Braid group actions and quasi K-matrices for quantum symmetric pairs

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Abstract

Quantum groups arose in the early 80's in the investigation of integrable systems in mathematical physics. Quantum groups are a family of non-commutative, non-cocommutative Hopf algebras which arise through deformation quantisation of universal enveloping algebras of Lie algebras or of coordinate rings of affine algebraic groups. In this thesis, we focus on quantum groups coming from universal enveloping algebras, known as 'quantised enveloping algebras'.

One of the fundamental properties of quantised enveloping algebras is that they give rise to a universal R-matrix which provides solutions of the quantum Yang-Baxter equation for each representation. The universal R-matrix allows applications of quantum groups in the construction of invariants of knots and links. The main component of the universal R-matrix is a quasi R-matrix, which has applications in other areas of representation theory, for instance in Lusztig's and Kashiwara's theory of canonical bases. Also essential to the theory of quantised enveloping algebras is the existence of a braid group action by algebra automorphisms, due to Lusztig. This braid group action allows the definition of root vectors and PBW bases.

Parallel to quantised enveloping algebras is the notion of quantum symmetric pair coideal subalgebras, developed by G. Letzter in a series of papers from 1999 to 2004. These are quantum group analogues of Lie subalgebras which are fixed under an involution. Over the past five years it has become increasingly clear that many of the results for quantised enveloping algebras have analogues in the quantum symmetric pair setting. An important example of this is the construction of a universal K-matrix for quantum symmetric pairs by Balagović and Kolb following earlier work by Bao and Wang. The universal K-matrix provides solutions to the reflection equation, which is an analogue of the quantum Yang-Baxter equation. The main ingredient of the universal K-matrix is a quasi K-matrix which is an analogue of the quasi R-matrix. The quasi K-matrix recently played a crucial role in the theory of canonical bases for quantum symmetric pairs, developed by Bao and Wang.

Until recently, only a recursive formula for the quasi K-matrix was known. The first main result of this thesis is to give an explicit formula for the quasi K-matrix in many cases. This formula closely resembles the known formula for the quasi R-matrix, which admits a factorisation as a product of rank one quasi R-matrices. In particular, the quasi K-matrix has a factorisation into a product of quasi K-matrices for Satake diagrams of rank one. This factorisation depends on the restricted Weyl group of the symmetric Lie algebra similarly to how the quasi R-matrix depends on the Weyl group of the Lie algebra. The key idea is to calculate the quasi K-matrix explicitly in rank one and in rank two.

Lusztig's braid group action is then used to build the quasi K-matrix in higher rank. We conjecture that the resulting formula holds in full generality.

In the second part of this thesis, we investigate the analogue of Lusztig's braid group action in the quantum symmetric pair setting. It was conjectured by Molev and Ragoucy, and more generally by Kolb and Pellegrini that there are two braid group actions on the quantum symmetric pair coideal subalgebras, one of which comes from the restricted Weyl group. This is known to be true in many cases where the underlying Satake diagram has either no black nodes or a trivial diagram automorphism. Lusztig's automorphisms for the restricted Weyl group do not leave the quantum symmetric pair coideal subalgebra invariant. Nevertheless, they can still be used as a useful guide in the constructions. Here, we consider Satake diagrams of type AIII, which is the first instance involving black nodes and a non-trivial diagram automorphism. We show that the braid group of the restricted Weyl group acts on the quantum symmetric pair coideal subalgebra by algebra automorphisms if the underlying Satake diagram has at most two black nodes. To assist in the verification of braid group and algebra relations, we rely on the package `QUAGROUP` of the computer algebra program `GAP`.

Declaration of collaborative work

My thesis contains collaborative work with my supervisor Dr S. Kolb. We have one joint paper [16]. This project started out when I explicitly calculated the quasi K-matrix in type AI for $\mathfrak{sl}_3$. I observed a factorisation in this case which led my supervisor to suggest a connection to the restricted Weyl group. I have performed the calculations of the rank one and rank two quasi K-matrices in Section 5.3 and Chapter 6 independently. We developed the theoretical statements of Sections 5.3 and 5.4 jointly in our weekly meetings. My supervisor wrote Section 5.5. I wrote the first draft of the paper [16] after which we iteratively revised during our weekly meetings.

Contents

Chapter 1

Introduction

1.1 Quantum symmetric pairs

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $U_q(\mathfrak{g})$ the corresponding Drinfeld-Jimbo quantised enveloping algebra. Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be an involutive Lie algebra automorphism and $\mathfrak{k} = \{x \in \mathfrak{g} \mid \theta(x) = x\}$ the corresponding fixed Lie subalgebra. We call the pair $(\mathfrak{g}, \mathfrak{k})$ a *symmetric pair*. Involutive automorphisms are parameterised up to conjugation by combinatorial data attached to the Dynkin diagram of $\mathfrak{g}$ known as *Satake diagrams* (I, X, τ). Here I denotes the nodes of the Dynkin diagram, X denotes a subset of I and τ denotes a diagram automorphism. Through Satake diagrams we obtain a classification for symmetric pairs, see [1] and also [38].

Quantum symmetric pairs provide quantum group analogues of the universal enveloping algebra $U(\mathfrak{k})$. In particular, families of subalgebras $B_{\mathbf{c},\mathbf{s}} \subset U_q(\mathfrak{g})$ are constructed which depend on parameters $\mathbf{c}$ and $\mathbf{s}$, see [42], [44] and [38]. Such subalgebras are quantum analogues of $U(\mathfrak{k})$ in the sense that $B_{\mathbf{c},\mathbf{s}}$ specialises to $U(\mathfrak{k})$ as q tends to 1. The crucial property of $B_{\mathbf{c},\mathbf{s}}$ is that it is a *right coideal subalgebra* of $U_q(\mathfrak{g})$, meaning

$$\Delta(B_{\mathbf{c},\mathbf{s}}) \subseteq B_{\mathbf{c},\mathbf{s}} \otimes U_q(\mathfrak{g})$$

where Δ denotes the coproduct of $U_q(\mathfrak{g})$. We call the pair $(U_q(\mathfrak{g}), B_{\mathbf{c},\mathbf{s}})$ a quantum symmetric pair.

The origin of quantum symmetric pairs lies in the theory of quantum integrable systems with boundary. The integrability of such systems required solutions of the so-called *reflection equation*, an analogue of the well-known quantum Yang-Baxter equation. The pioneers in this field were I. Cherednik [12] and E. Sklyanin [61] who studied factorised scattering on a half-line and lattice models with boundary conditions using the quantum inverse scattering method.

From the perspective of noncommutative geometry, quantum symmetric pairs give rise to quantum homogeneous spaces. In particular, instead of studying quantum groups that arise as deformations of universal enveloping algebras of Lie algebras, one looks at deformations of coordinate rings of affine algebraic groups. The first examples of quantum homogeneous spaces to be studied in detail were due to P. Podleś [56] and L. Vaksman and Y. Soibelman [64]. Podleś investigated quantum 2-spheres, that is, quantum homogeneous spaces of the quantum group $SU_q(2)$, while Vaksman and Soibelman considered higher dimensional spheres. A historical overview from this viewpoint can be found in [36, Section 11.7].

In order to perform harmonic analysis on quantum analogues of compact symmetric spaces, M. Noumi, T. Sugitani and M. Dijkhuizen developed a theory of quantum symmetric pairs for classical Lie algebras, see [54], [15] and [55] which was based on solutions of the reflection equation. The starting point for these constructions was T. Koornwinder's observation in [41] that the Podleś quantum sphere can be realised as infinitesimal invariants for a twisted primitive element in the quantised enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$. A comprehensive theory of quantum symmetric pairs avoiding casework was developed by G. Letzter in [42], [43] and [44], also with harmonic analysis in mind. In this setting, the construction of quantum symmetric pairs only relies on the Drinfeld-Jimbo presentation of quantised enveloping algebras and on involutive automorphisms of $\mathfrak{g}$. An extension of the theory to the Kac-Moody case was developed in [38].

Recently, quantum symmetric pairs have appeared in many different contexts. In [5], H. Bao and W. Wang constructed canonical bases for quantum symmetric pair coideal subalgebras of type AIII/AIV. The theory of canonical bases has many applications, including category $\mathcal{O}$, algebraic combinatorics, categorification and geometric representation theory. Independently, M. Ehrig and C. Stroppel observed a link between quantum symmetric pairs of type AIII/AIV and the type D category $\mathcal{O}$ in [18]. This places quantum symmetric pairs in a much broader representation theoretic context.

Both of the papers [5] and [18] consider a bar involution for quantum symmetric pair coideal subalgebras of type AIII/AIV. Moreover, Bao and Wang construct an intertwiner $\mathfrak{X}$ (denoted by Υ in [5]) between the bar involutions on $B_{\mathbf{c},\mathbf{s}}$ and on $U_q(\mathfrak{g})$. The intertwiner $\mathfrak{X}$ is an analogue of the quasi R-matrix for $U_q(\mathfrak{g})$. Using the intertwiner $\mathfrak{X}$, Bao and Wang show that large parts of Lusztig's theory of canonical bases [51, Part IV] extend to the theory of quantum symmetric pairs. More recently, Bao and Wang have extended the theory of canonical bases to all quantum symmetric pairs of finite type, [6].

Following the program of [5], the existence of a bar involution for $B_{\mathbf{c},\mathbf{s}}$ was established in full generality by Balagović and Kolb in [3]. In the sequel [4] it was proved that the intertwiner $\mathfrak{X}$ exists for general quantum symmetric pairs. This was used in [4] to construct a universal K-matrix for $B_{\mathbf{c},\mathbf{s}}$, which is an analogue of the universal R-matrix for $U_q(\mathfrak{g})$. It is for this reason that Balagović and Kolb call the intertwiner $\mathfrak{X}$ the *quasi K-matrix* for $B_{\mathbf{c},\mathbf{s}}$. It has recently been discovered in [39] that the universal K-matrix gives suitable categories of $B_{\mathbf{c},\mathbf{s}}$-modules the structure of a braided monoidal category and hence has applications in low-dimensional topology.

1.2 Quasi K-matrices for quantum symmetric pairs

Let U^+ and U^- denote the positive and negative parts of the Drinfeld-Jimbo quantised enveloping algebra $U_q(\mathfrak{g})$, respectively. The quasi R-matrix for $U_q(\mathfrak{g})$ is a canonical element in a completion of $U^- \otimes U^+$ that plays a pivotal role in many applications for quantum groups. In the theory of canoncial or crystal bases developed by G. Lusztig [50] and M. Kashiwara [31], the quasi R-matrix appears as an intertwiner of two bar involutions on $\Delta(U_q(\mathfrak{g}))$. The quasi R-matrix is used to define canonical bases of tensor products of $U_q(\mathfrak{g})$-modules, see [51, Part IV].

We denote the quasi R-matrix of $U_q(\mathfrak{g})$ by R as in the paper [4]. For each $i \in I$ let $U_q(\mathfrak{sl}_2(i))$ denote the subalgebra of $U_q(\mathfrak{g})$ generated by E_i, F_i and $K_i^{\pm 1}$. One of the key properties of the quasi R-matrix is that it admits a factorisation as a product of quasi R-matrices for $U_q(\mathfrak{sl}_2(i))$. Let $\{\alpha_i \mid i \in I\}$ denote the set of simple roots of $\mathfrak{g}$. Then the quasi R-matrix corresponding to $U_q(\mathfrak{sl}_2(i))$ is given by

$$R_i = \sum_{r \geq 0} (-1)^r q_i^{-r(r-1)/2} \frac{(q_i - q_i^{-1})^r}{[r]_{q_i}!} F_i^r \otimes E_i^r \tag{1.1}$$

where $q_i = q^{(\alpha_i, \alpha_i)/2}$. In order to build the quasi R-matrix in general from quasi R-matrices for $U_q(\mathfrak{sl}_2(i))$ we use the braid group action on $U_q(\mathfrak{g})$ by algebra automorphisms. Let σ_i for $i \in I$ denote the generators of the Weyl group W of $\mathfrak{g}$ and let $T_i : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ denote the corresponding Lusztig automorphisms. For any reduced expression $w_0 = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_t}$ of the longst word $w_0 \in W$ define

$$R^{[j]} = (T_{i_1} \cdots T_{i_{j-1}} \otimes T_{i_1} \cdots T_{i_{j-1}})(R_{i_j}) \quad \text{for } j = 1, \ldots, t. \tag{1.2}$$

The quasi R-matrix for $U_q(\mathfrak{g})$ is then given by

$$R = R^{[t]} \cdot R^{[t-1]} \cdots R^{[2]} \cdot R^{[1]}, \tag{1.3}$$

see [35], [48], [34] and [27, 8.30]. This is independent of the chosen reduced expression for w_0.

In Chapters 5 and 6 we construct an analogue of (1.3) in the setting of quantum symmetric pairs. In particular we provide a general closed formula for the quasi K-matrix $\mathfrak{X}$ in many cases, and conjecture that our formula holds in general for all quantum symmetric pairs of finite type. We take the construction of the quasi R-matrix (1.3) as a guide. The Weyl group W plays a crucial role in the construction of R so it is expected that a subgroup of W should be the key ingredient for the quasi K-matrix. Here, we take the subgroup $\widetilde{W} = W(\Sigma)$ which is the Weyl group associated to the restricted root system Σ of the symmetric Lie algebra $(\mathfrak{g}, \theta)$. The Coxeter generators $\widetilde{\sigma}_i$ of $\widetilde{W}$ are parameterised by the τ-orbits of $I \setminus X$. We introduce the notion of a *partial quasi K-matrix* $\mathfrak{X}_{\widetilde{w}}$ for any $\widetilde{w} \in \widetilde{W}$ with reduced expression $\widetilde{w} = \widetilde{\sigma}_{i_1} \cdots \widetilde{\sigma}_{i_t}$. More precisely, for $j = 1, \ldots, t$ let $\mathfrak{X}_j$ denote the rank one quasi K-matrix for the rank one Satake subdiagram $(X \cup \{i, \tau(i)\}, X, \tau|_{X \cup \{i,\tau(i)\}})$ of (I, X, τ). This is the analogue of the rank one quasi R-matrix R_i corresponding to $i \in I$ given by (1.1). Using the Lusztig automorphisms $T_i : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ we define automorphisms $\widetilde{T}_i := T_{\widetilde{\sigma}_i} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ for all $i \in I \setminus X$. This provides us with an action of the braid group $Br(\widetilde{W})$ corresponding to $\widetilde{W}$ on $U_q(\mathfrak{g})$ by algebra automorphisms. For $j = 1, \cdots, t$ we define

$$\mathfrak{X}^{[j]}_{\widetilde{w}} = \Psi \circ \widetilde{T}_{i_1} \cdots \widetilde{T}_{i_{j-1}} \circ \Psi^{-1}(\mathfrak{X}_{i_j}). \tag{1.4}$$

Here, Ψ denotes an algebra automorphism of an extension $\widetilde{U}^+ = \oplus_{\mu \in Q^+(2\Sigma)} U^+_\mu$ defined in (5.32). In analogy to (1.3) we define

$$\mathfrak{X}_{\widetilde{w}} = \mathfrak{X}^{[t]}_{\widetilde{w}} \cdots \mathfrak{X}^{[1]}_{\widetilde{w}}. \tag{1.5}$$

The following theorem is the the first main result of this thesis. It gives an explicit formula for $\mathfrak{X}$ for many examples when $\mathbf{s} = (0, \ldots, 0)$.

Theorem A. (Corollary 5.29) *Let $\mathfrak{g}$ be of type A_n or $X = \emptyset$. Then the quasi K-matrix for $B_{\mathbf{c},\mathbf{0}}$ is given by $\mathfrak{X} = \mathfrak{X}_{\widetilde{w}_0}$ for any reduced expression of the longest element $\widetilde{w}_0 \in \widetilde{W}$.*

We conjecture that Theorem A holds true for all quantum symmetric pairs of finite type. There are three key steps to prove Theorem A which proceed similarly to the construction of the quasi R-matrix found in [34]. First we construct the quasi K-matrix corresponding to rank one Satake subdiagrams of type A_n in the case where $\mathbf{s} = (0, \ldots, 0)$. The difficulty here is that there are many rank one cases, see Table 5.1. Next, we verify Theorem A in rank two by direct calculation. The key idea here is that in rank two the longest element $\widetilde{w}_0 \in \widetilde{W}$ has only two reduced expressions. In each case we show that $\mathfrak{X}_{\widetilde{w}_0}$ coincides with the quasi K-matrix by showing that $\mathfrak{X}_{\widetilde{w}_0}$ satisfies the defining recursive relations for the quasi K-matrix. All of the rank two calculations are completed in Chapter 6. In order to do these calculations, one is required to know explicitly the quasi K-matrices corresponding to rank one Satake subdiagrams. It is for this reason that Theorem A is stated with the restriction that $\mathfrak{g}$ is of type A_n or $X = \emptyset$. The calculations of Chapter 6 suggest that the partial quasi K-matrix $\mathfrak{X}_{\widetilde{w_0}}$ is independent of the chosen reduced expression for $\widetilde{w_0}$. We conjecture that this is true in general.

Conjecture B. (Conjecture 5.22) *Assume that (I, X, τ) is a Satake diagram of rank two. Then the element $\mathfrak{X}_{\widetilde{w}}$ only depends on $\widetilde{w} \in \widetilde{W}$ and not on the chosen reduced expression.*

Conjecture B is all that is needed in order to prove Theorem A. Assuming that Conjecture B holds, we use the fact that the automorphisms $\widetilde{T}_i$ satisfy braid relations in order to show that $\mathfrak{X}_{\widetilde{w}}$ is independent of the chosen reduced expression of $\widetilde{w} \in \widetilde{W}$ in higher rank.

In the case of the longest element $\mathfrak{X}_{\widetilde{w}_0}$, we choose different reduced expressions for $\widetilde{w}_0$ in order to show that the partial quasi K-matrix in this case satisfies the defining relations for the quasi K-matrix for $B_{\mathbf{c},\mathbf{0}}$. In summary, we obtain the following result in the case $\mathbf{s} = \mathbf{0}$.

Theorem C. (Theorems 5.25, 5.28) *Let (I, X, τ) be a Satake diagram such that all rank two Satake subdiagrams satisfy Conjecture B. Then the following hold:*

1. *The partial quasi K-matrix $\mathfrak{X}_{\widetilde{w}}$ depends only on $\widetilde{w} \in \widetilde{W}$ and not on the chosen reduced expression.*
2. *The quasi K-matrix $\mathfrak{X}$ for $B_{\mathbf{c},\mathbf{0}}$ is given by $\mathfrak{X} = \mathfrak{X}_{\widetilde{w}_0}$ where $\widetilde{w}_0 \in \widetilde{W}$ denotes the longest element.*

In the case $\mathbf{s} \neq \mathbf{0}$ it is harder to give an explicit formula for the quasi K-matrix $\mathfrak{X}_{\mathbf{c},\mathbf{s}}$ of $B_{\mathbf{c},\mathbf{s}}$. However, we can make use of the fact that $B_{\mathbf{c},\mathbf{s}}$ is obtained from $B_{\mathbf{c},\mathbf{0}}$ via a twist by a character $\chi_{\mathbf{s}}$ of $B_{\mathbf{c},\mathbf{0}}$. We consider the element $R^{\theta}_{\mathbf{c},\mathbf{s}} = \Delta(\mathfrak{X}_{\mathbf{c},\mathbf{s}})R(\mathfrak{X}^{-1}_{\mathbf{c},\mathbf{s}} \otimes 1)$ which was introduced in [5] under the name quasi R-matrix for $B_{\mathbf{c},\mathbf{s}}$, and which lives in a completion of $B_{\mathbf{c},\mathbf{s}} \otimes U^+$, see also [39, Section 3.3]. We show that the quasi K-matrix $\mathfrak{X}_{\mathbf{c},\mathbf{s}}$ for $B_{\mathbf{c},\mathbf{s}}$ satisfies the relation

$$\mathfrak{X}_{\mathbf{c},\mathbf{s}} = (\chi_{\mathbf{s}} \otimes \mathrm{id})(R^{\theta}_{\mathbf{c},\mathbf{0}}). \tag{1.6}$$

Hence the explicit formulas (1.3) and (1.5) for R and $\mathfrak{X}_{\mathbf{c},\mathbf{0}}$, respectively, provide a formula for the quasi K-matrix of $B_{\mathbf{c},\mathbf{s}}$ also in the case $\mathbf{s} \neq \mathbf{0}$. However, in this case we do not obtain a factorisation as in Equation (1.5). The results of Chapters 5 and 6 are available in [16].

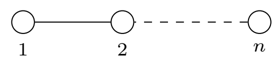
1.3 Braid group actions for $B_{\mathbf{c},\mathbf{s}}$

Recall the the Lusztig automorphisms $T_i : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ give rise to a representation of $Br(\mathfrak{g})$ on $U_q(\mathfrak{g})$, where $Br(\mathfrak{g})$ denotes the associated braid group corresponding to $\mathfrak{g}$, see [51]. This is the quantum analogue of the braid group action on $\mathfrak{g}$ by Lie algebra automorphisms, see Section 2.2.4. In the theory of quantum groups, the automorphisms T_i play a crucial role in the definition of root vectors and in the constructions of PBW bases for $U_q(\mathfrak{g})$. Further, they are the fundamental objects needed in the construction of canonical bases for $U_q(\mathfrak{g})$, as well as in the construction of quantum symmetric pairs.

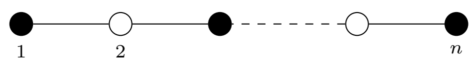
Classically we can also build a braid group action on the fixed Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. In this instance, we obtain an action of $Br(W_X) \rtimes Br(\widetilde{W})$ on $\mathfrak{k}$ by Lie algebra automorphisms. Here W_X is a parabolic subgroup of W corresponding to the subset X. It is natural to ask if there is an analogous braid group action for quantum symmetric pairs. The following theorem implies that we have a braid group action of $Br(W_X)$ on $B_{\mathbf{c},\mathbf{s}}$ by algebra automorphisms.

Theorem D. (Proposition 7.13) *For any $i \in X$ the Lusztig automorphism $T_i : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ restricts to an automorphism of $B_{\mathbf{c},\mathbf{s}}$.*

Theorem D was also independently proved in [6, Theorem 4.2]. A more difficult problem is establishing a braid group action of $Br(\widetilde{W})$ on $B_{\mathbf{c},\mathbf{s}}$ by algebra automorphisms. Many cases have already been checked in the literature. Initially, Molev and Ragoucy [53] constructed an action of $Br(\widetilde{W})$ on $B_{\mathbf{c},\mathbf{s}}$ if the underlying Satake diagram is of type AI.



This case was also checked independently by Chekov [10]. It was conjectured in [53, Conjecture 4.7] that there is a braid group action if the Satake diagram is of type AII.



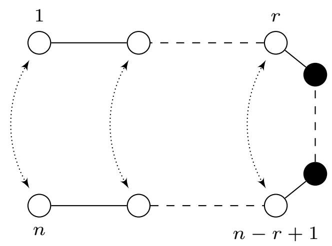
This was confirmed by Kolb and Pellegrini in [40] where actions of $Br(\widetilde{W})$ on $B_{\mathbf{c},\mathbf{s}}$ were constructed for numerous examples. In particular, they considered the following three classes of Satake diagram.

1. $\mathfrak{g}$ arbitrary, $X = \emptyset$, and $\tau = \mathrm{id}$,

2. $\mathfrak{g}$ arbitrary, $X = \emptyset$, and $\tau \neq \mathrm{id}$,

3. $\mathfrak{g} = \mathfrak{sl}_{2n}(\mathbb{C}),\ X = \{1, 3, 5, \ldots, 2n-1\}$, and $\tau = \mathrm{id}$.

However, so far braid group actions of $Br(\widetilde{W})$ on $B_{\mathbf{c},\mathbf{s}}$ have not been known for Satake diagrams with black dots and non-trivial diagram automorphisms. In Chapter 7 we consider the case AIII/AIV with black dots and a non-trivial diagram automorphism.



In this case the parameters $\mathbf{s}$ always satisfy $\mathbf{s} = (0, 0, \ldots, 0)$, see (3.34). Additionally $Br(\widetilde{W})$ is isomorphic to the braid group of type B_r. It is necessary for two or more black dots to consider a larger Hopf algebra than $U_q(\mathfrak{g})$ which we construct by enlarging the group algebra U^0. This is done by using the weight lattice instead of the root lattice. The resulting coideal subalgebra we obtain is denoted by $\check{B}_{\mathbf{c}}$.

Following the methods of Kolb and Pellegrini, the main tool we use is the package `QUAGROUP` [14] of the computer algebra program `GAP`, [22]. In order to use `GAP` we have to specify parameters $\mathbf{c}$ and $\mathbf{s}$ for the coideal subalgebra $B_{\mathbf{c},\mathbf{s}}$. We assume that the parameters $\mathbf{c}$ only take values that are integer powers of q. This assumption has appeared earlier, for instance in the construction of canonical bases from [6]. The following general theorem tells us that it does not matter what choice we make for the parameters.

Proposition E. (Proposition 7.16) *Let (X, τ) be any Satake diagram and suppose $\mathbf{c}, \mathbf{c}' \in (\pm q^{\mathbb{Z}})^{I \setminus X}$. Then there is an algebra isomorphism $\mathcal{A}_{\mathbf{c},\mathbf{c}'} : B_{\mathbf{c},\mathbf{s}} \to B_{\mathbf{c}',\mathbf{s}}$.*

One of the limitations of using `GAP` is that as we increase the number of black dots the running time for calculations increase massively. For this reason we can only use `GAP` to find a braid group action in the cases where we have one or two black dots. We describe the general procedure for constructing a braid group action of $Br(\widetilde{W})$ on $\check{B}_{\mathbf{c}}$ in these two cases when $n = 7$ and $n = 8$. For $i \in I \setminus X$ the algebra automorphisms $\widetilde{T}_i := T_{\widetilde{\sigma}_i}$ of $U_q(\mathfrak{g})$ do not restrict to $B_{\mathbf{c},\mathbf{s}}$. Despite this, they are still useful as a guide for our constructions. In particular we construct algebra automorphisms $\mathcal{T}_i$ of $\check{B}_{\mathbf{c}}$ for $1 \leq i \leq r$ such that $\mathcal{T}_i(B_j)$ and $\widetilde{T}_i(B_j)$ have identical terms containing maximal powers of the generators F_k, $k \in I$, for $j \in I \setminus X$. An inverse automorphism $\mathcal{T}_i^{-1}$ is similarly constructed and we check that $\mathcal{T}_i$ and $\mathcal{T}_i^{-1}$ are mutually inverse algebra automorphisms for each $i \in I \setminus X$. We then check that the automorphisms $\mathcal{T}_i$ satisfy the braid relations for $Br(\widetilde{W})$. For $1 \leq i < r$ the algebra automorphisms $\mathcal{T}_i$ correspond directly to those constructed in [40, Theorem 4.6].

It follows that we only need to construct $\mathcal{T}_r$ and check that the corresponding braid relations hold. With the assistance of `GAP` we construct algebra automorphisms $\mathcal{T}_r$ when $n = 7$ and $n = 8$ and show that these satisfy the corresponding braid relations.

Theorem F. (Theorems 7.22, 7.24) *Let (X, τ) be a Satake diagram of type AIII with $|X| = 1$ or $|X| = 2$. Then there exists a braid group action of $Br(\widetilde{W})$ on $\check{B}_{\mathbf{c}}$ by algebra automorphisms. This action is explicitly given by the formulas $\mathcal{T}_i$ for $1 \leq i < r$ given in* (7.57) *and $\mathcal{T}_r$ given in* (7.51) *and* (7.55).

In the general case, we can no longer rely on `GAP` for assistance due to memory issues. Based on the constructions when $|X| = 1$ and $|X| = 2$ we suggest a formula for $\mathcal{T}_r$ for any $|X| \geq 1$, see Equation (7.62). We make the following conjecture.

Conjecture G. (Conjecture 7.27) For any $|X| \geq 1$ the formulas for $\mathcal{T}_r$ given in (7.62) define algebra automorphisms of $\check{B}_{\mathbf{c}}$.

The difficulty in showing this in general comes from proving that $\mathcal{T}_r(B_i)$ satisfy the quantum Serre relations for $\check{B}_{\mathbf{c}}$. However, we show that if $\mathcal{T}_r$ is an algebra automorphism, then the correct braid relations are satisfied.

In Section 7.4.5 we prove Theorem F in full generality, assuming that the formulas for $\mathcal{T}_r$ define algebra automorphisms of $\check{B}_{\mathbf{c}}$. This proceeds by case-by-case computations and does not require the use of `GAP`. In particular, the results here suggest that many of the cases in [40] can be addressed without use of computer packages. In the final section of Chapter 7 we show that the two braid group actions of the preceding sections commute.

Theorem H. (Theorem 7.35) *Let (X, τ) be a Satake diagram of type AIII with $|X| = 1$ or $|X| = 2$. Then there exists an action of $Br(W_X) \times Br(\widetilde{W})$ on $\check{B}_{\mathbf{c}}$ by algebra automorphisms.*

We give a proof of Theorem H that does not make any assumptions on the size of X. As a result, if Conjecture G holds, then we immediately obtain an action of $Br(W_X) \times Br(\widetilde{W})$ on $\check{B}_{\mathbf{c}}$ for any $|X| \geq 1$.

1.4 Organisation

This thesis is organised as follows. In Chapter 2 we provide background material on Hopf algebras and quantised enveloping algebras. In particular, we recall the factorisation (1.3) of the quasi R-matrix in Section 2.2.8. Chapter 3 provides an overview of the theory of quantum symmetric pairs and give a presentation in terms of generators and relations for $B_{\mathbf{c},\mathbf{s}}$. This follows [45] and [38]. The connection between the symmetric Lie algebra $(\mathfrak{g}, \theta)$ and the restricted Weyl group is reviewed in Chapter 4, following the work of [60].

The main results of this thesis are contained in Chapters 5, 6 and 7. In Chapter 5 we recall the recursive definition of the quasi K-matrix and use this to explicitly compute the quasi K-matrix in many rank one cases in Section 5.3. In Section 5.4 we develop the theory of partial quasi K-matrices. Chapter 6 contains the rank two calculations for the quasi K-matrix. Finally in Chapter 7 we construct a braid group action on quantum symmetric pairs of type AIII/AIV. In particular, we use `GAP` to construct a braid group action for one or two black nodes and using this we suggest a general construction. In Section 7.4.5 we prove that our general construction satisfies the braid relations for $Br(\widetilde{W})$. This requires many additional relations, which we prove in Appendix A.

Chapter 2

Background

In this chapter we introduce the theory of quantised enveloping algebras that is required for the remainder of this thesis. We set the scene in section 2.1 by recalling the definition of Hopf algebras, of which quantised enveloping algebras form an important family of examples. This is shown in Section 2.2.2. Notation for semisimple Lie algebras and quantised enveloping algebras is set up in Sections 2.2.1 and 2.2.2. We establish a braid group action on semisimple simple Lie algebras following [62] in Section 2.2.4. This is the starting point for Chapter 7. The quasi R-matrix is introduced in Section 2.2.8. In particular we exhibit an explicit formula for the quasi R-matrix, due to [48], [35] and [34].

2.1 Hopf algebras

In the theory of quantum groups, a crucial role is played by algebras which have an additional coalgebra structure. Such objects are called *Hopf algebras*. We review the construction of Hopf algebras in this section. In order to do this, we require the notion of a tensor product.

2.1.1 Tensor products

Fix a field $\mathbb{K}$. Given vector spaces U and V, a naïve definition of the *tensor product* $U \otimes V$ is to take the vector space generated by symbols $u \otimes v$ with $u \in U$ and $v \in V$ such that the operator $\otimes$ is a bilinear operator. A more precise characterisation is given through a universal property regarding bilinear maps.

Definition/Theorem 2.1 ([32, Theorem II.1.1]). *Given vector spaces U and V there exists a vector space denoted $U \otimes V$ and a bilinear map $\phi : U \times V \to U \otimes V$ such that for every vector space W and for any bilinear map $f : U \times V \to W$, there is a unique linear map $g : U \otimes V \to W$ such that $f = g \circ \phi$. In other words, the following diagram commutes.*

$$\begin{array}{ccc} U \times V & \xrightarrow{\ \phi\ } & U \otimes V \\ & \searrow^{f} & \downarrow g \\ & & W \end{array} \tag{2.1}$$

Further, $U \otimes V$ is the unique vector space up to isomorphism satisfying this property.

Proof. Consider the vector space $\mathbb{K}[U \times V]$ whose basis is the set $U \times V$. Define $U \otimes V$ as the quotient space $\mathbb{K}[U \times V]/_R$ where R is the subspace generated by the elements

$$(u + u', v) - (u, v) - (u', v), \qquad (u, v + v') - (u, v) - (u, v'),$$
$$(\lambda u, v) - \lambda(u, v), \qquad (u, \lambda v) - \lambda(u, v)$$

where $u \in U$, $v \in V$ and $\lambda \in \mathbb{K}$. We define $\phi : U \times V \to U \otimes V$ to be the canonical map that sends $(u, v) \in U \times V$ to the associated equivalence class in the quotient. By construction, this map is bilinear.

Let W be a vector space and $f : U \times V \to W$ a bilinear map. We show that there is a unique linear map $g : U \otimes V \to W$ such that $f = g \circ \phi$. To show the existence of such a map, we define $g(u \otimes v) = f(u, v)$ and extend this linearly. Hence

$$g(\lambda_1(u_1 \otimes v_1) + \lambda_2(u_2 \otimes v_2)) = \lambda_1 g(u_1 \otimes v_1) + \lambda_2 g(u_2 \otimes v_2)$$

for all $u_1, u_2 \in U$, $v_1, v_2 \in V$ and $\lambda_1, \lambda_2 \in \mathbb{K}$. This satisfies $f = g \circ \phi$ and further g is a well-defined linear map. The uniqueness of g comes from the fact that any other choice for g would contradict $f = g \circ \phi$. Finally suppose there is a vector space Z also satisfying the universal property. Then there are unique linear maps $\psi_1 : U \otimes V \to Z$ and $\psi_2 : Z \to U \otimes V$ such that the diagram

$$\begin{array}{ccc} U \times V & \xrightarrow{\ \phi\ } & U \otimes V \\ & \searrow^{f} & \downarrow \psi_1 \\ & & Z \\ & \searrow^{\psi} & \downarrow \psi_2 \\ & & U \otimes V \end{array}$$

commutes. By the universal property for $U \otimes V$ it follows that $\psi_2 \circ \psi_1 = \mathrm{id}_{U \otimes V}$. Similarly, we also have $\psi_1 \circ \psi_2 = \mathrm{id}_Z$ and hence $U \otimes V \cong Z$. □

Tensor products are useful objects to study in algebra since the universal property reduces the study of bilinear maps to that of linear maps. The following lemma provides three canonical isomorphisms for tensor products of vector spaces.

Lemma 2.2 ([32, Proposition II.1.3]). *Let U, V and W be vector spaces. Then there are isomorphisms*

(1) $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$ *defined by*

$$(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w).$$

(2) $\mathbb{K} \otimes V \cong V \cong V \otimes \mathbb{K}$ *defined by*

$$\lambda \otimes v \mapsto \lambda v \mapsto v \otimes \lambda.$$

(3) $U \otimes V \cong V \otimes U$ *defined by*

$$u \otimes v \mapsto v \otimes u.$$

Proof. We only prove (1) as the other isomorphisms are similar. Fix $u \in U$ and define a bilinear map $f : V \times W \to (U \otimes V) \otimes W$ by $f(v, w) = (u \otimes v) \otimes w$. By the universal property, this induces a linear map $g_u : V \otimes W \to (U \otimes V) \otimes W$ such that the diagram

$$\begin{array}{ccc} V \times W & \xrightarrow{\phi} & V \otimes W \\ & {\scriptstyle f} \searrow & \downarrow {\scriptstyle g_u} \\ & & (U \otimes V) \otimes W \end{array}$$

commutes i.e. $g_u(v \otimes w) = (u \otimes v) \otimes w$ for each $v \in V, w \in W$. Using this, we define a bilinear map $f' : U \times (V \otimes W) \to (U \otimes V) \otimes W$ by

$$(u, v \otimes w) \mapsto g_u(v \otimes w).$$

The universal property implies that there is a unique linear map $g : U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ such that the diagram

$$\begin{array}{ccc} U \times (V \otimes W) & \xrightarrow{\phi'} & U \otimes (V \otimes W) \\ & {\scriptstyle f'} \searrow & \downarrow {\scriptstyle g} \\ & & (U \otimes V) \otimes W \end{array}$$

commutes. Similarly, a linear map $g' : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ can be constructed in the opposite direction. Using the universal property again implies that g and g' are inverses to one another and hence $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$ as required. $\square$

If U, U', V and V' are vector spaces and $f : U \to U'$ and $g : V \to V'$ are two linear maps, then we can construct the tensor product of the linear maps f and g, denoted by $f \otimes g : U \otimes V \to U' \otimes V'$, by defining

$$(f \otimes g)(u \otimes v) = f(u) \otimes g(v) \tag{2.2}$$

for all $u \in U$ and $v \in V$. This map arises naturally through the universal property. This is so since there is a bilinear map $f \times g : U \times V \to U \otimes V$ defined by

$$(f \times g)(u, v) = f(u) \otimes g(v)$$

which induces the map (2.2).

2.1.2 Algebras and coalgebras

Recall that we can define an algebra through commutative diagrams [32, Section III.1]. In particular, an algebra is a triple (A, μ, η) where A is a $\mathbb{K}$-vector space and $\mu : A \otimes A \to A$ and $\eta : \mathbb{K} \to A$ are linear maps satisfying the following two axioms.

(Assoc): The square

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu\otimes\mathrm{id}} & A \otimes A \\ {\scriptstyle \mathrm{id}\otimes\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow[\mu]{} & A \end{array} \tag{2.3}$$

commutes i.e. $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$. This expresses the usual requirement that multiplication is associative.

(Unit): The diagram

$$\begin{array}{ccccc} \mathbb{K} \otimes A & \xrightarrow{\eta\otimes\mathrm{id}} & A \otimes A & \xleftarrow{\mathrm{id}\otimes\eta} & A \otimes \mathbb{K} \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \cong} & \\ & & A & & \end{array} \tag{2.4}$$

commutes i.e. $\mu \circ (\eta \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \eta)$. This is equivalent to the requirement that $\eta(1)$ is a left and right unit for the multiplication map μ.

Additionally, the algebra A is *commutative* if the triangle

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mathrm{flip}} & A \otimes A \\ & {\scriptstyle \mu}\searrow \quad \swarrow{\scriptstyle \mu} & \\ & A & \end{array} \tag{2.5}$$

commutes. Here flip : $A \otimes A \to A \otimes A$ is the unique linear map that maps $a \otimes b$ to $b \otimes a$.

A *morphism* of algebras $f : (A, \mu, \eta) \to (A', \mu', \eta')$ is a linear map f from A to A' such that the diagrams

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f\otimes f} & A' \otimes A' \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu'} \\ A & \xrightarrow[f]{} & A' \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{K} & \xrightarrow{\eta} & A \\ & {\scriptstyle \eta'}\searrow & \downarrow{\scriptstyle f} \\ & & A' \end{array} \tag{2.6}$$

commute. In other words,

$$f \circ \mu = \mu' \circ (f \otimes f), \qquad f \circ \eta = \eta'. \tag{2.7}$$

This is just another way of stating the usual properties of morphisms that $f(ab) = f(a)f(b)$ and $f(\mathrm{id}_A) = \mathrm{id}_{A'}$.

To obtain the definition of a coalgebra, we systematically reverse all of the arrows in diagrams (2.3)–(2.6).

Definition 2.3 ([32, Definition III.1.1(a)]). A *coalgebra* is a triple (C, Δ, ε) where C is a

$\mathbb{K}$-vector space and $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \mathbb{K}$ are linear maps satisfying the follow axioms.

(Coass): The square

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta} \\ C \otimes C & \xrightarrow[\Delta\otimes\mathrm{id}]{} & C \otimes C \otimes C \end{array} \tag{2.8}$$

commutes i.e. $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$.

(Coun): The diagram

$$\begin{array}{ccccc} C \otimes \mathbb{K} & \xleftarrow{\mathrm{id}\otimes\varepsilon} & C \otimes C & \xrightarrow{\varepsilon\otimes\mathrm{id}} & \mathbb{K} \otimes C \\ & {\scriptstyle \cong}\nwarrow & {\scriptstyle \Delta}\uparrow & \nearrow{\scriptstyle \cong} & \\ & & C & & \end{array} \tag{2.9}$$

commutes i.e. $(\mathrm{id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta$. The map Δ is called the *comultiplication* and the map ε is called the *counit*. Additionally, the coalgebra is said to be *cocommuative* if the following triangle commutes

$$\begin{array}{ccc} & C & \\ {\scriptstyle \Delta}\swarrow & & \searrow{\scriptstyle \Delta} \\ C \otimes C & \xrightarrow{\mathrm{flip}} & C \otimes C \end{array} \tag{2.10}$$

The intuition behind the (Coun) axiom is that if we apply Δ to an element $c \in C$ and collapse either the left or right tensor components, then we retrieve c again.

Definition 2.4 ([32, Definition III.1.1(b)])**.** If (C, Δ, ε) and $(C', \Delta', \varepsilon')$ are two coalgebras, then $f : (C, \Delta, \varepsilon) \to (C', \Delta', \varepsilon')$ is a *coalgebra morphism* if the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\otimes f} \\ C' & \xrightarrow[\Delta']{} & C' \otimes C' \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\varepsilon} & \mathbb{K} \\ {\scriptstyle f}\downarrow & \nearrow{\scriptstyle \varepsilon'} & \\ C' & & \end{array} \tag{2.11}$$

commutes. In other words,

$$\Delta' \circ f = (f \otimes f) \circ \Delta, \qquad \varepsilon' \circ f = \varepsilon. \tag{2.12}$$

Example 2.5. The field $\mathbb{K}$ has a natural coalgebra structure with $\Delta(1) = 1 \otimes 1$ and $\varepsilon(1) = 1$. For any coalgebra (C, Δ, ε) the map $\varepsilon : C \to \mathbb{K}$ is a coalgebra morphism.

Example 2.6. Let S be a set and $C = \mathbb{K}[S]$ be the $\mathbb{K}$-vector space with basis S. Then C admits a cocommutative coalgebra structure with $\Delta(s) = s \otimes s$ and $\varepsilon(s) = 1$ for any $s \in S$.

Example 2.7 ([32, Section III.8, 2.])**.** Consider the polynomial ring $C = \mathbb{K}[x]$ in one variable. Then C obtains a coalgebra structure, called the *divided power coalgebra* by setting

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}, \quad \varepsilon(x^n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

This is again a cocommutative coalgebra, since $\binom{n}{k} = \binom{n}{n-k}$.

Another important concept that we will make use of is that of a *coideal.*

Definition 2.8 ([32, Definition III.1.5])**.** Let (C, Δ, ε) be a coalgebra. A subspace I of C is called a *coideal* if $\Delta(I) \subset I \otimes C + C \otimes I$ and $\varepsilon(I) = 0$.

Given a coalgebra (C, Δ, ε) and a coideal I, we can construct a new coalgebra, called the *quotient coalgebra*, in the following way. First, the comultiplication Δ factors through a map $\overline{\Delta}$ from C/I to

$$(C \otimes C)/(I \otimes C + C \otimes I) = C/I \otimes C/I \, .$$

The counit map factors similarly through a map $\overline{\varepsilon} : C/I \to \mathbb{K}$. This gives a coalgebra structure on C/I.

We also have the notions of a *left coideal* and *right coideal.* In particular, a subspace I of C is a left coideal if

$$\Delta(I) \subseteq C \otimes I \tag{2.13}$$

and a right coideal if

$$\Delta(I) \subseteq I \otimes C. \tag{2.14}$$

Note that we do not require $\varepsilon(I) = 0$ for left or right coideals. Right coideals play a major role in the construction of quantum symmetric pairs in Chapter 3.

The definition of coalgebra suggests that algebras and coalgebras should be dual to one another. Recall that for a $\mathbb{K}$-vector space V, we define the dual vector space $V^* = \mathrm{Hom}(V, \mathbb{K})$ consisting of linear functions $f : V \to \mathbb{K}$. The following proposition, given without proof, provides the link between algebras and coalgebras.

Proposition 2.9 ([32, Proposition III.1.2/1.3])**.** (1) *The dual vector space of a coalgebra is an algebra.*

(2) *The dual vector space of a finite-dimensional algebra has a coalgebra structure.*

Remark 2.10. For any vector space V, there is an injective homomorphism $\lambda : V^* \otimes V^* \to (V \otimes V)^*$ defined by

$$\lambda(f \otimes g)(v_1 \otimes v_2) = f(v_2) \otimes g(v_1). \tag{2.15}$$

This is an isomorphism if V is finite-dimensional, see [32, Corollary II.2.2]. The requirement in Proposition 2.9 (2) that the algebra A is finite-dimensional comes about since we define the comultiplication on A^* using λ^{-1}. If A is infinite dimensional, with basis $\{e_\beta\}_{\beta\in B}$ indexed by an infinite set B, then we proceed using the *finite dual* $A^\circ \subset A^*$ defined by

$$A^\circ := \{f \in A^* \mid \mu^*(f) \in A^* \otimes A^*\}, \tag{2.16}$$

see [32, Section III.9]. Then the alternative statement is that the finite dual has a coalgebra structure.

2.1.3 Bialgebras and Hopf algebras

We now let H be a $\mathbb{K}$-vector space equipped simultaneously with an algebra structure (H,μ,η) and with a coalgebra structure (H,Δ,ε).

Definition 2.11 ([32, Definition III.2.2])**.** A *bialgebra* is a 5-tuple $(H,\mu,\eta,\Delta,\varepsilon)$ such that (H,μ,η) is an algebra, (H,Δ,ε) is a coalgebra and the maps Δ and ε are morphisms of algebras.

The condition that Δ and ε are morphisms of algebras is equivalent to the commutativity of the following diagrams.

$$\begin{array}{ccc} H\otimes H & \xrightarrow{\Delta\otimes\Delta} & (H\otimes H)\otimes(H\otimes H) \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle(\mu\otimes\mu)(\mathrm{id}\otimes\mathrm{flip}\otimes\mathrm{id})} \\ H & \xrightarrow[\Delta]{} & H\otimes H \end{array} \qquad \begin{array}{ccc} \mathbb{K} & \xrightarrow{\eta} & H \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\Delta} \\ \mathbb{K}\otimes\mathbb{K} & \xrightarrow[\eta\otimes\eta]{} & H\otimes H \end{array}$$

and

$$\begin{array}{ccc} H\otimes H & \xrightarrow{\varepsilon\otimes\varepsilon} & \mathbb{K}\otimes\mathbb{K} \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\cong} \\ H & \xrightarrow[\varepsilon]{} & \mathbb{K} \end{array} \qquad \begin{array}{ccc} \mathbb{K} & \xrightarrow{\eta} & H \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle\varepsilon} \\ & & \mathbb{K} \end{array}$$

These four commutative diagrams are equivalent to the statement that μ and η are morphisms of coalgebras [32, Theorem III.2.1] which gives a compatability between the algebra and coalgebra structures on H.

We say that a *morphism of bialgebras* is a morphism for both of the underlying algebra and coalgebra structures.

Example 2.12. If H is a finite-dimensional bialgebra, then by Proposition 2.9 the dual vector space H^* has a natural bialgebra structure.

Example 2.13 ([32, Section III.3, Example 2.])**.** Following Example 2.6, assume instead that the set S comes with a unital monoid structure with a binary operation $\mu : S\times S \to S$ and a left and right unit e. Then the map μ induces an algebra structure on $\mathbb{K}[S]$. Further, the maps Δ and ε are morphisms of algebras since

$$\begin{aligned} \Delta(xy) &= xy \otimes xy = (x \otimes x)(y \otimes y) = \Delta(x)\Delta(y), \\ \varepsilon(xy) &= 1 = \varepsilon(x)\varepsilon(y) \end{aligned}$$

and hence $\mathbb{K}[S]$ has the structure of a bialgebra.

We are now ready to give the following important definition.

Definition 2.14 ([32, Definition III.3.2])**.** A *Hopf algebra* H is a 6-tuple $(H, \mu, \eta, \Delta, \varepsilon, S)$ such that $(H, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra and $S : H \to H$ is a $\mathbb{K}$-linear map which makes the following diagram commute.

$$\begin{array}{ccccccc} & & H \otimes H & \xrightarrow{S \otimes \mathrm{id}} & H \otimes H & & \\ & \nearrow^{\Delta} & & & & \searrow^{\mu} & \\ H & & \xrightarrow{\varepsilon} & \mathbb{K} & \xrightarrow{\eta} & & H \\ & \searrow_{\Delta} & & & & \nearrow_{\mu} & \\ & & H \otimes H & \xrightarrow[\mathrm{id} \otimes S]{} & H \otimes H & & \end{array} \tag{2.17}$$

The map $S : H \to H$ is called the *antipode*.

The following example suggests that the antipode can be thought of as a generalisation of the inverse map of a group.

Example 2.15. Let G be a monoid and consider the bialgebra $\mathbb{K}[G]$ of Example 2.13. If the antipode S exists, then by the antipode law (2.17) it must satisfy

$$S(x)x = xS(x) = \varepsilon(x)1 = 1.$$

Hence the antipode exists if and only if each $x \in G$ is invertible i.e. G is a group and then $S(x) = x^{-1}$ for all $x \in G$.

Definition 2.16 ([9, p. 103])**.** A *Hopf ideal* of a Hopf algebra H is a subspace I which is simultaneously an ideal and coideal, and satisfies $S(I) \subseteq I$.

For a Hopf ideal I of H, the quotient H/I obtains the structure of a Hopf algebra in the same way as the construction of the quotient coalgebra.

Given a Hopf algebra H, we define the *left adjoint representation* of H on itself in the following way, see [32, Section IX.3]. We make H into a $(H \otimes H)$-module by setting $(x \otimes y) \cdot z = xzS(y)$ for all $x, y, z \in H$. Via the coproduct, we then make H into a H-module where we denote the action by ad. Hence for any $x, y \in H$ we have

$$\mathrm{ad}(x)(y) = \Delta(x) \cdot y = \sum_i x_i y S(x_i') \tag{2.18}$$

where $\Delta(x) = \sum_i x_i \otimes x_i'$. There is a corresponding right adjoint representation, where instead we set $(x \otimes y) \cdot z = S(x)zy$ for any $x, y, z \in H$.

2.1.4 Universal enveloping algebras

The crucial example of Hopf algebra that we study comes from Lie theory. Any Lie algebra $\mathfrak{g}$ can be embedded into a larger associative algebra $\mathcal{A}$ in such a way that the Lie bracket $[x, y]$ in $\mathfrak{g}$ corresponds to taking the commutator $xy - yx$ in $\mathcal{A}$. The idea behind the universal enveloping algebra is to take the associative algebra obtained by forming all formal products and sums of elements in $\mathfrak{g}$, subject to the relations of $\mathfrak{g}$.

Since every Lie algebra is a vector space, we can construct the tensor algebra $T(\mathfrak{g})$ from it. This is the key ingredient in the construction of the universal enveloping algebra. We recall how $T(V)$ is constructed for any vector space V.

Definition 2.17 ([25, Section 17.1])**.** Let V be a vector space over a field $\mathbb{K}$ and for any non-negative integer k, let $T^kV = V^{\otimes k} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ factors}}$. The *tensor algebra* $T(V)$ is defined as

$$T(V) = \bigoplus_{k=0}^{\infty} T^kV = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus \cdots . \tag{2.19}$$

The multiplication in $T(V)$ is determined by the canonical isomorphism

$$\varphi : T^kV \otimes T^lV \to T^{k+l}V$$

which extends linearly to all of $T(V)$. This further implies that $T(V)$ is a graded algebra where T^kV is the k^{th}-graded component. Let $i : V \to T(V)$ denote the canonical embedding of V into $T(V)$. The tensor algebra has the following universal property.

Proposition 2.18 ([32, Proposition II.5.1])**.** *For any algebra A and linear map $f : V \to A$, there exist a unique algebra morphism $g : T(V) \to A$ such that the following diagram commutes.*

$$\begin{array}{ccc} V & \xrightarrow{\ i\ } & T(V) \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle g} \\ & & A \end{array} \tag{2.20}$$

The following proposition endows the tensor algebra with a Hopf algebra structure, which is proved by checking all of the necessary axioms are satisfied.

Proposition 2.19 ([32, Theorem III.2.4])**.** *Given a vector space V, there exists a unique cocommutative Hopf algebra structure on $T(V)$ such that*

$$\Delta(v) = 1 \otimes v + v \otimes 1,$$
$$\varepsilon(v) = 0,$$
$$S(v) = -v$$

for all $v \in V$.

We can now define the universal enveloping algebra $U(\mathfrak{g})$ in the following way by quotienting the tensor algebra by a suitable subspace.

Definition 2.20 ([32, Section V.2])**.** The *universal enveloping algebra* $U(\mathfrak{g})$ is defined as the quotient space

$$U(\mathfrak{g}) = {}^{T(\mathfrak{g})}\!\big/_{J}$$

where J is the two-sided ideal generated by elements of the form $x \otimes y - y \otimes x - [x, y]$ for $x, y \in \mathfrak{g}$.

