules*, denoted $\mathrm{DMod}(R^{\mathrm{op}})$, as the category whose objects are DG right R-modules (or DG left R^{op}-modules) and whose morphisms are the morphisms of DG right R-modules.

Since a DG module has an underlying graded module structure we can define a natural functor on DG modules analogous to that which we defined on DGAs.

Definition 3.1.11. We define the *forgetful functor on DG R-modules*, $(-)^\natural$, as the functor which sends a DG R-module to the underlying graded $R^\natural$-module by forgetting the differential.

Definition 3.1.12. We define the *suspension (or shift) functor* of DG R-modules to be the functor $\Sigma : \mathrm{DMod}(R) \to \mathrm{DMod}(R)$ which sends a DG R-module, M, to the DG R-module ΣM with entries $(\Sigma M)_i = M_{i-1}$ (i.e. Σ shifts the entries to the left) together with the differential $\partial_i^{\Sigma M} = -\partial_{i-1}^M$. Thus for a element $m \in M_i$ we have that $\Sigma m \in (\Sigma M)_{i-1}$. The R-action on ΣM is defined by $\Sigma(rm) = (-1)^{|r|} r \Sigma m$.

As with a DGAs, a DG R-module ${}_R M$ can be thought of a complex of k-modules

$$\cdots \xrightarrow{\partial^M} M_1 \xrightarrow{\partial^M} M_0 \xrightarrow{\partial^M} M_{-1} \xrightarrow{\partial^M} \cdots$$

and thus a setting for homological algebra.

In particular, as with DGAs, we can define the homology groups of a DG R-module, M, in the normal way. Set $Z_i(M) = \operatorname{Ker} \partial_i^M$ and $B_i(M) = \operatorname{Im} \partial_{i+1}^M$, then the homology groups are $\mathrm{H}_i(M) = \frac{Z_i(M)}{B_i(M)}$.

For $r_i \in Z_i(R)$ and $m_j \in Z_j(M)$ we have, by the Leibniz rule, that $\partial_{i+j}^M(r_i m_j) = 0$ so $r_i m_j \in Z_{i+j}(R)$. Thus we have that $Z(M) = \bigoplus_{i \in \mathbb{Z}} Z_i(M)$ is a graded $Z(R)$-module.

Let $r_i \in Z_i(R)$ and $m_j \in M_j$. Then we have, by the Leibniz rule, that $r_i \partial_j^M(m_j) = (-1)^i \partial_{i+j}^M(r_i m_j)$ so $r_i B_{j-1}(M)$ is contained in $B_{i+j-1}(M)$. Similarly for $r_i \in R_j$ and $m_j \in Z_j(M)$ we have that $\partial_i^R(r_i) m_j = \partial_{i+j}^M(r_i m_j)$ so $B_{i-1}(R) m_j$ is contained in $B_{i+j-1}(M)$. Thus for $\overline{r_i} \in \mathrm{H}_i(R)$ and $\overline{m_j} \in \mathrm{H}_j(M)$, we have that

$$\overline{r_i}\,\overline{m_j} = \overline{r_i m_j} \in \mathrm{H}_{i+j}(M).$$

Hence $\mathrm{H}(M) = \bigoplus_{i \in \mathbb{Z}} H_i(M)$ is a graded $H(R)$-module .

Definition 3.1.13. As with complexes over an abelian group a DG R-module, M, with $\mathrm{H}_i\, M = 0$ for all i is said to be *exact*.

In the theory of modules over rings we can have a bimodule: an object with two different yet compatible module structures at once. We can extend this concept to DG-modules to obtain DG-bimodules as detailed in the following definition.

Definitions 3.1.14. We define a *DG left R-right S-module (or DG R-S^{op}-bimodule)*, M, as a DG left R-module and a DG right S-module, such that the DG module structures are compatible, that is , $r(ms) = (rm)s$ for all $r \in R$, $s \in S$ and $m \in M$. We denote such a DG bimodule by ${}_RM_S$. We refer to DG R-R^{op}-bimodules simply as DG R-bimodules.

A *DG left R-left S-module (DG R-S-bimodule)*, M, is a DG left R-module and a DG left S-module, such that the module structures are compatible, i.e, $s(rm) = (-1)^{|r||s|}r(sm)$. We denote such a DG bimodule by ${}_{R,S}M$. Similarly a *DG right R- right S-module (DG R^{op}-S^{op}-bimodule)*, N, is a DG right R-module and a DG right S-module, such that the module structures are compatible, i.e, $(nr)s = (-1)^{|r||s|}(ns)r$. We denote such a DG bimodule by $N_{R,S}$.

Definition 3.1.15. A DG R-bimodule ${}_RM_R$ over a commutative DGA R is said to be *symmetric* if, for $r \in R$ and $m \in M$, that $rm = (-1)^{|r||m|}mr$. In other words, the left and right structures are determined by each other.

Definition 3.1.16. Let R be a DGA and M an DG-R-module. Then,

(i) M is *homologically bounded to the right* if $\mathrm{H}_i(M) = 0$ for $i \ll 0$.

(ii) M is *homologically bounded to the left* if $\mathrm{H}_i(M) = 0$ for $i \gg 0$.

(iii) M is *homologically bounded* if it is homologically bounded to both the left and to the right.

(iv) M is *degreewise finite* over R if $\mathrm{H}_i(M)$ is finitely generated as an $\mathrm{H}_0(R)$-module, for each i.

(v) M is *finite* over R if it is both degreewise finite and homologically bounded.

The following examples of DG modules build upon the examples of DGAs given in Examples 3.1.6.

Examples 3.1.17. (i) Consider an algebra A as a DGA concentrated in degree zero. Then a DG A-module is the same thing as a complex of A-modules in the ordinary sense.

(ii) Let $R = \oplus R_i$ be a graded k-algebra considered as a DGA with trivial differential. Then a DG left-R-module consists of a graded R-module $M = \oplus M_i$ together with a differential ∂^M which satisfies the property $\partial^M(rm) = (-1)^{|r|}r\partial^M(m)$.

3.1.3 Tensor products and homomorphisms

In classical homological algebra the adjoint pair of functors $-\underset{R}{\otimes}-$ and $\mathrm{Hom}_R(-,-,)$ play very important roles. It is therefore extremely useful to have a corresponding pair of functors for the DG-modules over DGAs. We give the definition of such functors below. Throughout this section R and S will denote DGAs over the base ring k.

Definition 3.1.18. Let M_R be a DG R^{op}-module and ${}_RN$ a DG R-module. Then by applying the forgetful functor we have a graded $(R^\natural)^{\mathrm{op}}$-module $M^\natural$ and a graded $R^\natural$-module $N^\natural$. We define the graded k-module $(M\underset{R}{\otimes}N)^\natural$ by $(M\underset{R}{\otimes}N)^\natural = M^\natural \underset{R^\natural}{\otimes} N^\natural$ where

$$(M^\natural \underset{R^\natural}{\otimes} N^\natural)_n = \{\sum_{i+j=n} m_i\otimes n_j \mid m_i\in M_i \text{ and } n_j\in N_j\}.$$

We can now use this to define the *tensor product*, $M\underset{R}{\otimes}N$, by defining the differential

$$\partial_i^{M\underset{R}{\otimes}N}(m\otimes n) = \partial^M m\otimes n + (-1)^{|m|} m\otimes \partial^N n.$$

If M has a DG S-R^{op}-bimodule structure then the tensor product $M\underset{R}{\otimes}N$ obtains a DG S-module structure via the action $s(m\otimes n) = sm\otimes n$. Similarly if N has a DG R-S^{op}-structure then $M\underset{R}{\otimes}N$ obtains a DG S^{op}-structure via the action $(m\otimes n)s = m\otimes ns$.

Definition 3.1.19. Let M and N be DG R-modules. By applying the forgetful functor we have graded $R^\natural$-modules $M^\natural$ and $N^\natural$. We define the graded k-module $(\mathrm{Hom}_R(M,N))^\natural$ by

$$(\mathrm{Hom}_R(M,N))_i^\natural = \{f : M^\natural \to N^\natural \mid f\ R^\natural \text{ linear with } f(M_j^\natural)\subseteq N_{i+j}^\natural\}.$$

We can now use this to define $\mathrm{Hom}_R(M,N)$, *the complex of DG R-morphisms from M to N*, by defining the differential

$$\partial^{\mathrm{Hom}_R(M,N)}(\beta) = \partial^N\circ\beta - (-1)^{|\beta|}\beta\circ\partial^M.$$

If M has a DG R-S^{op}-bimodule structure then $\mathrm{Hom}_R(M,N)$ obtains the a DG S-module structure via the action $(s.\theta)(m) = (-1)^{|s|(|\theta|+|m|)}\theta(ms)$. Similarly if N has a DG R-S^{op}-structure then $\mathrm{Hom}_R(M,N)$ obtains a DG S^{op}-structure via the action $(\theta.s)(m) = (-1)^{|m||s|}\theta(m)s$.

As with modules over rings, the previous two definitions give us a pair of bifunctors:

$$- \underset{R}{\otimes} - : \mathrm{DMod}(R^{\mathrm{op}}) \times \mathrm{DMod}(R) \to \mathrm{DMod}(k)$$

and

$$\mathrm{Hom}_R(-,-) : \mathrm{DMod}(R) \times \mathrm{DMod}(R) \to \mathrm{DMod}(k).$$

These bifunctors gain DG-module structures when one or both of the entries is a DG-bimodule in the following ways: For $- \underset{R}{\otimes} -$ let ${}_RM_S$ be a DG R-S^{op}-bimodule, N_R a DG R^{op}-module and ${}_SL$ a DG S-module. Then $M \underset{S}{\otimes} L$ has a DG R-module structure via the action $r(m \otimes n) = rm \otimes n$ and $N \underset{R}{\otimes} M$ has a DG S^{op}-module structure via the action $(n \otimes m)s = n \otimes ms$.

Similarly, for the bifunctor $\mathrm{Hom}_R(-,-)$, let ${}_RM_S$ be a DG R-S^{op}-bimodule and ${}_RN$ a DG R-module. Then $\mathrm{Hom}_R(M,N)$ has a DG S-module structure via the action $(s\theta)(m) = \theta(ms)$ and $\mathrm{Hom}_R(N,M)$ has a DG S^{op}-module structure via the action $\theta s(m) = (\theta(m))s$.

We now conclude this section with some useful standard isomorphisms involving these bifunctors, for proofs that these are indeed isomorphisms see [6, A.2.7-A.2.11].

Adjointness.
Let ${}_RM_S$ be a a DG R-S^{op}-module. Then the functor $M \underset{S}{\otimes} -$ is the left adjoint of the functor $\mathrm{Hom}_R(M,-)$. Hence, for a DG S-module ${}_SL$ and a DG R-module ${}_RN$ we have an adjunction isomorphism:

$$\mathrm{Hom}_S(L, \mathrm{Hom}_R(M,N)) \cong \mathrm{Hom}_R(M \underset{S}{\otimes} L, N),$$

described by $\gamma \mapsto (m \otimes l \mapsto \gamma(l)(m))$ and $(l \mapsto (m \mapsto \beta(l \otimes m))) \leftarrow\!\shortmid \beta$.

Associativity of tensor products.
Let L be a DG R^{op}-module, M a DG R-S^{op}-module and N a DG S-module. Then we have the associativity isomorphism:

$$L \underset{R}{\otimes} (M \underset{S}{\otimes} N) \cong (L \underset{R}{\otimes} M) \underset{S}{\otimes} N, \text{ given by } l \otimes (m \otimes n) \mapsto (l \otimes m) \otimes n.$$

Interaction with the suspension functor.
Let L be a DG R^{op}-module and let M and N be DG R-modules. Then for $n \in \mathbb{Z}$ we have the following isomorphisms:

$$\mathrm{Hom}_R(M,N) \cong \mathrm{Hom}_R(\Sigma^n M, \Sigma^n N).$$

Given by $\phi \mapsto \Sigma^n \phi$, where, for $\phi \in (\mathrm{Hom}_R(M,N))_i$ and $m \in M_j = (\Sigma^n M)_{j-n}$. While $\Sigma^n \phi$ is given by

$$(\Sigma^n \phi)(m) = (-1)^{in} \phi(m) \in (\Sigma^n N)_{i+j-n}.$$

$$\mathrm{Hom}_R(M, \Sigma^n N) \cong \Sigma^n \mathrm{Hom}_R(M, N).$$

This is in fact an equality. To see this let $m \in M_j$. Then

$$\theta \in (\mathrm{Hom}_R(M, \Sigma^n N)_i \Leftrightarrow \theta(m) \in (\Sigma^n N)_{i+j} = N_{i+j-n} \Leftrightarrow \theta \in (\Sigma^n \mathrm{Hom}_R(M, N))_i$$

so $\mathrm{Hom}_R(M^\natural, \Sigma^n N^\natural) = \Sigma^n \mathrm{Hom}_R(M^\natural, N^\natural)$. It is straightforward to check the the differentials are also equal.

$$\mathrm{Hom}_R(\Sigma^n M, N) \cong \Sigma^{-n} \mathrm{Hom}_R(M, N),$$

This is essentially the composition of the previous two isomorphisms.

$$(\Sigma^n L) \underset{R}{\otimes} M \cong \Sigma^n (L \underset{R}{\otimes} M).$$

This isomorphism is actually an equality.

$$L \underset{R}{\otimes} (\Sigma^n M) \cong \Sigma^n (L \underset{R}{\otimes} M).$$

Given by $l \otimes m \mapsto (-1)^{|l|n} l \otimes m$.

Where Σ above denotes the suspension functor for the appropriate categories.

Swap Isomorphism.
Let M be a DG R-module, N a DG S^{op}-module and L a DG R-S^{op}-bimodule. Then we have the swap isomorphism:

$$\mathrm{Hom}_R(M, \mathrm{Hom}_{S^{\mathrm{op}}}(N, L)) \cong \mathrm{Hom}_{S^{\mathrm{op}}}(N, \mathrm{Hom}_R(M, L))$$

where $\phi \mapsto (n \mapsto (m \mapsto (-1)^{|m||n|} \phi(m)(n)))$.

3.1.4 Examples of DGAs

Two common examples of differential graded algebras are the endomorphism DGAs of perfect complexes of modules and the Koszul complexes. We now give outlines of the constructions of both of these examples.

Example 3.1.20 (Endomorphism DGAs)**.** The following paragraphs, which mirror those of [12, Setup 4.1], give the definition and some properties of the endomorphism DGAs of perfect complexes of modules.

Let A be a noetherian local commutative ring and L a bounded complex of finitely generated projective A-modules with $H(L) \neq 0$. Consider $\mathscr{E} = \mathrm{Hom}_A(L, L)$ as a complex of A-modules.

We can define multiplication on $\mathscr{E}$ by composition: Let $\epsilon \in \mathscr{E}_i$. Then ϵ is a A-linear map $L \xrightarrow{\epsilon} \Sigma^{-i}L$. Also let $\epsilon' \in \mathscr{E}_j$. Then we define the product $\epsilon\epsilon'$ as the composition $\Sigma^{-j}(\epsilon) \circ \epsilon'$. This is an A-linear map $L \xrightarrow{\epsilon\epsilon'} \Sigma^{-(i+j)}L$, hence $\epsilon\epsilon' \in \mathscr{E}_{i+j}$. With this multiplication we have that $\mathscr{E}$ is a DGA.

Furthermore the A-complex L becomes a DG left $\mathscr{E}$-module via the scalar multiplication $\epsilon l = \epsilon(l)$ for $\epsilon \in \mathscr{E}$ and $l \in L$. This $\mathscr{E}$-structure is compatible with the A-structure, so L is a DG left A-left $\mathscr{E}$-module. Moreover the identification map

$$ {}_{\mathscr{E}}\mathscr{E}_{\mathscr{E}} \xrightarrow{\cong} \mathrm{Hom}_A({}_{A,\mathscr{E}}L, {}_{A,\mathscr{E}}L) $$

is an isomorphism.

Finally, since A is commutative, each element $a \in A$ gives a chain map $L \xrightarrow{a} L$ given by multiplication by a. This chain map is in $\mathscr{E}_0$. Hence we can define the morphism of DGAs,

$$ A \xrightarrow{\phi_{\mathscr{E}}} \mathscr{E}, \ a \mapsto (L \xrightarrow{a} L). $$

Example 3.1.21 (Koszul Complexes)**.** The following paragraphs give the definition and some properties of Koszul complexes.

Let A be a noetherian local commutative ring, and $\mathbf{a} = (a_1, \ldots, a_n)$ be a sequence of elements of the maximal ideal of A. The Koszul complex, $K(\mathbf{a})$, of $\mathbf{a}$ is the DGA which is the exterior algebra $\bigwedge F$ on the free module $F = Ae_1 \oplus \cdots \oplus Ae_n$, together with the differential

$$ \partial_j^{K(\mathbf{a})}(e_{s_1} \wedge \cdots \wedge e_{s_j}) = \sum_i (-1)^{i+1} a_{s_i} e_{s_1} \wedge \cdots \wedge \widehat{e_{s_i}} \wedge \cdots \wedge e_{s_j}, $$

where the hat indicates that e_{s_i} is left out of the wedge product.

$K(\mathbf{a})$ is a commutative DGA.

Since the degree zero component of $K(\mathbf{a})$ is A itself we can define a morphism DGAs

$$ A \xrightarrow{\phi_{K(\mathbf{a})}} K(\mathbf{a}). $$

3.1.5 The Derived Category for DGAs

In this section we construct the derived category of a DGA. The results in this section correspond to the analogous results for the construction of the derived category of an abelian category. We begin by obtaining the homotopy category for a DGA and show that it is a triangulated category. We then construct the derived category from the homotopy category by inverting the multiplicative set of all quasi-isomorphisms.

Throughout this section let R denote a DGA.

Remark 3.1.22. Note that if we consider a DG module as a complex, i.e. if we forget about the DG module structure, then the following definitions and results match the results for the construction of the derived category of abelian categories in Chapter 2 Sections 2.2.2 and 2.2.4.

As with morphisms of complexes of abelian groups we can define when two morphisms of DG-modules are homotopic.

Definition 3.1.23. Let $f, g : M \to N$ be morphisms of DG-R-modules. Then f and g are said to be *homotopic* if there is a morphisms of graded modules $h = (h_n)$ of degree 1, i.e. $h_n : M_n \to N_{n+1}$, such that

$$f_n - g_n = d^N_{n+1} h_n + h_{n-1} d^M_n$$

for all $n \in \mathbb{Z}$. Such a collection of maps h is called a *homotopy*.

A morphism of DG R-modules f is called *null homotopic* if it is homotopic to the zero map. We denote all the null homotopic maps from M to N by $\mathrm{Null}(M, N)$.

This allows us to define the homotopy category of a DGA.

Definition 3.1.24. We define the *homotopy category* of R, denoted $K(R)$, to be the category whose objects are DG R-modules and whose morphisms are of the form

$$\mathrm{Hom}_{K(R)}(M, N) = \frac{\mathrm{Hom}_{\mathrm{DMod}(R)}(M, N)}{\mathrm{Null}(M, N)}.$$

So, as with the case of abelian categories, the objects of the homotopy category for a DGA, R, are simply the DG R-modules and the morphisms are equivalence classes of the form $f + \mathrm{Null}(X, Y)$ where $f \in \mathrm{Hom}_{\mathrm{DMod}(R)}(M, N)$. As before we shall denote these equivalence classes of morphisms by $\overline{f}$ and the composition of these classes, $\overline{g} \circ \overline{f} = \overline{g \circ f}$, is well defined.

As was the case with the homotopy category of an abelian category, the homotopy category for a DGA is a triangulated category. We show this in an analogous way to the proof for the abelian category case. We thus need to define the mapping cone for a morphism of DG-modules.

Definition 3.1.25. Let $f : M \to N$ be a morphism of DG-R-modules. The *mapping cone* of f, which we shall denote by $\mathrm{C}(f)$, is the DG R-module:

$$\cdots \to \mathrm{C}(f)_{n+1} \xrightarrow{d_{n+1}^{\mathrm{C}(f)}} \mathrm{C}(f)_n \xrightarrow{d_n^{\mathrm{C}(f)}} \mathrm{C}(f)_{n-1} \longrightarrow \cdots$$

where $(\mathrm{C}(f))^\natural = \Sigma M^\natural \oplus N^\natural$ and $d_n^{\mathrm{C}(f)} = (-d_{n-1}^M, d_n^N + f_{n-1})$.

Theorem 3.1.26. *The homotopy category $K(R)$ defined above is a triangulated category.*

Proof. This is analogous to the proof of 2.2.11. We define distinguished triangles of $K(R)$ to be those triangles isomorphic in $K(R)$ to triangles of the form

$$M \xrightarrow{f} N \xrightarrow{i} \mathrm{C}(f) \xrightarrow{p} \Sigma M.$$

The suspension functor is simply the suspension functor for DG R-Modules as defined in 3.1.12.

It remains to verify that the axioms for triangulated categories hold. For the details see [14, Chapter IV, Theorem 1.9]. □

There is a link between the morphisms of the homotopy category and the bifunctor $\mathrm{Hom}_R(-,-)$ as the following theorem shows.

Theorem 3.1.27. *Let M and N be DG R-modules. Then*

$$\mathrm{H}_0 \mathrm{Hom}_R(M,N) = \mathrm{Hom}_{K(R)}(M,N).$$

Proof. Recall that

$$\mathrm{H}_0 \mathrm{Hom}_R(M,N) = \frac{Z_0 \mathrm{Hom}_R(M,N)}{B_0 \mathrm{Hom}(M,N)},$$

where $Z_0 \mathrm{Hom}_R(M,N) = \mathrm{Ker}\, \partial_0^{\mathrm{Hom}_R(M,N)}$ and $B_0 \mathrm{Hom}_R(M,N) = \mathrm{Im}\, \partial_1^{\mathrm{Hom}_R(M,N)}$. $\mathrm{Ker}\, \partial_0^{\mathrm{Hom}_R(M,N)}$ consists of all $R^\natural$-homomorphisms $M^\natural \to N^\natural$ such that

$$\partial^{\mathrm{Hom}_R(M,N)}(\mu) = 0,$$

i.e. $\partial^N \circ \mu - \mu \circ \partial^M = 0$ so $\partial^N \mu = \mu \partial^M$. Thus $Z_0 \operatorname{Hom}_R(M, N)$ consists of all morphisms of DG R-modules from M to N.

$\operatorname{Im} \partial_1^{\operatorname{Hom}_R(M,N)}$ consists of all morphisms $\mu : M^\natural \to N^\natural$ for which

$$\mu = \partial(\rho)$$

for some $\rho : M \to \Sigma N$, i.e $\mu = \partial^N \circ \rho + \rho \circ \partial^M$. Thus $B_0 \operatorname{Hom}_R(M, N)$ consists of all nullhomotopic morphisms from M to N.

Therefore

$$\mathrm{H}_0 \operatorname{Hom}_R(M, N) = \frac{\operatorname{Hom}_{\mathrm{DMod}(R)}(M, N)}{\operatorname{Null}(M, N)} = \operatorname{Hom}_{K(R)}(M, N).$$

□

We now look to build the derived category of a DGA from its homotopy category. We do this in the same manner as for the the derived category of an abelian category, by localisation at the multiplicative set of all quasi-isomorphisms.

Definition 3.1.28. A morphism $f : M \to N$ in $K(R)$ is called a *quasi-isomorphism* if it induces an isomorphism on the cohomology, that is if $\mathrm{H}(f) : \mathrm{H}(M) \to \mathrm{H}(N)$ is an isomorphism.

Proposition 3.1.29. *The collection of all quasi-isomorphisms in $K(R)$ forms a multiplicative set which is compatible with the triangulation.*

Proof. This is essentially the same as the proof of Proposition 2.2.25. □

We can now use the process of localisation given in Section 2.2.3 to obtain the definition of the derived category of a DGA.

Definition 3.1.30. Let S be the multiplicative system of all quasi-isomorphisms in $K(R)$. Then the *derived category* of R, denoted $D(R)$, is the triangulated category $S^{-1}K(R)$.

Remark 3.1.31. The set theoretic issues surrounding the existence of the derived category mentioned in Remark 2.2.17 are answered for the case of the derived category of a DGA in [18].

Definition 3.1.32. We can also define two subcategories of the derived category $D(R)$, namely the *bounded derived category*

$$D^b(R) = \{M \in D(R) \mid M \text{ is a bounded DG } R\text{-module}\}$$

and the *finite derived category*

$$D^f(R) = \{M \in D(R) \mid M \text{ is a finite DG } R\text{-module}\}.$$

3.1.6 The functors $\mathrm{RHom}_R(-,-)$ and $-\overset{\mathrm{L}}{\otimes}_R -$.

In Section 3.1.3 we defined the bifunctors $-\underset{R}{\otimes}-$ and $\mathrm{Hom}_R(-,-)$ for DG R-modules. By following the techniques of 2.3.2 we now construct the associated derived functors

$$-\underset{R}{\overset{\mathrm{L}}{\otimes}}- : D(R^{\mathrm{op}}) \times D(R) \to D(k)$$

and

$$\mathrm{RHom}_R(-,-) : D(R) \times D(R) \to D(k).$$

Throughout this section let R and S denote DGAs.

Before we can construct the derived functor we need that $-\underset{R}{\otimes}-$ and $\mathrm{Hom}_R(-,-)$ extend to bifunctors on the homotopy category.

Proposition 3.1.33. *The bifunctors* $-\underset{\mathrm{R}}{\otimes}-$ *and* $\mathrm{Hom}_R(-,-)$ *preserve homotopies and hence induce bifunctors*

$$-\underset{\mathrm{R}}{\otimes}- : K(R^{\mathrm{op}}) \times K(R) \to K(k)$$

and

$$\mathrm{Hom}_R(-,-) : K(R) \times K(R) \to K(k).$$

Proof. In the case of the bifunctor $-\underset{R}{\otimes}-$ it suffices to show that the functors $L\underset{R}{\otimes}-$ and $-\underset{R}{\otimes}K$ send nullhomotopic maps to nullhomotopic maps.

Let $f : {}_RM \to {}_RN$ be a nullhomotopic morphism of DG R-modules. Then there exists a morphism of graded modules h such that $f_n = \partial^N_{i+1}h_i + h_{i-1}\partial^M_i$. Applying the functor $L\underset{R}{\otimes}-$ gives a morphism $\mathrm{id}_L\underset{R}{\otimes}f : L\underset{R}{\otimes}M \to L\underset{R}{\otimes}N$. We now need to show that this morphism is also nullhomotopic.

Consider the family of morphisms of graded modules $\mathrm{id}_L\underset{R}{\otimes}h_i$. Then for $m \in M_p$ and

$l \in L_q$ with $p+q=i$ we have that:

$$(\partial_{i+1} \circ \mathrm{id}_L \underset{R}{\otimes} h + \mathrm{id}_L \underset{R}{\otimes} h \circ \partial_i)(l \otimes m)$$

$$= (\partial_{i+1} \circ \mathrm{id}_L \underset{R}{\otimes} h)(l \otimes m) + (\mathrm{id}_L \underset{R}{\otimes} h \circ \partial_i)(l \otimes m)$$

$$= \partial_{i+1}((-1)^q(l \otimes h(m)) + (\mathrm{id}_L \underset{R}{\otimes} h)(\partial_q(l) \otimes m + (-1)^q l \otimes \partial_p(m))$$

$$= (-1)^q \partial_q(l) \otimes h(m) + l \otimes \partial_{p+1}(h(m)) + (-1)^{q+1}\partial_q(l) \otimes h(m) + l \otimes h(\partial_p(m))$$

$$= l \otimes \partial_{p+1}(h(m)) + l \otimes h(\partial_p(m)) = l \otimes (\partial_{p+1}(h(m)) + h(\partial_p(m)))$$

$$= (\mathrm{id}_L \underset{R}{\otimes} f_p)(l \otimes m)$$

Thus

$$(\mathrm{id}_L \underset{R}{\otimes} f)_i = (\partial_{i+1}^{L \underset{R}{\otimes} N} \circ \mathrm{id}_L \underset{R}{\otimes} h + \mathrm{id}_L \underset{R}{\otimes} h \circ \partial_i^{L \underset{R}{\otimes} M})$$

and so $L \underset{R}{\otimes} -$ sends nullhomotopic maps to nullhomotopic maps. We can show that $- \underset{R}{\otimes} K$ sends nullhomotopic maps to nullhomotopic maps in a similar fashion.

Thus the bifunctor $- \underset{R}{\otimes} -$ preserves homotopies.

The proof for the bifunctor $\mathrm{Hom}_R(-,-)$ is similar. □

We can look to apply the methods of section 2.3.2 to obtain the derived functors $- \overset{\mathrm{L}}{\underset{R}{\otimes}} -$ and $\mathrm{RHom}_R(-,-)$. We begin by observing that the homotopy category of a DGA has enough K-projectives and K-injectives.

Proposition 3.1.34. *The homotopy category of a DGA R, denoted $K(R)$ has enough K-projective and K-injective objects.*

Proof. For K-projectives see [4, 10.12.2.4 to 10.12.2.6] or [18, Section 3.1] and for K-injectives see [18, Section 3.2]. □

The following definition deals with the case where we have DG bimodules.

Definition 3.1.35. Let ${}_RM_R$ be a DG R-bimodule. A *biprojective resolution* of ${}_RM_R$ is a quasi-isomorphism of DG R-bimodules ${}_RP_R \to {}_RM_R$ such that ${}_RP_R$ is K-projective as both a DG R-module and a DG R^{op}-module.

Similarly a *biinjective resolution* of ${}_RM_R$ is a quasi-isomorphism of DG R-bimodules ${}_RM_R \to {}_RI_R$ such that ${}_RI_R$ is K-injective as both a DG R-module and a DG R^{op}-module.

While K-projective resolutions always exist the same can not be said for biprojective resolutions, however the following proposition provides some situations in which they do.

Proposition 3.1.36. *Let ${}_RM_R$ be a DG R-bimodule. Then ${}_RM_R$ has a biprojective resolution if one of the following conditions hold:*

(i) *R is commutative and M is symmetric.*

(ii) *$R_i = 0$ for $i \ll 0$ and $R^\natural$ is a projective k-module.*

(iii) *k is a field.*

Proof. See [10, Proposition 1.3]. $\square$

Now define the functors $P(-)$ and $I(-)$ which send a DG R-module to an arbitrary K-projective resolution and K-injective resolution respectively. We also recall from Definition 2.2.18 the functor $Q : K(R) \to D(R)$ which sends a DG R-module to itself and a morphism, $f : M \to N$ in $K(R)$ to $\left[\begin{tikzcd}[row sep=small, column sep=small] & M \arrow[dl, equal] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}\right]$ in $D(R)$.

We can now construct the bifunctor

$$- \overset{\mathrm{L}}{\underset{R}{\otimes}} - : D(R^{\mathrm{op}}) \times D(R) \to D(k).$$

For a given $X \in D(R^{\mathrm{op}})$ we have a functor

$$X \underset{R}{\otimes} - : K(R) \to K(k).$$

The left derived functor of this is

$$Q(X \underset{R}{\otimes} P(-)) : D(R) \to D(k),$$

where $Q(- \underset{R}{\otimes} P(-))$ is a bifunctor.

Similarly, for a given $Y \in D(R)$ we have a functor

$$- \underset{R}{\otimes} Y : K(R^{\mathrm{op}}) \to K(k)$$

. The left derived functor of this is

$$Q(P(-) \underset{R}{\otimes} Y) : D(R^{\mathrm{op}}) \to D(k),$$

where $Q(P(-) \underset{R}{\otimes} -)$ is also a bifunctor.