The tensor algebra is infinite-dimensional, which implies that $U(\mathfrak{g})$ is also infinite-dimensional. Further, the algebra structure on $U(\mathfrak{g})$ is induced by the algebra structure on $T(\mathfrak{g})$. Since $U(\mathfrak{g})$ is generated by the Lie algebra $\mathfrak{g}$, we often use the following notational convention. Restricting to the case where $\mathfrak{g}$ is semisimple, the generators of $U(\mathfrak{g})$ are denoted by $\{E_i, F_i, H_i \mid i \in I\}$ which correspond to the Chevalley generators $\{e_i, f_i, h_i \mid i \in I\}$ of $\mathfrak{g}$. The relations satisfied in $U(\mathfrak{g})$ are induced by the relations in $\mathfrak{g}$. For instance, for all $i, j \in I$ the relation $[h_i, e_j] = a_{ji}e_j$ holds in $\mathfrak{g}$ which corresponds to the relation

$$H_iE_j - E_jH_i = a_{ji}E_j$$

in $U(\mathfrak{g})$, where now we omit the tensor product when multiplying elements in $T(\mathfrak{g})$. As one may expect, the universal enveloping algebra satisfies a universal property. Recall that any associative algebra $\mathcal{A}$ is made into a Lie algebra by defining the Lie bracket as $[x, y] = xy - yx$, see [19, Section 1.5].

Proposition 2.21 ([25, 17.2])**.** *Suppose we have a Lie algebra map $\phi : \mathfrak{g} \to \mathcal{A}$ such that*

$$\phi([x, y]) = \phi(x)\phi(y) - \phi(y)\phi(x) \quad \textit{for all } x, y \in \mathfrak{g}$$

and $\mathcal{A}$ is a unital, associative algebra over $\mathbb{K}$. Then there exists a unique unital algebra homomorphism $\varphi : U(\mathfrak{g}) \to \mathcal{A}$ such that the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\;j\;} & U(\mathfrak{g}) \\ & \underset{\phi}{\searrow} & \downarrow{\scriptstyle\varphi} \\ & & \mathcal{A} \end{array}$$

commutes, where $j : \mathfrak{g} \to U(\mathfrak{g})$ is the canonical embedding of $\mathfrak{g}$ into $T(\mathfrak{g})$, composed with the quotient map. Further, $U(\mathfrak{g})$ is the unique algebra satisfying this property, up to isomorphism.

Proof. By the universal property 2.18 of the tensor algebra, the Lie algebra map $\phi : \mathfrak{g} \to \mathcal{A}$ extends to a morphism of algebras $\overline{\phi} : T(\mathfrak{g}) \to \mathcal{A}$ such that

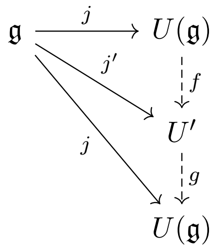
$$\overline{\phi}(x_1x_2 \cdots x_n) = \phi(x_1)\phi(x_2) \cdots \phi(x_n)$$

for $x_1, x_2, \ldots, x_n \in \mathfrak{g}$. The existence of a unital algebra homomorphism $\varphi : U(\mathfrak{g}) \to \mathcal{A}$ follows since

$$\overline{\phi}(xy - yx - [x, y]) = \phi(x)\phi(y) - \phi(y)\phi(x) - \phi([x, y]) = 0.$$

Hence $\overline{\phi}(J) = 0$. The uniqueness follows from the fact that $T(\mathfrak{g})$, and hence $U(\mathfrak{g})$, is generated by $\mathfrak{g}$.

Suppose U' is another associative algebra satisfying the universal property with a canonical embedding $j' : \mathfrak{g} \to U'$. By the universal properties for $U(\mathfrak{g})$ and U', there exists unique algebra homomorphisms $f : U(\mathfrak{g}) \to U'$ and $g : U' \to U(\mathfrak{g})$ such that the following diagram commutes.



It follows from the commutativity of the above diagram and the universal property for $U(\mathfrak{g})$ that $g \circ f = \mathrm{id}_{U'}$. Similarly, the universal property for U' implies that $f \circ g = \mathrm{id}_{U(\mathfrak{g})}$. This gives $U' \cong U(\mathfrak{g})$ as required. □

In particular, Proposition 2.21 implies that $U(\mathfrak{g})$ does not depend on the chosen basis for $\mathfrak{g}$. We now give a basis for $U(\mathfrak{g})$. The proof is technical, so we skip the details here.

Theorem 2.22 ([25, 17.3, Corollary C])**.** *Let $x_1, x_2, \ldots, x_n$ be a basis for $\mathfrak{g}$. Then $U(\mathfrak{g})$ has basis*

$$\{X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n} \mid a_1, a_2, \ldots, a_n \geq 0\} \tag{2.21}$$

where X_i is the element in $U(\mathfrak{g})$ corresponding to $x_i \in \mathfrak{g}$.

Showing that the basis elements span $U(\mathfrak{g})$ is done by using the defining relations; the difficulty comes from showing that the described elements are linearly independent. Such a basis is called a *PBW basis*, due to the constructions of Poincaré, Birkhoff and Witt, see [7] and [65] for example.

Example 2.23. Let $\mathfrak{g} = \mathfrak{sl}_3$ which has a basis

$$\{e_1, [e_1, e_2], e_2, h_1, h_2, f_1, [f_1, f_2], f_2\}.$$

Then the universal enveloping algebra $U(\mathfrak{g})$ has basis given by

$$\left\{E_1^{a_1}(E_1E_2 - E_2E_1)^{a_2} E_2^{a_3} H_1^{b_1} H_2^{b_2} F_1^{c_1} (F_1F_2 - F_2F_1)^{c_2} F_2^{c_3} \mid a_i, b_i, c_i \geq 0\right\}.$$

So any element of $U(\mathfrak{g})$ can be written as a linear combination of these elements. For instance,

$$\begin{aligned} F_2F_1E_1 &= F_2(E_1F_1 - H_1) \\ &= E_1F_2F_1 - F_2H_1 \\ &= E_1\big(F_1F_2 - (F_1F_2 - F_2F_1)\big) - (H_1F_2 - F_2) \\ &= E_1F_1F_2 - E_1(F_1F_2 - F_2F_1) - H_1F_2 + F_2. \end{aligned}$$

An important corollary to the Theorem 2.22 is the following, which follows from the linear independence of the elements $X_1, X_2, \dots, X_n$.

Corollary 2.24 ([19, Section 15.2])**.** *The Lie algebra $\mathfrak{g}$ can be viewed as a subspace of $U(\mathfrak{g})$. More generally, if $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$, then $U(\mathfrak{k})$ is a Hopf subalgebra of $U(\mathfrak{g})$.*

We finish this section by assigning a Hopf algebra structure to $U(\mathfrak{g})$. This in essence comes for free from the Hopf algebra structure on $T(\mathfrak{g})$ from Proposition 2.19. Since $\mathfrak{g}$ generates $U(\mathfrak{g})$ as an algebra, we only need to determine the structure on generators of $\mathfrak{g}$.

Proposition 2.25 ([9, Example 4.1.8])**.** *The universal enveloping algebra $U(\mathfrak{g})$ admits a unique cocommutative Hopf algebra structure such that*

$$\begin{aligned} \Delta(X) &= 1 \otimes X + X \otimes 1, \\ \varepsilon(X) &= 0, \\ S(X) &= -X \end{aligned}$$

for all $X \in \mathfrak{g}$.

Proof. We check that the ideal J is a Hopf ideal, from which it follows that $U(\mathfrak{g})$ has the structure of a Hopf algebra. Recall that J is generated by elements of the form

$$XY - YX - [X,Y]$$

for $X, Y \in \mathfrak{g}$. Using Proposition 2.19 we obtain $\varepsilon(XY - YX - [X,Y]) = 0$ and

$$\begin{aligned} &\Delta\big(XY - YX - [X,Y]\big) \\ &\quad = \Delta(X)\Delta(Y) - \Delta(Y)\Delta(X) - \Delta([X,Y]) \\ &\quad = (1 \otimes X + X \otimes 1)(1 \otimes Y + Y \otimes 1) - (1 \otimes Y + Y \otimes 1)(1 \otimes X + X \otimes 1) \\ &\qquad - (1 \otimes [X,Y] + [X,Y] \otimes 1) \\ &\quad = 1 \otimes XY + Y \otimes X + X \otimes Y + XY \otimes 1 - 1 \otimes YX - Y \otimes X \\ &\qquad - X \otimes Y - YX \otimes 1 - 1 \otimes [X,Y] - [X,Y] \otimes 1 \\ &\quad = 1 \otimes (XY - YX - [X,Y]) + (XY - YX - [X,Y]) \otimes 1 \end{aligned}$$

which lies in $T(\mathfrak{g}) \otimes J + J \otimes T(\mathfrak{g})$ as required. Hence $U(\mathfrak{g})$ has a bialgebra structure. It is an easy check to show that the antipode is defined by $S(X) = -X$ for all $X \in \mathfrak{g}$. Indeed, we have

$$\mu \circ (S \otimes \mathrm{id})(1 \otimes X + X \otimes 1) = 0 = \mu \circ (\mathrm{id} \otimes S)(1 \otimes X + X \otimes 1)$$

for all $X \in U(\mathfrak{g})$ and hence S satisfies the antipode law. This implies $U(\mathfrak{g})$ has a Hopf algebra structure. The cocommutativity of the comultiplication follows from the fact that $\mathrm{flip} \circ \Delta(X) = \Delta(X)$ for all $X \in \mathfrak{g}$. □

2.2 Quantised enveloping algebras

2.2.1 Semisimple Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and $\Phi \subset \mathfrak{h}^*$ the corresponding root system. Choose a set of simple roots $\Pi = \{\alpha_i \mid i \in I\}$ where I denotes an indexing set for the nodes of the Dynkin diagram of $\mathfrak{g}$. Let Φ^+ be the corresponding set of positive roots and set $V = \mathbb{R}\Phi$. Recall that $\mathfrak{g}$ admits a root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \tag{2.22}$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$ is the root space corresponding to $\alpha \in \Phi$ [19, Section 10.3].

For $i \in I$, let $\sigma_i : V \to V$ denote the reflection in the hyperplane H_i orthogonal to α_i. We write W to denote the Weyl group generated by the reflections σ_i. Fix a W-invariant scalar product $(-,-)$ on V such that $(\alpha, \alpha) = 2$ for all short roots $\alpha \in \Phi$ in each component. With this notation, the reflection of $\lambda \in V$ in the hyperplane H_i is given by the formula

$$\sigma_i(\lambda) = \lambda - \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i, \tag{2.23}$$

see [26, Section 1.1]. Equation (2.23) implies in particular that there is a natural action of the Weyl group W on $\mathfrak{h}^*$. Let $\{e_i, f_i, h_i \mid i \in I\}$ denote the Chevalley generators for $\mathfrak{g}$ where the elements h_i correspond to the generators of the Cartan subalgebra $\mathfrak{h}$. Let $\mathfrak{n}^+$ and $\mathfrak{n}^-$ denote the Lie subalgebras of $\mathfrak{g}$ generated by elements of the sets $\{e_i \mid i \in I\}$ and $\{f_i \mid i \in I\}$, respectively. Then the Lie algebra $\mathfrak{g}$ has a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-, \tag{2.24}$$

see [19, Section 15.1].

2.2.2 Definition of quantised enveloping algebras

Semisimple Lie algebras over $\mathbb{C}$ are rigid objects meaning that any formal deformation as an algebra is trivial. By passing to the universal enveloping algebra, any formal deformation as an algebra remains trivial. However, $U(\mathfrak{g})$ admits non-trivial deformations as a coalgebra. The theory of rigidity requires some cohomology, which is not discussed here, but details can be found in [9, Section 6.1] and [32, Sections XVIII.1 & XVIII.2].

The resulting algebra due to Drinfeld, [17] and Jimbo, [28] is denoted $U_q(\mathfrak{g})$ where q is an indeterminate. We give a definition in terms of generators and relations. Let $\mathbb{K}(q)$ be the field of rational functions in q with coefficients in $\mathbb{K}$ and let $q_i = q^{\frac{(\alpha_i,\alpha_i)}{2}}$ for any $i \in I$.

Recall from [27, Chapter 0] the definition of the q-number

$$[n]_{q_i} = [n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}} \tag{2.25}$$

for any $n \in \mathbb{Z}$ and $i \in I$. Using this, we define the q-factorial and q-binomial coefficients in the natural way

$$[n]_i! = [n]_i[n-1]_i \cdots [1]_i, \quad \begin{bmatrix} n \\ m \end{bmatrix}_i = \frac{[n]_i!}{[m]_i![n-m]_i!}\ . \tag{2.26}$$

If all roots $\alpha \in \Phi$ are of the same length, then we write $[n]$, $[n]!$ and $\begin{bmatrix} n \\ m \end{bmatrix}$.

Definition 2.26. ([27, Definition 4.3]) The *quantised enveloping algebra* $U_q(\mathfrak{g})$ is defined as the associative $\mathbb{K}(q)$-algebra with generators $\{E_i, F_i, K_i^{\pm 1} \mid i \in I\}$ subject to relations

(Q1) $K_iK_i^{-1} = K_i^{-1}K_i, \quad K_iK_j = K_jK_i,$

(Q2) $K_iE_jK_i^{-1} = q^{(\alpha_i,\alpha_j)}E_j,$

(Q3) $K_iF_jK_i^{-1} = q^{-(\alpha_i,\alpha_j)}F_j,$

(Q4) $E_iF_j - F_jE_i = \delta_{ij}\dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ where δ_{ij} is the Kroenecker delta function,

(Q5) $\displaystyle\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i E_i^{1-a_{ij}-r}E_jE_i^r = 0 \quad \text{for } i \neq j,$

(Q6) $\displaystyle\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i F_i^{1-a_{ij}-r}F_jF_i^r = 0 \quad \text{for } i \neq j.$

The relations (Q5) and (Q6) are known as the *quantum Serre relations.* The quantised enveloping algebra $U_q(\mathfrak{g})$ inherits a deformed Hopf algebra structure from that of $U(\mathfrak{g})$, which is given in the following proposition.

Proposition 2.27 ([27, Proposition 4.11])**.** *There is a unique structure of a Hopf algebra on $U_q(\mathfrak{g})$ such that*

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \qquad \varepsilon(E_i) = 0, \qquad S(E_i) = -K_i^{-1}E_i, \tag{2.27}$$

$$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \qquad \varepsilon(F_i) = 0, \qquad S(F_i) = -F_iK_i, \tag{2.28}$$

$$\Delta(K_i) = K_i \otimes K_i, \qquad \varepsilon(K_i) = 1, \qquad S(K_i) = K_i^{-1} \tag{2.29}$$

for all $i \in I$.

We don't give a proof here, but instead give the general strategy. A full detailed proof can be found in [27, Sections 4.8–4.11]. We let $\overline{U}$ be the algebra generated by the same elements $\{E_i, F_i, K_i^{\pm 1} \mid i \in I\}$ but only satisfying relations (Q1)–(Q4). One shows that $\overline{U}$ has a Hopf algebra structure given by the same formulas by showing that the relations are preserved under Δ, ε and S. To then show $U_q(\mathfrak{g})$ has a Hopf algebra structure, we quotient $\overline{U}$ by a suitable two-sided ideal I in $\overline{U}$ which adds in the relations (Q5) and (Q6). Such an ideal satisfies

$$\Delta(I) \subseteq \overline{U} \otimes I + I \otimes \overline{U}, \quad \varepsilon(I) = 0, \quad S(I) \subseteq I$$

and hence I is a Hopf ideal, and induces a Hopf structure on $U_q(\mathfrak{g})$.

Let U^+, U^0 and U^- denote the subalgebras of $U_q(\mathfrak{g})$ generated by $\{E_i \mid i \in I\}$, $\{K_i^{\pm 1} \mid i \in I\}$ and $\{F_i \mid i \in I\}$, respectively. The following lemma is proved by a simple check using relations (Q1)-(Q6).

Lemma 2.28 ([27, Lemma 4.6])**.**

1) *There is a unique algebra automorphism γ of $U_q(\mathfrak{g})$ such that $\gamma(E_i) = F_i$, $\gamma(F_i) = E_i$ and $\gamma(K_i) = K_i^{-1}$ for all $i \in I$.*

2) *There is a unique algebra antiautomorphism σ of $U_q(\mathfrak{g})$ such that $\sigma(E_i) = E_i$, $\sigma(F_i) = F_i$ and $\sigma(K_i) = K_i^{-1}$ for all $i \in I$.*

Let $Q = \mathbb{Z}\Phi$ be the root lattice for $\mathfrak{g}$ and $Q^+ = \mathbb{N}_0\Phi \subset Q$ the positive part of Q. For $\lambda = \sum_{i \in I} n_i \alpha_i \in Q$, we write

$$K_\lambda = \prod_{i \in I} K_i^{n_i}. \tag{2.30}$$

The elements K_λ for $\lambda \in Q$ form a vector space basis for U^0, see [27, 4.17/4.21]. Recall from Equation (2.18) that there is a left adjoint representation of $U_q(\mathfrak{g})$ on itself. We make this explicit using Proposition 2.27. In particular, we have

$$\mathrm{ad}(E_i)(u) = E_i u - K_i u K_i^{-1} E_i, \tag{2.31}$$

$$\mathrm{ad}(F_i)(u) = (F_i u - u F_i) K_i, \tag{2.32}$$

$$\mathrm{ad}(K_i)(u) = K_i u K_i^{-1} \tag{2.33}$$

for any $u \in U_q(\mathfrak{g})$. For any U^0-module M and $\lambda \in Q$, let

$$M_\lambda = \{m \in M \mid K_i m = q^{(\lambda, \alpha_i)} m \text{ for all } i \in I\} \tag{2.34}$$

denote the corresponding weight space [27, 5.1]. Note that all weight spaces encountered in this thesis correspond to weights in the root lattice and not in the weight lattice. Both of the subalgebras U^+ and U^- are U^0-modules with respect to the left adjoint action so we can apply the above notation. We hence obtain algebra gradings

$$U^+ = \bigoplus_{\mu \in Q^+} U^+_\mu, \quad U^- = \bigoplus_{\mu \in Q^+} U^-_{-\mu}, \tag{2.35}$$

see [27, Section 5.2]. Additionally, one can show that there is a triangular decomposition for $U_q(\mathfrak{g})$ similar to the triangular decomposition of $\mathfrak{g}$ from (2.24)

$$U_q(\mathfrak{g}) \cong U^+ \otimes U^0 \otimes U^-, \tag{2.36}$$

see [27, 4.21].

2.2.3 Completion of quantised enveloping algebras

It will be necessary in Chapter 5 to consider a completion $\mathscr{U}$ of $U_q(\mathfrak{g})$. We recall the construction of $\mathscr{U}$, following [4, Section 3.1].

Let $\mathcal{O}$ be the category of all finitely-generated $U_q(\mathfrak{g})$ modules M which decompose as a direct sum of weight spaces $M = \oplus_{\lambda \in Q} M_\lambda$ and on which the action of U^+ is locally finite. Let $\mathcal{O}_{\text{int}}$ be the subcategory of $\mathcal{O}$ consisting of all finite-dimensional $U_q(\mathfrak{g})$-modules.

Let $\mathcal{V}ect$ denote the category of $\mathbb{K}(q)$-vector spaces. Both of the categories $\mathcal{O}_{\text{int}}$ and $\mathcal{V}ect$ can be equipped with a tensor product, which makes these examples of *monoidal categories*. In the case of $\mathcal{V}ect$, we take the ordinary tensor product of vector spaces and linear maps as in Section 2.1.1. A similar construction is used for $\mathcal{O}_{\text{int}}$, see [32, Section III.5]. Denote by $\mathcal{F}or$ the forgetful functor $\mathcal{F}or : \mathcal{O}_{\text{int}} \to \mathcal{V}ect$. This is a monoidal functor since it preserves tensor products.

We let $\mathscr{U} = \text{End}(\mathcal{F}or)$ be the set of all natural transformations from the functor $\mathcal{F}or$ to itself. So elements of $\mathscr{U}$ are families of vector space endomorphisms

$$\big(\varphi_M : \mathcal{F}or(M) \to \mathcal{F}or(M)\big)_{M \in \text{Ob}(\mathcal{O}_{\text{int}})}$$

such that for any $U_q(\mathfrak{g})$-module homomorphism $\phi : M \to N$, the diagram

$$\begin{array}{ccc} \mathcal{F}or(M) & \xrightarrow{\mathcal{F}or(\phi)} & \mathcal{F}or(N) \\ {\scriptstyle \varphi_M}\downarrow & & \downarrow{\scriptstyle \varphi_N} \\ \mathcal{F}or(M) & \xrightarrow[\mathcal{F}or(\phi)]{} & \mathcal{F}or(N) \end{array} \tag{2.37}$$

commutes. The composition of natural transformations equips $\mathscr{U}$ with a multiplication and hence we may consider $\mathscr{U}$ as a $\mathbb{K}(q)$-algebra.

Lemma 2.29 ([58, Section 1.3]). *The algebra $U_q(\mathfrak{g})$ is a subalgebra of $\mathscr{U}$.*

Proof. Let $u \in U_q(\mathfrak{g})$. For each $M \in \text{Ob}(\mathcal{O}_{\text{int}})$ the action of u on M gives rise to a $U_q(\mathfrak{g})$-module homomorphism $\varphi_{M,u} : M \to M$ such that $\varphi_{M,u}(m) = u \cdot m$ for all $m \in M$. The family $\varphi_u = (\varphi_{M,u})_{M \in \text{Ob}(\mathcal{O}_{\text{int}})}$ is a natural transformation of the functor $\mathcal{F}or$ to itself.

This is so since

$$\mathcal{F}or(\phi)(u \cdot m) = u \cdot \mathcal{F}or(\phi)(m)$$

for any $U_q(\mathfrak{g})$-module homomorphism $\phi : M \to N$. This gives an injective algebra homomorphism $U_q(\mathfrak{g}) \to \mathscr{U}$ by [51, Proposition 3.5.4] and [27, 5.11]. Hence $U_q(\mathfrak{g})$ is a subalgebra of $\mathscr{U}$. □

Let $\widehat{U^+} = \prod_{\mu \in Q^+} U^+_\mu$. This is an algebra with multiplication given by component-wise multiplication.

Lemma 2.30 ([4, Example 3.2]). *The algebra $\widehat{U^+} = \prod_{\mu \in Q^+} U^+_\mu$ is a subalgebra of $\mathscr{U}$.*

Proof. Let $(X_\mu)_{\mu \in Q^+} \in \widehat{U^+}$. Let $M \in \mathrm{Ob}(\mathcal{O}_{\mathrm{int}})$ and $m \in M$. We can decompose M into weight spaces $M = \oplus_{\lambda \in Q} M_\lambda$ such that $M_\lambda \neq 0$ for finitely many λ.

Since $E_i M_\lambda \subset M_{\lambda + \alpha_i}$ for all $i \in I$, $\lambda \in Q$ it follows that there are only finitely many $\mu \in Q^+$ such that $X_\mu m = 0$. Hence the expression $\sum_{\mu \in Q^+} X_\mu m$ is well-defined.

This gives rises to a map $\varphi_M : M \to M$ such that $\varphi_M(m) = \sum_{\mu \in Q^+} X_\mu m$ and the commutative diagram 2.37 commutes. Hence we can view $(X_\mu)_{\mu \in Q^+}$ as an endomorphism of $\mathcal{F}or$. This implies that $\widehat{U^+}$ is a subalgebra of $\mathscr{U}$. □

2.2.4 Braid group action on semisimple Lie algebras

Recall that there is a symmetric, non-degenerate bilinear form on $\mathfrak{g}$ called the *Killing form*, defined by

$$\kappa(x, y) := \mathrm{tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y) \quad \text{for } x, y \in \mathfrak{g} \tag{2.38}$$

where $\mathrm{tr} : \mathrm{gl}(\mathfrak{g}) \to \mathbb{C}$ denotes the trace map and $\mathrm{ad} : \mathfrak{g} \to \mathrm{gl}(\mathfrak{g})$ denotes the adjoint action [19, Definition 9.5]. This is an associative bilinear form, meaning

$$\kappa([x, y], z) = \kappa(x, [y, z]) \quad \text{for all } x, y, z \in \mathfrak{g}.$$

The Killing form induces a $\mathfrak{g}$-module isomorphism

$$\phi : \mathfrak{g} \to \mathfrak{g}^*, \quad \phi(x) = \kappa(x, -). \tag{2.39}$$

Here, the $\mathfrak{g}$-module structure on $\mathfrak{g}$ is determined by the adjoint map. We make the dual Lie algebra $\mathfrak{g}^*$ into a $\mathfrak{g}$-module by defining

$$(x \cdot \psi)(y) = -\psi(x \cdot y) \quad \text{for } x, y \in \mathfrak{g}, \psi \in \mathfrak{g}^*, \tag{2.40}$$

see [19, Exercise 7.12]. Using the $\mathfrak{g}$-module isomorphism ϕ, we can identify the Cartan subalgebra $\mathfrak{h}$ with its dual $\mathfrak{h}^*$. This gives a natural action of the Weyl group W on $\mathfrak{h}$. We would like to extend the action of W on $\mathfrak{h}$ to an action on $\mathfrak{g}$, but this fails in general.

Example 2.31. Suppose $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ and let $\mathfrak{h}$ have generators $\{h_i \mid i = 1, \ldots, n\}$ such that $h_i = e_{i,i} - e_{i+1,i+1}$ where $e_{i,j}$ denotes the matrix with a 1 in the ij-th position and 0 elsewhere. The Weyl group for $\mathfrak{g}$ is given by the symmetric group $\mathcal{S}_{n+1}$ on $n+1$ elements. Since any element of $\mathfrak{h}$ is an $(n+1)\times(n+1)$ diagonal matrix, we may identify $\mathfrak{h}$ with vectors in $\mathbb{C}^{n+1}$. Hence we see that $\mathcal{S}_{n+1}$ acts on $h \in \mathfrak{h}$ by permuting the diagonal entries of h. More precisely, for $i, j \in \{1, \ldots, n+1\}$ we have

$$\sigma_i(h_j) = \begin{cases} -h_i & \text{if } i = j, \\ h_i + h_j & \text{if } i = j \pm 1, \\ h_j & \text{otherwise.} \end{cases} \tag{2.41}$$

This representation has a description in terms of matrices. Let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For each $i \in \{1, \ldots, n\}$ let M_i be the $(n+1)\times(n+1)$ block matrix defined by

$$M_i = \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n-i} \end{pmatrix} \tag{2.42}$$

where I_k denotes the $k \times k$ identity matrix and $\mathbf{0}$ denotes the zero matrix of the correct size. Then for any $h \in \mathfrak{h}$ and any $i \in \{1, \ldots, n\}$, we have $\sigma_i(h) = M_i h M_i^{-1}$. For each $i \in \{1, \ldots, n\}$, we extend σ_i to a Lie algebra homomorphism $\varphi_i : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\varphi_i(x) = M_i x M_i^{-1} \quad \text{for } x \in \mathfrak{g}. \tag{2.43}$$

By Equation (2.41) each σ_i has order two. However, since $J^2 = -I_2$ it follows that each φ_i has order four. Hence the map $\rho : \mathcal{S}_n \to \mathrm{Aut}(\mathfrak{g})$ given by $\rho(\sigma_i) = \varphi_i$ is not a group homomorphism and we can not extend the action of W on $\mathfrak{h}$ given by (2.41) to an action on $\mathfrak{g}$.

More generally, it is known that the action of W on $\mathfrak{h}$ does not extend to an action on $\mathfrak{g}$, see [63]. Let G denote the Lie group of $\mathfrak{g}$ and let T be the maximal torus of G. Then the Weyl group has a realisation as the quotient N_T/C_T of the normaliser and centraliser of T, respectively. In the current setting, $C_T = T = \mathrm{diag}_{n+1} \cap G$.

Now, the quotient group N_T/T acts on $\mathfrak{h}$ via the adjoint representation of G. This is so since N_T is a subgroup of G and T acts trivially on $\mathfrak{h}$. However, T does not act trivially on $\mathfrak{g}$. To see this, note that for any $D = \mathrm{diag}(d_1, \ldots, d_{n+1}) \in T$ we have

$$D e_{i,j} D^{-1} = d_i d_j^{-1} e_{i,j}$$

for all $i, j \in \{1, \ldots, n+1\}$. As a result N_T/T does not act on $\mathfrak{g}$.

In the above example, we come across a problem by requiring that the extension of σ_i to a Lie algebra homomorphism $\varphi_i : \mathfrak{g} \to \mathfrak{g}$ should have order two. This suggests that we should instead consider the action of the Artin braid group $Br(\mathfrak{g})$ on $\mathfrak{g}$.

Definition 2.32 ([33, Section 6.6.2])**.** The *Artin braid group* $Br(\mathfrak{g})$ corresponding to $\mathfrak{g}$ is the group generated by elements $\{\varsigma_i \mid i \in I\}$ subject to relations

$$\underbrace{\varsigma_i\varsigma_j\varsigma_i\cdots}_{m_{ij}\text{ factors}} = \underbrace{\varsigma_j\varsigma_i\varsigma_j\cdots}_{m_{ij}\text{ factors}} \tag{2.44}$$

where m_{ij} denotes the order of $\sigma_i\sigma_j$ in W.

The difference between the Weyl group W and the Artin braid group $Br(\mathfrak{g})$ is that we have omitted the condition $\sigma_i^2 = 1$ in W for each $i \in I$. For an element $w \in W$ which is reduced, we write m to denote the corresponding element in $Br(\mathfrak{g})$ in order to distinguish between the Weyl group and the braid group.

Following [62], we give a description of the action of $Br(\mathfrak{g})$ on $\mathfrak{g}$. Let $\exp : \mathrm{gl}(\mathfrak{g}) \to \mathrm{gl}(\mathfrak{g})$ denote the exponential power series for linear transformations which is defined in the usual way by

$$\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!} \quad \text{for } X \in \mathrm{gl}(\mathfrak{g}). \tag{2.45}$$

In the case where X is a nilpotent map (i.e. $X^n = 0$ for some $n \geq 1$), the exponential $\exp(X)$ makes sense since it has only finitely many terms. Recall that a *derivation* of $\mathfrak{g}$ is a linear map $\delta : \mathfrak{g} \to \mathfrak{g}$ such that

$$\delta(xy) = x\delta(y) + \delta(x)y \quad \text{for all } x, y \in \mathfrak{g}. \tag{2.46}$$

The following lemma is taken from [25, Section 2.3].

Lemma 2.33 ([25, Section 2.3])**.** *Suppose that $\delta : \mathfrak{g} \to \mathfrak{g}$ is a nilpotent derivation of $\mathfrak{g}$. Then $\exp(\delta)$ is an automorphism of $\mathfrak{g}$.*

Proof. Since $\mathfrak{g}$ is semisimple, it is isomorphic to a linear Lie algebra and hence for $x, y \in \mathfrak{g}$, the product xy can be given by matrix multiplication. Recall the Leibniz rule for derivations [25, Section 2.3]:

$$\delta^k(xy) = \sum_{l=0}^{k} \binom{k}{l} \delta^l(x)\delta^{k-l}(y) \quad \text{for all } x, y \in \mathfrak{g}. \tag{2.47}$$

Using this and $\delta^n = 0$, we obtain

$$\begin{aligned}
\exp(\delta)(x)\exp(\delta)(y) &= \left(\sum_{k=0}^{n-1} \frac{\delta^k(x)}{k!}\right)\left(\sum_{k=0}^{n-1} \frac{\delta^k(y)}{k!}\right) \\
&= \sum_{l=0}^{2n-2} \sum_{k=0}^{l} \left(\frac{\delta^k(x)}{k!}\right)\left(\frac{\delta^{l-k}(y)}{(l-k)!}\right) \\
&\overset{(2.47)}{=} \sum_{l=0}^{2n-2} \frac{\delta^l(xy)}{l!} = \exp(\delta)(xy).
\end{aligned}$$

It follows from this that $[\exp(\delta)(x), \exp(\delta)(y)] = \exp(\delta)([x, y])$ and hence $\exp(\delta)$ is a Lie algebra homomorphism. Set $\eta = \exp(\delta) - 1$. This is also nilpotent since

$$\eta^n = \left(\sum_{k=1}^{n-1} \frac{\delta^k}{k!}\right)^n = 0$$

and $\delta^n = 0$. Additionally, since $(1+\eta)^{-1} = 1 - \eta + \eta^2 - \eta^3 + \cdots \pm \eta^{n-1}$ it follows that $\exp(\delta)$ is invertible. □

We recall the link between representations of $\mathfrak{g}$ with representations of the corresponding simply connected Lie group G with $\mathrm{Lie}(G) = \mathfrak{g}$, see [20, Section 8.3] for example. For a finite dimensional vector space W, there are representations

$$\rho_{\mathfrak{g}} : \mathfrak{g} \to \mathrm{End}(W) = \mathfrak{gl}(W) \tag{2.48}$$

$$\rho_G : G \to \mathrm{Aut}(W) = GL(W). \tag{2.49}$$

We always pass from a representation of the Lie group to a representation of the Lie algebra by taking the derivative at the identity. Recall that there is an exponential map $\exp : \mathfrak{g} \to G$ which is also given by Equation (2.45) where instead $X \in \mathfrak{g}$ since G is a matrix group [37, Chapter 1, Section 17]. Then the following diagram commutes [20, pg 116].

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\rho_{\mathfrak{g}}} & \mathrm{End}(W) \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\rho_G]{} & \mathrm{Aut}(W) \end{array}$$

In other words, $\exp(\rho_{\mathfrak{g}}(x)) = \rho_G(\exp(x))$ for all $x \in \mathfrak{g}$. In particular, if we take the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ then the corresponding representation of the Lie algebra is the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$.

The adjoint map is a nilpotent derivation and hence by Lemma 2.33, the map $\exp(\mathrm{ad}(x))$ is an automorphism of $\mathfrak{g}$ for all $x \in \mathfrak{g}$. Let $\pi_G : Br(\mathfrak{g}) \to G$ be the map such that

$$\pi_G(\varsigma_i) = \exp(e_i)\exp(-f_i)\exp(e_i). \tag{2.50}$$

By [30, Remark 3.8], π_G is a group homomorphism. For each $i \in I$ let

$$\mathrm{Ad}(\pi_G(\varsigma_i)) = \exp(\mathrm{ad}(e_i))\exp(\mathrm{ad}(-f_i))\exp(\mathrm{ad}(e_i)) \in \mathrm{Aut}(\mathfrak{g}). \tag{2.51}$$

To shorten notation, we write $\mathrm{Ad}(\varsigma_i)$ instead of $\mathrm{Ad}(\pi_G(\varsigma_i))$.

The first observation about the automorphisms $\mathrm{Ad}(\varsigma_i)$ is that we can recover the action of W on $\mathfrak{h}$ by a direct calculation. Let $A = (a_{ij})$ denote the Cartan matrix of $\mathfrak{g}$, with entries given by $a_{ij} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$.

Lemma 2.34 ([30, Lemma 3.8]). *For any $i, j \in I$, we have*

$$\mathrm{Ad}(\varsigma_i)(h_j) = h_j - a_{ij}h_i. \tag{2.52}$$

Proof. We proceed by direct calculation. First note that the Serre relations (see [19, Lemma 14.5]) yield

$$\begin{aligned}\exp(\mathrm{ad}(e_i))(h_j) &= h_j - a_{ij}e_i,\\ \exp(\mathrm{ad}(-f_i))(h_j) &= h_j - a_{ij}f_i,\\ \exp(\mathrm{ad}(-f_i))(e_i) &= e_i + h_i - f_i.\end{aligned}$$

Substituting these identities into the formula for $\mathrm{Ad}(\varsigma_i)$, we obtain

$$\begin{aligned}\mathrm{Ad}(\varsigma_i)(h_j) &= \exp(\mathrm{ad}(e_i))\exp(\mathrm{ad}(-f_i))\exp(\mathrm{ad}(e_i))(h_j)\\ &= \exp(\mathrm{ad}(e_i))\exp(\mathrm{ad}(-f_i))\big(h_j - a_{ij}e_i\big)\\ &= \exp(\mathrm{ad}(e_i))\big(h_j - a_{ij}h_i - a_{ij}e_i\big)\\ &= h_j - a_{ij}h_i\end{aligned}$$

as required. □

Viewing the elements of W as acting on the Cartan subalgebra $\mathfrak{h}$, we have

$$\mathrm{Ad}(\varsigma_i)|_{\mathfrak{h}} = \sigma_i.$$

Additionally by [30, Lemma 3.8], the automorphisms $\mathrm{Ad}(\varsigma_i)$ satisfy $\mathrm{Ad}(\varsigma_i)(x) \in \mathfrak{g}_{\sigma_i(\alpha)}$ for any $i \in I$ and $\alpha \in \Phi$. Indeed, let $x \in \mathfrak{g}_\alpha$ and $h \in \mathfrak{h}$. Then

$$[h, \mathrm{Ad}(\varsigma_i)(x)] = \mathrm{Ad}(\varsigma_i)[\sigma_i^{-1}(h), x] = \mathrm{Ad}(\varsigma_i)(\alpha(\sigma_i^{-1}(h))x) = \sigma_i(\alpha)(h)\mathrm{Ad}(\varsigma_i)(x)$$

and hence $\mathrm{Ad}(\varsigma_i)(x) \in \mathfrak{g}_{\sigma_i(\alpha)}$. It follows from this that $\mathrm{Ad}(\varsigma_i)(\mathfrak{g}_\alpha) \subseteq \mathfrak{g}_{\sigma_i(\alpha)}$. By a similar argument, the reverse inclusion also holds which implies $\mathrm{Ad}(\varsigma_i)(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma_i(\alpha)}$.

The following lemma requires an $\mathfrak{sl}_2$-argument which will occur again in Lemma 3.5 and Lemma 3.6. In particular for $i \in I$ let $\mathfrak{sl}_2(i)$ denote the Lie subalgebra generated by the set $\{e_i, f_i, h_i\}$. Further, let $SL_2(i)$ denote the corresponding Lie group with $\mathfrak{sl}_2(i) = \mathrm{Lie}(SL_2(i))$.

Lemma 2.35. *The relation*

$$\mathrm{Ad}(\varsigma_i) = \exp(\mathrm{ad}(-f_i))\exp(\mathrm{ad}(e_i))\exp(\mathrm{ad}(-f_i)) \tag{2.53}$$

holds in $\mathrm{Aut}(\mathfrak{g})$ *for each* $i \in I$.

Proof. As elements of $\mathfrak{sl}_2(i)$ we have

$$e_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h_i = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}.$$

Since $\mathrm{ad}^2(e_i) = \mathrm{ad}^2(f_i) = 0$, we obtain matrix representations

$$\exp(e_i) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \exp(-f_i) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{2.54}$$

This implies that

$$\exp(e_i)\exp(-f_i)\exp(e_i) = \exp(-f_i)\exp(e_i)\exp(-f_i).$$

These two elements coincide in the corresponding Lie group $\mathrm{SL}_2(i)$ and hence they coincide under any representation also. In particular, taking the adjoint representation, it follows that

$$\begin{aligned}\exp(\mathrm{ad}(e_i))\exp(\mathrm{ad}(-f_i))\exp(\mathrm{ad}(e_i)) &= \mathrm{Ad}(\exp(e_i))\mathrm{Ad}(\exp(-f_i))\mathrm{Ad}(\exp(e_i)) \\ &= \mathrm{Ad}(\exp(-f_i))\mathrm{Ad}(\exp(e_i))\mathrm{Ad}(\exp(-f_i)) \\ &= \exp(\mathrm{ad}(-f_i))\exp(\mathrm{ad}(e_i))\exp(\mathrm{ad}(-f_i))\end{aligned}$$

from which the result follows.

□

Lemma 2.36. *For each $i \in I$ the relation*

$$\mathrm{Ad}(\varsigma_i)^{-1} = \exp(\mathrm{ad}(-e_i))\exp(\mathrm{ad}(f_i))\exp(\mathrm{ad}(-e_i)) \tag{2.55}$$

holds in $\mathrm{Aut}(\mathfrak{g})$.

Proof. The result follows from the same argument as in the proof of Lemma 2.35, by noting that the relations

$$\begin{aligned}\exp(e_i)^{-1} &= \exp(-e_i), \\ \exp(f_i)^{-1} &= \exp(-f_i)\end{aligned}$$

hold in $\mathfrak{sl}_2(i)$. It then follows that $\mathrm{Ad}(\varsigma_i)\mathrm{Ad}(\varsigma_i)^{-1} = \mathrm{id} = \mathrm{Ad}(\varsigma_i)^{-1}\mathrm{Ad}(\varsigma_i)$. □

Using Equation (2.53), we also see that the relation

$$\mathrm{Ad}(\varsigma_i)^{-1} = \exp(\mathrm{ad}(f_i))\exp(\mathrm{ad}(-e_i))\exp(\mathrm{ad}(f_i)) \tag{2.56}$$

holds in $\mathrm{Aut}(\mathfrak{g})$. We are now ready to give the main result of this Section. The proof follows that of Steinberg [62] given in the setting of Chevalley groups.

Theorem 2.37 ([62, Lemma 56])**.** *There is a group homomorphism*

$$\mathrm{Ad} : Br(\mathfrak{g}) \to \mathrm{Aut}(\mathfrak{g}) \tag{2.57}$$

such that $\mathrm{Ad}(\varsigma_i)$ *is given by* (2.51).

Proof. We only need to show that

$$\underbrace{\mathrm{Ad}(\varsigma_i)\mathrm{Ad}(\varsigma_j)\mathrm{Ad}(\varsigma_i)\cdots}_{n \text{ factors}} = \underbrace{\mathrm{Ad}(\varsigma_j)\mathrm{Ad}(\varsigma_i)\mathrm{Ad}(\varsigma_j)\cdots}_{n \text{ factors}}$$

where $\sigma_i\sigma_j$ has order n in W. For ease of notation, we assume that $n = 3$. It follows from a direct $\mathfrak{sl}_3$-calculation that

$$\mathrm{Ad}(\varsigma_i)\mathrm{Ad}(\varsigma_j)(e_i) = e_j,$$
$$\mathrm{Ad}(\varsigma_i)\mathrm{Ad}(\varsigma_j)(f_i) = f_j.$$

Let

$$X = \mathrm{Ad}(\varsigma_i)\mathrm{Ad}(\varsigma_j)\mathrm{Ad}(\varsigma_i)\mathrm{Ad}(\varsigma_j)^{-1}\mathrm{Ad}(\varsigma_i)^{-1}\mathrm{Ad}(\varsigma_j)^{-1}.$$

For any automorphism ϕ of $\mathfrak{g}$ and $x \in \mathfrak{g}$, we have

$$\phi \circ \mathrm{ad}(x) \circ \phi^{-1} = \mathrm{ad}(\phi(x)) \tag{2.58}$$

by definition of the adjoint map $\mathrm{ad}(x)$. We hence have

$$\mathrm{Ad}(\varsigma_i)\mathrm{Ad}(\varsigma_j)\exp(\mathrm{ad}(e_i))\mathrm{Ad}(\varsigma_j)^{-1}\mathrm{Ad}(\varsigma_i)^{-1} = \exp(\mathrm{ad}(e_j)),$$
$$\mathrm{Ad}(\varsigma_i)\mathrm{Ad}(\varsigma_j)\exp(\mathrm{ad}(-f_i))\mathrm{Ad}(\varsigma_j)^{-1}\mathrm{Ad}(\varsigma_i)^{-1} = \exp(\mathrm{ad}(-f_j)).$$

Using this we obtain

$$\begin{aligned} X\mathrm{Ad}(\varsigma_j) &= \mathrm{Ad}(\varsigma_i)\mathrm{Ad}(\varsigma_j)\mathrm{Ad}(\varsigma_i)\mathrm{Ad}(\varsigma_j)^{-1}\mathrm{Ad}(\varsigma_i)^{-1} \\ &= \exp(\mathrm{ad}(e_j))\exp(\mathrm{ad}(-f_j))\exp(\mathrm{ad}(e_j)) \\ &= \mathrm{Ad}(\varsigma_j). \end{aligned}$$

This implies that $X = \mathrm{id}$ as required. □

2.2.5 The Lusztig automorphisms on $U_q(\mathfrak{g})$

Recall from Equations (2.31)-(2.33) the adjoint action on $U_q(\mathfrak{g})$. By [29], the adjoint action of $U_q(\mathfrak{g})$ on itself is not locally finite. This means that there exists $x \in U_q(\mathfrak{g})$ such that $\dim(\mathrm{ad}(U_q(\mathfrak{g}))(x)) = \infty$. Hence in order to obtain an analogue of Theorem 2.37 for $U_q(\mathfrak{g})$ we require a new construction. In this section, we recall Lusztig's braid group action on $U_q(\mathfrak{g})$ by algebra automorphisms, as in [51, Part VI].

In order to ease notation, we introduce the divided powers

$$E_i^{(n)} = \frac{E_i^n}{[n]_i!}, \quad F_i^{(n)} = \frac{F_i^n}{[n]_i!} \tag{2.59}$$

for each $n \in \mathbb{N}_0$. For any $M \in \mathrm{Ob}(\mathcal{O}_{\mathrm{int}})$ and $i \in I$, let T_i be the linear isomorphism of M defined by

$$T_i(m) = \sum_{a,b,c\geq 0;\ a-b+c=\lambda(h_i)} (-1)^b q_i^{(b-ac)} E_i^{(a)} F_i^{(b)} E_i^{(c)} m \tag{2.60}$$

for $m \in M_\lambda, \lambda \in Q$. These linear isomorphisms are denoted by $T''_{i,1}$ in [51, 5.2.1]. By [51, Proposition 5.2.3], the inverse of T_i is given by

$$T_i^{-1}(m) = \sum_{a,b,c\geq 0;\ a-b+c=-\lambda(h_i)} (-1)^b q_i^{(ac-b)} F_i^{(a)} E_i^{(b)} F_i^{(c)} m. \tag{2.61}$$

The following theorem, given without proof, shows that the linear maps $T_i : M \to M$ satisfy braid relations.

Theorem 2.38 ([51, Theorem 39.4.3]). *For any $i \neq j$ in I the isomorphisms T_i and T_j of M satisfy the equality*

$$\underbrace{T_iT_jT_i\cdots}_{m_{ij}\ factors} = \underbrace{T_jT_iT_j\cdots}_{m_{ij}\ factors} \tag{2.62}$$

where m_{ij} denotes the order of $\sigma_i\sigma_j \in W$.

The above theorem implies that for any $w \in W$ with reduced expression $w = \sigma_{i_1}\cdots\sigma_{i_t}$, there is a well-defined isomorphism denoted $T_w : M \to M$ such that

$$T_w = T_{i_1}\cdots T_{i_t}, \tag{2.63}$$

see [27, 8.14, (1)]. The isomorphism $T_i : M \to M$ induces an automorphism of $U_q(\mathfrak{g})$, also denoted by T_i such that for all $u \in U_q(\mathfrak{g}), m \in M$ we have

$$T_i(um) = T_i(u)T_i(m). \tag{2.64}$$

By Theorem 2.38, the algebra automorphisms $T_i : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ also satisfy braid relations.

Corollary 2.39 ([51, Theorem 39.4.3]). *For any $i \neq j$ in I the algebra automorphisms T_i and T_j of $U_q(\mathfrak{g})$ satisfy the equality*

$$\underbrace{T_iT_jT_i\cdots}_{m_{ij}\ factors} = \underbrace{T_jT_iT_j\cdots}_{m_{ij}\ factors} \tag{2.65}$$

where m_{ij} denotes the order of $\sigma_i\sigma_j \in W$.

Proof. Let $u \in U_q(\mathfrak{g})$. Set $u_1 = (T_iT_jT_i\cdots)(u) \in U_q(\mathfrak{g})$ and $u_2 = (T_jT_iT_j\cdots)(u) \in U_q(\mathfrak{g})$. Using Equation (2.64) and Theorem 2.38 twice, we see that

$$\begin{aligned} u_1 \cdot (T_jT_iT_j\cdots)(m) &= u_1 \cdot (T_iT_jT_i\cdots)(m) \\ &= (T_iT_jT_i\cdots)(um) \\ &= (T_jT_iT_j\cdots)(um) = u_2 \cdot (T_jT_iT_j\cdots)(m). \end{aligned}$$

It follows from this that $u_1 - u_2$ acts as zero on M since $T_jT_iT_j\cdots$ is an isomorphism of M. Further, since M is chosen arbitrarily, we must have $(u_1 - u_2)M = 0$ for all $M \in \mathrm{Ob}(\mathcal{O}_{\mathrm{int}})$. It follows from [27, Proposition 5.11] that $u_1 = u_2$. □

Hence for any $w \in W$ with reduced expression $w = \sigma_{i_1}\cdots\sigma_{i_t}$ we also obtain a well-defined algebra automorphism $T_w = T_{i_1}\cdots T_{i_t}$ of $U_q(\mathfrak{g})$. Using Equation (2.64), we have the following frequently used formulas for the actions of T_i and T_i^{-1} on the generators of

$U_q(\mathfrak{g})$ [51, 37.1.3], [27, 8.14]

$$T_i(K_\lambda) = K_{\sigma_i(\lambda)} = T_i^{-1}(K_\lambda), \tag{2.66}$$

and

$$T_i(E_i) = -F_iK_i, \quad T_i^{-1}(E_i) = -K_i^{-1}F_i, \tag{2.67}$$

$$T_i(F_i) = -K_i^{-1}E_i, \quad T_i^{-1}(F_i) = -E_iK_i. \tag{2.68}$$

For $i \neq j$ we have

$$T_i(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^r q_i^{-r} E_i^{(-a_{ij}-r)} E_j E_i^{(r)}, \tag{2.69}$$

$$T_i^{-1}(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^r q_i^{-r} E_i^{(r)} E_j E_i^{(-a_{ij}-r)}, \tag{2.70}$$

$$T_i(F_j) = \sum_{r=0}^{-a_{ij}} (-1)^r q_i^{r} F_i^{(r)} F_j F_i^{(-a_{ij}-r)}, \tag{2.71}$$

$$T_i^{-1}(F_j) = \sum_{r=0}^{-a_{ij}} (-1)^r q_i^{r} F_i^{(-a_{ij}-r)} F_j F_i^{(r)}. \tag{2.72}$$

From these formulas, it follows that for any $w \in W$ and $\lambda \in Q$ we have

$$T_w(K_\lambda) = K_{w(\lambda)} \tag{2.73}$$

and $T_i^{-1} = \sigma \circ T_i \circ \sigma$, where σ is the antiautomorphism from Lemma 2.28. A harder check is showing that

$$\gamma(T_i(u)) = (-q_i)^{2\frac{(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}} T_i(\gamma(u)) \tag{2.74}$$

for $i \in I$ and $u \in U_\lambda$ where γ is the automorphism from Lemma 2.28. This is also observed through the formulas for the action of T_i on the generators of $U_q(\mathfrak{g})$. For instance,

$$\begin{aligned}
\gamma(T_i(E_j)) &= \gamma\Big(\sum_{r=0}^{(-a_{ij})} (-1)^r q_i^{-r} E_i^{(-a_{ij}-r)} E_j E_i^{(r)} \Big) \\
&= (-q_i)^{a_{ij}} \sum_{r=0}^{(-a_{ij})} (-1)^{r-a_{ij}} q_i^{-r-a_{ij}} F_i^{(-a_{ij}-r)} F_j F_i^{(r)} \\
&= (-q_i)^{a_{ij}} T_i(F_j) \\
&= (-q_i)^{a_{ij}} T_i(\gamma(E_j))
\end{aligned}$$

and similarly for the other generators of $U_q(\mathfrak{g})$.

Using the Lusztig automorphisms T_i for $i \in I$, we can now construct a PBW basis for $U_q(\mathfrak{g})$, similar to that in Theorem 2.22. In particular, we construct a basis for U^+ and a corresponding basis for U^-. The following proposition is the first step in this construction.

Proposition 2.40 ([27, Proposition 8.20]). *Let $w \in W$ and $\alpha_i \in \Pi$. If $w\alpha_i > 0$, then $T_w(E_i) \in U^+_{w(\alpha_i)}$. If $w\alpha_i \in \Pi$, then $T_w(E_i) = E_{w(\alpha_i)}$.*

As a consequence of this, if $w \in W$ has reduced expression $w = \sigma_{i_1} \cdots \sigma_{i_t}$, then all products of the form

$$T_{i_1} T_{i_2} \cdots T_{i_{t-1}}(E_{i_t})^{a_t} \cdots T_{i_1}(E_{i_2})^{a_2} E_{i_1}^{a_1} \tag{2.75}$$

such that $a_i \in \mathbb{N}_0$ lie in U^+. This is so because the sequence

$$\alpha_{i_1}, \sigma_{i_1}(\alpha_{i_2}), \ldots, \sigma_{i_1} \cdots \sigma_{i_{t-1}}(\alpha_{i_t})$$

consists of t positive roots [26, 5.6 Exercise 1]. For any $w \in W$ let $U^+[w]$ be the subspace of U^+ spanned by all elements of the form (2.75). Similarly, we obtain subspaces $U^-[w]$ by replacing E_i with F_i in (2.75). By [13, 2.2] the subspace $U^+[w]$ is always a subalgebra of U^+. We are interested in the structure of $U^+[w]$ particularly in the case where we take a reduced expression for the longest element of W. This is given by the following theorem.

Theorem 2.41 ([27, Proposition 8.22 a), Theorem 8.24]).

1) *The subalgebras $U^+[w]$ of U^+ depend only on $w \in W$ and not on a chosen reduced expression.*
2) *If $w_0 \in W$ is the longest element with reduced expression $w_0 = \sigma_{i_1} \cdots \sigma_{i_t}$ then $U^+[w_0] = U^+$ and all elements of the form* (2.75) *form a basis for U^+.*

Using (2.74), we immediately acquire a a PBW basis for U^- by applying γ to the PBW basis of U^+. This gives the following corollary.

Corollary 2.42 ([27, Remark 8.24]). *Let $w_0 \in W$ be the longest element with reduced expression $w_0 = \sigma_{i_1} \cdots \sigma_{i_t}$. Then all products*

$$T_{i_1} T_{i_2} \cdots T_{i_{t-1}}(F_{i_t})^{a_t} \cdots T_{i_2}(F_{i_2})^{a_2} F_{i_1}^{a_1} \tag{2.76}$$

with $a_i \in \mathbb{N}_0$ form a basis for U^-.