For all $X \in R^{\mathrm{op}}$ and $Y \in R$ we have natural quasi-isomorphisms $P(X) \xrightarrow{\pi_X} X$ and $P(Y) \xrightarrow{\pi_Y} Y$ in $K(R^{\mathrm{op}})$ and $K(R)$ respectively.

This gives us, for all X in $D(R^{\mathrm{op}})$ and Y in $D(R)$, the following diagram

$$\begin{tikzcd}
Q(X \underset{R}{\otimes} P(Y)) & & Q(P(X) \underset{R}{\otimes} Y) \\
& Q(P(X) \underset{R}{\otimes} P(Y)) \arrow[ul, "Q(\pi_X \underset{R}{\otimes} \mathrm{id})"] \arrow[ur, "Q(\mathrm{id} \underset{R}{\otimes} \pi_Y)"'] &
\end{tikzcd}$$

where $Q(- \underset{R}{\otimes} P(-))$, $Q(P(-) \underset{R}{\otimes} -)$ and $Q(P(-) \underset{R}{\otimes} P(-))$ are all bifunctors and the morphisms are natural.

Furthermore for all $X \in D(R^{\mathrm{op}})$ and $Y \in D(R)$ we have that $Q(\pi_X \underset{R}{\otimes} P(Y))$ and $Q(P(X) \underset{R}{\otimes} \pi_Y)$ are isomorphisms and so give rise to natural equivalences of bifunctors. Thus we have that the bifunctors $Q(- \underset{R}{\otimes} P(-))$ and $Q(P(-) \underset{R}{\otimes} -)$ are naturally equivalent.

Thus we define the left derived bifunctor $- \overset{\mathrm{L}}{\underset{R}{\otimes}} -$ by

$$X \overset{\mathrm{L}}{\underset{R}{\otimes}} - = Q(X \underset{R}{\otimes} P(-))$$

and

$$- \overset{\mathrm{L}}{\otimes} Y = Q(P(-) \underset{R}{\otimes} Y).$$

So we have a bifunctor $- \overset{\mathrm{L}}{\underset{R}{\otimes}} - : D(R^{\mathrm{op}}) \times D(R) \to D(k)$ which can be calculated by replacing either the first or second term with its K-projective resolution.

We can by a similar process construct the bifunctor

$$\mathrm{RHom}_R(-,-) : D(R) \times D(R) \to D(k).$$

For a given $X \in D(R)$ we have a functor

$$\mathrm{Hom}_R(X, -) : K(R) \to K(k),$$

with right derived functor

$$RF_X(-) = Q(\mathrm{Hom}_R(X, I(-))) : D(R) \to D(k),$$

where $Q(\mathrm{Hom}_R(-, I(-)))$ is a bifunctor. Similarly, for a given $Y \in D(R)$ we have a functor

$$\mathrm{Hom}_R(-, Y) : K(R) \to K(k),$$

with the right derived functor

$$RG_Y(-) = Q(\mathrm{Hom}_R(P(-), Y)) : D(R) \to D(k),$$

where $Q(\mathrm{Hom}_R(P(-), -))$ is a bifunctor.

For all $X \in R$ we have natural quasi-isomorphisms $P(X) \xrightarrow{\pi_X} X$ and $X \xrightarrow{\iota_X} I(X)$ in $K(R^{\mathrm{op}})$ and $K(R)$ respectively.

So, for all X and Y in $D(R)$ we have the following diagram:

$$\begin{tikzcd}
Q(\mathrm{Hom}_R(X,I(Y))) \arrow[dr, "Q(\mathrm{Hom}_R(\pi_X,\mathrm{id}))"'] & & Q(\mathrm{Hom}_R(P(X),Y)) \arrow[dl, "Q(\mathrm{Hom}_R(\mathrm{id},\iota_Y))"] \\
& Q(\mathrm{Hom}_R(P(X),I(Y))) &
\end{tikzcd}$$

where $Q(\mathrm{Hom}_R(X, I(Y)))$, $Q(\mathrm{Hom}_R(P(X), Y))$ and $Q(\mathrm{Hom}_R(P(X), I(Y)))$ are all bifunctors and the morphisms are natural.

Furthermore, for all X and Y in $D(R)$ we have that the morphisms $Q(\mathrm{Hom}_R(\pi_X, I(Y)))$ and $Q(\mathrm{Hom}_R(P(X), \iota_Y))$ are isomorphisms and so give rise to natural equivalences of bifunctors. Thus, the bifunctors $Q(\mathrm{Hom}_R(-, I(-)))$ and $Q(\mathrm{Hom}_R(P(-), -))$ are naturally equivalent.

Thus we can define the right derived bifunctor $\mathrm{RHom}_R(-, -)$ by

$$\mathrm{RHom}_R(X, -) = Q(\mathrm{Hom}_R(X, I(-)))$$

and

$$\mathrm{RHom}_R(-, Y) = Q(\mathrm{Hom}_R(P(-), Y)).$$

Hence we have a bifunctor $\mathrm{RHom}_R(-, -) : D(R) \times D(R) \to D(k)$ which can be calculated by replacing the first term with a K-projective resolution or the second term by a K-injective resolution.

We can now use the standard isomorphisms involving $\mathrm{Hom}_R(-, -)$ and $- \underset{R}{\otimes} -$, given in Section 3.1.3, to obtain corresponding versions for the derived functors $\mathrm{RHom}_R(-, -)$ and $- \overset{\mathrm{L}}{\underset{R}{\otimes}} -$. Thus we have the following isomorphisms:

Adjointness.

Let ${}_RM_S$ be a DG R-S^{op}-bimodule, ${}_SL$ a DG S-module and ${}_RN$ a DG R-module.

We know that the functor $M \underset{S}{\otimes} -$ is the left adjoint of the functor $\operatorname{Hom}_R(M, -)$. Let $P \to L$ be a K-projective resolution of L and $N \to I$ be a K-injective resolution of N. Then we have that

$$\operatorname{RHom}_S(L, \operatorname{RHom}_R(M,N)) \cong \operatorname{Hom}_S(P, \operatorname{Hom}_R(M,I)) \cong \operatorname{Hom}_R(M \underset{S}{\otimes} P, I)$$

$$\cong \operatorname{RHom}_R(M \overset{\mathrm{L}}{\underset{S}{\otimes}} L, N).$$

So we have a version of the adjointness isomorphism for the derived functors $\operatorname{RHom}_R(-,-)$ and $- \overset{\mathrm{L}}{\underset{R}{\otimes}} -$.

We can use a the same method with the other standard isomorphisms given in section 3.1.3 to obtain the derived versions below.

Associativity of tensor products.

Let L_R be a DG R^{op}-module, ${}_RM_S$ a DG R-S^{op}-module and ${}_SN$ a DG S-module. Then we have the associativity isomorphism,

$$L \overset{\mathrm{L}}{\underset{R}{\otimes}} (M \overset{\mathrm{L}}{\underset{S}{\otimes}} N) \cong (L \overset{\mathrm{L}}{\underset{R}{\otimes}} M) \overset{\mathrm{L}}{\underset{S}{\otimes}} N.$$

Interaction with the suspension functor.

Let L_R be a DG R^{op}-module and let ${}_RM$ and ${}_RN$ be DG R-module. Then for $n \in \mathbb{Z}$ we have the following isomorphisms,

$$\operatorname{RHom}_R(M,N) \cong \operatorname{RHom}_R(\Sigma^n M, \Sigma^n N),$$

$$\operatorname{RHom}_R(M, \Sigma^n N) \cong \Sigma^n \operatorname{RHom}_R(M,N),$$

$$\operatorname{RHom}_R(\Sigma^n M, N) \cong \Sigma^{-n} \operatorname{RHom}_R(M,N),$$

$$(\Sigma^n L) \overset{\mathrm{L}}{\underset{R}{\otimes}} M \cong \Sigma^n (L \overset{\mathrm{L}}{\underset{R}{\otimes}} M),$$

$$L \overset{\mathrm{L}}{\underset{R}{\otimes}} (\Sigma^n M) \cong \Sigma^n (L \overset{\mathrm{L}}{\underset{R}{\otimes}} M).$$

Where Σ above denotes the suspension functor for the appropriate category.

Swap Isomorphism.
Let ${}_RM$ be a DG R-module, N_S a DG S^{op}-module and ${}_RL_S$ a DG R-S^{op}-bimodule. Then we have the swap isomorphism,

$$\mathrm{RHom}_R(M, \mathrm{RHom}_{S^{\text{op}}}(N, L)) \cong \mathrm{RHom}_{S^{\text{op}}}(N, \mathrm{RHom}_R(M, L)).$$

3.2 Properties of Differential Graded Algebras and Differential Graded Modules

3.2.1 Compact Differential Graded Modules

In this section we give the definition of a compact object of an arbitrary triangulated category and show that in the case of the derived category of a DGA, R, that the compact objects of $D(R)$ are those objects which are finitely built from R. Having done this we also give two examples of standard isomorphisms involving compact objects as well as the statement of Keller's Theorem which gives some criteria under which two DGAs are derived equivalent.

Notation 3.2.1. For a triangulated category, $\mathscr{T}$, with set indexed coproducts, and an object, B, in $\mathscr{T}$ we denote by $\langle B\rangle$ the triangulated subcategory of $\mathscr{T}$ which consists of the objects built from B using distinguished triangles, suspensions, direct summands and set indexed coproducts.

Definition 3.2.2. Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. An object A in $\mathscr{T}$ is called *compact* if the functor $\mathrm{Hom}_{\mathscr{T}}(A, -)$ respects set indexed coproducts. That is, for any coproduct, $\coprod X_i$, of objects in $\mathscr{T}$ the canonical isomorphism

$$\coprod \mathrm{Hom}_{\mathscr{T}}(A, X_i) \to \mathrm{Hom}_{\mathscr{T}}(A, \coprod X_i).$$

is a bijection.

An object B in $\mathscr{T}$ is called *self compact* if it is compact in the subcategory $\langle B\rangle$, i.e. if the restricted functor $\mathrm{Hom}_{\mathscr{T}}(B, -)|_{\langle B\rangle}$ respects set indexed coproducts.

Proposition 3.2.3. *Let R be a DGA and A and B be DG R-modules such that A is a compact object and B a self compact object in $D(R)$. Then the derived functors* $\mathrm{RHom}_R(A, -)$ *and* $\mathrm{RHom}_R(B, -)|_{\langle B\rangle}$ *respect set indexed coproducts.*

Proof. We want to show that $\mathrm{RHom}_R(A, \coprod X_i) \cong \coprod \mathrm{RHom}_R(A, X_i)$ in $D(k)$, i.e that $\mathrm{H}_n \mathrm{RHom}_R(A, \coprod X_i) \cong \mathrm{H}_n \coprod \mathrm{RHom}_R(A, X_i)$.

Consider $H_n \mathrm{RHom}_R(A, \coprod X_i)$, by replacing A with a K-projective resolution $P \to A$, we get

$$\mathrm{H}_n \mathrm{RHom}_R(P, \coprod X_i) \cong \mathrm{H}_n \mathrm{Hom}_R(P, \coprod X_i) \cong \mathrm{H}_0 \mathrm{Hom}_R(P, \coprod \Sigma^n X_i)$$

$$\cong \mathrm{Hom}_{K(R)}(P, \coprod \Sigma^n X_i) \cong \mathrm{Hom}_{D(R)}(P, \coprod \Sigma^n X_i) \cong \coprod \mathrm{Hom}_{D(R)}(P, \Sigma^n X_i)$$

$$\cong \coprod \mathrm{Hom}_{K(R)}(P, \Sigma^n X_i) \cong \coprod \mathrm{H}_0 \mathrm{Hom}_R(P, \Sigma^n X_i) \cong \mathrm{H}_n \coprod \mathrm{Hom}_R(P, X_i)$$

$$\mathrm{H}_n \coprod \mathrm{RHom}_R(P, \coprod X_i).$$

Hence $\mathrm{H}_n \mathrm{RHom}_R(A, \coprod X_i) \cong \mathrm{H}_n \coprod \mathrm{RHom}_R(A, X_i)$ as required.

The proof for $\mathrm{RHom}_R(B, -)|_{\langle B \rangle}$ is essentially the same. $\square$

Definition 3.2.4. Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Then $\mathscr{T}$ is called *compactly generated* if there exists a set, $\mathscr{S}$, of compact objects of $\mathscr{T}$, for which $\mathrm{Hom}(S, X) = 0 \Rightarrow X = 0$ for all $S \in \mathscr{S}$.

Definition 3.2.5. Let $\mathscr{T}$ be a compactly generated triangulated category, then a set $\mathscr{S}$ of compact objects of $\mathscr{T}$ is called a *generating set* if

(i) $\mathrm{Hom}_{\mathscr{T}}(\mathscr{S}, X) = 0 \Rightarrow X = 0$,

(ii) $\mathscr{S}$ is closed under suspension, i.e. $S \in \mathscr{S} \Rightarrow \Sigma S \in \mathscr{S}$.

Example 3.2.6. For a DGA, R, the derived category $D(R)$ is compactly generated with generating set $\{\Sigma^i R \mid i \in \mathbb{Z}\}$. Alternatively, the set of all compact objects in $D(R)$ is also a generating set.

Definition 3.2.7. Let R be a DGA. A DG R-module M is *finitely built from* ${}_R R$ in $D(R)$ if M can be obtained from ${}_R R$ using finitely many distinguished triangles, suspensions, direct summands and finite coproducts.

Proposition 3.2.8. *Let R be a DGA and let M be finitely built from ${}_R R$ in $D(R)$. Then M is a compact object of $D(R)$.*

Proof. This follows from the fact that R is compact and each of the constructions preserve compactness. $\square$

Theorem 3.2.9. *Let R be a DGA. Then a a DG R-module, M, is finitely built from ${}_RR$ in $D(R)$ if and only if it is a compact object of $D(R)$.*

Proof. Let $\mathscr{S}$ be the set of all objects which are finitely built from ${}_RR$. Then by Proposition 3.2.8, $\mathscr{S}$ consists entirely of compact objects in $D(R)$ and is closed under suspension. We can therefore use Thomason's localisation theorem [23, Theorem 2.1].

Since $\mathscr{S}$ is a generating set for $D(R)$ we have, from [23, Theorem 2.1.2], that the smallest subcategory of $D(R)$ which contains $\mathscr{S}$ that is closed with respect to coproducts and distinguished triangles is $D(R)$ itself.

Now from [23, Theorem 2.1.3] since $\mathscr{S}$ is closed under the formation of triangles and direct summands we have that $\mathscr{S}$ consists of all of the compact objects of $D(R)$. $\square$

We can now present two well known isomorphisms.

Tensor Evaluation
Let ${}_RM$ a DG R-module, ${}_RN_S$ a DG R-S^{op}-bimodule and ${}_SL$ a DG S-module.

Then we have a morphism

$$\mathrm{Hom}_R({}_RM, {}_RN_S) \underset{S}{\otimes} {}_SL \to \mathrm{Hom}_R({}_RM, {}_RN_S \underset{S}{\otimes} {}_SL)$$

given by $\phi \otimes l \mapsto (m \mapsto (-1)^{|m||l|}\phi(m) \otimes l)$.

This gives a morphism involving the derived version of the functors

$$\mathrm{RHom}_R({}_RM, {}_RN_S) \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SL \to \mathrm{RHom}_R({}_RM, {}_RN_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SL).$$

Note that if we set ${}_RM$ to be ${}_RR$ then these morphisms become isomorphisms. Furthermore, since the morphism respects distinguished triangles, suspensions, taking direct summands and finite coproducts, we have for ${}_RM$ finitely built from R in $D(R)$ the Tensor Evaluation isomorphism:

$$\mathrm{RHom}_R({}_RM, {}_RN_S) \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SL \cong \mathrm{RHom}_R({}_RM, {}_RN_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SL).$$

Hom Evaluation
Let ${}_RM$ a DG R-module, ${}_RN_S$ a DG R-S^{op}-bimodule and L_S a DG S^{op}-module.

Then we have a morphism

$$\mathrm{Hom}_{S^{\mathrm{op}}}({}_RN_S, L_S) \underset{R}{\otimes} {}_RM \to \mathrm{Hom}_{S^{\mathrm{op}}}(\mathrm{Hom}_R({}_RM, {}_RN_S), L_S)$$

via $\theta \otimes m \mapsto (\eta \mapsto (-1)^{|\eta||m|}\theta\eta(m))$.

This gives a morphism involving the derived version of the functors

$$\mathrm{RHom}_{S^{\mathrm{op}}}({}_RN_S, L_S) \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_RM \to \mathrm{RHom}_{S^{\mathrm{op}}}(\mathrm{RHom}_R({}_RM, {}_RN_S), L_S).$$

As with the Tensor Evaluation above these morphisms become isomorphisms when we set ${}_RM$ to be ${}_RR$ and, since the morphism respects distinguished triangles, suspensions, taking direct summands and finite coproducts, we have for ${}_RM$ finitely built from R in $D(R)$ the Hom Evaluation isomorphism:

$$\mathrm{RHom}_{S^{\mathrm{op}}}({}_RN_S, L_S) \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_RM \cong \mathrm{RHom}_{S^{\mathrm{op}}}(\mathrm{RHom}_R({}_RM, {}_RN_S), L_S).$$

The following theorem, due to Keller, gives the conditions under which the derived categories of two specific DGAs, are equivalent.

Theorem 3.2.10 (Keller's Theorem)**.** *Let A be a DGA and let N be a K-projective DG A-module which is compact in $D(A)$ such that $\langle N \rangle = D(A)$ and let $\mathscr{H} = \mathrm{End}_A(N)$. Then $D(A) \simeq D(\mathscr{H}^{\mathrm{op}})$.*

Proof. See [18, Theorem 8.2]. □

3.2.2 Recollement of DGAs

The notion behind recollement is that, in certain situations, we can view a triangulated category as being "glued together" from two other triangulated categories. In this section we give the definition of a recollement of triangulated categories followed by a theorem which gives conditions under which we have a recollement of derived categories of DGAs. Both the definition and the theorem are taken from [27].

Definition 3.2.11. A recollement of triangulated categories is a diagram

$$\begin{tikzcd} \mathscr{T}' \arrow[r, "i_*"] & \mathscr{T} \arrow[l, bend right=60, "i^*"'] \arrow[l, bend left=60, "i^!"] \arrow[r, "j^*"] & \mathscr{T}'' \arrow[l, bend right=60, "j_!"'] \arrow[l, bend left=60, "j_*"] \end{tikzcd},$$

consisting of triangulated categories and triangulated functors, satisfying

(i) (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$ and (j^*, j_*) are adjoint pairs.

(ii) $j^* i_* = 0$

(iii) i_*, $j_!$ and j_* are full embeddings.

(iv) For every object $X \in \mathscr{T}$ we have distinguished triangles

  (a) $i_* i^! X \to X \to j_* j^* X \to \Sigma i_* i^! X$

  (b) $j_! j^* X \to X \to i_* i^* X \to \Sigma j_! j^* X$
  where the arrows to and from X are the counit and unit morphisms respectively.

Before we proceed we need the following piece of notation.

Notation 3.2.12. Let $\mathscr{T}$ be a triangulated category and let X be an object of $\mathscr{T}$ then we have a full subcategory

$$X^{\perp} = \{Y \in \mathscr{T} \,|\, \mathrm{Hom}_{\mathscr{T}}(\Sigma^i X, Y) = 0 \text{ for all } i\}.$$

Theorem 3.2.13. *Let R be a DGA and let B and C be DG R-modules. Then the following are equivalent.*

(i) There is a recollement

$$\begin{tikzcd} D(S) \arrow[r, "i_*"] & D(R) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D(T) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} ,$$

where S and T are DGAs which satisfy $i_*({}_S S) \cong B$ and $j_!({}_T T) \cong C$.

(ii) In the derived category $D(R)$, the DG R-modules B and C satisfy the following.

  (a) B is self compact,

  (b) C is compact,

  (c) $B^{\perp} \cap C^{\perp} = 0$ and

  (d) $B \in C^{\perp}$.

Proof. See [27, Theorem 3.3] □

Furthermore, from [27, Remark 3.4], if R, B and C in the theorem above are known we can construct the DGAs S and T. We do this by replacing B and C with K-projective resolutions and setting $\mathscr{E}$ and $\mathscr{F}$ to be the endomorphism DGAs of B and C respectively, then we set $S = \mathscr{E}^{\mathrm{op}}$ and $T = \mathscr{F}^{\mathrm{op}}$.

We can also have that the functors i_*, $i^!$, $j_!$, j^* and j_*, are as follows,

$$\begin{aligned} & & j_!(-) &= RC_T \overset{\mathrm{L}}{\underset{T}{\otimes}} -, \\ i_*(-) &= {}_RB_S \overset{\mathrm{L}}{\underset{S}{\otimes}} -, & j^*(-) &= \mathrm{RHom}_R({}_RC_T, -), \\ i^!(-) &= \mathrm{RHom}_R({}_RB_S, -), & j_*(-) &= \mathrm{RHom}_T({}_TC^*_R, -) \end{aligned}$$

where ${}_TC^*_R = \mathrm{RHom}_R({}_RC_T, {}_RR_R)$.

3.2.3 Dualising DG-Modules

We now introduce the concept of dualising DG-modules. These are DG versions of the dualising complexes for local noetherian rings and display many of the properties of dualising complexes. The main reference for this section is [10] from which the definitions and results are taken.

Before we give the definition of a dualising module it is useful first to have the following notion.

Definition 3.2.14. Let R be a DGA and ${}_RX_R$ a DG R-bimodule. A DG R-module ${}_RM$ is *X-reflexive* if the biduality morphism

$$M \longrightarrow \mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R(M, X), X)$$

is an isomorphism.

Similarly a DG R^{op}-module N_R is *X-reflexive* if the biduality morphism

$$N \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_{R^{\mathrm{op}}}(N, X), X)$$

is an isomorphism.

Definition 3.2.15. Let R be a DGA. We say that a DG R-bimodule ${}_RD_R$ is *dualising* if for any finite DG R-module ${}_RM$ and finite DG R^{op}-bimodule N_R, the following conditions hold:

(i) D has both a biprojective and a biinjective resolution.

(ii) The DG R-module $\mathrm{RHom}_R(M, D)$ and DG R^{op}-module $\mathrm{RHom}_{R^{\mathrm{op}}}(N, D)$ are both finite.

(iii) The DG R-modules M and $D \overset{\mathrm{L}}{\underset{R}{\otimes}} M$ are D-reflexive, as are the DG R^{op}-modules N and $N \overset{\mathrm{L}}{\underset{R}{\otimes}} D$.

(iv) R is D-reflexive as both a DG R-module and a DG R^{op}-module.

Remarks 3.2.16. It is worth noting that a dualising complex for a commutative noetherian ring of finite Krull dimension is also a dualising DG-module when considered in the DG context.

Unlike in the ring theoretical case, where a dualising complex for a commutative noetherian local ring is unique up to the taking of shifts, we do not necessarily have that a dualising DG-module for a DGA is unique.

The following result ,establishes the existence of dualising modules for a number of DGAs.

Proposition 3.2.17. *Let R be a DGA which is finite over the base ring, k. Let C be a dualising complex for k and set $D = \mathrm{RHom}_k(R, C)$. Suppose the following conditions hold.*

(i) D has biprojective and biinjective resolutions.

(ii) For any finite DG R-module ${}_RM$ (respectively finite DG R^{op}-module N_R), the DG k-module $D \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} M$ (respectively $N \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} D$) is C-reflexive.

Then D is a dualising module for R.

Proof. See [10, Proposition 2.6]. □

Proposition 3.2.18. *Let R be a connective DGA which is finite over the base ring k and let C be a dualising complex for k. If either $R_i = 0$ for $i \ll 0$ and $R^\natural$ is a projective k-module or R is commutative then the DG R-bimodule $\mathrm{RHom}_k(R, C)$ is dualising.*

Proof. See [10, Proposition 2.7]. □

Proposition 3.2.19. *Let R be a DGA over a field, k, with the following properties* *(i) R is concentrated in non-positive degrees, i.e. $R_i = 0$ for $i > 0$.*

(ii) $R_0 = k$ and $\mathrm{H}_{-1}(R) = 0$

Then the DG R-bimodule $\mathrm{Hom}_k(R, k)$ is dualising.

Proof. See [10, Proposition 5.2]. □

In particular we have the following two theorems, [10, Theorems 2.1 and 2.4], which establish the existence of dualising modules for both the Koszul complex and endomorphism DGAs.

Theorem 3.2.20. *Let k be a commutative noetherian ring and C a dualising complex for k. Then for K, the Koszul complex on a finite sequence of elements in k, we have that the DG K-bimodule $\mathrm{Hom}_k(K, C)$ is dualising.*

Theorem 3.2.21. *Let k be a commutative noetherian ring and C a dualising complex for k. Let P be a perfect complex of k-modules with $\mathrm{H}(P) \neq 0$ and $\mathscr{E}$ the endomorphism DGA $\mathrm{Hom}_A(P, P)$. Then the DG $\mathscr{E}$-bimodule $\mathrm{Hom}_A(\mathscr{E}, C)$ is dualising.*

3.2.4 Gorenstein Differential Graded Algebras

As with dualising DG modules being a generalisation of the ring theory concept of dualising complexes, the idea of a Gorenstein DGA is a generalisation of the ring theoretical concept of a Gorenstein ring.

There have been a number of attempts to define Gorenstein conditions for some special cases of augmented DGAs, firstly by Felix, Halperin and Thomas in [8], this was followed by Avramov and Foxby who in [2] gave Gorenstein conditions for finite commutative local DGAs.

The definition for a Gorenstein DGA given below is due to Frankild and Jørgensen and was first introduced in [12]. Not only does this definition have the advantage that it does not require any augmentation on the DGAs; it also does not require us to work with only chain or cochain DGAs.

Definition 3.2.22. Let R be a DGA such that $\mathrm{H}_0(R)$ is a noetherian ring. We call R a *Gorenstein* DGA if it satisfies the following properties.

(G1) For $M \in D^f(R)$ and $N \in D^f(R^{\mathrm{op}})$ the biduality morphisms

$${}_RM \to \mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R({}_RM, {}_RR_R), {}_RR_R)$$

and

$$N_R \to \mathrm{RHom}_R(\mathrm{RHom}_{R^{\mathrm{op}}}(N_R, {}_RR_R), {}_RR_R)$$

are isomorphisms.

(G2) The functors $\mathrm{RHom}_R(-, {}_RR_R)$ and $\mathrm{RHom}_{R^{\mathrm{op}}}(-, {}_RR_R)$ send $D^f(R)$ to $D^f(R^{\mathrm{op}})$ and $D^f(R^{\mathrm{op}})$ to $D^f(R)$ respectively.

Remark 3.2.23. Note that if a DGA R is a Gorenstein DGA then, when considered as a DG R-bimodule, R is a dualising DG R-module.

The following results, taken from [10], illustrate that Frankild and Jørgensen's definition of a Gorenstein DGA which we give above coincides with the previous definitions of Gorenstein.

Theorem 3.2.24. *Let R be a commutative DGA which satisfies the following conditions.*

(i) R is concentrated in non-negative degrees, i.e. $R_i = 0$ for $i < 0$.

(ii) R_0 is a local ring with residue field k.

(iii) The $\mathrm{H}_0(R)$-module $\mathrm{H}(R)$ is finitely generated.

Then R is Gorenstein if and only if $\mathrm{rank}_k \mathrm{Ext}_R(k, R) = 1$.

This is [10, Theorem I] and shows that, for the appropriate class of DGAs, Frankild and Jørgensen's definition of a Gorenstein DGA coincides with that given by Avramov and Foxby in [2].

The following result, [10, Theorem 5.3], provides one direction of the corresponding cochain version.

Theorem 3.2.25. *Let R be a DGA over a field, k, with the following properties.*

(i) R is concentrated in non-positive degrees, i.e. $R_i = 0$ for $i > 0$.

(ii) $R_0 = k$ and $\mathrm{H}_{-1}(R) = 0$

(iii) $\mathrm{H}(R)$ is commutative.

If $\mathrm{rank}_k \mathrm{Ext}_R(k, R) = 1$ then R is Gorenstein.

Additionally this also shows that, again for the appropriate class of DGAs, that a DGA which satisfies the Gorenstein conditions of Felix, Halperin and Thomas as given in [8] also satisfies Frankild and Jørgensen's definition.

The next theorem, [10, Theorem 5.5], gives a partial converse.

Theorem 3.2.26. *Let R be a commutative DGA over a field, k, with the following properties.*

(i) R is concentrated in non-positive degrees, i.e. $R_i = 0$ for $i > 0$.

(ii) $R_0 = k$ and $\mathrm{H}_{-1}(R) = 0$.

If R is Gorenstein then $\operatorname{rank}_k \operatorname{Ext}_R(k, R) = 1$.

Chapter 4

Derived Equivalence of Upper Triangular DGAs

The question of when two derived categories of rings are equivalent has been studied extensively. Morita theory answered the question of when two module categories of rings are equivalent. In [28], Rickard applied the concept of tilting modules to develop a version of Morita theory for the derived categories of rings. The question of when the derived categories of DGAs are derived equivalent was answered by Keller, see Theorem 3.2.10.

The approach of Rickard was applied by Ladkani, in [20], to the situation of derived equivalences of upper triangular matrix rings. We will extend the main results from [20] to the more general case of upper triangular matrix DGAs. For this we will make extensive use of the tool of recollements and in particular Theorem 3.2.13 due to Jørgensen. The results in this chapter are an expansion on those presented in [22]

We begin in section 4.1 by introducing the upper triangular matrix DGA

$$\Lambda = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix},$$

where R and S are DGAs and ${}_R M_S$ is a DG R-S^{op}-bimodule. Following from this we define the DG Λ-modules

$$B = \begin{bmatrix} R \\ 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} M \\ S \end{bmatrix}.$$

In Section 4.2, we are able to apply 3.2.13 to construct the recollement of the form

$$\begin{tikzcd} D(R) \arrow[r, "i_*"] & D(\Lambda) \arrow[l, bend right=40, "i^*"'] \arrow[l, bend left=40, "i^!"] \arrow[r, "j^*"] & D(S) \arrow[l, bend right=40, "j_!"'] \arrow[l, bend left=40, "j_*"] \end{tikzcd} ,$$

where $i_*({}_RR) \cong B$ and $j_!({}_SS) \cong C$. By studying this recollement construction we obtain a number of results which we shall utilise in the rest of the chapter.

In section 4.3 we turn our attention to the chapter's main aim, a generalisation of the main theorem from Ladkani to DGA's. To do so we follow a method similar to that used by Ladkani in the proof of [20, Theorem 4.5], by considering the DG-module $T = \Sigma i_*X \oplus j_*j^*\Lambda$ where X is compact and $\langle X \rangle = D(R)$. We can apply Keller's Theorem 3.2.10 to prove the following theorem, our "first attempt" at a generalisation of [20, Theorem 4.5].

Theorem. *Let ${}_RX$ be a compact DG R-module such that $\langle X \rangle = D(R)$. Let ${}_RM_S$ be a DG R-S^{op}-bimodule which is compact as a DG R-module. Then for the DG Λ-module $T = \Sigma i_*X \oplus j_*j^*\Lambda$ set $\mathscr{E} = \mathrm{End}_\Lambda(P)$, where P is a K-projective resolution of T. Then $\mathscr{E}$ is a DGA with $D(\Lambda) \simeq D(\mathscr{E}^{\mathrm{op}})$.*

To improve upon this first attempt we turn our attention to considering the structure of P, the K-projective resolution of T. By doing this we are able to explicitly calculate its endomorphism DGA. This leads to our main result, a complete generalisation of [20, Theorem 4.5] for DGAs.

Theorem. *Let X be a DG R-module such that ${}_RX$ is compact and $\langle {}_RX \rangle = D(R)$. Let ${}_RM_S$ be compact as a DG R-module and let U and V be K-projective resolutions of X and M respectively. Then for the upper triangular differential graded algebras*

$$\Lambda = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix} \quad \text{and} \quad \tilde{\Lambda} = \begin{bmatrix} S & \mathrm{Hom}_R(V, U) \\ 0 & \mathrm{Hom}_R(U, U)^{\mathrm{op}} \end{bmatrix}$$

we have that $D(\Lambda) \simeq D(\tilde{\Lambda})$.

A specific advantage of considering the DGA case, rather than the ring case, is that we can do without a number of constraints required in the case of rings to ensure that the derived equivalence was between two rings. By working with DGAs rather than rings we can do away with such artificial restriction.

We conclude the chapter with a look at some special cases. In the first we reconsider the restriction to the original case considered by Ladkani, involving just rings. We show that by making the same assumptions in our general theorem we obtain the same equivalence as in [20, Theorem 4.5], so we do indeed have a generalisation.

Our second example briefly considers the special case where ${}_RX = {}_RR$. The final example looks at DGAs over a field k with R self dual, in the sense that, $\mathrm{Hom}_k(R, k) \cong R$ as DG R-bimodules. This gives us the following result.

Corollary. *Let R be a self dual finite dimensional DGA and S be a DGA, both over a field k. Let ${}_RM_S$ be compact as a DG R-module. Then*

$$\Lambda = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix} \text{ and } \tilde{\Lambda} = \begin{bmatrix} S & DM \\ 0 & R \end{bmatrix}$$

are derived equivalent.

4.1 Upper Triangular DGAs and their DG Modules

Definition 4.1.1. Let R and S be DGAs and let ${}_RM_S$ be a DG R-S^{op}-bimodule. An *upper triangular matrix DGA*, Λ, takes the form

$$\Lambda = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}.$$

The elements of an upper triangular matrix DGA of degree i have the form

$$\begin{bmatrix} r_i & m_i \\ 0 & s_i \end{bmatrix},$$

where $r_i \in R_i$, $s_i \in S_i$ and $m_i \in M_i$, whilst multiplication and addition are simply given by the matrix multiplication and addition. Finally the differential is defined by

$$\partial^\Lambda \left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \right) = \begin{bmatrix} \partial^R r & \partial^M m \\ 0 & \partial^S s \end{bmatrix}.$$

Notation 4.1.2. Throughout this chapter, unless specified otherwise, let R and S be DGAs over a commutative base ring k and let ${}_RM_S$ be a DG R-S^{op}-bimodule for which there exist a quasi-isomorphism ${}_RV_S \to {}_RM_S$ of DG R-S^{op}-bimodules where V is K-projective as a DG R-module.

Furthermore, let Λ denote the upper triangular matrix DGA $\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$.

With regard to the conditions on the DG R-S^{op}-bimodule ${}_RM_S$, it is worth observing that when the base ring k is a field we have that any DG-bimodule ${}_RM_S$ is quasi-isomorphic to some DG-bimodule ${}_RV_S$, where V is K-projective as a DG-R-module.

Definition 4.1.3. Let $e_R = \begin{bmatrix} \mathrm{id}_R & 0 \\ 0 & 0 \end{bmatrix}$ and $e_S = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{id}_S \end{bmatrix}$. We shall define B and C to be the DG Λ-Modules generated by e_R and e_S respectively. Thus,

$$B = \Lambda e_R \cong \begin{bmatrix} R \\ 0 \end{bmatrix} \text{ and } C = \Lambda e_S \cong \begin{bmatrix} M \\ S \end{bmatrix}.$$

Where B has the differential

$$\partial\left(\begin{bmatrix} r \\ 0 \end{bmatrix}\right) = \left(\begin{bmatrix} \partial^R r \\ 0 \end{bmatrix}\right)$$

and C has the differential

$$\partial\left(\begin{bmatrix} m \\ s \end{bmatrix}\right) = \left(\begin{bmatrix} \partial^M m \\ \partial^S s \end{bmatrix}\right).$$

The aim is to construct a recollement involving these objects. To do this we shall use Theorem 3.2.13 but before we can do this we need to show that the DG-modules B and C satisfy the conditions of the theorem.

Lemma 4.1.4. *$\Lambda \cong B \oplus C$ in $D(\Lambda)$ and hence both B and C are K-projective DG Λ-modules which are compact in $D(\Lambda)$.*

Proof. Define $\Theta : \Lambda \to B \oplus C$ and $\Phi : B \oplus C \to \Lambda$ by

$$\Theta\left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix}\right) = \left(\begin{bmatrix} r \\ 0 \end{bmatrix}, \begin{bmatrix} m \\ s \end{bmatrix}\right)$$

and

$$\Phi\left(\left(\begin{bmatrix} r \\ 0 \end{bmatrix}, \begin{bmatrix} m \\ s \end{bmatrix}\right)\right) = \begin{bmatrix} r & m \\ 0 & s \end{bmatrix}.$$

It is plain to see that Θ and Φ are inverses of each other so we just need to check that they are homomorphisms of DG-modules which we can do directly, beginning with Θ.

Addition:

$$\Theta\left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} + \begin{bmatrix} r' & m' \\ 0 & s' \end{bmatrix}\right) = \Theta\left(\begin{bmatrix} r + r' & m + m' \\ 0 & s + s' \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} r+r' \\ 0 \end{bmatrix}, \begin{bmatrix} m+m' \\ s+s' \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} r \\ 0 \end{bmatrix}, \begin{bmatrix} m \\ s \end{bmatrix} \right) + \left(\begin{bmatrix} r' \\ 0 \end{bmatrix}, \begin{bmatrix} m' \\ s' \end{bmatrix} \right)$$

$$= \Theta \left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \right) + \Theta \left(\begin{bmatrix} r' & m' \\ 0 & s' \end{bmatrix} \right).$$

Λ-linearity:

$$\Theta \left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \begin{bmatrix} r' & m' \\ 0 & s' \end{bmatrix} \right) = \Theta \left(\begin{bmatrix} rr' & rm'+ms' \\ 0 & ss' \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} rr' \\ 0 \end{bmatrix}, \begin{bmatrix} rm'+ms' \\ ss' \end{bmatrix} \right) = \begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \left(\begin{bmatrix} r' \\ 0 \end{bmatrix}, \begin{bmatrix} m' \\ s \end{bmatrix} \right)$$

$$= \begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \Theta \left(\begin{bmatrix} r' & m' \\ 0 & s' \end{bmatrix} \right).$$

Finally we need to show that Θ commutes with the differentials:

$$\partial^{B\oplus C} \circ \Theta \left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \right) = \partial^{B\oplus C} \left(\begin{bmatrix} r \\ 0 \end{bmatrix}, \begin{bmatrix} m \\ s \end{bmatrix} \right) = \left(\begin{bmatrix} \partial^R r \\ 0 \end{bmatrix}, \begin{bmatrix} \partial^M m \\ \partial^S s \end{bmatrix} \right)$$

$$= \Theta \left(\begin{bmatrix} \partial^R r & \partial^M m \\ 0 & \partial^S s \end{bmatrix} \right) = \Theta \circ \partial^\Lambda \left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \right).$$

Φ can be shown to be a homomorphism in a similar way. Hence both Θ and Φ are isomorphisms and so $\Lambda \cong B \oplus C$. □

Lemma 4.1.5. *$B \in C^\perp$ as DG-modules and hence in $D(\Lambda)$.*

Proof. Let $C \xrightarrow{f} B$ be a morphism of DG-modules. It suffices to show that $f = 0$ and since $\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}$ generates C we only need to show that $f \left(\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix} \right) = 0$.

Let $f \left(\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix} \right) = \begin{bmatrix} r \\ 0 \end{bmatrix}$ for some $r \in R$. Then

$$f \left(\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix} \right) = f \left(e_S . \begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix} \right) = e_S f \left(\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{id}_S \end{bmatrix} \begin{bmatrix} r \\ 0 \end{bmatrix} = 0$$

as required, hence $f = 0$ and so $B \in C^{\perp}$. $\square$

Lemma 4.1.6. *$B^{\perp} \cap C^{\perp} = 0$ in $D(\Lambda)$.*

Proof. Let $X \in B^{\perp} \cap C^{\perp}$, then $\mathrm{Hom}_{D(\Lambda)}(\Sigma^i B, X) = 0$ and $\mathrm{Hom}_{D(\Lambda)}(\Sigma^i C, X) = 0$ for each i.

$$H^i X \cong H^i \,\mathrm{Hom}_{\Lambda}(\Lambda, X) \cong \mathrm{Hom}_{K(\Lambda)}(\Lambda, \Sigma^i X)$$

$$\cong \mathrm{Hom}_{D(\Lambda)}(\Lambda, \Sigma^i X) \cong \mathrm{Hom}_{D(\Lambda)}(B \oplus C, \Sigma^i X)$$

$$\cong \mathrm{Hom}_{D(\Lambda)}(B, \Sigma^i X) \oplus \mathrm{Hom}_{D(\Lambda)}(C, \Sigma^i X)$$

$$\cong 0 \oplus 0 = 0$$

for all i. Hence we have that $X \cong 0$ in $D(\Lambda)$ and so $B^{\perp} \cap C^{\perp} = 0$. $\square$

Lemma 4.1.7. *Let $\mathscr{F} = \mathrm{End}_{\Lambda}(B)$ and $\mathscr{G} = \mathrm{End}_{\Lambda}(C)$. Then $\mathscr{F}^{\mathrm{op}} \cong R$ and $\mathscr{G}^{\mathrm{op}} \cong S$ as Differential Graded Algebras.*

Proof. Since $\begin{bmatrix} \mathrm{id}_R \\ 0 \end{bmatrix}$ is a generator of B each element of $\mathrm{End}_{\Lambda}(B)$ depends entirely on where it sends $\begin{bmatrix} \mathrm{id}_R \\ 0 \end{bmatrix}$. For each $r \in R$ define the homomorphism f_r as the element of $\mathscr{F}$ which sends $\begin{bmatrix} \mathrm{id}_R \\ 0 \end{bmatrix}$ to $\begin{bmatrix} r \\ 0 \end{bmatrix}$.

We can now define $\phi : R^{\mathrm{op}} \to \mathscr{F}$ by $\phi(r) = f_r$. Since elements of $\mathscr{F}$ depend entirely on where they send $\begin{bmatrix} \mathrm{id}_R \\ 0 \end{bmatrix}$ this is obviously a bijection. It remains to show that ϕ is a homomorphism.

It is obvious that $\phi(r + r') = \phi(r) + \phi(r')$ and

$$\phi(r)\phi(r')\left(\begin{bmatrix} \mathrm{id}_R \\ 0 \end{bmatrix}\right) = f_r f_{r'}\left(\begin{bmatrix} \mathrm{id}_R \\ 0 \end{bmatrix}\right) = f_r\left(\begin{bmatrix} r' \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} r' & 0 \\ 0 & 0 \end{bmatrix} f_r\left(\begin{bmatrix} \mathrm{id}_R \\ 0 \end{bmatrix}\right) = \begin{bmatrix} r' & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} r \\ 0 \end{bmatrix} = \begin{bmatrix} r'r \\ 0 \end{bmatrix}$$

$$= f_{r'r}\left(\begin{bmatrix} \mathrm{id}_R \\ 0 \end{bmatrix}\right) = \phi(r'r)(e_R) = \phi(r.r')\left(\begin{bmatrix} \mathrm{id}_R \\ 0 \end{bmatrix}\right),$$

where . indicates multiplication in R^{op}.

Furthermore $\partial^{\mathscr{E}} \circ \phi(r) = \partial^{\mathscr{E}}(f_r)$ and for all $b \in B$,

$$\partial^{\mathscr{E}}(f_r)(b) = \partial^B f_r(b) - (-1)^{|r|} f_r \partial^B(b)$$

$$= \partial^B((-1)^{|b||r|} br) - (-1)^{|r|}(-1)^{(|b|-1)|r|}\partial^B(b)r$$

$$= (-1)^{|b||r|}(\partial^B(b)r + (-1)^{|b|} b\partial^R(r)) - (-1)^{|b||r|}\partial^B(b)r$$

$$= (-1)^{|b|(|r|+1)} b\partial^R(r)$$

$\phi \circ \partial^R(r) = f_{\partial^R r}$ and $f_{\partial^R r}(b) = (-1)^{|b|(|r|-1)} b\partial^R(r) = \partial^{\mathscr{F}}(f_r)(b)$.

So we have $\partial^{\mathscr{F}} \circ \phi(r) = \phi \circ \partial^R(r)$.

We therefore have that ϕ is an isomorphism and so $R^{\mathrm{op}} \cong \mathscr{F}$.

Now let $g \in \mathscr{G}$. Since C is generated by $\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}$ we know that g depends entirely on where it sends $\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}$. Let $g\left(\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}\right) = \begin{bmatrix} m \\ s \end{bmatrix}$.

However $g\left(e_S . \begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}\right) = g\left(\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}\right) = e_S g\left(\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}\right) = e_S \begin{bmatrix} m \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ s \end{bmatrix}$, so $m = 0$ and hence $g\left(\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}\right) = \begin{bmatrix} 0 \\ s \end{bmatrix}$. So for each $s \in S$ we can define the homomorphism $g_s \in \mathscr{G}$ as the element of $\mathscr{G}$ which sends $\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}$ to $\begin{bmatrix} 0 \\ s \end{bmatrix}$.

So we can define a map $\theta : S^{\mathrm{op}} \to \mathscr{G}$ sending $s \mapsto g_s$ which is an isomorphism by a proof similar to that for ϕ above and therefore $S^{\mathrm{op}} \cong \mathscr{G}$. $\square$

4.2 A Recollement Situation

In the previous section we showed that the DG Λ-modules B and C satisfy the conditions of Theorem 3.2.13. By applying the theorem we obtain the following recollement.

$$\begin{tikzcd} D(R) \arrow[r, "i_*"] & D(\Lambda) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D(S) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} .$$

Five of the functors are given by

$$\begin{aligned} & & j_!(-) &= {}_{\Lambda}C_S \overset{\mathrm{L}}{\underset{S}{\otimes}} -, \\ i_*(-) &= {}_{\Lambda}B_R \overset{\mathrm{L}}{\underset{R}{\otimes}} -, & j^*(-) &= \mathrm{RHom}_{\Lambda}({}_{\Lambda}C_S, -), \\ i^!(-) &= \mathrm{RHom}_{\Lambda}({}_{\Lambda}B_R, -), & j_*(-) &= \mathrm{RHom}_S({}_SC^*_{\Lambda}, -). \end{aligned}$$

Here ${}_SC^*_{\Lambda} = \mathrm{RHom}_{\Lambda}({}_{\Lambda}C_S, \Lambda)$.

We shall give a number of results, obtained from this recollement, which we will find to be of great use in the next section.

Remarks 4.2.1. We have from Theorem 3.2.13 that $i_*(R) \cong B$ and $j_!(S) \cong C$. Furthermore it is easy to see that the functor $i_*(-) = {}_{\Lambda}B_R \overset{\mathrm{L}}{\underset{R}{\otimes}} -$ sends a DG R-module, X, to the DG Λ-module, $\begin{bmatrix} X \\ 0 \end{bmatrix}$.

Proposition 4.2.2. *The DG S-Λ^{op}-bimodule ${}_SC^*_{\Lambda} = \mathrm{RHom}_{\Lambda}({}_{\Lambda}C_S, \Lambda)$ has the following properties.*

(i) $C^* \cong {}_S\begin{bmatrix} 0 & S \end{bmatrix}_{\Lambda}$ *as a DG-left-S-right-Λ-module where* $\begin{bmatrix} 0 & S \end{bmatrix}$ *has the differential*

$$\partial^{[0\ S]}\left(\begin{bmatrix} 0 & s \end{bmatrix}\right) = \begin{bmatrix} 0 & \partial^S s \end{bmatrix}.$$

(ii) ${}_SC^*_{\Lambda}$ *is a K-projective object over both S and Λ.*

Proof. (i) Note that, since C is K-projective over Λ,

$$\mathrm{RHom}_{\Lambda}({}_{\Lambda}C_S, \Lambda) \cong \mathrm{Hom}_{\Lambda}({}_{\Lambda}C_S, \Lambda).$$

Let $\theta \in \mathrm{Hom}_{\Lambda}(C, \Lambda)$, then $\theta\left(\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}\right) = \begin{bmatrix} r & m \\ 0 & s \end{bmatrix}$.

Then

$$\theta\left(\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}\right) = \theta\left(e_S.\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}\right) = e_S.\theta\left(\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}\right) = e_S\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix},$$

so $\theta\left(\begin{bmatrix}0\\ \mathrm{id}_S\end{bmatrix}\right)=\begin{bmatrix}0&0\\0&s\end{bmatrix}$.

Therefore for every $s\in S$ we can define $\theta_s\in\mathrm{Hom}_\Lambda(C,\Lambda)$ to be the element which sends $\begin{bmatrix}0\\ \mathrm{id}_S\end{bmatrix}$ to $\begin{bmatrix}0&0\\0&s\end{bmatrix}$.

We can now use this to define a map $\Theta:\mathrm{Hom}_\Lambda(C,\Lambda)\to\begin{bmatrix}0&S\end{bmatrix}$ given by $\Theta(\theta_s)=\begin{bmatrix}0&s\end{bmatrix}$. This map is obviously a bijection so we only need to check that it is a homomorphism DG S-Λ^{op}-bimodules.

Addition:

$$\begin{aligned}\Theta(\theta_s+\theta_{s'})=\Theta(\theta_{s+s'})&=\begin{bmatrix}0&s+s'\end{bmatrix}\\&=\begin{bmatrix}0&s\end{bmatrix}+\begin{bmatrix}0&s'\end{bmatrix}\\&=\Theta(\theta_s)+\Theta(\theta_{s'}).\end{aligned}$$

S-linearity:

$$\begin{aligned}(s'\theta_s)\left(\begin{bmatrix}0\\ \mathrm{id}_S\end{bmatrix}\right)&=\theta_s\left(\begin{bmatrix}0\\ \mathrm{id}_S\end{bmatrix}.s'\right)=\theta_s\left(\begin{bmatrix}0\\ s'\end{bmatrix}\right)\\&=\theta\left(\begin{bmatrix}0&0\\0&s'\end{bmatrix}\begin{bmatrix}0\\ \mathrm{id}_S\end{bmatrix}\right)=\begin{bmatrix}0&0\\0&s'\end{bmatrix}\theta_s\left(\begin{bmatrix}0\\ \mathrm{id}_S\end{bmatrix}\right)\\&=\begin{bmatrix}0&0\\0&s'\end{bmatrix}\begin{bmatrix}0&0\\0&s\end{bmatrix}=\begin{bmatrix}0&0\\0&s's\end{bmatrix}=\theta_{s's}\left(\begin{bmatrix}0\\ \mathrm{id}_S\end{bmatrix}\right)\end{aligned}$$

so $s'\theta_s=\theta_{s's}$, and hence

$$\begin{aligned}\Theta(s'\theta_s)=\Theta(\theta_{s's})&=\begin{bmatrix}0&s's\end{bmatrix}\\&=s'\begin{bmatrix}0&s\end{bmatrix}=s'\Theta(\theta_s).\end{aligned}$$

Λ-linearity:

$$\begin{aligned}\left(\theta_s\begin{bmatrix}r&m\\0&s'\end{bmatrix}\right)\left(\begin{bmatrix}0\\ \mathrm{id}_S\end{bmatrix}\right)&=\left(\theta_s\left(\begin{bmatrix}0\\ \mathrm{id}_S\end{bmatrix}\right)\right)\begin{bmatrix}r&m\\0&s'\end{bmatrix}\\&=\begin{bmatrix}0&0\\0&s\end{bmatrix}\begin{bmatrix}r&m\\0&s'\end{bmatrix}=\begin{bmatrix}0&0\\0&ss'\end{bmatrix}=\theta_{ss'}\end{aligned}$$

so $\theta_s \begin{bmatrix} r & m \\ 0 & s' \end{bmatrix} = \theta_{ss'}$ and hence

$$\Theta\left(\theta_s \begin{bmatrix} r & m \\ 0 & s' \end{bmatrix}\right) = \Theta(\theta_{ss'}) = \begin{bmatrix} 0 & ss' \end{bmatrix}$$

$$= \begin{bmatrix} 0 & s \end{bmatrix} \begin{bmatrix} r & m \\ 0 & s' \end{bmatrix} = \Theta(\theta_s) \begin{bmatrix} r & m \\ 0 & s' \end{bmatrix}.$$

Finally we need to show that Θ respects the differentials:

First we observe that

$$\partial^{\mathrm{Hom}_\Lambda(C,\Lambda)}(\theta_s)\left(\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}\right) = (\partial^\Lambda \theta_s - (-1)^{|s|}\theta_s \partial^C)\left(\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}\right)$$

$$= \partial^\Lambda\left(\begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & \partial^S s \end{bmatrix} = \theta_{(\partial^S s)}$$

Using this gives us that

$$\Theta \circ \partial^{\mathrm{Hom}_\Lambda(C,\Lambda)}(\theta_s) = \Theta(\theta_{(\partial^S s)}) = \begin{bmatrix} 0 & \partial^S s \end{bmatrix}$$

$$= \partial^{[\,0\ S\,]}\left(\begin{bmatrix} 0 & s \end{bmatrix}\right) = \partial^{[\,0\ S\,]} \circ \Theta(\theta_s)$$

so $\Theta \circ \partial^{\mathrm{Hom}_\Lambda(C,\Lambda)} = \partial^{[\,0\ S\,]} \circ \Theta(\theta_s)$ as required.

Hence Θ is an isomorphism of DG S-Λ^{op}-bimodules.

(ii) To see that C^* is K-projective as a DG S-module it suffices to observe that $C^* \cong S$ as DG S-modules. It remains to show now that C^* is also K-projective over Λ. We do this by showing that $C^* \cong \begin{bmatrix} 0 & S \end{bmatrix}$ is a direct summand of Λ as a DG-right-Λ-module.

First observe that $\begin{bmatrix} R & M \end{bmatrix}$ is a DG-right-Λ-module with the differential

$$\partial^{[\,R\ M\,]}\left(\begin{bmatrix} r & m \end{bmatrix}\right) = \begin{bmatrix} \partial^R r & \partial^M m \end{bmatrix}.$$

Now define $\Phi : \Lambda_\Lambda \to \begin{bmatrix} R & M \end{bmatrix}_\Lambda \oplus \begin{bmatrix} 0 & S \end{bmatrix}_\Lambda$ such that

$$\Phi\left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix}\right) = \left(\begin{bmatrix} r & m \end{bmatrix}, \begin{bmatrix} 0 & s \end{bmatrix}\right).$$

It is clear that Φ is bijective. It remains to show that it is a homomorphism of DG-modules.

Addition:

$$\begin{aligned}
\Phi\left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} + \begin{bmatrix} r' & m' \\ 0 & s' \end{bmatrix}\right) &= \Phi\left(\begin{bmatrix} r+r' & m+m' \\ 0 & s+s' \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} r+r' & m+m' \end{bmatrix}, \begin{bmatrix} 0 & s+s' \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} r & m \end{bmatrix}, \begin{bmatrix} 0 & s \end{bmatrix}\right) + \left(\begin{bmatrix} r' & m' \end{bmatrix}, \begin{bmatrix} 0 & s' \end{bmatrix}\right) \\
&= \Phi\left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix}\right) + \Phi\left(\begin{bmatrix} r' & m' \\ 0 & s' \end{bmatrix}\right).
\end{aligned}$$

Λ-linearity:

$$\begin{aligned}
\Phi\left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \cdot \begin{bmatrix} r' & m' \\ 0 & s' \end{bmatrix}\right) &= \Phi\left(\begin{bmatrix} rr' & rm'+ms' \\ 0 & ss' \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} rr' & rm'+ms' \end{bmatrix}, \begin{bmatrix} 0 & ss' \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} r & m \end{bmatrix}, \begin{bmatrix} 0 & s \end{bmatrix}\right) \cdot \begin{bmatrix} r' & m' \\ 0 & s' \end{bmatrix} \\
&= \Phi\left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix}\right) \cdot \begin{bmatrix} r' & m' \\ 0 & s' \end{bmatrix}.
\end{aligned}$$

So Φ is a homomorphism of DG-modules. Hence $\Lambda \cong \begin{bmatrix} R & M \end{bmatrix} \oplus \begin{bmatrix} 0 & S \end{bmatrix}$ as right Λ-modules. Therefore $C^*_\Lambda \cong \begin{bmatrix} 0 & S \end{bmatrix}_\Lambda$ is a direct summand of Λ_Λ and thus a K-projective DG Λ^{op}-module.

□

Lemma 4.2.3. *In the set up of the recollement we have that:*

(i) $j^*(\Lambda) \cong {}_S S$ *in* $D(S)$*,*

(ii) $j_*({}_S S) \cong \dfrac{{}_\Lambda C}{{}_\Lambda \begin{bmatrix} M \\ 0 \end{bmatrix}}$ *in* $D(\Lambda)$*.*

Proof. (i) $j^*(\Lambda) = \mathrm{RHom}_\Lambda({}_\Lambda C_S, \Lambda) = {}_S C^* \cong {}_S S$.

(ii) Since C^* is a K-projective DG S-module we have that

$$j_*({}_S S) = \mathrm{RHom}_S({}_S C^*_\Lambda, {}_S S) \cong \mathrm{Hom}_S({}_S C^*_\Lambda, {}_S S) \cong \mathrm{Hom}_S\left({}_S \begin{bmatrix} 0 & S \end{bmatrix}_\Lambda, {}_S S\right).$$

However, the DG S-module ${}_S\begin{bmatrix} 0 & S \end{bmatrix}$ is generated by $\begin{bmatrix} 0 & \mathrm{id}_S \end{bmatrix}$ so, for all $s \in S$, we can define $\phi_s \in \mathrm{Hom}_S({}_S C^*_\Lambda, {}_S S)$ to be the element which sends $\begin{bmatrix} 0 & \mathrm{id}_S \end{bmatrix}$ to s. We can now define the map

$$\Phi : \mathrm{Hom}_S({}_S C^*_\Lambda, {}_S S) \to \frac{{}_\Lambda C}{{}_\Lambda \left[\begin{smallmatrix} M \\ 0 \end{smallmatrix}\right]}$$

given by $\Phi(\phi_s) = \overline{\begin{bmatrix} 0 \\ s \end{bmatrix}}$, where $\overline{\begin{bmatrix} 0 \\ s \end{bmatrix}}$ denotes the element $\begin{bmatrix} 0 \\ s \end{bmatrix} + \begin{bmatrix} M \\ 0 \end{bmatrix}$ in $\frac{{}_\Lambda C}{{}_\Lambda \left[\begin{smallmatrix} M \\ 0 \end{smallmatrix}\right]}$. It is plain that Φ is a bijection so it just remains to show that it is also a homomorphism of DG Λ-modules.

Addition.

$$\Phi(\phi_s + \phi_{s'}) = \Phi(\phi_{s+s'}) = \overline{\begin{bmatrix} 0 \\ s+s' \end{bmatrix}}$$

$$= \overline{\begin{bmatrix} 0 \\ s \end{bmatrix}} + \overline{\begin{bmatrix} 0 \\ s' \end{bmatrix}} = \Phi(\phi_s) + \Phi(\phi_{s'})$$

Λ-linearity.

Note that

$$\left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \phi_{s'}\right)\left(\begin{bmatrix} 0 & 1 \end{bmatrix}\right) = \phi_{s'}\left(\begin{bmatrix} 0 & \mathrm{id}_S \end{bmatrix}\begin{bmatrix} r & m \\ 0 & s \end{bmatrix}\right)$$
$$= \phi_{s'}\left(\begin{bmatrix} 0 & s \end{bmatrix}\right) = s\phi_{s'}\left(\begin{bmatrix} 0 & \mathrm{id}_S \end{bmatrix}\right)$$
$$= \begin{bmatrix} 0 & ss' \end{bmatrix} = \phi_{ss'}\left(\begin{bmatrix} 0 & \mathrm{id}_S \end{bmatrix}\right).$$

Thus $\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \phi_{s'} = \phi_{ss'}$ and so we have that

$$\Phi\left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \phi_{s'}\right) = \Phi(\phi_{ss'}) = \overline{\begin{bmatrix} 0 \\ ss' \end{bmatrix}}$$

$$= \begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \overline{\begin{bmatrix} 0 \\ s' \end{bmatrix}} = \begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \Phi(\phi_{s'}).$$

□

4.3 Derived Equivalences of Upper Triangular DGA's

We are now almost in the position where we can make a start on what is the main aim of this paper, to obtain a generalised version of [20, Theorem 4.5] for upper triangular DGAs. In Theorem 4.3.1, which is the first major step towards our goal, we obtain a derived equivalence between $D(\Lambda)$ and $D(\mathscr{E})$, where $\mathscr{E}$ is the endomorphism DGA of a K-projective resolution of the DG-module $T = \Sigma i_* X \oplus j_* j^* \Lambda$. We then follow this up by constructing a K-projective resolution for T in proposition 4.3.4 which in turn is followed by the structure of the endomorphism DGA $\mathscr{E}$ in Proposition 4.3.8. The remainder of this section is involved with the details of computing a quasi-isomorphism between the DGA $\mathscr{E}$ and the upper triangular matrix DGA $\tilde{\Lambda} = \begin{bmatrix} S & \operatorname{Hom}_R(V,U) \\ 0 & \operatorname{Hom}_R(U,U)^{\mathrm{op}} \end{bmatrix}$ with the final result being Theorem 4.3.13, the main result of this chapter which gives us a derived equivalence between the upper triangular matrix DGAs Λ and $\tilde{\Lambda}$.

We now make our first attempt at generalising [20, Theorem 4.5] for DGAs. For this we follow a similar method as Ladkani, by introducing the DG Λ-module $T = \Sigma i_* X \oplus j_* j^* \Lambda$.

Theorem 4.3.1. *Let X be a DG R-module such that ${}_RX$ is compact and $\langle {}_RX \rangle = D(R)$. Let ${}_RM_S$ be compact as a DG R-module. Let $\mathscr{E} = \operatorname{End}_\Lambda(P)$, where P is a K-projective resolution of $T = \Sigma i_* X \oplus j_* j^* \Lambda$. Then $\mathscr{E}$ is a DGA with $D(\Lambda) \simeq D(\mathscr{E}^{\mathrm{op}})$.*

Proof. Our aim is to apply Keller's theorem 3.2.10. To do this we need to show that T is compact and that $\langle T \rangle \cong D(\Lambda)$. We begin with the compactness of T.

Since T is a direct sum it is sufficient to show that both its direct summands $i_* X$ and $j_* j^* \Lambda$ are compact.

To show that $i_* X$ is compact we first note that by adjointness

$$\operatorname{Hom}_{D(\Lambda)}(i_* X, \coprod A_k) \simeq \operatorname{Hom}_{D(R)}(X, i^!(\coprod A_k))$$

and that $i^!(\coprod A_k) = \operatorname{RHom}_\Lambda(B, \coprod A_k)$. Also, since B is compact it is straightforward to show that $i^!(\coprod A_k) \cong \coprod i^!(A_k)$ in $D(\Lambda)$.

We therefore have that

$$\begin{aligned} \operatorname{Hom}_{D(\Lambda)}(i_* X, \coprod A_k) &\simeq \operatorname{Hom}_{D(R)}(X, i^!(\coprod A_k)) \\ &\cong \operatorname{Hom}_{D(R)}(X, \coprod i^! A_k) \cong \coprod \operatorname{Hom}_{D(R)}(X, i^! A_k) \\ &\simeq \coprod \operatorname{Hom}_{D(\Lambda)}(i_* X, A_k) \end{aligned}$$

so i_*X is compact as required.