2.2.6 Lusztig's skew derivations

Let $'\mathbf{f}$ denote the free associaive $\mathbb{K}(q)$-algebra generated by elements f_i for $i \in I$ as in [51, 1.2.1]. The algebra $'\mathbf{f}$ is a U^0-module algebra with $K_\lambda \cdot \mathrm{f}_i = q^{(\lambda,\alpha_i)}\mathrm{f}_i$ for any $i \in I$ and $\lambda \in Q^+$. Hence $'\mathbf{f}$ is a Q^+-graded algebra with

$$'\mathbf{f} = \bigoplus_{\lambda \in Q^+} {'\mathbf{f}_\lambda}.$$

The natural projection map $\pi : {}'\mathbf{f} \to U^+$ with $\mathrm{f}_i \mapsto E_i$ respects the Q^+-grading. There are uniquely determined $\mathbb{K}(q)$-linear maps $r_i : {}'\mathbf{f} \to {}'\mathbf{f}$ and ${}_ir : {}'\mathbf{f} \to {}'\mathbf{f}$ such that

$$r_i(\mathrm{f}_j) = \delta_{ij}, \qquad r_i(xy) = q^{(\alpha_i,\nu)} r_i(x)y + xr_i(y), \tag{2.77}$$

$${}_ir(\mathrm{f}_j) = \delta_{ij}, \qquad {}_ir(xy) = {}_ir(x)y + q^{(\alpha_i,\mu)} x_ir(y) \tag{2.78}$$

for all $i, j \in I$, $x \in {}'\mathbf{f}_\mu$ and $y \in {}'\mathbf{f}_\nu$ [51, 1.2.13]. In particular, these equations imply that that $r_i(1) = 0 = {}_ir(1)$ for all $i \in I$ since for any $j \neq i$ we have

$$0 = r_i(\mathrm{f}_j) = r_i(\mathrm{f}_j \cdot 1) = \mathrm{f}_j r_i(1)$$

and similarly for ${}_ir$. Using the projection π, there exist linear maps $r_i, {}_ir : U^+ \to U^+$ satisfying Equations (2.77) and (2.78) with $x \in U^+_\mu, y \in U^+_\nu$ and f_j replaced by E_j. Such maps are called *skew derivations* due to the similarity between (2.77) and (2.78) and the usual notion of a derivation, see (2.46).

Example 2.43. We use Equation (2.77) to find $r_i(E_i^n)$ for $n \geq 1$, [27, Section 8.26,(3)]. We claim that

$$r_i(E_i^n) = \frac{q_i^{2n} - 1}{q_i^2 - 1}(E_i^{2n-1}). \tag{2.79}$$

If $n = 1$ then $r_i(E_i) = 1$. Proceeding by induction on n we have

$$\begin{aligned} r_i(E_i^{n+1}) &= q^{(\alpha_i, n\alpha_i)} r_i(E_i)E_i^n + E_i r_i(E_i^n) \\ &= q^{2n}E_i^n + \frac{q_i^{2n} - 1}{q_i^2 - 1}E_i^n = \frac{q_i^{2(n+1)} - 1}{q_i^2 - 1}E_i^n \end{aligned}$$

as required. Similarly, using Equation (2.78) one obtains

$${}_ir(E_i^n) = \frac{q_i^{2n} - 1}{q_i^2 - 1}E_i^{n-1}. \tag{2.80}$$

In view of the q-number $[n]_i$ we introduce the following modification which will appear in Section 2.2.8 and the calculations of Chapter 5 and Chapter 6, see [16, Equations (3.36)-(3.38)]. For $n \geq 1$ define

$$\{n\}_i = q_i^{n-1}[n]_i = 1 + q_i^2 + q_i^4 + \ldots + q_i^{2(n-1)}. \tag{2.81}$$

Using this notation we have

$$r_i(E_i^n) = {}_ir(E_i^n) = \{n\}_i E_i^{n-1} \tag{2.82}$$

for $n \geq 1$ and $i \in I$. Additionally define $\{n\}_i! = \prod_{k=1}^n \{k\}_i$ and let $\{n\}_i!!$ denote the double factorial of $\{n\}_i$ defined by

$$\{n\}_i!! = \prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} \{n - 2k\}_i. \tag{2.83}$$

For $n = 0$ we set $\{0\}_i! = 1$ and $\{0\}_i!! = 1$. As with the q-number, we omit the subscript i

if all roots are of the same length.

The maps r_i and ${}_ir$ appear in many different contexts; here we give two equivalent properties regarding the algebra and coalgebra structure of $U_q(\mathfrak{g})$.

Proposition 2.44 ([51, Proposition 3.1.6])**.** *For all $x \in U^+$ and $i \in I$ we have*

$$[x, F_i] = (q_i - q_i^{-1})^{-1}\big(r_i(x)K_i - K_i^{-1}{}_ir(x)\big). \tag{2.84}$$

Proposition 2.45 ([27, 6.14])**.** *For all $x \in U_\mu^+$ we have*

$$\Delta(x) = x \otimes 1 + \sum_{i\in I} r_i(x)K_i \otimes E_i + (\text{rest})_1, \tag{2.85}$$

$$\Delta(x) = K_\mu \otimes x + \sum_{i\in I} E_iK_{\mu-\alpha_i} \otimes {}_ir(x) + (\text{rest})_2 \tag{2.86}$$

where $(\text{rest})_1 \in \sum_{\alpha\notin\Pi\cup\{0\}} U_{\mu-\alpha}^+K_\alpha \otimes U_\alpha^+$ *and* $(\text{rest})_2 \in \sum_{\alpha\notin\Pi\cup\{0\}} U_\alpha^+K_{\mu-\alpha} \otimes U_{\mu-\alpha}^+$.

In Section 2.2.7, we give one additional equivalent property which is useful for inductive arguments. The original definitions (2.77) and (2.78) are also effective for inductive arguments; the value of this will be made clear in Chapter 5.

Remark 2.46. We could instead factor $r_i, {}_ir : {}'\mathbf{f} \to {}'\mathbf{f}$ over U^-. Then the linear maps $r_i, {}_ir : U^- \to U^-$ satisfy (2.77) and (2.78) with f_i replaced by F_i and $x \in U_{-\mu}^-, y \in U_{-\nu}^-$. They also satisfy corresponding versions of Propositions 2.44 and 2.45.

By an inductive argument, one shows that the following lemma holds.

Lemma 2.47 ([27, Lemma 10.1])**.** *For all $i, j \in I$, the relation*

$$r_i \circ {}_jr(x) = {}_jr \circ r_i(x) \tag{2.87}$$

holds for all $x \in U_q(\mathfrak{g})$.

Proof. Both sides of (2.87) clearly coincide for $x = 1$ and $x = E_k$ for $k \in I$. By linearity, we only need to show that if the claim holds for $x \in U_\mu^+$ and $y \in U_\nu^+$, then it holds for element xy.

Indeed, we have

$$\begin{aligned}(r_i \circ {}_jr)(xy) &= r_i\big({}_jr(x)y + q^{(\alpha_j,\mu)}x{}_jr(y)\big)\\ &= q^{(\alpha_i,\nu)}(r_i \circ {}_jr)(x)y + {}_jr(x){}_ir(y) + q^{(\alpha_j,\mu)}q^{(\alpha_i,\nu-\alpha_j)}r_i(x){}_jr(y)\\ &\quad + q^{(\alpha_j,\mu)}x(r_i \circ {}_jr)(y)\\ &= q^{(\alpha_i,\nu)}({}_jr \circ r_i)(x)y + {}_jr(x){}_ir(y) + q^{(\alpha_j,\mu)}q^{(\alpha_i,\nu-\alpha_j)}r_i(x){}_jr(y)\\ &\quad + q^{(\alpha_j,\mu)}x({}_jr \circ r_i)(y).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}({}_j r \circ r_i)(xy) &= {}_j r\big(q^{(\alpha_i,\nu)} r_i(x) y + x r_i(y)\big)\\ &= q^{(\alpha_i,\nu)}({}_j r \circ r_i)(x) y + q^{(\alpha_i,\nu)} q^{(\alpha_j,\mu-\alpha_i)} r_i(x)\, {}_j r(y) + {}_j r(x) r_i(y)\\ &\quad + q^{(\alpha_j,\mu)} x({}_j r \circ r_i)(y).\end{aligned}$$

It follows by comparison that $r_i \circ {}_j r(xy)$ coincides with ${}_j r \circ r_i(xy)$ as required. □

Using the antiautomorphism σ of $U_q(\mathfrak{g})$ from Lemma 2.28 we can relate the skew derivations r_i and ${}_i r$ by a similar inductive argument as in the proof of Lemma 2.47.

Lemma 2.48 ([27, Lemma 6.14 c)]). *The map σ intertwines the skew derivations r_i and ${}_i r$, i.e.*

$$\sigma \circ r_i(x) = {}_i r \circ \sigma(x) \tag{2.88}$$

for all $i \in I$ and $x \in U^+$.

Recall that there is a $\mathbb{K}$-algebra automorphism $\overline{}^U : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$, called the *bar involution*, that is defined by

$$\overline{q}^U = q^{-1}, \quad \overline{E_i}^U = E_i, \quad \overline{F_i}^U = F_i, \quad \overline{K_\lambda}^U = K_{-\lambda} \tag{2.89}$$

for $i \in I$ and $\lambda \in Q$, see [51, Section 3.1.12]. The bar involution satisfies the following property.

Lemma 2.49 ([51, Lemma 1.2.14]). *The bar involution intertwines the skew derivations r_i and ${}_i r$ for each $i \in I$*

$${}_i r(\overline{x}^U) = q^{(\alpha_i,\mu-\alpha_i)} \overline{r_i(x)}^U \quad \text{for all } x \in U^+_\mu, \mu \in Q^+. \tag{2.90}$$

Let $i \in I$ and let $w_0 \in W$ denote the longest element. Then the subspaces $U^+[\sigma_i w_0]$ and $U^-[\sigma_i w_0]$ can be determined using the skew derivations. Here, we allow both r_i and ${}_i r$ to act on elements of U^- as in Remark 2.46.

Lemma 2.50 ([27, Lemma 8.26]). *For $i \in I$ and $w_0 \in W$ we have*

$$T_i(U^+[\sigma_i w_0]) = \{x \in U^+ \mid r_i(x) = 0\}. \tag{2.91}$$

To obtain the following corollary, we use Lemma 2.48 and the relation $T_i^{-1} = \sigma \circ T_i \circ \sigma$.

Corollary 2.51 ([27, Remark 8.26]). *For $i \in I$ and $w_0 \in W$ the following equalities hold.*

$$U^+[\sigma_i w_0] = \{x \in U^+ \mid {}_i r(x) = 0\}, \tag{2.92}$$

$$U^-[\sigma_i w_0] = \{x \in U^- \mid r_i(x) = 0\}, \tag{2.93}$$

$$T_i(U^-[\sigma_i w_0]) = \{x \in U^- \mid {}_i r(x) = 0\}. \tag{2.94}$$

2.2.7 The bilinear pairing on $U_q(\mathfrak{g})$

The PBW bases for U^+ and U^- from Theorem 2.41 and Corollary 2.42 are dual to one another, which we make more explicit here.

Let A and B be $\mathbb{K}$-algebras with a bilinear pairing $\langle -,-\rangle : A \times B \to \mathbb{K}$. Using the tensor product, we may extend this to a new bilinear pairing $\langle -,-\rangle : A^{\otimes n} \times B^{\otimes n} \to \mathbb{K}$ by setting

$$\langle a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n\rangle = \prod_{i=1}^{n} \langle a_i, b_i\rangle. \tag{2.95}$$

We use this in the current setting as in [27, Proposition 6.12]. Let $U^{\geq 0} = U^+U^0$ and $U^{\leq 0} = U^-U^0$. Then there exists a unique bilinear pairing $\langle -,-\rangle : U^{\leq 0} \times U^{\geq 0} \to \mathbb{K}(q)$ such that for any $x, x' \in U^{\geq 0}, y, y' \in U^{\leq 0}$, $\mu, \nu \in Q$ and $i, j \in I$ we have

$$\langle y, xx'\rangle = \langle \Delta(y), x' \otimes x\rangle, \qquad \langle yy', x\rangle = \langle y \otimes y', \Delta(x)\rangle, \tag{2.96}$$

$$\langle K_\mu, K_\nu\rangle = q^{-(\mu,\nu)}, \qquad \langle F_i, E_j\rangle = -\delta_{ij}(q_i - q_i^{-1})^{-1}, \tag{2.97}$$

$$\langle K_\mu, E_i\rangle = 0, \qquad \langle F_i, K_\mu\rangle = 0. \tag{2.98}$$

Equation (2.97) implies that the elements F_i and E_i are dual to one another with respect to $\langle -,-\rangle$, up to a scalar. We extend the duality to the bases for U^- and U^+. First, using the skew derivations r_i and ${}_ir$ from Section 2.2.6 we obtain an inductive formula for the bilinear pairing.

Proposition 2.52 ([51, 1.2.13])**.** *For all $x \in U^+, y \in U^-$ and $i \in I$ we have*

$$\langle F_i y, x\rangle = \langle F_i, E_i\rangle\langle y, {}_ir(x)\rangle, \quad \langle yF_i, x\rangle = \langle F_i, E_i\rangle\langle y, r_i(x)\rangle. \tag{2.99}$$

Proof. We only prove the first equality; the second is obtained similarly. The bilinearity of $\langle -,-\rangle$ implies that we only need consider $x \in U^+_\mu$ for $\mu \in Q^+ \setminus \{0\}$. Using Proposition 2.45 and Equation (2.96) we have

$$\begin{aligned}
\langle F_i y, x\rangle &= \langle F_i \otimes y, \Delta(x)\rangle \\
&= \langle F_i \otimes y, K_\mu \otimes x + \sum_j E_j K_{\mu-\alpha_j} \otimes {}_jr(x) + (\text{rest})_2\rangle \\
&= \langle F_i \otimes y, K_\mu \otimes x\rangle + \sum_j \langle F_i \otimes y, E_j K_{\mu-\alpha_j} \otimes {}_jr(x)\rangle + \langle F_i \otimes y, (\text{rest})_2\rangle \\
&= \sum_j \langle F_i, E_j K_{\mu-\alpha_j}\rangle\langle y, {}_jr(x)\rangle + \langle F_i \otimes y, (\text{rest})_2\rangle
\end{aligned}$$

since $\langle F_i, K_\mu \rangle = 0$ for all $i \in I$ and $\mu \in Q^+$. Again using Equation (2.96) it follows that

$$\begin{aligned}\langle F_i, E_j K_{\mu-\alpha_j} \rangle &= \langle \Delta(F_i), K_{\mu-\alpha_j} \otimes E_j \rangle \\ &= \langle F_i \otimes K_i^{-1} + 1 \otimes F_i, K_{\mu-\alpha_j} \otimes E_j \rangle \\ &= \langle F_i, K_{\mu-\alpha_j} \rangle \langle K_i^{-1}, E_j \rangle + \langle 1, K_{\mu-\alpha_j} \rangle \langle F_i, E_j \rangle \\ &= \delta_{ij} \langle F_i, E_j \rangle\end{aligned}$$

for all $i, j \in I$. By a similar argument, one shows that

$$\langle F_i \otimes y, (\text{rest})_2 \rangle = 0.$$

Hence we obtain

$$\langle F_i y, x \rangle = \langle F_i, E_i \rangle \langle y, {}_j r(x) \rangle$$

as required. □

Example 2.53. We claim that for any $i \in I$ and $n \geq 1$ we have

$$\langle F_i^n, E_i^n \rangle = (-1)^n \frac{\{n\}_i!}{(q_i - q_i^{-1})^n}, \tag{2.100}$$

see [27, Section 3.16,(4)]. By (2.97) we have $\langle F_i, E_i \rangle = -(q_i - q_i^{-1})^{-1}$. Using Proposition 2.52 and Equation (2.82) we have

$$\langle F_i^{n+1}, E_i^{n+1} \rangle = \langle F_i, E_i \rangle \langle F_i^n, \{n+1\}_i E_i^n \rangle = (-1)^{n+1} \frac{\{n+1\}_i!}{(q_i - q_i^{-1})^{n+1}}$$

which proves the claim.

The following is a technical proposition that is not proved here, but the details can be found in [27, Chapter 8A].

Proposition 2.54 ([27, Lemma 8.27, Proposition 8.28])**.** *Let $i \in I, x \in U^+[\sigma_i w_0]$ and $y \in U^-[\sigma_i w_0]$. Then*

$$\langle T_i(y) F_i^n, T_i(x) E_i^m \rangle = \delta_{nm} \langle y, x \rangle \langle F_i^n, E_i^n \rangle \quad \textit{for all } n, m \in \mathbb{N}_0. \tag{2.101}$$

Using Proposition 2.54, we now show that PBW bases for U^+ and U^- are dual to one another with respect to $\langle -, - \rangle$. This follows [27, Proposition 8.29].

Proposition 2.55 ([27, Proposition 8.29])**.** *Let $w \in W$ and let $w = \sigma_{i_1} \cdots \sigma_{i_t}$ be a reduced expression. Then the expression*

$$\langle T_{i_1} T_{i_2} \cdots T_{i_{t-1}}(F_{i_t}^{a_t}) \cdots T_{i_1}(F_{i_2}^{a_2}) F_{i_1}^{a_1}, T_{i_1} T_{i_2} \cdots T_{i_{t-1}}(E_{i_t}^{b_t}) \cdots T_{i_1}(E_{i_2}^{b_2}) E_{i_1}^{b_1} \rangle \tag{2.102}$$

is 0 if there exists an index $1 \leq k \leq t$ with $a_k \neq b_k$; otherwise it is equal to $\prod_{k=1}^t \langle F_{i_k}^{a_k}, E_{i_k}^{a_k} \rangle$.

Proof. We proceed by induction on the length $\ell(w)$ where $\ell : W \to \mathbb{N}_0$ denotes the length function with respect to w. The result is clear when $\ell(w) \leq 1$. For $t = \ell(w) > 1$ suppose

$w = \sigma_{i_1} \cdots \sigma_{i_t}$ is a reduced expression. Set

$$x = T_{i_2} \cdots T_{i_{t-1}}(E_{i_t}^{a_t}) \cdots T_{i_2}(E_{i_3}^{a_3}) E_{i_2}^{a_2},$$
$$y = T_{i_2} \cdots T_{i_{t-1}}(F_{i_t}^{b_t}) \cdots T_{i_2}(F_{i_3}^{b_3}) F_{i_2}^{b_2}.$$

Then $x \in U^+[\sigma_{i_1} w_0]$ and $y \in U^-[\sigma_{i_1} w_0]$. By Proposition 2.54 we have

$$\langle T_{i_1}(y) F_{i_1}^{b_1}, T_{i_1}(x) E_{i_1}^{a_1} \rangle = \delta_{a_1 b_1} \langle y, x \rangle \langle F_{i_1}^{a_1}, E_{i_1}^{a_1} \rangle.$$

We can now apply the inductive hypothesis to x and y since $\sigma_{i_1} w = \sigma_{i_2} \cdots \sigma_{i_t}$ is reduced and $\ell(\sigma_{i_1} w) < \ell(w)$. The result follows from this. □

The following corollary shows that the pairing $\langle -, - \rangle$ respects weights.

Corollary 2.56 ([27, Corollary 8.30])**.** *The restriction of $\langle -, - \rangle$ to $U^-_{-\mu} \times U^+_{\nu}$ vanishes for any $\mu, \nu \in Q^+$ with $\mu \neq \nu$. If $\mu = \nu$, then the restriction of the bilinear pairing to $U^-_{-\mu} \times U^+_{\mu}$ is non-degenerate.*

The non-degeneracy of $\langle -, - \rangle : U^-_{-\mu} \times U^+_{\mu} \to \mathbb{K}(q)$ and Proposition 2.52 imply that for any $x \in U^+_{\mu}$ with $\mu \in Q^+ \setminus \{0\}$ we have

$$r_i(x) = 0 \text{ for all } i \in I \iff x = 0 \iff {}_i r(x) = 0 \text{ for all } i \in I, \tag{2.103}$$

as in [51, 1.2.15].

2.2.8 The quasi R-matrix

Let $\mu \in Q^+$ and let $\{u_i^{\mu}\}_i$ be a basis of $U^-_{-\mu}$. By Corollary 2.56 we can find a dual basis $\{v_i^{\mu}\}_i$ of U^+_{μ} with respect to the bilinear pairing $\langle -, - \rangle$. Define

$$R_{\mu} = \sum_i u_i^{\mu} \otimes v_i^{\mu} \in U^-_{-\mu} \otimes U^+_{\mu}, \tag{2.104}$$

see [27, Section 7.1]. The element R_{μ} does not depend on the chosen basis for $U^-_{-\mu}$.

For $M, M' \in \mathrm{Ob}(\mathcal{O}_{\mathrm{int}})$ we have

$$R_{\mu}(M_{\lambda} \otimes M'_{\lambda'}) \subset M_{\lambda-\mu} \otimes M'_{\lambda'+\mu} \quad \text{for all } \lambda, \lambda' \in Q, \mu \in Q^+.$$

Hence there are only finitely many $\mu \in Q^+$ such that R_{μ} acts non-trivially on $M \otimes M'$. This allows us to define a linear transformation

$$R = R_{M,M'} : M \otimes M' \to M \otimes M', \quad R = \sum_{\mu \in Q^+} R_{\mu} \in \prod_{\mu \in Q^+} U^-_{-\mu} \otimes U^+_{\mu} \tag{2.105}$$

which we call the *quasi R-matrix*.

Example 2.57 ([27, Section 3.11])**.** The fundamental example which is crucial to our later constructions is the quasi R-matrix for $U_q(\mathfrak{sl}_2)$. Here we have $Q = \mathbb{Z}\alpha_1$ and hence any $\mu \in Q^+$ must be of the form $n\alpha_1$ for $n \geq 0$. The subspace $U^-_{-n\alpha_1}$ has a basis given by the element F_1^n, whilst by Equation (2.100) the element

$$(-1)^n \frac{(q-q^{-1})^n}{\{n\}!} E_1^n$$

generates the dual basis of $U^+_{n\alpha_1}$. It hence follows that

$$R_{n\alpha_1} = (-1)^n \frac{(q-q^{-1})^n}{\{n\}!} F_1^n \otimes E_1^n$$

which implies that the quasi R-matrix is given by

$$R = \sum_{n\geq 0} R_{n\alpha_1} = \sum_{n\geq 0} (-1)^n \frac{(q-q^{-1})^n}{\{n\}!} F_1^n \otimes E_1^n. \tag{2.106}$$

For any $i \in I$, let R_i denote the quasi R-matrix corresponding to the copy of $U_q(\mathfrak{sl}_2(i))$ labelled by i. Then we have

$$R_i = \sum_{n\geq 0} (-1)^n \frac{(q_i-q_i^{-1})^n}{\{n\}_i!} F_i^n \otimes E_i^n. \tag{2.107}$$

By [51, Theorem 4.1.2] there is a second important characterisation of the quasi R-matrix which uses the bar involution from Equation (2.89). We define a bar involution $\overline{}^{U\otimes U}$ on $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ by setting

$$\overline{u \otimes v}^{U\otimes U} = \overline{u}^U \otimes \overline{v}^U \tag{2.108}$$

for all $u, v \in U_q(\mathfrak{g})$. We normally omit the superscripts U and $U \otimes U$ if it is clear which space the bar involution is acting on. Generally for $u \in U_q(\mathfrak{g})$ we have $\Delta(\overline{u}) \neq \overline{\Delta(u)}$ and further, the element $\overline{\Delta(u)}$ is not an element of $\Delta(U_q(\mathfrak{g}))$. However, the quasi R-matrix allows one to intertwine between the two bar involutions.

Theorem 2.58 ([51, Theorem 4.2.1])**.** *The quasi R-matrix is the uniquely determined element $R = \sum_{\mu\in Q^+} R_\mu \in \prod_{\mu \in Q^+} U^-_{-\mu} \otimes U^+_\mu$ with $R_0 = 1 \otimes 1$ and $R_\mu \in U^-_{-\mu} \otimes U^+_\mu$ which intertwines the bar involution in the sense that*

$$\Delta(\overline{u})R = R\overline{\Delta(u)} \quad \text{for any } u \in U_q(\mathfrak{g}). \tag{2.109}$$

A crucial property of the quasi R-matrix is that it admits a factorisation as a product of quasi R-matrices for $U_q(\mathfrak{sl}_2)$, see [47], [35], [27, Remark 8.30] and [34] for example.

Let $w_0 = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_t}$ denote a reduced expression for the longest element $w_0 \in W$. For $1 \leq j \leq t$ set $\gamma_j = \sigma_{i_1}\cdots\sigma_{i_{j-1}}(\alpha_{i_j})$ and define

$$E_{\gamma_j} = T_{i_1}\cdots T_{i_{j-1}}(E_{i_j}), \quad F_{\gamma_j} = T_{i_1}\cdots T_{i_{j-1}}(F_{i_j}). \tag{2.110}$$

The elements E_{γ_j} and F_{γ_j} are the root vectors used in the construction of the PBW basis elements corresponding to U^+ and U^-, respectively. Whilst root vectors in $\mathfrak{g}$ have simple commutator formulas, e.g. [25, Proposition 8.4(d)], this is not true for $U_q(\mathfrak{g})$.

Lemma 2.59 ([48, Lemma 1]). *For all $i < j$ we have*

$$E_{\gamma_j} E_{\gamma_i} - q^{-(\gamma_i,\gamma_j)} E_{\gamma_i} E_{\gamma_j} = \sum_{i<k_1<k_2<\cdots<k_s<j} C_{\mathbf{k},\mathbf{n}} E^{n_1}_{\gamma_{k_1}} E^{n_2}_{\gamma_{k_2}} \cdots E^{n_s}_{\gamma_{k_s}} \tag{2.111}$$

where $C_{\mathbf{k},\mathbf{n}} = C_{k_1,k_2,\dots,k_s,n_1,n_2,\dots,n_s}$ are constants.

The above lemma states that for $i < j$ the element $E_{\gamma_j} E_{\gamma_i} - q^{-(\gamma_i,\gamma_j)} E_{\gamma_i} E_{\gamma_j}$ is a linear combination of basis elements as in (2.75) only involving E_{γ_k} with $i < k < j$, see [27, Remark 8.24]. We make use of Theorem 2.59 in Section 5.3.

For $\mu \in Q^+$ a basis for $U^-_{-\mu}$ is given by the elements

$$F^{a_t}_{\gamma_t} F^{a_{t-1}}_{\gamma_{t-1}} \cdots F^{a_1}_{\gamma_1}$$

for $a_i \in \mathbb{N}_0$ and $\mu = \sum_{i=1}^t a_i \gamma_i$. By Equation (2.100) and Proposition 2.55 the elements

$$\left(\prod_{i=1}^t (-1)^{a_i} \frac{(q_i - q_i^{-1})^{a_i}}{\{a_i\}_i!} \right) E^{a_t}_{\gamma_t} E^{a_{t-1}}_{\gamma_{t-1}} \cdots E^{a_1}_{\gamma_1}$$

such that $\mu = \sum_{i=1}^t a_i \gamma_i$ form the dual basis of U^+_μ with respect to the bilinear pairing $\langle -, - \rangle$. For $1 \le j \le t$ define

$$R^{[j]} = (T_{i_1} \cdots T_{i_{j-1}} \otimes T_{i_1} \cdots T_{i_{j-1}})(R_{i_j}) = \sum_{r \ge 0} (-1)^r \frac{(q_{i_j} - q_{i_j}^{-1})^r}{\{r\}_{i_j}!} F^r_{\gamma_j} \otimes E^r_{\gamma_j}. \tag{2.112}$$

Theorem 2.60 ([34],[47],[35]). *The quasi R-matrix is given by*

$$R = R^{[t]} \cdot R^{[t-1]} \cdots R^{[2]} \cdot R^{[1]} \tag{2.113}$$

Remark 2.61. The quasi R-matrix plays a pivotal role in many deep applications of quantum qroups, which we outline here. By Theorem 2.58 the quasi R-matrix is the unique element that intertwines between two bar involutions on $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$. This relation first appeared in the development of *canonical* (or *crystal*) *bases* for $U_q(\mathfrak{g})$, established by G. Lusztig [50] and M. Kashiwara [31].

For $M, M' \in \mathrm{Ob}(\mathcal{O}_{\mathrm{int}})$ let $\kappa : M \otimes M' \to M \otimes M'$ be the linear map defined by

$$\kappa(m \otimes m') = q^{(\mu,\nu)} m \otimes m' \quad \text{for } m \in M_\mu, m' \in M'_\nu. \tag{2.114}$$

The quasi R-matrix gives rise to a *universal R-matrix* $\mathfrak{R}$ such that

$$\mathfrak{R} = R \circ \kappa^{-1} \circ \mathrm{flip} : M \otimes M' \to M' \otimes M \tag{2.115}$$

is an isomorphism of $U_q(\mathfrak{g})$-modules, see [27, Theorem 7.3]. In the theory of integrable systems, the R-matrix is a crucial tool for providing solutions to the quantum Yang-Baxter equation (or QYBE).

Chapter 3

Quantum symmetric pairs

Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be an involutive automorphism. We recall the construction of involutive automorphisms of $\mathfrak{g}$ in Section 3.1. Up to conjugation, these are classified by Satake diagrams (X, τ). This allows one to construct fixed Lie subalgebras $\mathfrak{k}$ of $\mathfrak{g}$. In order to define quantum symmetric pairs, a quantum analogue of $U(\mathfrak{k})$ is constructed in Section 3.2. Recall that the construction depends on additional parameters $\mathbf{c} \in \mathcal{C}$ and $\mathbf{s} \in \mathcal{S}$, see Definition 3.21. Importantly, we obtain a family of right coideal subalgebras instead of Hopf subalgebras. In Section 3.6 we give relations for quantum symmetric pairs. The results of this chapter follow mostly the papers [45] and [38].

3.1 Involutive automorphisms of semisimple Lie algebras

For any subset $X \subseteq I$, let $\mathfrak{g}_X$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\{e_i, f_i, h_i \mid i \in X\}$. Let Q_X denote the subgroup of Q generated by $\{\alpha_i \mid i \in X\}$. This is the root lattice for $\mathfrak{g}_X$. Let $\rho_X \in V$ and $\rho_X^\vee \in V^*$ denote the half sum of positive roots and coroots for $\mathfrak{g}_X$, respectively. Let $W_X \subseteq W$ be the corresponding parabolic subgroup of W generated by $\{\sigma_i \mid i \in X\}$. This is the Weyl group for $\mathfrak{g}_X$. Let $w_X \in W_X$ denote the longest element of W_X. Let $\tau : I \to I$ denote a diagram automorphism for the Dynkin diagram of $\mathfrak{g}$. This can be viewed as an automorphism of $\mathfrak{g}$ by setting

$$\tau(e_i) = e_{\tau(i)}, \quad \tau(f_i) = f_{\tau(i)}, \quad \tau(h_i) = h_{\tau(i)}, \quad \text{for } i \in I. \tag{3.1}$$

The induced map of $\mathfrak{h}^*$ further satisfies $\tau(\alpha_i) = \alpha_{\tau(i)}$ for all $i \in I$.

Definition 3.1 ([59, p. 109], see also [38, Definition 2.3]). Let $X \subseteq I$ and let $\tau : I \to I$ be a diagram automorphism such that $\tau(X) = X$. The pair (X, τ) is called a *Satake diagram* if it satisfies the following properties:

(1) $\tau^2 = \mathrm{id}_I$.

(2) The action of τ on X coincides with the action of $-w_X$.

(3) If $j \in I \setminus X$ and $\tau(j) = j$, then $\alpha_j(\rho_X^\vee) \in \mathbb{Z}$.

Graphically, the components of a Satake diagram are recorded in the Dynkin diagram of $\mathfrak{g}$. The nodes labelled by X are coloured black and a double sided arrow is used to denote the diagram automorphism.

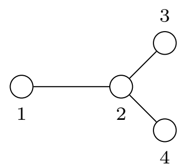
Remark 3.2. We denote a Satake diagram by a triple (I, X, τ) if we need to identify the underlying Lie algebra. With this notation, if (I, X, τ) is a Satake diagram and $i \in I \setminus X$, then $(X \cup \{i, \tau(i)\}, X, \tau|_{X \cup \{i,\tau(i)\}})$ is also a Satake diagram. This notation will be extensively used in Chapter 5.

Recall that there exists a diagram automorphism $\tau_0 : I \to I$ such that the longest element $w_0 \in W$ satisfies

$$w_0(\alpha_i) = -\alpha_{\tau_0(i)}. \tag{3.2}$$

It follows from this and the definition that the pair $(X = I, \tau = -w_X)$ is always a Satake diagram. A complete list of Satake diagrams for simple $\mathfrak{g}$ can be found in [1, pp. 32/33]. Additionally, by inspection of the list of Satake diagrams one sees that the set X is τ_0-invariant.

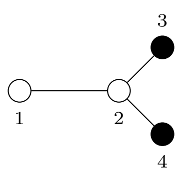
Example 3.3. Consider $\mathfrak{g} = \mathfrak{so}_8(\mathbb{C})$ which is of Dynkin type D_4 and the standard choice of simple roots and coroots.

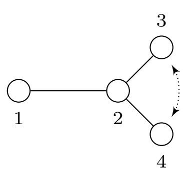


Then the pair $(\{3, 4\}, \mathrm{id})$ is a Satake diagram since $-w_X$ acts as the identity on X and $\rho_X^\vee = \frac{1}{2}(h_3 + h_4)$ satisfies

$$\alpha_1(\rho_X^\vee) = 0, \quad \alpha_2(\rho_X^\vee) = -1.$$

The pairs $(\{1, 3\}, \mathrm{id})$ and $(\{1, 4\}, \mathrm{id})$ are also Satake diagrams for similar reasons. The pair $(\emptyset, (3, 4))$ is a Satake diagram since conditions (2) and (3) are empty. However, the pair $(\{2\}, \mathrm{id})$ is not a Satake diagram, since in this case $\rho_X^\vee = \frac{1}{2}h_2$ satisfies $\alpha_1(\rho_X^\vee) = -\frac{1}{2}$. The Satake diagrams $(\{3, 4\}, \mathrm{id})$ and $(\emptyset, (3, 4))$ are represented graphically by





respectively.

Let $\omega : \mathfrak{g} \to \mathfrak{g}$ be the Chevalley involution given by

$$\omega(e_i) = -f_i, \quad \omega(h_i) = -h_i, \quad \omega(f_i) = -e_i \tag{3.3}$$

for all $i \in I$. Denote by $m_X \in Br(\mathfrak{g})$ the element of the braid group that corresponds to the longest element $w_X \in W_X$.

Lemma 3.4 ([38, Proposition 2.2, Part 3)])**.** *Let $\tau : I \to I$ be a diagram automorphism such that $\tau(X) = X$. Then the automorphism $\mathrm{Ad}(m_X)$ of $\mathfrak{g}$ commutes with both τ and ω.*

Proof. By definition the relation

$$\tau(\mathrm{Ad}(\varsigma_i)(x)) = \mathrm{Ad}(\varsigma_{\tau(i)})(\tau(x)) \tag{3.4}$$

holds for any $i \in I$ and $x \in \mathfrak{g}$. Since $w_X \in W_X$ is τ-invariant, it follows that

$$\tau(\mathrm{Ad}(m_X)(x)) = \mathrm{Ad}(m_X)(\tau(x))$$

and hence τ and $\mathrm{Ad}(m_X)$ commute. Equation (2.53) immediately implies that ω commutes with $\mathrm{Ad}(\varsigma_i)$ for any $i \in I$ and hence also with $\mathrm{Ad}(m_X)$. □

The following Lemma from [2, Lemme 4.9] has been rewritten for the convenience of the reader. Recall from the proof of Lemma 2.35 the Lie subalgebra $\mathfrak{sl}_2(i)$ of $\mathfrak{g}$ and the corresponding Lie subgroup $SL_2(i)$ of G for each $i \in I$.

Lemma 3.5 ([2, Lemme 4.9])**.** *The automorphism $\mathrm{Ad}(m_0)$ of $\mathfrak{g}$ satisfies*

$$\mathrm{Ad}(m_0) = \tau_0 \circ \omega. \tag{3.5}$$

Proof. We only need to verify the lemma on the generators e_i, f_i and h_i for $i \in I$. The result follows for the elements h_i by using Lemma 2.34 and Equation (3.2). Since $\mathrm{Ad}(m_0)(e_i) \in \mathfrak{g}_{-\tau_0(\alpha_i)}$ and $\mathrm{Ad}(m_0)(f_i) \in \mathfrak{g}_{\tau_0(\alpha_i)}$ there exist scalars $c_{\tau_0(i)}, d_{\tau_0(i)}$ such that

$$\mathrm{Ad}(m_0)(e_i) = -c_{\tau_0(i)} f_{\tau_0(i)},$$
$$\mathrm{Ad}(m_0)(f_i) = -d_{\tau_0(i)} e_{\tau_0(i)}.$$

In fact, since

$$-h_{\tau_0(i)} = \mathrm{Ad}(m_0)(h_i) = \mathrm{Ad}(m_0)([e_i, f_i]) = c_{\tau_0(i)} d_{\tau_0(i)} [f_{\tau_0(i)}, e_{\tau_0(i)}] = -c_{\tau_0(i)} d_{\tau_0(i)} h_{\tau_0(i)},$$

it follows that $c_{\tau_0(i)} = d^{-1}_{\tau_0(i)}$ for all $i \in I$. We show that $c_{\tau_0(i)} = 1$ for all $i \in I$ from which the statement of the lemma follows.

The relation $w_0 \sigma_i w_0^{-1} = \sigma_{\tau_0(i)}$ holds in W. This implies that

$$\pi_G(m_0 \varsigma_i m_0^{-1}) = \pi_G(\varsigma_{\tau_0(i)}) = \exp(e_{\tau_0(i)}) \exp(-f_{\tau_0(i)}) \exp(e_{\tau_0(i)}).$$

On the otherhand, since the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ acts by conjugation we have

$$\begin{aligned}\pi_G(m_0\varsigma_i m_0^{-1}) &= \pi_G(m_0)\exp(e_i)\exp(-f_i)\exp(e_i)\pi_G(m_0^{-1})\\ &= \mathrm{Ad}(m_0)\big(\exp(e_i)\exp(-f_i)\exp(e_i)\big)\\ &= \exp(\mathrm{Ad}(m_0)(e_i))\exp(\mathrm{Ad}(m_0)(-f_i))\exp(\mathrm{Ad}(m_0)(e_i))\\ &= \exp(-c_{\tau_0(i)}f_{\tau_0(i)})\exp(c_{\tau_0(i)}^{-1}e_{\tau_0(i)})\exp(-c_{\tau_0(i)}f_{\tau_0(i)}).\end{aligned}$$

For any $t\in\mathbb{C}$ let $\pi_G^t : Br(\mathfrak{g})\to G$ be the map that sends

$$\varsigma_i\mapsto \exp(te_i)\exp(-t^{-1}f_i)\exp(te_i).$$

In this way, we have $\pi_G=\pi_G^1$. Considering e_i and f_i as elements of $\mathfrak{sl}_2(i)$ we have

$$\pi_G^t(\varsigma_i)=\begin{pmatrix}0 & t\\ -t^{-1} & 0\end{pmatrix}\in \mathrm{SL}_2(i).$$

It follows that the map π_G^t is injective for all $t\in\mathbb{C}$. By comparison of the two expressions for $\pi_G(m_0\varsigma_i m_0^{-1})$ above, we hence obtain $c_{\tau_0(i)}=1$ as required. □

Let τ_X denote the diagram automorphism of $\mathfrak{g}_X$ corresponding to the longest element $w_X\in W_X$ and let ω_X be the restriction of the Chevalley involution to $\mathfrak{g}_X$. As a consequence of the previous lemma, the automorphism $\mathrm{Ad}(m_X)$ of $\mathfrak{g}$ leaves $\mathfrak{g}_X$ invariant and satisfies

$$\mathrm{Ad}(m_X)|_{\mathfrak{g}_X}=\tau_X\circ\omega_X. \tag{3.6}$$

We now give an expression for $\mathrm{Ad}(m_X)^2$ which will be needed in the proof of Theorem 3.9. This follows [2, Lemme 4.10] where we fill in some of the details.

Lemma 3.6 ([2, Lemme 4.10])**.** *The relation*

$$\mathrm{Ad}(\varsigma_i^2)=\exp(\mathrm{ad}(i\pi\alpha_i^\vee)) \tag{3.7}$$

holds in $\mathrm{Aut}(\mathfrak{g})$.

Proof. Recall from Equation (2.50) that there is a group homomorphism $\pi_G : Br(\mathfrak{g})\to G$ such that $\varsigma_i\mapsto\exp(e_i)\exp(-f_i)\exp(e_i)$ for each $i\in I$. Over the Lie subgroup $SL_2(i)$ we have

$$\pi_{SL_2(i)}(\varsigma_i)=\exp(e_i)\exp(-f_i)\exp(e_i)=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

which implies that $\pi_{SL_2(i)}(\varsigma_i^2)=-\mathrm{id}$. Recall from [25, Chapter 7] that the simple $\mathfrak{sl}_2$ representations are given by finite dimensional modules V_d of dimension $d+1$ with basis $v_d, v_{d-2},\dots,v_{-d}$ such that $hv_j=jv_j$ for $j=d,d-2,\dots,-d$. The vector v_d is the highest weight vector for V_d. We want to find the action of $\pi_{SL_2(i)}(\varsigma_i^2)$ on V_d. Let $V=V_1=\mathbb{C}^2$.

Then

$$V_d \subseteq \underbrace{V \otimes V \otimes \cdots \otimes V}_{d \text{ times}} = V^{\otimes d}.$$

Indeed, since $V = \mathbb{C}v_1 \oplus \mathbb{C}v_{-1}$ and v_1 is a highest weight vector for V of weight 1, it follows that $v_1 \otimes v_1 \otimes \cdots \otimes v_1 \in V^{\otimes d}$ is a highest weight vector of weight d. This implies that to understand the action of $\pi_{SL_2(i)}(\varsigma_i^2)$ on V_d, we should consider the action on $V^{\otimes d}$ instead.

To keep track of the underlying vector space W, we write $\rho_{\mathfrak{g},W} : \mathfrak{g} \to \mathrm{End}(W)$ and $\rho_{G,W} : G \to \mathrm{Aut}(W)$ for representations of $\mathfrak{g}$ and the corresponding simply connected Lie group G, respectively. Then for any vector space W we have

$$\begin{aligned}
&\big(\rho_{G,V} \otimes \rho_{G,W}\big)(\pi_{SL_2(i)}(\varsigma_i^2)) \\
&\quad = \big(\rho_{G,V} \otimes \rho_{G,W}\big)\big((\exp(e_i)\exp(-f_i)\exp(e_i))^2\big) \\
&\quad = \big((\rho_{G,V} \otimes \rho_{G,W})(\exp(e_i)) \cdot (\rho_{G,V} \otimes \rho_{G,W})(\exp(-f_i)) \cdot (\rho_{G,V} \otimes \rho_{G,W})(\exp(e_i))\big)^2.
\end{aligned}$$

For $x \in \mathfrak{g}$ we have

$$\big(\rho_{\mathfrak{g},V} \otimes \rho_{\mathfrak{g},W}\big)(x) = \rho_{\mathfrak{g},V}(x) \otimes 1 + 1 \otimes \rho_{\mathfrak{g},W}(x)$$

and hence

$$\begin{aligned}
\big(\rho_{G,V} \otimes \rho_{G,W}\big)(\exp(x)) &= \exp(\rho_{\mathfrak{g},V} \otimes 1 + 1 \otimes \rho_{\mathfrak{g},W}(x)) \\
&= \exp(\rho_{\mathfrak{g},V}(x) \otimes 1)\exp(1 \otimes \rho_{\mathfrak{g},W}(x)) \\
&= \exp(\rho_{\mathfrak{g},V}(x) \otimes \rho_{\mathfrak{g},W}(x)) = \rho_{G,V\otimes W}(\exp(x))
\end{aligned}$$

where the second equality follows from the fact that $\rho_{\mathfrak{g},V}(x) \otimes 1$ and $1 \otimes \rho_{\mathfrak{g},W}(x)$ commute. This implies that

$$\begin{aligned}
\big(\rho_{G,V} \otimes \rho_{G,W}\big)(\pi_{SL_2(i)}(\varsigma_i^2)) &= \big(\rho_{G,V\otimes W}(\exp(e_i))\rho_{G,V\otimes W}(\exp(-f_i))\rho_{G,V\otimes W}(\exp(e_i))\big)^2 \\
&= \rho_{G,V\otimes W}\big((\exp(e_i)\exp(-f_i)\exp(e_i))^2\big) \\
&= \rho_{G,V\otimes W}(\pi_{SL_2(i)}(\varsigma_i^2)).
\end{aligned}$$

It follows that $\pi_{SL_2(i)}(\varsigma_i^2)$ acts on $V \otimes W$ diagonally. Since $\pi_{SL_2(i)}(\varsigma_i^2)$ acts on V as $-\mathrm{id}_V$, it follows by extension that $\pi_{SL_2(i)}(\varsigma_i^2)$ acts on $V^{\otimes d}$ as $(-1)^d \mathrm{id}_{V^{\otimes d}}$ and therefore ς_i^2 acts on V_d by multiplication by $(-1)^d$.

On the otherhand the element $\exp(i\pi\alpha_i^\vee)$ acts on $v_j \in V_d$ by $\exp^{i\pi j} = (-1)^j = (-1)^d$. These two actions coincide and hence they also coincide under any representation of G. In particular, we see that

$$\mathrm{Ad}(\varsigma_i^2) = \mathrm{Ad}(\exp(i\pi\alpha_i^\vee)) = \exp(\mathrm{ad}(i\pi\alpha_i^\vee))$$

as required. □

Proposition 3.7 ([2, Proposition 4.10.1])**.** *Let $w \in W$ and let $w = \sigma_{i_1} \cdots \sigma_{i_t}$ be a reduced* *expression. Then in* $\mathrm{Aut}(\mathfrak{g})$ *the relation*

$$\mathrm{Ad}\big((\varsigma_{i_t}\cdots\varsigma_{i_1})(\varsigma_{i_1}\cdots\varsigma_{i_t})\big) = \exp(\mathrm{ad}(i\pi H_w)) \tag{3.8}$$

holds, where $H_w = \sum_{\alpha\in\Phi(w)} \alpha^\vee$ *and* $\Phi(w) = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}$.

Proof. We proceed by induction on the length $\ell(w)$ of $w \in W$, where $\ell : W \to \mathbb{N}_0$ denotes the length function with respect to W. If $\ell(w) = 1$, then $\mathrm{Ad}(\varsigma_j^2) = \exp(\mathrm{ad}(i\pi\alpha_j^\vee))$ by Lemma 3.6.

Suppose that $w' = \sigma_{i_1}\cdots\sigma_{i_{t-1}}$ is reduced with $\ell(w') = \ell(w) - 1$ such that

$$\mathrm{Ad}\big((\varsigma_{i_{t-1}}\cdots\varsigma_{i_1})(\varsigma_{i_1}\cdots\varsigma_{i_{t-1}})\big) = \exp(\mathrm{ad}(i\pi H_{w'})) = \mathrm{Ad}(\exp(i\pi H_{w'})).$$

From this, we deduce that

$$\begin{aligned}\mathrm{Ad}\big((\varsigma_{i_t}\cdots\varsigma_{i_1})(\varsigma_{i_1}\cdots\varsigma_{i_t})\big) &= \mathrm{Ad}(\varsigma_{i_t})\mathrm{Ad}(\exp(i\pi H_{w'}))\mathrm{Ad}(\varsigma_{i_t})^{-1}\mathrm{Ad}(\varsigma_{i_t}^2)\\ &= \mathrm{Ad}\big(\exp(i\pi(\sigma_{i_t}(H_{w'}) + \alpha_{i_t}^\vee))\big).\end{aligned}$$

It remains to show that $H_w = \sigma_{i_t}(H_{w'}) + \alpha_{i_t}^\vee$, or equivalently, $\Phi(w) = \sigma_{i_t}(\Phi(w')) \cup \{\alpha_{i_t}\}$. Since $\ell(w) = t$, we have $w(\alpha_{i_t}) \in \Phi^-$ and thus $\alpha_{i_t} \in \Phi(w)$. Let $\alpha \in \Phi(w) - \{\alpha_{i_t}\}$, then $\sigma_{i_t}(\alpha) \in \Phi^+ - \{\alpha_{i_t}\}$. Since $\alpha \in \Phi(w)$, it follows that $w(\alpha) = w'(\sigma_{i_t}(\alpha)) \in \Phi^-$ which implies $\sigma_{i_t}(\alpha) \in \Phi(w')$. Hence we have $\Phi(w) - \{\alpha_{i_t}\} \subseteq \sigma_{i_t}(\Phi(w'))$.

On the other hand, let $\alpha \in \Phi(w')$. Since $\ell(w'\sigma_{i_t}) > \ell(w')$ we have $\alpha \neq \alpha_{i_t}$ and hence $\sigma_{i_t}(\alpha) \in \Phi^+ - \{\alpha_{i_t}\}$. As $w\sigma_{i_t}(\alpha) = w'(\alpha) \in \Phi^-$, it follows that $\sigma_{i_t}(\alpha) \in \Phi(w)$ which implies $\sigma_{i_t}(\Phi(w')) \subseteq \Phi(w) - \{\alpha_{i_t}\}$.

Therefore we have $\Phi(w) - \{\alpha_{i_t}\} = \sigma_{i_t}(\Phi(w'))$ as required. □

Corollary 3.8 ([2, Corollaire 4.10.3]). *The relation*

$$\mathrm{Ad}(m_X^2) = \exp(\mathrm{ad}(i\pi 2\rho_X^\vee)) \tag{3.9}$$

holds in $\mathrm{Aut}(\mathfrak{g})$.

Proof. Let $w_X \in W_X$ have reduced expression $w_X = \sigma_{i_1}\cdots\sigma_{i_t}$. Since $w_X^2 = 1$, the element $\sigma_{i_t}\cdots\sigma_{i_1}$ is another reduced expression for w_X. By Proposition 3.7 we have

$$\mathrm{Ad}(m_X^2) = \mathrm{Ad}\big((\sigma_{i_t}\cdots\sigma_{i_1})(\sigma_{i_1}\cdots\sigma_{i_t})\big) = \mathrm{Ad}(\exp(i\pi H_{w_X}).$$

Since $w_X(\Phi^-) \cap \Phi^+ = \Phi_X^+$, it follows that $\Phi(w_X) = \Phi_X^+$ and hence $H_{w_X} = 2\rho_X^\vee$ as required. □

It follows from Corollary 3.8 that if $x \in \mathfrak{g}_{\alpha_i}$ for $i \in I$ then

$$\begin{aligned}\mathrm{Ad}(m_X)^2(x) &= \exp(\mathrm{ad}(i\pi 2\rho_X^\vee))(x)\\ &= \sum_{k=0}^{\infty} \frac{\mathrm{ad}^k(i\pi 2\rho_X^\vee)(x)}{k!}\\ &= \left(\sum_{k=0}^{\infty} \frac{(i\pi\alpha_i(2\rho_X^\vee))^k}{k!}\right)x\\ &= (-1)^{\alpha_i(2\rho_X^\vee)}x.\end{aligned}$$

Let $s : I \to \mathbb{C}^\times$ be a function satisfying

$$s(i) = 1 \qquad \text{if } i \in X \text{ or } \tau(i) = i, \tag{3.10}$$

$$\frac{s(i)}{s(\tau(i))} = (-1)^{\alpha_i(2\rho_X^\vee)} \qquad \text{if } i \notin X \text{ and } \tau(i) \neq i. \tag{3.11}$$

This function extends to a group homomorphism $s_Q : Q \to \mathbb{C}^\times$ such that $s_Q(\alpha_i) = s(i)$ for each simple root α_i. Using the group homomorphism s_Q, we define a Lie algebra automorphism $\mathrm{Ad}(s) : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathrm{Ad}(s)|_{\mathfrak{h}} = \mathrm{id}|_{\mathfrak{h}}, \quad \mathrm{Ad}(s)(x) = s_Q(\alpha)x \text{ for } \alpha \in \Phi,\, x \in \mathfrak{g}_\alpha. \tag{3.12}$$

We now associate an involutive automorphism to any Satake diagram (X, τ), following [38, Theorem 2.5].

Theorem 3.9 ([38, Theorem 2.5])**.** *Let (X, τ) be a Satake diagram. Then*

$$\theta(X,\tau) = \theta = \mathrm{Ad}(s) \circ \mathrm{Ad}(m_X) \circ \tau \circ \omega \tag{3.13}$$

is an involutive automorphism of $\mathfrak{g}$.