To show that $j_*j^*\Lambda$ is compact we observe from Lemma 4.2.3 that $j_*j^*\Lambda \cong \dfrac{C}{\left[\begin{smallmatrix} M \\ 0 \end{smallmatrix}\right]}$. We know that C is compact and since there is a distinguished triangle $\begin{bmatrix} M \\ 0 \end{bmatrix} \to C \to \dfrac{C}{\left[\begin{smallmatrix} M \\ 0 \end{smallmatrix}\right]}$ in $D(\Lambda)$ it is sufficient to show that $\begin{bmatrix} M \\ 0 \end{bmatrix}$ is compact.

Since $\begin{bmatrix} M \\ 0 \end{bmatrix} \cong i_*M = B \overset{L}{\otimes}_R M$ and both B and M are compact we have that

$$\operatorname{Hom}_{D(\Lambda)}(B \overset{\mathrm{L}}{\underset{R}{\otimes}} M, \coprod A_k) \cong H^0 \operatorname{RHom}_\Lambda(B \overset{\mathrm{L}}{\underset{R}{\otimes}} M, \coprod A_k)$$

$$\cong H^0 \operatorname{RHom}_R(M, \operatorname{RHom}_\Lambda(B, \coprod A_k)) \cong H^0 \operatorname{RHom}_R(M, \coprod \operatorname{RHom}_\Lambda(B, A_k))$$

$$\cong \coprod H^0 \operatorname{RHom}_R(M, \operatorname{RHom}_\Lambda(B, A_k)) \cong \coprod H^0 \operatorname{RHom}_\Lambda(B \overset{\mathrm{L}}{\underset{R}{\otimes}} M, A_k)$$

$$\cong \coprod \operatorname{Hom}_{D(\Lambda)}(i_*M, A_k).$$

So $\begin{bmatrix} M \\ 0 \end{bmatrix}$ is compact and since C is also compact we have that $j_*j^*\Lambda \cong \dfrac{C}{\left[\begin{smallmatrix} M \\ 0 \end{smallmatrix}\right]}$ is compact, and so $T = \Sigma i_*X \oplus j_*j^*\Lambda$ is compact.

It remains to show that $\langle T \rangle = D(\Lambda)$. Since $\langle \Lambda \rangle = D(\Lambda)$ it is sufficient to show that $\Lambda \in \langle T \rangle$.

Since $\Lambda \cong B \oplus C$ we only have to show that both B and C are in $\langle T \rangle$.

To show that B is contained in $\langle T \rangle$ we first observe that the functor $i_*(-)$ respects the operations of taking distinguished triangles, set indexed coproducts, quotients and suspensions. This gives us that $i_*(\langle X \rangle) \subseteq \langle i_*(X) \rangle$ for all $X \in D(R)$. Hence

$$B = i_*R \in i_*(D(R)) = i_* \langle X \rangle \subseteq \langle i_*X \rangle \subseteq \langle T \rangle .$$

To show that $C \in \langle T \rangle$ we first observe that $\dfrac{C}{\left[\begin{smallmatrix} M \\ 0 \end{smallmatrix}\right]} \cong j_*j^*\Lambda \in \langle T \rangle$ so if we can show that $\begin{bmatrix} M \\ 0 \end{bmatrix} \in \langle T \rangle$ then C is in $\langle T \rangle$. To show this we first observe that $\langle X \rangle = D(R)$ so ${}_RM$ can be built from X. Since i_* preserves the possible constructions, we can build $\begin{bmatrix} M \\ 0 \end{bmatrix} = i_*M$ from $i_*X \in \langle T \rangle$.

Hence we have that both B and $C \in \langle T \rangle$ and therefore that $\Lambda \in \langle T \rangle$ so $\langle T \rangle = D(\Lambda)$.

We are now in a position to apply Theorem 3.2.10 to get that $D(\Lambda) \simeq D(\mathscr{E}^{op})$. $\square$

Our aim now is to find a K-projective resolution of T in the above theorem so that we can calculate $\mathscr{E}$. For this we first need the following lemmas.

Lemma 4.3.2. *Let U be a K-projective resolution of a DG R-module X. Then $\begin{bmatrix} U \\ 0 \end{bmatrix}$ is a K-projective resolution of the DG Λ-module $\begin{bmatrix} X \\ 0 \end{bmatrix}$.*

Proof. Let J be an exact DG Λ-module. Then

$$\mathrm{Hom}_\Lambda\left(\begin{bmatrix} U \\ 0 \end{bmatrix}, J\right) \cong \mathrm{Hom}_\Lambda(B \otimes_R U, J) \cong \mathrm{Hom}_R(U, \mathrm{Hom}_\Lambda(B, J)).$$

Since both U and B are K-projective we have that this is exact and hence $\begin{bmatrix} U \\ 0 \end{bmatrix}$ is K-projective. $\square$

Remark 4.3.3. From definition of ${}_R M_S$ we have a quasi-isomorphism ${}_R V_S \xrightarrow{f} {}_R M_S$ where V is K-projective over R. Also for the DG R-module ${}_R X$ we can choose a quasi-isomorphism ${}_R U \xrightarrow{g} {}_R X$ where U is a K-projective resolution.

We can now prove the following proposition about the structure of P, a K-projective resolution of T.

Proposition 4.3.4. *Let $T = \Sigma i_* X \oplus j_* j^* \Lambda$ as defined in Theorem 4.3.1. Then T has a K-projective resolution $P = \Sigma \begin{bmatrix} U \\ 0 \end{bmatrix} \oplus W$ over Λ where W is the mapping cone associated with the morphism $\begin{bmatrix} V \\ 0 \end{bmatrix} \xrightarrow{\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]} \begin{bmatrix} M \\ S \end{bmatrix}$.*

Proof. By lemma 4.3.2 we have that $\begin{bmatrix} U \\ 0 \end{bmatrix}$ is a K-projective resolution of $i_* X = \begin{bmatrix} X \\ 0 \end{bmatrix}$ and that $\begin{bmatrix} V \\ 0 \end{bmatrix}$ is a K-projective resolution of $\begin{bmatrix} M \\ 0 \end{bmatrix}$ over Λ.

We now wish to find a K-projective resolution of $j_*j^*\Lambda$. To do this we first recall that $j_*j^*\Lambda = \dfrac{C}{\left[\begin{smallmatrix} M \\ 0 \end{smallmatrix}\right]}$.

We now consider the morphism

$$\begin{bmatrix} V \\ 0 \end{bmatrix} \xrightarrow{\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]} \begin{bmatrix} M \\ S \end{bmatrix}$$

of DG Λ-modules. We can use this to obtain a distinguished triangle

$$\begin{bmatrix} V \\ 0 \end{bmatrix} \xrightarrow{\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]} \begin{bmatrix} M \\ S \end{bmatrix} \longrightarrow W \longrightarrow \Sigma \begin{bmatrix} V \\ 0 \end{bmatrix}$$

in $D(\Lambda)$.

We can now use this to obtain the diagram

$$\begin{tikzcd}
\begin{bmatrix} M \\ 0 \end{bmatrix} \arrow[r] & C \arrow[r] & C/\left[\begin{smallmatrix} M \\ 0 \end{smallmatrix}\right] \arrow[r] & \Sigma \begin{bmatrix} M \\ 0 \end{bmatrix} \\
\begin{bmatrix} V \\ 0 \end{bmatrix} \arrow[u, "\simeq"] \arrow[r] & C \arrow[u, equal] \arrow[r] & W \arrow[u, dotted, "\exists"] \arrow[r] & \Sigma \begin{bmatrix} V \\ 0 \end{bmatrix} \arrow[u, "\simeq"]
\end{tikzcd}$$

of distinguished triangles in $D(\Lambda)$ so there exists a quasi-isomorphism $W \to C/\left[\begin{smallmatrix} M \\ 0 \end{smallmatrix}\right]$.

By Lemma 2.3.3 we also have that W is K-projective and hence a K-projective resolution of $C/\left[\begin{smallmatrix} M \\ 0 \end{smallmatrix}\right] \cong j_*j^*\Lambda$.

We now have K-projective resolutions for both direct summands of T and hence T has the K-projective resolution, $P = \Sigma \begin{bmatrix} U \\ 0 \end{bmatrix} \oplus W$.

□

Now that we have a K-projective resolution for T in Theorem 4.3.1 we can try to calculate the endomorphism DGA, $\mathscr{E} = \mathrm{End}_\Lambda(T)$. However before we do so we need to take a closer look at the mapping cone W.

Remark 4.3.5. W is the mapping cone of $\begin{bmatrix} V \\ 0 \end{bmatrix} \xrightarrow{\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]} \begin{bmatrix} M \\ S \end{bmatrix}$ so

$$W^\natural = \begin{bmatrix} M \\ S \end{bmatrix}^\natural \oplus \Sigma \begin{bmatrix} V \\ 0 \end{bmatrix}^\natural$$

i.e any element $w \in W^\natural$ is of the form $w = \begin{bmatrix} \left[\begin{smallmatrix} m \\ s \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right] \end{bmatrix}$. In addition W is equipped with the differential

$$\partial^W = \begin{bmatrix} \partial^C & \left[\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right] \\ 0 & -\partial^{\left[\begin{smallmatrix} V \\ 0 \end{smallmatrix}\right]} \end{bmatrix}.$$

and the quasi-isomorphism $W \to C/\left[\begin{smallmatrix} M \\ 0 \end{smallmatrix}\right]$ in the proof of proposition 4.3.4 is given by

$$\begin{bmatrix} \left[\begin{smallmatrix} m \\ s \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right] \end{bmatrix} \mapsto \overline{\begin{bmatrix} 0 \\ s \end{bmatrix}}.$$

Lemma 4.3.6. *The mapping cone W is isomorphic, in $K(\Lambda)$, to $\begin{bmatrix} Z \\ S \end{bmatrix}$ where Z is exact and the mapping cone of f.*

Proof. Since Z is the mapping cone of f we have a distinguished triangle in of the form

$$V \xrightarrow{f} M \xrightarrow{h} Z \xrightarrow{k} \Sigma V.$$

We can use this to construct the following distinguished triangle in $K(\Lambda)$

$$\begin{bmatrix} V \\ 0 \end{bmatrix} \xrightarrow{\overline{\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]}} \begin{bmatrix} M \\ S \end{bmatrix} \xrightarrow{\overline{\left[\begin{smallmatrix} h \\ 1 \end{smallmatrix}\right]}} \begin{bmatrix} Z \\ S \end{bmatrix} \xrightarrow{\overline{\left[\begin{smallmatrix} k \\ 0 \end{smallmatrix}\right]}} \Sigma \begin{bmatrix} V \\ 0 \end{bmatrix}.$$

Which in turn we can use to obtain the diagram of distinguished triangles in $K(R)$:

$$\begin{tikzcd}
\begin{bmatrix} V \\ 0 \end{bmatrix} \arrow[r, "\overline{\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]}"] \arrow[d, equal] & \begin{bmatrix} M \\ S \end{bmatrix} \arrow[r, "\overline{\left[\begin{smallmatrix} h \\ 1 \end{smallmatrix}\right]}"] \arrow[d, equal] & \begin{bmatrix} Z \\ S \end{bmatrix} \arrow[r, "\overline{\left[\begin{smallmatrix} k \\ 0 \end{smallmatrix}\right]}"] \arrow[d, dashed, "\exists"] & \Sigma \begin{bmatrix} V \\ 0 \end{bmatrix} \arrow[d, equal] \\
\begin{bmatrix} V \\ 0 \end{bmatrix} \arrow[r, "\overline{\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]}"] & \begin{bmatrix} M \\ S \end{bmatrix} \arrow[r] & W \arrow[r] & \Sigma \begin{bmatrix} V \\ 0 \end{bmatrix}.
\end{tikzcd}$$

Hence there exists an isomorphism $\begin{bmatrix} Z \\ S \end{bmatrix} \to W$.

Finally since Z is the mapping cone of a quasi-isomorphism it is exact. $\square$

Lemma 4.3.7. *Let A and B be DG R-modules. Then*

$$\mathrm{Hom}_R(A,B) \cong \mathrm{Hom}_\Lambda\left(\begin{bmatrix} A \\ 0 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}\right)$$

as complexes of abelian groups.

Furthermore $\mathrm{End}_R(A) \cong \mathrm{End}_\Lambda\left(\begin{bmatrix} A \\ 0 \end{bmatrix}\right)$ *as DGA's.*

Proof. Define $\Theta : \mathrm{Hom}_R(A,B) \to \mathrm{Hom}_\Lambda\left(\begin{bmatrix} A \\ 0 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}\right)$ by $\Theta(\phi) = \begin{bmatrix} \phi & 0 \\ 0 & 0 \end{bmatrix}$.
It is easy to see that Θ is an isomorphism of complexes of abelian groups. In addition for the case $B = A$, Θ becomes an isomorphism of DGA's. □

The following proposition gives the structure of $\mathscr{E}$ which by Theorem 4.3.1 is derived equivalent to the upper triangular matrix DGA Λ.

Proposition 4.3.8. *Let X be a DG R-module such that ${}_RX$ is compact and $\langle {}_RX \rangle = D(R)$. Let ${}_RM_S$ be compact as a DG R-module. Let $\mathscr{E} = \mathrm{End}_\Lambda(P)$, where P is a K-projective resolution of $T = \Sigma i_* X \oplus j_* j^* \Lambda$. Then*

$$\mathscr{E} \cong \begin{bmatrix} \mathrm{Hom}_R(U,U) & \mathrm{Hom}_\Lambda(W, \Sigma \left[\begin{smallmatrix} U \\ 0 \end{smallmatrix}\right]) \\ \mathrm{Hom}_\Lambda(\Sigma \left[\begin{smallmatrix} U \\ 0 \end{smallmatrix}\right], W) & \mathrm{Hom}_\Lambda(W,W) \end{bmatrix}.$$

Where U is a K-projective resolution of X and W is the mapping cone of the morphism $\begin{bmatrix} V \\ 0 \end{bmatrix} \xrightarrow{\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]} \begin{bmatrix} M \\ S \end{bmatrix}$ *where ${}_RV_S$ is a K-projective resolution of ${}_RM_S$ over R.*

Proof. Since $P = \Sigma U \oplus W$ consists of a direct sum we have that

$$\mathscr{E} = \mathrm{Hom}_\Lambda(P,P) = \begin{bmatrix} \mathrm{Hom}_\Lambda(\Sigma \left[\begin{smallmatrix} U \\ 0 \end{smallmatrix}\right], \Sigma \left[\begin{smallmatrix} U \\ 0 \end{smallmatrix}\right]) & \mathrm{Hom}_\Lambda(W, \Sigma \left[\begin{smallmatrix} U \\ 0 \end{smallmatrix}\right]) \\ \mathrm{Hom}_\Lambda(\Sigma \left[\begin{smallmatrix} U \\ 0 \end{smallmatrix}\right], W) & \mathrm{Hom}_\Lambda(W,W) \end{bmatrix}.$$

Furthermore from Lemma 4.3.7 above we have that

$$\mathrm{Hom}_\Lambda\left(\Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}, \Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}\right) \cong \mathrm{Hom}_\Lambda\left(\begin{bmatrix} U \\ 0 \end{bmatrix}, \begin{bmatrix} U \\ 0 \end{bmatrix}\right) \cong \mathrm{Hom}_R(U,U).$$

□

Our attention now turns to obtaining a quasi-isomorphism between the entries of $\mathscr{E}^{\mathrm{op}}$

and the corresponding entries of $\tilde{\Lambda} = \begin{bmatrix} S & \operatorname{Hom}_R(V,U) \\ 0 & \operatorname{Hom}_R(U,U)^{\mathrm{op}} \end{bmatrix}$. This will allow us to construct an isomorphism between the two DGAs.

Lemma 4.3.9. *The complex of abelian groups* $\operatorname{Hom}_\Lambda\left(\Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}, W\right)$ *is exact.*

Proof. For all i we have that

$$\mathrm{H}_i \operatorname{Hom}_\Lambda\left(\Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}, W\right) \cong \mathrm{H}_0 \operatorname{Hom}_\Lambda\left(\begin{bmatrix} U \\ 0 \end{bmatrix}, \Sigma^{i-1} W\right)$$

$$\cong \operatorname{Hom}_{K(\Lambda)}\left(\begin{bmatrix} U \\ 0 \end{bmatrix}, \Sigma^{i-1} W\right) \cong \operatorname{Hom}_{K(\Lambda)}\left(\begin{bmatrix} U \\ 0 \end{bmatrix}, \Sigma^{i-1} \begin{bmatrix} Z \\ S \end{bmatrix}\right).$$

However, for $\theta \in \operatorname{Hom}_\Lambda\left(\begin{bmatrix} U \\ 0 \end{bmatrix}, \Sigma^{i-1} \begin{bmatrix} Z \\ S \end{bmatrix}\right)$ such that $\theta\left(\begin{bmatrix} u \\ 0 \end{bmatrix}\right) = \begin{bmatrix} z \\ s \end{bmatrix}$ for some $u \in U, z \in Z$ and $s \in S$, we have

$$\begin{bmatrix} z \\ s \end{bmatrix} = \theta\left(\begin{bmatrix} u \\ 0 \end{bmatrix}\right) = \theta\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \theta\left(\begin{bmatrix} u \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z \\ s \end{bmatrix} = \begin{bmatrix} z \\ 0 \end{bmatrix}.$$

So $s = 0$ and so $\theta\left(\begin{bmatrix} u \\ 0 \end{bmatrix}\right) = \begin{bmatrix} z \\ 0 \end{bmatrix}$.

Hence $\operatorname{Hom}_\Lambda\left(\Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}, \Sigma^{i-1} \begin{bmatrix} Z \\ S \end{bmatrix}\right) \cong \operatorname{Hom}_\Lambda\left(\Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}, \Sigma^{i-1} \begin{bmatrix} Z \\ 0 \end{bmatrix}\right)$ and by Lemma 4.3.7 this is isomorphic to $\operatorname{Hom}_R(\Sigma U, \Sigma^{i-1} Z)$.

Taking this together with U being K-projective and the exactness of Z gives us that

$$\operatorname{Hom}_{K(\Lambda)}\left(\begin{bmatrix} U \\ 0 \end{bmatrix}, \Sigma^{i-1} \begin{bmatrix} Z \\ S \end{bmatrix}\right) \cong \operatorname{Hom}_{K(\Lambda)}\left(U, \Sigma^{i-1} Z\right)$$

$$\cong \operatorname{Hom}_{D(\Lambda)}\left(U, \Sigma^{i-1} Z\right) \cong 0.$$

Hence $\mathrm{H}^i \operatorname{Hom}_\Lambda\left(\Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}, W\right) \cong 0$ for all i and so $\operatorname{Hom}_\Lambda\left(\Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}, W\right)$ is exact. $\square$

Lemma 4.3.10. *There is a quasi-isomorphism of DGAs*

$$\alpha : S^{\mathrm{op}} \to \mathrm{Hom}_\Lambda(W, W).$$

Proof. Define $\alpha : S^{\mathrm{op}} \to \mathrm{Hom}_\Lambda(W, W)$ by $\alpha(\tilde{s}) = \begin{bmatrix} g_{\tilde{s}} & 0 \\ 0 & l_{\tilde{s}} \end{bmatrix}$ where $g_{\tilde{s}}\left(\begin{bmatrix} m \\ s \end{bmatrix}\right) = (-1)^{|\tilde{s}||s|}\begin{bmatrix} m\tilde{s} \\ s\tilde{s} \end{bmatrix}$ and $l_{\tilde{s}}\left(\begin{bmatrix} v \\ 0 \end{bmatrix}\right) = (-1)^{|\tilde{s}|(|v|+1)}\begin{bmatrix} v\tilde{s} \\ 0 \end{bmatrix}$.

We first want to show that α is a homomorphism of Differential Graded Algebras.

It is straightforward to check that α respects the operations of addition and multiplication. So it remains to check that α is compatible with the differential,

$$\partial^{\mathrm{Hom}_\Lambda(W,W)}(\alpha(\tilde{s})) = \partial^{\mathrm{Hom}_\Lambda(W,W)}\left(\begin{bmatrix} g_{\tilde{s}} & 0 \\ 0 & l_{\tilde{s}} \end{bmatrix}\right)$$

$$= \partial^W \begin{bmatrix} g_{\tilde{s}} & 0 \\ 0 & l_{\tilde{s}} \end{bmatrix} - (-1)^{|\tilde{s}|}\begin{bmatrix} g_{\tilde{s}} & 0 \\ 0 & l_{\tilde{s}} \end{bmatrix}\partial^W$$

$$= \begin{bmatrix} \partial^C & \left[\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right] \\ 0 & -\partial^{\left[\begin{smallmatrix} V \\ 0 \end{smallmatrix}\right]} \end{bmatrix}\begin{bmatrix} g_{\tilde{s}} & 0 \\ 0 & l_{\tilde{s}} \end{bmatrix} - (-1)^{|\tilde{s}|}\begin{bmatrix} g_{\tilde{s}} & 0 \\ 0 & l_{\tilde{s}} \end{bmatrix}\begin{bmatrix} \partial^C & \left[\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right] \\ 0 & -\partial^{\left[\begin{smallmatrix} V \\ 0 \end{smallmatrix}\right]} \end{bmatrix}$$

$$= \begin{bmatrix} \partial^C g_{\tilde{s}} & \left[\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right] l_{\tilde{s}} \\ 0 & -\partial^{\left[\begin{smallmatrix} V \\ 0 \end{smallmatrix}\right]} l_{\tilde{s}} \end{bmatrix} - (-1)^{|\tilde{s}|}\begin{bmatrix} g_{\tilde{s}}\partial^C & g_{\tilde{s}}\left[\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right] \\ 0 & -l_{\tilde{s}}\partial^{\left[\begin{smallmatrix} V \\ 0 \end{smallmatrix}\right]} \end{bmatrix}$$

$$= \begin{bmatrix} \partial^C g_{\tilde{s}} - (-1)^{|\tilde{s}|}g_{\tilde{s}}\partial^C & \left[\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right] l_{\tilde{s}} - (-1)^{|\tilde{s}|}g_{\tilde{s}}\left[\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right] \\ 0 & -\partial^{\left[\begin{smallmatrix} V \\ 0 \end{smallmatrix}\right]} l_{\tilde{s}} + (-1)^{|\tilde{s}|}l_{\tilde{s}}\partial^{\left[\begin{smallmatrix} V \\ 0 \end{smallmatrix}\right]} \end{bmatrix}.$$

Considering each of the terms in this matrix separately, starting with the upper left term, we get

$$\left(\partial^C g_{\tilde{s}} - (-1)^{|\tilde{s}|}g_{\tilde{s}}\partial^C\right)\left(\begin{bmatrix} m \\ s \end{bmatrix}\right)$$

$$= \partial^C\left((-1)^{|\tilde{s}||s|}\begin{bmatrix} m\tilde{s} \\ s\tilde{s} \end{bmatrix}\right) - (-1)^{|\tilde{s}|}g_{\tilde{s}}\left(\begin{bmatrix} \partial^M(m) \\ \partial^S(s) \end{bmatrix}\right)$$

$$= (-1)^{|\tilde{s}||s|}\begin{bmatrix} \partial^M(m\tilde{s}) \\ \partial^S(s\tilde{s}) \end{bmatrix} - (-1)^{|\tilde{s}|}(-1)^{|\tilde{s}|(|s|-1)}\begin{bmatrix} \partial^M(m)\tilde{s} \\ \partial^S(s)\tilde{s} \end{bmatrix}$$

$$= (-1)^{|\tilde{s}||s|}\begin{bmatrix} \partial^M(m)\tilde{s} + (-1)^{|s|}m\partial^S(\tilde{s}) \\ \partial^S(s)\tilde{s} + (-1)^{|s|}s\partial^S(\tilde{s}) \end{bmatrix} - (-1)^{|\tilde{s}||s|}\begin{bmatrix} \partial^M(m)\tilde{s} \\ \partial^S(s)\tilde{s} \end{bmatrix}$$

$$= (-1)^{|\tilde{s}||s|} \begin{bmatrix} (-1)^{|s|} m\partial^S(\tilde{s}) \\ (-1)^{|s|} s\partial^S(\tilde{s}) \end{bmatrix} = (-1)^{|\partial^S(\tilde{s})||s|} \begin{bmatrix} m\partial^S(\tilde{s}) \\ s\partial^S(\tilde{s}) \end{bmatrix}$$

$$= g_{\partial^S(\tilde{s})} \left(\begin{bmatrix} m \\ s \end{bmatrix} \right),$$

So $\partial^C g_{\tilde{s}} - (-1)^{|\tilde{s}|} g_{\tilde{s}} \partial^C = g_{\partial^S(\tilde{s})}$.

By a similar argument we also have for the lower right entry that

$$-\partial^{\left[\begin{smallmatrix} V \\ 0 \end{smallmatrix}\right]} l_{\tilde{s}} + (-1)^{|\tilde{s}|} l_{\tilde{s}} \partial^{\left[\begin{smallmatrix} V \\ 0 \end{smallmatrix}\right]} = l_{\partial^S(\tilde{s})}.$$

Finally for the upper right entry we have

$$\left(\begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix} l_{\tilde{s}} - (-1)^{|\tilde{s}|} g_{\tilde{s}} \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix} \right) \left(\begin{bmatrix} v \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix} \left((-1)^{|\tilde{s}|(|v|+1)} \begin{bmatrix} v\tilde{s} \\ 0 \end{bmatrix} \right) - (-1)^{|\tilde{s}|} g_{\tilde{s}} \left(\begin{bmatrix} f(v) \\ 0 \end{bmatrix} \right)$$

$$(-1)^{|\tilde{s}|(|v|+1)} \begin{bmatrix} f(v\tilde{s}) \\ 0 \end{bmatrix} - (-1)^{|\tilde{s}|} (-1)^{|\tilde{s}||v|} \begin{bmatrix} f(v)\tilde{s} \\ 0 \end{bmatrix}$$

$$(-1)^{|\tilde{s}|(|v|+1)} \begin{bmatrix} f(v)\tilde{s} \\ 0 \end{bmatrix} - (-1)^{|\tilde{s}|(|v|+1)} \begin{bmatrix} f(v)\tilde{s} \\ 0 \end{bmatrix} = 0.$$

Substituting these values back into the matrix gives us that

$$\partial^{\mathrm{Hom}_\Lambda(W,W)}(\alpha(\tilde{s})) = \begin{bmatrix} g_{\partial^S(\tilde{s})} & 0 \\ 0 & l_{\partial^S(\tilde{s})} \end{bmatrix} = \alpha(\partial^S(\tilde{s})).$$

Hence we have that α is a homomorphism of Differential Graded Algebras. It remains to show that it is also a quasi-isomorphism.

Let $\theta \in \mathrm{Hom}_\Lambda\big(W, C/\left[\begin{smallmatrix} M \\ 0 \end{smallmatrix}\right]\big)$ such that $\theta\left(\begin{bmatrix} \left[\begin{smallmatrix} m \\ s \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right] \end{bmatrix}\right) = \overline{\begin{bmatrix} 0 \\ s \end{bmatrix}}$.
This quasi-isomorphism gives us the homomorphism of complexes of abelian groups

$$\mathrm{Hom}_\Lambda(W, \theta) : \mathrm{Hom}_\Lambda(W, W) \to \mathrm{Hom}_\Lambda\left(W, C/\left[\begin{smallmatrix} M \\ 0 \end{smallmatrix}\right]\right).$$

Define $\beta = \mathrm{Hom}_\Lambda(W, \theta) \circ \alpha : S^{\mathrm{op}} \to \mathrm{Hom}_\Lambda\left(W, C/\left[\begin{smallmatrix} M \\ 0 \end{smallmatrix}\right]\right)$. Then β is a homomorphism of complexes of abelian groups with

$$\beta(\tilde{s})\left(\begin{bmatrix}\left[\begin{smallmatrix} m\\ s\end{smallmatrix}\right]\\ \left[\begin{smallmatrix} v\\ 0\end{smallmatrix}\right]\end{bmatrix}\right)=\theta\left((-1)^{|s||\tilde{s}|}\begin{bmatrix}\left[\begin{smallmatrix} m\tilde{s}\\ s\tilde{s}\end{smallmatrix}\right]\\ \left[\begin{smallmatrix} v\\ 0\end{smallmatrix}\right]\end{bmatrix}\right)=(-1)^{|s||\tilde{s}|}\overline{\begin{bmatrix}0\\ s\tilde{s}\end{bmatrix}}.$$

Now let $\psi\in\mathrm{Hom}_{\Lambda}\left(W,C/\left[\begin{smallmatrix} M\\ 0\end{smallmatrix}\right]\right)$. Then for any $m\in M,s\in S$ and $v\in V$ we have that $\psi\left(\begin{bmatrix}\left[\begin{smallmatrix} m\\ s\end{smallmatrix}\right]\\ \left[\begin{smallmatrix} v\\ 0\end{smallmatrix}\right]\end{bmatrix}\right)=\overline{\begin{bmatrix}0\\ \tilde{s}\end{bmatrix}}$ for some $\tilde{s}\in S$. Hence we have

$$\psi\left(\begin{bmatrix}\left[\begin{smallmatrix} m\\ s\end{smallmatrix}\right]\\ \left[\begin{smallmatrix} v\\ 0\end{smallmatrix}\right]\end{bmatrix}\right)=\overline{\begin{bmatrix}0\\ \tilde{s}\end{bmatrix}}=\begin{bmatrix}0&0\\0&1\end{bmatrix}\overline{\begin{bmatrix}0\\ \tilde{s}\end{bmatrix}}=\begin{bmatrix}0&0\\0&1\end{bmatrix}\psi\left(\begin{bmatrix}\left[\begin{smallmatrix} m\\ s\end{smallmatrix}\right]\\ \left[\begin{smallmatrix} v\\ 0\end{smallmatrix}\right]\end{bmatrix}\right)=\psi\left(\begin{bmatrix}\left[\begin{smallmatrix} 0\\ s\end{smallmatrix}\right]\\ \left[\begin{smallmatrix} 0\\ 0\end{smallmatrix}\right]\end{bmatrix}\right).$$

Furthermore

$$\psi\left(\begin{bmatrix}\left[\begin{smallmatrix} m\\ s\end{smallmatrix}\right]\\ \left[\begin{smallmatrix} v\\ 0\end{smallmatrix}\right]\end{bmatrix}\right)=\psi\left(\begin{bmatrix}\left[\begin{smallmatrix} 0\\ s\end{smallmatrix}\right]\\ \left[\begin{smallmatrix} 0\\ 0\end{smallmatrix}\right]\end{bmatrix}\right)=\begin{bmatrix}0&0\\0&s\end{bmatrix}\psi\left(\begin{bmatrix}\left[\begin{smallmatrix} 0\\ 1\end{smallmatrix}\right]\\ \left[\begin{smallmatrix} 0\\ 0\end{smallmatrix}\right]\end{bmatrix}\right).$$

So each element of $\mathrm{Hom}_{\Lambda}\left(W,C/\left[\begin{smallmatrix} M\\ 0\end{smallmatrix}\right]\right)$ depends entirely on where it sends $\begin{bmatrix}\left[\begin{smallmatrix} 0\\ 1\end{smallmatrix}\right]\\ \left[\begin{smallmatrix} 0\\ 0\end{smallmatrix}\right]\end{bmatrix}$.