Proof. Suppose $x \in \mathfrak{g}_X$. By Equation (3.6) and Equation (3.10) we have

$$\theta^2(x) = \big(\tau_X \circ \omega_X \circ \tau \circ \omega\big)^2(x) = x$$

as required. If instead $x \in \mathfrak{h}$, then since $\mathrm{Ad}(w_X) \circ \tau \circ \omega(\mathfrak{h}) \subseteq \mathfrak{h}$ it follows that

$$\theta^2(x) = \big(\mathrm{Ad}(m_X) \circ \tau \circ \omega\big)^2(x).$$

By Lemma 3.4 and Corollary 3.8, we obtain $\theta^2(x) = x$. Hence we may assume that $x \in \mathfrak{g}_{\alpha_i}$ for some $i \in I \setminus X$. Note that applying $\mathrm{Ad}(s)$ to an element $y \in \mathfrak{g}_\alpha$ has the effect of multiplying by a scalar which depends on the given $\alpha \in \Phi$. Hence

$$\begin{aligned}\theta^2(x) &= \big(\mathrm{Ad}(s) \circ \mathrm{Ad}(m_X) \circ \tau \circ \omega\big)^2(x)\\ &= s_Q\big((w_X \circ \tau \circ \omega)^2(\alpha_i)\big)s_Q\big(w_X \circ \tau \circ \omega(\alpha_i)\big)\big(\mathrm{Ad}(m_X) \circ \tau \circ \omega\big)^2(x).\end{aligned}$$

Viewed as involutive automorphisms of $\mathfrak{h}^*$, both τ and ω commute with w_X. This and

Lemma 3.4 imply

$$\theta^2(x) = s_Q(w_X(-\alpha_{\tau(i)}))s_Q(\alpha_i)\mathrm{Ad}(m_X)^2(x).$$

It follows from Equation (3.10) that $s_Q(w_X(-\alpha_{\tau(i)})) = s_Q(-\alpha_{\tau(i)})$. Using this, we obtain

$$\theta^2(x) = \frac{s(i)}{s(\tau(i))}\mathrm{Ad}(m_X)^2(x).$$

By Corollary 3.8, we have $\mathrm{Ad}(m_X)^2(x) = (-1)^{\alpha_i(2\rho_X^\vee)}x$. The result follows from Condition (3) of Definition 3.1 and Equation (3.11). □

Remark 3.10. A full classification of involutive automorphisms of $\mathfrak{g}$ is approached in [38, Appendix A]. In particular, it is shown that given any involutive automorphism ϑ, there is a Satake diagram (X,τ) such that ϑ is $\mathrm{Aut}(\mathfrak{g})$-conjugate to $\theta = \theta(X,\tau)$, see [38, Proposition A.6].

For any Satake diagram (X,τ), the automorphism $\theta = \theta(X,\tau)$ satisfies $\theta(\mathfrak{h}) = \mathfrak{h}$. More explicitly, Equation (3.13) implies that

$$\theta(h) = \mathrm{Ad}(w_X)\circ\omega\circ\tau(h) = -w_X\circ\tau(h) \tag{3.14}$$

for $h \in \mathfrak{h}$. This restriction defines a dual map $\Theta : \mathfrak{h}^* \to \mathfrak{h}^*$ which is given by the same expression

$$\Theta = -w_X \circ \tau \tag{3.15}$$

where now both w_X and τ act on $\mathfrak{h}^*$. For later use, we note the following lemma.

Lemma 3.11 ([3, Lemma 3.2])**.** *Let (X,τ) be a Satake diagram. For all $i \in I$, we have*

$$\alpha_i - \Theta(\alpha_i) = \alpha_{\tau(i)} - \Theta(\alpha_{\tau(i)}). \tag{3.16}$$

Proof. We have $w_X(\alpha_i) - \alpha_i \in Q_X$ for any $i \in I$. The claim of the lemma follows from Property (2) of Definition 3.1 by observing that

$$w_X(w_X(\alpha_i) - \alpha_i) = -\tau(w_X(\alpha_i) - \alpha_i))$$

holds for any $i \in I$. □

3.2 The fixed Lie subalgebra

Let (X,τ) be a Satake diagram and let $\theta = \theta(X,\tau) : \mathfrak{g} \to \mathfrak{g}$ be the corresponding involutive automorphism, as defined in Equation (3.13). Then the Lie algebra $\mathfrak{g}$ has a decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{3.17}$$

into the $+1$ and -1 eigenspace of θ with $\mathfrak{k} = \{x \in \mathfrak{g} \mid \theta(x) = x\}$. Note that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$. On the other hand, for any $x, y \in \mathfrak{p}$ with $[x,y] \neq 0$ we have

$$\theta([x,y]) = [\theta(x), \theta(y)] = [-x,-y] = [x,y]$$

and hence $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$.

The following lemma gives a description of the generators of the Lie subalgebra $\mathfrak{k}$.

Lemma 3.12 ([38, Lemma 2.8]). *The Lie algebra $\mathfrak{k}$ is generated by the elements*

$$e_i, f_i \quad \text{for } i \in X, \tag{3.18}$$

$$h \in \mathfrak{h} \quad \text{with } \theta(h) = h, \tag{3.19}$$

$$f_i + \theta(f_i) \quad \text{for } i \in I \setminus X. \tag{3.20}$$

Proof. Let $\overline{\mathfrak{k}}$ denote the Lie subalgebra of $\mathfrak{g}$ generated by all elements of the form (3.18), (3.19) and (3.20). By Equation (3.6), the generators (3.18) are invariant under θ. The remaining generators are invariant under θ by definition, hence $\overline{\mathfrak{k}} \subseteq \mathfrak{k}$.

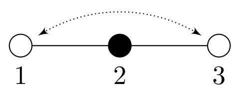
Conversely, suppose $x \in \mathfrak{k}$. By the triangular decomposition (2.24) we write

$$x = x^+ + x^0 + x^-$$

with $x^+ \in \mathfrak{n}^+$, $x^0 \in \mathfrak{h}$ and $x^- \in \mathfrak{n}^-$. Since $\theta(f_i) \in \mathfrak{n}^+$ for $i \in I \setminus X$, it follows that there exists an element $y \in \overline{\mathfrak{k}}$ such that $x - y \in \mathfrak{n}^+ \oplus \mathfrak{h}$. Such an element is a linear combination of elements of the form (3.18), (3.20) and all possible Lie brackets between these elements. We may hence assume that $x^- = 0$. Further, since $\theta(x^0) = x^0$ we have $x^0 \in \overline{\mathfrak{k}}$. We can hence assume that $x^0 = 0$.

It follows that we can write $x = x^+ \in \mathfrak{n}^+$ as a sum of weight vectors $x = \sum_{\alpha \in Q^+} x_\alpha$. As $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-w_X \tau(\alpha)}$, it follows from (2) that $x_\alpha \neq 0$ implies $\alpha = \sum_{i \in X} n_i \alpha_i$ for $n_i \in \mathbb{N}_0$. Hence $x \in \overline{\mathfrak{k}}$ as required. □

Example 3.13. Let $\mathfrak{g} = \mathfrak{sl}_4$ with Satake diagram $(\{2\}, (13))$.



Here, the fixed Lie subalgebra $\mathfrak{k}$ is isomorphic to $\mathfrak{sl}_4 \cap (\mathfrak{gl}_3 \oplus \mathfrak{gl}_1)$. In this case, the corresponding involutive automorphism θ is of the form

$$\theta = \mathrm{Ad}(s) \circ \mathrm{Ad}(\varsigma_2) \circ \tau \circ \omega.$$

By relation (3.11) we have $s(1) = -s(3)$ so we assume $s(1) = 1$ and $s(3) = -1$. Hence θ acts on the Chevalley generators by

$$\begin{aligned} \theta(e_1) &= [f_3, f_2], & \theta(h_1) &= -h_2 - h_3, & \theta(f_1) &= [e_2, e_3], \\ \theta(e_2) &= e_2, & \theta(h_2) &= h_2, & \theta(f_2) &= f_2, \\ \theta(e_3) &= -[f_1, f_2], & \theta(h_3) &= -h_1 - h_2, & \theta(f_3) &= -[e_2, e_1]. \end{aligned}$$

The generators of the fixed Lie subalgebra $\mathfrak{k}$ are then given by the elements

$$\begin{gathered} e_2, \quad f_2, \\ h_2, \quad h_1 - h_3, \\ f_1 - s(3)[e_2, e_3], \quad f_3 - s(1)[e_2, e_1]. \end{gathered}$$

3.3 Quantum involutions

By Corollary 2.24, the universal enveloping algebra of $\mathfrak{k}$ is a Hopf subalgebra of $U(\mathfrak{g})$. Further, using Lemma 3.12 we can write down the generators of $U(\mathfrak{k})$ by modifying by constant terms.

Lemma 3.14 ([38, Corollary 2.9])**.** *Let* $\mathbf{s} = (s_i)_{i \in I \setminus X} \in \mathbb{C}^{I \setminus X}$*. The universal enveloping algebra* $U(\mathfrak{k})$ *is generated by the elements*

$$\begin{aligned} &E_i, F_i \quad \text{for } i \in X, \\ &H \in \mathfrak{h} \quad \text{with } \theta(H) = H, \\ &F_i + \theta(F_i) + s_i \quad \text{for } i \in I \setminus X. \end{aligned} \tag{3.21}$$

as a unital algebra.

We would like to construct a subalgebra of $U_q(\mathfrak{g})$ which is a quantum analogue of $U(\mathfrak{k})$.

$$\begin{array}{ccc} U(\mathfrak{g}) & \longrightarrow & U_q(\mathfrak{g}) \\ \uparrow & & \uparrow \\ U(\mathfrak{k}) & \dashrightarrow & ? \end{array}$$

Even if $\mathfrak{k}$ is semisimple, the natural candidate of taking the quantised enveloping algebra $U_q(\mathfrak{k})$ turns out to be incorrect; it is not even a Hopf subalgebra of $U_q(\mathfrak{g})$, see [8] so this can never work. Instead, we construct a new algebra, denoted by $B_{\mathbf{c},\mathbf{s}}$, which has the desired properties. We recall this, following the work of G. Letzter [42] and the conventions of S. Kolb [38].

The first step in the construction is to deform the involutive automorphism $\theta(X,\tau) : \mathfrak{g} \to \mathfrak{g}$ to an automorphism $\theta_q(X,\tau) : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$. To do this we use the Lusztig automorphism corresponding to the longest element w_X of the parabolic subgroup W_X. Let $\kappa : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ denote the algebra automorphism defined by

$$\kappa(E_i) = E_i K_i, \quad \kappa(F_i) = K_i^{-1} F_i, \quad \kappa(K_i) = K_i \tag{3.22}$$

for all $i \in I$. Let $\omega : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ be the algebra automorphism defined by

$$\omega(E_i) = -F_i, \quad \omega(F_i) = -E_i, \quad \omega(K_\lambda) = K_{-\lambda} \tag{3.23}$$

for all $i \in I$. This is the quantum analogue of the Chevalley involution of $\mathfrak{g}$, denoted by the same symbol. The diagram automorphism τ induces an algebra automorphism of $U_q(\mathfrak{g})$, also denoted by τ such that

$$\tau(E_i) = E_{\tau(i)}, \quad \tau(F_i) = F_{\tau(i)}, \quad \tau(K_i) = K_{\tau(i)} \tag{3.24}$$

for all $i \in I$. Recall the Lusztig automorphisms T_w for $w \in W$ from Section 2.2.5.

Definition 3.15 ([38, Definition 4.3])**.** The automorphism $\theta_q(X,\tau) : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined by

$$\theta_q(X,\tau) = \theta_q = \mathrm{Ad}(s) \circ T_{w_X} \circ \kappa \circ \omega \circ \tau \tag{3.25}$$

is called the *quantum involution* corresponding to (X,τ).

The map θ_q is not an involutive automorphism of $U_q(\mathfrak{g})$, but it does retain the crucial properties of θ, motivating the use of the name 'quantum involution'. Let $\mathcal{M}_X$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_i, F_i, K_i^{\pm 1} \mid i \in X\}$. Using Proposition 2.40 we can determine the action of T_{w_X} on the generators of $\mathcal{M}_X$.

Lemma 3.16 ([38, Lemma 3.4])**.** *For all $i \in X$ one has*

$$T_{w_X}(E_i) = -F_{\tau(i)}K_{\tau(i)}, \qquad T_{w_X}(F_i) = -K_{\tau(i)}^{-1}E_{\tau(i)}, \qquad T_{w_X}(K_i) = K_{\tau(i)}^{-1}, \tag{3.26}$$

$$T_{w_X}^{-1}(E_i) = -K_{\tau(i)}F_{\tau(i)}, \qquad T_{w_X}^{-1}(F_i) = -E_{\tau(i)}K_{\tau(i)}, \qquad T_{w_X}^{-1}(K_i) = K_{\tau(i)}^{-1}. \tag{3.27}$$

Proof. Since $w_X(\alpha_i) = -\alpha_{\tau(i)}$ by Condition (2) of Definition 3.1, it follows that $T_{w_X}(K_i) = K_{\tau(i)}^{-1} = T_{w_X}^{-1}(K_i)$. Write $w_X = w'\sigma_i$ for some $w' \in W_X$. Again, Condition (2) of 3.1 implies that $w'(\alpha_i) = \alpha_{\tau(i)}$. By Proposition 2.40 it follows that $T_{w'}(E_i) = E_{\tau(i)}$ and hence by Equation 2.68 we have

$$T_{w_X}(F_i) = T_{w'}T_i(F_i) = T_{w'}(-K_i^{-1}E_i) = -K_{\tau(i)}E_{\tau(i)}.$$

The expression for $T_{w_X}^{-1}(F_i)$ follows by conjugation with the antiautomorphism σ from Lemma 2.28. The expressions for $T_{w_X}(E_i)$ and $T_{w_X}(E_{\tau(i)})$ follow smilarly. $\square$

Recall from (3.15) the dual map $\Theta : Q \to Q$. The algebra automorphism $\kappa : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ introduced in (3.22) is needed in order for the following proposition to hold.

Proposition 3.17 ([38, Theorem 4.4], [4, Section 5.2])**.** *The automorphism θ_q satisfies the following properties:*

(1) $\theta_q|_{\mathcal{M}_X} = \mathrm{id}|_{\mathcal{M}_X}$.

(2) $\theta_q(K_\mu) = K_{\Theta(\mu)}$ *for all* $\mu \in Q$.

(3) $\theta_q(K_i^{-1}E_i) = -s(\tau(i))^{-1}T_{w_X}(F_{\tau(i)}) \in U^-_{\Theta(\alpha_i)}$ *for all* $i \in I \setminus X$.

(4) $\theta_q(F_iK_i) = -s(\tau(i))T_{w_X}(E_{\tau(i)}) \in U^+_{-\Theta(\alpha_i)}$ *for all* $i \in I \setminus X$.

Proof. Parts (1) and (2) follow from Lemma 3.16 and the definition of θ_q. Parts (3) and (4) are similar so we only consider (4). We have

$$\begin{aligned}\theta_q(F_iK_i) &= \mathrm{Ad}(s)\circ T_{w_X}\circ\kappa\circ\omega\circ\tau(F_iK_i)\\ &= \mathrm{Ad}(s)\circ T_{w_X}\circ\kappa\circ\omega(F_{\tau(i)}K_{\tau(i)})\\ &= \mathrm{Ad}(s)\circ T_{w_X}\circ\kappa(-E_{\tau(i)}K_{-\tau(i)})\\ &= -\mathrm{Ad}(s)\circ T_{w_X}(E_{\tau(i)}) = -s(\tau(i))T_{w_X}(E_{\tau(i)})\end{aligned}$$

as required. □

3.4 Quantum symmetric pair coideal subalgebras

The elements $\theta(F_iK_i)$ for $i\in I\setminus X$ are a major ingredient in our constructions of a quantum analogue of $U(\mathfrak{k})$. As in [4, (5.4)] to shorten notation we write

$$X_i = \theta_q(F_iK_i) = -s(\tau(i))T_{w_X}(E_{\tau(i)})\quad \text{for } i\in I\setminus X. \tag{3.28}$$

Let $Q^\Theta = \{\lambda\in Q \mid \Theta(\lambda)=\lambda\}$ and denote by U^0_Θ the subalgebra of U^0 generated by $\{K_\lambda \mid \lambda\in Q^\Theta\}$. By Condition (2) of Definition 3.1 and Lemma 3.11, the subalgebra U^Θ_0 is generated by the elements $K_i^{\pm1}$ for $i\in X$ and $K_jK^{-1}_{\tau(j)}$ for $j\in I\setminus X$.

Definition 3.18 ([38, Definition 5.1]). Let (X,τ) be a Satake diagram, $\mathbf{c}=(c_i)_{i\in I\setminus X}\in\mathbb{K}(q)^{I\setminus X}$ and $\mathbf{s}=(s_i)_{i\in I\setminus X}\in\mathbb{K}(q)^{I\setminus X}$. Define $B_{\mathbf{c},\mathbf{s}}$ to be the subalgebra generated by $\mathcal{M}_X, U^0_\Theta$ and elements of the form

$$B_i = F_i + c_iX_iK_i^{-1} + s_iK_i^{-1} \tag{3.29}$$

for all $i\in I\setminus X$.

The key property of $B_{\mathbf{c},\mathbf{s}}$ is that it is a *right coideal subalgebra* of $U_q(\mathfrak{g})$ [42, Theorem 4.9], meaning

$$\Delta(B_{\mathbf{c},\mathbf{s}})\subseteq B_{\mathbf{c},\mathbf{s}}\otimes U_q(\mathfrak{g}). \tag{3.30}$$

Indeed, it is clear that $\mathcal{M}_X$ and U^0_Θ are Hopf subalgebras of $U_q(\mathfrak{g})$. Hence we only need to find $\Delta(B_i)$ for $i\in I\setminus X$.

Lemma 3.19 ([38, Propoistion 5.2]). *For any $i\in I\setminus X$ we have*

$$\Delta(B_i) - B_i\otimes K_i^{-1}\in\mathcal{M}_XU^0_\Theta\otimes U_q(\mathfrak{g}). \tag{3.31}$$

Proof. Using Equation (2.85) for $i\in I\setminus X$ we have

$$\Delta(T_{w_X}(E_{\tau(i)})) = T_{w_X}(E_{\tau(i)})\otimes 1 + \sum_{j\in I} r_j(T_{w_X}(E_{\tau(i)}))K_j\otimes E_j + (\mathrm{rest})_1$$

where $(\mathrm{rest})_1\in\sum_{\alpha\notin\Pi\cup\{0\}}U^+_{w_X(\alpha_{\tau(i)})-\alpha}K_\alpha\otimes U^+_\alpha$. By Lemma 2.50 it follows that the element $r_j(T_{w_X}(E_{\tau(i)}))$ is nonzero only if $j=\tau(i)$. Since $r_{\tau(i)}(T_{w_X}(E_{\tau(i)}))\in\mathcal{M}_X^+$ it follows that

$$(\text{rest})_1 \in \sum_{\alpha > \alpha_{\tau(i)}} U^+_{w_X(\alpha_{\tau(i)})-\alpha} K_\alpha \otimes U^+_\alpha \subseteq \mathcal{M}_X K_{\tau(i)} \otimes U_q(\mathfrak{g}).$$

This implies

$$\begin{aligned}\Delta(B_i) - B_i \otimes K_i^{-1} &= 1 \otimes F_i - c_i s(\tau(i)) r_{\tau(i)}\big(T_{w_X}(E_{\tau(i)})\big) K_{\tau(i)} K_i^{-1} \otimes E_{\tau(i)} K_i^{-1} + Y \\ &\in \mathcal{M}_X U^0_\Theta \otimes U_q(\mathfrak{g})\end{aligned}$$

as required, where $Y = (\text{rest})_1 (K_i^{-1} \otimes K_i^{-1})$. □

By Lemma 3.14 one of the properties that $U(\mathfrak{k})$ satisfies is $U(\mathfrak{k}) \cap \mathfrak{h} = \mathfrak{h}^\theta$ where $\mathfrak{h}^\theta = \{h \in \mathfrak{h} \mid \theta(h) = h\}$. For $B_{\mathbf{c},\mathbf{s}}$ to be a quantum analogue of $U(\mathfrak{k})$ a similar property should be satisfied, namely

$$B_{\mathbf{c},\mathbf{s}} \cap U^0 = U^0_\Theta.$$

For this to hold, restrictions are imposed on the parameters $\mathbf{c}$ and $\mathbf{s}$, as in [38, Lemma 5.3–5.5]. Let

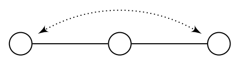
$$I_{ns} = \{i \in I \setminus X \mid \tau(i) = i \text{ and } a_{ij} = 0 \text{ for all } j \in X\} \tag{3.32}$$

and let

$$\mathcal{C} = \{\mathbf{c} \in (\mathbb{K}(q)^\times)^{I \setminus X} \mid c_i = c_{\tau(i)} \text{ if } \tau(i) \neq i \text{ and } (\alpha_i, \Theta(\alpha_i)) = 0\}, \tag{3.33}$$

$$\mathcal{S} = \{\mathbf{s} \in (\mathbb{K}(q)^\times)^{I \setminus X} \mid s_j \neq 0 \Rightarrow (j \in I_{ns} \text{ and } a_{ij} \in -2\mathbb{N}_0\ \forall i \in I_{ns} \setminus \{j\})\}. \tag{3.34}$$

Example 3.20. We give an example to show that it is necessary to have restrictions on the parameters $\mathbf{c}$ and $\mathbf{s}$. Consider the Satake diagram



with $\mathbf{c} \notin \mathcal{C}$ and $\mathbf{s} = \mathbf{0}$. By (3.11), we assume that $s(1) = s(3) = 1$. Then

$$\begin{aligned}[B_1, B_3] &= [F_1 - c_1 E_3 K_1^{-1}, F_3 - c_3 E_1 K_3^{-1}] \\ &= -c_3 [F_1, E_1 K_3^{-1}] - c_1 [E_3 K_1^{-1}, F_3] \\ &= c_3 \frac{K_1 - K_1^{-1}}{q - q^{-1}} K_3^{-1} - c_1 \frac{K_3 - K_3^{-1}}{q - q^{-1}} K_1^{-1} \\ &= (q - q^{-1})^{-1} \big(c_3 K_1 K_3^{-1} - c_1 K_3 K_1^{-1} - (c_3 - c_1) K_1^{-1} K_3^{-1}\big).\end{aligned}$$

Since $\mathbf{c} \notin \mathcal{C}$ it follows that $K_1^{-1} K_3^{-1} \in B_{\mathbf{c},\mathbf{s}}$. However, $K_1^{-1} K_3^{-1} \notin U^0_\Theta$.

On the other hand suppose that $s_1 \neq 0$ which implies that $\mathbf{s} \notin \mathcal{S}$. Then

$$q^2 (K_1 K_3^{-1}) B_1 (K_3 K_1^{-1}) - B_1 = (q^2 - 1) K_1^{-1}$$

which implies $K_1^{-1} \in B_{\mathbf{c},\mathbf{s}}$. However, $K_1^{-1} \notin U^0_\Theta$.

From now on, we will only consider $B_{\mathbf{c},\mathbf{s}}$ with $\mathbf{c} \in \mathcal{C}$ and $\mathbf{s} \in \mathcal{S}$.

Definition 3.21 ([38, Definition 5.6])**.** The subalgebra $B_{\mathbf{c},\mathbf{s}}$ for $\mathbf{c} \in \mathcal{C}$ and $\mathbf{s} \in \mathcal{S}$ is called a *quantum symmetric pair coideal subalgebra* of $U_q(\mathfrak{g})$.

3.5 Specialisation

It can be shown that $B_{\mathbf{c},\mathbf{s}}$ specialises to $U(\mathfrak{k})$, see [42, Theorem 4.9]. We give a brief overview here. Details are omitted but can be found in [38, Section 10], for example. Let $A = \mathbb{K}[q]_{(q-1)}$ denote the localization of the polynomial ring $\mathbb{K}[q]$ with respect to the prime ideal generated by $(q-1)$. Define the A-form U_A of $U_q(\mathfrak{g})$ to be the A-subalgebra of $U_q(\mathfrak{g})$ generated by $E_i, F_i, K_i^{\pm 1}$ and elements

$$\frac{K_i - 1}{q - 1}$$

for $i \in I$. The *specialization* of $U_q(\mathfrak{g})$ at $q = 1$ is the algebra

$$U_1 = \mathbb{K} \otimes_A U_A.$$

Here, $\mathbb{K}$ is viewed as an A-module via the evaluation at 1. By [38, Theorem 10.1], U_1 is isomorphic to $U(\mathfrak{g})$ as an algebra. We extend this notion to subalgebras B of $U_q(\mathfrak{g})$ by defining the specialization at $q = 1$ to be the algebra

$$B_1 = \mathbb{K} \otimes_A (U_A \cap B).$$

It follows from the algebra isomorphism $\phi : U_1 \to U(\mathfrak{g})$ that B_1 is isomorphic to a subalgebra of $U(\mathfrak{g})$.

The notion of specialization can also be applied to algebra automorphisms of $U_q(\mathfrak{g})$. In particular if ζ is an automorphism of $U_q(\mathfrak{g})$ such that $\zeta(U_A) \subset U_A$ and ζ' is an automorphism of $U(\mathfrak{g})$ such that

$$U_1 \ni 1 \otimes \zeta(x) = \zeta'(\phi(x)) \quad \text{for all } x \in U_A$$

then we say that ζ specializes to ζ'. In Section 3.3 we defined the quantum involution θ_q as a deformation of the involution $\theta : \mathfrak{g} \to \mathfrak{g}$. One can make this more precise by showing that θ_q specializes at $q = 1$ to θ [38, Proposition 10.2].

3.6 Relations for $B_{\mathbf{c},\mathbf{s}}$

The algebra $B_{\mathbf{c},\mathbf{s}}$ can be described explicitly by generators and relations. Using [44, Section 7] and [38, Section 7] we give a summary of the relations. To unify notation we let $B_i = F_i$ and $c_i = s_i = 0$ if $i \in X$. For $i, j \in I$ let F_{ij} denote the function in two variables defined by

$$F_{ij}(x,y) = \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} x^{1-a_{ij}-r} y x^r. \tag{3.35}$$

With this notation, the quantum Serre relations (Q5) and (Q6) can be rewritten as

$$F_{ij}(E_i, E_j) = F_{ij}(F_i, F_j) = 0 \quad \text{for all } i \in I.$$

In the below theorem, the 'order' of an expression refers to the maximal number of F_i's appearing over all summands.

Theorem 3.22 ([44, Theorem 7.1], [38, Theorem 7.1])**.** *The algebra $B_{\mathbf{c},\mathbf{s}}$ is generated over $\mathcal{M}_X^+ U_\Theta^0$ by the elements $\{B_i \mid i \in I\}$ subject to the following relations:*

$$K_\lambda B_i = q^{-(\lambda,\alpha_i)} B_i K_\lambda \qquad \textit{for all } \lambda \in Q^\Theta, i \in I, \tag{3.36}$$

$$E_i B_j - B_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \qquad \textit{for all } i \in X, j \in I, \tag{3.37}$$

$$F_{ij}(B_i, B_j) = C_{ij}(\mathbf{c}) \qquad \textit{for all } i,j \in I \textit{ with } i \neq j \tag{3.38}$$

where $C_{ij}(\mathbf{c})$ is a formal expression independent of $\mathbf{s} \in \mathcal{S}$ with lower order than $F_{ij}(B_i, B_j)$.

Remark 3.23. The expressions $C_{ij}(\mathbf{c})$ can be determined in all cases. In [44, Theorem 7.1] expressions for $C_{ij}(\mathbf{c})$ are found when $\mathfrak{g}$ is of finite type. This was generalised to the Kac-Moody case in [38, Theorems 7.4 and 7.8] for $a_{ij} \in \{0,-1,-2\}$ and [3, Section 3.2] which includes the case $a_{ij} = -3$. Further, in [3, Theorem 3.6] a closed formula for $C_{i\tau(i)}(\mathbf{c})$ is given for $i \neq \tau(i)$. More recently, in [11, Theorems 3.1 and 3.7] expressions for $C_{i\tau(i)}(\mathbf{c})$ are found for $i = \tau(i)$.

Following the method of Letzter [44, Section 7], we recall how to determine $C_{ij}(\mathbf{c})$ for $i \notin \{\tau(i), \tau(j)\}$ or $a_{ij} \in \{0,-1\}$ as seen in [38, Theorems 7.3, 7.4, 7.8]. By (2.27) and (2.36) there is a direct sum decomposition

$$U_q(\mathfrak{g}) = \bigoplus_{\lambda \in Q} U^+ K_\lambda S(U^-).$$

For $\lambda \in Q$, let $P_\lambda : U_q(\mathfrak{g}) \to U^+ K_\lambda S(U^-)$ denote the corresponding projection map. The formula for the coproduct implies

$$\Delta \circ P_\lambda(x) = (\mathrm{id} \otimes P_\lambda)\Delta(x) \quad \text{for all } x \in U_q(\mathfrak{g}). \tag{3.39}$$

There is a second direct sum decomposition

$$U_q(\mathfrak{g}) = \bigoplus_{\mu,\nu \in Q^+} U_\mu^+ U^0 U_{-\nu}^-$$

that we consider. With respect to this decomposition we obtain projections $Q_{\mu,\nu} : U_q(\mathfrak{g}) \to U_\mu^+ U^0 U_{-\nu}^-$ for all $\mu,\nu \in Q^+$. Let $\gamma_{ij} = (1-a_{ij})\alpha_i + \alpha_j$ for $i,j \in I$. The following technical proposition is given without proof, but is the main tool for computing $C_{ij}(\mathbf{c})$.

Proposition 3.24 ([44, Proof of Theorem 7.4], [38, Equation (7.8)])**.** *For any $i, j \in I$ we have*

$$C_{ij}(\mathbf{c}) = -(\mathrm{id} \otimes \varepsilon)\big(\mathrm{id} \otimes (P_{-\gamma_{ij}} \circ Q_{0,0})\big)\big(\Delta(F_{ij}(B_i, B_j)) - F_{ij}(B_i, B_j) \otimes K_{-\gamma_{ij}}\big). \quad (3.40)$$

Remark 3.25. Note that for s_i to be non-zero for $i \in I \setminus X$ we must have $i \in I_{ns}$. In this case we have

$$B_i = F_i - c_i E_i K_i^{-1} + s_i K_i^{-1}.$$

It follows from this that

$$\Delta(B_i) = B_i \otimes K_i^{-1} + 1 \otimes (F_i - c_i E_i K_i^{-1}).$$

By Equation (3.40) this implies that C_{ij} is indeed independent of $\mathbf{s} \in \mathcal{S}$.

In Chapter 7 we focus on a particular Satake diagram of type A. For this reason, we only give the expressions $C_{ij}(\mathbf{c})$ explicitly for $a_{ij} = 0$ and $a_{ij} = -1$. The same method can be used to find $C_{ij}(\mathbf{c})$ for $a_{ij} \leq -2$ but the calculations become more involved, see [38, Sections 7.2 and 7.3] and [3, Theorems 3.7 and 3.8]. One can show directly that $C_{ij}(\mathbf{c}) = 0$ if $i \in X$, [38, Lemma 5.11]. Let

$$\mathcal{Z}_i = -s(\tau(i)) r_{\tau(i)}(T_{w_X}(E_{\tau(i)})) K_{\tau(i)} K_i^{-1} \quad (3.41)$$

for $i \in I \setminus X$. For all $i, j \in I \setminus X$ we have

$$r_{\tau(i)}\big(T_{w_X}(E_{\tau(i)})\big) B_j = B_j r_{\tau(i)}\big(T_{w_X}(E_{\tau(i)})\big)$$

by Equation (3.37). It follows from this and Equation (3.36) that

$$\mathcal{Z}_i B_j = q^{(\alpha_i - \alpha_{\tau(i)}, \alpha_j)} B_j \mathcal{Z}_i \quad (3.42)$$

for all $i, j \in I \setminus X$. Recall from the proof of Lemma 3.19 that for $i \in I \setminus X$ we have

$$\Delta(B_i) = B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i \mathcal{Z}_i \otimes E_{\tau(i)} K_i^{-1} + Y \quad (3.43)$$

where $Y \in \mathcal{M}_X U^0_\Theta \otimes \sum_{\alpha > \alpha_{\tau(i)}} U^+_\alpha K_i^{-1}$, [38, Lemma 7.2]. This implies the following.

Lemma 3.26 ([38, Theorem 7.3])**.** *For any $i, j \in I$ with $i \notin \{\tau(i), \tau(j)\}$ we have $C_{ij}(\mathbf{c}) = 0$.*

Proof. The statement of the lemma follows from the fact that many terms of $(\mathrm{id} \otimes Q_{0,0})(\Delta(F_{ij}(B_i, B_j)))$ vanish. Indeed, $Q_{0,0}$ only acts non-trivially on elements of $U_q(\mathfrak{g})$ if they contain an equal number of E_i's and F_i's for each $i \in I$. By (3.43) this can only happen in the second tensor factor if $i = \tau(i)$ or $i = \tau(j)$. □

There are two cases to consider. If $j \in I \setminus X$ then all summands of $\Delta(F_{ij}(B_i, B_j))$ involving Y from (3.43) are killed off under $(\mathrm{id} \otimes Q_{0,0})$.

Theorem 3.27 ([38, Theorem 7.4])**.** *Assume that $i, j \in I \setminus X$. Then if $a_{ij} = 0$ we have*

$$C_{ij}(\mathbf{c}) = \delta_{i,\tau(j)}(q_i - q_i^{-1})^{-1}(c_i\mathcal{Z}_i - c_j\mathcal{Z}_j). \tag{3.44}$$

If $a_{ij} = -1$ we have

$$C_{ij}(\mathbf{c}) = \delta_{i,\tau(i)}q_i c_i \mathcal{Z}_i B_j - \delta_{i,\tau(j)}(q_i + q_i^{-1})(q_i c_j \mathcal{Z}_j + q_i^{-2}c_i\mathcal{Z}_i)B_i. \tag{3.45}$$

Proof. Suppose that $a_{ij} = 0$. Then $F_{ij}(B_i, B_j) = B_iB_j - B_jB_i$. Let $X_{ij} = (\mathrm{id} \otimes (P_{-\gamma_{ij}} \circ Q_{0,0}))$. By Proposition 3.24 we have

$$C_{ij}(\mathbf{c}) = -(\mathrm{id} \otimes \varepsilon)X_{ij}(\Delta(B_iB_j - B_jB_i) - (B_iB_j - B_jB_i) \otimes K_{-\gamma_{ij}}).$$

By Equation (3.43) we have

$$\begin{aligned}\Delta(B_iB_j) = &\big(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i\mathcal{Z}_i \otimes E_{\tau(i)}K_i^{-1} + Y\big)\\ &+ \big(B_j \otimes K_j^{-1} + 1 \otimes F_j + c_j\mathcal{Z}_j \otimes E_{\tau(j)}K_i^{-j} + Y\big).\end{aligned}$$

It follows that X_{ij} acts non-trivially on the terms $B_iB_j \otimes K_i^{-1}K_j^{-1}$ and $c_i\mathcal{Z}_j \otimes F_iE_{\tau(i)}K_j^{-1}$. This implies

$$\begin{aligned}X_{ij}(\Delta(B_iB_j) - B_iB_j \otimes K_{-\gamma_{ij}}) &= X_{ij}(c_j\mathcal{Z}_j \otimes F_iE_{\tau(i)}K_j^{-1})\\ &= X_{ij}\left(c_j\mathcal{Z}_j \otimes \left(E_{\tau(j)}F_i - \delta_{i,\tau(j)}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}\right)K_j^{-1}\right)\\ &= X_{ij}(-\delta_{i,\tau(j)}(q_i - q_i^{-1})^{-1}c_j\mathcal{Z}_j \otimes (K_i - K_i^{-1})K_j^{-1})\\ &= \delta_{i,\tau(j)}(q_i - q_i^{-1})^{-1}c_j\mathcal{Z}_j \otimes K_{-\gamma_{ij}}.\end{aligned}$$

Similarly we have

$$X_{ij}(\Delta(B_jB_i) - B_jB_i \otimes K_{-\gamma_{ij}}) = \delta_{i,\tau(j)}(q_i - q_i^{-1})^{-1}c_i\mathcal{Z}_i \otimes K_{-\gamma_{ij}}.$$

Substituting both into the expression for $C_{ij}(\mathbf{c})$ we obtain

$$C_{ij}(\mathbf{c}) = \delta_{i,\tau(j)}(q_i - q_i^{-1})^{-1}(c_i\mathcal{Z}_i - c_j\mathcal{Z}_j)$$

as required.

Now suppose $a_{ij} = -1$. Then $F_{ij} = B_i^2B_j - (q_i + q_i^{-1})B_iB_jB_i + B_jB_i^2$. Calculating as above and using Equation (3.42) when necessary, we find

$$\begin{aligned}
&X_{ij}(\Delta(B_i^2B_j) - B_i^2B_j \otimes K_{-\gamma_{ij}})\\
&\quad = \delta_{i,\tau(j)}\frac{q_i^3(1+q_i^{-2})}{q_i - q_i^{-1}}c_j\mathcal{Z}_jB_i \otimes K_{-\gamma_{ij}} + \delta_{i,\tau(i)}\frac{1}{q_i - q_i^{-1}}c_i\mathcal{Z}_iB_j \otimes K_{-\gamma_{ij}},\\
&X_{ij}(\Delta(B_iB_jB_i) - B_iB_jB_i \otimes K_{-\gamma_{ij}})\\
&\quad = \delta_{i,\tau(j)}\frac{q_i^{-3}}{q_i - q_i^{-1}}c_i\mathcal{Z}_iB_i \otimes K_{-\gamma_{ij}} + \delta_{i,\tau(j)}\frac{1}{q_i - q_i^{-1}}c_j\mathcal{Z}_jB_i \otimes K_{-\gamma_{ij}}\\
&\qquad + \delta_{i,\tau(i)}\frac{q_i}{q_i - q_i^{-1}}c_i\mathcal{Z}_iB_j \otimes K_{-\gamma_{ij}},\\
&X_{ij}(\Delta(B_jB_i^2) - B_jB_i^2 \otimes K_{-\gamma_{ij}})\\
&\quad = \delta_{i,\tau(j)}\frac{1+q_i^{-2}}{q_i - q_i^{-1}}c_i\mathcal{Z}_iB_i \otimes K_{-\gamma_{ij}} + \delta_{i,\tau(i)}\frac{1}{q_i - q_i^{-1}}c_i\mathcal{Z}_iB_j \otimes K_{-\gamma_{ij}}.
\end{aligned}$$

Equation (3.45) follows from this by substituting these expressions into (3.40). □

If instead $j \in X$ then any summands of $\Delta(F_{ij}(B_i, B_j))$ involving Y are killed off unless they include terms of weight $\alpha_i + \alpha_j$. We expand the expression for $\Delta(B_i)$ from Equation (3.43) by including the summand belonging to $\mathcal{M}_XU^0_\Theta \otimes U^+_{\alpha_i+\alpha_j}K_i^{-1}$. Recall the definition of the left adjoint representation of $U_q(\mathfrak{g})$ on itself from (2.31).

Lemma 3.28 ([38, Lemma 7.7]). *Assume $i \in I \setminus X, j \in X$ and $\tau(i) = i$. Then there exist elements $\mathcal{W}_{ij} \in \mathcal{M}_X$ such that*

$$\Delta(B_i) = B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i\mathcal{Z}_i \otimes E_iK_i^{-1} + c_i\mathcal{W}_{ij}K_j \otimes \mathrm{ad}(E_j)(E_i)K_i^{-1} + Y \quad (3.46)$$

where $Y \in \mathcal{M}_XU^0_\Theta \otimes \sum_{\alpha>\alpha_i;\alpha\neq\alpha_i+\alpha_j} U^+_\alpha K_i^{-1}$.

The elements $\mathcal{W}_{ij}$ can be expressed in terms of the skew derivations.

Lemma 3.29 ([3, Lemma 3.4]). *Let $i \in I \setminus X$ such that $\tau(i) = i$ and $j \in X$. If $a_{ij} \neq 0$ then the relation*

$$\mathcal{W}_{ij} = (1 - q^{2(\alpha_i,\alpha_j)})^{-1}r_j(\mathcal{Z}_i) \quad (3.47)$$

holds.

The proof of the following theorem is similar to that of Theorem 3.27 and is seen in [38, Theorem 7.8]. For this reason, we omit the proof here.

Theorem 3.30 ([38, Theorem 7.8]). *Assume $i \in I \setminus X$ and $j \in X$. Then if $a_{ij} = 0$ we have*

$$C_{ij}(\mathbf{c}) = 0. \quad (3.48)$$

If $a_{ij} = -1$ we have

$$C_{ij}(\mathbf{c}) = \delta_{i,\tau(i)}c_i\left(\frac{1}{q_i - q_i^{-1}}(q_i^2B_j\mathcal{Z}_i - \mathcal{Z}_iB_j) + \frac{q_i + q_i^{-1}}{q_j - q_j^{-1}}\mathcal{W}_{ij}K_j\right). \quad (3.49)$$

Chapter 4

The restricted Weyl group

Involutive automorphisms of $\mathfrak{g}$ allows one to construct a subgroup W^Θ of the Weyl group W consisting of elements fixed under the corresponding group automorphism of W. Of particular importance is a subgroup $\widetilde{W}$ of W^Θ which has an interpretation as the Weyl group of the corresponding restricted root system. In Section 4.1 we define $\widetilde{W}$ and give an alternative description of this subgroup which is useful for many of the arguments in Section 4.2. More explicitly we show that W^Θ is a semidirect product of the subgroup W_X with the subgroup $\widetilde{W}$. In order to provide the connection between $\widetilde{W}$ and the restricted root system established in Section 4.4, we first show that $\widetilde{W}$ is realised as a Coxeter subgroup of W. The results in this chapter do not claim originality. Most of the results can be found in [60], which is influenced by the results of [49], [52] and [21].

4.1 The subgroup $\widetilde{W}$

For any subset $J \subseteq I$, write w_J to denote the longest element in the parabolic subgroup W_J of W. For $i \in I \setminus X$, define

$$\widetilde{\sigma}_i = w_{X\cup\{i,\tau(i)\}} w_X^{-1}. \tag{4.1}$$

By Remark 3.2, the triple $(X\cup\{i,\tau(i)\}, X, \tau|_{X\cup\{i,\tau(i)\}})$ is a Satake diagram for any $i \in I\setminus X$. Let $W_{X\cup\{i,\tau(i)\}}$ denote the Weyl group of the corresponding Dynkin diagram with nodes labelled by the set $X \cup \{i,\tau(i)\}$. We can hence consider W_X as a subgroup of $W_{X\cup\{i,\tau(i)\}}$.

Similar to Equation (3.2), there exists a diagram automorphism $\tau_{0,i} : X \cup \{i,\tau(i)\} \to X \cup \{i,\tau(i)\}$ such that $X \cup \{i,\tau(i)\}$ is $\tau_{0,i}$-invariant and

$$w_{X\cup\{i,\tau(i)\}}(\alpha_j) = -\alpha_{\tau_{0,i}(j)} \tag{4.2}$$

for $j \in X \cup \{i,\tau(i)\}$.

Lemma 4.1. *The elements w_X and $w_{X\cup\{i,\tau(i)\}}$ commute for any $i \in I \setminus X$.*

Proof. By the notation (4.2), it follows that

$$w_{X\cup\{i,\tau(i)\}}\sigma_j = \sigma_{\tau_{0,i}(j)}w_{X\cup\{i,\tau(i)\}}. \tag{4.3}$$

Hence we obtain

$$w_{X\cup\{i,\tau(i)\}}w_X = \tau_{0,i}(w_X)w_{X\cup\{i,\tau(i)\}} = w_X w_{X\cup\{i,\tau(i)\}} \tag{4.4}$$

as required. □

Denote by $\widetilde{W} \subset W^\Theta$ the subgroup of W generated by $\widetilde{\sigma}_i$ for $i \in I \setminus X$. We study the subgroup $\widetilde{W}$ in more detail, following the Weyl group combinatorics of [21] and [60] and also taking guidance from [49] and [52] which we do not pursue here. Recall that $\ell : W \to \mathbb{N}_0$ denotes the length function with respect to W. For any subset $J \subseteq I$ let

$$W^J = \{w \in W \mid \ell(\sigma_i w) > \ell(w) \text{ for all } i \in J\} \tag{4.5}$$

denote the set of minimal length left coset representatives of W/W_J. Similarly, define

$${}^JW = \{w \in W \mid \ell(w\sigma_i) > \ell(w) \text{ for all } i \in J\} \tag{4.6}$$

to be the set of minimal length right coset representatives of $W_J\backslash W$. Since $\ell(\sigma_i w) = \ell(w^{-1}\sigma_i)$ for any $w \in W$ and $i \in I$, it follows that $w \in W^J$ if and only if $w^{-1} \in {}^JW$. This implies that any properties for W^J have a corresponding version for ${}^JW$.

Lemma 4.2 ([60, Proposition 2.7.2])**.** *Any element $w \in W$ is in W^J if and only if all reduced expressions for w begin with a σ_i with $i \in I \setminus J$. Correspondingly, any element $w \in W$ is in ${}^JW$ if and only if all reduced expressions for w end with a σ_i with $i \in I \setminus J$.*

Proposition 4.3 ([26, Proposition p. 19])**.** *Any $w \in W$ can be written uniquely as $w = uv$, where $u \in W_J$ and $v \in W^J$ such that the lengths satisfy*

$$\ell(w) = \ell(u) + \ell(v).$$

We now consider the subset

$$\mathcal{W} = \{w \in W^X \mid wW_X = W_X w\} \tag{4.7}$$

of W. We show that any element of $\mathcal{W}$ belongs to ${}^XW$.

Lemma 4.4. *The set $\mathcal{W}$ is a subset of $W^X \cap {}^XW$.*

Proof. Since $w \in \mathcal{W}$ it follows that $\ell(\sigma_i w) = \ell(w) + 1$ for all $i \in X$. Let $\varphi : W_X \to W_X$ be the map that sends $s \in W_X$ to the element $\varphi(s) \in W_X$ such that

$$sw = w\varphi(s)$$

holds. The map φ is a group homomorphism since for any $s, t \in W_X$ we have

$$w\varphi(st) = (st)w = sw\varphi(t) = w\varphi(s)\varphi(t)$$

and hence $\varphi(st) = \varphi(s)\varphi(t)$. For any $s \in W_X$ we have $\ell(\varphi(s)) = \ell(s)$ since

$$\ell(w) + \ell(\varphi(s)) \geq \ell(w\varphi(s)) = \ell(sw) = \ell(s) + \ell(w)$$

where the last equality follows since $wW_X = W_Xw$. This implies that φ is surjective, hence also a group isomorphism. For any $s \in W_X$ we obtain $\ell(ws) = \ell(\varphi^{-1}(s)w) = \ell(s) + \ell(w)$ and thus $w \in {}^XW$ as required. □

A consequence of this is that $\mathcal{W}$ is a subgroup of W, see [52, 25.1]. In order to prove this we require use of the Deletion Condition, see [26, pg. 14], which we state here.

Deletion Condition ([26, pg. 14]). *Given an expression $w = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_t}$ that is not reduced, there exist indices $1 \leq i_j < i_k \leq n$ such that $w = \sigma_{i_1}\cdots\hat{\sigma}_{i_j}\cdots\hat{\sigma}_{i_k}\cdots\sigma_{i_t}$ where $\hat{\sigma}_{i_j}$ denotes the omission of the factor σ_{i_j}.*

Corollary 4.5. *The subset $\mathcal{W}$ is a subgroup of W.*

Proof. We need only show that if $w_1, w_2 \in \mathcal{W}$ then $w_1w_2 \in \mathcal{W}$. It is clear from the definition of $\mathcal{W}$ that $w_1w_2W_X = W_Xw_1w_2$ so we only need to show that $w_1w_2 \in W^X$. Suppose for a contradiction that for some $i \in X$ we have $\ell(\sigma_iw_1w_2) = \ell(w_1w_2) - 1$. Then $\sigma_iw_1w_2$ is not a reduced expression. By the Deletion Condition a reduced expression for $\sigma_iw_1w_2$ must be obtained by omitting the factor σ_i and a factor from w_1w_2.

Since $w_1W_X = W_Xw_1$ we have $\sigma_iw_1 = w_1\sigma_j$ for some $j \in X$. By Lemma 4.4 both w_1 and w_2 are elements of $W^X \cap {}^XW$ hence both $w_1\sigma_j$ and σ_jw_2 are reduced expressions. This gives a contradiction since a reduced expression for $\sigma_iw_1w_2 = w_1\sigma_jw_2$ can only be obtained by omitting a factor from both w_1 and w_2, which is not possible in view of $\ell(\sigma_iw_1w_2) = \ell(w_1w_2) - 1$. This implies that $\ell(\sigma_iw_1w_2) > \ell(w_1w_2)$ for all $i \in X$ as required. □

By (4.1) and (4.3) we have $\widetilde{\sigma}_i \in \mathcal{W}$ for all $i \in I \setminus X$. Let

$$\mathcal{W}^\tau = \{w \in \mathcal{W} \mid \tau(w) = w\}. \tag{4.8}$$

Then since $\tau(X) = X$, it follows that $\widetilde{\sigma}_i \in \mathcal{W}^\tau$ for all $i \in I \setminus X$ and hence $\widetilde{W} \subseteq \mathcal{W}^\tau$. The following Lemma is a generalisation of [52, A1(a)] and [21, Lemma 2] as indicated by [21, Remark 8]. The argument appears in [60, Lemma 3.4.2], so we repeat it here.

Lemma 4.6 ([60, Lemma 3.4.2]). *Any $w \in \mathcal{W}^\tau$ can be written as $w = \widetilde{\sigma}_{i_1}\widetilde{\sigma}_{i_2}\cdots\widetilde{\sigma}_{i_t}$ such that $\widetilde{\sigma}_{i_1}, \ldots, \widetilde{\sigma}_{i_t} \in \widetilde{W}$ and $\ell(w) = \ell(\widetilde{\sigma}_{i_1}) + \cdots \ell(\widetilde{\sigma}_{i_t})$.*

Proof. We use induction on the length of $w \in W$. When $\ell(w) = 0$ there is nothing to show so suppose $\ell(w) > 0$. Since $w \in \mathcal{W}$, the element $u = ww_X$ satisfies

$$u = ww_X = w_Xw$$

where $\ell(u) = \ell(w) + \ell(w_X)$. As $\ell(w) > 0$ and $w \in W^X$, all reduced expressions for w begin with an σ_i where $i \in I \setminus X$. Hence there exists an $i \in I \setminus X$ such that $\ell(\sigma_i u) < \ell(u)$. Since $\tau(u) = u$ and τ is a length-preserving function we also have $\ell(\sigma_{\tau(i)} u) < \ell(u)$. Further, for all $i \in X$ we have $\ell(\sigma_i u) < \ell(u)$ since $\ell(\sigma_i w_X) = \ell(w_X) - 1$. By Proposition 4.3 u can be written uniquely as $u = vx$ where $v \in W_{X\cup\{i,\tau(i)\}}, x \in W^{X\cup\{i,\tau(i)\}}$ and $\ell(u) = \ell(v) + \ell(x)$. For all $j \in X \cup \{i, \tau(i)\}$ we have $\sigma_j v \in W_{X\cup\{i,\tau(i)\}}$ and hence $\ell(\sigma_j vx) = \ell(\sigma_j v) + \ell(x)$. Since $\ell(\sigma_j u) < \ell(u)$, this implies that $\ell(\sigma_j v) < \ell(v)$ for all $j \in X \cup \{i, \tau(i)\}$. It follows from this that $v = w_{X\cup\{i,\tau(i)\}}$ and

$$w = w_X u = w_X w_{X\cup\{i,\tau(i)\}} x = \widetilde{\sigma}_i x.$$

We have

$$\ell(w) = \ell(u) - \ell(w_X) = \ell(w_{X\cup\{i,\tau(i)\}}) + \ell(x) - \ell(w_X) = \ell(\widetilde{\sigma}_i) + \ell(x)$$

and so we can apply the inductive hypothesis to the element x to obtain the result. □

As a result of Lemma 4.6, we see that $\mathcal{W}^\tau = \widetilde{W}$. This gives an alternative description of the subgroup $\widetilde{W}$ which will be used in the next section.

4.2 The subgroup W^Θ

Using the involutive automorphism $\Theta : \mathfrak{h}^* \to \mathfrak{h}^*$ from (3.15), we obtain a group automorphism

$$\Theta_W : W \to W, \quad w \mapsto \Theta \circ w \circ \Theta. \tag{4.9}$$

Let $W^\Theta = \{w \in W \mid \Theta_W(w) = w\}$ denote the subgroup of elements fixed by Θ_W. For $i \in I$ we have

$$\Theta_W(\sigma_i) = (-w_X \circ \tau) \circ \sigma_i \circ (-w_X \circ \tau) = w_X \sigma_{\tau(i)} w_X. \tag{4.10}$$

It follows from this and Condition (2) of Definition 3.1 that $\Theta_W(\sigma_i) = \sigma_i$ for $i \in X$ and hence W_X is a subgroup of W^Θ.

It follows from Lemma 4.1 and Equation (4.10) that for all $i \in I \setminus X$,

$$\begin{aligned}\Theta_W(\widetilde{\sigma}_i) &= (-w_X\tau) w_X w_{X\cup\{i,\tau(i)\}} (-w_X\tau)\\ &= w_X \tau w_{X\cup\{i,\tau(i)\}} \tau\\ &= w_X w_{X\cup\{i,\tau(i)\}} = \widetilde{\sigma}_i.\end{aligned}$$

Hence $\widetilde{\sigma}_i \in W^\Theta$ for all $i \in I \setminus X$ which implies that $\widetilde{W}$ is a subgroup of W^Θ. The following proposition implies that the subgroup W^Θ is generated by W_X and $\widetilde{W}$, as shown in [60, Proposition 3.5.3, Corollary 3.5.4]. Recall from Lemma 4.6 that $\widetilde{W} = \mathcal{W}^\tau = \{w \in W^X \mid wW_X = W_X w, \tau(w) = w\}$.

Proposition 4.7 ([60, Proposition 3.5.3]). *Any element $w \in W^\Theta$ can be written as $w = uv$ with $u \in W_X$ and $v \in \widetilde{W}$. In other words, $W^\Theta = W_X\widetilde{W}$.*

Proof. Since W_X and $\widetilde{W}$ are subgroups of W^Θ, the containment $W_X\widetilde{W} \subseteq W^\Theta$ is immediate. We show that the containment $W^\Theta \subseteq W_X\widetilde{W}$ holds.