Therefore, for every $s\in S$ we have that $\beta(s)$ is the element of $\mathrm{Hom}_{\Lambda}\left(W,C/\left[\begin{smallmatrix} M\\ 0\end{smallmatrix}\right]\right)$ which sends $\begin{bmatrix}\left[\begin{smallmatrix} 0\\ 1\end{smallmatrix}\right]\\ \left[\begin{smallmatrix} 0\\ 0\end{smallmatrix}\right]\end{bmatrix}$ to $\overline{\begin{bmatrix}0\\ s\end{bmatrix}}$.

Since elements of $\mathrm{Hom}_{\Lambda}\left(W,C/\left[\begin{smallmatrix} M\\ 0\end{smallmatrix}\right]\right)$ depend entirely on where they send $\begin{bmatrix}\left[\begin{smallmatrix} 0\\ 1\end{smallmatrix}\right]\\ \left[\begin{smallmatrix} 0\\ 0\end{smallmatrix}\right]\end{bmatrix}$ and since $\overline{\begin{bmatrix}0\\ s\end{bmatrix}}\neq\overline{\begin{bmatrix}0\\ s'\end{bmatrix}}$ for all $s,s'\in S$ with $s\neq s'$ we have that β is a bijection and so an isomorphism of complexes of abelian groups.

Furthermore, since W is K-projective and θ is a quasi-isomorphism we have that $\mathrm{Hom}_{\Lambda}(W,\theta)$ is a quasi-isomorphism and therefore since β is an isomorphism we have that α must also be a quasi-isomorphism. □

Lemma 4.3.11. *There exists a quasi-isomorphism of complexes of abelian groups*

$$\Psi:\mathrm{Hom}_{R}(V,U)\to\mathrm{Hom}_{\Lambda}\left(W,\Sigma\begin{bmatrix}U\\ 0\end{bmatrix}\right),$$

such that

$$\Psi(\theta)\left(\begin{bmatrix}\left[\begin{smallmatrix} m\\ s\end{smallmatrix}\right]\\ \left[\begin{smallmatrix} v\\ 0\end{smallmatrix}\right]\end{bmatrix}\right)=(-1)^{|\theta|}\begin{bmatrix}\theta(v)\\ 0\end{bmatrix}.$$

Proof. Consider the distinguished triangle

$$\begin{bmatrix} V \\ 0 \end{bmatrix} \xrightarrow{\left[\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right]} \begin{bmatrix} M \\ S \end{bmatrix} \xrightarrow{\iota} W \xrightarrow{\pi} \Sigma \begin{bmatrix} V \\ 0 \end{bmatrix}$$

in $K(\Lambda)$.

Since W is the mapping cone of $\begin{bmatrix} f \\ 0 \end{bmatrix}$ we have that π is given by $\pi\left(\begin{bmatrix} \left[\begin{smallmatrix} m \\ s \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right] \end{bmatrix}\right) = \begin{bmatrix} v \\ 0 \end{bmatrix}$.

By applying the functor $\operatorname{Hom}_\Lambda\left(-, \Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}\right)$ we get a distinguished triangle

$$\leftarrow \operatorname{Hom}_\Lambda\left(\begin{bmatrix} M \\ S \end{bmatrix}, \Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}\right) \leftarrow \operatorname{Hom}_\Lambda\left(W, \Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}\right)$$
$$\xleftarrow{\pi^*} \operatorname{Hom}_\Lambda\left(\Sigma \begin{bmatrix} V \\ 0 \end{bmatrix}, \Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}\right) \leftarrow \operatorname{Hom}_\Lambda\left(\Sigma \begin{bmatrix} M \\ S \end{bmatrix}, \Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}\right)$$

in $K(\mathrm{Ab})$.

Now let $\theta \in \operatorname{Hom}_\Lambda\left(\begin{bmatrix} M \\ S \end{bmatrix}, \Sigma^i \begin{bmatrix} U \\ 0 \end{bmatrix}\right)$. Then, since $\begin{bmatrix} M \\ S \end{bmatrix}$ is generated by $\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}$ as a DG-Λ-module, we have that θ depends entirely upon where it sends $\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}$.

Let $\theta\left(\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}\right) = \begin{bmatrix} u \\ 0 \end{bmatrix}$. Then

$$\begin{bmatrix} u \\ 0 \end{bmatrix} = \theta\left(\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}\right) = \theta\left(\begin{bmatrix} 0 & 0 \\ 0 & \mathrm{id}_S \end{bmatrix}\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{id}_S \end{bmatrix}\theta\left(\begin{bmatrix} 0 \\ \mathrm{id}_S \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{id}_S \end{bmatrix}\begin{bmatrix} u \\ 0 \end{bmatrix} = 0,$$

so $\theta = 0$ and hence $\operatorname{Hom}_\Lambda\left(\begin{bmatrix} M \\ S \end{bmatrix}, \Sigma^i \begin{bmatrix} U \\ 0 \end{bmatrix}\right) = 0$ for all i.

Hence the distinguished triangle above shows that

$$\pi^* : \operatorname{Hom}_\Lambda\left(\Sigma \begin{bmatrix} V \\ 0 \end{bmatrix}, \Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}\right) \to \operatorname{Hom}_\Lambda\left(W, \Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}\right)$$

is a quasi-isomorphism.

We can now use this along with the suspension Σ and the isomorphism Θ defined in the proof of Lemma 4.3.7 to obtain the diagram

$$\mathrm{Hom}_R(V,U) \xrightarrow{\Theta} \mathrm{Hom}_\Lambda\left(\begin{bmatrix} V \\ 0 \end{bmatrix}, \begin{bmatrix} U \\ 0 \end{bmatrix}\right) \xrightarrow{\Sigma(-)}$$

$$\mathrm{Hom}_\Lambda\left(\Sigma\begin{bmatrix} V \\ 0 \end{bmatrix}, \Sigma\begin{bmatrix} U \\ 0 \end{bmatrix}\right) \xrightarrow{\pi^*} \mathrm{Hom}_\Lambda\left(W, \Sigma\begin{bmatrix} U \\ 0 \end{bmatrix}\right).$$

Since each of the maps in the diagram is a quasi-isomorphism we can use them to define the quasi-isomorphism $\Psi : \mathrm{Hom}_R(V,U) \to \mathrm{Hom}_\Lambda\left(W, \Sigma\begin{bmatrix} U \\ 0 \end{bmatrix}\right)$ by the composition

$$\Psi = \pi^* \circ \Sigma(-) \circ \Theta.$$

Finally for $\theta \in \mathrm{Hom}_R(V,U)$ we have that

$$\begin{aligned} \Psi(\theta) &= \pi^* \circ \Sigma \circ \Theta(\theta) \\ &= \pi^* \Sigma\left(\begin{bmatrix} \theta & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &= \pi^*\left((-1)^{|\theta|}\begin{bmatrix} \theta & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &= (-1)^{|\theta|}\begin{bmatrix} \theta & 0 \\ 0 & 0 \end{bmatrix} \circ \pi. \end{aligned}$$

So for $\begin{bmatrix} \left[\begin{smallmatrix} m \\ s \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right] \end{bmatrix} \in W$, we have that

$$\begin{aligned} \Psi(\theta)\left(\begin{bmatrix} \left[\begin{smallmatrix} m \\ s \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right] \end{bmatrix}\right) &= (-1)^{|\theta|}\begin{bmatrix} \theta & 0 \\ 0 & 0 \end{bmatrix} \circ \pi\left(\begin{bmatrix} \left[\begin{smallmatrix} m \\ s \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right] \end{bmatrix}\right) \\ &= (-1)^{|\theta|}\begin{bmatrix} \theta & 0 \\ 0 & 0 \end{bmatrix}\left(\begin{bmatrix} v \\ 0 \end{bmatrix}\right) = (-1)^{|\theta|}\begin{bmatrix} \theta(v) \\ 0 \end{bmatrix}. \end{aligned}$$

□

Remark 4.3.12. The right DG S-module structure on V gives us a left DG S-module structure on $\mathrm{Hom}_R(V,U)$. In addition $\mathrm{Hom}_R(V,U)$ is a left DG-$\mathrm{Hom}_R(U,U)$-module. Hence we have that $\begin{bmatrix} S & \mathrm{Hom}_R(V,U) \\ 0 & \mathrm{Hom}_R(U,U)^{\mathrm{op}} \end{bmatrix}$ is a DGA.

We can now combine the previous lemmas to obtain the principle result of this chapter, a version of [20, Theorem 4.5] for DGAs.

Theorem 4.3.13. *Let X be a DG R-module such that ${}_RX$ is compact with $\langle {}_RX\rangle = D(R)$ and let ${}_RM_S$ be compact as a DG R-module.*

Then for the upper triangular differential graded algebras

$$\Lambda = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix} \text{ and } \tilde{\Lambda} = \begin{bmatrix} S & \mathrm{Hom}_R(V,U) \\ 0 & \mathrm{Hom}_R(U,U)^{\mathrm{op}} \end{bmatrix}$$

where U is a K-projective resolution of X and ${}_RV_S$ is a K-projective resolution over R of ${}_RM_S$. We have that $D(\Lambda) \simeq D(\tilde{\Lambda})$.

Proof. From Theorem 4.3.1 and Proposition 4.3.8 we have that $D(\Lambda) \simeq D(\mathscr{E}^{\mathrm{op}})$ where

$$\mathscr{E} = \begin{bmatrix} \mathrm{Hom}_R(U,U) & \mathrm{Hom}_\Lambda(W, \Sigma \left[\begin{smallmatrix} U \\ 0 \end{smallmatrix}\right]) \\ \mathrm{Hom}_R(\Sigma \left[\begin{smallmatrix} U \\ 0 \end{smallmatrix}\right], W) & \mathrm{Hom}_\Lambda(W,W) \end{bmatrix}.$$

We therefore only need to show that there is a quasi-isomorphism of DGA's from $\tilde{\Lambda}^{\mathrm{op}}$ to $\mathscr{E}$.

From Lemma 4.3.10 we have that there exists a quasi-isomorphism

$$\alpha : S^{\mathrm{op}} \to \mathrm{Hom}_\Lambda(W,W).$$

Hence we can define the map

$$\Phi : \begin{bmatrix} \mathrm{Hom}_R(U,U) & \mathrm{Hom}_R(V,U) \\ 0 & S^{\mathrm{op}} \end{bmatrix} \to \begin{bmatrix} \mathrm{Hom}_R(U,U) & \mathrm{Hom}_\Lambda(W, \Sigma \left[\begin{smallmatrix} U \\ 0 \end{smallmatrix}\right]) \\ \mathrm{Hom}_R(\Sigma \left[\begin{smallmatrix} U \\ 0 \end{smallmatrix}\right], W) & \mathrm{Hom}_\Lambda(W,W) \end{bmatrix},$$

by $\Phi\left(\begin{bmatrix} \phi & \theta \\ 0 & s \end{bmatrix}\right) = \left(\begin{bmatrix} \phi & (-1)^{|\theta|}\Psi(\theta) \\ 0 & \alpha(s) \end{bmatrix}\right)$

where $\Psi : \mathrm{Hom}_R(V,U) \to \mathrm{Hom}_\Lambda\left(W, \Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}\right)$ is the quasi-isomorphism from Lemma 4.3.11.

We now need to show that Φ is a morphism of DGA's.

Both addition and compatibility with the differential follow from the fact that α is a morphism of differential graded algebras.

Let . denote multiplication in S^{op}. Let $\begin{bmatrix} \phi & \theta \\ 0 & s \end{bmatrix} \in \tilde{\Lambda}_i$ and $\begin{bmatrix} \phi' & \theta' \\ 0 & s' \end{bmatrix} \in \tilde{\Lambda}_j$; then we have

$$\begin{aligned} \Phi\left(\begin{bmatrix} \phi & \theta \\ 0 & s \end{bmatrix}\right)\Phi\left(\begin{bmatrix} \phi' & \theta' \\ 0 & s' \end{bmatrix}\right) &= \begin{bmatrix} \phi & (-1)^i\Psi(\theta) \\ 0 & \alpha(s) \end{bmatrix}\begin{bmatrix} \phi' & (-1)^j\Psi(\theta') \\ 0 & \alpha(s') \end{bmatrix} \\ &= \begin{bmatrix} \phi\phi' & (-1)^j\phi\Psi(\theta') + (-1)^i\Psi(\theta)\alpha(s') \\ 0 & \alpha(s)\alpha(s') \end{bmatrix} \\ &= \begin{bmatrix} \phi\phi' & (-1)^j\phi\Psi(\theta') + (-1)^i\Psi(\theta)\alpha(s') \\ 0 & \alpha(s.s') \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \Phi\left(\begin{bmatrix} \phi & \theta \\ 0 & s \end{bmatrix}\begin{bmatrix} \phi' & \theta' \\ 0 & s' \end{bmatrix}\right) &= \Phi\left(\begin{bmatrix} \phi\phi' & \phi\theta + \theta.s' \\ 0 & s.s' \end{bmatrix}\right) \\ &= \begin{bmatrix} \phi\phi' & (-1)^{(i+j)}\Psi(\phi\theta' + (-1)^{ij}s'\theta) \\ 0 & \alpha(s.s') \end{bmatrix}. \end{aligned}$$

However

$$(-1)^{(i+j)}\Psi((\phi\theta' + (-1)^{ij}s'\theta))\left(\begin{bmatrix} \left[\begin{smallmatrix} m \\ s \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right] \end{bmatrix}\right)$$

$$\begin{aligned} &= (-1)^{(i+j)}\Psi(\phi\theta')\left(\begin{bmatrix} \left[\begin{smallmatrix} m \\ s \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right] \end{bmatrix}\right) + (-1)^{(i+j)}(-1)^{ij}\Psi(s'\theta)\left(\begin{bmatrix} \left[\begin{smallmatrix} m \\ s \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right] \end{bmatrix}\right) \\ &= (-1)^{(i+j)}(-1)^{(i+j)}\begin{bmatrix} \phi\theta'(v) \\ 0 \end{bmatrix} + (-1)^{(i+j)}(-1)^{ij}(-1)^{(i+j)}\left(\begin{bmatrix} (s'\theta)(v) \\ 0 \end{bmatrix}\right) \\ &= \phi\begin{bmatrix} \theta'(v) \\ 0 \end{bmatrix} + (-1)^{ij}(-1)^{j(i+(i+1))}\begin{bmatrix} \theta(vs') \\ 0 \end{bmatrix} \\ &= \phi\begin{bmatrix} \theta'(v) \\ 0 \end{bmatrix} + (-1)^{j(i+1)}\begin{bmatrix} \theta(vs') \\ 0 \end{bmatrix} \\ &= (-1)^j\phi\Psi(\theta')\left(\begin{bmatrix} \left[\begin{smallmatrix} m \\ s \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right] \end{bmatrix}\right) + (-1)^{j(i+1)}(-1)^i\Psi(\theta)\left(\begin{bmatrix} \left[\begin{smallmatrix} ms' \\ ss' \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} vs' \\ 0 \end{smallmatrix}\right] \end{bmatrix}\right) \\ &= ((-1)^j\phi\Psi(\theta') + (-1)^i\Psi(\theta)\alpha(s'))\left(\begin{bmatrix} \left[\begin{smallmatrix} m \\ s \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right] \end{bmatrix}\right). \end{aligned}$$

So $(-1)^j \phi \Psi(\theta') + (-1)^i \Psi(\theta)\alpha(s') = (-1)^{(i+j)}\Psi((\phi\theta' + (-1)^{ij} s'\theta))$ and therefore

$$\Phi\left(\begin{bmatrix} \phi & \theta \\ 0 & s \end{bmatrix}\right) \Phi\left(\begin{bmatrix} \phi' & \theta' \\ 0 & s' \end{bmatrix}\right) = \Phi\left(\begin{bmatrix} \phi & \theta \\ 0 & s \end{bmatrix}\begin{bmatrix} \phi' & \theta' \\ 0 & s' \end{bmatrix}\right).$$

We therefore have that Φ is a morphism of Differential Graded Algebras. Furthermore, from lemma 4.3.9, we have that $\mathrm{Hom}_\Lambda\left(\Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}, W\right)$ is exact, and so the map

$$0 \to \mathrm{Hom}_\Lambda\left(\Sigma \begin{bmatrix} U \\ 0 \end{bmatrix}, W\right)$$

is a quasi-isomorphism.

Taking these together with the fact that $\alpha : S^{\mathrm{op}} \to \mathrm{Hom}_\Lambda(W, W)$ is a quasi-isomorphism we have that Φ is a quasi-isomorphism.

Hence $\mathscr{E} \simeq \tilde{\Lambda}^{\mathrm{op}}$ and so

$$D(\Lambda) \simeq D(\mathscr{E}^{\mathrm{op}}) \simeq D(\tilde{\Lambda}).$$

□

4.4 Examples

We shall conclude with some examples. In the first example we will show that by taking R and S to be k-algebras and making the same assumptions as in [20], we obtain what is in essence the same result.

Definition 4.4.1. An R-module X is called rigid if $\mathrm{Ext}^i_R(X, X) = 0$ for all $i \neq 0$.

Theorem 4.4.2. *Let R and S be rings and ${}_RM_S$ a R-S-bimodule such that ${}_RM$ is compact in $D(R)$. Assume that when R and S are considered as DGAs then the DG-bimodule ${}_RM_S$ is quasi-isomorphic to ${}_RV_S$, which is a K-projective DG R-module. Let ${}_RX$ be a compact and rigid R-module with $\langle X \rangle = D(R)$ and $\mathrm{Ext}^n_R({}_RM, {}_RX) = 0$ for all $n \neq 0$. Then the triangular matrix rings*

$$\Lambda = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix} \text{ and } \tilde{\Lambda} = \begin{bmatrix} S & \mathrm{Hom}_R(M, X) \\ 0 & \mathrm{End}_R(X)^{\mathrm{op}} \end{bmatrix}$$

are derived equivalent.

Proof. By considering the rings R and S and modules M and X to be DGA's and DG-modules respectively we can apply Theorem 4.3.13 to get that the DGA's

$$\begin{bmatrix} R & M \\ 0 & S \end{bmatrix} \text{ and } \begin{bmatrix} S & \operatorname{Hom}_R(V,U) \\ 0 & \operatorname{Hom}_R(U,U)^{\mathrm{op}} \end{bmatrix}$$

are derived equivalent, where U is a K-projective resolution of X.

Since $\operatorname{Hom}_R(U,U) = \operatorname{RHom}_R(X,X)$ we have that

$$H^i \operatorname{Hom}_R(U,U) = H^i \operatorname{RHom}_R(X,X) = \operatorname{Ext}^i_R(X,X) = 0,$$

for all $i \neq 0$, since X is rigid, and

$$H^0 \operatorname{Hom}_R(U,U) = H^0 \operatorname{RHom}_R(X,X) = \operatorname{End}_R(X).$$

Similarly, since $\operatorname{Hom}_R(V,U) = \operatorname{RHom}_R(M,X)$, we have that

$$H^i \operatorname{Hom}_R(V,U) = H^i \operatorname{RHom}_R(M,X) = \operatorname{Ext}^i_R(M,X) = 0,$$

for all $i \neq 0$, and

$$H^0 \operatorname{Hom}_R(V,U) = H^0 \operatorname{RHom}_R(M,X) = \operatorname{Hom}_R(M,X).$$

Hence we have that $H^i \begin{bmatrix} S & \operatorname{Hom}_R(V,U) \\ 0 & \operatorname{Hom}_R(U,U)^{\mathrm{op}} \end{bmatrix} = 0$ for all $i \neq 0$ and

$$H^0 \begin{bmatrix} S & \operatorname{Hom}_R(V,U) \\ 0 & \operatorname{Hom}_R(U,U)^{\mathrm{op}} \end{bmatrix} = \begin{bmatrix} S & \operatorname{Hom}_R(M,X) \\ 0 & \operatorname{End}_R(X)^{\mathrm{op}} \end{bmatrix}.$$

We therefore have that the matrix ring $\begin{bmatrix} S & \operatorname{Hom}_R(M,X) \\ 0 & \operatorname{End}_R(X)^{\mathrm{op}} \end{bmatrix}$ is derived equivalent to the DGA $\begin{bmatrix} S & \operatorname{Hom}_R(V,U) \\ 0 & \operatorname{Hom}_R(U,U)^{\mathrm{op}} \end{bmatrix}$, and so derived equivalent to the matrix ring $\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$. $\square$

Our next example considers the special case obtained when we take ${}_RX = {}_RR$.

Corollary 4.4.3. *Let ${}_RM_S$ be compact as a DG R-module. Then the triangular matrix*

DGAs

$$\Lambda = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix} \text{ and } \tilde{\Lambda} = \begin{bmatrix} S & \mathrm{Hom}_R(V,R) \\ 0 & R \end{bmatrix}$$

where V is K-projective over R and is quasi-isomorphic to ${}_R M_S$, are derived equivalent.

For the next example we require the idea of the duality on $\mathrm{DMod}\, R$ which we define next.

Definition 4.4.4. Let R be a finite dimensional DGA over a field k. Then we can define the duality on $\mathrm{DMod}\, R$ by $D : \mathrm{DMod}\, R \to \mathrm{DMod}\, R^{\mathrm{op}}$ where $D(-) = \mathrm{Hom}_k(-, k)$.

The final example below considers the case where the DGA's R and S are over some field k and R is self dual in the sense of the above definition.

Theorem 4.4.5. *Let R be a finite dimensional and self dual in the sense that $DR \cong R$ in the derived category of DG-bi-R-modules and let ${}_R M_S$ be compact as a DG R-module. Then*

$$\Lambda = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix} \text{ and } \tilde{\Lambda} = \begin{bmatrix} S & DM \\ 0 & R \end{bmatrix}$$

are derived equivalent.

Proof. From Corollary 4.4.3 we have that

$$\begin{bmatrix} R & M \\ 0 & S \end{bmatrix} \text{ and } \begin{bmatrix} S & \mathrm{Hom}_R(V,R) \\ 0 & R \end{bmatrix}$$

are derived equivalent, where ${}_R V_S$ is quasi-isomorphic to ${}_R M_S$ and ${}_R V$ is K-projective. Since R is self dual we have that

$$\mathrm{Hom}_R(V,R) \cong \mathrm{Hom}_R(V,DR) = \mathrm{Hom}_R(V, \mathrm{Hom}_k(R,k))$$

$$\cong \mathrm{Hom}_k(R \otimes_R V, k) \cong \mathrm{Hom}_k(V,k) = DV.$$

Furthermore, applying the functor $D(-)$ to the quasi-isomorphism $V \to M$ gives us the quasi-isomorphism $DM \to DV$. This in turn allows us to define a quasi-isomorphism

$$\begin{bmatrix} S & DM \\ 0 & R \end{bmatrix} \to \begin{bmatrix} S & DV \\ 0 & R \end{bmatrix}.$$

Thus $\begin{bmatrix} S & DM \\ 0 & R \end{bmatrix}$ and $\begin{bmatrix} S & DV \\ 0 & R \end{bmatrix}$ are derived equivalent and hence

$$\begin{bmatrix} R & M \\ 0 & S \end{bmatrix} \text{ and } \begin{bmatrix} S & DM \\ 0 & R \end{bmatrix}$$

are derived equivalent. $\square$

Chapter 5

Gorenstein DGAs and Generalised Gorenstein Morphisms

In [29], Sharp showed that for a Cohen-Macaulay local ring, A, with a dualising module, Ω, there are quasi-inverse equivalences

$$\mathscr{P}(A) \underset{\mathrm{Hom}_A(\Omega,-)}{\overset{\Omega \underset{A}{\otimes} -}{\rightleftarrows}} \mathscr{I}(A)$$

where $\mathscr{I}(A)$ is the category of finitely generated A-modules of finite injective dimension and $\mathscr{P}(A)$ is the category of finitely generated A-modules of finite projective dimension.

This was then greatly expanded upon in the work of Avramov and Foxby in [3], with their theory of dualising (or Foxby) equivalence. They showed that for a local ring R with a dualising complex D the adjoint functors

$$D(R) \underset{\mathrm{RHom}_R(D,-)}{\overset{D \overset{\mathrm{L}}{\underset{R}{\otimes}} -}{\rightleftarrows}} D(R)$$

restrict to give quasi-inverse equivalences between the Auslander and Bass classes:

$$\begin{gathered}\mathscr{A}(R) = \{M \in D^b(R) : D \overset{\mathrm{L}}{\underset{R}{\otimes}} M \in D^b(R) \text{ and the unit morphism} \\ \eta_M : M \to \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\underset{R}{\otimes}} M) \text{ is an isomorphism } \}\end{gathered}$$

and

$$\mathscr{B}(R) = \{M \in D^b(R) : \mathrm{RHom}_R(D, M) \in D^b(R) \text{ and the counit morphism}$$
$$\epsilon_M : D \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathrm{RHom}_R(D, M) \to M \text{ is an isomorphism } \}.$$

They also showed that this restricts further to an equivalence between the subcategories of bounded complexes of flat modules and bounded complexes of injective modules.

Frankild and Jørgensen, in [11], considered dualising equivalences in a generalised setting, in which they define the Auslander and Bass classes in such a way as to ensure that there is a quasi-inverse equivalence of categories between them. By taking this approach they are then able to apply it to DGAs R and S and a DG-bimodule ${}_{R,S}M$. Then the adjoint pair of functors

$$\begin{tikzcd} D(R^{\mathrm{op}}) \arrow[r, shift left, "- \overset{\mathrm{L}}{\underset{R}{\otimes}} M"] & D(S) \arrow[l, shift left, "\mathrm{RHom}_S(M,-)"] \end{tikzcd}$$

restricts to quasi-inverse equivalences between the associated Auslander and Bass categories:

$$\mathscr{A}_M(R^{\mathrm{op}}) = \{L \in D(R^{\mathrm{op}}) : \eta_L \text{ is an isomorphism}\}$$

and

$$\mathscr{B}_M(S) = \{N \in D(S) : \epsilon_N \text{ is an isomorphism}\}$$

where η_L and ϵ_N denote the unit morphism, $\eta_L : L \to \mathrm{RHom}_R(M, M \overset{\mathrm{L}}{\underset{R}{\otimes}} L)$, and counit morphism, $\epsilon_N : M \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathrm{RHom}_R(M, L) \to L$, of the adjunction respectively.

In the first section of this chapter we follow the approach of Frankild and Jørgensen to obtain, for a DGA, R, and a dualising DG-R-module, D, the associated Auslander and Bass classes $\mathscr{A}_D(R)$ and $\mathscr{B}_D(R)$. Then we introduce bounded and finite versions, the bounded versions being the closest equivalent for DGAs to the Auslander and Bass classes for rings as given by Avramov and Foxby. We show that, as well as the quasi-inverse equivalence of categories between $\mathscr{A}_D(R)$ and $\mathscr{B}_D(R)$, we also have quasi-inverse equivalences between the bounded and finite versions of the Auslander and Bass classes. In the rest of the section we introduce two types of DGA, which are referred to as the connective and coconnective cases. A number of the results in this chapter will be for these two types of DGA. We conclude the section by extending some results of Avramov and Foxby to both the connective and coconnective cases of DGAs.

In the second section we turn our attention to obtaining a number of Gorenstein theorems for DGAs. In ring theory there exist a number of Gorenstein Theorems which generally take the form of showing for a given type of ring, R, that R being Gorenstein is equivalent to certain other properties of R. In [6] Christensen gives a number of these Gorenstein theorems for rings. We are particularly interested in [6, Theorems 3.1.12 and 3.2.10]; these theorems tell us that for a local ring R which admits a dualising complex, R being Gorenstein is equivalent to the Auslander and Bass classes of R being maximal. In the second section we generalise these results to DGAs. We show that, in both the connective and coconnective cases, for a DGA R to be Gorenstein is equivalent to the bounded and finite versions of the Auslander and Bass classes being maximal.

In the third section we look to extend the ring theory concept of a Gorenstein morphism. A definition of a Gorenstein morphism for DGAs was given by Frankild and Jørgensen in [12]. Unfortunately this definition does not appear to allow some useful ring theory results to be extended to the general situation of DGAs, in particular base change for the Auslander class. The definition of a Gorenstein morphism that we present here is also a further generalisation in that we shall work in the situation of DG-bimodules considered as generalised morphisms of DGAs. This approach was used by Keller in [19] and Pauksztello in [24]. For DGAs R and S we consider a DG-bimodule ${}_SM_R$ to be a generalised morphism from S to R via the functor

$$D(R) \xrightarrow{{}_SM_R \overset{\mathrm{L}}{\underset{R}{\otimes}} -} D(S)\ .$$

Once we have given our definition of a generalised Gorenstein morphism we then show that this definition allows the base change for the Auslander class to be generalised to DGAs and also that the main result for Gorenstein morphisms of DGAs from [12], the ascent theorem of Gorenstein DGAs, still holds with our new definition. The section concludes by looking at the case where we have a homomorphism of DGAs, $\rho : R \to S$, and giving conditions under which the DG-bimodule ${}_SS_R$ is a generalised Gorenstein homomorphism.

The fourth and final section gives some examples of generalised Gorenstein morphisms. In the first example we consider the almost trivial case of a commutative DGA R with a symmetric bimodule ${}_RM_R$ and a symmetric dualising module ${}_RD_R$ and show that if M is a compact DG-module then it is a generalised Gorenstein morphism from R to itself. The second example considers the case of the endomorphism DGA, $\mathscr{E}$, of a perfect complex of A-modules, where A is a noetherian local commutative ring. We show that the DG-bimodule ${}_A\mathscr{E}_{\mathscr{E}}$ is a generalised Gorenstein morphism from A to $\mathscr{E}$. We also apply the ascent theorem to recover the known result that when A is a Gorenstein ring, $\mathscr{E}$ is a Gorenstein DGA. The third example deals with the Koszul complex $K(\mathbf{a})$ over a noetherian local commutative ring. Again we show that the DG A-$K(\mathbf{a})$-bimodule, ${}_AK(\mathbf{a})_{K(\mathbf{a})}$, is a generalised Gorenstein morphism from A to $K(\mathbf{a})$ and also apply the ascent theorem to recover the result of Avramov and Foxby that A is a Gorenstein ring $\Rightarrow K(\mathbf{a})$ is a Gorenstein DGA. The fourth and final example looks at the case where Λ is a finite dimensional algebra over a field k and $\mathscr{E}$ is the endomorphism DGA of L, a bounded complex of finitely generated projective Λ-modules. We show that ${}_{\Lambda,\mathscr{E}}L$ is a generalised Gorenstein morphism from $\mathscr{E}$ to Λ^{op}.

Convention. Throughout this chapter we shall, unless specified otherwise, assume that all DGAs are over a commutative Notherian base ring k, which has a dualising complex.

5.1 The Auslander and Bass Classes

In this section we shall consider some properties of the Auslander and Bass classes of DGAs. In particular we shall consider the idea of Foxby equivalence between the Auslander and Bass classes of a DGA. This was developed for the classical case of noetherian, local commutative rings by Avramov and Foxby in [3] and generalised to derived categories of DGAs by Frankild and Jørgensen in [11].

We shall follow the method of [11] to define the Auslander and Bass classes for a DGA, R, with respect to a dualising DG-R-module, D. We then define the respective bounded and finite subcategories. This is followed by showing that Foxby equivalence for DGAs restricts further to give quasi-inverse equivalences between both the bounded and finite Auslander and Bass categories.

We conclude the section by generalising some further results of [3] to two types of DGAs.

Let R be a DGA and let ${}_RD_R$ be a dualising DG-module for R. Then the canonical morphism $\rho : R \to \mathrm{RHom}_R(D, D)$ is an isomorphism.

There is an adjoint pair of functors between homotopy categories of DG-modules:

$$K(R) \underset{\mathrm{Hom}_R(D,-)}{\overset{D \underset{R}{\otimes} -}{\rightleftarrows}} K(R)$$

with the unit morphism

$$\mathrm{id}_{K(R)}(M) \xrightarrow{\eta_M} \mathrm{Hom}_R(D, D \underset{R}{\otimes} M)\ ,$$

$$\eta_M(m) = (-1)^{|m||d|}(d \mapsto d \otimes m).$$

and counit morphism

$$D \underset{R}{\otimes} \mathrm{Hom}_R(D, M) \xrightarrow{\epsilon_M} \mathrm{id}_{K(R)}(M)\ ,$$

$$\epsilon_M(d \otimes \mu) = (-1)^{|d||\mu|}\mu(d).$$

All DG-modules have K-projective and K-injective resolutions and this induces an adjoint pair of functors between derived categories of DG-modules

$$\begin{tikzcd} D(R) \arrow[r, shift left, "D \overset{\mathrm{L}}{\underset{R}{\otimes}} -"] & D(R) \arrow[l, shift left, "\mathrm{RHom}_R(D,-)"] \end{tikzcd}.$$

We can now give the definition of the Auslander and Bass classes for DGAs.

Definition 5.1.1. The *Auslander class* for a dualising DG R-module D is defined as

$$\mathscr{A}_D(R) = \{M \in D(R) \mid \eta_M \text{ is an isomorphism}\}.$$

The *Bass class* for a dualising DG R-module D is defined as

$$\mathscr{B}_D(R) = \{M \in D(R) \mid \epsilon_M \text{ is an isomorphism}\}.$$

Remark 5.1.2. Note that in the case of commutative noetherian local rings the uniqueness of dualising complexes means that the Auslander and Bass classes associated with a ring are unique. In the DGA situation where it is possible to have more than one dualising DG-module we do not necessarily have a single Auslander or Bass class.

We can also define the bounded and finite Auslander and Bass classes by imposing the appropriate restrictions on the DG-modules involved.

Definition 5.1.3. For a dualising DG R-module, D, we define

(i) The *bounded Auslander class*

$$\mathscr{A}^b_D(R) = \{M \in D(R) \mid \eta_M \text{ is an isomorphism; } M \text{ and } D \overset{\mathrm{L}}{\underset{R}{\otimes}} M \text{ are homologically bounded}\},$$

(ii) The *bounded Bass class*

$$\mathscr{B}_D^b(R) = \{M \in D(R) \mid \epsilon_M \text{ is an isomorphism; } M \text{ and } \mathrm{RHom}_R(D,M) \text{ are homologically bounded}\},$$

(iii) The *finite Auslander class*

$$\mathscr{A}_D^f(R) = \{M \in D(R) \mid \eta_M \text{ is an isomorphism; } M, D \overset{\mathrm{L}}{\underset{R}{\otimes}} M \in D^f(R)\},$$

(iv) The *finite Bass class*

$$\mathscr{B}_D^f(R) = \{M \in D(R) \mid \epsilon_M \text{ is an isomorphism; } M, \mathrm{RHom}_R(D,M) \in D^f(R)\}.$$

It is the bounded Auslander and Bass classes which most closely mirror the Auslander and Bass categories which Avramov and Foxby used in their theory of dualising equivalence for rings and the larger Auslander and Bass categories can be though of as a generalisation of the bounded case. As such some of the results later in the chapter which generalise to DGAs for the bounded and finite cases do not for the 'larger' unbounded case, in particular the Gorenstein theorems of the next section.

From the Generalised Foxby equivalence in [11] we already know that there are quasi-inverse equivalences of categories between the Auslander and Bass categories of a DGA. The next proposition restates that result and extends it to show that, as expected, there are also quasi-equivalences between both the bounded and finite versions of the Auslander and Bass classes.

Theorem 5.1.4. *Let R be a DGA and D a dualising DG R-module. Then the adjoint pair of functors*

$$D(R) \underset{G=\mathrm{RHom}_R(D,-)}{\overset{F=D\overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}}-}{\rightleftarrows}} D(R)$$

restrict to the following quasi-inverse equivalences,

(i) $\mathscr{A}_D(R) \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}_D(R)$,

(ii) $\mathscr{A}_D^b(R) \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}_D^b(R)$,

(iii) $\mathscr{A}_D^f(R) \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}_D^f(R)$.

Proof. (i) Let $M \in \mathscr{A}_D(R)$. Then

$$M \xrightarrow{\eta_M} GFM$$

is an isomorphism.

Now consider the diagram:

$$\begin{tikzcd} FM \arrow[r, "F(\eta_M)"] \arrow[dr, "1_{FM}"'] & FGFM \arrow[d, "\epsilon_{FM}"] \\ & FM \end{tikzcd} .$$

Since the diagram commutes we have that ϵ_{FM} is an isomorphism $\Leftrightarrow$ $F(\eta_M)$ is an isomorphism. However since η_M is an isomorphism we have that $F(\eta_M)$ is an isomorphism and so ϵ_{FM} is an isomorphism. This gives us that $FM \in \mathscr{B}_D(R)$. So F restricts to a functor

$$\mathscr{A}_D(R) \xrightarrow{F} \mathscr{B}_D(R).$$

Now let $N \in \mathscr{B}_D(R)$. Then

$$FGN \xrightarrow{\epsilon_N} N$$

is an isomorphism.

Now consider the diagram

$$\begin{tikzcd} GN \arrow[r, "\eta_{GN}"] \arrow[dr, "1_{GN}"'] & GFGN \arrow[d, "G(\epsilon_N)"] \\ & GN. \end{tikzcd}$$

Since the diagram commutes we have that η_{GN} is an isomorphism $\Leftrightarrow$ $G(\epsilon_N)$ is an isomorphism. However since ϵ_N is an isomorphism we have that $G(\epsilon_N)$ is an isomorphism and hence η_{GN} is an isomorphism. This gives us that $GN \in \mathscr{A}_D(R)$. So G restricts to a functor

$$\mathscr{A}_D(R) \xleftarrow{G} \mathscr{B}_D(R).$$

Finally the fact that F and G are quasi-inverse equivalent follows from the definitions of $\mathscr{A}_D(R)$ and $\mathscr{B}_D(R)$.

(ii) Let $M \in \mathscr{A}^b_D(R)$. Then since $\mathscr{A}^b_D(R) \subseteq \mathscr{A}_D(R)$ we have that $FM \in \mathscr{B}_D(R)$. Furthermore $FM = D \overset{\mathrm{L}}{\underset{R}{\otimes}} M \in D^b(R)$ and since $M \overset{\eta_M}{\to} GFM$ is an isomorphism we have that $GFM = \mathrm{RHom}_R(D, FM) \in D^b(R)$. This gives us that $FM \in \mathscr{B}^b_D(R)$ and so F restricts to a functor

$$\mathscr{A}_D^b(R) \xrightarrow{F} \mathscr{B}_D^b(R).$$

Now let $N \in \mathscr{B}_D^b(R)$. Then since $N \in \mathscr{B}_D^b(R) \subseteq \mathscr{B}_D(R)$ we have that $GM \in \mathscr{A}_D(R)$. Furthermore $GN = \mathrm{RHom}_R(D, N) \in D^b(R)$ and since $FGN \overset{\epsilon_N}{\to} N$ is an isomorphism we have that $FGN = D \overset{\mathrm{L}}{\underset{R}{\otimes}} GN \in D^b(R)$. This gives us that $GN \in \mathscr{A}_D^b(R)$ and so G restricts to a functor

$$\mathscr{B}_D^b(R) \xrightarrow{G} \mathscr{A}_D^b(R).$$

Finally the fact that F and G are quasi-inverse equivalent follows from the definitions of $\mathscr{A}_D^b(R)$ and $\mathscr{B}_D^b(R)$.

(iii) Let $M \in \mathscr{A}_D^f(R)$. Then since $\mathscr{A}_D^f(R) \subseteq \mathscr{A}_D(R)$ we have that $FM \in \mathscr{B}_D(R)$. Furthermore $FM = D \overset{\mathrm{L}}{\underset{R}{\otimes}} M \in D^f(R)$ and since $M \overset{\eta_M}{\to} GFM$ is an isomorphism we have that $GFM = \mathrm{RHom}_R(D, FM) \in D^f(R)$. This gives us that $FM \in \mathscr{B}_D^f(R)$ and so F restricts to a functor

$$\mathscr{A}_D^f(R) \xrightarrow{F} \mathscr{B}_D^f(R).$$

Now let $N \in \mathscr{B}_D^f(R)$. Then since $N \in \mathscr{B}_D^f(R) \subseteq \mathscr{B}_D(R)$ we have that $GM \in \mathscr{A}_D(R)$. Furthermore $GN = \mathrm{RHom}_R(D, N) \in D^f(R)$ and since $FGN \overset{\epsilon_N}{\to} N$ is an isomorphism we have that $FGN = D \overset{\mathrm{L}}{\underset{R}{\otimes}} GN \in D^f(R)$. This gives us that $GN \in \mathscr{A}_D^f(R)$ and so G restricts to a functor

$$\mathscr{B}_D^f(R) \xrightarrow{G} \mathscr{A}_D^f(R).$$

Finally the fact that F and G are quasi-inverse equivalent follows from the definitions of $\mathscr{A}_D^f(R)$ and $\mathscr{B}_D^f(R)$. $\square$

From now on a number of results will be given in two versions which will correspond to one of the two types of DGA that we shall mostly be dealing with. Namely the cases where:

(i) R is a connective and degreewise finite DG k-algebra, with $\mathrm{H}_0(R)$ local.

(ii) R is a coconnective and degreewise finite DG k-algebra, where k is field, such that $\mathrm{H}_0(R) = k$ and $H_{-1}(R) = 0$.

These will frequently be referred to as the connective and coconnective cases respectively. The reason for our interest in these two types of DGA is due to them having a number of useful properties as the next few lemmas demonstrate.

The following two lemmas, which mirror [10, Facts 1.5 and 1.6], are well know facts for the connective and coconnective cases. The proofs make use of the Eilenberg-Moore spectral sequence. More information on spectral sequences can be found in [32, Chapter 5].

Lemma 5.1.5. *Let R be connective and degreewise finite over k. Let L_R be a DG R^{op}-module and let ${}_RM$ be a DG R-module. If L and M are degreewise finite and homologically bounded to the right, then the complex of k-modules $L \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} M$ is also degreewise finite and homologically bounded to the right.*

Proof. Since L is right homologically bounded we have that $\mathrm{H}_i L = 0$ for all $i < i_0$, for some i_0. Similarly since M is also right homologically bounded $\mathrm{H}_j M = 0$ for $j < j_0$, for some j_0.

Since R is connective we have that there is a graded free resolution, F, of $\mathrm{H}L$ over $\mathrm{H}R$ such that each F_i is finitely generated non zero only in degrees $\geq i_o$.

Now $\mathrm{Tor}_p^{\mathrm{H}R}(\mathrm{H}L, \mathrm{H}M)_q = \mathrm{H}_p(F \underset{\mathrm{H}R}{\otimes} \mathrm{H}N)_q$ and so each $\mathrm{Tor}_p^{\mathrm{H}R}(\mathrm{H}L, \mathrm{H}M)_q$ is finitely generated over k. Moreover,

$$\mathrm{Tor}_p^{\mathrm{H}R}(\mathrm{H}L, \mathrm{H}M)_q = 0 \text{ for } q < i_0 + j_0 = q_0.$$

Furthermore $\mathrm{Tor}_p^{\mathrm{H}\,R}(\mathrm{H}L, \mathrm{H}M)_q = 0$ when $p < 0$.

Now consider the Eilenberg-Moore spectral sequence

$$E^2_{pq} = \mathrm{Tor}_p^{\mathrm{H}R}(\mathrm{H}L, \mathrm{H}M)_q \Rightarrow \mathrm{H}_{p+q}(L \overset{\mathrm{L}}{\underset{R}{\otimes}} M).$$

From above we have that each entry of E^2_{pq} is finitely generated over k and also $E^2_{pq} = 0$ whenever either $p < 0$ or $q < q_0$. The same properties follow for E^∞_{pq}.

Since the E^i_{pq}'s converge to $\mathrm{H}_{p+q}(L \overset{\mathrm{L}}{\underset{R}{\otimes}} M)$ we have that each $\mathrm{H}_{p+q}(L \overset{\mathrm{L}}{\underset{R}{\otimes}} M)$ has a finite filtration with $E^\infty_{pq} \cong F_p \,\mathrm{H}_{p+q} / F_{p-1} \,\mathrm{H}_{p+q}$.

When $n = p + q < q_0$ then $F_p \,\mathrm{H}_n / F_{p-1} \,\mathrm{H}_n = E^\infty_{pq} = 0$ and this in turn gives us that $H_n(L \overset{\mathrm{L}}{\underset{R}{\otimes}} M) = 0$. Hence $L \overset{\mathrm{L}}{\underset{R}{\otimes}} M$ is right homologically bounded. Furthermore for $p + q > q_0$ each finite filtration consists of entirely of finitely generated k-modules and so $\mathrm{H}_{p+q}(L \overset{\mathrm{L}}{\otimes}_R N)$ is finitely generated. Therefore $L \overset{\mathrm{L}}{\underset{R}{\otimes}} M$ is degreewise finite. □

Lemma 5.1.6. *Let R be coconnective and degreewise finite over a field k with $\mathrm{H}_0(R) = k$ and $H_{-1}(R) = 0$. Let L_R be a DG R^{op}-module and let ${}_RM$ be a DG R-module. If L*

and M are degreewise finite and homologically bounded to the left, then the complex of k-modules $L \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} M$ is also degreewise finite and homologically bounded to the left.

Proof. Since $\mathrm{H}\,L$ is left bounded it is only non-zero in degrees $\leq i$. Similarly $\mathrm{H}\,M$ is left bounded and so is only non-zero in degrees $\leq j$. Since $\mathrm{H}\,R$ is noetherian we can construct a graded free resolution F of $\mathrm{H}\,L$ in such a way that each F_n is finitely generated and is only non-zero in degrees $\leq i - 2n$.

Hence $\mathrm{Tor}_p^{\mathrm{H}A}(\mathrm{H}L, \mathrm{H}M) = \mathrm{H}^p(F \underset{\mathrm{H}R}{\otimes} \mathrm{H}N)$ is only non-zero in degrees $\leq i + j - 2p$ and each $\mathrm{Tor}_p^{\mathrm{H}\,A}(\mathrm{H}\,L, \mathrm{H}\,M)_q$ is finitely generated over k.

Thus E^2_{pq} is non-zero only in the wedge where $p \geq 0$ and $q \leq i + j - 2p$. Furthermore each entry is finitely generated. Hence E^∞_{pq} is also non-zero only within the same wedge and each entry is finitely generated over k.

Since the E^i_{pq}'s converge to $\mathrm{H}_{p+q}(L \overset{\mathrm{L}}{\otimes}_R N)$ we have that each $\mathrm{H}_{p+q}(L \overset{\mathrm{L}}{\otimes}_R N)$ has a finite filtration with $E^\infty_{pq} \cong F_p \,\mathrm{H}_{p+q} / F_{p-1} \,\mathrm{H}_{p+q}$. For $p + q > i + j$ each entry of this filtration is 0 so $\mathrm{H}_{p+q}(L \overset{\mathrm{L}}{\otimes}_R N)$, we therefore have that $L \overset{\mathrm{L}}{\underset{R}{\otimes}} M$ is homologically bounded to the left. Furthermore for $p + q \leq i + j$ each finite filtration consists of entirely of finitely generated k-modules and so $\mathrm{H}_{p+q}(L \overset{\mathrm{L}}{\otimes}_R N)$ is finitely generated, therefore $L \overset{\mathrm{L}}{\underset{R}{\otimes}} M$ is degreewise finite. □

Lemma 5.1.7. *Let R be connective and degreewise finite over k with $\mathrm{H}_0(R)$ local and let D be a right homologically bounded and degreewise finite dualising DG-module for R. Then for $C \in D^f(R)$ if $C \ncong 0$ then $D \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} C \ncong 0$.*

Proof. Since $C \ncong 0$, we have $\mathrm{H}(C) \neq 0$. Also, since $R \cong \mathrm{RHom}_R(D, D)$ we have that $D \ncong 0$ and so $H(D) \neq 0$. Thus, by [25, Lemma 2.3], $H(D \overset{\mathrm{L}}{\underset{R}{\otimes}} C) \neq 0$ and so $D \overset{\mathrm{L}}{\underset{R}{\otimes}} C \ncong 0$. □

Lemma 5.1.8. *Let R be connective and degreewise finite over k with $\mathrm{H}_0(R)$ local and let D be a right homologically bounded and degreewise finite dualising DG-module for R. Then if $C \in D^f(R)$ with $C \ncong 0$ we have that $\mathrm{RHom}_R(D, C) \ncong 0$.*

Proof. Since $C \in D^f(R)$ we have that it is D-reflexive so

$$C \cong \mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R(C, D), D).$$

Therefore

$$\mathrm{RHom}_R(D,C) \cong \mathrm{RHom}_R(D, \mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R(C,D),D))$$

$$\cong \mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R(C,D) \overset{\mathrm{L}}{\underset{R}{\otimes}} D, D)$$

by adjointness. Since $C \ncong 0$ we have $\mathrm{RHom}_R(C,D) \ncong 0$ as otherwise we would have that $C \cong \mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R(C,D),D) \cong 0$. Moreover, $\mathrm{RHom}_R(C,D) \in D^f(R)$.

Lemma 5.1.7 gives us that

$$\mathrm{RHom}_R(C,D) \overset{\mathrm{L}}{\underset{R}{\otimes}} D \ncong 0.$$

Finally since $\mathrm{RHom}_R(C,D)$ is finite we have that $\mathrm{RHom}_R(C,D) \overset{\mathrm{L}}{\otimes}_R D$ is D-reflexive and thus $\mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R(C,D) \overset{\mathrm{L}}{\underset{R}{\otimes}} D, D) \ncong 0$. □

The next two lemmas give coconnective versions of the previous two lemmas.

Lemma 5.1.9. *Let R be coconnective and degreewise finite over a field k with $\mathrm{H}_0(R) = k$ and $H_{-1}(R) = 0$ and let D be a left homologically bounded and degreewise finite dualising DG-module for R. Then for $C \in D^f(R)$ if $C \ncong 0$ then $D \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} C \ncong 0$.*

Proof. The proof is essentially the same as for 5.1.7, the single exception being that we use [13, Proposition 1.5], rather that [25, Lemma 2.3]. □

Lemma 5.1.10. *Let R be coconnective and degreewise finite over the field k with $\mathrm{H}_0(R) = k$ and $H_{-1}(R) = 0$ and let D be a left homologically bounded and degreewise finite dualising DG-module for R. Then if $C \in D^f(R)$ with $C \ncong 0$ we have that $\mathrm{RHom}_R(D,C) \ncong 0$.*

Proof. Again this is essentially the same as the proof for 5.1.8. The only difference is that we have to use Lemma 5.1.9 rather than 5.1.7. □

We end this section with the following two theorems, generalisations of [3, Theorem 3.2(a) and (b)] to the connective and coconnective DGA cases.

Theorem 5.1.11. *Let R be a DG k-algebra which is connective and degreewise finite over k with $H_0(R)$ local. Then for a dualising DG-R-module, D, which is right homologically bounded and degreewise finite,* *(i) If $M \in D^f(R)$ with $D \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} M \in \mathscr{B}_D^f(R)$, then $M \in \mathscr{A}_D^f(R)$.*

(ii) If $M \in D^f(R)$ with $\mathrm{RHom}_R(D, M) \in \mathscr{A}_D^f(R)$, then $M \in \mathscr{B}_D^f(R)$.

Proof. (i) Let $M \in D^f(R)$ and let F and G be the functors $D \overset{\mathrm{L}}{\underset{R}{\otimes}} -$ and $\mathrm{RHom}_R(D, -)$ respectively.

First note that $FM = D \overset{\mathrm{L}}{\otimes}_R M \in \mathscr{B}_D^f(R) \subseteq D^f(R)$.

It remains to show that η_M is an isomorphism.

We have the commutative diagram:

$$\begin{tikzcd} FM \arrow[r, "F(\eta_M)"] \arrow[dr, "1_{FM}"'] & FGFM \arrow[d, "\epsilon_{FM}"] \\ & FM \end{tikzcd}$$

where ϵ_{FM} is a isomorphism, since $FM \in \mathscr{B}_D^f(R)$. Therefore since the diagram commutes $F(\eta_M)$ is an isomorphism.

Now consider the distinguished triangle:

$$M \xrightarrow{\eta_M} GFM \longrightarrow C \longrightarrow \Sigma M$$

where C is the mapping cone of η_M.

Applying the functor F to this gives the distinguished triangle:

$$FM \xrightarrow{F(\eta_M)} FGFM \longrightarrow FC \longrightarrow \Sigma FM.$$

As $F(\eta_M)$ is an isomorphism we have that $FC = D \overset{\mathrm{L}}{\underset{R}{\otimes}} C \cong 0$.

We have $M \in D^f(R)$ and since $FM = D \overset{\mathrm{L}}{\underset{R}{\otimes}} M \in \mathscr{B}_D^f(R)$ we have that $GFM = \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\underset{R}{\otimes}} M) \in D^f(R)$. This gives us that the mapping cone $C \in D^f(R)$.

But now $D \overset{\mathrm{L}}{\underset{R}{\otimes}} C \cong 0$ implies $C \cong 0$ by Lemma 5.1.7, so η_M is an isomorphism.

(ii) Let $M \in D^f(R)$.

First note that $GM = \mathrm{RHom}_R(D, M) \in \mathscr{A}_D^f(R) \subseteq D^f(R)$.

It remains to show that ϵ_M is an isomorphism.

We have the commutative diagram

$$\begin{tikzcd} GM \arrow[r, "\eta_{GM}"] \arrow[dr, "1_{GM}"'] & GFGM \arrow[d, "G(\epsilon_M)"] \\ & GM \end{tikzcd}$$

where η_{GM} is a isomorphism, since $GM \in \mathscr{A}_D^f(R)$. Therefore since the diagram commutes $G(\epsilon_M)$ is an isomorphism.

Now consider the distinguished triangle:

$$FGM \xrightarrow{\epsilon_M} M \longrightarrow C \longrightarrow \Sigma FGM$$

where C is the mapping cone of ϵ_M.

Applying the functor G to this gives the distinguished triangle:

$$GFGM \xrightarrow{G(\epsilon_M)} GM \longrightarrow GC \longrightarrow \Sigma GFGM,$$

As $G(\epsilon_M)$ is an isomorphism we have that $GC = \mathrm{RHom}_R(D, C) \cong 0$.

We have $M \in D^f(R)$ and, since $GM = \mathrm{RHom}_R(D, M) \in \mathscr{A}_D^f(R)$,

$$FGM = D \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(D, M) \in D^f(R).$$

This gives us that the mapping cone $C \in D^f(R)$.

But now $\mathrm{RHom}_R(D, C) \cong 0$ implies $C \cong 0$ by Lemma 5.1.8, so ϵ_M is an isomorphism. □

Theorem 5.1.12. *Let k be a field and let R be a degreewise finite coconnective DG-k-algebra such that $H_0(R) = k$ and $H_{-1}(R) = 0$. Then for a dualising DG R-module, D, that is left homologically bounded and degreewise finite,*

(i) If $M \in D^f(R)$ with $D \overset{\mathrm{L}}{\otimes}_R M \in \mathscr{B}_D^f(R)$, then $M \in \mathscr{A}_D^f(R)$.

(ii) If $M \in D^f(R)$ with $\mathrm{RHom}_R(D, M) \in \mathscr{A}_D^f(R)$, then $M \in \mathscr{B}_D^f(R)$.

Proof. This is essentially the same as the proof for the above theorem with the exception that we use the coconnective versions of the required lemmas. □

5.2 Gorenstein Theorems

We now turn our attention to developing a number of Gorenstein theorems involving the bounded and finite Auslander classes for DGAs of both the connective and coconnective cases. In these we shall again generalise results from ring theory to show that a DGA being Gorenstein is equivalent to the existence of maximal bounded and finite Auslander and Bass classes.

We begin with the following proposition which gives a condition under which a DG R-module is R-reflexive.

Proposition 5.2.1. *For $M \in D^f(R)$, we have that $M \in \mathscr{A}_D(R) \Leftrightarrow$ The double duality morphism*

$$M \to \mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R(M,R),R)$$

is an isomorphism.

Proof. Since $R \cong \mathrm{RHom}_R(D,D)$ we can replace R in the double duality map with $\mathrm{RHom}_R(D,D)$. This allows us to construct the following diagram

$$\begin{tikzcd}
M \arrow[r, "\delta"] \arrow[dd, "\eta"] & \mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R(M,\mathrm{RHom}_R(D,D)),\mathrm{RHom}_R(D,D)) \arrow[d, "\sigma"] \\
 & \mathrm{RHom}_R(D,\mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R(M,\mathrm{RHom}_R(D,D)),D)) \\
\mathrm{RHom}_R(D,D \overset{\mathrm{L}}{\underset{R}{\otimes}} M) \arrow[r, "\mathrm{RHom}_R(D{,}\delta)"] & \mathrm{RHom}_R(D,\mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R(D \overset{\mathrm{L}}{\underset{R}{\otimes}} M,D),D)) \arrow[u, "\mathrm{RHom}_R(D{,}\mathrm{RHom}_{R^{\mathrm{op}}}(\alpha{,}D))"']
\end{tikzcd}$$

where δ is the double duality morphism, σ is the swap isomorphism and α is the adjointness isomorphism.

Since ${}_RD_R$ is a dualising DG-module it has a biprojective resolution, ${}_RP_R$, and a biinjective resolution, ${}_RI_R$. Hence we have $\mathrm{RHom}_R(D,D) \cong \mathrm{Hom}_R(P,I)$ where $\mathrm{Hom}_R(P,I)$ is a K-injective DG R-module. We can now, in the diagram above, replace $\mathrm{RHom}_R(D,D)$ by $\mathrm{Hom}(P,I)$ and D by either P or I as appropriate to give us the diagram

$$\begin{tikzcd}
M \arrow[r, "\delta"] \arrow[dd, "\eta"] & \mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Hom}_R(M,\mathrm{Hom}_R(P,I)),\mathrm{Hom}_R(P,I)) \arrow[d, "\sigma"] \\
 & \mathrm{Hom}_R(P,\mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Hom}_R(M,\mathrm{Hom}_R(P,I)),I)) \\
\mathrm{Hom}_R(P,P \underset{R}{\otimes} M) \arrow[r, "\delta'_*"] & \mathrm{Hom}_R(P,\mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Hom}_R(P \underset{R}{\otimes} M,I),I)) \arrow[u, "(\alpha_*)^*"']
\end{tikzcd}$$

where δ and δ', are the double duality morphisms, given by

$$\delta(m)(\mu) = (-1)^{|m||\mu|}\mu(m)$$

and

$$\delta'(p \underset{R}{\otimes} m)(\xi) = (-1)^{(|p|+|m|)|\xi|}\xi(p \underset{R}{\otimes} m)$$

where $m \in M$, $p \in P$, $\mu \in \mathrm{Hom}(M, \mathrm{Hom}_R(P, I))$ and $\xi \in \mathrm{Hom}_R(P \underset{R}{\otimes} M, I)$,

σ, the swap isomorphism, is given by

$$\sigma(\theta)(p)(\mu) = (-1)^{|\mu||p|}\theta(\mu)(p)$$

where $\theta \in \mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Hom}_R(M, \mathrm{Hom}_R(P, I)), \mathrm{Hom}_R(P, I))$, $p \in P$ and $\mu \in \mathrm{Hom}(M, \mathrm{Hom}_R(P, I))$,

$\alpha : \mathrm{Hom}_R(M, \mathrm{Hom}_R(P, I)) \to \mathrm{Hom}_R(P \underset{R}{\otimes} M, I)$, the adjointness isomorphism, is given by

$$\alpha(\mu)(p \otimes m) = (-1)^{|p||m|}\mu(m)(p)$$

where $m \in M$, $p \in P$ and $\mu \in \mathrm{Hom}(M, \mathrm{Hom}_R(P, I))$,

and finally η is given by

$$\eta(m)(p) = (-1)^{|m||p|}p \otimes m$$

where $m \in M$ and $p \in P$.

So

$$[\sigma \circ \delta](m)(p)(\mu) = (-1)^{|\mu||p|}\delta(m)(\mu)(p) = (-1)^{|\mu|(|p|+|m|)}\mu(m)(p)$$

and

$$\begin{aligned}[(\alpha_*)^* \circ \delta'_* \circ \eta(m)](p)(\mu) &= (\delta'_* \circ \eta(m))(p)(\alpha(\mu)) = \delta'(\eta(m)(p))(\alpha(\mu)) \\ &= (-1)^{|p||m|}\delta(p \otimes m)(\alpha(\mu)) = (-1)^{|p||m|+|\mu|(|p|+|m|)}\alpha(\mu)(p \otimes m) \\ &= (-1)^{|\mu|(|p|+|m|)}\mu(m)(p).\end{aligned}$$

Hence the diagram commutes. Furthermore α and σ are isomorphisms and since M is finite we have that δ_* is an isomorphism by 3.2.15. We therefore have that the double duality map δ is an isomorphism iff η is an isomorphism which is the same as $M \in \mathscr{A}_D(R)$. □

An immediate consequence of this proposition is that if $D^f(R)$ is contained in $\mathscr{A}_D(R)$ then condition [G1] of the definition of a Gorenstein DGA (Definition 3.2.22) is satisfied. Conversely if, for a DGA R, a finite DG R-module is R-reflexive then we have that it is contained in every possible Auslander class of R.

We can now look at obtaining the various Gorenstein Theorems. We shall start with the connective case and the following lemma.

Lemma 5.2.2. *Let R be a DG k-algebra which is connective and degreewise finite over k. If there exists a right homologically bounded and degreewise finite dualising DG-R-module, D, such that $D^f(R) \subseteq \mathscr{A}^b_D(R)$ and $D^f(R^{\mathrm{op}}) \subseteq \mathscr{A}^b_D(R^{\mathrm{op}})$ then R is Gorenstein.*

Proof. We need to show that R satisfies the properties [G1] and [G2] of Definition 3.2.22.

Firstly for [G1], let $M \in D^f(R)$ and $N \in D^f(R^{\mathrm{op}})$. Since $D^f(R) \subseteq \mathscr{A}^b_D(R) \subseteq \mathscr{A}_D(R)$ and $D^f(R^{\mathrm{op}}) \subseteq \mathscr{A}^b_D(R^{\mathrm{op}})$ we have by Proposition 5.2.1 that the double duality morphisms

$$M \to \mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R(M,R),R)$$

and

$$N \to \mathrm{RHom}_R(\mathrm{RHom}_{R^{\mathrm{op}}}(N,R),R)$$

are isomorphisms.

Secondly for [G2], let $M \in D^f(R)$. We consider the DG R^{op}-module $\mathrm{RHom}_R(M,R)$, this is isomorphic to $\mathrm{RHom}_R(D \overset{\mathrm{L}}{\underset{R}{\otimes}} M, D)$ via the isomorphism $R \cong \mathrm{RHom}_R(D,D)$ and adjointness. Since both D and M are degreewise finite and homologically bounded to the right, we have, by 5.1.5, that $D \overset{\mathrm{L}}{\underset{R}{\otimes}} M$ is degreewise finite over k and homologically bounded to the right. Also since $M \in D^f(R) \subseteq \mathscr{A}^b_D(R)$ we have that $D \overset{\mathrm{L}}{\underset{R}{\otimes}} M$ is homologically bounded and since $\mathrm{H}_0(R)$ is finitely generated over k we have that $D \overset{\mathrm{L}}{\underset{R}{\otimes}} M$ is degreewise finite as an DG R-module. Hence $D \overset{\mathrm{L}}{\underset{R}{\otimes}} M \in D^f(R)$. Finally since D is a dualising DG R-module we have that $\mathrm{RHom}_R(-,D)$ sends $D^f(R)$ to $D^f(R^{\mathrm{op}})$, Hence $\mathrm{RHom}_R(D \overset{\mathrm{L}}{\underset{R}{\otimes}} M, D) \in D^f(R^{\mathrm{op}})$. We can similarly show for $N \in D^f(R^{\mathrm{op}})$ that $\mathrm{RHom}_{R^{\mathrm{op}}}(N,R) \in D^f(R)$ □

We are now able to present our first Gorenstein Theorem which deals with the case of the bounded Auslander class.