Let $w \in W^\Theta$. By Proposition 4.3 we can write w uniquely as $w = uv$ where $u \in W_X$ and $v \in W^X$. Since Θ_W is a group automorphism we have $\Theta_W(w) = \Theta_W(u)\Theta_W(v)$. The elements w and u are Θ_W-invariant which implies that v must also satisfy $\Theta_W(v) = v$. By Equation (4.9) we have $w_X v w_X = \tau(v)$. Since the left-hand side of this expression is non-reduced and W is a Coxeter group we may apply the Deletion Condition $\ell(w_X)$ times. As τ is a length-preserving function we have

$$\ell(vw_X) = \ell(\tau(v)w_X) = \ell(w_X v) = \ell(w_X) + \ell(v)$$

which implies $v \in {}^X W$. Hence the elements $w_X v$ and vw_X are reduced. As a result, each time the Deletion condition is applied to $w_X v w_X$ we remove one factor from each w_X. It follows from this that $\tau(v) = v$.

Let x be a reduced expression for $\sigma_i w_X$ where $i \in X$. Then we have

$$xvw_X = \sigma_i v.$$

In the same way as above, we use the Deletion Condition $\ell(x) = \ell(w_X) - 1$ times to find a reduced expression for xvw_X. This leaves the equation

$$v\sigma_j = \sigma_i v$$

for some $j \in X$. Since this equation holds for all $i \in X$ it follows that

$$vW_X = W_X v$$

and hence $v \in \mathcal{W}^\tau = \widetilde{W}$ as required. □

Theorem 4.8 ([60, Corollary 3.5.4]). *The subgroup W^Θ is a semidirect product of the subgroups W_X and $\widetilde{W}$*

$$W^\Theta = W_X \rtimes \widetilde{W}. \tag{4.11}$$

Proof. By Proposition 4.7 we have seen that $W^\Theta = W_X\widetilde{W}$. It remains to show that $W_X \cap \widetilde{W} = \{\mathrm{id}\}$ and W_X is a normal subgroup of W^Θ.

Let $w \in \widetilde{W} = \mathcal{W}^\tau$. Then $w \in W^X$ and by Lemma 4.2 all reduced expressions for w do not begin with an σ_i with $i \in X$. Hence the only way for w to be an element of W_X is if $w = 1$. This implies that $W_X \cap \widetilde{W} = \{\mathrm{id}\}$. By definition of $\mathcal{W}^\tau$, it follows that $wW_X w^{-1} = W_X$ for $w \in \widetilde{W}$ which implies that $W_X \lhd W^\Theta$ as required. □

4.3 A Coxeter subgroup of W

The results of this section do not claim any originality, but are contained in the MMath project of Sarah Sigley, [60]. The author feels it is to the benefit of the reader to include these results here.

The subgroup W_X is realised as the Weyl group of the Lie subalgebra $\mathfrak{g}_X$ in the natural way. The subgroup $\widetilde{W}$ also has an interpretation as a Weyl group. In particular it is realised as the Weyl group of the restricted root system of the symmetric Lie algebra $(\mathfrak{g}, \theta)$. In order to make this connection more explicit, we first show that $\widetilde{W}$ is a Coxeter group. We write $x \bullet y$ to denote xy with $\ell(xy) = \ell(x) + \ell(y)$.

Lemma 4.9 ([60, Lemma 3.4.3], [21, Lemma 4 (Special case)]). *Let $w \in \mathcal{W}^\tau$ and assume we have two expressions*

$$w = \widetilde{\sigma}_{i_1} \bullet \ldots \bullet \widetilde{\sigma}_{i_p} = \widetilde{\sigma}_{j_1} \bullet \ldots \bullet \widetilde{\sigma}_{j_r}$$

for $i_k, j_l \in I \setminus X$. Then $p = r$.

Proof. We proceed by induction on the length of w. If $\ell(w) = 0$, then $w = 1$ and there is nothing to show. Suppose that $\ell(w) > 0$. Then $p \geq 1$ and $r \geq 1$.

If $i_1 = j_1$ then

$$w' = \widetilde{\sigma}_{i_1} w = \widetilde{\sigma}_{i_2} \bullet \ldots \bullet \widetilde{\sigma}_{i_p} = \widetilde{\sigma}_{j_2} \bullet \ldots \bullet \widetilde{\sigma}_{j_r}$$

so by the induction hypothesis we have $p - 1 = r - 1$ and hence $p = r$.

Suppose instead that $i_1 \neq j_1$. Let $K = X \cup \{i_1, \tau(i_1)\} \cup \{j_1, \tau(j_1)\}$. Consider the element

$$w' = w_X w = w_{X \cup \{i_1, \tau(i_1)\}} \bullet \widetilde{\sigma}_{i_2} \bullet \ldots \bullet \widetilde{\sigma}_{i_p} = w_{X \cup \{j_1, \tau(j_1)\}} \bullet \widetilde{\sigma}_{j_2} \bullet \ldots \bullet \widetilde{\sigma}_{j_r}.$$

By Proposition 4.3 we can uniquely write $w' = v \bullet x$ where $v \in W_K$ and $x \in W^K$. We have $\ell(\sigma_i w') < \ell(w')$ for all $i \in K$ so it follows that $\ell(\sigma_i v) < \ell(v)$ for all $i \in K$. Hence v must be the longest element of W_K i.e. $v = w_K$. Consider the subgroup

$$\mathcal{W}^\tau_K := \{w \in W_K \cap W^X \mid w W_X = W_X w, \tau(w) = w\} = \mathcal{W}^\tau \cap W_K.$$

We claim that the element $w_X w_K$ is in $\mathcal{W}^\tau_K$. Since $\tau(X) = X$ and $\tau(K) = K$, it follows that $\tau(w_X w_K) = w_X w_K$. Since we can consider w_X as an element of W_K, we have $\ell(w_X w_K) = \ell(w_K) - \ell(w_X)$. Hence for $i \in X$ we have

$$\ell(\sigma_i w_X w_K) \geq \ell(w_K) - \ell(\sigma_i w_X) = \ell(w_K) - \ell(w_X) + 1 = \ell(w_X w_K) + 1$$

from which it follows that $w_X w_K \in W^X$. Let u be a reduced expression for $w_X w_K$. Consider the element $u \sigma_i u$ for $i \in X$. Since $\sigma_i u \in W_K$ and $\ell(u \sigma_i u) = \ell(\sigma_i)$, it follows that $u \sigma_i u$ is not a reduced expression in W. As the pair $(W, S = \{\sigma_i \mid i \in I\})$ is a Coxeter system, we may apply the Deletion condition to remove factors and obtain a reduced expression. The expressions $u\sigma_i$ and $\sigma_i u$ are both reduced. Hence the Deletion condition forces a factor from each u to be deleted. It follows that $u\sigma_i u = \sigma_i$ which implies that $w_X w_K W_X = W_X w_X w_K$ and hence $w_X w_K \in \mathcal{W}_K^\tau$.

As $w = w_X w' = w_X w_K \bullet x$ and both w and $w_X w_K$ are elements of $\mathcal{W}^\tau$, it follows that $x \in \mathcal{W}^\tau$. By Lemma 4.6 we may write $x = \widetilde{\sigma}_{l_1} \bullet \cdots \bullet \widetilde{\sigma}_{l_q}$ for $l_i \in I \setminus X$. Further, $\widetilde{W} = \mathcal{W}^\tau$ implies $\mathcal{W}^\tau = \langle \widetilde{\sigma}_i \mid i \in I \setminus X \rangle$ and hence $\mathcal{W}_K^\tau = \langle \widetilde{\sigma}_{i_1}, \widetilde{\sigma}_{j_1} \rangle$. Since $\widetilde{\sigma}_{i_1}$ and $\widetilde{\sigma}_{j_1}$ are involutions, it follows that $\mathcal{W}_K^\tau$ is the dihedral group of order $2m$. As $w_X w_K$ is an involution and is an element of $\mathcal{W}_K^\tau$ it follows that

$$w_X w_K = \underbrace{\widetilde{\sigma}_{i_1} \bullet \widetilde{\sigma}_{j_1} \bullet \widetilde{\sigma}_{i_1} \bullet \dots}_{m \text{ terms}} = \underbrace{\widetilde{\sigma}_{j_1} \bullet \widetilde{\sigma}_{i_1} \bullet \widetilde{\sigma}_{j_1} \bullet \dots}_{m \text{ terms}}$$

Using this, we obtain

$$\widetilde{\sigma}_{j_1} \bullet \dots \bullet \widetilde{\sigma}_{j_r} = w = w_X w_K \bullet x = \underbrace{\widetilde{\sigma}_{j_1} \bullet \widetilde{\sigma}_{i_1} \bullet \widetilde{\sigma}_{j_1} \bullet \dots}_{m \text{ terms}} \bullet \widetilde{\sigma}_{l_1} \bullet \dots \bullet \widetilde{\sigma}_{l_q}.$$

Cancelling $\widetilde{\sigma}_{j_1}$ from both sides gives

$$\widetilde{\sigma}_{j_2} \bullet \dots \bullet \widetilde{\sigma}_{j_r} = \underbrace{\widetilde{\sigma}_{i_1} \bullet \widetilde{\sigma}_{j_1} \bullet \widetilde{\sigma}_{i_1} \bullet \dots}_{m-1 \text{ terms}} \bullet \widetilde{\sigma}_{l_1} \bullet \dots \bullet \widetilde{\sigma}_{l_q}$$

from which it follows that $(r-1) = (m-1) + q$. Similarly, we also have

$$\widetilde{\sigma}_{i_2} \bullet \dots \bullet \widetilde{\sigma}_{i_r} = \underbrace{\widetilde{\sigma}_{j_1} \bullet \widetilde{\sigma}_{i_1} \bullet \widetilde{\sigma}_{j_1} \bullet \dots}_{m-1 \text{ terms}} \bullet \widetilde{\sigma}_{l_1} \bullet \dots \bullet \widetilde{\sigma}_{l_q}$$

which implies $(p-1) = (m-1) + q$. This gives $p = r$, as required. □

Let $\lambda : \widetilde{W} \to \mathbb{N}_0$ denote the length function with respect to $\widetilde{W}$.

Lemma 4.10 ([60, Lemma 3.4.4]). *Let $w \in \widetilde{W}$ and $w = \widetilde{\sigma}_{i_1} \cdots \widetilde{\sigma}_{i_t}$ with $\lambda(w) = t$. Then $w = \widetilde{\sigma}_{i_1} \bullet \dots \bullet \widetilde{\sigma}_{i_t}$.*

Proof. We use induction on t. If $t = 0$ or 1, then there is nothing to show so suppose $t \geq 2$. Let $\widetilde{w}' = \widetilde{\sigma}_{i_1} w$ with $\lambda(w') = t - 1$. By the inductive hypothesis we have $w' = \widetilde{\sigma}_{i_2} \bullet \dots \bullet \widetilde{\sigma}_{i_t}$. There are two cases to consider.

If $\ell(\sigma_i w') > \ell(w')$ for all $i \in \{i_1, \tau(i_1)\}$, then $w' \in W^{\{i_1, \tau(i_1)\}} \cap W^X$. This implies that $\ell(\widetilde{\sigma}_{i_1} w') = \ell(\widetilde{\sigma}_{i_1}) + \ell(w')$ and hence $w = \widetilde{\sigma}_{i_1} \bullet w'$ as required.

On the other hand, suppose $\ell(\sigma_i w') < \ell(w')$ for some $i \in \{i_1, \tau(i_1)\}$. We will show that this gives a contradiction. Since the length function is τ-invariant, it follows that $\ell(\sigma_{\tau(i)} w') < \ell(w')$ also. As in the proof of Lemma 4.6 this implies that we may write $w' = \widetilde{\sigma}_{j_1} \bullet \dots \bullet \widetilde{\sigma}_{j_r}$ for $j_k \in I \setminus X$ and $j_1 = i_1$. By Lemma 4.9, we have $t - 1 = r$ and hence we obtain

$$w = \widetilde{\sigma}_{i_1} w' = \widetilde{\sigma}_{j_2} \bullet \dots \bullet \widetilde{\sigma}_{j_r}$$

where $\lambda(w) = r - 1 < \lambda(w')$ which gives a contradiction. □

As a corollary to this, we see that reduced expressions in $\widetilde{W}$ are also reduced in W.

Corollary 4.11 ([60, Corollary 3.4.5]). *Let $w, w' \in \widetilde{W}$. Then $\ell(ww') = \ell(w) + \ell(w')$ if and only if $\lambda(ww') = \lambda(w) + \lambda(w')$.*

Proof. Suppose $\lambda(w) = p$ and $\lambda(w') = q$. By Lemma 4.6 we have

$$w = \widetilde{\sigma}_{i_1} \bullet \ldots \bullet \widetilde{\sigma}_{i_p},$$
$$w' = \widetilde{\sigma}_{j_1} \bullet \ldots \bullet \widetilde{\sigma}_{j_q}.$$

If $\ell(ww') = \ell(w) + \ell(w')$, then $ww' = \widetilde{\sigma}_{i_1} \bullet \ldots \bullet \widetilde{\sigma}_{i_p} \bullet \widetilde{\sigma}_{j_1} \bullet \ldots \bullet \widetilde{\sigma}_{j_q}$. Let $r \leq \lambda(ww') = p+q$. By Lemma 4.6 we have $ww' = \widetilde{\sigma}_{l_1} \bullet \ldots \bullet \widetilde{\sigma}_{l_r}$ where $r \leq \lambda(ww') = p + q$. By Lemma 4.9 we have $r = p + q$ as required.

Suppose instead that $\lambda(ww') = \lambda(w) + \lambda(w') = p+q$. Then $ww' = \widetilde{\sigma}_{i_1} \cdots \widetilde{\sigma}_{i_p} \widetilde{\sigma}_{j_1} \cdots \widetilde{\sigma}_{j_q}$ and Lemma 4.10 implies that $ww' = \widetilde{\sigma}_{i_1} \bullet \ldots \bullet \widetilde{\sigma}_{i_p} \bullet \widetilde{\sigma}_{j_1} \bullet \ldots \bullet \widetilde{\sigma}_{j_q}$. It follows from this that $\ell(ww') = \ell(w) + \ell(w')$. □

Let $\widetilde{S} = \{\widetilde{\sigma}_i \mid i \in I \setminus X\}$. Recall from [26, 1.7, 1.9] that a pair (W, S) is a Coxeter system if and only if the Exchange condition holds for (W, S). For reference, we restate the Exchange condition for (W, S).

Exchange Condition ([26, pg. 14]). *Let $w \in W$ and suppose $w = \sigma_{i_1} \cdots \sigma_{i_t}$ is a reduced expression. If $\ell(\sigma_j w) < \ell(w)$ for some $j \in I$, then there exists some $k \in \{1, \ldots, t\}$ such that $\sigma_j w = \sigma_{i_1} \cdots \sigma_{i_{k-1}} \sigma_{i_{k+1}} \cdots \sigma_{i_t}$.*

The following theorem is proved in [49, Theorem 5.9(i)] and [52, 25.1]. We give a combinatorial proof as indicated in [21, Remark 8].

Theorem 4.12 ([60, Theorem 3.4.6, Corollary 3.4.7]). *The pair $(\widetilde{W}, \widetilde{S})$ is a Coxeter system.*

Proof. We show that the Exchange condition is satisfied for $(\widetilde{W}, \widetilde{S})$ from which the result follows. Let $w \in \widetilde{W}$ and $w = \widetilde{\sigma}_{i_1} \cdots \widetilde{\sigma}_{i_t}$ be a reduced expression. Suppose $\lambda(\widetilde{\sigma}_j w) < \lambda(w)$ for some $j \in I \setminus X$. If $\ell(\sigma_k w) > \ell(w)$ for all $k \in X \cup \{j, \tau(j)\}$, then $w \in W^{X \cup \{j, \tau(j)\}}$ and hence $\ell(\widetilde{\sigma}_j w) = \ell(\widetilde{\sigma}_j) + \ell(w)$ by Proposition 4.3. Corollary 4.11 implies that $\lambda(\widetilde{\sigma}_j w) = \lambda(\widetilde{\sigma}_j) + \lambda(w)$ which contradicts the assumption that $\lambda(\widetilde{\sigma}_j w) < \lambda(w)$.

Hence there exists some $k \in X \cup \{j, \tau(j)\}$ such that $\ell(\sigma_k w) < \ell(w)$. Since $w \in \widetilde{W} = \mathcal{W}^{\tau}$, it follows that $k \in \{j, \tau(j)\}$. As the Exchange condition holds for (W, S), there exists some index $l \in \{1, \ldots, t\}$ such that

$$\sigma_k w = \widetilde{\sigma}_{i_1} \cdots \widetilde{\sigma}_{i_{l-1}} x \widetilde{\sigma}_{i_{l+1}} \cdots \widetilde{\sigma}_{i_t}$$

where $x \in W_{X \cup \{i_l, \tau(i_l)\}}$ is the reduced expression for $\widetilde{\sigma}_{i_l}$ minus one factor. Let $z = \widetilde{\sigma}_{i_1} \cdots \widetilde{\sigma}_{i_{l-1}}$. Then

$$z^{-1} \sigma_k z = x \widetilde{\sigma}_{i_l} \in W_{X \cup \{i_l, \tau(i_l)\}}.$$

Since $z \in \widetilde{W} = \mathcal{W}^\tau$, we have $\tau(z^{-1}\sigma_k z) = z^{-1}\sigma_{\tau(k)}z = \tau(x\widetilde{\sigma}_{i_l}) \in W_{X\cup\{i_l,\tau(i_l)\}}$. Further, by Condition (2) of Definition 3.1 and (4.3) we have $z\sigma_i = \sigma_i z$ for all $i \in X$. Hence $z^{-1}\sigma_i z = \sigma_i \in W_{X\cup\{i_l,\tau(i_l)\}}$ for all $i \in X$. As a result we have $z^{-1}\widetilde{\sigma}_k z \in \widetilde{W} \cap W_{X\cup\{i_l,\tau(i_l)\}}$. Hence $z^{-1}\widetilde{\sigma}_k z \in \langle\widetilde{\sigma}_{i_l}\rangle = \{1, \widetilde{\sigma}_{i_l}\}$ so we must have $z^{-1}\widetilde{\sigma}_k z = \widetilde{\sigma}_{i_l}$. Using this we obtain

$$\widetilde{\sigma}_k w = z\widetilde{\sigma}_{i_l}z^{-1}w = \widetilde{\sigma}_{i_1}\cdots\widetilde{\sigma}_{i_{l-1}}\widetilde{\sigma}_{i_{l+1}}\cdots\widetilde{\sigma}_{i_t}$$

as required. □

4.4 The restricted Weyl group

We now explain the connection between $\widetilde{W}$ and the restricted root system of the symmetric Lie algebra $(\mathfrak{g}, \theta)$ in some detail. This is a fact implicit in [49], but here we avoid Lusztig's more sophisticated setting and give a more pedestrian approach.

Since $\theta(\mathfrak{h}) = \mathfrak{h}$ we may decompose $\mathfrak{h}$ as a direct sum

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{a} \tag{4.12}$$

where $\mathfrak{h}_1 = \{x \in \mathfrak{h} \mid \theta(x) = x\}$ and $\mathfrak{a} = \{x \in \mathfrak{h} \mid \theta(x) = -x\}$ denote the $+1$– and -1–eigenspaces of $\mathfrak{h}$, respectively.

Definition 4.13. The *restricted root system* $\Sigma \subset \mathfrak{a}^*$ is the set obtained by restricting all roots in Φ to $\mathfrak{a}$. In other words,

$$\Sigma = \Phi|_{\mathfrak{a}} \setminus \{0\}. \tag{4.13}$$

Recall from Section 2.2.1 that $V = \mathbb{R}\Phi$. As $\Theta(\Phi) = \Phi$, we have $\Theta(V) = V$. Any element $\alpha \in V$ can be written as

$$\alpha = \frac{\alpha + \Theta(\alpha)}{2} + \frac{\alpha - \Theta(\alpha)}{2}. \tag{4.14}$$

Since the inner product $(-,-)$ is Θ-invariant we hence obtain a direct sum decomposition

$$V = V_{+1} \oplus V_{-1} \tag{4.15}$$

where $V_\lambda = \{v \in V \mid \Theta(v) = \lambda v\}$ and $1/2(\alpha + \lambda\Theta(\alpha)) \in V_\lambda$ for $\lambda \in \{\pm 1\}$. If $\beta \in V_{-1}$ and $h \in \mathfrak{h}_1$, then $\beta(h) = 0$ since

$$\beta(h) = \beta(\Theta(h)) = \Theta(\beta)(h) = -\beta(h).$$

This allows us to consider V_{-1} as a a subspace of $\mathfrak{a}^*$, with $V_{-1} = \mathbb{R}\Sigma$. For any $\beta \in V$, define

$$\widetilde{\beta} = \frac{\beta - \Theta(\beta)}{2}, \tag{4.16}$$

see [46, Equation (1.4)]. Equation (4.14) implies that $\Sigma = \{\widetilde{\beta} \mid \beta \in \Phi, \widetilde{\beta} \neq 0\}$. We write $\widetilde{\Pi} = \{\widetilde{\alpha}_i \mid i \in I \setminus X\}$ and we define $Q(\Sigma) = \mathbb{Z}\Sigma = \mathbb{Z}\widetilde{\Pi}$ and $Q^+(\Sigma) = \mathbb{N}_0\widetilde{\Pi}$.

Lemma 4.14. *The group W^Θ acts on Σ.*

Proof. We calculate directly. For any $w \in W^{\Theta}$ and $\widetilde{\beta} \in \Sigma$ we have

$$\begin{aligned} w(\widetilde{\beta}) = w\left(\frac{\beta - \Theta(\beta)}{2}\right) &= \frac{w(\beta) - w(\Theta(\beta))}{2} \\ &= \frac{w(\beta) - \Theta(w(\beta))}{2} = \widetilde{w(\beta)} \end{aligned}$$

as required. □

By restricting the inner product on V, we obtain an inner product on V_{-1}. Since the inner product is W-invariant and V_{-1} is a W^{Θ}-invariant subspace, it follows that the inner product on V_{-1} is W^{Θ}-invariant. For any $i \in X$, we have $\alpha_i \in V_{+1}$. The decomposition (4.15) implies that $\sigma_i(\widetilde{\beta}) = \widetilde{\beta}$ for all $i \in X$ and $\widetilde{\beta} \in \Sigma$. On the other hand, we can interpret the subgroup $\widetilde{W}$ using the restricted root system Σ.

Proposition 4.15 ([16, Proposition 2.7(1)]). *The reflections at the hyperplanes perpendicular to elements of Σ generate a finite reflection group $W(\Sigma)$.*

Proof. For any $i \in I \setminus X$ we have

$$\begin{aligned} \widetilde{\sigma}_i(\widetilde{\alpha}_i) &= \left(w_X^{-1} w_{X\cup\{i,\tau(i)\}}\right)(\widetilde{\alpha}_i) \\ &= w_{X\cup\{i,\tau(i)\}}(\widetilde{\alpha}_i) \\ &= \left(w_{X\cup\{i,\tau(i)\}}(\alpha_i)\right)|_{\mathfrak{a}}. \end{aligned}$$

It hence follows from (4.2) that

$$\widetilde{\sigma}_i(\widetilde{\alpha}_i) = -\widetilde{\alpha}_i. \tag{4.17}$$

Now suppose $\widetilde{\beta} \in V_{-1}$ such that $(\widetilde{\beta}, \widetilde{\alpha}_i) = 0$. Using the W^{Θ}-invariance of the bilinear form on V_{-1}, we obtain

$$\begin{aligned} (\widetilde{\sigma}_i(\widetilde{\beta}), \widetilde{\alpha}_i) &= (\widetilde{\beta}, \widetilde{\sigma}_i(\widetilde{\alpha}_i)) \\ &= -(\widetilde{\beta}, \widetilde{\alpha}_i) \\ &= 0. \end{aligned}$$

On the other hand, by the definition of $\widetilde{\sigma}_i$ we have

$$\widetilde{\sigma}_i(\beta) = \beta + n_i\alpha_i + n_{\tau(i)}\alpha_{\tau(i)} + \sum_{j\in X} n_j\alpha_j$$

for some $n_j \in \mathbb{Q}$. From this we obtain

$$\widetilde{\sigma}_i(\widetilde{\beta}) = \widetilde{\beta} + m_i\widetilde{\alpha}_i$$

where $m_i = n_i + n_{\tau(i)}$. Hence,

$$\begin{aligned} 0 = (\widetilde{\sigma}_i(\widetilde{\beta}), \widetilde{\alpha}_i) &= (\widetilde{\beta} + m_i\widetilde{\alpha}_i, \widetilde{\alpha}_i) \\ &= (\widetilde{\beta}, \widetilde{\alpha}_i) + m_i(\widetilde{\alpha}_i, \widetilde{\alpha}_i) \\ &= m_i(\widetilde{\alpha}_i, \widetilde{\alpha}_i). \end{aligned}$$

The inner product is positive definite so it follows that $m_i = 0$. Hence $\widetilde{\sigma}_i(\widetilde{\beta}) = \widetilde{\beta}$. This together with (4.17) implies that $\widetilde{\sigma}_i$ is the reflection at the hyperplane orthogonal to $\widetilde{\alpha}_i$. □

By the above, the action of $\widetilde{W}$ on V_{-1} gives a group homomorphism

$$\rho : \widetilde{W} \to W(\Sigma) \tag{4.18}$$

which sends $\widetilde{\sigma}_i$ to the reflection at the hyperplane perpendicular to the element $\widetilde{\alpha}_i$ for any $i \in I \setminus X$. We check that ρ is a group isomorphism. Adapting the proof of [26, Theorem 1.5] one shows that ρ is surjective. Showing that ρ is injective is a consequence of Lemma 4.16 below.

Lemma 4.16 ([16, Proposition 2.8]). *The action of $\widetilde{W}$ on Σ is faithful.*

Proof. Assume that there exists $w \in \widetilde{W}$ such that $w \neq 1_{\widetilde{W}}$ and

$$w(\widetilde{\alpha}_i) = \widetilde{\alpha}_i \quad \text{for all } i \in I.$$

We can rewrite this formula as

$$w(\alpha_i) - w(\Theta(\alpha_i)) = \alpha_i - \Theta(\alpha_i). \tag{4.19}$$

For all $i \in X$ we have $w(\alpha_i) > 0$ as $l(w\sigma_i) = l(w) + 1$. Hence there exists $i \in I \setminus X$ such that $w(\alpha_i) < 0$. In this case also $w(\alpha_{\tau(i)}) < 0$ since elements of $\widetilde{W}$ are fixed under τ. Consider Equation (4.19) for this i: The right hand side lies in Q^+ and is of the form

$$\alpha_i + \alpha_{\tau(i)} + \sum_{j \in X} n_j \alpha_j \tag{4.20}$$

where $n_j \in \mathbb{N}_0$ for each $j \in X$. We can write the left hand side as

$$w(\alpha_i) - w(\Theta(\alpha_i)) = w(\alpha_i) + w(\alpha_{\tau(i)}) + \sum_{j \in X} m_j w(\alpha_j) \tag{4.21}$$

where $m_j \in \mathbb{N}_0$ for each $j \in X$. Hence inserting (4.20) and (4.21) into (4.19), we get

$$w(\alpha_i) + w(\alpha_{\tau(i)}) + \sum_{j \in X} m_j w(\alpha_j) = \alpha_i + \alpha_{\tau(i)} + \sum_{j \in X} n_j \alpha_j.$$

Now we apply the tilde map to the above equation. The terms involving α_j for $j \in X$ vanish, because the tilde map is zero on Q_X and w commutes with Θ. We get

$$\widetilde{w(\alpha_i)} + \widetilde{w(\alpha_{\tau(i)})} = \widetilde{\alpha}_i + \widetilde{\alpha}_{\tau(i)}.$$

The right hand side lies in $Q^+(\Sigma)$. The left hand side lies in $-Q^+(\Sigma)$ because $w(\alpha_i)$ and $w(\alpha_{\tau(i)})$ lie in $-Q^+$. Hence both sides of the equation must vanish. However, this is not possible, in particular for the right hand side which is $2\widetilde{\alpha}_i$. We have a contradiction. □

Proposition 4.15 has the following consequence, which will be used in Section 5.4.

Corollary 4.17. *For any $i \in I \setminus X$ and $\mu \in Q(\Sigma)$ the relation*

$$\widetilde{\sigma}_i(\mu) = \mu - 2\frac{(\mu, \widetilde{\alpha}_i)}{(\widetilde{\alpha}_i, \widetilde{\alpha}_i)}\widetilde{\alpha}_i \tag{4.22}$$

holds and $2\frac{(\mu, \widetilde{\alpha}_i)}{(\widetilde{\alpha}_i, \widetilde{\alpha}_i)} \in \mathbb{Z}$.

Chapter 5

Factorisation of quasi K-matrices for quantum symmetric pairs

The quasi K-matrix $\mathfrak{X}$ is an element lying in a completion of $U_q(\mathfrak{g})$ that is an analogue of the quasi R-matrix R. In particular $\mathfrak{X}$ satisfies an intertwiner property similar to the property (2.109) of the quasi R-matrix. In order to construct the quasi K-matrix we first define a bar involution for $B_{\mathbf{c},\mathbf{s}}$ in Section 5.1, following [3]. In Section 5.2 we recall many of the notions and known properties of the quasi K-matrix following [4].

Recall by Theorem 2.60 the quasi R-matrix has a deep connection to the Weyl group W. In the remainder of this chapter, we establish a similar connection between the quasi K-matrix and the restricted Weyl group $\widetilde{W}$. In particular, we will see that in many cases the quasi K-matrix factorises into a product of quasi K-matrices for Satake diagrams of rank one. In Sections 5.3 and 5.4 we establish explicit formulas for $\mathfrak{X}$ in the case $\mathbf{s} = (0, \dots, 0)$. The case with general parameters $\mathbf{s}$ is then considered in Section 5.5. The results of Sections 5.3, 5.4 and 5.5 are joint with Stefan Kolb and can be found in [16, Section 3].

5.1 Bar involution for $B_{\mathbf{c},\mathbf{s}}$

Recall the bar involution $\overline{}^U : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ given by (2.89). The bar involution is a crucial ingredient in the theory of quantum groups. For instance, the quasi R-matrix is characterised by the bar involution, see Theorem 2.58. The bar involution hence has applications in low-dimensional topolgy, see for example [57].

The papers of Bao and Wang [5] and Ehrig and Stroppel [18] suggest that the bar involution plays an important role in the context of quantum symmetric pairs. In Section 5.2 we consider an analogue of Theorem 2.58 in the setting of quantum symmetric pairs. In order to do this we require a bar involution for $B_{\mathbf{c},\mathbf{s}}$. Since $\overline{}^U$ does not leave $B_{\mathbf{c},\mathbf{s}}$

invariant we construct a new automorphism of $B_{\mathbf{c},\mathbf{s}}$ which is an analogue of $\overline{}^U$. We denote this automorphism by $\overline{}^B : B_{\mathbf{c},\mathbf{s}} \to B_{\mathbf{c},\mathbf{s}}$. The first examples were constructed in [6] and [18] for specific quantum symmetric pair coideal subalgebras of type AIII/AIV. Here, we recall the general constructions of Balagović and Kolb in [3, Section 3.3]. Recall from Section 3.4 the subalgebras $\mathcal{M}_X$ and U^0_Θ of $U_q(\mathfrak{g})$ and also the elements $\mathcal{Z}_i$ from Equation (3.41).

Theorem 5.1 ([3, Theorem 3.11]). *The following are equivalent:*

(1) *There exists a $\mathbb{K}(q)$-algebra automorphism $\overline{}^B : B_{\mathbf{c},\mathbf{s}} \to B_{\mathbf{c},\mathbf{s}}$ such that*

$$\overline{B_i}^B = B_i \quad \text{for all } i \in I \setminus X \tag{5.1}$$

and $\overline{}^B$ coincides with $\overline{}^U$ on $\mathcal{M}_X U^0_\Theta$.

(2) *The relation*

$$\overline{c_i \mathcal{Z}_i}^B = q^{(\alpha_i, \alpha_{\tau(i)})} c_{\tau(i)} \mathcal{Z}_{\tau(i)} \tag{5.2}$$

holds for all $i \in I \setminus X$ for which $\tau(i) \neq i$ or for which there exists $j \in I \setminus \{i\}$ such that $a_{ij} = 0$.

In order to construct the quasi K-matrix in Section 5.2 it is necessary to assume that (5.2) holds for all $i \in I \setminus X$, see [4, Lemma 6.7]. We will hence make the assumption that (5.2) holds for the remainder of this thesis. One can show that

$$\overline{\mathcal{Z}_i}^B = q^{(\alpha_i, \alpha_i - w_X(\alpha_i) - 2\rho_X)} \mathcal{Z}_{\tau(i)}, \tag{5.3}$$

see [3, Proposition 3.5]. Using this and Lemma 3.11, it follows that Part (2) of Theorem 5.1 is equivalent to the condition

$$c_{\tau(i)} = q^{(\alpha_i, \Theta(\alpha_i) - 2\rho_X)} \overline{c_i} \quad \text{for all } i \in I \setminus X. \tag{5.4}$$

In [4, Section 5.4] the assumption

$$s_i = \overline{s_i}^U \tag{5.5}$$

for all $i \in I \setminus X$ is introduced. We also require this assumption in Section 5.5.

5.2 Quasi K-matrices

The bar involution $\overline{}^U$ on $U_q(\mathfrak{g})$ and the bar involution $\overline{}^B$ on $B_{\mathbf{c},\mathbf{s}}$ do not coincide when restricted to elements of $B_{\mathbf{c},\mathbf{s}}$. However, as in Section 2.2.8 we can find an element which intertwines between the two bar involutions. Recall the root lattice $Q = \mathbb{Z}\Phi$ for $\mathfrak{g}$ and its positive part $Q^+ = \mathbb{N}_0\Phi \subset Q$.

Theorem 5.2 ([4, Theorem 6.10])**.** *There exists a uniquely determined element* $\mathfrak{X} = \sum_{\mu\in Q^+} \mathfrak{X}_\mu \in \widehat{U^+}$ *with* $\mathfrak{X}_0 = 1$ *and* $\mathfrak{X}_\mu \in U^+_\mu$ *such that*

$$\overline{x}^B \mathfrak{X} = \mathfrak{X}\overline{x}^U \tag{5.6}$$

holds for all $x \in B_{\mathbf{c},\mathbf{s}}$.

This is a direct analogue to Theorem 2.58 in which we stated that the quasi R-matrix is an intertwiner between two bar involutions on $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$. It is for this reason that we call the element $\mathfrak{X}$ the *quasi K-matrix* for $B_{\mathbf{c},\mathbf{s}}$. In [5, Theorems 2.10, 6.4] the first known examples of $\mathfrak{X}$ were constructed in types AIII/AIV. There, Bao and Wang denoted the quasi K-matrix as the intertwiner Υ.

Recall from Section 2.2.6 the skew derivations $r_i, {}_ir : U^+ \to U^+$ for $i \in I$. Further recall from (3.28) the elements X_i for $i \in I$. The crucial property of the quasi K-matrix is that it satisfies a recursive formula. The proof makes use of Proposition 2.44 and is given in [4, Proposition 6.1].

Proposition 5.3 ([4, Proposition 6.1])**.** *The following are equivalent.*

(1) *The quasi K-matrix satisfies* $\overline{x}^B \mathfrak{X} = \mathfrak{X}\overline{x}^U$ *for all* $x \in B_{\mathbf{c},\mathbf{s}}$.

(2) *For any* $\mu \in Q^+$ *and all* $i \in I$ *we have*

$$r_i(\mathfrak{X}_\mu) = -(q_i - q_i^{-1})\big(\mathfrak{X}_{\mu+\Theta(\alpha_i)-\alpha_i}\overline{c_iX_i}^U + \overline{s_i}^U \mathfrak{X}_{\mu-\alpha_i}\big), \tag{5.7}$$

$${}_ir(\mathfrak{X}_\mu) = -(q_i - q_i^{-1})\Big(q^{-(\Theta(\alpha_i),\alpha_i)} c_iX_i\mathfrak{X}_{\mu+\Theta(\alpha_i)-\alpha_i} + s_i\mathfrak{X}_{\mu-\alpha_i}\Big). \tag{5.8}$$

Proof. By the definition of B_i the property (5.6) implies

$$\big(F_i + c_iX_iK_i^{-1} + s_iK_i^{-1}\big)\mathfrak{X} = \mathfrak{X}\big(F_i + \overline{c_iX_i}^U K_i + \overline{s_i}^U K_i\big)$$

for any $i \in I \setminus X$. Hence by Equation (2.84) we have

$$[\mathfrak{X}, F_i] = \big(c_iX_iK_i^{-1} + s_iK_i^{-1}\big)\mathfrak{X} - \mathfrak{X}\big(\overline{c_iX_i}^U K_i + \overline{s_i}^U K_i\big).$$

Note that $[\mathfrak{X}_\mu, F_i]$ is an element of $U^+_{\mu-\alpha_i}$ so we compare the $(\mu - \alpha_i)$ homogeneous components in the equation above for all $\mu \in Q^+$. Since X_i has weight $-\Theta(\alpha_i)$ we have

$$\begin{aligned}[\mathfrak{X}_\mu, F_i] = &-\big(\mathfrak{X}_{\mu-\alpha_i+\Theta(\alpha_i)}\overline{c_iX_i}^U + \overline{s_i}^U \mathfrak{X}_{\mu-\alpha_i}\big)K_i\\ &+ K_i^{-1}\big(q^{-(\Theta(\alpha_i),\alpha_i)}c_iX_i\mathfrak{X}_{\mu-\alpha_i+\Theta(\alpha_i)} + s_i\mathfrak{X}_{\mu-\alpha_i}\big).\end{aligned}$$

By Proposition 2.44 we have

$$[X_\mu, F_i] = (q_i - q_i^{-1})^{-1}(r_i(\mathfrak{X}_\mu)K_i - K_i^{-1}\,{}_ir(\mathfrak{X}_\mu))$$

which implies the statement of the proposition. □

Remark 5.4. The proof of Theorem 5.2 requires the use of the recursive formulas (5.7) and (5.8). In order to show that the quasi K-matrix exists, one is required to show that solutions to the recursions (5.7) and (5.8) exist. This is addressed in [4, Sections 6.2,6.4].

In view of [4, Proposition 6.3] and Remark 5.4 we need only consider one system of recursions. In order to avoid any complications with the bar involution, we choose to only consider the system (5.8). Recalling that $s_i = c_i = 0$ for $i \in X$, Equation (5.8) implies that

$$ {}_ir(\mathfrak{X}_\mu) = 0 \quad \text{for all } i \in X. \tag{5.9}$$

We can extend the skew derivation ${}_ir : U^+ \to U^+$ to a linear map

$$ {}_ir : \widehat{U^+} \to \widehat{U^+}, \qquad \sum_{\mu \in Q^+} u_\mu \mapsto \sum_{\mu\in Q^+} {}_ir(u_\mu) \tag{5.10}$$

where ${}_ir(u_\mu)$ is the component in $U^+_{\mu-\alpha_i}$ for all $\mu \in Q^+$ with $\mu \geq \alpha_i$, see [16, Equation (3.12)]. Similarly we extend the skew derivation $r_i : U^+ \to U^+$ to a linear map $r_i : \widehat{U^+} \to \widehat{U^+}$. Using this we can rewrite the recursions (5.7) and (5.8) more compactly as

$$ r_i(\mathfrak{X}) = -(q_i - q_i^{-1})\big(\mathfrak{X}\overline{c_iX_i}^U + \overline{s_i}^U\mathfrak{X}\big), \tag{5.11}$$

$$ {}_ir(\mathfrak{X}) = -(q_i - q_i^{-1})\Big(q^{-(\Theta(\alpha_i),\alpha_i)}c_iX_i\mathfrak{X} + s_i\mathfrak{X}\Big). \tag{5.12}$$

The compact form for ${}_ir(\mathfrak{X})$ is used to perform calculations in Section 5.3 and Chapter 6.

As a consequence of Equation (5.9) we obtain the following result which was already observed in [6, Proposition 4.15]. We give an alternative proof, as in [16, Lemma 3.2]. For any $w \in W$ recall the definition of the subalgebra $U^+[w]$ of $U_q(\mathfrak{g})$ from Section 2.2.5.

Lemma 5.5 ([16, Lemma 3.2])**.** *For any $\mu \in Q^+$ the relation $\mathfrak{X}_\mu \in U^+[w_Xw_0]$ holds.*

Proof. By Equations (5.9) and (2.92) we have $\mathfrak{X}_\mu \in U^+[\sigma_jw_0]$ for all $j \in X$. By [23, Theorem 7.3] we have

$$\bigcap_{j\in X} U^+[\sigma_jw_0] = U^+[w_Xw_0]$$

as required. □

Using the involution $\Theta : \mathfrak{h}^* \to \mathfrak{h}^*$ we can determine many of the components $\mathfrak{X}_\mu$. Recall the height function $\mathrm{ht} : Q^+ \to \mathbb{N}_0$ which sends $\mu = \sum_{i\in I} n_i\alpha_i \mapsto \sum_{i\in I} n_i$, see [26, pg.11]. The following lemma is **(3)** $\Rightarrow$ **(4)** in [4, Proposition 6.1].

Lemma 5.6 ([4, Proposition 6.1, (4)])**.** *For any $\mu \in Q^+$ such that $\mathfrak{X}_\mu \neq 0$, we have $\Theta(\mu) = -\mu$.*

Proof. We proceed by induction of the height of μ. If $\mathrm{ht}(\mu) = 0$ then there is nothing to show. Assume that $\mathrm{ht}(\mu) > 0$. If $\mathfrak{X}_\mu \neq 0$ then by (2.103) there exists $i \in I$ such that

${}_ir(\mathfrak{X}_\mu) \neq 0$. By (5.8) we either have $\mathfrak{X}_{\mu+\Theta(\alpha_i)-\alpha_i} \neq 0$ or $s_i\mathfrak{X}_{\mu-\alpha_i} \neq 0$. If $\mathfrak{X}_{\mu+\Theta(\alpha_i)-\alpha_i} \neq 0$, then since $\mathrm{ht}(\mu + \Theta(\alpha_i) - \alpha(i)) < \mathrm{ht}(\mu)$ the induction hypothesis implies

$$\Theta(\mu + \Theta(\alpha_i) - \alpha_i) = -(\mu + \Theta(\alpha_i) - \alpha_i).$$

Rearranging gives $\Theta(\mu) = -\mu$.

On the other hand if $s_i\mathfrak{X}_{\mu-\alpha_i} \neq 0$ then $s_i \neq 0$. By (3.34) it follows that $\tau(i) = i$ and $a_{ij} = 0$ for all $j \in X$, hence $\Theta(\alpha_i) = -\alpha_i$. Again using the inductive hypothesis we have

$$\Theta(\mu - \alpha_i) = -(\mu - \alpha_i)$$

from which we obtain $\Theta(\mu) = -\mu$ as required. □

The above lemma implies that if $\Theta(\mu) \neq -\mu$, then $\mathfrak{X}_\mu = 0$. We write $\mathfrak{X}_{\mathbf{c},\mathbf{s}}$ if we need to specify the dependence on the parameters $\mathbf{c}$ and $\mathbf{s}$. Any diagram automorphism $\eta : I \to I$ induces a map $\eta : \mathbb{K}(q)^{I\setminus X} \to \mathbb{K}(q)^{I\setminus X}$ by

$$\eta\big((c_i)\big) = (d_i) \quad \text{with } d_i = c_{\eta^{-1}(i)}, \tag{5.13}$$

see [16, Equation (3.28)]. We can record the effect of the diagram automorphism on the quasi K-matrix using this notation. This will be used in Proposition 5.26.

Lemma 5.7 ([16, Lemma 3.3])**.** *Let $\eta : I \to I$ be a diagram automorphism and $\mathbf{c} \in \mathcal{C}$, $\mathbf{s} \in \mathcal{S}$. Then $\eta(\mathbf{c}) \in \mathcal{C}, \eta(\mathbf{s}) \in \mathcal{S}$ and*

$$\eta(\mathfrak{X}_{\mathbf{c},\mathbf{s}}) = \mathfrak{X}_{\eta(\mathbf{c}),\eta(\mathbf{s})}. \tag{5.14}$$

Proof. The relations $\eta(\mathbf{c}) \in \mathcal{C}, \eta(\mathbf{s}) \in \mathcal{S}$ follow from the definitions (3.33) and (3.34), respectively. By [4, Proposition 6.1], relation (5.6) is equivalent to

$$B_i^{\mathbf{c},\mathbf{s}}\mathfrak{X} = \mathfrak{X}\overline{B_i^{\mathbf{c},\mathbf{s}}}^U \quad \text{for all } i \in I \setminus X. \tag{5.15}$$

Here we write $B_i^{\mathbf{c},\mathbf{s}}$ to also denote the dependence of the elements B_i on the parameters $\mathbf{c}$ and $\mathbf{s}$. By construction we have $\eta(B_i^{\mathbf{c},\mathbf{s}}) = B_{\eta(i)}^{\eta(\mathbf{c}),\eta(\mathbf{s})}$. Further the bar involution $\overline{}^U$ commutes with η. Applying η to (5.15) we have

$$B_{\eta(i)}^{\eta(\mathbf{c}),\eta(\mathbf{s})}\eta(\mathfrak{X}_{\mathbf{c},\mathbf{s}}) = \eta(\mathfrak{X}_{\mathbf{c},\mathbf{s}})\overline{B_{\eta(i)}^{\eta(\mathbf{c}),\eta(\mathbf{s})}}^U.$$

It hence follows that $\eta(\mathfrak{X}_{\mathbf{c},\mathbf{s}}) = \mathfrak{X}_{\eta(\mathbf{c}),\eta(\mathbf{s})}$ as required. □

5.3 Rank one quasi K-matrices

For the remainder of this chapter, following Remark 3.2, we denote Satake diagrams as triples (I, X, τ) to also indicate the underlying Lie algebra. The remainder of this chapter is taken from the author's recent paper [16, Section 3.3].

Definition 5.8 ([16, Definition 3.4]). A *subdiagram* of a Satake diagram (I, X, τ) is a triple $(J, X \cap J, \tau|_J)$ such that $J \subset I$ and $(J, X \cap J, \tau|_J)$ is a Satake diagram for the subdiagram of the Dynkin diagram of $\mathfrak{g}$ indexed by J.

We only consider subdiagrams $(J, X \cap J, \tau|_J)$ with the property that any connected component of $X \cap J$ is connected to a white node of J. Let $\widetilde{I}$ be the set of τ-orbits of $I \setminus X$. There is a projection map

$$\pi\colon I \setminus X \longrightarrow \widetilde{I} \tag{5.16}$$

that takes any white node to the τ-orbit it belongs to.

Definition 5.9 ([16, Definition 3.6]). The *rank* of a Satake diagram (I, X, τ) is defined by $\operatorname{rank}(I, X, \tau) = |\pi(I \setminus X)|$.

In other words, a Satake diagram has rank n if there are n distinct orbits of white nodes. By Proposition 4.15 the rank of a Satake diagram coincides with the rank of the corresponding restricted root system Σ.

Given a Satake diagram (I, X, τ), any $i \in I \setminus X$ determines a subdiagram $(\{i, \tau(i)\} \cup X, X, \tau|_{\{i,\tau(i)\}\cup X})$ of rank one. Let $\mathfrak{X}_i$ be the quasi K-matrix corresponding to this rank one subdiagram. For any $w \in W$ we define $\widehat{U^+[w]} = \prod_{\mu \in Q^+} U^+[w]_\mu$. As $U[w]^+$ is a subalgebra of U^+ we obtain that $\widehat{U^+[w]}$ is a subalgebra of $\widehat{U^+}$ and hence of $\mathscr{U}$ by Lemma 2.30. Formulating Lemma 5.5 in the present setting we obtain

$$\mathfrak{X}_i \in \widehat{U^+[\widetilde{\sigma}_i]}. \tag{5.17}$$

In the following lemma we consider the case $\tau(i) = i$ and make the dependence of $\mathfrak{X}_i$ on the parameters c_i more explicit.

Lemma 5.10 ([16, Lemma 3.7]). *Assume that* $\mathbf{s} = (0, \ldots, 0)$ *and* $i \in I \setminus X$ *satisfies* $\tau(i) = i$. *Then*

$$\mathfrak{X}_i = \sum_{n \in \mathbb{N}_0} c_i^n E_{n(\alpha_i - \Theta(\alpha_i))} \tag{5.18}$$

where $E_{n(\alpha_i - \Theta(\alpha_i))} \in U^+_{n(\alpha_i - \Theta(\alpha_i))}$ *is independent of* $\mathbf{c}$.

Proof. It follows from the recursion (2.78) and the assumption $s_i = 0$ that

$$\mathfrak{X}_i = \sum_{n \in \mathbb{N}_0} \mathfrak{X}_{n(\alpha_i - \Theta(\alpha_i))}$$

with $\mathfrak{X}_{n(\alpha_i - \Theta(\alpha_i))} \in U^+_{n(\alpha_i - \Theta(\alpha_i))}$. Again by (2.78), the elements

$$E_{n(\alpha_i - \Theta(\alpha_i))} = c_i^{-n} \mathfrak{X}_{n(\alpha_i - \Theta(\alpha_i))}$$

for $n \in \mathbb{N}$ satisfy the relations

$${}_ir(E_{n(\alpha_i - \Theta(\alpha_i))}) = -(q - q^{-1}) q^{-(\Theta(\alpha_i), \alpha_i)} X_i E_{(n-1)(\alpha_i - \Theta(\alpha_i))} \quad \text{for } i \in I \setminus X \tag{5.19}$$

and

$$ {}_jr(E_{n(\alpha_i-\Theta(\alpha_i))}) = 0 \quad \text{for } j \in X. \tag{5.20} $$

The relations (5.19) and (5.20) are independent of $\mathbf{c}$ and determine $E_{n(\alpha_i-\Theta(\alpha_i))}$ uniquely if we additionally impose $E_0 = 1$. $\square$

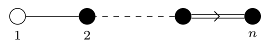
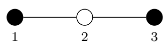
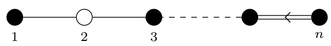
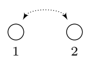
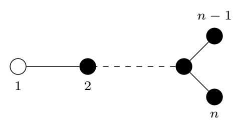
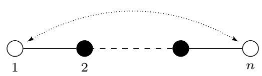
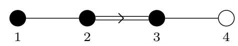
The quasi K-matrices of rank one are the building blocks for quasi K-matrices of higher rank. In the following we give explicit formulas for rank one quasi K-matrices of type A shown on the left hand side of Table 5.1 in the case $\mathbf{s} = (0, \dots, 0)$. These were calculated in [16, Lemmas 3.8–3.10]. Additionally, we present the rank one quasi K-matrices for Satake diagrams of types BII and DII not contained in [16].

Recall from Equation (2.81) and Equation (2.83) the modified q-number $\{n\}_i$, the factorial $\{n\}_i!$ and the double factorial $\{n\}_i!!$. Further, we use the following conventions. For any $x, y \in U_q(\mathfrak{g})$, $a \in \mathbb{K}(q)$ we denote by $[x, y]_a$ the element $xy - ayx$. For any $i, j \in I$ we write $T_{ij} = T_i \circ T_j : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ and we extend this definition recursively. In Sections 5.3.5 and 5.3.6 we shorten notation further by writing

$$ \begin{aligned} T_{i--j} &= T_i T_{i+1} \cdots T_{j-1} T_j, \\ T_{j--i} &= T_j T_{j-1} \cdots T_{i+1} T_i \end{aligned} \tag{5.21} $$

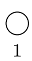
for $1 \leq i \leq j \leq n$.

Table 5.1: Satake diagrams of symmetric pairs of rank one

AI_1		BII, $n \geq 2$	
AII_3		CII, $n \geq 3$	
AIII_{11}		DII, $n \geq 4$	
AIV, $n \geq 2$		FII	

5.3.1 Type AI_1

Consider the Satake diagram of type AI_1.



Lemma 5.11 ([16, Lemma 3.7]). *The quasi K-matrix $\mathfrak{X}$ of type* AI_1 *is given by*

$$\mathfrak{X} = \sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{2n\}!!}(q^2c_1)^n E_1^{2n}. \tag{5.22}$$

Proof. By Equation 2.78, we need to show that

$${}_1r(\mathfrak{X}) = (q-q^{-1})(q^2c_1)E_1\mathfrak{X}.$$

Recall from (2.82) the relation

$${}_1r(E_1^n) = \{n\}E_1^{n-1} \qquad \text{for all } n \in \mathbb{N}.$$

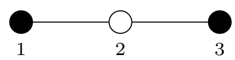
Hence

$$\begin{aligned}
{}_1r(\mathfrak{X}) &= \sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{2n\}!!}(q^2c_1)^n {}_1r(E_1^{2n}) \\
&= \sum_{n\geq 1} \frac{(q-q^{-1})^n}{\{2n\}!!}(q^2c_1)^n \{2n\} E_1^{2n-1} \\
&= \sum_{n\geq 0} \frac{(q-q^{-1})^{n+1}}{\{2n\}!!}(q^2c_1)^{n+1} E_1^{2n+1} \\
&= (q-q^{-1})(q^2c_1)E_1\mathfrak{X}
\end{aligned}$$

as required. □

5.3.2 Type AII_3

Consider the Satake diagram of type AII_3.



Lemma 5.12 ([16, Lemma 3.8]). *The quasi K-matrix $\mathfrak{X}$ of type* AII_3 *is given by*

$$\mathfrak{X} = \sum_{n\geq 0} \frac{(qc_2)^n}{\{n\}!}[E_2, T_{13}(E_2)]_{q^{-2}}^n. \tag{5.23}$$

Proof. Since $T_{13}(E_2) = [E_1, T_3(E_2)]_{q^{-1}}$, Property (2.78) of the skew derivative ${}_1r$ implies that ${}_1r(T_{13}(E_2)) = (1-q^{-2})T_3(E_2)$. Again by Property (2.78), it follows that ${}_1r([E_2, T_{13}(E_2)]_{q^{-2}}) = 0$. Hence ${}_1r(\mathfrak{X}) = 0$. By symmetry, we also have ${}_3r(\mathfrak{X}) = 0$.