Theorem 5.2.3. *Let R be a DG k-algebra which is connective and degreewise finite over k. Then R is Gorenstein $\Leftrightarrow \exists$ a dualising DG R-module D which is degreewise finite and homologically bounded to the right such that $\mathscr{A}^b_R(R) = D^b(R)$ and $\mathscr{A}^b_R(R^{\mathrm{op}}) = D^b(R^{\mathrm{op}})$.*

Proof. ($\Rightarrow$) Let R be Gorenstein, then R is a dualising DG R-module. Furthermore since R is connective and degreewise finite over k, we have that it is right homologically bounded and degreewise finite over itself. Finally it has the bounded Auslander class

$\mathscr{A}^b_R(R) = \{M \in D(R) \mid \eta_M \text{ is an isomorphism, } M \text{ is bounded}\}$.

However in this case η_M is the identity isomorphism for all M. Hence $\mathscr{A}^b_D(R) = D^b(R)$. Similarly we also have that $\mathscr{A}^b_D(R^{\mathrm{op}}) = D^b(R^{\mathrm{op}})$

($\Leftarrow$) This follows from Lemma 5.2.2, as $D^f(R) \subseteq D^b(R) = \mathscr{A}^b_D(R)$ and $D^f(R^{\mathrm{op}}) \subseteq D^b(R^{\mathrm{op}}) = \mathscr{A}^b_D(R^{\mathrm{op}})$. $\square$

Before we can produce our second Gorenstein Theorem which deals with the bounded Bass class we first need to prove some technical results.

Lemma 5.2.4. *Let R be a DG k-algebra which is connective and degreewise finite over k. Let D be a right homologically bounded and degreewise finite dualising DG R-module with $\mathscr{B}^b_D(R) = D^b(R)$ and $\mathscr{B}^b_D(R^{\mathrm{op}}) = D^b(R^{\mathrm{op}})$. Then, for ${}_RM \in D^f(R)$ and $N_R \in D^f(R^{\mathrm{op}})$ we have the following:*

(i) $\mathrm{RHom}_R(D, M) \in D^f(R)$ *and* $\mathrm{RHom}_{R^{\mathrm{op}}}(D, N) \in D^f(R^{\mathrm{op}})$.

(ii) $\mathrm{RHom}_R(M, R) \in D^f(R^{\mathrm{op}})$ *and* $\mathrm{RHom}_{R^{\mathrm{op}}}(N, R) \in D^f(R)$.

(iii) $D \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} M \in D^f(R)$ *and* $N \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} D \in D^f(R^{\mathrm{op}})$.

Proof. (i) Let C be a dualising k-module then, since M is finite, we have that

$${}_RM \cong \mathrm{RHom}_k(\mathrm{RHom}_k({}_RM, C), C),$$

and hence

$$\mathrm{RHom}_R(D, M) \cong \mathrm{RHom}_R(D, \mathrm{RHom}_k(\mathrm{RHom}_k(M, C), C))$$

$$\cong \mathrm{RHom}_k(\mathrm{RHom}_k(M, C) \overset{\mathrm{L}}{\underset{R}{\otimes}} D, C) = (*).$$

Now since ${}_RM$ is finite we have that $\mathrm{RHom}_k({}_RM, C)$ is a finite DG-right-R-module and so, by 5.1.5, we have that

$$\mathrm{RHom}_k(M, C) \overset{\mathrm{L}}{\underset{R}{\otimes}} D$$

is right homologically bounded and degreewise finite.

Hence $(*)$ is degreewise finite by [16, Page 257].

Furthermore, since $D^f(R) \subseteq D^b(R) = \mathscr{B}_D^b(R)$ we have that

$$M \in \mathscr{B}_D^b(R).$$

Therefore

$$\mathrm{RHom}_R(D, M) \in \mathscr{A}_D^b(R) \subseteq D^b(R).$$

Thus $\mathrm{RHom}_R(D, M) \in D^f(R^{\mathrm{op}})$.

The proof that $\mathrm{RHom}_{R^{\mathrm{op}}}(D, N) \in D^f(R^{\mathrm{op}})$ is similar.

(ii) By the swap isomorphism we have that

$$\mathrm{RHom}_R(M, R) \cong \mathrm{RHom}_R(M, \mathrm{RHom}_R(D, D))$$

$$\cong \mathrm{RHom}_R(D, \mathrm{RHom}_R(M, D)).$$

Since D is a dualising module and $M \in D^f(R)$ then $\mathrm{RHom}_R(M, D) \in D^f(R^{\mathrm{op}})$.

Hence, by (i), we have that $\mathrm{RHom}_R(D, \mathrm{RHom}_R(M, D)) \in D^f(R^{\mathrm{op}})$

The proof that $\mathrm{RHom}_{R^{\mathrm{op}}}(N, R) \in D^f(R)$ is similar.

(iii) Since M is finite we have that $D \overset{\mathrm{L}}{\underset{R}{\otimes}} M$ is D-reflexive, i.e

$$D \overset{\mathrm{L}}{\underset{R}{\otimes}} M \cong \mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R(D \overset{\mathrm{L}}{\underset{R}{\otimes}} M, D), D)$$

$$\cong \mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R(M, \mathrm{RHom}_R(D, D)), D)$$

$$\cong \mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R(M, R), D).$$

By (ii) we have that $\mathrm{RHom}_R(M, R) \in D^f(R^{\mathrm{op}})$ and so, since D is a dualising DG-R-module, we have that $D \overset{\mathrm{L}}{\otimes}_R M \in D^f(R)$.

Again the proof that $N \overset{\mathrm{L}}{\underset{R}{\otimes}} D \in D^f(R^{\mathrm{op}})$ is similar. $\square$

Lemma 5.2.5. *Let R be a DG k-algebra which is connective and degreewise finite over k. Let D be a right homologically bounded and degreewise finite dualising DG R-module with $\mathscr{B}_D^b(R) = D^b(R)$ and $\mathscr{B}_D^b(R^{\mathrm{op}}) = D^b(R^{\mathrm{op}})$. Then for $M \in D^f(R)$ we have, for the unit morphism η_M, that $F(\eta_M)$ is an isomorphism.*

Proof. Let $M \in D^f(R) \subseteq D^b(R) = \mathscr{B}_D^b(R)$.

We have the commutative diagram:

$$\begin{tikzcd} FM \arrow[r, "F(\eta_M)"] \arrow[dr, "1_{FM}"'] & FGFM \arrow[d, "\epsilon_{FM}"] . \\ & FM \end{tikzcd}$$

Since $M \in D^f(R)$ we have by Lemma 5.2.4(iii) that

$$FM = D \overset{\mathrm{L}}{\underset{R}{\otimes}} M \in D^f(R) \subseteq \mathscr{B}^b_D(R)$$

and so ϵ_{FM} is an isomorphism.

Since the diagram above commutes this then gives us that $F(\eta_M)$ is an isomorphism. □

We can now prove our Gorenstein Theorem for the bounded Bass class.

Theorem 5.2.6. *Let R be a DG k-algebra which is connective and degreewise finite over k with $H_0(R)$ local. Then R is Gorenstein $\Leftrightarrow \exists$ a dualising DG R-module D which is degreewise finite and homologically bounded to the right such that $\mathscr{B}^b_D(R) = D^b(R)$ and $\mathscr{B}^b_D(R^{\mathrm{op}}) = D^b(R^{\mathrm{op}})$.*

Proof. ($\Rightarrow$) Let R be Gorenstein. Then R is a dualising DG R-module. Furthermore since R is connective and degreewise finite over k, we have that it is right homologically bounded and degreewise finite over itself. Finally it has the bounded Bass classes

$\mathscr{B}^b_D(R) = \{M \in D(R) \mid \epsilon_M$ is an isomorphism, M is bounded$\}$ and $\mathscr{B}^b_D(R^{\mathrm{op}}) = \{N \in D(R^{\mathrm{op}}) \mid \epsilon_N$ is an isomorphism, N is bounded$\}$.

However, in this case ϵ_M and ϵ_N are the identity isomorphisms for all M and N respectively. Hence $\mathscr{B}^b_D(R) = D^b(R)$ and $\mathscr{B}^b_D(R^{\mathrm{op}}) = D^b(R^{\mathrm{op}})$.

($\Leftarrow$) Let $M \in D^f(R)$.

Consider the distinguished triangle:

$$M \xrightarrow{\eta_M} GFM \longrightarrow C \longrightarrow \Sigma M$$

where C is the mapping cone of η_M.

Applying the functor F to this gives the distinguished triangle:

$$FM \xrightarrow{F(\eta_M)} FGFM \longrightarrow FC \longrightarrow \Sigma FM.$$

From Lemma 5.2.5 we have that $F(\eta_M)$ is an isomorphism and so $FC = D \overset{\mathrm{L}}{\underset{R}{\otimes}} C \cong 0$.

Since $M \in D^f(R)$ we have that $GFM = \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R M)$ is finite, by Lemma 5.2.4(i) and (iii). This in turn gives us that the mapping cone C is finite.

Since $D \overset{\mathrm{L}}{\underset{R}{\otimes}} C \cong 0$, Lemma 5.1.7 gives $C \cong 0$ and hence η_M is an isomorphism.

Hence $M \in \mathscr{A}^b_D(R)$ and so $D^f(R) \subseteq \mathscr{A}^b_D(R)$.

We can similarly show that for $N \in D^f(R^{\mathrm{op}})$ that $N \in \mathscr{A}^b_D(R^{\mathrm{op}})$ and so $D^f(R^{\mathrm{op}}) \subseteq \mathscr{A}^b_D(R^{\mathrm{op}})$.

Therefore by Lemma 5.2.2 we have that R is Gorenstein. □

As with the bounded Auslander and Bass classes we can also produce Gorenstein theorems which deal with the cases of the finite Auslander and Bass classes.

Theorem 5.2.7. *Let R be a DG k-algebra which is connective and degreewise finite over k. Then R is Gorenstein $\Leftrightarrow$ $\exists$ a dualising DG R-module D which is degreewise finite and homologically bounded to the right such that $\mathscr{A}^f_D(R) = D^f(R)$ and $\mathscr{A}^f_D(R^{\mathrm{op}}) = D^f(R^{\mathrm{op}})$.*

Proof. ($\Rightarrow$) Let R be Gorenstein, then R is a dualising DG R-module. Furthermore since R is connective and degreewise finite over k, we have that it is right homologically bounded and degreewise finite over itself. Finally it has the finite Auslander class

$\mathscr{A}^f_D(R) = \{M \in D(R) \mid \eta_M \text{ is an isomorphism, } M \text{ is finite}\}$.

However in this case η_M is the identity isomorphism for all M. Hence $\mathscr{A}^f_D(R) = D^f(R)$. The proof that $\mathscr{A}^f_D(R^{\mathrm{op}}) = D^f(R^{\mathrm{op}})$ is very similar.

($\Leftarrow$) This follows from Lemma 5.2.2, as $D^f(R) = \mathscr{A}^f_D(R) \subseteq \mathscr{A}^b_D(R)$ and $D^f(R^{\mathrm{op}}) = \mathscr{A}^f_D(R^{\mathrm{op}}) \subseteq \mathscr{A}^b_D(R^{op})$. □

Theorem 5.2.8. *Let R be a DG k-algebra which is connective and degreewise finite over k with $H_0(R)$ local. Then R is Gorenstein $\Leftrightarrow$ $\exists$ a dualising DG R-module D which is degreewise finite and homologically bounded to the right such that $\mathscr{B}^f_D(R) = D^f(R)$ and $\mathscr{B}^f_D(R^{\mathrm{op}}) = D^f(R^{\mathrm{op}})$.*

Proof. ($\Rightarrow$) Let R be Gorenstein, then R is a dualising DG R-module. Furthermore since R is connective and degreewise finite over k, we have that it is right homologically bounded and degreewise finite over itself. Finally it has the finite Bass class

$\mathscr{B}_D^f(R) = \{M \in D(R) \mid \epsilon_M \text{ is an isomorphism, } M \text{ is finite}\}$

However in this case ϵ_M is the identity isomorphism for all M. Hence $\mathscr{B}_D^f(R) = D^f(R)$. The proof that $\mathscr{B}_D^f(R^{\text{op}}) = D^f(R^{\text{op}})$ is similar.

($\Leftarrow$) Let $M \in D^f(R)$. First note that since $M \in D^f(R) = \mathscr{B}_D^f(R)$ we have that $D \overset{\mathrm{L}}{\otimes}_R M \in D^f(R)$.

Now consider the distinguished triangle:

$$M \xrightarrow{\eta_M} GFM \longrightarrow C \longrightarrow \Sigma M$$

where C is the mapping cone of η_M.

Applying the functor F to this gives the distinguished triangle:

$$FM \xrightarrow{F(\eta_M)} FGFM \longrightarrow FC \longrightarrow \Sigma FM.$$

From Lemma 5.2.5 we have that $F(\eta_M)$ is an isomorphism and so $FC = D \underset{R}{\overset{\mathrm{L}}{\otimes}} C \cong 0$.

Since $D \overset{\mathrm{L}}{\otimes}_R M \in D^f(R) = \mathscr{B}_D^f(R)$ we have that

$$GFM = \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R M) \in \mathscr{A}_D^f(R) \subseteq D^f(R).$$

This in turn gives us that the mapping cone C is finite.

But $D \underset{R}{\overset{\mathrm{L}}{\otimes}} C \not\cong 0$. so by Lemma 5.1.7 $C \cong 0$ and hence η_M is an isomorphism.

Hence $M \in \mathscr{A}_D^f(R) \subseteq \mathscr{A}_D^b(R)$ and so $D^f(R) \subseteq \mathscr{A}_D^b(R)$. Also we can similarly show that $D^f(R^{\text{op}}) \subseteq \mathscr{A}_D^b(R^{\text{op}})$. Therefore by Lemma 5.2.2 we have that R is Gorenstein. $\square$

We can also, by using the same methods as above in the results for the case of connective DGAs, obtain the corresponding results for the case of coconnective DGAs by replacing the connective versions of previous results with the corresponding coconnective results.

We begin again with the bounded Auslander class and first a lemma for the coconnective case which corresponds to Lemma 5.2.2 in the connective case.

Lemma 5.2.9. *Let k be a field and let R be a degreewise finite coconnective DG k-algebra such that $\mathrm{H}_0(R) = k$ and $\mathrm{H}_{-1}(R) = 0$. If there exists a left homologically bounded and degreewise finite dualising DG-R-module, D, such that $D^f(R) \subseteq \mathscr{A}_D^b(R)$ and $D^f(R^{\text{op}}) \subseteq \mathscr{A}_D^b(R^{\text{op}})$ then R is Gorenstein.*

Proof. This essentially the same as the proof of Lemma 5.2.2 above with the exception that we use 5.1.6 rather than 5.1.5. $\square$

This in turn leads to the coconnective version of the Gorenstein Theorem for the bounded Auslander class.

Theorem 5.2.10. *Let k be a field and let R be a degreewise finite coconnective DG k-algebra such that $\mathrm{H}_0(R) = k$ and $\mathrm{H}_{-1}(R) = 0$. Then R is Gorenstein $\Leftrightarrow \exists$ a dualising DG R-module D which is degreewise finite and homologically bounded to the left such that $\mathscr{A}_D^b(R) = D^b(R)$ and $\mathscr{A}_D^b(R^{\mathrm{op}}) = D^b(R^{\mathrm{op}})$.*

Proof. This is essentially the same as the proof of Theorem 5.2.3 above, the main difference is that it uses Lemma 5.2.9 instead of Lemma 5.2.2. $\square$

The following two Lemmas provide coconnective versions of Lemmas 5.2.4 and 5.2.5.

Lemma 5.2.11. *Let k be a field and let R be a degreewise finite coconnective DG k-algebra such that $H_0(R) = k$ and $H_{-1}(R) = 0$. Let D be a left homologically bounded and degreewise finite dualising DG R-module with $\mathscr{B}_D^b(R) = D^b(R)$ and $\mathscr{B}_D^b(R^{\mathrm{op}}) = D^b(R^{\mathrm{op}})$. Then for ${}_RM \in D^f(R)$ and $N_R \in D^f(R^{\mathrm{op}})$ we have that*

(i) $\mathrm{RHom}_R(D, M) \in D^f(R)$ *and* $\mathrm{RHom}_{R^{\mathrm{op}}}(D, N) \in D^f(R^{\mathrm{op}})$.

(ii) $\mathrm{RHom}_R(M, R) \in D^f(R^{\mathrm{op}})$ *and* $\mathrm{RHom}_{R^{\mathrm{op}}}(N, R) \in D^f(R)$.

(iii) $D \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} M \in D^f(R)$ *and* $N \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} D \in D^f(R^{\mathrm{op}})$

Proof. This proof is essentially identical to that of Lemma 5.2.4 with the sole exception that we use Lemma 5.1.6 in the proof of part (i) in place of Lemma 5.1.5. $\square$

Lemma 5.2.12. *Let k be a field and let R be a degreewise finite coconnective DG k-algebra such that $H_0(R) = k$ and $H_{-1}(R) = 0$. Let D be a left homologically bounded and degreewise finite dualising DG R-module with $D^f(R) \subseteq \mathscr{B}_D^b(R)$. Then for $M \in D^f(R)$ we have that $F(\eta_M)$ is an isomorphism.*

With the above two Lemmas we can now prove a coconnective version of the Gorenstein theorem for the bounded Bass class.

Theorem 5.2.13. *Let k be a field and let R be a degreewise finite coconnective DG k-algebra such that $H_0(R) = k$ and $H_{-1}(R) = 0$. Then R is Gorenstein $\Leftrightarrow \exists$ a dualising DG R-module D which is degreewise finite and homologically bounded to the left such that $\mathscr{B}^b_D(R) = D^b(R)$ and $\mathscr{B}^b_D(R^{\text{op}}) = D^b(R^{\text{op}})$.*

Proof. The proof is essentially the same as that of Theorem 5.2.6 with the main difference being that it makes use of the coconnective versions of various Lemmas involved. □

As one would expect it is also possible to produce for the coconnective case Gorenstein theorems for the finite Auslander and Bass classes.

Theorem 5.2.14. *Let k be a field and let R be a degreewise finite coconnective DG k-algebra such that $\mathrm{H}_0(R) = k$ and $\mathrm{H}_{-1}(R) = 0$. Then R is Gorenstein $\Leftrightarrow \exists$ a dualising DG R-module D which is degreewise finite and homologically bounded to the left such that $\mathscr{A}^b_D(R) = D^b(R)$ and $\mathscr{A}^b_D(R^{\text{op}}) = D^b(R^{\text{op}})$.*

Proof. This is essentially the same as the proof of Theorem 5.2.7 above, the main difference is that it uses Lemma 5.2.9 instead of Lemma 5.2.2. □

Theorem 5.2.15. *Let k be a field and let R be a degreewise finite coconnective DG-k-algebra such that $H_0(R) = k$ and $H_{-1}(R) = 0$. Then R is Gorenstein $\Leftrightarrow \exists$ a dualising DG R-module D which is degreewise finite and homologically bounded to the left such that $\mathscr{B}^f_D(R) = D^f(R)$ and $\mathscr{B}^f_D(R^{\text{op}}) = D^f(R^{\text{op}})$.*

Proof. Again this is essentially the same as in the connective case. □

5.3 Generalised Gorenstein Morphisms

We now look to extend the ring theory concept of a Gorenstein morphism to the DG case. A previous attempt at this was made by Frankild and Jørgensen in [12], however here we shall work in a more general setting by considering DG-bimodules as generalised morphisms of DGAs.

Given two DGAs R and S and a morphism $R \xrightarrow{\rho} S$ of DGAs we can obtain a DG R-module structure on S via $r.s = \rho(r)s$. This in turn leads to the the functor

$$ {}_RS_S \overset{\mathrm{L}}{\underset{S}{\otimes}} - : D(S) \to D(R). $$

We can replace ${}_RS_S$ in the above functor with any DG R-S^{op}-bimodule ${}_RM_S$ to give a functor

$$ {}_RM_S \overset{\mathrm{L}}{\underset{S}{\otimes}} - : D(S) \to D(R). $$

Thus we can view DG-bimodules, in this setting, as being generalised morphisms of DGAs. This approach has been adopted by Keller in [19] and Pauksztello in [24].

Our aim now is to give a definition for a DG version of a Gorenstein morphism for a DG-bimodule considered as a generalised morphism of DGA. We will then use this definition to extend the ring theory result that a Gorenstein morphism allows a base change for the Auslander class and also give a new version of Frankild and Jørgensen's ascent theorem for Gorenstein morphisms, [12, Theorem 3.6].

Notation 5.3.1. Throughout this section unless specified otherwise R and S will denote DGAs over a Noetherian commutative ring k. In addition ${}_RM_S$ will denote a DG R-S^{op}-bimodule, which is compact in $D(S^{\mathrm{op}})$. Also we let ${}_RU_S \xrightarrow{\simeq} {}_RM_S$ be a K-projective resolution in both $D(R)$ and $D(S^{\mathrm{op}})$. Finally

$$ {}_SZ_R = \mathrm{RHom}_{S^{\mathrm{op}}}({}_RM_S, {}_SS_S) $$

will denote the dual of ${}_RM_S$ over S.

Note that U is also a K-projective object in both $D(R)$ and $D(S^{\mathrm{op}})$.

Lemma 5.3.2. *Let the DG-bimodule ${}_RM_S$ be compact in $D(S^{\mathrm{op}})$. Then*

(i) $\mathrm{Hom}_{S^{\mathrm{op}}}({}_RU_S, {}_SS_S)$ *is a K-projective resolution of ${}_SZ_R$ in both $D(S)$ and $D(R^{\mathrm{op}})$.*

(ii) We have the following pair of relations:

(a) ${}_RM_S \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} - \cong \mathrm{RHom}_S({}_SZ_R, -)$,

(b) $- \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} {}_SZ_R \cong \mathrm{RHom}_{S^{\mathrm{op}}}({}_RM_S, -)$.

Proof. (i) Since ${}_RM_S$ is compact in $D(S^{\mathrm{op}})$ we have that

$$ \mathrm{RHom}_S(\mathrm{RHom}_{S^{\mathrm{op}}}({}_RM_S, {}_SS_S), -) \cong {}_RM_S \overset{\mathrm{L}}{\underset{S}{\otimes}} \mathrm{RHom}_S({}_SS_S, -) \cong {}_RM_S \overset{\mathrm{L}}{\underset{S}{\otimes}} -. $$

By replacing ${}_RM_S$ with the K-projective resolution ${}_RU_S$ we obtain that

$$ \mathrm{Hom}_S(\mathrm{Hom}_{S^{\mathrm{op}}}({}_RU_S, {}_SS_S), -) \cong {}_RU_S \underset{S}{\otimes} -. $$

Since ${}_RU_S$ is K-projective the functor ${}_RU_S \underset{S}{\otimes} -$ preserves quasi-isomorphisms, therefore the functor $\operatorname{Hom}_S(\operatorname{Hom}_{S^{\mathrm{op}}}({}_RU_S, {}_SS_S), -)$ also preserves quasi-isomorphisms and so $\operatorname{Hom}_{S^{\mathrm{op}}}({}_RU_S, {}_SS_S)$ is K-projective in $D(S)$. For (ii) See [24, Remark 2.2] and [27, Setup 2.1]. □

Notation 5.3.3. We shall denote by τ the canonical map

$$Z \overset{\mathrm{L}}{\underset{R}{\otimes}} M \xrightarrow{\tau} S$$

which corresponds to the map $\operatorname{Hom}_{S^{\mathrm{op}}}(U, S) \underset{R}{\otimes} U \to S$ given by $\mu \otimes u \mapsto \mu(u)$.

We now give our definition of a generalised Gorenstein morphism for a DG-bimodule.

Definition 5.3.4. The bimodule ${}_RM_S$ is a *generalised Gorenstein morphism from S to R* if it satisfies the following conditions:

(i) M_S is compact in $D(S^{\mathrm{op}})$.

(ii) There exist dualising DG-modules ${}_RD_R$ and ${}_SE_S$ for R and S respectively such that there exist isomorphisms

$$\phi : {}_RD_R \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_RM_S \xrightarrow{\cong} {}_RM_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SE_S$$

and

$$\theta : {}_SZ_R \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_RD_R \xrightarrow{\cong} {}_SE_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SZ_R.$$

(iii) The isomorphisms ϕ and θ are compatible in the sense that the following diagram commutes

$$\begin{tikzcd}
Z \overset{\mathrm{L}}{\underset{R}{\otimes}} D \overset{\mathrm{L}}{\underset{R}{\otimes}} M \arrow[r, "1\otimes\phi"] \arrow[d, "\theta\otimes 1"] & Z \overset{\mathrm{L}}{\underset{R}{\otimes}} M \overset{\mathrm{L}}{\underset{S}{\otimes}} E \arrow[r, "\tau\otimes 1"] & S \overset{\mathrm{L}}{\underset{S}{\otimes}} E \arrow[d, "\cong"] \\
E \overset{\mathrm{L}}{\underset{S}{\otimes}} Z \overset{\mathrm{L}}{\underset{R}{\otimes}} M \arrow[r, "1\otimes\tau"] & E \overset{\mathrm{L}}{\underset{S}{\otimes}} S \arrow[r, "\cong"] & E.
\end{tikzcd}$$

Remark 5.3.5. There is a symmetry between M and Z in the above definition. Combining this with the fact that $\operatorname{RHom}_S({}_SZ_R, {}_SS_S) \cong {}_RM_S$ gives us that when ${}_RM_S$ is a generalised gorenstein morphism from S to R we also have that ${}_SZ_R$ is a generalised gorenstein morphism from S^{op} to R^{op}.

Remark 5.3.6. Let ${}_RM_S$ be a generalised Gorenstein morphism from R to S. If we replace ${}_RM_S$ with ${}_RU_S$ and ${}_RD_R$ and ${}_SE_S$ with the biprojective resolutions ${}_RP_R$ and ${}_SQ_S$ respectively, then the diagram in the definition above becomes:

$$\begin{tikzcd}
\mathrm{Hom}_{S^{\mathrm{op}}}(U,S) \underset{R}{\otimes} P \underset{R}{\otimes} U \arrow[r, "1\otimes\phi"] \arrow[d, "\theta\otimes 1"] & \mathrm{Hom}_{S^{\mathrm{op}}}(U,S) \underset{R}{\otimes} U \underset{S}{\otimes} Q \arrow[r, "\tau\otimes 1"] & S \underset{S}{\otimes} Q \arrow[d] \\
Q \underset{S}{\otimes} \mathrm{Hom}_{S^{\mathrm{op}}}(U,S) \underset{R}{\otimes} U \arrow[r, "1\otimes\tau"] & Q \underset{S}{\otimes} S \arrow[r] & Q
\end{tikzcd}.$$

We can make use of this definition to obtain a DG version of [3, Proposition 3.7(b)], a base change for the Auslander class.

Theorem 5.3.7. *Let the DG R-S^{op}-bimodule ${}_RM_S$ be a generalised Gorenstein morphism from R to S. Then*

$${}_SN \in \mathscr{A}_E(S) \Rightarrow {}_RM_S \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} {}_SN \in \mathscr{A}_D(R),$$

and

$$N'_S \in \mathscr{A}_E(S^{\mathrm{op}}) \Rightarrow N'_S \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} {}_SZ_R \in \mathscr{A}_D(R^{\mathrm{op}}),$$

where ${}_RD_R$ and ${}_SE_S$ are the dualising modules which satisfy conditions (ii) and (iii) in Definition 5.3.4.

Furthermore if the functors $M \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} -$ or $- \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} Z$ reflect isomorphisms then the appropriate reverse implication also holds.

Proof. Since ${}_SN \in \mathscr{A}_E(S)$ we have that the unit morphism

$${}_SN \xrightarrow{\psi_N} \mathrm{RHom}_S({}_SE_S, {}_SE_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SN)$$

is an isomorphism. In order to show that ${}_RM_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SN \in \mathscr{A}_D(R)$ we need to show that the unit morphism

$$M \overset{\mathrm{L}}{\underset{S}{\otimes}} N \xrightarrow{\sigma} \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\underset{R}{\otimes}} M \overset{\mathrm{L}}{\underset{S}{\otimes}} N)$$

is an isomorphism.

Since ${}_RM_S$ is a generalised Gorenstein morphism from R to S we have that M_S is compact in $D(S^{\mathrm{op}})$ so by Lemma 5.3.2 we have the following isomorphisms:

$$ {}_SE_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SZ_R \xrightarrow{\gamma} \mathrm{RHom}_{S^{\mathrm{op}}}({}_RM_S, {}_SE_S) $$

and

$$ {}_RM_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SE_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SN \xrightarrow{\delta} \mathrm{RHom}_S({}_SZ_R, {}_SE_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SN). $$

The compactness of M_S also gives the following isomorphism:

$$ {}_RM_S \overset{\mathrm{L}}{\underset{S}{\otimes}} \mathrm{RHom}_S({}_SE_S, {}_SE_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SN) $$
$$ \xrightarrow{\beta} \mathrm{RHom}_S(\mathrm{RHom}_{S^{\mathrm{op}}}({}_RM_S, {}_SE_S), {}_SE_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SN). $$

Taking these maps together with the the maps θ and ϕ, M being a generalised Gorenstein morphism from R to S, and the adjointness map α allows us to construct the following diagram:

$$
\begin{tikzcd}
M \overset{\mathrm{L}}{\underset{S}{\otimes}} N \arrow[r, "\sigma"] \arrow[d, "1_M \overset{\mathrm{L}}{\underset{S}{\otimes}} \psi"] & \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\underset{R}{\otimes}} M \overset{\mathrm{L}}{\underset{S}{\otimes}} N) \arrow[d, "(\phi \overset{\mathrm{L}}{\underset{S}{\otimes}} 1_N)_*"] \\
M \overset{\mathrm{L}}{\underset{S}{\otimes}} \mathrm{RHom}_S(E, E \overset{\mathrm{L}}{\underset{S}{\otimes}} N) \arrow[d, "\beta"] & \mathrm{RHom}_R(D, M \overset{\mathrm{L}}{\underset{S}{\otimes}} E \overset{\mathrm{L}}{\underset{S}{\otimes}} N) \arrow[d, "\delta_*"] \\
\mathrm{RHom}_S(\mathrm{RHom}_{S^{\mathrm{op}}}(M, E), E \overset{\mathrm{L}}{\underset{S}{\otimes}} N) \arrow[d, "\gamma^*"] & \mathrm{RHom}_R(D, \mathrm{RHom}_S(Z, E \overset{\mathrm{L}}{\underset{S}{\otimes}} N)) \arrow[d, "\alpha"] \\
\mathrm{RHom}_S(E \overset{\mathrm{L}}{\underset{S}{\otimes}} Z, E \overset{\mathrm{L}}{\underset{S}{\otimes}} N) \arrow[r, "\theta^*"] & \mathrm{RHom}_S(Z \overset{\mathrm{L}}{\underset{R}{\otimes}} D, E \overset{\mathrm{L}}{\underset{S}{\otimes}} N)
\end{tikzcd}
.
$$

Hence in order to show that σ is an isomorphism it is enough to show that this diagram commutes.

To do this we begin by replacing D and E with the biprojective resolutions P and Q respectively, ${}_RM_S$ with ${}_RU_S$ a K-projective resolution of M over S and Z with the corresponding K-projective resolution $\mathrm{Hom}_{S^{\mathrm{op}}}(U, S)$. Hence the diagram becomes:

$$\begin{tikzcd}
U \underset{S}{\otimes} N \arrow[r, "\sigma"] \arrow[d, "1\otimes\psi"] & \mathrm{Hom}_R(P, P \underset{R}{\otimes} U \underset{S}{\otimes} N) \arrow[d, "(\phi\otimes 1)_*"] \\
U \underset{S}{\otimes} \mathrm{Hom}_S(Q, Q \underset{S}{\otimes} N) \arrow[d, "\beta"] & \mathrm{Hom}_R(P, U \underset{S}{\otimes} Q \underset{S}{\otimes} N) \arrow[d, "\delta_*"] \\
\mathrm{Hom}_S(\mathrm{Hom}_{S^{\mathrm{op}}}(U,Q), Q \underset{S}{\otimes} N) \arrow[d, "\gamma^*"] & \mathrm{Hom}_R(P, \mathrm{Hom}_S(\mathrm{Hom}_{S^{\mathrm{op}}}(U,S), Q \underset{S}{\otimes} N)) \arrow[d, "\alpha"] \\
\mathrm{Hom}_S(Q \underset{S}{\otimes} \mathrm{Hom}_{S^{\mathrm{op}}}(U,S), Q \underset{S}{\otimes} N) \arrow[r, "\theta^*"] & \mathrm{Hom}_S(\mathrm{Hom}_{S^{\mathrm{op}}}(U,S) \underset{R}{\otimes} P, Q \underset{S}{\otimes} N)
\end{tikzcd}$$

where the morphisms ψ and σ are defined as:

$\psi(n)(q) = (-1)^{|n||q|} q \otimes n$ and $\sigma(u \otimes n)(p) = (-1)^{|p|(|u|+|n|)} p \otimes u \otimes n$.

The isomorphisms β, γ and δ are given by:

$$\beta(u \otimes \psi)(\xi) = (-1)^{|u|(|\psi|+|\xi|)} \psi(\xi(u)),$$

$$\gamma(q \otimes \mu)(u) = q\mu(u)$$

and

$$\delta(u \otimes q \otimes n)(\mu) = (-1)^{|\mu|(|u|+|q|+|n|)} \mu(u) q \otimes n.$$

The adjointness isomorphism α is given by:

$$\alpha(\nu)(\mu \otimes p) = (-1)^{|\mu||p|} \nu(p)(\mu)$$

while for $\mu \in \mathrm{Hom}_{S^{\mathrm{op}}}(U, Q)$, $u \in U$ and $p \in P$, let

$$\theta(\mu \otimes p) = \tilde{q} \otimes \tilde{\mu}$$

and

$$\phi(p \otimes u) = \hat{u} \otimes \hat{q},$$

for some $\hat{u} \in U$, $\tilde{\mu} \in \mathrm{Hom}_{S^{\mathrm{op}}}(U, Q)$ and $\hat{q}, \tilde{q} \in Q$.

We can now check the commutativity:

$$\theta^*(\gamma^*(\beta(1 \otimes \psi(u \otimes n))))(\mu \otimes p)$$

$$= \beta(u \otimes \psi(n))(\gamma(\theta(\mu \otimes p))) = \beta(u \otimes \psi_N(n))(\gamma(\tilde{q} \otimes \tilde{\mu}))$$

$$= (-1)^{|u|(|n|+|\tilde{q}|+|\tilde{mu}|)}\psi(n)(\gamma(\tilde{q}\otimes\tilde{\mu}))(u) = (-1)^{|u|(|n|+|\tilde{q}|+|\tilde{mu}|)}\psi(n)(\tilde{q}\tilde{\mu}(u))$$
$$= (-1)^{|u|(|n|+|\tilde{q}|+|\tilde{mu}|)}(-1)^{|n|(|\tilde{q}|+|\tilde{\mu}|)}\tilde{q}\tilde{\mu}(u)\otimes n$$
$$= (-1)^{(|u|+|n|)(|\tilde{q}|+|\tilde{\mu}|)}\tilde{q}\tilde{\mu}(u)\otimes n.$$

Whilst in the other direction we get:

$$\alpha(\delta_*((\phi\otimes 1)_*(\sigma(u\otimes n))))(\mu\otimes p)$$
$$= (-1)^{|\mu||p|}\delta_*((\phi\otimes 1)_*(\sigma(u\otimes n)))(p)(\mu)$$
$$= (-1)^{|\mu||p|}\delta((\phi\otimes 1)(\sigma(u\otimes n)(p)))(\mu)$$
$$= (-1)^{|\mu||p|}(-1)^{|p|(|u|+|n|)}\delta((\phi\otimes 1)(p\otimes u\otimes n))(\mu)$$
$$= (-1)^{|p|(|\mu|+|u|+|n|)}\delta(\hat{u}\otimes\hat{q}\otimes n)(\mu)$$
$$= (-1)^{|p|(|\mu|+|u|+|n|)}(-1)^{|\mu|(|n|+|\hat{q}|+|\hat{u}|)}\mu(\hat{u})\hat{q}\otimes n$$
$$= (-1)^{|p|(|\mu|+|u|+|n|)+|\mu|(|n|+|\hat{q}|+|\hat{u}|)}\mu(\hat{u})\hat{q}\otimes n.$$

However $|\hat{q}|+|\hat{u}| = |u|+|p|$ and $|\tilde{q}|+|\tilde{\mu}| = |p|+|\mu|$ since $\phi(p\otimes u) = \hat{u}\otimes\hat{q}$ and $\theta(\mu\otimes p) = \tilde{q}\otimes\tilde{\mu}$ so

$$(-1)^{|p|(|\mu|+|u|+|n|)+|\mu|(|n|+|\hat{q}|+|\hat{u}|)} = (-1)^{|p|(|\mu|+|u|+|n|)+|\mu|(|n|+|u|+|p|)}$$
$$= (-1)^{(|p|+|\mu|)(|n|+|u|)} = (-1)^{(|\tilde{q}|+|\tilde{\mu}|)(|n|+|u|)}.$$

Also from Remark 5.3.6 we know that $\mu(\hat{u})\hat{q}\cong\tilde{q}\tilde{\mu}(u)$.

Therefore

$$\alpha(\delta_*((\phi\otimes 1)_*(\sigma(u\otimes n))))(\mu\otimes p) \cong (-1)^{(|u|+|n|)(|\tilde{q}|+|\tilde{\mu}|)}\tilde{q}\tilde{\mu}(u)\otimes n.$$

So the diagram commutes and we have that σ is an isomorphism and hence $M\overset{\mathrm{L}}{\otimes}_S N\in\mathscr{A}_D(R)$.

Similarly, for the reverse implication, when $M\overset{\mathrm{L}}{\otimes}_S N\in\mathscr{A}_D(R)$ we have that σ is an isomorphism and therefore since the diagram commutes we have that $1_M\overset{\mathrm{L}}{\otimes}\psi$ is an isomorphism. So when $M\underset{S}{\overset{\mathrm{L}}{\otimes}}-$ reflects isomorphisms this gives us that ψ is an isomorphism and therefore $N\in\mathscr{A}_E(S)$.

The proof for the right sided version is essentially the same with ${}_SZ_R$ used in place of ${}_RM_S$. $\square$

We now turn our attention to constructing a theorem analogous to the ascent theorem of Frankild and Jørgensen, [12, Theorem 3.6].

Lemma 5.3.8. *Let R be a Gorenstein DGA. Then for any dualising DG R-module, ${}_RD_R$, and ${}_RM \in D^f(R)$ we have that ${}_RD_R \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} {}_RM \in D^f(R)$.*

Proof. Let ${}_RM \in D^f(R)$ and let ${}_RD_R$ be a dualising DG-R-module. Then $D \overset{\mathrm{L}}{\underset{R}{\otimes}} M$ is D-reflexive, i.e.

$$\begin{aligned} D \overset{\mathrm{L}}{\underset{R}{\otimes}} M &\cong \mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R(D \overset{\mathrm{L}}{\underset{R}{\otimes}} M, D), D) \\ &\cong \mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R(M, \mathrm{RHom}_R(D, D)), D) \\ &\cong \mathrm{RHom}_{R^{\mathrm{op}}}(\mathrm{RHom}_R(M, R), D). \end{aligned}$$

Since R is Gorenstein, ${}_RR_R$ is a dualising DG-R-module. Hence

$$\mathrm{RHom}_R(M, R) \in D^f(R^{\mathrm{op}})$$

and therefore, since D is dualising, $D \overset{\mathrm{L}}{\underset{R}{\otimes}} M \in D^f(R)$. ☐

Theorem 5.3.9. ***(Ascent)*** *Let R and S be DGAs. Suppose that there exists a DG R-S^{op}-bimodule ${}_RM_S$ satisfying the following properties:*

(i) *${}_RM_S$ is a generalised Gorenstein morphism from R to S,*

(ii) *The functors $M \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} -$ and $- \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} Z$ reflects isomorphisms,*

(iii) *${}_SN \in D^f(S) \Leftrightarrow M \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} N \in D^f(R)$ and $N'_S \in D^f(S^{\mathrm{op}}) \Leftrightarrow N' \overset{\mathrm{L}}{\underset{\mathrm{S}}{\otimes}} Z \in D^f(R^{\mathrm{op}})$.*

Then R is a Gorenstein DGA $\Rightarrow$ S is a Gorenstein DGA.

Proof. Let ${}_SN \in D^f(S)$. We want to show that the conditions [G1] and [G2] of the definition of a Gorenstein DGA are satisfied.

For condition [G1] we need to show that ${}_SN$ is S-reflexive.

By our assumptions we have that ${}_RM_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SN \in D^f(R)$. Then since R is Gorenstein we have that ${}_RM_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SN$ is R-reflexive and hence by Prop. 5.2.1 that ${}_RM_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SN \in \mathscr{A}_D(R)$ for any dualising DG R-module, D.

Since ${}_RM_S \overset{\mathrm{L}}{\underset{S}{\otimes}} -$ reflects isomorphisms we have from Theorem 5.3.7 that ${}_SN \in \mathscr{A}_E(S)$. Thus, since ${}_SN \in D^f(S)$, we have from Prop. 5.2.1 that ${}_SN$ is S-reflexive as required.

The right sided part of [G1] is proved in a similar way, except that we use ${}_SZ_R$ rather than ${}_RM_S$.

Now to show condition [G2] we need that $\mathrm{RHom}_S(N,S) \in D^f(S^{\mathrm{op}})$.

To do this we first note that since ${}_SE_S$ is a dualising DG S-module we have that

$$\begin{aligned}\mathrm{RHom}_S(N,S) &\cong \mathrm{RHom}_S(N,\mathrm{RHom}_S(E,E))\\ &\overset{\text{adj.}}{\cong} \mathrm{RHom}_S(E \overset{\mathrm{L}}{\underset{S}{\otimes}} N, E).\end{aligned}$$

Therefore, as ${}_SE_S$ is dualising, it is sufficient to show that ${}_SE_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SN \in D^f(S)$.

By our assumptions we have that

$$E \overset{\mathrm{L}}{\underset{S}{\otimes}} N \in D^f(S) \text{ iff } M \overset{\mathrm{L}}{\underset{S}{\otimes}} E \overset{\mathrm{L}}{\underset{S}{\otimes}} N \in D^f(R).$$

However, M being a generalised Gorenstein morphism gives us that,

$$M \overset{\mathrm{L}}{\underset{S}{\otimes}} E \overset{\mathrm{L}}{\underset{S}{\otimes}} N \cong D \overset{\mathrm{L}}{\underset{R}{\otimes}} M \overset{\mathrm{L}}{\underset{S}{\otimes}} N.$$

Since $M \overset{\mathrm{L}}{\underset{S}{\otimes}} N \in D^f(R)$ we have by Lemma 5.3.8 that $D \overset{\mathrm{L}}{\underset{R}{\otimes}} M \overset{\mathrm{L}}{\underset{S}{\otimes}} N \in D^f(R)$, so $M \overset{\mathrm{L}}{\underset{S}{\otimes}} E \overset{\mathrm{L}}{\underset{S}{\otimes}} N \in D^f(R)$ and hence $E \overset{\mathrm{L}}{\underset{S}{\otimes}} N \in D^f(R)$ as required.

Again the prove of the right sided condition of [G2] is similar.

□

We now end this section with the following theorem which deals with the special case where we have a morphism $\rho : R \longrightarrow S$ of DGAs and gives a situation where the DG-bimodule ${}_RS_S$, obtained from the morphism, is a generalised Gorenstein morphism from R to S. The related corollary deals with the yet more specific case in which we have a morphism of DGAs from a commutative base ring, A, to a DG A-algebra, S, and lays out criteria under which the DG-bimodule ${}_AS_S$ is a generalised Gorenstein morphism from A to S.

Theorem 5.3.10. *Let R and S be DGAs with R commutative. Let $\rho : R \to S$ be a morphism of DGAs such that the image of ρ is in the centre of S. If there exists a* *symmetric dualising DG R-module ${}_RD_R$ and a dualising DG S-module ${}_SE_S$ such that there exists an isomorphism:*

$$\Phi : {}_SS_{S,R} \overset{\mathrm{L}}{\underset{\mathrm{R}}{\otimes}} {}_RD \longrightarrow {}_SE_S$$

of DG S-bimodules, then the bimodule ${}_RS_S$ is a generalised Gorenstein morphism from R to S.

Proof. Let $\pi : {}_SP_R \to {}_SS_R$ be K-projective resolution for S when considered as DG R-module and let ${}_RQ_R$ be a symmetric biprojective resolution for ${}_RD_R$.

Since R is commutative we have that there exist isomorphisms:

$$\upsilon : {}_SS_R \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_RD_R \longrightarrow ({}_SS_{S,R} \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_RD) \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SS_R$$

and

$$\nu : {}_RD_R \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_RS_S \longrightarrow {}_RS_S \overset{\mathrm{L}}{\underset{S}{\otimes}} ({}_SS_{S,R} \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_RD).$$

These correspond to the isomorphisms:

$$\upsilon : {}_SP_R \underset{R}{\otimes} {}_RQ_R \longrightarrow ({}_SS_{S,R} \underset{R}{\otimes} {}_RQ) \underset{S}{\otimes} {}_SS_R$$

and

$$\nu : {}_RQ_R \underset{R}{\otimes} {}_RP_S \longrightarrow {}_RS_S \underset{S}{\otimes} ({}_SS_{S,R} \underset{R}{\otimes} {}_RQ),$$

given by $\upsilon(p \otimes q) = \pi(p) \otimes q \otimes 1_S$ and $\nu(q \otimes p) = (-1)^{|p||q|} 1_S \otimes \pi(p) \otimes q$.

These together with the isomorphism Φ allow us to construct the following two isomorphisms:

$$\theta = (\Phi \otimes 1) \circ \upsilon : {}_SS_S \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_RD_R \longrightarrow {}_SE_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SS_R$$

and

$$\phi = (1 \otimes \Phi) \circ \nu : {}_RD_R \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_RS_S \longrightarrow {}_RS_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_SE_S.$$

It now remains to check the commutativity of the diagram:

$$\begin{tikzcd}
{}_S S_R \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_R D_R \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_R S_S \arrow[r, "1\otimes\phi"] \arrow[d, "\theta\otimes 1"] & {}_S S_R \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_R S_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_S E_S \arrow[r, "\tau\otimes 1"] & {}_S S_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_S E_S\ . \arrow[d] \\
{}_S E_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_S S_R \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_R S_S \arrow[r, "1\otimes\tau"] & {}_S E_S \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_S S_S \arrow[r] & {}_S E_S.
\end{tikzcd}$$

Replacing the appropriate Modules with their K-projective resolutions gives us following corresponding diagram:

$$\begin{tikzcd}
{}_S P_R \underset{R}{\otimes} {}_R Q_R \underset{R}{\otimes} {}_R P_S \arrow[r, "1\otimes\phi"] \arrow[d, "\theta\otimes 1"] & {}_S P_R \underset{R}{\otimes} {}_R S_S \underset{S}{\otimes} {}_S E_S \arrow[r, "\tau\otimes 1"] & {}_S S_S \underset{S}{\otimes} {}_S E_S \arrow[d] \\
{}_S E_S \underset{S}{\otimes} {}_S S_R \underset{R}{\otimes} {}_R P_S \arrow[r, "1\otimes\tau"] & {}_S E_S \underset{S}{\otimes} {}_S S_S \arrow[r] & {}_S E_S.
\end{tikzcd}$$

Going from the top left to the bottom right in one direction gives us:

$$(\tau \otimes 1) \circ (1 \otimes \phi)(p \otimes q \otimes \tilde{p}) = (-1)^{|\tilde{p}||q|}(\tau \otimes 1)(p \otimes 1_S \otimes \Phi(\pi(\tilde{p}) \otimes q))$$
$$= (-1)^{|\tilde{p}||q|}\pi(p) \otimes \Phi(\pi(\tilde{p}) \otimes q) \cong (-1)^{|\tilde{p}||q|}\pi(p)\Phi(\pi(\tilde{p}) \otimes q)$$
$$= \pi(p)\Phi(1_S \otimes q)\pi(\tilde{p}).$$

Whilst in the other direction we get that

$$(1 \otimes \tau) \circ (\theta \otimes 1)(p \otimes q \otimes \tilde{p}) = \Phi(\pi(p) \otimes d) \otimes 1_S \otimes \tilde{p}$$
$$= \Phi(\pi(p) \otimes d) \otimes \pi(\tilde{p}) = \Phi(\pi(p) \otimes d)\pi(\tilde{p})$$
$$= \pi(p)\Phi(1_S \otimes d)\pi(\tilde{p}).$$

Hence the diagram commutes and therefore ${}_R S_S$ is a generalised Gorenstein morphism from R to S.

□

Corollary 5.3.11. *Let A be a commutative ring and let S be a DG A-algebra such that there exists a morphism $A \xrightarrow{\rho} S$, which satisfy the following properties:*

(i) ${}_A S$ is compact as a DG A-module.

(ii) There exists, D, a symmetric dualising complex for A such that $\mathrm{RHom}_A(S, D)$ *is a dualising DG S-module.*

(iii) The image of ρ is contained inside the centre of S.

(iv) There is an isomorphism of DG S-bimodules,

$$_SS_S \longrightarrow \mathrm{RHom}_A({_SS_S}, \Sigma^n A).$$

Then the DG-bimodule $_AS_S$ is a generalised Gorenstein morphism from R to S.

Proof. We begin by noting that as $\mathrm{RHom}_A(S, D)$ is a dualising DG S-module, we also have that $\Sigma^n \mathrm{RHom}_A(S, D)$ is a dualising DG-S-module as well.

Also

$$_SS_{S,A} \overset{\mathrm{L}}{\underset{A}{\otimes}} {_AD} \overset{(a)}{\cong} \mathrm{RHom}_A({_SS_S}, \Sigma^n A) \overset{\mathrm{L}}{\underset{A}{\otimes}} {_AD}$$

$$\overset{(b)}{\cong} \mathrm{RHom}_A({_SS_S}, \Sigma^n A \overset{\mathrm{L}}{\underset{A}{\otimes}} {_AD}) \cong \Sigma^n \mathrm{RHom}_A({_SS_S}, D),$$

where (a) is due to the isomorphism $_SS_S \to \mathrm{RHom}_A({_SS_S}, \Sigma^n A)$ and (b) is due to the compactness of $_AS$.

We therefore have by Theorem 5.3.10 that $_AS_S$ is a generalised Gorenstein morphism from R to S. $\square$

5.4 Examples of Generalised Gorenstein Morphisms

Our first example of a generalised gorenstein morphism below is the almost trivial case of a generalised gorenstein morphism from a commutative DGA R to itself.

Theorem 5.4.1. *Let R be a commutative DGA with a symmetric bimodule $_RM_R$ and a symmetric dualising module $_RD_R$. If M is a compact R module then M is a generalised gorenstein morphism from R to itself.*

Proof. We want to show that when M is compact it satisfies the conditions of definition 5.3.4 with $R = S$.

Condition (i) holds by assumption.

For condition (ii), since $R = S$ we can set $D = E$ and since both M and D are symmetric we have that the morphisms

$$\phi : {}_R D_R \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_R M_R \xrightarrow{\cong} {}_R M_R \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_R D_R$$

and

$$\theta : {}_R Z_R \overset{\mathrm{L}}{\underset{R}{\otimes}} {}_R D_R \xrightarrow{\cong} {}_R D_R \overset{\mathrm{L}}{\underset{S}{\otimes}} {}_R Z_R.$$

Finally for condition (iii), we need to show that the diagram below commutes,

$$\begin{tikzcd}
Z \overset{\mathrm{L}}{\underset{R}{\otimes}} D \overset{\mathrm{L}}{\underset{R}{\otimes}} M \arrow[r, "1\otimes\phi"] \arrow[d, "\theta\otimes 1"] & Z \overset{\mathrm{L}}{\underset{R}{\otimes}} M \overset{\mathrm{L}}{\underset{R}{\otimes}} D \arrow[r, "\tau\otimes 1"] & R \overset{\mathrm{L}}{\underset{R}{\otimes}} D \arrow[d, "\cong"] \\
D \overset{\mathrm{L}}{\underset{R}{\otimes}} Z \overset{\mathrm{L}}{\underset{R}{\otimes}} M \arrow[r, "1\otimes\tau"] & D \overset{\mathrm{L}}{\underset{R}{\otimes}} R \arrow[r, "\cong"] & D.
\end{tikzcd}$$

Let $z \in Z$, $d \in D$ and $m \in M$ then

$$(\tau \otimes 1) \circ (1 \otimes \phi)(z \otimes d \otimes m) = z(m) \otimes d = z(m).d$$

while

$$(1 \otimes \tau) \circ (\theta \otimes 1) = d \otimes z(m) = d.z(m),$$

However since D and M are symmetric we have that $z(m).d = d.z(m)$ and the diagram commutes as required. □

The next example involves The Endomorphism DGA from example 3.1.20 and the notation used in the results below is the same as that used in the example.

Theorem 5.4.2. *The DG-bimodule ${}_A\mathscr{E}_{\mathscr{E}}$ is a generalised Gorenstein morphism from A to $\mathscr{E}$.*

Proof. To do this, it suffices to show that, for the DGAs A and $\mathscr{E}$ and the morphism $\phi_{\mathscr{E}}$, the the conditions of Corollary 5.3.11 are satisfied.

Since $\mathscr{E}$ is a bounded complex of finitely generated projective A-modules we have that $\mathscr{E}$ is compact in $D(A)$.

By 3.2.21 we have that for a dualising complex C of A that $\mathrm{RHom}_A(\mathscr{E}, C)$ is a dualising module for $\mathscr{E}$.

The image of $\phi_{\mathscr{E}}$ in contained in the centre of $\mathscr{E}$, since for an element $e \in \mathscr{E}_i$ we have that

$$(\phi_{\mathscr{E}}(a)e)(l) = (\Sigma^{-}i(\phi_{\mathscr{E}}(a)) \circ e)(l) = ae(l)$$
$$= e(al) = (e \circ \phi_{\mathscr{E}}(a))(l) = (e\phi_{\mathscr{E}}(a))(l).$$

Finally we need to show that there is an isomorphism

$${}_{\mathscr{E}}\mathscr{E}_{\mathscr{E}} \to \mathrm{RHom}_A({}_{\mathscr{E}}\mathscr{E}_{\mathscr{E}}, \Sigma^n A).$$

Since $\mathscr{E}$ is K-projective over A we have that $\mathrm{RHom}_A({}_{\mathscr{E}}\mathscr{E}_{\mathscr{E}}, \Sigma^n A) \cong \mathrm{Hom}_A({}_{\mathscr{E}}\mathscr{E}_{\mathscr{E}}, \Sigma^n A)$ and by the compactness of ${}_A\mathscr{E}$ we have that all the morphisms in the diagram,

$$\begin{tikzcd}
{}_{\mathscr{E}}\mathscr{E}_{\mathscr{E}} \arrow[r, "\cong"] & \mathrm{Hom}_A({}_{A,\mathscr{E}}L, {}_{A,\mathscr{E}}L) \arrow[d, "\cong"] & \\
& \mathrm{Hom}_A({}_{A,\mathscr{E}}L, A \underset{A}{\otimes} {}_{A,\mathscr{E}}L) & \\
& \mathrm{Hom}_A({}_{A,\mathscr{E}}L, A) \underset{A}{\otimes} {}_{A,\mathscr{E}}L \arrow[u, "\cong"] \arrow[d, "\cong"] & \\
& \mathrm{Hom}_A(\mathrm{Hom}_A({}_{A,\mathscr{E}}L, {}_{A,\mathscr{E}}L), A) \arrow[r, "\cong"] & \mathrm{Hom}_A({}_{\mathscr{E}}\mathscr{E}_{\mathscr{E}}, A)
\end{tikzcd},$$

are isomorphisms

Hence we have an isomorphism, ${}_{\mathscr{E}}\mathscr{E}_{\mathscr{E}} \to \mathrm{RHom}_A({}_{\mathscr{E}}\mathscr{E}_{\mathscr{E}}, \Sigma^n A)$, as required.

□

Finally we can apply the ascent theorem to this example.

Theorem 5.4.3. *A is a Gorenstein ring $\Rightarrow$ $\mathscr{E}$ is a Gorenstein DGA.*

Proof. By [12, proposition 2.5] we have that A when viewed as a DGA concentrated in degree 0 is a Gorenstein DGA. We can now use Theorem 5.3.9 with ${}_RM_S = {}_A\mathscr{E}_{\mathscr{E}}$. We have shown in Theorem 5.4.2 above that ${}_A\mathscr{E}_{\mathscr{E}}$ is a Gorenstein morphism from A to $\mathscr{E}$ thus satisfying condition (i).

Conditions (ii) and (iii) hold since A being the base ring. □

The following example involves The Koszul complex as defined in example 3.1.21, yet again the notation used in the results below is the same as that used in the example.

Theorem 5.4.4. *The DG A-$K(\boldsymbol{a})$-bimodule $K(\boldsymbol{a})$ is a generalised Gorenstein morphism from A to $K(\boldsymbol{a})$.*

Proof. To do this, it suffices to show that the conditions of Corollary 5.3.11 are satisfied.

Since $K(\mathbf{a})$ is a bounded complex of finitely generated projective A-modules we have that $K(\mathbf{a})$ is compact in $D(A)$.

By 3.2.20 we have that for a dualising complex C of A that $\mathrm{RHom}_A(K(\mathbf{a}), C)$ is a dualising module for $K(\mathbf{a})$.

The morphism $\phi_{K(\mathbf{a})}$ has image in the centre of $K(\mathbf{a})$, since, as $K(\mathbf{a})$ is commutative, the centre is all of $K(\mathbf{a})$.

Finally it remains to show that there is an isomorphism $K(\mathbf{a}) \longrightarrow \mathrm{RHom}_A(K(\mathbf{a}), \Sigma^n A)$. The degree n component of $K(\mathbf{a})$ is A itself the projection of $K(\mathbf{a})$ onto its degree n component is of the form $K(\mathbf{a}) \xrightarrow{\pi} \Sigma^n A$. We can now define an isomorphism

$$K(\mathbf{a}) \xrightarrow{\cong} \mathrm{Hom}_A(K(\mathbf{a}), \Sigma^n A),$$

given by $k \mapsto (l \mapsto \pi(k \wedge l))$.

Since $K(\mathbf{a})$ is K-projective over A we have that

$$\mathrm{RHom}_A(K(\mathbf{a}), \Sigma^n A) \cong \mathrm{Hom}_A(K(\mathbf{a}), \Sigma^n A)$$

and hence we have an isomorphism

$$K(\mathbf{a}) \longrightarrow \mathrm{RHom}_A(K(\mathbf{a}), \Sigma^n A).$$

□

We can now apply our ascent theorem to this situation.

Theorem 5.4.5. *A is a Gorenstein ring $\Rightarrow$ $K(\boldsymbol{a})$ is a Gorenstein DGA.*

Proof. This is almost identical to the proof of Theorem 5.4.3, We have from Theorem 5.4.4 above that condition (i) of Theorem 5.3.9 holds. Again conditions (ii) and (iii) hold due to the fact that A is the base ring. □

The final example again involves endomorphism DGAs with the following setup.

Setup 5.4.6. Let k be a field and let Λ be a finite dimensional k-algebra. Let ${}_{\Lambda}L$ be a complex of finitely generated projective Λ-modules. Define the Endomorphism DGA $\mathscr{E} = \mathrm{Hom}_{\Lambda}(L, L)$.

Lemma 5.4.7. *${}_{\Lambda}\Lambda_{\Lambda}$ is a dualising Λ-module.*

Proof. Since k is a field we have that it is a dualising complex over itself, so by 3.2.18 we have that $\mathrm{RHom}_k(\Lambda, k)$ is a dualising Λ-module. However since Λ is symmetric we have that $\Lambda \cong \mathrm{RHom}_k(\Lambda, k)$ so Λ is itself a dualising Λ-module. $\square$

Lemma 5.4.8. *${}_{\mathscr{E}}\mathscr{E}_{\mathscr{E}}$ is a dualising $\mathscr{E}$-module.*

Proof. By [10, Prop 2.6] we have that $\mathrm{RHom}_k(\mathscr{E}, k)$ is a dualising $\mathscr{E}$-module. Since $\mathscr{E}$ is K-projective over k we have that

$$\mathrm{RHom}_k(\mathscr{E}, k) \cong \mathrm{Hom}_k(\mathscr{E}, k)$$

and

$$\begin{aligned}\mathrm{Hom}_k(\mathscr{E}, k) \cong \mathrm{Hom}_k(\mathrm{Hom}_{\Lambda}(L, L), k) &\overset{(a)}{\cong} \mathrm{Hom}_k(L, k) \overset{\mathrm{L}}{\underset{\Lambda}{\otimes}} L \\ &\overset{(b)}{\cong} \mathrm{Hom}_{\Lambda}(L, \Lambda) \overset{\mathrm{L}}{\underset{\Lambda}{\otimes}} L \overset{(c)}{\cong} \mathrm{Hom}_{\Lambda}(L, L) \cong \mathscr{E},\end{aligned}$$

where (a) and (c) are due to L being a bounded complex of projective modules and (b) is from

$$\mathrm{Hom}_k(L, \Lambda) \cong \mathrm{Hom}(L, \mathrm{Hom}_k(\Lambda, k)) \overset{\mathrm{adj}}{\cong} \mathrm{Hom}_k(L, k).$$

Hence $\mathscr{E}$ is a dualising $\mathscr{E}$-module. $\square$

Theorem 5.4.9. *${}_{\Lambda,\mathscr{E}}L$ is a generalised Gorenstein morphism from $\mathscr{E}$ to Λ^{op}.*

Proof. We do this by showing directly that ${}_{\Lambda,\mathscr{E}}L$ satisfies the conditions of Definition 5.3.4, with $R = \mathscr{E}$ and $S = \Lambda^{\mathrm{op}}$.

For condition (i), since ${}_{\Lambda}L$ is a bounded complex of projective modules it is compact in $D(\Lambda)$.

For conditions (ii) and (iii), by Lemmas 5.4.7 and 5.4.8 we can take ${}_{\Lambda}\Lambda_{\Lambda}$ and ${}_{\mathscr{E}}\mathscr{E}_{\mathscr{E}}$ to be the dualising modules for Λ^{op} and $\mathscr{E}$, the isomorphisms ϕ and θ are then the identity isomorphisms and the commutativity of the diagram in condition (iii) becomes trivial. $\square$

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