We want to show that

$${}_2r(\mathfrak{X}) = (q-q^{-1})c_2T_{13}(E_2)\mathfrak{X}.$$

Since ${}_2r(T_{13}(E_2)) = 0$ by (2.92), the relation

$${}_2r([E_2, T_{13}(E_2)]_{q^{-1}}) = (1-q^{-2})T_{13}(E_2)$$

holds in $U_q(\mathfrak{sl}_4)$.

Moreover, the element $T_{13}(E_2)$ commutes with the element $[E_2, T_{13}(E_2)]_{q^{-2}}$. Indeed, this follows from the fact that E_2 commutes with $[T_{213}(E_2), E_2]_{q^{-2}}$ by applying the automorphism T_{13}. This implies that the relation

$$ {}_2r([E_2, T_{13}(E_2)]^n_{q^{-2}}) = (1-q^{-2})\{n\}T_{13}(E_2)[E_2, T_{13}(E_2)]^{n-1}_{q^{-2}} $$

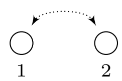
holds in $U_q(\mathfrak{sl}_4)$. Using this, we obtain

$$ \begin{aligned} {}_2r(\mathfrak{X}) &= \sum_{n\geq 0} \frac{(qc_2)^n}{\{n\}!} {}_2r([E_2, T_{13}(E_2)]^n_{q^{-2}}) \\ &= (1-q^{-2})T_{13}(E_2) \sum_{n\geq 1} \frac{(qc_2)^n}{\{n-1\}!} [E_2, T_{13}(E_2)]^{n-1}_{q^{-2}} \\ &= (q-q^{-1})c_2 T_{13}(E_2)\mathfrak{X} \end{aligned} $$

as required. □

5.3.3 Type AIII$_{11}$

Consider the Satake diagram of type AIII$_{11}$.



Note that $s(1) = s(2) = 1$ by (3.11) and $c_1 = c_2$ by (3.33) and (5.4).

Lemma 5.13 ([16, Lemma 3.9])**.** *The quasi K-matrix $\mathfrak{X}$ of type* AIII$_{11}$ *is given by*

$$ \mathfrak{X} = \sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{n\}!} c_1^n (E_1E_2)^n. \tag{5.24} $$

Proof. By symmetry, we only need to show that

$$ {}_1r(\mathfrak{X}) = (q-q^{-1})c_1 E_2 \mathfrak{X}. $$

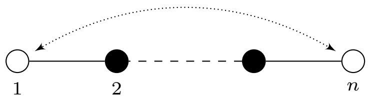
By (2.82), we have

$$ \begin{aligned} {}_1r(\mathfrak{X}) &= \sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{n\}!} c_1^n {}_1r((E_1E_2)^n) \\ &= \sum_{n\geq 1} \frac{(q-q^{-1})^n}{\{n\}!} \{n\} c_1^n E_1^{n-1} E_2^n \\ &= (q-q^{-1})c_1 E_2 \mathfrak{X} \end{aligned} $$

as required. □

5.3.4 Type AIV for $n \geq 2$

Consider the Satake diagram of type AIV for $n \geq 2$.



By (3.11), we have $s(1) = -s(n)$ and by (5.4), we have $c_1 = q^{-2}\overline{c_n}$.

Lemma 5.14 ([16, Lemma 3.10])**.** *The quasi K-matrix $\mathfrak{X}$ of type* AIV *is given by*

$$\mathfrak{X} = \left(\sum_{k\geq 0} \frac{(c_1 s(n))^k}{\{k\}!} T_1 T_{w_X}(E_n)^k\right)\left(\sum_{k\geq 0} \frac{(c_n s(1))^k}{\{k\}!} T_n T_{w_X}(E_1)^k\right). \tag{5.25}$$

Proof. We have ${}_i r(\mathfrak{X}) = 0$ for $i \in X$. Hence by symmetry we only need to show that

$${}_1 r(\mathfrak{X}) = (q - q^{-1})q^{-1} c_1 s(n) T_{w_X}(E_n)\mathfrak{X}$$

since $T_1 T_{w_X}(E_n)$ and $T_n T_{w_X}(E_1)$ commute. We have

$$\begin{aligned}
&{}_1 r(T_n T_{w_X}(E_1)^k) = 0,\\
&{}_1 r(T_1 T_{w_X}(E_n)^k) = q^{-1}(q - q^{-1})\{k\} T_{w_X}(E_n) T_1 T_{w_X}(E_n)^{k-1}.
\end{aligned}$$

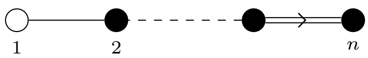
Using this, we obtain

$$\begin{aligned}
{}_1 r(\mathfrak{X}) &= \left(\sum_{k\geq 0} \frac{(c_1 s(n))^k}{\{k\}!} {}_1 r(T_1 T_{w_X}(E_n)^k)\right)\left(\sum_{k\geq 0} \frac{(c_n s(1))^k}{\{k\}!} T_n T_{w_X}(E_1)^k\right)\\
&= q^{-1}(q - q^{-1}) c_1 s(n) T_{w_X}(E_n)\mathfrak{X}
\end{aligned}$$

as required. □

5.3.5 Type BII for $n \geq 3$

Consider the Satake diagram of type BII for $n \geq 3$.



In this case we have

$$w_0 = \sigma_n(\sigma_{n-1}\sigma_n\sigma_{n-1})\cdots(\sigma_1\cdots\sigma_n\cdots\sigma_1).$$

The following lemma is necessary for Lemma 5.16.

Lemma 5.15. *The element E_1 commutes with $[T_1 T_{w_X}(E_1), E_1]_{q^{-4}}$.*

Proof. We have

$$\begin{aligned}
T_1 T_{w_X}(E_1) = T_1 T_{2--n--2}(E_1) &= T_{1--n--3}([E_2, E_1]_{q^{-2}})\\
&= [T_{1--n--3}(E_2), T_{12}(E_1)]_{q^{-2}}\\
&= [T_{1--n--3}(E_2), E_2]_{q^{-2}}.
\end{aligned}$$

The fact that $T_{12}(E_1) = E_2$ follows from Proposition 2.40 by noting that $\sigma_1\sigma_2(\alpha_1) = \alpha_2$. The element $E_1 T_{1--n--3}(E_2)$ has weight $2\alpha_1 + \alpha_2 + 2(\alpha_3 + \cdots + \alpha_n)$. Hence by Theorem 2.59 it follows that

$$E_1 T_{1--n--3}(E_2) = q^2 T_{1--n--3}(E_2) E_1. \tag{5.26}$$

Since $T_1(E_2) = E_1E_2 - q^{-2}E_2E_1$ we obtain

$$\begin{aligned}
E_1 T_1 T_{w_X}(E_1) &= E_1 T_{1--n--3}(E_2) E_2 - q^{-2} E_1 E_2 T_{1--n--3}(E_2)\\
&= q^2 T_{1--n--3}(E_2) E_1 E_2 - q^{-2}\big(T_1(E_2) + q^{-2}E_2E_1\big) T_{1--n--3}(E_2)\\
&= q^2 T_{1--n--3}(E_2)\big(T_1(E_2) + q^{-2}E_2E_1\big) - q^{-2} T_1(E_2) T_{1--n--3}(E_2)\\
&\quad - q^{-2} E_2 T_{1--n--3}(E_2) E_1\\
&= q^2 [T_{1--n--3}(E_2), T_1(E_2)]_{q^{-4}} + T_1 T_{w_X}(E_1) E_1.
\end{aligned}$$

This implies

$$\begin{aligned}
E_1[T_1T_{w_X}(E_1), E_1]_{q^{-4}} &= [E_1 T_1 T_{w_X}(E_1), E_1]_{q^{-4}}\\
&= [T_1 T_{w_X}(E_1), E_1]_{q^{-4}} E_1 + q^2\big[[T_{1--n--3}(E_2), T_1(E_2)]_{q^{-4}}, E_1\big]_{q^{-4}}.
\end{aligned}$$

By Equation (5.26) and the fact that $T_1(E_2)E_1 = q^{-2}E_1T_1(E_2)$ it follows that

$$\big[[T_{1--n--3}(E_2), T_1(E_2)]_{q^{-4}} E_1\big]_{q^{-4}} = 0.$$

Hence we have $E_1[T_1T_{w_X}(E_1), E_1]_{q^{-4}} = [T_1T_{w_X}(E_1), E_1]_{q^{-4}}E_1$ as required. □

Lemma 5.16. *The quasi K-matrix $\mathfrak{X}$ in type* BII *is given by*

$$\mathfrak{X} = \sum_{m\geq 0} \frac{(q^2c_1)^m}{\{m\}_1!}[E_1, T_{w_X}(E_1)]^m_{q^{-4}}. \tag{5.27}$$

Proof. By Equation (5.12) we want to show that ${}_1r(\mathfrak{X}) = (q^2 - q^{-2})c_2 T_{w_X}(E_1)\mathfrak{X}$. By (2.92) we have ${}_1r(T_{w_X}(E_1)) = 0$ so it follows that

$${}_1r([E_1, T_{w_X}(E_1)]_{q^{-4}}) = (1 - q^{-4})T_{w_X}(E_1).$$

Since $\sigma_1 w_X(\alpha_1) = w_X\alpha_1$ we have $w_X\sigma_1 w_X(\alpha_1) = \alpha_1$ and hence Proposition 2.40 implies that $T_{w_X}T_1T_{w_X}(E_1) = E_1$. Applying T_{w_X} to the relation $E_1[T_1T_{w_X}(E_1), E_1]_{q^{-4}} = [T_1T_{w_X}(E_1), E_1]_{q^{-4}}E_1$ from Lemma 5.15 we obtain

$$T_{w_X}(E_1)[E_1, T_{w_X}(E_1)]_{q^{-4}} = [E_1, T_{w_X}(E_1)]_{q^{-4}}T_{w_X}(E_1).$$

Using Property (2.78) of the skew derivation ${}_1r$ we have

$${}_1r([E_1, T_{w_X}(E_1)]^2_{q^{-4}}) = \{2\}_1(1 - q^{-4})T_{w_X}(E_1)[E_1, T_{w_X}(E_1)]_{q^{-4}}.$$

Continuing inductively it follows that

$${}_1r([E_1, T_{w_X}(E_1)]^m_{q^{-4}}) = (1 - q^{-4})\{m\}_1 T_{w_X}(E_1)[E_1, T_{w_X}(E_1)]^{m-1}_{q^{-4}}.$$

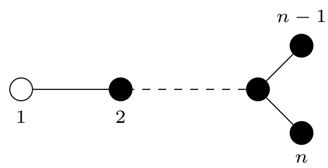
This implies that

$$\begin{aligned}
{}_1r(\mathfrak{X}) &= \sum_{m\geq 0} \frac{(q^2c_1)^m}{\{m\}_1!}\, {}_1r([E_1, T_{w_X}(E_1)]^m_{q^{-4}}) \\
&= (1-q^{-4})T_{w_X}(E_1) \sum_{m\geq 1} \frac{(q^2c_1)^m}{\{m-1\}_1!}[E_1, T_{w_X}(E_1)]^{m-1}_{q^{-4}} \\
&= (q^2-q^{-2})c_1 T_{w_X}(E_1)\mathfrak{X}
\end{aligned}$$

as required. □

5.3.6 Type DII for $n \geq 4$

Consider the Satake diagram of type DII for $n \geq 4$.



In this case we have

$$w_0 = (\sigma_n\sigma_{n-1})(\sigma_{n-2}\sigma_n\sigma_{n-1}\sigma_{n-2})\dots(\sigma_1\dots\sigma_{n-2}\sigma_{n-1}\sigma_n\sigma_{n-2}\dots\sigma_1). \tag{5.28}$$

Before constructing the quasi K-matrix in this case, we need the following lemma, similar to Lemma 5.15.

Lemma 5.17. *The element E_1 commutes with $[T_1T_{w_X}(E_1), E_1]_{q^{-2}}$.*

Proof. We have

$$\begin{aligned}
T_1T_{w_X}(E_1) &= T_1T_{2--n}T_{n-2--2}(E_1) \\
&= T_{1--n}T_{n-2--3}([E_2, E_1]_{q^{-1}}) \\
&= [T_{1--n}T_{n-2--3}(E_2), E_2]_{q^{-1}}
\end{aligned}$$

where we use $T_{12}(E_1) = E_2$ by Proposition 2.40. The element $E_1T_{1--n}T_{n-2--3}(E_2)$ has weight $2\alpha_1 + \alpha_2 + 2(\alpha_3 + \dots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$. Hence by Theorem 2.59 it follows that

$$E_1T_{1--n}T_{n-2--3}(E_2) = qT_{1--n}T_{n-2--3}(E_2)E_1. \tag{5.29}$$

Since $T_1(E_2) = E_1E_2 - q^{-1}E_2E_1$ we obtain

$$\begin{aligned}
E_1T_1T_{w_X}(E_1) &= E_1T_{1--n}T_{n-2--3}(E_2)E_2 - q^{-1}E_1E_2T_{1--n}T_{n-2--3}(E_2) \\
&= qT_{1--n}T_{n-2--3}(E_2)\big(T_1(E_2) + q^{-1}E_2E_1\big) \\
&\quad - q^{-1}\big(T_1(E_2) + q^{-1}E_2E_1\big)T_{1--n}T_{n-2--3}(E_2) \\
&= q[T_{1--n}T_{n-2--3}(E_2), T_1(E_2)]_{q^{-2}} + T_1T_{w_X}(E_1)E_1.
\end{aligned}$$

This implies that

$$E_1[T_1T_{w_X}(E_1), E_1]_{q^{-2}} = [T_1T_{w_X}(E_1), E_1]_{q^{-2}}E_1 \\ + q\big[[T_{1--n}T_{n-2--3}(E_2), T_1(E_2)]_q^{-2}, E_1\big]_{q^{-2}}.$$

By Equation (5.29) and the fact $T_1(E_2)E_1 = q^{-1}E_1T_1(E_2)$, it follows that

$$\big[[T_{1--n}T_{n-2--3}(E_2), T_1(E_2)]_q^{-2}, E_1\big]_{q^{-2}} = 0.$$

Hence $E_1[T_1T_{w_X}(E_1), E_1]_{q^{-2}} = [T_1T_{w_X}(E_1), E_1]_{q^{-2}}E_1$ as required. □

Lemma 5.18. *The quasi K-matrix $\mathfrak{X}$ in type* DII *is given by*

$$\mathfrak{X} = \sum_{m\geq 0} \frac{(qc_1)^m}{\{m\}!}[E_1, T_{w_X}(E_1)]_{q^{-2}}^m. \tag{5.30}$$

Proof. By Equation (5.12) we want to show that ${}_1r(\mathfrak{X}) = (q-q^{-1})c_1T_{w_X}(E_1)\mathfrak{X}$. By (2.92) we have ${}_1r(T_{w_X}(E_1)) = 0$ so it follows that

$${}_1r([E_1, T_{w_X}(E_1)]_{q^{-2}}) = (1-q^{-2})T_{w_X}(E_1).$$

Since $\sigma_1 w_X(\alpha_1) = w_X\alpha_1$ we have $w_X\sigma_1w_X(\alpha_1) = \alpha_1$ and hence Proposition 2.40 implies that $T_{w_X}T_1T_{w_X}(E_1) = E_1$. Applying T_{w_X} to the relation $E_1[T_1T_{w_X}(E_1), E_1]_{q^{-2}} = [T_1T_{w_X}(E_1), E_1]_{q^{-2}}E_1$ from Lemma 5.17 we obtain

$$T_{w_X}(E_1)[E_1, T_{w_X}(E_1)]_{q^{-2}} = [E_1, T_{w_X}(E_1)]_{q^{-2}}T_{w_X}(E_1).$$

Using Property (2.78) of the skew derivation ${}_1r$ we have

$${}_1r([E_1, T_{w_X}(E_1)^2_{q^{-2}}) = \{2\}(1-q^{-2})T_{w_X}(E_1)[E_1, T_{w_X}(E_1)]_{q^{-2}}.$$

Continuing inductively it follows that

$${}_1r([E_1, T_{w_X}(E_1)]^m_{q^{-2}}) = (1-q^{-2})\{m\}T_{w_X}(E_1)[E_1, T_{w_X}(E_1)]^{m-1}_{q^{-2}}.$$

This implies that

$$\begin{aligned} {}_1r(\mathfrak{X}) &= \sum_{m\geq 0}\frac{(qc_1)^m}{\{m\}!}\,{}_1r([E_1, T_{w_X}(E_1)]^m_{q^{-2}}) \\ &= (1-q^{-2})T_{w_X}(E_1)\sum_{m\geq 1}\frac{(qc_1)^m}{\{m-1\}!}[E_1, T_{w_X}(E_1)]^{m-1}_{q^{-2}} \\ &= (q-q^{-1})c_1T_{w_X}(E_1)\mathfrak{X} \end{aligned}$$

as required. □

Remark 5.19. Let $\mathscr{A} = \mathbb{Z}[q, q^{-1}]$ and let ${}_{\mathscr{A}}U^+$ be the $\mathscr{A}$-subalgebra of U^+ generated by $E_i^{(n)} = \frac{E_i^n}{[n]!}$ for all $n \in \mathbb{N}_0$, $i \in I$. Set ${}_{\mathscr{A}}\widehat{U^+} = \prod_{\mu\in Q^+} {}_{\mathscr{A}}U^+_\mu$ where ${}_{\mathscr{A}}U^+_\mu = {}_{\mathscr{A}}U^+ \cap U^+_\mu$ for all $\mu \in Q^+$. By [6, Theorem 5.3] we have $\mathfrak{X} \in {}_{\mathscr{A}}\widehat{U^+}$ if $c_is(\tau(i)) \in \pm q^{\mathbb{Z}}$ for all $i \in I \setminus X$.

This integrality property is crucial for the theory of canonical bases of quantum symmetric pairs developed in [6].

We observe that the integrality of the quasi K-matrix in rank one can in some cases be read off the explicit formulas given in this section. Indeed, Lemma 5.11, 5.13 and 5.14 imply that $\mathfrak{X} \in {}_{\mathscr{A}}\widehat{U^+}$ in the rank one cases of type AI, $AIII$ and AIV. The rank one cases AII_3, BII and DII are more complicated, and Lemmas 5.12, 5.16 and 5.18 does not give an obvious way to see that $\mathfrak{X} \in {}_{\mathscr{A}}\widehat{U^+}$. Nevertheless, $\mathfrak{X}$ is also integral in this case, as shown in [6, A.5]. Based on the present remark, the integrality of $\mathfrak{X}$ in higher rank is discussed in Remark 5.31.

5.4 Partial quasi K-matrices

All through this section we make the assumption that $\mathbf{s} = \mathbf{0} = (0, 0, \ldots, 0) \in \mathcal{S}$. In Section 5.5 we discuss the case of general parameters $\mathbf{s} \in \mathcal{S}$. We provide a construction for the quasi K-matrix analogous to the construction of the quasi R-matrix in Theorem 2.60.

Recall from Section 4.4 the restricted Weyl group $\widetilde{W}$ and the set of simple roots $\widetilde{\Pi} = \{\widetilde{\alpha_i} \mid i \in I \setminus X\}$. By Corollary 2.39 the Lusztig automorphisms T_i of $U_q(\mathfrak{g})$ for all $i \in I$ give rise to a representation of $Br(W)$ on $U_q(\mathfrak{g})$. Since $\widetilde{W}$ is a subgroup of W, we hence obtain algebra automorphisms of $U_q(\mathfrak{g})$ defined by

$$\widetilde{T_i} := T_{\widetilde{\sigma}_i} \qquad \text{for each } i \in I \setminus X.$$

By Theorem 4.12 and Corollary 4.11 the algebra automorphisms $\widetilde{T_i}$ give rise to a representation of $Br(\widetilde{W})$ on $U_q(\mathfrak{g})$.

Define $Q(2\Sigma) = 2\mathbb{Z}\widetilde{\Pi}$ and $Q^+(2\Sigma) = 2\mathbb{N}_0\widetilde{\Pi}$. By Equation (5.8) and the assumption $\mathbf{s} = \mathbf{0}$ we have

$${}_ir(\mathfrak{X}_\mu) = -(q_i - q_i^{-1})q^{-(\Theta(\alpha_i),\alpha_i)}c_i X_i \mathfrak{X}_{\mu - 2\widetilde{\alpha}_i} \qquad \text{for any } \mu \in Q^+.$$

By Lemma 5.18 it follows that $\mathfrak{X}_\mu \neq 0$ only if $\mu \in Q^+(2\Sigma)$. Hence by Lemma 2.30 we may consider the quasi K-matrix $\mathfrak{X}$ as an element in $\prod_{\mu \in Q^+(2\Sigma)} U^+_\mu \subset \widehat{U^+} \subset \mathscr{U}$. For any $w \in W$ define

$$\widetilde{U}^+[w] = \bigoplus_{\mu \in Q^+(2\Sigma)} U^+[w]_\mu$$

and set $\widetilde{U}^+ = \bigoplus_{\mu \in Q^+(2\Sigma)} U^+_\mu$. Then $\widetilde{U}^+$ and $\widetilde{U}^+[w]$ are $\mathbb{K}(q)$-subalgebras of U^+ and $U^+[w]$, respectively. In particular by Equation (5.17) we have

$$\mathfrak{X}_i \in \widehat{\widetilde{U}^+[\widetilde{\sigma}_i]} = \prod_{\mu \in Q^+(2\Sigma)} \widetilde{U}^+[\widetilde{\sigma}_i]_\mu \qquad \text{for any } i \in I \setminus X.$$

Let $\mathbb{K}'$ be a field extension of $\mathbb{K}(q)$ which contains $q^{1/2}$ and elements $\widetilde{c}_i$ such that

$$\widetilde{c}_i^{\,2} = c_i c_{\tau(i)} s(i) s(\tau(i)) \qquad \text{for all } i \in I \setminus X. \tag{5.31}$$

We extend $\widetilde{U}^+$ and $\widetilde{U}^+[w]$ for $w \in W$ to $\mathbb{K}'$-algebras $\widetilde{U}^+_{1/2} = \mathbb{K}' \otimes_{\mathbb{K}(q)} \widetilde{U}^+$ and $\widetilde{U}^+_{1/2}[w] = \mathbb{K}' \otimes_{\mathbb{K}(q)} \widetilde{U}^+[w]$. Define an algebra automorphism $\Psi : \widetilde{U}^+_{1/2} \to \widetilde{U}^+_{1/2}$ by

$$\Psi(E_{2\widetilde{\alpha}_i}) = q^{(\widetilde{\alpha}_i, \widetilde{\alpha}_i)} \widetilde{c}_i E_{2\widetilde{\alpha}_i} \qquad \text{for all } E_{2\widetilde{\alpha}_i} \in U^+_{2\widetilde{\alpha}_i}. \tag{5.32}$$

For each $i \in I \setminus X$ define an algebra homomorphism

$$\Omega_i = \Psi \circ \widetilde{T}_i \circ \Psi^{-1} : \widetilde{U}^+_{1/2}[\widetilde{\sigma}_i w_0] \to \widetilde{U}^+_{1/2}. \tag{5.33}$$

We consider the restriction of the algebra homomorphism Ω_i to the subalgebra $\widetilde{U}^+[\widetilde{\sigma}_i w_0]$, and we denote this restriction also by Ω_i. Crucially, by the following proposition, the image of the restriction Ω_i belongs to $\widetilde{U}^+$ and does not involve any of the adjoined square roots.

Proposition 5.20 ([16, Proposition 3.12])**.** *For every $i \in I \setminus X$ the map $\Omega_i : \widetilde{U}^+[\widetilde{\sigma}_i w_0] \to \widetilde{U}^+$ is a well defined algebra homomorphism.*

Proof. It remains to show that the image of Ω_i is contained in $\widetilde{U}^+$. Observe that $\widetilde{T}_i(\widetilde{U}^+_\mu) \subseteq \widetilde{U}^+_{\widetilde{\sigma}_i(\mu)}$ for all $\mu \in Q^+(2\Sigma)$. By Corollary 4.17 we have

$$\widetilde{\sigma}_i(\mu) = \mu - \frac{2(\mu, \widetilde{\alpha}_i)}{(\widetilde{\alpha}_i, \widetilde{\alpha}_i)} \widetilde{\alpha}_i \qquad \text{for all } \mu \in Q^+(2\Sigma).$$

Hence Equation (5.32) implies that

$$\Omega_i|_{\widetilde{U}^+_\mu} = q^{-(\mu, \widetilde{\alpha}_i)} \widetilde{c}_i^{\,-(\mu,\widetilde{\alpha}_i)/(\widetilde{\alpha}_i,\widetilde{\alpha}_i)} \widetilde{T}_i|_{\widetilde{U}^+_\mu}.$$

Since $\mu \in Q^+(2\Sigma)$ it follows that the exponent $-(\mu, \widetilde{\alpha}_i)$ is an integer. Moreover, Corollary 4.17 implies that the exponent $-(\mu, \widetilde{\alpha}_i)/(\widetilde{\alpha}_i, \widetilde{\alpha}_i)$ is an integer.

If $i = \tau(i)$ then Equation (5.31) and Condition (3.10) imply that $\widetilde{c}_i = \pm c_i$. This implies that the image of Ω_i is contained in $\widetilde{U}^+$ in this case.

Suppose instead that $i \in I \setminus X$ satisfies $i \neq \tau(i)$. If additionally $(\alpha_i, \Theta(\alpha_i)) = 0$, then (3.33) implies that $c_i = c_{\tau(i)}$. Moreover in this case $\Theta(\alpha_i) = -\alpha_{\tau(i)}$ by [38, Lemma 5.3] and hence $s(i) = s(\tau(i))$ by (3.11). Hence we get $\widetilde{c}_i = \pm c_i s(i)$ in the case $i \neq \tau(i)$, $(\alpha_i, \Theta(\alpha_i)) = 0$ which implies that the image of Ω_i is contained in $\widetilde{U}^+$ in this case.

Finally, we consider the case that $i \neq \tau(i)$ and $(\alpha_i, \Theta(\alpha_i)) \neq 0$. We are then in Case 3 in [46, p. 17] and hence the restricted root system Σ is of type $(BC)_n$ for $n \geq 1$ and $(\widetilde{\alpha}_i, \widetilde{\alpha}_i) = \frac{1}{4}(\alpha_i, \alpha_i)$. Since $\mu \in Q^+(2\Sigma) \subset Q$ we have

$$\frac{(\mu, \widetilde{\alpha}_i)}{(\widetilde{\alpha}_i, \widetilde{\alpha}_i)} = 4 \frac{(\mu, \alpha_i)}{(\alpha_i, \alpha_i)} \in 2\mathbb{Z}.$$

Hence the image of Ω_i is contained in $\widetilde{U}^+$ in all cases as required. □

Consider $\widetilde{w} \in \widetilde{W}$ and let $\widetilde{w} = \widetilde{\sigma}_{i_1}\widetilde{\sigma}_{i_2}\dots\widetilde{\sigma}_{i_t}$ be a reduced expression. For $k = 1, \dots, t$ let

$$\mathfrak{X}^{[k]}_{\widetilde{w}} = \Omega_{i_1}\Omega_{i_2}\cdots\Omega_{i_{k-1}}(\mathfrak{X}_{i_k}) = \Psi \circ \widetilde{T}_{i_1}\cdots\widetilde{T}_{i_{k-1}} \circ \Psi^{-1}(\mathfrak{X}_{i_k}). \tag{5.34}$$

By Corollary 4.11 we have $U^+[\widetilde{\sigma}_{i_k}] \subset U^+[\widetilde{\sigma}_{i_{k-1}}(w_0)]$ for $k = 2, \dots, t$, and

$$\widetilde{T}_{i_l}\cdots\widetilde{T}_{i_{k-1}}(U^+[\widetilde{\sigma}_{i_k}]) \subset U^+[\widetilde{\sigma}_{i_{l-1}}(w_0)] \qquad \text{for } l = 2, \dots, k-1$$

and hence the elements $\mathfrak{X}^{[k]}_{\widetilde{w}}$ are well-defined. Moreover, by Proposition 5.20 we have

$$\mathfrak{X}^{[k]}_{\widetilde{w}} \in \widehat{\widetilde{U}^+[\widetilde{w}]} = \prod_{\mu \in Q^+(2\Sigma)} U^+[\widetilde{w}]_\mu \qquad \text{for } k = 1, \dots, t.$$

When clear, we omit the subscript $\widetilde{w}$ and write $\mathfrak{X}^{[k]}$ instead of $\mathfrak{X}^{[k]}_{\widetilde{w}}$.

Definition 5.21 ([16, Definition 3.13]). Let $\widetilde{w} \in \widetilde{W}$ and let $\widetilde{w} = \widetilde{\sigma}_{i_1}\widetilde{\sigma}_{i_2}\dots\widetilde{\sigma}_{i_t}$ be a reduced expression. The *partial quasi K-matrix* $\mathfrak{X}_{\widetilde{w}}$ associated to $\widetilde{w}$ and the given reduced expression is defined by

$$\mathfrak{X}_{\widetilde{w}} = \mathfrak{X}^{[k]}\mathfrak{X}^{[k-1]}\dots\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}. \tag{5.35}$$

We expect that the partial quasi K-matrix $\mathfrak{X}_{\widetilde{w}}$ only depends on $\widetilde{w} \in \widetilde{W}$ and not on the chosen reduced expression. As we will see in Theorem 5.25 it suffices to check the independence of the reduced expression in rank two. If the Satake diagram is of rank two then the restricted Weyl group $\widetilde{W}$ is of one of the types $A_1 \times A_1$, A_2, B_2 or G_2. In each case, only the longest word for $\widetilde{W}$ has distinct reduced expressions.

Conjecture 5.22 ([16, Conjecture 3.14]). *Assume that (I, X, τ) is a Satake diagram of rank two. Then the element $\mathfrak{X}_{\widetilde{w}} \in \mathscr{U}$ defined by (5.35) depends only on $\widetilde{w} \in \widetilde{W}$ and not on the chosen reduced expression.*

In Chapter 6, we prove the following Theorem which confirms Conjecture 5.22 in many cases. The proof is performed by showing that for both reduced expressions of the longest word in $\widetilde{W}$ the resulting elements $\mathfrak{X}_w$ satisfy the relations (5.12).

Theorem 5.23 ([16, Theorem 3.15]). *Assume that $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ or $X = \emptyset$. Then Conjecture 5.22 holds.*

Remark 5.24. The Hopf algebra automorphism Ψ in the definition of Ω_i turns out to be necessary for the rank two calculations in Chapter 6 which prove Theorem 5.23. The conjugation by Ψ affects the coefficients in the partial quasi K-matrix associated to a reduced expression of an element $\widetilde{w} \in \widetilde{W}$. In rank two the two partial quasi K-matrices associated to the longest word $\widetilde{w}_0 \in \widetilde{W}$ coincide only after this change of coefficients. The effect of the conjugation by Ψ can be seen in particular in Sections 6.3 and 6.4 which treat type $AIII_n$ for $n \geq 3$.

Once the rank two case is established, we can generalise to higher rank cases.

Theorem 5.25 ([16, Theorem 4.16]). *Suppose that (I, X, τ) is a Satake diagram such that all subdiagrams $(J, X \cap J, \tau|_J)$ of rank two satisfy Conjecture 5.22. Then the element $\mathfrak{X}_{\widetilde{w}} \in \mathscr{U}$ depends on $\widetilde{w} \in \widetilde{W}$ and not on the chosen reduced expression.*

Proof. Let $\widetilde{w}$ and $\widetilde{w}'$ be reduced expressions which represent the same element in $\widetilde{W}$. Assume that $\widetilde{w}$ and $\widetilde{w}'$ differ by a single braid relation. The following are the possible braid relations:

$$\begin{aligned}
\widetilde{\sigma}_p\widetilde{\sigma}_r &= \widetilde{\sigma}_r\widetilde{\sigma}_p,\\
\widetilde{\sigma}_p\widetilde{\sigma}_r\widetilde{\sigma}_p &= \widetilde{\sigma}_r\widetilde{\sigma}_p\widetilde{\sigma}_r,\\
(\widetilde{\sigma}_p\widetilde{\sigma}_r)^2 &= (\widetilde{\sigma}_r\widetilde{\sigma}_p)^2,\\
(\widetilde{\sigma}_p\widetilde{\sigma}_r)^3 &= (\widetilde{\sigma}_r\widetilde{\sigma}_p)^3.
\end{aligned} \tag{5.36}$$

The argument for each relation is the same, so we only consider the second case. Assume that $\widetilde{w}$ and $\widetilde{w}'$ differ by relation (5.36), that is

$$\begin{aligned}
\widetilde{w} &= \widetilde{\sigma}_{i_1}\cdots\widetilde{\sigma}_{i_{k-1}}\big(\widetilde{\sigma}_p\widetilde{\sigma}_r\widetilde{\sigma}_p\big)\widetilde{\sigma}_{i_{k+3}}\cdots\widetilde{\sigma}_{i_t},\\
\widetilde{w}' &= \widetilde{\sigma}_{i_1}\cdots\widetilde{\sigma}_{i_{k-1}}\big(\widetilde{\sigma}_r\widetilde{\sigma}_p\widetilde{\sigma}_r\big)\widetilde{\sigma}_{i_{k+3}}\cdots\widetilde{\sigma}_{i_t}
\end{aligned}$$

for some $k = 1, \ldots, t-2$. For $l = 1, \ldots, k-1$, we have

$$\mathfrak{X}_{\widetilde{w}}^{[l]} = \Psi \circ \widetilde{T}_{i_1}\cdots\widetilde{T}_{i_{l-1}} \circ \Psi^{-1}(\mathfrak{X}_{i_l}) = \mathfrak{X}_{\widetilde{w}'}^{[l]}.$$

Since the algebra automorphisms $\widetilde{T}_i$ satisfy braid relations, we have

$$\begin{aligned}
\mathfrak{X}_{\widetilde{w}}^{[l]} &= \Psi \circ \widetilde{T}_{i_1}\cdots\widetilde{T}_{i_{k-1}}\big(\widetilde{T}_p\widetilde{T}_r\widetilde{T}_p\big)\widetilde{T}_{i_{k+3}}\cdots\widetilde{T}_{i_{l-1}} \circ \Psi^{-1}(\mathfrak{X}_{i_l})\\
&= \Psi \circ \widetilde{T}_{i_1}\cdots\widetilde{T}_{i_{k-1}}\big(\widetilde{T}_r\widetilde{T}_p\widetilde{T}_r\big)\widetilde{T}_{i_{k+3}}\cdots\widetilde{T}_{i_{l-1}} \circ \Psi^{-1}(\mathfrak{X}_{i_l}) = \mathfrak{X}_{\widetilde{w}'}^{[l]}
\end{aligned}$$

for $l = k+3, \ldots, t$. Finally, consider the rank two subdiagram $(J, X \cap J, \tau|_J)$ obtained by taking $J = J_1 \cup J_2$, where $J_1 = \{r, p, \tau(r), \tau(p)\}$ and $J_2 \subset X$ is the union of connected components of X which are connected to a node of J_1. By assumption,

$$\begin{aligned}
\mathfrak{X}_{\widetilde{\sigma}_p\widetilde{\sigma}_r\widetilde{\sigma}_p} &= \mathfrak{X}_p \cdot \Psi\widetilde{T}_p\Psi^{-1}(\mathfrak{X}_r) \cdot \Psi\widetilde{T}_p\widetilde{T}_r\Psi^{-1}(\mathfrak{X}_p)\\
&= \mathfrak{X}_r \cdot \Psi\widetilde{T}_r\Psi^{-1}(\mathfrak{X}_p) \cdot \Psi\widetilde{T}_r\widetilde{T}_p\Psi^{-1}(\mathfrak{X}_r) = \mathfrak{X}_{\widetilde{\sigma}_r\widetilde{\sigma}_p\widetilde{\sigma}_r}.
\end{aligned}$$

It follows from this that

$$\begin{aligned}
\mathfrak{X}_{\widetilde{w}}^{[k]}\mathfrak{X}_{\widetilde{w}}^{[k+1]}\mathfrak{X}_{\widetilde{w}}^{[k+2]} &= \Psi\widetilde{T}_{i_1}\cdots\widetilde{T}_{i_{k-1}}\Psi^{-1}\big(\mathfrak{X}_{\widetilde{\sigma}_p\widetilde{\sigma}_r\widetilde{\sigma}_p}\big)\\
&= \Psi\widetilde{T}_{i_1}\cdots\widetilde{T}_{i_{k-1}}\Psi^{-1}\big(\mathfrak{X}_{\widetilde{\sigma}_r\widetilde{\sigma}_p\widetilde{\sigma}_r}\big)\\
&= \mathfrak{X}_{\widetilde{w}'}^{[k]}\mathfrak{X}_{\widetilde{w}'}^{[k+1]}\mathfrak{X}_{\widetilde{w}'}^{[k+2]}
\end{aligned}$$

Hence we have $\mathfrak{X}_{\widetilde{w}} = \mathfrak{X}_{\widetilde{w}'}$ as required.

If $\widetilde{w}$ and $\widetilde{w}'$ differ by more than a single relation, then we can find a sequence of reduced expressions

$$\widetilde{w} = \widetilde{w}_1, \widetilde{w}_2, \ldots, \widetilde{w}_n = \widetilde{w}'$$

such that for each $i = 1, \ldots, n-1$, the expressions $\widetilde{w}_i$ and $\widetilde{w}_{i+1}$ differ by a single relation. We repeat the above argument at each step and obtain $\mathfrak{X}_{\widetilde{w}} = \mathfrak{X}_{\widetilde{w}'}$. □

Recall from Equation (3.2) that there exists a diagram automorphism $\tau_0 : I \to I$ such that the longest element $w_0 \in W$ satisfies

$$w_0(\alpha_i) = -\alpha_{\tau_0(i)} \tag{5.37}$$

for all $i \in I$.

Proposition 5.26 ([16, Proposition 3.18])**.** *Let $\widetilde{w}_0 \in \widetilde{W}$ be the longest element with reduced expression $\widetilde{w}_0 = \widetilde{\sigma}_{i_1} \cdots \widetilde{\sigma}_{i_t}$. Then*

$$\mathfrak{X}^{[t]}_{\widetilde{w}_0} = \mathfrak{X}_{\tau_0(i_t)}. \tag{5.38}$$

Proof. To simplify notation we write $i_t = i$. By construction we have

$$\widetilde{w}_0 w_X = w_0,$$
$$w_X \widetilde{\sigma}_i = w_{\{i,\tau(i)\}\cup X} \quad \text{for all } i \in I \setminus X.$$

By Lemma 4.1 the elements w_X and $w_{\{i,\tau(i)\}\cup X}$ commute so we get

$$\begin{aligned}
\mathfrak{X}^{[t]}_{\widetilde{w}_0} &= \Psi \circ T_{\widetilde{w}_0} T^{-1}_{\widetilde{\sigma}_i} \circ \Psi^{-1}(\mathfrak{X}_i) \\
&= \Psi \circ T_{w_0} T^{-1}_{w_X} T_{w_X} T^{-1}_{w_{\{i,\tau(i)\}\cup X}} \circ \Psi^{-1}(\mathfrak{X}_i) \\
&= \Psi \circ T_{w_0} T^{-1}_{w_{\{i,\tau(i)\}\cup X}} \circ \Psi^{-1}(\mathfrak{X}_i).
\end{aligned}$$

Recall that

$$T_{w_0} = \mathrm{tw}^{-1} \circ \tau_0 \tag{5.39}$$

where $\mathrm{tw} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the algebra automorphism defined by

$$\mathrm{tw}(E_i) = -K_i^{-1} F_i, \quad \mathrm{tw}(F_i) = -E_i K_i, \quad \mathrm{tw}(K_i) = K_i^{-1}$$

for $i \in I$, see [4, Section 7.1]. Analogously we have on $U_q(\mathfrak{g}_{\{i,\tau(i)\}\cup X})$ the relation

$$T_{w_{\{i,\tau(i)\}\cup X}} = \mathrm{tw}^{-1} \circ \tau_{0,i} = \tau_{0,i} \circ \mathrm{tw}^{-1} \tag{5.40}$$

where $\tau_{0,i} : \{i, \tau(i)\} \cup X \to \{i, \tau(i)\} \cup X$ is the diagram automorphism satisfying (4.2). We obtain

$$\begin{aligned}
\mathfrak{X}^{[t]}_{\widetilde{w}_0} &= \Psi \circ \mathrm{tw}^{-1} \circ \tau_0 \circ \mathrm{tw} \circ \tau_{0,i} \circ \Psi^{-1}(\mathfrak{X}_i) \\
&= \Psi \circ \tau_0 \tau_{0,i} \circ \Psi^{-1}(\mathfrak{X}_i).
\end{aligned} \tag{5.41}$$

Case 1. $\tau(i) = i$.

In this case Lemma 5.7 implies that

$$\tau_{0,i}(\mathfrak{X}_i) = \mathfrak{X}_i. \tag{5.42}$$

Moreover $s(\tau(i)) = s(i) = 1$ and hence by Lemma 5.10 and by definition of Ψ we have

$$\begin{aligned}
\mathfrak{X}^{[t]}_{\widetilde{w}_0} &= \sum_{n\in\mathbb{N}_0} \Psi \circ \tau_0\tau_{0,i} \circ \Psi^{-1}(c_i^n E_{n(\alpha_i-\Theta(\alpha_i))}) \\
&= \sum_{n\in\mathbb{N}_0} q^{-n/2(\alpha_i-\Theta(\alpha_i),\alpha_i)} \Psi \circ \tau_0\tau_{0,i}(E_{n(\alpha_i-\Theta(\alpha_i))}) \\
&\overset{(5.42)}{=} \sum_{n\in\mathbb{N}_0} q^{-n/2(\alpha_i-\Theta(\alpha_i),\alpha_i)} \Psi \circ \tau_0(E_{n(\alpha_i-\Theta(\alpha_i))}) \\
&= \sum_{n\in\mathbb{N}_0} q^{-n/2(\alpha_i-\Theta(\alpha_i),\alpha_i)} \Psi(E_{n(\alpha_{\tau_0(i)}-\Theta(\alpha_{\tau_0(i)}))})
\end{aligned}$$

where we use the notation from Lemma 5.10 also for $\mathfrak{X}_{\tau_0(i)}$.

As $(\alpha_{\tau_0(i)} - \Theta(\alpha_{\tau_0(i)}), \alpha_{\tau_0(i)}) = (\alpha_i - \Theta(\alpha_i), \alpha_i)$ and $s(\tau_0(i)) = 1$, formula (5.32) gives us

$$\mathfrak{X}^{[t]}_{\widetilde{w}_0} = \sum_{n\in\mathbb{N}_0} c^n_{\tau_0(i)} E_{n(\alpha_{\tau_0(i)}-\Theta(\alpha_{\tau_0(i)}))} = \mathfrak{X}_{\tau_0(i)} \tag{5.43}$$

which proves the Lemma in this case.

Case 2. $\tau(i) \neq i$.

In this case the rank one Satake subdiagram is either of type $AIII_{11}$ or of type AIV for $n \geq 2$ as in Table 5.1.

If the rank one Satake subdiagram is of type AIV for $n \geq 2$ then $\tau = \tau_0$ and $\tau_{0,i}$ coincide on $\{i, \tau(i)\} \cup X$ and hence (5.41) implies that

$$\mathfrak{X}^{[t]}_{\widetilde{w}_0} = \mathfrak{X}_i = \mathfrak{X}_{\tau\tau_0(i)} = \mathfrak{X}_{\tau_0(i)}. \tag{5.44}$$

If the rank one subdiagram is of type $AIII_{11}$ then $\tau_{0,i}(i) = i$. If additionally $\tau_0(i) = i$ then $\tau_0(\tau(i)) = \tau(i)$ and hence (5.41) implies that $\mathfrak{X}^{[t]}_{\widetilde{w}_0} = \mathfrak{X}_i = \mathfrak{X}_{\tau_0(i)}$ in this case.

If on the other hand $\tau_0(i) \neq i$ then $\tau_0 = \tau$ and we invoke the fact that

$$s(i) = s(\tau(i)), \quad c_i = c_{\tau(i)} \tag{5.45}$$

which holds by (3.11) and (3.33). Relation (5.45) and $\tau = \tau_0$ imply that $\tau_0 \circ \Psi^{-1}(\mathfrak{X}_i) = \Psi^{-1} \circ \tau_0(\mathfrak{X}_i)$. Hence Equation (5.41) implies that $\mathfrak{X}^{[t]}_{\widetilde{w}_0} = \mathfrak{X}_{\tau_0(i)}$ also in this case. □

Lemma 5.27 ([16, Lemma 3.19]). *Let $\widetilde{w}_0 = \widetilde{\sigma}_{i_1} \cdots \widetilde{\sigma}_{i_t}$ be a reduced expression for the longest word in $\widetilde{W}$. Then $\mathfrak{X}^{[i]}_{\widetilde{w}_0} \in U^+[\widehat{\widetilde{\sigma}_k\widetilde{w}_0}]$ for $i = 1, \ldots, t-1$ and $k = \tau_0(i_t)$.*

Proof. We have

$$\widetilde{\sigma}_k\widetilde{w}_0 = \widetilde{\sigma}_k w_0 w_X = w_0\widetilde{\sigma}_{\tau_0(k)} w_X = w_0 w_X \widetilde{\sigma}_{\tau_0(k)} = \widetilde{w}_0\widetilde{\sigma}_{i_t} = \widetilde{\sigma}_{i_1} \cdots \widetilde{\sigma}_{i_{t-1}}.$$

By definition of $U^+[w]$ for each $w \in W$ and Corollary 4.11 we have

$$\widetilde{T}_{i_1} \cdots \widetilde{T}_{i_{j-1}}(U^+[\widetilde{\sigma}_{i_j}]) \subseteq U^+[\widetilde{\sigma}_k \widetilde{w}_0]$$

for $j = 1, \ldots, t-1$. Now the claim of the lemma follows from Equation (5.17), Proposition 5.20 and the fact that

$$\mathfrak{X}^{[j]}_{\widetilde{w}_0} = \Psi \circ \widetilde{T}_{i_1} \cdots \widetilde{T}_{i_{j-1}} \circ \Psi^{-1}(\mathfrak{X}_{i_j})$$

for $j = 1, \ldots, t-1$. □

With the above preparations we are ready to prove the main result of the chapter, cf. Theorem 2.60.

Theorem 5.28 ([16, Theorem 3.20]). *Suppose that (I, X, τ) is a Satake diagram such that all subdiagrams $(J, X \cap J, \tau|_J)$ of rank two satisfy Conjecture 5.22. Then $\mathfrak{X}_{\widetilde{w}_0}$ coincides with the quasi K-matrix $\mathfrak{X}$.*

Proof. It suffices to show that

$$_ir(\mathfrak{X}_{\widetilde{w}_0}) = (q - q^{-1}) q^{-(\Theta(\alpha_i), \alpha_i)} c_i s(\tau(i)) T_{w_X}(E_{\tau(i)}) \mathfrak{X}_{\widetilde{w}_0} \tag{5.46}$$

for all $i \in I \setminus X$. By Theorem 5.25, we can choose any reduced expression $\widetilde{w}_0 = \widetilde{\sigma}_{i_1} \cdots \widetilde{\sigma}_{i_t}$ of the longest element of $\widetilde{W}$. Proposition 5.26 implies that

$$\mathfrak{X}_{\widetilde{w}_0} = \mathfrak{X}_{\tau_0(i_t)} \mathfrak{X}^{[t-1]} \cdots \mathfrak{X}^{[2]} \mathfrak{X}^{[1]}.$$

Suppose $\tau_0(i_t) \in I$ is a representative of the τ-orbit $\{k, \tau(k)\}$ for some $k \in I \setminus X$.

By the previous lemma we have $\mathfrak{X}^{[i]} \in \widehat{U^+[\widetilde{\sigma}_k \widetilde{w}_0]}$ for $i = 1, \ldots, t-1$. By [27, 8.26, (4)] this implies that $_kr(\mathfrak{X}^{[i]}) = 0$ for $i = 1, \ldots, t-1$. By Equation 5.12, we have

$$_kr(\mathfrak{X}_{\tau_0(i_t)}) = (q - q^{-1}) q^{-(\Theta(\alpha_k), \alpha_k)} c_k s(\tau(k)) T_{w_X}(E_{\tau(k)}) \mathfrak{X}_{\tau_0(i_t)}$$

and similarly an expression for $_{\tau(k)}r(\mathfrak{X}_{\tau_0(i_t)})$. Equation (5.46) for $k, \tau(k)$ follows from the above and the skew derivation property (2.78). Since we can arbitrarily choose the reduced expression for $\widetilde{w}_0$, the result follows. □

Combining Theorems 5.23 and 5.28 we obtain the following result.

Corollary 5.29 ([16, Corollary 3.21]). *Let $\mathfrak{g}$ be of type A or $X = \emptyset$. Then the quasi K-matrix $\mathfrak{X}$ is given by $\mathfrak{X} = \mathfrak{X}_{\widetilde{w}_0}$ for any reduced expression of the longest word $\widetilde{w}_0 \in \widetilde{W}$.*

Conjecture 5.30 ([16, Conjecture 3.22]). *The statement of Corollary 5.29 holds for any Satake diagram of finite type.*

Remark 5.31. We continue the discussion of the integrality of the quasi K-matrix $\mathfrak{X}$ from Remark 5.19 under the assumption that $c_i s(\tau(i)) \in \pm q^{\mathbb{Z}}$ for all $i \in I \setminus X$. In this case $\mathfrak{X}^{[k]}_{\widetilde{w}} \in {}_{\mathscr{A}}\widehat{U^+}$ for $k = 1, \ldots, t$ if $\widetilde{w} \in \widetilde{W}$ has a reduced expression $\widetilde{w} = \widetilde{\sigma}_{i_1} \widetilde{\sigma}_{i_2} \ldots \widetilde{\sigma}_{i_t}$. Indeed, the discussion in the proof of Proposition 5.20 shows that $\mathfrak{X}^{[k]}_{\widetilde{w}}$ differs from $\widetilde{T}_{i_1}\dots\widetilde{T}_{i_1}(\mathfrak{X}_{i_k})$ by a factor in $\pm q^{\mathbb{Z}}$. Hence we obtain $\mathfrak{X}_{\widetilde{w}} \in {}_{\mathscr{A}}\widehat{U^+}$ for all $\widetilde{w} \in \widetilde{W}$. By Corollary 5.29, choosing $\widetilde{w} = \widetilde{w}_0$, we obtain $\mathfrak{X} \in {}_{\mathscr{A}}\widehat{U^+}$ whenever $\mathfrak{g}$ is of type A or $X = \emptyset$. In these cases we have hence reproduced [6, Theorem 5.3] for $\mathbf{s} = \mathbf{0}$ without the use of canonical bases. The case of general Satake diagrams hinges on Conjecture 5.30 and the integrality in rank one from [6, Appendix A].

5.5 Quasi K-matrices for general parameters

We now give a description of the quasi K-matrix $\mathfrak{X}$ for general parameters $\mathbf{s} \in \mathcal{S}$ from [16, Section 3.5]. Recall from the proof of Lemma 5.7 that we denote the generators B_i by $B_i^{\mathbf{c},\mathbf{s}}$ if we need to specify the dependence on the parameters.

The following lemma provides an algebra isomorphism between the subalgebras $B_{\mathbf{c},\mathbf{s}}$ for different parameters $\mathbf{s}, \mathbf{s}' \in \mathcal{S}$. This follows from Theorem 3.22 since none of the defining relations for $B_{\mathbf{c},\mathbf{s}}$ depend on the parameters $\mathbf{s}$.

Lemma 5.32. *Let $\mathbf{s}, \mathbf{s}' \in \mathcal{S}$. Then the map $\varphi_{\mathbf{s},\mathbf{s}'} : B_{\mathbf{c},\mathbf{s}} \to B_{\mathbf{c},\mathbf{s}'}$ given by*

$$\varphi_{\mathbf{s},\mathbf{s}'}(B_i^{\mathbf{c},\mathbf{s}}) = B_i^{\mathbf{c},\mathbf{s}'}, \quad \varphi_{\mathbf{s},\mathbf{s}'}(b) = b \qquad \textit{for all } i \in I \setminus X,\ b \in \mathcal{M}_X U^0_\Theta \tag{5.47}$$

is an algebra isomorphism.

We write $\varphi_{\mathbf{s}}$ to denote the isomorphism $\varphi_{\mathbf{0},\mathbf{s}}$. This algebra isomorphism allows us to define a one dimensional representation $\chi_{\mathbf{s}} : B_{\mathbf{c},\mathbf{0}} \to \mathbb{K}(q)$ by $\chi_{\mathbf{s}} = \varepsilon \circ \varphi_{\mathbf{s}}$. By definition we have

$$\chi_{\mathbf{s}}(B_i^{\mathbf{c},\mathbf{0}}) = s_i \quad \text{for all } i \in I \setminus X, \qquad \chi_{\mathbf{s}}|_{\mathcal{M}_X U^0_\Theta} = \varepsilon|_{\mathcal{M}_X U^0_\Theta}.$$

By Lemma 3.19 we have

$$\Delta(B_i) - B_i \otimes K_i^{-1} \in \mathcal{M}_X U^0_\Theta \otimes U_q(\mathfrak{g}) \tag{5.48}$$

which implies that

$$\varphi_{\mathbf{s}} = (\chi_{\mathbf{s}} \otimes \mathrm{id}) \circ \Delta \tag{5.49}$$

on $B_{\mathbf{c},\mathbf{0}}$. For later use we observe the following compatibility with the bar involution.

Lemma 5.33 ([16, Lemma 3.24]). *For all $b \in B_{\mathbf{c},\mathbf{0}}$ we have*

$$(\chi_{\mathbf{s}} \otimes \mathrm{id}) \circ (\overline{}^{B_{\mathbf{c},\mathbf{0}}} \otimes \overline{}^{U}) \circ \Delta(b) = \overline{\varphi_{\mathbf{s}}(b)}^{U}. \tag{5.50}$$

Proof. As $(\chi_{\mathbf{s}} \otimes \mathrm{id}) \circ (\overline{}^{B_{\mathbf{c},\mathbf{0}}} \otimes \overline{}^{U}) \circ \Delta$ and $\overline{}^{U} \circ \varphi_{\mathbf{s}}$ are $\mathbb{K}$-algebra homomorphisms, it suffices to check Equation (5.50) on the generators $B_i^{\mathbf{c},\mathbf{0}}$ for $i \in I \setminus X$ and on $\mathcal{M}_X U^0_\Theta$. If $b \in \mathcal{M}_X U^0_\Theta$ then both sides of (5.50) coincide with $\overline{b}^{U}$. If $b = B_i^{\mathbf{c},\mathbf{0}}$ for some $i \notin \{j \in I_{ns} \,|\, a_{jk} \in -2\mathbb{N}_0 \text{ for all } k \in I_{ns} \setminus \{j\}\}$ then $s_i = 0$ by the definition of $\mathcal{S}$ in (3.34) and hence $B_i^{\mathbf{c},\mathbf{0}} = B_i^{\mathbf{c},\mathbf{s}}$. Using the membership property (5.48) we get

$$((\chi_{\mathbf{s}} \circ \overline{}^{B_{\mathbf{c},\mathbf{0}}}) \otimes \overline{}^{U}) \circ \Delta(B_i^{\mathbf{c},\mathbf{0}}) = ((\overline{}^{U} \circ \varepsilon) \otimes \overline{}^{U}) \circ \Delta(B_i^{\mathbf{c},\mathbf{0}}) = \overline{B_i^{\mathbf{c},\mathbf{0}}}^{U} = \overline{\varphi_{\mathbf{s}}(b)}^{U}$$

which proves (5.50) in this case. Finally, if $i \in \{j \in I_{ns} \,|\, a_{jk} \in -2\mathbb{N}_0 \text{ for all } k \in I_{ns} \setminus \{j\}\}$, then the definition (3.32) of I_{ns} implies that

$$B_i^{\mathbf{c},\mathbf{s}} = F_i - c_i E_i K_i^{-1} + s_i K_i^{-1}.$$

Hence, using $s_i = \overline{s_i}^U$ from (5.5), we get

$$\begin{aligned}((\chi_{\mathbf{s}} \circ \overline{}^{B_{\mathbf{c},\mathbf{0}}}) \otimes \overline{}^{U}) \circ \Delta(B_i^{\mathbf{c},\mathbf{0}}) &= ((\chi_{\mathbf{s}} \circ \overline{}^{B_{\mathbf{c},\mathbf{0}}}) \otimes \overline{}^{U})(B_i^{\mathbf{c},\mathbf{0}} \otimes K_i^{-1} + 1 \otimes B_i^{\mathbf{c},\mathbf{0}}) \\ &= s_i K_i + \overline{B_i^{\mathbf{c},\mathbf{0}}}^{U} \\ &= \overline{B_i^{\mathbf{c},\mathbf{s}}}^{U} \\ &= \overline{\varphi_{\mathbf{s}}(B_i^{\mathbf{c},\mathbf{0}})}^{U}\end{aligned}$$

which completes the proof of the lemma. □

As in [4, 3.2] we consider the algebra

$$\mathscr{U}_0^{(2)} = \mathrm{End}(\mathcal{F}or \circ \otimes : \mathcal{O}_{int} \times \mathcal{O}_{int} \to \mathcal{V}ect)$$

and observe that $\prod_{\mu \in Q^+} U_\mu^- \otimes U_\mu^+$ is a subalgebra of $\mathscr{U}_0^{(2)}$. Recall from Section 2.2.8 the quasi R-matrix $R \in \prod_{\mu \in Q^+} U_\mu^- \otimes U_\mu^+$. Following [5, 3.1] we define an element

$$R^\theta = \Delta(\mathfrak{X}) \cdot R \cdot (\mathfrak{X}^{-1} \otimes 1) \in \mathscr{U}_0^{(2)}, \tag{5.51}$$

see also [39, Section 3.3]. In [5] the element R^θ is called the quasi R-matrix for $B_{\mathbf{c},\mathbf{s}}$. By [5, Proposition 3.2] it satisfies the following intertwiner property

$$\Delta(\overline{b}^B) \cdot R^\theta = R^\theta \cdot (\overline{}^{B} \otimes \overline{}^{U}) \circ \Delta(b) \qquad \text{for all } b \in B = B_{\mathbf{c},\mathbf{s}} \tag{5.52}$$

in $\mathscr{U}_0^{(2)}$. Moreover, by [5, Proposition 3.5], [39, Proposition 3.6] we can write R^θ as an infinite sum

$$R^\theta = \sum_{\mu \in Q^+} R^\theta_\mu \qquad \text{with } R^\theta_\mu \in B_{\mathbf{c},\mathbf{s}} \otimes U_\mu^+. \tag{5.53}$$

Similarly to the notation $\mathfrak{X}_{\mathbf{c},\mathbf{s}}$ introduced in Lemma 5.7, we write $R^\theta_{\mathbf{c},\mathbf{s}}$ if we need to specify the dependence on the parameters. Observe that once we have an explicit formula for $\mathfrak{X}_{\mathbf{c},\mathbf{0}}$, Equation (5.51) provides us with an explicit formula for $R^\theta_{\mathbf{c},\mathbf{0}}$. This in turn provides a formula for the quasi K-matrix $\mathfrak{X}_{\mathbf{c},\mathbf{s}}$ for general parameters $\mathbf{s} \in \mathcal{S}$. Indeed, by Equation (5.53) we can apply the character $\chi_{\mathbf{s}}$ to the first tensor factor of $R^\theta_{\mathbf{c},\mathbf{0}}$ to obtain an element $\mathfrak{X}' = (\chi_{\mathbf{s}} \otimes \mathrm{id})(R^\theta_{\mathbf{c},\mathbf{0}})$ which can be written as

$$\mathfrak{X}' = \sum_{\mu \in Q^+} \mathfrak{X}'_\mu \qquad \text{with } \mathfrak{X}'_\mu \in U_\mu^+.$$

Moreover, Equation (5.53) implies that $\mathfrak{X}'_0 = 1$. By the following proposition the element $\mathfrak{X}' \in \mathscr{U}$ is the quasi K-matrix for $B_{\mathbf{c},\mathbf{s}}$.

Proposition 5.34 ([16, Proposition 3.25]). *For any* $\mathbf{c} \in \mathcal{C}$, $\mathbf{s} \in \mathcal{S}$ *we have* $\mathfrak{X}_{\mathbf{c},\mathbf{s}} = (\chi_{\mathbf{s}} \otimes \mathrm{id})(R^{\theta}_{\mathbf{c},\mathbf{0}})$.

Proof. We keep the notation $\mathfrak{X}' = (\chi_{\mathbf{s}} \otimes \mathrm{id})(R^{\theta}_{\mathbf{c},\mathbf{0}})$ from above. By Equation (5.52) we have

$$\Delta(\overline{b}^{B_{\mathbf{c},\mathbf{0}}}) \cdot R^{\theta}_{\mathbf{c},\mathbf{0}} = R^{\theta}_{\mathbf{c},\mathbf{0}} \cdot (\overline{}^{B_{\mathbf{c},\mathbf{0}}} \otimes \overline{}^{U}) \circ \Delta(b) \qquad \text{for all } b \in B_{\mathbf{c},\mathbf{0}}.$$

Applying $\chi_{\mathbf{s}} \otimes \mathrm{id}$ to both sides of this relation, we obtain in view of Equation (5.49) the relation

$$\varphi_{\mathbf{s}}(\overline{b}^{B_{\mathbf{c},\mathbf{0}}}) \cdot \mathfrak{X}' = \mathfrak{X}'(\chi_{\mathbf{s}} \otimes \mathrm{id}) \circ (\overline{}^{B_{\mathbf{c},\mathbf{0}}} \otimes \overline{}^{U}) \circ \Delta(b) \qquad \text{for all } b \in B_{\mathbf{c},\mathbf{0}}.$$

By Lemma 5.33 the above relation implies that

$$\varphi_{\mathbf{s}}(\overline{b}^{B_{\mathbf{c},\mathbf{0}}})\mathfrak{X}' = \mathfrak{X}'\overline{\varphi_{\mathbf{s}}(b)}^{U} \qquad \text{for all } b \in B_{\mathbf{c},\mathbf{0}}.$$

This gives in particular $B_i^{\mathbf{c},\mathbf{s}}\mathfrak{X}' = \mathfrak{X}'\overline{B_i^{\mathbf{c},\mathbf{s}}}^{U}$ for all $i \in I$ and $b\mathfrak{X}' = \mathfrak{X}'b$ for all $b \in \mathcal{M}_X U_0^{\Theta}$. This means that $\mathfrak{X}'$ satisfies the defining relation (5.6) of $\mathfrak{X}_{\mathbf{c},\mathbf{s}}$ and hence, in view of the normalisation $\mathfrak{X}'_0 = 1 \otimes 1$ observed above, we get $\mathfrak{X}' = \mathfrak{X}_{\mathbf{c},\mathbf{s}}$. ☐

Remark 5.35. The existence of the quasi K-matrix $\mathfrak{X}_{\mathbf{c},\mathbf{s}}$ was established in [4, Theorem 6.10] by fairly involved calculations. It was noted in [4, Remark 6.9] that these calculations simplify significantly if one restricts to the case $\mathbf{s} = \mathbf{0}$. Proposition 5.34 now shows that in the presence of (5.53) the existence of $\mathfrak{X}_{\mathbf{c},\mathbf{0}}$ implies the existence of $\mathfrak{X}_{\mathbf{c},\mathbf{s}}$ for any $\mathbf{s} \in \mathcal{S}$ satisfying (5.5). Relation (5.53) was established in [39] for $\mathfrak{g}$ of finite type.

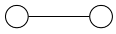
Chapter 6

Quasi K-matrices in rank two

In rank two, there are two distinct reduced expressions for the longest word $\widetilde{w}_0 \in \widetilde{W}$. All irreducible Satake diagrams of rank 2 are shown in Table 6.1. In this chapter we verify Theorem 5.23 using the explicit formulas from Section 5.4. We do this by confirming that the partial quasi K-matrices for the two reduced expressions for $\widetilde{w}_0$ coincide with the quasi K-matrix. With the exception of type G_2 in Section 6.6, all of the calculations of this chapter come from [16, Appendix A].

6.1 Type AI_2

Consider the Satake diagram of type AI_2.



Since $\Theta = -\mathrm{id}$ the restricted Weyl group $\widetilde{W}$ coincides with the Weyl group W. The longest word of the Weyl group has two reduced expressions given by

$$w_0 = \sigma_1\sigma_2\sigma_1,$$
$$w_0' = \sigma_2\sigma_1\sigma_2.$$

Proposition 6.1. *In this case, the partial quasi K-matrices $\mathfrak{X}_{w_0}$ and $\mathfrak{X}_{w_0'}$ coincide with the quasi K-matrix $\mathfrak{X}$. Hence $\mathfrak{X}_{w_0} = \mathfrak{X}_{w_0'}$.*

Before we prove this, we need to know how the Lusztig skew derivations ${}_1r$ and ${}_2r$ act on certain elements and their powers, and also some commutation relations. These are given in the following two lemmas, whose proofs are obtained by straightforward computation.

Lemma 6.2. *For any $n \in \mathbb{N}$, the relations*

$$E_2^n E_1 = q^n E_1 E_2^n - q\{n\} E_2^{n-1} T_1(E_2),$$

$$
\begin{aligned}
E_1^n E_2 &= q^n E_2 E_1^n - q\{n\} E_1^{n-1} T_2(E_1),\\
T_1(E_2)^n E_1 &= q^{-n} E_1 T_1(E_2)^n,\\
T_2(E_1)^n E_2 &= q^{-n} E_2 T_2(E_1)^n
\end{aligned}
$$

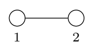
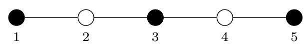
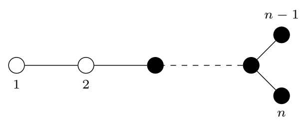
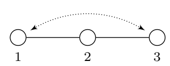
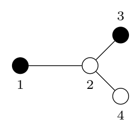
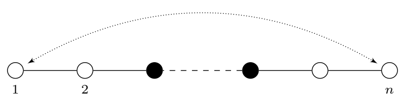
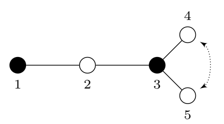
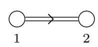
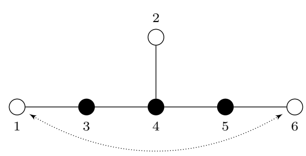
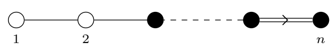
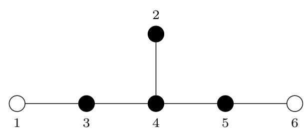
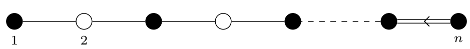
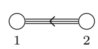
hold in $U_q(\mathfrak{sl}_3)$.

Lemma 6.3. *For any* $n \in \mathbb{N}$, *the relations*

$$
\begin{aligned}
&{}_1r(E_2^n) = {}_1r(T_2(E_1)^n) = {}_2r(E_1^n) = {}_2r(T_1(E_2)^n) = 0,\\
&{}_1r(E_1^n) = \{n\} E_1^{n-1},\\
&{}_2r(E_2^n) = \{n\} E_2^{n-1},\\
&{}_1r(T_1(E_2)^n) = (1 - q^{-2})\{n\} E_2 T_1(E_2)^{n-1},\\
&{}_2r(T_2(E_1)^n) = (1 - q^{-2})\{n\} E_1 T_2(E_1)^{n-1}
\end{aligned}
$$

hold in $U_q(\mathfrak{sl}_3)$.

Table 6.1: Irreducible Satake diagrams of rank two for simple $\mathfrak{g}$

AI_2		CII_4	
AII_5		DI_n, $n \geq 5$	
$AIII_3$		$DIII_4$	
$AIII_n$, $n \geq 4$		$DIII_5$	
$(BC)_2$		$EIII$	
BI_n, $n \geq 3$		EIV	
CII_n, $n \geq 5$		G	

Proof of Proposition 6.1. Consider first the element $\mathfrak{X}_{w_0}$. Using (5.35) and Lemma 5.11, we write

$$\mathfrak{X}_{w_0} = \mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \tag{6.1}$$

where

$$\begin{aligned}
\mathfrak{X}^{[3]} &\overset{(5.38)}{=} \mathfrak{X}_2 = \sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{2n\}!!}(q^2c_2)^n E_2^{2n},\\
\mathfrak{X}^{[2]} &= \Psi\circ T_1\circ\Psi^{-1}(\mathfrak{X}_2) = \sum_{n\geq 0}\frac{(q-q^{-1})^n}{\{2n\}!!}(q^2c_1)^n(q^2c_2)^nT_1(E_2)^{2n},\\
\mathfrak{X}^{[1]} &= \mathfrak{X}_1 = \sum_{n\geq 0}\frac{(q-q^{-1})^n}{\{2n\}!!}(q^2c_1)^nE_1^{2n}.
\end{aligned}$$

For $i=1,2,3$, let $\mathfrak{X}_{[i]} = K_1\mathfrak{X}^{[i]}K_1^{-1}$. The difference between $\mathfrak{X}_{[i]}$ and $\mathfrak{X}^{[i]}$ is the occurrence of a q-power in each summand of the infinite series. By Equation (5.12), to show that $\mathfrak{X}_{w_0}$ coincides with the quasi K-matrix $\mathfrak{X}$ we show that

$$\begin{aligned}
{}_1r(\mathfrak{X}_{w_0}) &= (q-q^{-1})(q^2c_1)E_1\mathfrak{X}_{w_0},\\
{}_2r(\mathfrak{X}_{w_0}) &= (q-q^{-1})(q^2c_2)E_2\mathfrak{X}_{w_0}.
\end{aligned}$$

By Lemma 6.3 and 5.11, we see that

$$\begin{aligned}
{}_2r(\mathfrak{X}_{w_0}) &= {}_2r(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}\\
&= (q-q^{-1})(q^2c_2)E_2\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}\\
&= (q-q^{-1})(q^2c_2)E_2\mathfrak{X}_{w_0}.
\end{aligned}$$

By the property (2.78) of the skew derivative ${}_1r$, we have

$${}_1r(\mathfrak{X}_{w_0}) = \mathfrak{X}_{[3]}\,{}_1r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]} + \mathfrak{X}_{[3]}\mathfrak{X}_{[2]}\,{}_1r(\mathfrak{X}^{[1]}) \tag{6.2}$$

Using Lemma 6.3, we have

$$\begin{aligned}
{}_1r(\mathfrak{X}^{[2]}) &= \sum_{n\geq}\frac{(q-q^{-1})^n}{\{2n\}!!}(q^2c_1)^n(q^2c_2)^n\,{}_1r(T_1(E_2)^{2n})\\
&= q^{-1}(q-q^{-1})E_2\sum_{n\geq 1}\frac{(q-q^{-1})^n}{\{2n-2\}!!}(q^2c_1)^n(q^2c_2)^nT_1(E_2)^{2n-1}\\
&= (q-q^{-1})^2q^{-1}(q^2c_1)(q^2c_2)E_2T_1(E_2)\mathfrak{X}^{[2]},\\
{}_1r(\mathfrak{X}^{[1]}) &= (q-q^{-1})(q^2c_1)E_1\mathfrak{X}^{[1]}.
\end{aligned}$$

The second summand of Equation (6.2) is of the form

$$(q-q^{-1})(q^2c_1)\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}E_1\mathfrak{X}^{[1]}.$$

We use Lemma 6.2 to bring the E_1 in the above summand to the front. We have

$$\begin{aligned}
\mathfrak{X}_{[2]}E_1 &= \sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{2n\}!!}(q^2c_1)^n(q^2c_2)^n q^{2n} T_1(E_2)^{2n}E_1\\
&= E_1 \sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{2n\}!!}(q^2c_1)^n(q^2c_2)^n T_1(E_2)^{2n}\\
&= E_1\mathfrak{X}^{[2]},\\
\mathfrak{X}_{[3]}E_1 &= \sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{2n\}!!}(q^2c_2)^n q^{-2n}E_2^{2n}E_1\\
&= \sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{2n\}!!}(q^2c_2)^n q^{-2n}(q^{2n}E_1E_2^{2n} - q\{2n\}E_2^{2n-1}T_1(E_2))\\
&= E_1\mathfrak{X}^{[3]} - q\sum_{n\geq 1} \frac{(q-q^{-1})^n}{\{2n-2\}!!}(q^2c_2)^n q^{-2n}E_2^{2n-1}T_1(E_2)\\
&= E_1\mathfrak{X}^{[3]} - (q-q^{-1})q^{-1}(q^2c_2)\mathfrak{X}_{[3]}E_2T_1(E_2).
\end{aligned}$$

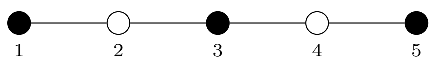
Hence,

$$\begin{aligned}
(q-q^{-1})(q^2c_1)\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}E_1\mathfrak{X}^{[1]} &= (q-q^{-1})(q^2c_1)\mathfrak{X}_{[3]}E_1\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}\\
&= (q-q^{-1})(q^2c_1)E_1\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}\\
&\quad - (q-q^{-1})^2q^{-1}(q^2c_1)(q^2c_2)\mathfrak{X}_{[3]}E_2T_1(E_2)\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}\\
&= (q-q^{-1})(q^2c_1)E_1\mathfrak{X}_{w_0} - \mathfrak{X}_{[3]}\,{}_1r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]}.
\end{aligned}$$

It follows from (6.2) that ${}_1r(\mathfrak{X}_{w_0}) = (q-q^{-1})(q^2c_1)E_1\mathfrak{X}_{w_0}$, so $\mathfrak{X}_{w_0}$ coincides with the quasi K-matrix $\mathfrak{X}$. Instead of repeating the same calculation for $\mathfrak{X}_{w_0'}$, we use the underlying symmetry in type AI_2, which implies that $\mathfrak{X}_{w_0'}$ also coincides with the quasi K-matrix $\mathfrak{X}$. ☐

6.2 Type AII$_5$

Consider the Satake diagram of type AII$_5$.



In this case the involutive automorphism $\Theta : \mathfrak{h}^* \to \mathfrak{h}^*$ is given by

$$\Theta = -\sigma_1\sigma_3\sigma_5.$$

There are two τ-orbits of white nodes given by the sets $\{2\}$ and $\{4\}$. The restricted root system is of type AI$_2$ since the restricted roots

$$\widetilde{\alpha}_2 = \frac{\alpha_1 + 2\alpha_2 + \alpha_3}{2}, \qquad \widetilde{\alpha}_4 = \frac{\alpha_3 + 2\alpha_4 + \alpha_5}{2}$$

have the same length. The subgroup $\widetilde{W} \subset W^\Theta$ is generated by the elements

$$\widetilde{\sigma}_2 = \sigma_2\sigma_1\sigma_3\sigma_2, \qquad \widetilde{\sigma}_4 = \sigma_4\sigma_3\sigma_5\sigma_4.$$

The longest word of the restricted Weyl group has two reduced expressions given by

$$\widetilde{w}_0 = \widetilde{\sigma}_2\widetilde{\sigma}_4\widetilde{\sigma}_2, \qquad \widetilde{w}_0' = \widetilde{\sigma}_4\widetilde{\sigma}_2\widetilde{\sigma}_4.$$

By Lemma 5.12 we have

$$\mathfrak{X}_2 = \sum_{n\geq 0} \frac{(qc_2)^n}{\{n\}!}[E_2, T_{13}(E_2)]^n_{q^{-2}},$$
$$\mathfrak{X}_4 = \sum_{n\geq 0} \frac{(qc_4)^n}{\{n\}!}[E_4, T_{35}(E_4)]^n_{q^{-2}}.$$

Proposition 6.4. *The partial quasi K-matrices $\mathfrak{X}_{\widetilde{w}_0}$ and $\mathfrak{X}_{\widetilde{w}_0'}$ coincide with the quasi K-matrix $\mathfrak{X}$.*

We have the following relations needed for the proof of Proposition 6.4. These are proved by induction.

Lemma 6.5. *For any $n \in \mathbb{N}$ the relations*

$$\begin{aligned}[E_4, T_{35}(E_4)]^n_{q^{-2}} T_{13}(E_2) &= q^n T_{13}(E_2)[E_4, T_{35}(E_4)]^n_{q^{-2}} \\ &\quad - q\{n\}[E_4, T_{35}(E_4)]^{n-1}_{q^{-2}}[T_3(E_4), T_{1235}(E_4)]_{q^{-2}},\end{aligned} \tag{6.3}$$

$$[T_{23}(E_4), T_{1235}(E_4)]^n_{q^{-2}} T_{13}(E_2) = q^{-n} T_{13}(E_2)[T_{23}(E_4), T_{1235}(E_4)]^n_{q^{-2}} \tag{6.4}$$

hold in $U_q(\mathfrak{sl}_6)$.

Lemma 6.6. *For any $n \in \mathbb{N}$ the relation*

$$\begin{aligned}&{}_2r([T_{23}(E_4), T_{1235}(E_4)]^n_{q^{-2}}) \\ &\quad = q^{-1}(q-q^{-1})\{n\}[T_3(E_4), T_{1235}(E_4)]_{q^{-2}}[T_{23}(E_4), T_{1235}(E_4)]^{n-1}_{q^{-2}}\end{aligned} \tag{6.5}$$

holds in $U_q(\mathfrak{sl}_6)$.

Proof of Proposition 6.4. We only confirm that $\mathfrak{X}_{\widetilde{w}_0}$ coincides with the quasi K-matrix $\mathfrak{X}$. By the underlying symmetry in type AII_5, the calculation for $\mathfrak{X}_{\widetilde{w}_0'}$ is the same up to a change of indices. By defintion, we have

$$\mathfrak{X}_{\widetilde{w}_0} = \mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}$$

where

$$\mathfrak{X}^{[3]} \overset{(5.38)}{=} \mathfrak{X}_4,$$

$$\mathfrak{X}^{[2]} = \Psi \circ T_{2132} \circ \Psi^{-1}(\mathfrak{X}_4) = \sum_{n\geq 0} \frac{(q^2c_2c_4)^n}{\{n\}!}[T_{23}(E_4), T_{1235}(E_4)]^n_{q^{-2}},$$

$$\mathfrak{X}^{[1]} = \mathfrak{X}_2.$$

Using Lemma 5.12 and Corollary 2.51 we see that

$$\begin{aligned} {}_4r(\mathfrak{X}_{\widetilde{w}_0}) &= {}_4r(\mathfrak{X}_4)\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\ &= (q-q^{-1})c_4T_{35}(E_4)\mathfrak{X}_4\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\ &= (q-q^{-1})c_4T_{35}(E_4)\mathfrak{X}_{\widetilde{w}_0}. \end{aligned}$$

We want to show that

$${}_2r(\mathfrak{X}_{\widetilde{w}_0}) = (q-q^{-1})c_2T_{13}(E_2)\mathfrak{X}_{\widetilde{w}_0}.$$

For $i = 1,2,3$, let $\mathfrak{X}_{[i]} = K_2\mathfrak{X}^{[i]}K_2^{-1}$. By property (2.78) of the skew derivative ${}_2r$ and Corollary 2.51 we have

$${}_2r(\mathfrak{X}_{\widetilde{w}_0}) = \mathfrak{X}_{[3]}\,{}_2r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]} + \mathfrak{X}_{[3]}\mathfrak{X}_{[2]}\,{}_2r(\mathfrak{X}^{[1]}). \tag{6.6}$$

By Lemma 5.12 we have

$${}_2r(\mathfrak{X}^{[1]}) = (q-q^{-1})c_2T_{13}(E_2)\mathfrak{X}^{[1]}.$$

By Lemma 6.6 we have

$$\begin{aligned} {}_2r(\mathfrak{X}^{[2]}) &= \sum_{n\geq 1}\frac{(q^2c_2c_4)^n}{\{n\}!}\,{}_2r([T_{23}(E_4),T_{1235}(E_4)]^n_{q^{-2}}) \\ &\overset{(6.5)}{=} q(q-q^{-1})c_2c_4[T_3(E_4),T_{1235}(E_4)]_{q^{-2}}\mathfrak{X}^{[2]}. \end{aligned}$$

The second summand of Equation (6.6) is of the form $(q-q^{-1})c_2\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}T_{13}(E_2)\mathfrak{X}^{[1]}$. Using Lemma 6.5, we bring the $T_{13}(E_2)$ term in this expression to the front. We have

$$\mathfrak{X}_{[2]}T_{13}(E_2) \overset{(6.4)}{=} T_{13}(E_2)\mathfrak{X}^{[2]}, \tag{6.7}$$

$$\mathfrak{X}_{[3]}T_{13}(E_2) \overset{(6.3)}{=} T_{13}(E_2)\mathfrak{X}^{[3]} - qc_4\mathfrak{X}_{[3]}[T_3(E_4),T_{1235}(E_4)]_{q^{-2}}. \tag{6.8}$$

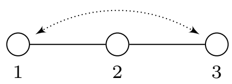
Substituting (6.7) and (6.8) into $(q-q^{-1})c_2\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}T_{13}(E_2)\mathfrak{X}^{[1]}$, we obtain

$$\begin{aligned} (q-q^{-1})c_2\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}T_{13}(E_2)\mathfrak{X}^{[1]} &\overset{(6.7)}{=} (q-q^{-1})c_2\mathfrak{X}_{[3]}T_{13}(E_2)\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\ &\overset{(6.8)}{=} (q-q^{-1})c_2T_{13}(E_2)\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\ &\quad - (q-q^{-1})qc_2c_4\mathfrak{X}^{[3]}[T_3(E_4),T_{1235}(E_4)]_{q^{-2}}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\ &= (q-q^{-1})c_2T_{13}(E_2)\mathfrak{X}_{\widetilde{w}_0} - \mathfrak{X}_{[3]}\,{}_2r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]}. \end{aligned}$$

It follows from (6.6) that ${}_2r(\mathfrak{X}_{\widetilde{w}_0}) = (q-q^{-1}c_2T_{13}(E_2)\mathfrak{X}_{\widetilde{w}_0}$ as required. □

6.3 Type AIII$_3$

We consider the diagram of type AIII$_3$ with non-trivial diagram automorphism τ and no black dots.



Here, we see that there are 2 nodes in the restricted Dynkin diagram, corresponding to the restricted roots

$$\widetilde{\alpha}_1 = \frac{\alpha_1 + \alpha_3}{2}, \qquad \widetilde{\alpha}_2 = \alpha_2.$$

A quick check confirms that $1/2(\alpha_1 + \alpha_3)$ is the short root, and hence the restricted root system is of type B_2. The subgroup $\widetilde{W}$ is generated by the elements

$$\widetilde{\sigma}_1 = \sigma_1\sigma_3, \qquad \widetilde{\sigma}_2 = \sigma_2.$$

The longest word of the restricted Weyl group has two reduced expressions given by

$$\widetilde{w}_0 = \widetilde{\sigma}_1\widetilde{\sigma}_2\widetilde{\sigma}_1\widetilde{\sigma}_2, \qquad \widetilde{w}_0' = \widetilde{\sigma}_2\widetilde{\sigma}_1\widetilde{\sigma}_2\widetilde{\sigma}_1.$$

The definition (3.33) and condition (5.4) imply that $c_1 = c_3 = \overline{c_1}$. By Lemmas 5.11 and 5.13 we have

$$\mathfrak{X}_1 = \sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{n\}!} c_1^n (E_1E_3)^n, \tag{6.9}$$

$$\mathfrak{X}_2 = \sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{2n\}!!} (q^2c_2)^n E_2^{2n}. \tag{6.10}$$

Proposition 6.7. *The partial quasi K-matrix $\mathfrak{X}_{\widetilde{w}_0}$ coincides with the quasi K-matrix $\mathfrak{X}$.*

The following relations are needed for the proof of Proposition 6.7. They are checked by induction.

Lemma 6.8. *For any $n \in \mathbb{N}$, the relations*

$$T_{13}(E_2)^n E_3 = q^{-n} E_3 T_{13}(E_2)^n, \tag{6.11}$$

$$\begin{aligned}\big(T_1(E_2)T_3(E_2)\big)^n E_3 = E_3\big(T_1(E_2)T_3(E_2)\big)^n \\ - q\{n\}\big(T_1(E_2)T_3(E_2)\big)^{n-1} T_3(E_2)T_{13}(E_2),\end{aligned} \tag{6.12}$$

$$E_2^n E_3 = q^n E_3 E_2^n - q\{n\} E_2^{n-1} T_3(E_2) \tag{6.13}$$

hold in $U_q(\mathfrak{sl}_4)$.

Lemma 6.9. *For any $n \in \mathbb{N}$, the relations*

$$_1r(T_{13}(E_2)^n) = q^{-1}(q-q^{-1})\{n\} T_3(E_2)T_{13}(E_2)^{n-1}, \tag{6.14}$$

$$_1r\big(\big(T_1(E_2)T_3(E_2)\big)^n\big) = q^{-1}(q-q^{-1})\{n\} E_2 T_3(E_2)\big(T_1(E_2)T_3(E_2)\big)^{n-1} \tag{6.15}$$

hold in $U_q(\mathfrak{sl}_4)$.

Proof of Proposition 6.7. Take $\widetilde{w}_0 = \widetilde{\sigma}_1\widetilde{\sigma}_2\widetilde{\sigma}_1\widetilde{\sigma}_1$. Then we have

$$\mathfrak{X}_{\widetilde{w}_0} = \mathfrak{X}^{[4]}\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}$$

where

$$\begin{aligned}
\mathfrak{X}^{[4]} &\overset{(5.38)}{=} \mathfrak{X}_{\tau_0(2)} = \mathfrak{X}_2,\\
\mathfrak{X}^{[3]} &= \Psi\circ\widetilde{T}_1\widetilde{T}_2\circ\Psi^{-1}(\mathfrak{X}_1) = \sum_{n\geq 0}\frac{(q-q^{-1})^n}{\{n\}!}(q^2c_1c_2)^n(T_3(E_2)T_1(E_2))^n,\\
\mathfrak{X}^{[2]} &= \Psi\circ\widetilde{T}_1\circ\Psi^{-1}(\mathfrak{X}_2) = \sum_{n\geq 0}\frac{(q-q^{-1})^n}{\{2n\}!!}(q^4c_1^2c_2)^nT_{13}(E_2)^{2n},\\
\mathfrak{X}^{[1]} &= \mathfrak{X}_1.
\end{aligned}$$

By Lemma 5.11, property (2.78) of the skew derivative ${}_2r$ and Corollary 2.51, we see that ${}_2r(\mathfrak{X}_{\widetilde{w}_0}) = (q-q^{-1})q^2c_2E_2\mathfrak{X}_{\widetilde{w}_0}$. Due to the underlying symmetry in this case, we only need to show that

$${}_1r(\mathfrak{X}_{\widetilde{w}_0}) = (q-q^{-1})c_1E_3\mathfrak{X}_{\widetilde{w}_0}.$$

For each $i = 1,2,3,4$, let $\mathfrak{X}_{[i]} = K_1\mathfrak{X}^{[i]}K_1^{-1}$. Then by the property (2.78) of the skew derivation ${}_1r$, we have

$${}_1r(\mathfrak{X}_w) = \mathfrak{X}_{[4]}{}_1r(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}{}_1r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_1r(\mathfrak{X}^{[1]}).$$

Using Lemmas 5.13 and 6.9, it follows that

$${}_1r(\mathfrak{X}^{[3]}) \overset{(6.15)}{=} q^{-1}(q-q^{-1})^2(q^2c_1c_2)E_2T_3(E_2)\mathfrak{X}^{[3]}, \tag{6.16}$$

$${}_1r(\mathfrak{X}^{[2]}) \overset{(6.14)}{=} q^{-1}(q-q^{-1})^2(q^4c_1^2c_2)T_3(E_2)T_{13}(E_2)\mathfrak{X}^{[2]}, \tag{6.17}$$

$${}_1r(\mathfrak{X}^{[1]}) \overset{(5.24)}{=} (q-q^{-1})E_3\mathfrak{X}^{[1]}. \tag{6.18}$$

Using Lemma 6.8, we look at the term $\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_1r(\mathfrak{X}^{[1]})$ in more detail. We have

$$\mathfrak{X}_{[2]}E_3 \overset{(6.11)}{=} E_3\mathfrak{X}^{[2]}, \tag{6.19}$$

$$\mathfrak{X}_{[3]}E_3 \overset{(6.12)}{=} E_3\mathfrak{X}^{[3]} - q(q-q^{-1})(q^2c_1c_2)\mathfrak{X}_{[3]}T_3(E_2)T_{13}(E_2), \tag{6.20}$$

$$\mathfrak{X}_{[4]}E_3 \overset{(6.13)}{=} E_3\mathfrak{X}^{[4]} - q^{-1}(q-q^{-1})(q^2c_2)\mathfrak{X}_{[4]}E_2T_3(E_2). \tag{6.21}$$

It follows that

$$\begin{aligned}
&\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_1r(\mathfrak{X}^{[1]})\\
&\overset{(6.19)}{=} (q-q^{-1})c_1\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}E_3\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}\\
&\overset{(6.20)}{=} (q-q^{-1})c_1\mathfrak{X}_{[4]}\big(E_3\mathfrak{X}^{[3]} - q(q-q^{-1})(q^2c_1c_2)\mathfrak{X}_{[3]}T_3(E_2)T_{13}(E_2)\big)\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}\\
&\overset{(6.21)}{=} (q-q^{-1})c_1\big(E_3\mathfrak{X}^{[4]} - q^{-1}(q-q^{-1})(q^2c_2)\mathfrak{X}_{[4]}E_2T_3(E_2)\big)\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}\\
&\quad - \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}{}_1r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]}\\
&= (q-q^{-1})c_1E_3\mathfrak{X}_{\widetilde{w}_0} - \mathfrak{X}_{[4]}{}_1r(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} - \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}{}_1r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]}
\end{aligned}$$

by equations (6.16), (6.17) and (6.18). Hence, it follows that ${}_1r(\mathfrak{X}_{\widetilde{w}_0}) = (q-q^{-1})c_1E_3\mathfrak{X}_{\widetilde{w}_0}$, as required.

□

Proposition 6.10. *The partial quasi K-matrix $\mathfrak{X}_{\widetilde{w}_0'}$ coincides with the quasi K-matrix $\mathfrak{X}$.*

The following relations are needed for the proof of Proposition 6.10. They are checked by induction.

Lemma 6.11. *For any $n \in \mathbb{N}$, the relations*

$$\big(T_2(E_3)T_2(E_1)\big)^n E_2 = q^{-2n} E_2 \big(T_2(E_3)T_2(E_1)\big)^n, \tag{6.22}$$

$$T_{213}(E_2)^n E_2 = E_2 T_{213}^n(E_2) - (q-q^{-1})\{n\} T_{213}(E_2)^{n-1} T_2(E_3)T_2(E_1), \tag{6.23}$$

$$\begin{aligned}\big(E_1E_3\big)^n E_2 = q^{2n} E_2 \big(E_1E_3\big)^n - q\{n\}\big(E_1E_3\big)^{n-1}(E_3T_2(E_1) + e_1T_2(E_3)) \\ - q^2\{n\}^2\big(E_1E_3\big)^{n-1} T_{213}(E_2)\end{aligned} \tag{6.24}$$

hold in $U_q(\mathfrak{sl}_4)$.

Lemma 6.12. *For any $n \in \mathbb{N}$, the relations*

$$ {}_2r(T_{213}(E_2)^n) = q^{-2}(q-q^{-1})^2\{n\} E_1E_3 T_{213}(E_2)^{n-1}, \tag{6.25}$$

$$\begin{aligned} {}_2r(T_2(E_3)^nT_2(E_1)^n) = {} & q^{-1}(q-q^{-1})\{n\} E_3 T_2(E_3)^{n-1} T_2(E_1)^n \\ & + q^{-1}(q-q^{-1})\{n\} E_1 T_2(E_3)^n T_2(E_1)^{n-1} \\ & + (q-q^{-1})\{n\}^2 T_{213}(E_2)\big(T_2(E_1)T_2(E_3)\big)^{n-1}\end{aligned} \tag{6.26}$$

hold in $U_q(\mathfrak{sl}_4)$.

Proof of Proposition 6.10. For $\widetilde{w}_0' = \widetilde{\sigma}_2\widetilde{\sigma}_1\widetilde{\sigma}_2\widetilde{\sigma}_1$ we have

$$\mathfrak{X}_{\widetilde{w}_0'} = \mathfrak{X}^{[4]}\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]},$$

where

$$\begin{aligned}
\mathfrak{X}^{[4]} &\overset{(5.38)}{=} \mathfrak{X}_{\tau_0(1)} = \mathfrak{X}_1, \\
\mathfrak{X}^{[3]} &= \Psi \circ \widetilde{T}_2\widetilde{T}_1 \circ \Psi^{-1}(\mathfrak{X}_2) = \sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{2n\}!!}(q^4c_1^2c_2)^n T_{213}(E_2)^{2n}, \\
\mathfrak{X}^{[2]} &= \Psi \circ \widetilde{T}_2 \circ \Psi^{-1}(\mathfrak{X}_1) = \sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{n\}!}(q^2c_1c_2)^n T_2(E_1)^n T_2(E_3)^n, \\
\mathfrak{X}^{[1]} &= \mathfrak{X}_2.
\end{aligned}$$

By Lemma 5.13, property (2.78) of the skew derivative ${}_1r$ and Corollary 2.51 we have ${}_1r(\mathfrak{X}_{\widetilde{w}_0'}) = (q-q^{-1})c_1E_3\mathfrak{X}_{\widetilde{w}_0'}$. We want to show that

$$ {}_2r(\mathfrak{X}_{\widetilde{w}_0'}) = (q-q^{-1})(q^2c_2)E_2\mathfrak{X}_{\widetilde{w}_0'}.$$

For $i = 1, 2, 3, 4$, we let $\mathfrak{X}_{[i]} = K_2 \mathfrak{X}^{[i]} K_2^{-1}$. Note that we have $\mathfrak{X}_{[3]} = \mathfrak{X}^{[3]}$. By the property (2.78) of the skew derivative ${}_2r$, we have

$$ {}_2r(\mathfrak{X}_{\widetilde{w}_0'}) = \mathfrak{X}_{[4]}\,{}_2r(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\,{}_2r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}\,{}_2r(\mathfrak{X}^{[1]}). $$

Using Lemma 6.12, we have

$$ \begin{aligned} {}_2r(\mathfrak{X}^{[3]}) &= \sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{2n\}!!}(q^4c_1^2c_2)^n\,{}_2r(T_{213}(E_2)^{2n}) \\ &\overset{(6.25)}{=} q^{-2}(q-q^{-1})^2 \sum_{n\geq 1} \frac{(q-q^{-1})^n}{\{2n\}!!}(q^4c_1^2c_2)^n\{2n\}E_1E_3T_{213}(E_2)^{2n-1} \\ &= q^{-2}(q-q^{-1})^2 E_1E_3 \sum_{n\geq 0} \frac{(q-q^{-1})^{n+1}}{\{2n\}!!}(q^4c_1^2c_2)^{n+1}T_{213}(E_2)^{2n+1} \\ &= (q-q^{-1})^3(q^2c_1^2c_2)E_1E_3T_{213}(E_2)\mathfrak{X}^{[3]}. \end{aligned} \tag{6.27} $$

Similarly, we have

$$ \begin{aligned} {}_2r(\mathfrak{X}^{[2]}) &= \sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{n\}!}(q^2c_1c_2)^n\,{}_2r(T_2(E_1)^nT_2(E_3)^n) \\ &\overset{(6.26)}{=} q^{-1}(q-q^{-1})^2(q^2c_1c_2)\Big(E_3T_2(E_1)+E_1T_2(E_3)\Big)\mathfrak{X}^{[2]} \\ &\quad + (q-q^{-1})T_{213}(E_2)\sum_{n\geq 1} \frac{(q-q^{-1})^n}{\{n\}!}(q^2c_1c_2)^n\{n\}^2T_2(E_1)^{n-1}T_2(E_3)^{n-1} \\ &= (q-q^{-1})^2(qc_1c_2)\Big(E_3T_2(E_1)+E_1T_2(E_3)\Big)\mathfrak{X}^{[2]} \\ &\quad + (q-q^{-1})^2(q^2c_1c_2)T_{213}(E_2)\sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{n\}!}(q^2c_1c_2)^n\{n+1\}T_2(E_1)^nT_2(E_3)^n. \end{aligned} $$

We want to write the last summand in terms of $\mathfrak{X}^{[2]}$. To do this, we use the fact that $\{n+1\} = 1 + q^2\{n\}$ for $n \geq 1$. This is a useful fact that will be used again in future calculations. Using this, we have

$$ \begin{aligned} &\sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{n\}!}(q^2c_1c_2)^n\{n+1\}T_2(E_1)^nT_2(E_3)^n \\ &\quad = 1 + \sum_{n\geq 1} \frac{(q-q^{-1})^n}{\{n\}!}(q^2c_1c_2)^n(1+q^2\{n\})T_2(E_1)^nT_2(E_3)^n \\ &\quad = \mathfrak{X}^{[2]} + q^2\sum_{n\geq 1} \frac{(q-q^{-1})^n}{\{n-1\}!}(q^2c_1c_2)^nT_2(E_1)^nT_2(E_3)^n \\ &\quad = \mathfrak{X}^{[2]} + q^2\sum_{n\geq 0} \frac{(q-q^{-1})^{n+1}}{\{n\}!}(q^2c_1c_2)^{n+1}T_2(E_1)^{n+1}T_2(E_3)^{n+1} \\ &\quad = \mathfrak{X}^{[2]} + (q-q^{-1})(q^4c_1c_2)T_2(E_1)T_2(E_3)\mathfrak{X}^{[2]}. \end{aligned} $$

Inserting this equation into the expression for ${}_2r(\mathfrak{X}^{[2]})$, we obtain

$$\begin{aligned} {}_2r(\mathfrak{X}^{[2]}) &= (q-q^{-1})^2(qc_1c_2)\Big(E_3T_2(E_1)+E_1T_2(E_3)\Big)\mathfrak{X}^{[2]} \\ &\quad + (q-q^{-1})^2(q^2c_1c_2)T_{213}(E_2)\mathfrak{X}^{[2]} \\ &\qquad + (q-q^{-1})^3(q^6c_1^2c_2^2)T_{213}(E_2)T_2(E_1)T_2(E_3)\mathfrak{X}^{[2]}. \end{aligned} \tag{6.28}$$

Finally, we have

$${}_2r(\mathfrak{X}^{[1]}) \overset{(5.24)}{=} (q-q^{-1})(q^2c_2)E_2\mathfrak{X}^{[1]}. \tag{6.29}$$

Now, we use Lemma 6.11 to rewrite the term $\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_2r(\mathfrak{X}^{[1]})$. By calculations similar to those leading to (6.27) and (6.28), we obtain

$$\mathfrak{X}_{[2]}E_2 = E_2\mathfrak{X}^{[2]}, \tag{6.30}$$

$$\mathfrak{X}_{[3]}E_2 = E_2\mathfrak{X}^{[3]} - (q-q^{-1})^2(q^4c_1^2c_2)\mathfrak{X}_{[3]}T_{213}(E_2)T_2(E_1)T_2(E_3), \tag{6.31}$$

$$\begin{aligned} \mathfrak{X}_{[4]}E_2 &= E_2\mathfrak{X}^{[4]} - q^{-1}(q-q^{-1})c_1\mathfrak{X}_{[4]}\big(E_1T_2(E_3)+E_3T_2(E_1)\big) \\ &\quad - (q-q^{-1})c_1\big(\mathfrak{X}_{[4]} + (q-q^{-1})c_1\mathfrak{X}_{[4]}E_1E_3\big)T_{213}(E_2). \end{aligned} \tag{6.32}$$

We use these to obtain the following.

$$\begin{aligned} &\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_2r(\mathfrak{X}^{[1]}) \\ &\quad \overset{(6.29)}{=} (q-q^{-1})(q^2c_2)\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}E_2\mathfrak{X}^{[1]} \\ &\quad \overset{(6.30)}{=} (q-q^{-1})(q^2c_2)\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}E_2\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\ &\quad \overset{(6.31)}{=} (q-q^{-1})(q^2c_2)\mathfrak{X}_{[4]}E_2\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\ &\qquad - (q-q^{-1})^3(q^6c_1^2c_2^2)\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}T_{213}(E_2)T_2(E_1)T_2(E_3)\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\ &\quad \overset{(6.32)}{=} (q-q^{-1})(q^2c_2)\Big(E_2\mathfrak{X}^{[4]} - q^{-1}(q-q^{-1})c_1\mathfrak{X}_{[4]}\big(E_1T_2(E_3)+E_3T_2(E_1)\big) \\ &\qquad - (q-q^{-1})c_1\big(\mathfrak{X}_{[4]} + (q-q^{-1})c_1\mathfrak{X}_{[4]}E_1E_3\big)T_{213}(E_2)\Big)\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\ &\qquad\quad - (q-q^{-1})^3(q^6c_1^2c_2^2)\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}T_{213}(E_2)T_2(E_1)T_2(E_3)\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}. \end{aligned}$$

We gather terms now and get

$$\begin{aligned} &\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_2r(\mathfrak{X}^{[1]}) \\ &\quad = (q-q^{-1})(q^2c_2)E_2\mathfrak{X}_{\widetilde{w}_0'} \\ &\qquad - (q-q^{-1})^2(qc_1c_2)\mathfrak{X}_{[4]}\big(E_1T_2(E_3)+E_3T_2(E_1)\big)\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\ &\qquad\quad - (q-q^{-1})^2(q^2c_1c_2)\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}T_{213}(E_2)\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\ &\qquad\qquad + (q-q^{-1})^3(q^6c_1^2c_2^2)\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}T_{213}(E_2)T_2(E_1)T_2(E_3)\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\ &\qquad\qquad\quad - (q-q^{-1})^3(q^2c_1^2c_2)\mathfrak{X}_{[4]}E_1E_3T_{213}(E_2)\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\ &\quad = (q-q^{-1})(q^2c_2)E_2\mathfrak{X}_{\widetilde{w}_0'} - \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}{}_2r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]} - \mathfrak{X}_{[4]}{}_2r(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \end{aligned}$$

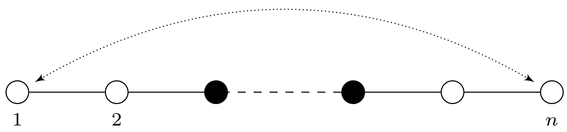
where we use the fact that $E_1T_2(E_3)$ and $E_3T_2(E_1)$ both commute with $T_{213}(E_2)$, and hence with $\mathfrak{X}^{[3]}$. It follows that

$$\begin{aligned} {}_2r(\mathfrak{X}_{\widetilde{w}_0'}) &= \mathfrak{X}_{[4]}\,{}_2r(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\,{}_2r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}\,{}_2r(\mathfrak{X}^{[1]}) \\ &= (q - q^{-1})(q^2 c_2)E_2\mathfrak{X}_{\widetilde{w}_0'} \end{aligned}$$

as required. □

6.4 Type AIII$_n$ for $n \geq 4$

Consider the Satake diagram of type AIII$_n$ for $n \geq 4$.



In this case the restricted root system is of type B$_2$ with

$$\widetilde{\alpha}_1 = \frac{\alpha_1 + \alpha_n}{2}, \qquad \widetilde{\alpha}_2 = \frac{\alpha_2 + \alpha_3 + \cdots + \alpha_{n-1}}{2}.$$

The subgroup $\widetilde{W} \subset W^\Theta$ is generated by the elements

$$\widetilde{\sigma}_1 = \sigma_1\sigma_n, \qquad \widetilde{\sigma}_2 = w_X w_{\{2,n-1\}\cup X} = \sigma_2\sigma_3 \dots \sigma_{n-1} \dots \sigma_3\sigma_2.$$

The longest word of the restricted Weyl group has two reduced expressions given by

$$\widetilde{w}_0 = \widetilde{\sigma}_1\widetilde{\sigma}_2\widetilde{\sigma}_1\widetilde{\sigma}_2, \qquad \widetilde{w}_0' = \widetilde{\sigma}_2\widetilde{\sigma}_1\widetilde{\sigma}_2\widetilde{\sigma}_1.$$

The definition (3.33) and condition (5.4) imply that $c_1 = c_n = \overline{c_1}$. By Lemmas 5.13 and 5.14 we have

$$\begin{aligned} \mathfrak{X}_1 &= \sum_{k\geq 0} \frac{(q-q^{-1})^k}{\{k\}!} c_1^k (E_1E_n)^k, \\ \mathfrak{X}_2 &= \left(\sum_{k\geq 0} \frac{(c_2 s(n-1))^k}{\{k\}!} T_2T_{w_X}(E_{n-1})^k\right)\left(\sum_{k\geq 0} \frac{(c_{n-1}s(2))^k}{\{k\}!} T_{n-1}T_{w_X}(E_2)^k\right). \end{aligned}$$

Proposition 6.13. *The partial quasi K-matrix $\mathfrak{X}_{\widetilde{w}_0}$ coincides with the quasi K-matrix $\mathfrak{X}$.*

We have the following relations needed in the proof of Proposition 6.13, proved by induction.

Lemma 6.14. *For any $k \in \mathbb{N}$ the relations*

$$\begin{aligned} &T_{1n(n-1)}T_{w_X}(E_2)^k E_n = q^{-k} E_n T_{1n(n-1)}T_{w_X}(E_2)^k, \\ &T_{12n}T_{w_X}(E_{n-1})^k E_n = q^{-k} E_n T_{12n}T_{w_X}(E_{n-1})^k, \\ &T_{n--3}(E_2)^k E_n = q^{-k} E_n T_{n--3}(E_2)^k, \\ &T_{1--(n-2)}(E_{n-1})^k E_n = q^k E_n T_{1--(n-2)}(E_{n-1})^k \end{aligned}$$

$$
\begin{aligned}
&\quad - q\{k\}T_{1--(n-2)}(E_{n-1})^{k-1}T_{12n}T_{w_X}(E_{n-1}),\\
&T_{n-1}T_{w_X}(E_2)^k E_n = q^k E_n T_{n-1}T_{w_X}(E_2)^k\\
&\quad - q\{k\}T_{n-1}T_{w_X}(E_2)^{k-1}T_{n--3}(E_2),\\
&T_2T_{w_X}(E_{n-1})^k E_n = q^k E_n T_2T_{w_X}(E_{n-1})^k\\
&\quad - q\{k\}T_2T_{w_X}(E_{n-1})^{k-1}T_{2n}T_{w_X}(E_{n-1}),\\
&T_{2n}T_{w_X}(E_{n-1})T_{n-1}T_{w_X}(E_2)^k = q^{-k}T_{n-1}T_{w_X}(E_2)^k T_{2n}T_{w_X}(E_{n-1}) \qquad (6.33)\\
&\quad + q^{1-k}(q-q^{-1})\{k\}T_{n-1}T_{w_X}(E_2)^{k-1}T_2T_{w_X}(E_{n-1})T_{n--3}(E_2),\\
&T_{12n}T_{w_X}(E_{n-1})^k T_{n--3}(E_2) \qquad (6.34)\\
&\quad = \big(q^{-k} + (q-q^{-1})q^{1-k}\{k\}\big)T_{n--3}(E_2)T_{12n}T_{w_X}(E_{n-1})^k
\end{aligned}
$$

hold in $U_q(\mathfrak{sl}_{n+1})$.

Lemma 6.15. *For any* $k \in \mathbb{N}$ *the relations*

$$
\begin{aligned}
&{}_1r(T_{1--(n-2)}(E_{n-1})^k) \qquad (6.35)\\
&\quad = q^{-1}(q-q^{-1})\{k\}T_{2--(n-2)}(E_{n-1})T_{1--(n-2)}(E_{n-1})^{k-1},\\
&{}_1r(T_{12n}T_{w_X}(E_{n-1})^k) \qquad (6.36)\\
&\quad = q^{-1}(q-q^{-1})\{k\}T_{2n}T_{w_X}(E_{n-1})T_{12n}T_{w_X}(E_{n-1})^{k-1},\\
&{}_1r(T_{1n(n-1)}T_{w_X}(E_2)^k) \qquad (6.37)\\
&\quad = q^{-1}(q-q^{-1})\{k\}T_{n(n-1)}T_{w_X}(E_2)T_{1n(n-1)}T_{w_X}(E_2)^{k-1}
\end{aligned}
$$

hold in $U_q(\mathfrak{sl}_{n+1})$.

One can show that the above Lemmas still hold if we consider the case where we have no black dots. In this situation, the calculations differ slightly but the results still hold.

Proof of Proposition 6.13. We write

$$\mathfrak{X}_{\widetilde{w}_0} = \mathfrak{X}^{[4]}\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}, \qquad (6.38)$$

where, recalling the notation $\widetilde{c}_2^2 = c_2c_{n-1}s(n-1)s(2)$, we have

$$
\begin{aligned}
\mathfrak{X}^{[4]} &\overset{(5.38)}{=} \mathfrak{X}_2,\\
\mathfrak{X}^{[3]} &= \sum_{k\geq 0}\frac{(q-q^{-1})^k}{\{k\}!}(qc_1\widetilde{c}_2^2)^k T_{n--3}(E_2)^k T_{1--(n-2)}(E_{n-1})^k,\\
\mathfrak{X}^{[2]} &= \left(\sum_{k\geq 0}\frac{(qc_1c_2s(n-1))^k}{\{k\}!}T_{12n}T_{w_X}(E_{n-1})^k\right)\\
&\qquad\qquad \left(\sum_{k\geq 0}\frac{(qc_1c_{n-1}s(2))^k}{\{k\}!}T_{1n(n-1)}T_{w_X}(E_2)^k\right),\\
\mathfrak{X}^{[1]} &= \mathfrak{X}_1.
\end{aligned}
$$

When necessary, since $\mathfrak{X}^{[4]}$ and $\mathfrak{X}^{[2]}$ are a product of two infinite sums, we write $\mathfrak{X}^{[4]} = \mathfrak{X}^{[4;1]}\mathfrak{X}^{[4;2]}$ and $\mathfrak{X}^{[2]} = \mathfrak{X}^{[2;1]}\mathfrak{X}^{[2;2]}$.

For each $i = 1, 2, 3, 4$, let $\mathfrak{X}_{[i]} = K_1\mathfrak{X}^{[i]}K_1^{-1}$. We write $\mathfrak{X}_{[4]} = \mathfrak{X}_{[4;1]}\mathfrak{X}_{[4;2]}$ and $\mathfrak{X}_{[2]} = \mathfrak{X}_{[2;1]}\mathfrak{X}_{[2;2]}$. Now, by the rank one case for $\mathfrak{X}_2$ given in Lemma 5.14 and Corollary 2.51, we obtain

$$ {}_2r(\mathfrak{X}_{\widetilde{w}_0}) = (q - q^{-1})c_2 s(n-1)T_{w_X}(E_{n-1})\mathfrak{X}_{\widetilde{w}_0}. \tag{6.39} $$

The underlying symmetry implies that we only need to show that

$$ {}_1r(\mathfrak{X}_{\widetilde{w}_0}) = (q - q^{-1})c_1 E_n \mathfrak{X}_{\widetilde{w}_0}. $$

By Property (2.78) of the skew derivative ${}_1r$ and Corollary 2.51 we have

$$ {}_1r(\mathfrak{X}_{\widetilde{w}_0}) = \mathfrak{X}_{[4]}{}_1r(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}{}_1r(\mathfrak{X})^{[2]}\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_1r(\mathfrak{X}^{[1]}). \tag{6.40} $$

Using Lemma 6.15 and Lemma 5.13 we find that

$$ {}_1r(\mathfrak{X}^{[3]}) \overset{(6.35)}{=} (q - q^{-1})^2 c_1 \widetilde{c}_2^{\,2} T_{n--3}(E_2)T_{2--(n-2)}(E_{n-1})\mathfrak{X}^{[3]}, \tag{6.41} $$

$$ {}_1r(\mathfrak{X}^{[1]}) \overset{(5.24)}{=} (q - q^{-1})c_1 E_n \mathfrak{X}^{[1]}. \tag{6.42} $$

To write an expression for ${}_1r(\mathfrak{X}^{[2]})$, we use the splitting of $\mathfrak{X}^{[2]}$ into a product of two infinite sums. By equations (6.36) and (6.37) of Lemma 6.15 we have

$$ \begin{aligned} {}_1r(\mathfrak{X}^{[2]}) &= {}_1r(\mathfrak{X}^{[2;1]})\mathfrak{X}^{[2;2]} + \mathfrak{X}_{[2;1]}{}_1r(\mathfrak{X}^{[2;2]}) \\ &= (q - q^{-1})c_1c_2 s(n-1)T_{2n}T_{w_X}(E_{n-1})\mathfrak{X}^{[2]} \\ &\qquad + (q - q^{-1})c_1c_{n-1}s(2)\mathfrak{X}_{[2;1]}T_{n(n-1)}T_{w_X}(E_2)\mathfrak{X}^{[2;2]}. \end{aligned} \tag{6.43} $$

We would like the last summand of this expression to be in terms of $\mathfrak{X}^{[2]}$. To this end, we use Equation (6.34) in Lemma 6.14 to obtain

$$ \begin{aligned} &\mathfrak{X}_{[2;1]}T_{n(n-1)}T_{w_X}(E_2) \\ &= \left(1 + \sum_{k\geq 1} \frac{(qc_1c_2s(n-1))^k}{\{k\}!} q^k T_{12n}T_{w_X}(E_{n-1})^k\right) T_{n(n-1)}T_{w_X}(E_2) \\ &\overset{(6.34)}{=} T_{n(n-1)}T_{w_X}(E_2)\mathfrak{X}^{[2;1]} \\ &\qquad + q^2(q - q^{-1})c_1c_2s(n-1)T_{n(n-1)}T_{w_X}(E_2)T_{12n}T_{w_X}(E_{n-1})\mathfrak{X}^{[2;1]}. \end{aligned} $$

Substituting this into (6.43) it follows that

$$ \begin{aligned} {}_1r(\mathfrak{X}^{[2]}) &= (q - q^{-1})c_1c_2s(n-1)T_{2n}T_{w_X}(E_{n-1})\mathfrak{X}^{[2]} \\ &\quad + (q - q^{-1})c_1c_{n-1}s(2)T_{n(n-1)}T_{w_X}(E_2)\mathfrak{X}^{[2]} \\ &\qquad + q^2(q - q^{-1})^2 c_1^2\widetilde{c}_2^{\,2} T_{n(n-1)}T_{w_X}(E_2)T_{12n}T_{w_X}(E_{n-1})\mathfrak{X}^{[2]}. \end{aligned} \tag{6.44} $$

When we calculate ${}_1r(\mathfrak{X}_{\widetilde{w}_0})$, we obtain a component of the form $\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_1r(\mathfrak{X}^{[1]})$. From Equation (6.42), we see that we obtain an E_n term that we want to pass to the front of this expression. We use Lemma 6.14 to do this. We have

$$\mathfrak{X}_{[2]}E_n = E_n\mathfrak{X}^{[2]}, \tag{6.45}$$

$$\mathfrak{X}_{[3]}E_n = E_n\mathfrak{X}_{[3]} - q^2(q-q^{-1})c_1\widetilde{c}_2^{\,2}T_{n--3}(E_2)\mathfrak{X}^{[3]}T_{12n}T_{w_X}(E_{n-1}), \tag{6.46}$$

$$\mathfrak{X}_{[4;2]}E_n = E_n\mathfrak{X}^{[4;2]} - c_1c_{n-1}s(2)\mathfrak{X}_{[4;2]}T_{n--3}(E_2), \tag{6.47}$$

$$\mathfrak{X}_{[4;1]}E_n = E_n\mathfrak{X}^{[4;1]} - c_1c_2s(n-1)\mathfrak{X}_{[4;1]}T_{2n}T_{w_X}(E_{n-1}). \tag{6.48}$$

Now, using Equation (6.33), we obtain the following expression for $\mathfrak{X}_{[4]}E_n$.

$$\begin{aligned}
\mathfrak{X}_{[4]}E_n &\overset{(6.47)}{=} \mathfrak{X}_{[4;1]}\big(E_n\mathfrak{X}^{[4;2]} - c_1c_{n-1}s(2)\mathfrak{X}_{[4;2]}T_{n--3}(E_2)\big)\\
&\overset{(6.48)}{=} \big(E_n\mathfrak{X}^{[4;1]} - c_1c_2s(n-1)\mathfrak{X}_{[4;1]}T_{2n}T_{w_X}(E_{n-1})\big)\mathfrak{X}^{[4;2]}\\
&\quad - c_1c_{n-1}s(2)\mathfrak{X}_{[4]}T_{n--3}(E_2)\\
&\overset{(6.33)}{=} E_n\mathfrak{X}^{[4]} - c_1c_{n-1}s(2)\mathfrak{X}_{[4]}T_{n--3}(E_2)\\
&\quad - c_1c_2s(n-1)\mathfrak{X}_{[4;1]}\mathfrak{X}_{[4;2]}T_{2n}T_{w_X}(E_{n-1})\\
&\qquad + (q-q^{-1})c_1^2c_2c_{n-1}s(2)s(n-1)\mathfrak{X}_{[4;1]}\mathfrak{X}_{[4;2]}T_2T_{w_X}(E_{n-1})T_{n--3}(E_2)\\
&= E_n\mathfrak{X}^{[4]} - c_1c_{n-1}s(2)\mathfrak{X}_{[4]}T_{n--3}(E_2) - c_1c_2s(n-1)\mathfrak{X}_{[4]}T_{2n}T_{w_X}(E_{n-1}) \qquad (6.49)\\
&\quad - (q-q^{-1})c_1^2\widetilde{c}_2^{\,2}\mathfrak{X}_{[4]}T_2T_{w_X}(E_{n-1})T_{n--3}(E_2).
\end{aligned}$$

Using equations (6.45), (6.46) and (6.49), and comparing with equations (6.41), (6.42) and (6.44), one finds that

$$\begin{aligned}
\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}\,{}_1r(\mathfrak{X}^{[1]}) &= (q-q^{-1})\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}E_n\mathfrak{X}^{[1]}\\
&= (q-q^{-1})E_n\mathfrak{X}_{\widetilde{w}_0} - \mathfrak{X}_{[4]}\,{}_1r(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} - \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\,{}_1r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]},
\end{aligned}$$

and hence it follows that

$${}_1r(\mathfrak{X}_{\widetilde{w}_0}) = (q-q^{-1})E_n\mathfrak{X}_{\widetilde{w}_0},$$

as required. This completes the proof. □

Proposition 6.16. *The partial quasi K-matrix $\mathfrak{X}_{\widetilde{w}_0'}$ coincides with the quasi K-matrix $\mathfrak{X}$.*

We have the following relations needed in the proof of Proposition 6.16, proved by induction.

Lemma 6.17. *For any $k\in\mathbb{N}$ the relations*

$$\begin{aligned}
&T_{2--(n-1)}(E_n)^kT_{w_X}(E_{n-1}) = T_{w_X}(E_{n-1})T_{2--(n-1)}(E_n)^k,\\
&T_{(n-1)--2}(E_1)^kT_{w_X}(E_{n-1}) = q^{-k}T_{w_X}(E_{n-1})T_{(n-1)--2}(E_1)^k,\\
&T_{2--(n-1)}T_{12n}T_{w_X}(E_{n-1})^kT_{w_X}(E_{n-1}) = T_{w_X}(E_{n-1})T_{2--(n-1)}T_{12n}T_{w_X}(E_{n-1})^k,\\
&T_{(n-1)--2}T_{1n(n-1)}T_{w_X}(E_2)^kT_{w_X}(E_{n-1}) = T_{w_X}(E_{n-1})T_{(n-1)--2}T_{1n(n-1)}T_{w_X}(E_2)^k\\
&\qquad - (q-q^{-1})\{k\}T_{(n-1)--2}T_{1n(n-1)}T_{w_X}(E_2)^{k-1}T_{3--(n-1)}(E_n)T_{(n-1)--2}(E_1),
\end{aligned}$$

$$E_1^k T_{w_X}(E_{n-1}) = T_{w_X}(E_{n-1})E_1^k,$$
$$E_n^k T_{w_X}(E_{n-1}) = q^k T_{w_X}(E_{n-1})E_n^k - q\{k\}E_n^{k-1}T_{w_X}T_{n-1}(E_n)$$

hold in $U_q(\mathfrak{sl}_{n+1})$.

Lemma 6.18. *For any* $k \in \mathbb{N}$ *the relations*

$$ {}_2r(T_{2--(n-1)}(E_n)^k) = q^{-1}(q-q^{-1})\{k\}T_{3--(n-1)}(E_n)T_{2--(n-1)}(E_n)^{k-1},$$
$$ {}_2r(T_{2--(n-1)}T_{12n}T_{w_X}(E_{n-1})^k)$$
$$ = q^{-2}(q-q^{-1})^2\{k\}E_1T_{3--(n-1)}(E_n)T_{2--(n-1)}T_{12n}T_{w_X}(E_{n-1})^{k-1}$$

hold in $U_q(\mathfrak{sl}_{n+1})$.

Proof of Proposition 6.16. We write

$$\mathfrak{X}_{\widetilde{w}_0'} = \mathfrak{X}^{[4]}\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]},$$

where we have

$$\mathfrak{X}^{[4]} \overset{(5.38)}{=} \mathfrak{X}_1,$$
$$\mathfrak{X}^{[3]} = \Big(\sum_{k_1\geq 0}\frac{(qc_1c_2s(n-1))^{k_1}}{\{k_1\}!}T_{2--(n-1)}T_{12n}T_{w_X}(E_{n-1})^{k_1}\Big)$$
$$\Big(\sum_{k_2\geq 0}\frac{(qc_1c_{n-1}s(2))^{k_2}}{\{k_2\}!}T_{(n-1)--2}T_{1n(n-1)}T_{w_X}(E_2)^{k_2}\Big),$$
$$\mathfrak{X}^{[2]} = \sum_{k\geq 0}\frac{(q-q^{-1})^k}{\{k\}!}(qc_1\widetilde{c}_2^2)^kT_{(n-1)--2}(E_1)^kT_{2--(n-1)}(E_n)^k,$$
$$\mathfrak{X}^{[1]} = \mathfrak{X}_2.$$

For $i=1,2,3,4$ let $\mathfrak{X}_{[i]} = K_2\mathfrak{X}^{[i]}K_2^{-1}$. By Lemma 5.13 and Corollary 2.51 it follows that

$$ {}_1r(\mathfrak{X}_{\widetilde{w}_0'}) = {}_1r(\mathfrak{X}^{[4]})\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}$$
$$ = (q-q^{-1})E_n\mathfrak{X}_{\widetilde{w}_0'}.$$

Hence, we only need to check that

$$ {}_2r(\mathfrak{X}_{\widetilde{w}_0'}) = q^{-1}(q-q^{-1})c_2s(n-1)T_{w_X}(E_{n-1})\mathfrak{X}_{\widetilde{w}_0'}.$$

By Corollary 2.51 and property (2.78) of the skew derivative ${}_2r$ we have

$$ {}_2r(\mathfrak{X}_{\widetilde{w}_0'}) = \mathfrak{X}_{[4]}\,{}_2r(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\,{}_2r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}\,{}_2r(\mathfrak{X}^{[1]}).$$

Using Lemmas 5.14 and 6.18, we have

$$ {}_2r(\mathfrak{X}^{[3]}) = q^{-1}(q-q^{-1})^2c_1c_2s(n-1)E_1T_{3--(n-1)}(E_n)\mathfrak{X}^{[3]},$$
$$ {}_2r(\mathfrak{X}^{[2]}) = (q-q^{-1})^2c_1\widetilde{c}_2^2T_{3--(n-1)}(E_n)T_{(n-1)--2}(E_1)\mathfrak{X}^{[2]},$$
$$ {}_2r(\mathfrak{X}^{[1]}) = q^{-1}(q-q^{-1})c_2s(n-1)T_{w_X}(E_{n-1})\mathfrak{X}^{[1]}.$$

By Lemma 6.17, we have

$$
\begin{aligned}
\mathfrak{X}_{[2]}T_{w_X}(E_{n-1}) &= T_{w_X}(E_{n-1})\mathfrak{X}^{[2]},\\
\mathfrak{X}_{[3]}T_{w_X}(E_{n-1}) &= T_{w_X}(E_{n-1})\mathfrak{X}^{[3]}\\
&\qquad - q(q-q^{-1})c_1c_{n-1}s(2)\mathfrak{X}_{[3]}T_{3--(n-1)}(E_n)T_{(n-1)--2}(E_1),\\
\mathfrak{X}_{[4]}T_{w_X}(E_{n-1}) &= T_{w_X}(E_{n-1})\mathfrak{X}^{[4]} - (q-q^{-1})c_1\mathfrak{X}_{[4]}E_1T_{3--(n-1)}(E_n).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}\,{}_2r(\mathfrak{X}^{[1]}) &= q^{-1}(q-q^{-1})c_2s(n-1)T_{w_X}(E_{n-1})\mathfrak{X}_{w'}\\
&\quad - \mathfrak{X}_{[4]}\,{}_2r(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} - \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\,{}_2r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]}
\end{aligned}
$$

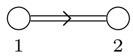
and therefore we obtain

$$
{}_2r(\mathfrak{X}_{\widetilde{w}_0'}) = q^{-1}(q-q^{-1})c_2s(n-1)T_{w_X}(E_{n-1})\mathfrak{X}_{\widetilde{w}_0'}
$$

as required. □

6.5 Type CI_2

Consider the Satake diagram of type CI_2.



Since $\Theta = -\mathrm{id}$ the subgroup $\widetilde{W}$ coincides with W. The longest word of the Weyl group has two reduced expressions given by

$$
w_0 = \sigma_1\sigma_2\sigma_1\sigma_2, \qquad w_0' = \sigma_2\sigma_1\sigma_2\sigma_1.
$$

By Lemma 5.11 we have

$$
\begin{aligned}
\mathfrak{X}_1 &= \sum_{n\geq 0}\frac{(q^2-q^{-2})^n}{\{2n\}_1!!}(q^4c_1)^nE_1^{2n},\\
\mathfrak{X}_2 &= \sum_{n\geq 0}\frac{(q-q^{-1})^n}{\{2n\}_2!!}(q^2c_2)^nE_2^{2n}.
\end{aligned}
$$

Proposition 6.19. *The partial quasi K-matrix $\mathfrak{X}_{w_0}$ coincides with the quasi K-matrix $\mathfrak{X}$.*

The following relations are necessary for the proof of Proposition 6.19, proved by induction.

Lemma 6.20. *For any $n \in \mathbb{N}$ the relations*

$$
E_2^nE_1 = q^{2n}E_1E_2^n - q^2\{n\}_2E_2^{n-1}T_1(E_2) - q^3\{n\}_2\{n-1\}_2E_2^{n-1}T_{12}(E_1), \tag{6.50}
$$

$$
T_1(E_2)^nE_1 = q^{-2n}E_1T_1(E_2)^n, \tag{6.51}
$$

$$T_{12}(E_1)^n E_1 = E_1 T_{12}(E_1)^n - \frac{(q^2-1)}{[2]_2}\{n\}_1 T_{12}(E_1)^{n-1} T_1(E_2)^2 \tag{6.52}$$

hold in $U_q(\mathfrak{so}_5)$.

Lemma 6.21. *For any $n \in \mathbb{N}$ the relations*

$$\begin{aligned} {}_1r(T_1(E_2)^{n+1}) &= q^{-2}(q^2-q^{-2})\{n+1\}_2 E_2 T_1(E_2)^n \\ &\quad + q^{-1}(q^2-q^{-2})\{n+1\}_2\{n\}_2 T_{12}(E_1)T_1(E_2)^{n-1}, \end{aligned} \tag{6.53}$$

$$ {}_1r(T_{12}(E_1)^n) = q^{-3}(q-q^{-1})^2\{n\}_1 E_2^2 T_{12}(E_1)^{n-1} \tag{6.54}$$

hold in $U_q(\mathfrak{so}_5)$.

Proof of Proposition 6.19. We have

$$\mathfrak{X}_{w_0} = \mathfrak{X}^{[4]}\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}$$

where

$$\begin{aligned} \mathfrak{X}^{[4]} &\overset{(5.38)}{=} \mathfrak{X}_2, \\ \mathfrak{X}^{[3]} &= \Psi \circ T_{12} \circ \Psi^{-1}(\mathfrak{X}_1) = \sum_{n\geq 0} \frac{(q^2-q^{-2})^n}{\{2n\}_1!!}(q^4c_1)^n(q^2c_2)^{2n} T_{12}(E_1)^{2n}, \\ \mathfrak{X}^{[2]} &= \Psi \circ T_1 \circ \Psi^{-1}(\mathfrak{X}_2) = \sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{2n\}_2!!}(q^4c_1)^n(q^2c_2)^n T_1(E_2)^{2n}, \\ \mathfrak{X}^{[1]} &= \mathfrak{X}_1. \end{aligned}$$

By Lemma 5.11 and Corollary 2.51 we have

$$\begin{aligned} {}_2r(\mathfrak{X}_{w_0}) &= {}_2r(\mathfrak{X}_2)\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\ &= (q-q^{-1})(q^2c_2)E_2\mathfrak{X}_2\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} \\ &= (q-q^{-1})(q^2c_2)E_2\mathfrak{X}_{w_0}. \end{aligned}$$

We want to show that

$${}_1r(\mathfrak{X}_{w_0}) = (q^2-q^{-2})(q^4c_1)E_1\mathfrak{X}_{w_0}.$$

For each $i=1,2,3,4$ let $\mathfrak{X}_{[i]} = K_1\mathfrak{X}^{[i]}K_1^{-1}$. Note that $\mathfrak{X}_{[3]} = \mathfrak{X}^{[3]}$. By property (2.78) of the skew derivative ${}_1r$ we see that

$${}_1r(\mathfrak{X}_{w_0}) = \mathfrak{X}_{[4]}\,{}_1r(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\,{}_1r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}\,{}_1r(\mathfrak{X}^{[1]}). \tag{6.55}$$

Lemma 6.21 gives

$${}_1r(\mathfrak{X}^{[3]}) = q^{-3}(q-q^{-1})^2(q^2-q^{-2})(q^4c_1)(q^2c_2)^2E_2^2T_{12}(E_1)\mathfrak{X}^{[3]}, \tag{6.56}$$

$${}_1r(\mathfrak{X}^{[1]}) = (q^2-q^{-2})(q^4c_1)E_1\mathfrak{X}^{[1]}. \tag{6.57}$$

By Equation (6.53) we have

$$\begin{aligned}{}_1r(\mathfrak{X}^{[2]}) &= q^{-2}(q-q^{-1})(q^2-q^{-2})(q^4c_1)(q^2c_2)E_2T_1(E_2)\mathfrak{X}^{[2]}\\ &\quad + q^{-1}(q-q^{-1})(q^2-q^{-2})(q^6c_1c_2)T_{12}(E_1)\widehat{\mathfrak{X}}^{[2]}\end{aligned} \tag{6.58}$$

where

$$\widehat{\mathfrak{X}}^{[2]} = \sum_{n\geq 0}\frac{(q-q^{-1})^n}{\{2n\}_2!!}(q^6c_1c_2)^n\{2n+1\}_2T_1(E_2)^{2n}.$$

For any $n\geq 1$ we have

$$\{2n+1\}_2 = 1+q^2\{2n\}_2. \tag{6.59}$$

It hence follows that

$$\begin{aligned}&\sum_{n\geq 0}\frac{(q-q^{-1})^n}{\{2n\}_2!!}(q^4c_1)^n(q^2c_2)^n\{2n+1\}_2T_1(E_2)^{2n}\\ &\qquad\qquad = \mathfrak{X}^{[2]} + q^2(q-q^{-1})(q^4c_1)(q^2c_2)T_1(E_2)^2\mathfrak{X}^{[2]}.\end{aligned}$$

Substituting this into Equation (6.58) we see that

$$\begin{aligned}{}_1r(\mathfrak{X}^{[2]}) &= q^{-2}(q-q^{-1})(q^2-q^{-2})(q^4c_1)(q^2c_2)E_2T_1(E_2)\mathfrak{X}^{[2]}\\ &\quad + q^{-1}(q-q^{-1})(q^2-q^{-2})(q^4c_1)(q^2c_2)T_{12}(E_1)\mathfrak{X}^{[2]}\\ &\quad + q(q-q^{-1})^2(q^2-q^{-2})(q^4c_1)^2(q^2c_2)^2T_{12}(E_1)T_1(E_2)^2\mathfrak{X}^{[2]}\end{aligned} \tag{6.60}$$

When we calculate ${}_1r(\mathfrak{X}_{w_0})$, we obtain a component of the form $\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_1r(\mathfrak{X}^{[1]})$. From Equation (6.57), this contains an E_1 term that we pass to the front using Lemma 6.20. We have

$$\mathfrak{X}_{[2]}E_1 \overset{(6.51)}{=} E_1\mathfrak{X}^{[2]}, \tag{6.61}$$

$$\mathfrak{X}_{[3]}E_1 \overset{(6.52)}{=} E_1\mathfrak{X}^{[3]} - q(q-q^{-1})^2(q^4c_1)(q^2c_2)^2\mathfrak{X}_{[3]}T_{12}(E_1)T_1(E_2)^2, \tag{6.62}$$

$$\begin{aligned}\mathfrak{X}_{[4]}E_1 &\overset{(6.50)}{=} E_1\mathfrak{X}^{[4]} - q^{-2}(q-q^{-1})(q^2c_2)E_2\mathfrak{X}_{[4]}T_1(E_2)\\ &\qquad - q^{-1}(q-q^{-1})(q^2c_2)\mathfrak{X}_{[4]}T_{12}(E_1)\\ &\qquad\quad - q^{-3}(q-q^{-1})^2(q^2c_2)^2\mathfrak{X}_{[4]}E_2^2T_{12}(E_1)\end{aligned} \tag{6.63}$$

where we also use (6.59) to obtain (6.63). Note that $\mathfrak{X}^{[3]}$ commutes with $E_2T_1(E_2)$. Using (6.61), (6.62) and (6.63), and comparing with (6.56) and (6.60) we obtain

$$\begin{aligned}&\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_1r(\mathfrak{X}^{[1]})\\ &\qquad = (q^2-q^{-2})(q^4c_1)E_1\mathfrak{X}_{w_0} - \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}{}_1r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]} - \mathfrak{X}_{[4]}{}_1r(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}.\end{aligned}$$

Hence by (6.55) we have

$${}_1r(\mathfrak{X}_{w_0}) = (q^2-q^{-2})(q^4c_1)E_1\mathfrak{X}_{w_0}$$

as required. □

We now consider the reduced expression $w_0' = \sigma_2\sigma_1\sigma_2\sigma_1$.

Proposition 6.22. *The partial quasi K-matrix $\mathfrak{X}_{w_0'}$ coincides with the quasi K-matrix $\mathfrak{X}$.*

The following relations are needed for the proof and are obtained by induction.

Lemma 6.23. *For any $n \in \mathbb{N}$ the relations*

$$E_1^n E_2 = q^{2n} E_2 E_1^n - q^2 \{n\}_1 E_1^{n-1} T_{21}(E_2), \tag{6.64}$$

$$T_2(E_1)^n E_2 = q^{-2n} E_2 T_2(E_1)^n, \tag{6.65}$$

$$T_{21}(E_2)^n E_2 = E_2 T_{21}(E_2)^n - [2]_2 \{n\}_2 T_{21}(E_2)^{n-1} T_2(E_1) \tag{6.66}$$

hold in $U_q(\mathfrak{so}_5)$.

Lemma 6.24. *For any $n \in \mathbb{N}$ the relations*

$${}_2r(T_2(E_1)^n) = (q - q^{-1})\{n\}_1 T_{21}(E_2) T_2(E_1)^{n-1}, \tag{6.67}$$

$${}_2r(T_{21}(E_2)^n) = q^{-2}(q^2 - q^{-2})\{n\}_2 E_1 T_{21}(E_2)^{n-1} \tag{6.68}$$

hold in $U_q(\mathfrak{so}_5)$.

Proof of Proposition 6.22. We have

$$\mathfrak{X}_{w_0'} = \mathfrak{X}^{[4]}\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}$$

where

$$\begin{aligned}
\mathfrak{X}^{[4]} &\overset{(5.38)}{=} \mathfrak{X}_1, \\
\mathfrak{X}^{[3]} &= \sum_{n\geq 0} \frac{(q-q^{-1})^n}{\{2n\}_2!!} (q^4 c_1)^n (q^2 c_2)^n T_{21}(E_2)^{2n}, \\
\mathfrak{X}^{[2]} &= \sum_{n\geq 0} \frac{(q^2-q^{-2})^n}{\{2n\}_1!!} (q^4 c_1)^n (q^2 c_2)^{2n} T_2(E_1)^{2n}, \\
\mathfrak{X}^{[1]} &= \mathfrak{X}_2.
\end{aligned}$$

By Lemma 5.11 and Corollary 2.51 we have ${}_1r(\mathfrak{X}_{w_0'}) = (q^2 - q^{-2})(q^4 c_1) E_1 \mathfrak{X}_{w_0'}$. We want to show that

$${}_2r(\mathfrak{X}_{w_0'}) = (q - q^{-1})(q^2 c_2) E_2 \mathfrak{X}_{w_0'}.$$

For each $i = 1, 2, 3, 4$ let $\mathfrak{X}_{[i]} = K_2 \mathfrak{X}^{[i]} K_2^{-1}$. By property (2.78) of the skew derivative ${}_2r$ we have

$${}_2r(\mathfrak{X}_{w_0'}) = \mathfrak{X}_{[4]}\,{}_2r(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\,{}_2r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]} + \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}\,{}_2r(\mathfrak{X}^{[1]}). \tag{6.69}$$

Using Lemma 6.24 we obtain

$${}_2r(\mathfrak{X}^{[3]}) \overset{(6.68)}{=} q^{-2}(q - q^{-1})(q^2 - q^{-2})(q^4 c_1)(q^2 c_2) E_1 T_{21}(E_2)\mathfrak{X}^{[3]}, \tag{6.70}$$

$${}_2r(\mathfrak{X}^{[2]}) \overset{(6.67)}{=} (q - q^{-1})(q^2 - q^{-2})(q^4 c_1)(q^2 c_2)^2 T_{21}(E_2) T_2(E_1)\mathfrak{X}^{[2]}, \tag{6.71}$$

$$ {}_2r(\mathfrak{X}^{[1]}) \overset{(5.22)}{=} (q-q^{-1})(q^2c_2)E_2\mathfrak{X}^{[1]}. \tag{6.72}$$

By Lemma 6.23 we have

$$\mathfrak{X}_{[2]}E_2 \overset{(6.65)}{=} E_2\mathfrak{X}^{[2]}, \tag{6.73}$$

$$\mathfrak{X}_{[3]}E_2 \overset{(6.66)}{=} E_2\mathfrak{X}^{[3]} - (q^2-q^{-2})(q^4c_1)(q^2c_2)\mathfrak{X}_{[3]}T_{21}(E_2)T_2(E_1), \tag{6.74}$$

$$\mathfrak{X}_{[4]}E_2 \overset{(6.64)}{=} E_2\mathfrak{X}^{[4]} - q^{-2}(q^2-q^{-2})(q^4c_1)\mathfrak{X}_{[4]}E_1T_{21}(E_2). \tag{6.75}$$

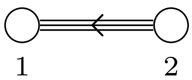
Using equations (6.73), (6.74) and (6.75), and comparing with equations (6.70) and (6.71) we can rewrite the term $\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_2r(\mathfrak{X}^{[1]})$ as

$$\begin{aligned}
&\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}{}_2r(\mathfrak{X}^{[1]})\\
&\quad\overset{(6.72)}{=} (q-q^{-1})(q^2c_2)\mathfrak{X}_{[4]}\mathfrak{X}_{[3]}\mathfrak{X}_{[2]}E_2\mathfrak{X}^{[1]}\\
&\quad= (q-q^{-1})(q^2c_2)E_2\mathfrak{X}_{w_0'} - \mathfrak{X}_{[4]}{}_2r(\mathfrak{X}^{[3]})\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} - \mathfrak{X}_{[4]}\mathfrak{X}_{[3]}{}_2r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]}.
\end{aligned}$$

It hence follows from (6.69) that ${}_2r(\mathfrak{X}_{w_0'}) = (q-q^{-1})(q^2c_2)E_2\mathfrak{X}_{w_0'}$ as required. □

6.6 Type G_2

Consider the Satake diagram of type G_2.



Here the subgroup $\widetilde{W}$ coincides with W and the longest element of the Weyl group has two reduced expressions given by

$$w_0 = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2, \quad w_0' = \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1.$$

We introduce the following terms in order to reduce the length of the proceeding calculations. In particular let

$$\begin{aligned}
Q(n) &= q^n - q^{-n} \quad \text{for } n\in\mathbb{N},\\
C_1 &= q^2c_1,\\
C_2 &= q^6c_2.
\end{aligned}$$

Further, we write $\mathfrak{X}^{[i]}\mathfrak{X}^{[j]} = \mathfrak{X}^{[ij]}$. By Lemma 5.11 we have

$$\begin{aligned}
\mathfrak{X}_1 &= \sum_{n\geq 0}\frac{Q(1)^n}{\{2n\}_1!!}C_1^nE_1^{2n},\\
\mathfrak{X}_2 &= \sum_{n\geq 0}\frac{Q(3)^n}{\{2n\}_2!!}C_2^nE_2^{2n}.
\end{aligned}$$

Proposition 6.25. *The partial quasi K-matrix $\mathfrak{X}_{w_0}$ coincides with the quasi K-matrix $\mathfrak{X}$.*

We have the following necessary relations needed for the proof of Proposition 6.25.

Lemma 6.26. *For any $n \in \mathbb{N}$ the relations*

$$T_1(E_2)^n E_1 = q^{-3n} E_1 T_1(E_2)^n, \tag{6.76}$$

$$T_{12}(E_1)^n E_1 = q^{-n} E_1 T_{12}(E_1)^n - q^{-1}[3]_1\{n\}_1 T_{12}(E_1)^{n-1} T_1(E_2), \tag{6.77}$$

$$T_{121}(E_2)^n E_1 = E_1 T_{121}(E_2)^n - qQ(1)\{n\}_2 T_{121}(E_2)^{n-1} T_{12}(E_1)^2, \tag{6.78}$$

$$\begin{aligned} T_{1212}(E_1)^{2n} E_1 = q^{2n} E_1 T_{1212}(E_1)^{2n} &- q[2]_1\{2n\}_1 T_{1212}(E_1)^{2n-1} T_{12}(E_1) \\ &- q[3]_1\{2n\}_1\{2n-1\}_1 T_{1212}(E_1)^{2n-2} T_{121}(E_2), \end{aligned} \tag{6.79}$$

$$E_2^n E_1 = q^{3n} E_1 E_2^n - q^3\{n\}_2 E_2^{n-1} T_{1212}(E_1) \tag{6.80}$$

hold in $U_q(\mathfrak{g}_2)$ where $\mathfrak{g}_2$ is the exceptional Lie algebra with root system of type G_2.

Lemma 6.27. *For any $n \in \mathbb{N}$ the relations*

$${}_1r(E_1^n) = \{n\}_1 E_1^{n-1}, \tag{6.81}$$

$${}_1r(T_{1212}(E_1)^n) = q^{-3} Q(3)\{n\}_1 E_2 T_{1212}(E_1)^{n-1}, \tag{6.82}$$

$$\begin{aligned} {}_1r(T_{12}(E_1)^{2n}) = {} & q^{-1}[2]_1 Q(1)\{2n\}_1 T_{1212}(E_1) T_{12}(E_1)^{2n-1} \\ & + q^{-1} Q(3)\{2n\}_1\{2n-1\}_1 T_{121}(E_2) T_{12}(E_1)^{2n-2}, \end{aligned} \tag{6.83}$$

$${}_1r(T_1(E_2)^n) = qQ(1)\{n\}_2 T_{12}(E_1) T_1(E_2)^{n-1}, \tag{6.84}$$

$${}_1r(T_{121}(E_2)^n) = q^{-1} Q(1)^2\{n\}_2 T_{1212}(E_1)^2 T_{121}(E_2)^{n-1} \tag{6.85}$$

hold in $U_q(\mathfrak{g}_2)$.

Proof of Proposition 6.25. We have

$$\mathfrak{X}_{w_0} = \mathfrak{X}^{[6]}\mathfrak{X}^{[5]}\mathfrak{X}^{[4]}\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]} = \mathfrak{X}^{[654321]}$$

where

$$\begin{aligned}
\mathfrak{X}^{[6]} &\overset{(5.38)}{=} \mathfrak{X}_2, \\
\mathfrak{X}^{[5]} &= \Psi \circ T_{1212} \circ \Psi^{-1}(\mathfrak{X}_1) = \sum_{n \geq 0} \frac{Q(1)^n}{\{2n\}_1!!} C_1^n C_2^n T_{1212}(E_1)^{2n}, \\
\mathfrak{X}^{[4]} &= \Psi \circ T_{121} \circ \Psi^{-1}(\mathfrak{X}_2) = \sum_{n \geq 0} \frac{Q(3)^n}{\{2n\}_2!!} C_1^{3n} C_2^{2n} T_{121}(E_2)^{2n}, \\
\mathfrak{X}^{[3]} &= \Psi \circ T_{12} \circ \Psi^{-1}(\mathfrak{X}_1) = \sum_{n \geq 0} \frac{Q(1)^n}{\{2n\}_1!!} C_1^{2n} C_2^n T_{12}(E_1)^{2n}, \\
\mathfrak{X}^{[2]} &= \Psi \circ T_1 \circ \Psi^{-1}(\mathfrak{X}_2) = \sum_{n \geq 0} \frac{Q(3)^n}{\{2n\}_2!!} C_1^{3n} C_2^n T_1(E_2)^{2n}, \\
\mathfrak{X}^{[1]} &= \mathfrak{X}_1.
\end{aligned}$$

By Lemma 5.11 and Corollary 2.51 we have

$$\begin{aligned} {}_2r(\mathfrak{X}_{w_0}) &= {}_2r(\mathfrak{X}_2)\mathfrak{X}^{[54321]} \\ &= Q(3)C_2E_2\mathfrak{X}_2\mathfrak{X}^{[54321]} \\ &= Q(3)C_2E_2\mathfrak{X}_{w_0}. \end{aligned}$$

We want to show that

$${}_1r(\mathfrak{X}_{w_0}) = Q(1)C_1E_1\mathfrak{X}_{w_0}.$$

For each $i = 1, \ldots, 6$ let $\mathfrak{X}_{[i]} = K_1\mathfrak{X}^{[i]}K_1^{-1}$ and write $\mathfrak{X}_{[i]}\mathfrak{X}_{[j]} = \mathfrak{X}_{[ij]}$. By property (2.78) of the skew derivative

$$\begin{aligned} {}_1r(\mathfrak{X}_{w_0}) = \mathfrak{X}_{[6]}\,{}_1r(\mathfrak{X}^{[5]})\mathfrak{X}^{[4321]} + \mathfrak{X}_{[65]}\,{}_1r(\mathfrak{X}^{[4]})\mathfrak{X}^{[321]} + \mathfrak{X}_{[654]}\,{}_1r(\mathfrak{X}^{[3]})\mathfrak{X}^{[21]} \\ + \mathfrak{X}_{[6543]}\,{}_1r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]} + \mathfrak{X}_{[65432]}\,{}_1r(\mathfrak{X}^{[1]}). \end{aligned}$$

By Lemma 6.27 we have

$${}_1r(\mathfrak{X}^{[1]}) \overset{(6.81)}{=} Q(1)C_1E_1\mathfrak{X}^{[1]}, \tag{6.86}$$

$${}_1r(\mathfrak{X}^{[2]}) \overset{(6.84)}{=} qQ(1)Q(3)C_1^3C_2T_{12}(E_1)T_1(E_2)\mathfrak{X}^{[2]}, \tag{6.87}$$

$${}_1r(\mathfrak{X}^{[4]}) \overset{(6.85)}{=} q^{-1}Q(1)^2Q(3)C_1^3C_2^2T_{1212}(E_1)^2T_{121}(E_2)\mathfrak{X}^{[4]}, \tag{6.88}$$

$${}_1r(\mathfrak{X}^{[5]}) \overset{(6.82)}{=} q^{-3}Q(1)Q(3)C_1C_2E_2T_{1212}(E_1)\mathfrak{X}^{[5]}. \tag{6.89}$$

By Equation (6.83) we have

$$\begin{aligned} {}_1r(\mathfrak{X}^{[3]}) = q^{-1}Q(1)^2[2]_1C_1^2C_2T_{1212}(E_1)T_{12}(E_1)\mathfrak{X}^{[3]} \\ + q^{-1}Q(1)Q(3)C_1^2C_2T_{121}(E_2)\widehat{\mathfrak{X}}^{[3]} \end{aligned} \tag{6.90}$$

where

$$\widehat{\mathfrak{X}}^{[3]} = \sum_{n\geq 0} \frac{Q(1)^n}{\{2n\}_1!!} C_1^{2n}C_2^n\{2n+1\}_1T_{12}(E_1)^{2n}.$$

Recall that for $n \geq 1$ and $i = 1, 2$ we have $\{2n+1\}_i = 1 + q_i^2\{2n\}_i$. It follows from this that

$$\widehat{\mathfrak{X}}^{[3]} = \mathfrak{X}^{[3]} + q^2Q(1)C_1^2C_2T_{12}(E_1)^2\mathfrak{X}^{[3]}.$$

Substituting this into Equation (6.90) we get

$$\begin{aligned} {}_1r(\mathfrak{X}^{[3]}) = q^{-1}Q(1)^2[2]_1C_1^2C_2T_{1212}(E_1)T_{12}(E_1)\mathfrak{X}^{[3]} + q^{-1}Q(1)Q(3)C_1^2C_2T_{121}(E_2)\mathfrak{X}^{[3]} \\ + qQ(1)^2Q(3)C_1^4C_2^2T_{121}(E_2)T_{12}(E_1)^2\mathfrak{X}^{[3]}. \end{aligned} \tag{6.91}$$

When we calculate ${}_1r(\mathfrak{X}_{w_0})$ we obtain a component of the form $\mathfrak{X}_{[65432]}\,{}_1r(\mathfrak{X}^{[1]})$. By Equation (6.86) this contains an E_1 term that we pass to the front using Lemma 6.26. We have

$$\mathfrak{X}_{[2]}E_1 \overset{(6.76)}{=} E_1\mathfrak{X}^{[2]}, \tag{6.92}$$

$$\mathfrak{X}_{[3]}E_1 \overset{(6.77)}{=} E_1\mathfrak{X}^{[3]} - qQ(3)C_1^2C_2\mathfrak{X}_{[3]}T_{12}(E_1)T_1(E_2), \tag{6.93}$$

$$\mathfrak{X}_{[4]}E_1 \overset{(6.78)}{=} E_1\mathfrak{X}^{[4]} - qQ(1)Q(3)C_1^3C_2^2\mathfrak{X}^{[4]}T_{121}(E_2)T_{12}(E_1)^2, \tag{6.94}$$

$$\begin{aligned}\mathfrak{X}_{[5]}E_1 \overset{(6.79)}{=} {} & E_1\mathfrak{X}^{[5]} - q^{-1}Q(1)[2]_1C_1C_2\mathfrak{X}^{[5]}T_{1212}(E_1)T_{12}(E_1) - q^{-1}Q(3)C_1C_2\mathfrak{X}_{[5]}T_{121}(E_2) \\ & - q^{-1}Q(1)Q(3)C_1^2C_2^2\mathfrak{X}_{[5]}T_{1212}(E_1)^2T_{121}(E_2),\end{aligned} \tag{6.95}$$

$$\mathfrak{X}^{[6]}E_1 \overset{(6.80)}{=} E_1\mathfrak{X}_{[6]} - q^{-3}Q(3)C_2\mathfrak{X}_{[6]}E_2T_{1212}(E_1) \tag{6.96}$$

where we also use $\{2n+1\}_1 = 1+q^2\{2n\}_1$ for $n \geq 1$ to show (6.95). Comparing Equations (6.86)–(6.89) and (6.91) with Equations (6.92)–(6.96) we see that

$$\begin{aligned}\mathfrak{X}_{[65432]}\,{}_1r(\mathfrak{X}^{[1]}) = {} & Q(1)C_1E_1\mathfrak{X}_{w_0} - \mathfrak{X}_{[6543]}\,{}_1r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]} - \mathfrak{X}_{[654]}\,{}_1r(\mathfrak{X}^{[3]})\mathfrak{X}^{[21]} \\ & - \mathfrak{X}_{[65]}\,{}_1r(\mathfrak{X}^{[4]})\mathfrak{X}^{[321]} - \mathfrak{X}_{[6]}\,{}_1r(\mathfrak{X}^{[5]})\mathfrak{X}^{[4321]}\end{aligned}$$

from which it follows that

$${}_1r(\mathfrak{X}_{w_0}) = Q(1)C_1E_1\mathfrak{X}_{w_0}$$

as required. □

We now consider the reduced expression $w_0' = \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1$.

Proposition 6.28. *The partial quasi K-matrix $\mathfrak{X}_{w_0'}$ coincides with the quasi K-matrix $\mathfrak{X}$.*

The following relations are needed for the proof of Proposition 6.28. We split such relations among three lemmas. In Lemma 6.29 we show how the element E_2 commutes past the PBW elements corresponding to the longest element w_0'. Lemma 6.30 gives additional relations required in the latter parts of the proof. Finally, we give the skew derivative ${}_2r$ of PBW basis elements in Lemma 6.31.

Lemma 6.29. *For any $n \in \mathbb{N}$ the relations*

$$T_2(E_1)^nE_2 = q^{-3n}E_2T_2(E_1)^n, \tag{6.97}$$

$$T_{21}(E_2)^nE_2 = q^{-3n}E_2T_{21}(E_2)^n - Q(1)^3Q(3)^{-1}\{n\}_2T_{21}(E_2)^{n-1}T_2(E_1)^3, \tag{6.98}$$

$$\begin{aligned}T_{212}(E_1)^nE_2 = {} & E_2T_{212}(E_1)^n - qQ(1)\{n\}_1T_{212}(E_1)^{n-1}T_2(E_1)^2 \\ & - q^2Q(3)\{n\}_1\{n-1\}_1T_{212}(E_1)^{n-2}T_{21}(E_2)T_2(E_1) \\ & - Q(3)\{n\}_1\{n-1\}_1\{n-2\}_1T_{212}(E_1)^{n-3}T_{21}(E_2)^2,\end{aligned} \tag{6.99}$$

$$\begin{aligned}T_{2121}(E_2)^nE_2 = {} & q^{3n}E_2T_{2121}(E_2)^n - q^4Q(1)\{n\}_2T_{2121}(E_2)^{n-1}T_{212}(E_1)T_2(E_1) \\ & - (q^6-q^4-q^2)\{n\}_2T_{2121}(E_2)^{n-1}T_{21}(E_2) \\ & - q^6Q(1)^3Q(3)^{-1}\{n\}_2\{n-1\}_2T_{2121}(E_2)^{n-2}T_{212}(E_1)^3,\end{aligned} \tag{6.100}$$

$$\begin{aligned}E_1^nE_2 = {} & q^{3n}E_2E_1^n - q^3\{n\}_1E_1^{n-1}T_2(E_1) - q^5\{n\}_1\{n-1\}_1E_1^{n-2}T_{212}(E_1) \\ & - q^6\{n\}_1\{n-1\}_1\{n-2\}_1E_1^{n-3}T_{2121}(E_2),\end{aligned} \tag{6.101}$$

hold in $U_q(\mathfrak{g}_2)$.

Lemma 6.30. *For any $n \in N$ the relations*

$$\begin{aligned} E_1T_2(E_1)T_{2121}(E_2)^n &= q^{-3n}T_{2121}(E_2)^nE_1T_2(E_1) \\ &\quad + q^{-3n+4}Q(1)T_{2121}(E_2)^{n-1}E_1T_{212}(E_1)^2, \qquad (6.102)\\ E_1T_2(E_1)T_{212}(E_1)^n &= T_{212}(E_1)^nE_1T_2(E_1) - [3]_1\{n\}_1T_{2121}(E_2)T_2(E_1)T_{212}(E_1)^{n-1} \\ &\quad + [3]_1\{n\}_1T_{212}(E_1)^{n-1}E_1T_{21}(E_2), \qquad (6.103)\\ E_1T_2(E_1)T_{21}(E_2)^n &= q^{3n}T_{21}(E_2)^nE_1T_2(E_1) \\ &\quad - q^4Q(1)\{n\}_2T_{212}(E_1)^2T_2(E_1)T_{21}(E_2)^{n-1} \qquad (6.104)\end{aligned}$$

hold in $U_q(\mathfrak{g}_2)$. Additionally, for $n \geq 1$ the relations

$$T_{212}(E_1)T_{2121}(E_2)^n = \big(q^{-3n} + q^{-3(n-1)}Q(3)\{n\}_2\big)T_{2121}(E_2)^nT_{212}(E_1), \qquad (6.105)$$

$$T_{212}(E_1)T_{21}(E_2)^n = \big(q^{3n} - q^{-3(n-1)}Q(3)\{n\}_2\big)T_{21}(E_2)^nT_{212}(E_1) \qquad (6.106)$$

hold in $U_q(\mathfrak{g}_2)$.

Lemma 6.31. *For any $n \in \mathbb{N}$ the relations*

$$\begin{aligned} {}_2r(E_2^n) &= \{n\}_2E_2^{n-1}, \qquad (6.107)\\ {}_2r(T_2(E_1)^n) &= q^{-3}Q(3)\{n\}_1E_1T_2(E_1)^{n-1} \\ &\quad + q^{-1}Q(3)\{n\}_1\{n-1\}_1T_{212}(E_1)T_2(E_1)^{n-2}, \qquad (6.108)\\ {}_2r(T_{212}(E_1)^n) &= q^{-5}Q(1)Q(3)\{n\}_1E_1^2T_{212}(E_1)^{n-1} \\ &\quad + q^{-4}Q(3)^2\{n\}_1\{n-1\}_1E_1T_{2121}(E_2)T_{212}(E_1)^{n-2} \\ &\qquad + q^{-6}Q(3)^2\{n\}_1\{n-1\}_1\{n-2\}_1T_{2121}(E_2)^2T_{212}(E_1)^{n-3}, \qquad (6.109)\\ {}_2r(T_{2121}(E_2)) &= q^{-6}Q(3)\{n\}_2E_1^3T_{2121}(E_2)^{n-1}, \qquad (6.110)\\ {}_2r(T_{21}(E_2)^n) &= q^{-2}Q(1)Q(3)\{n\}_2E_1T_{212}(E_1)T_{21}(E_2)^{n-1} \\ &\quad + q^{-2}(q^2-1-q^{-2})Q(3)\{n\}_2T_{2121}(E_2)T_{21}(E_2)^{n-1} \\ &\qquad + Q(1)^3\{n\}_2\{n-1\}_2T_{212}(E_1)^3T_{21}(E_2)^{n-2} \qquad (6.111)\end{aligned}$$

hold in $U_q(\mathfrak{g}_2)$.

Proof of Proposition 6.28. We have

$$\mathfrak{X}_{w_0'} = \mathfrak{X}^{[6]}\mathfrak{X}^{[5]}\mathfrak{X}^{[4]}\mathfrak{X}^{[3]}\mathfrak{X}^{[2]}\mathfrak{X}^{[1]}$$

where

$$\begin{aligned} \mathfrak{X}^{[6]} &\overset{(5.38)}{=} \mathfrak{X}_1, \\ \mathfrak{X}^{[5]} &= \Psi \circ T_{2121} \circ \Psi^{-1}(\mathfrak{X}_2) = \sum_{n\geq 0} \frac{Q(3)^n}{\{2n\}_2!!}C_1^{3n}C_2^nT_{2121}(E_2)^{2n}, \\ \mathfrak{X}^{[4]} &= \Psi \circ T_{212} \circ \Psi^{-1}(\mathfrak{X}_1) = \sum_{n\geq 0} \frac{Q(1)^n}{\{2n\}_1!!}C_1^{2n}C_2^nT_{212}(E_1)^{2n}, \end{aligned}$$

$$\begin{aligned}\mathfrak{X}^{[3]} &= \Psi \circ T_{21} \circ \Psi^{-1}(\mathfrak{X}_2) = \sum_{n\geq 0} \frac{Q(3)^n}{\{2n\}_2!!} C_1^{3n} C_2^{2n} T_{21}(E_2)^{2n},\\ \mathfrak{X}^{[2]} &= \Psi \circ T_2 \circ \Psi^{-1}(\mathfrak{X}_1) = \sum_{n\geq 0} \frac{Q(1)^n}{\{2n\}_1!!} C_1^{n} C_2^{n} T_2(E_1)^{2n},\\ \mathfrak{X}^{[1]} &= \mathfrak{X}_2.\end{aligned}$$

By Lemma 5.11 and Corollary 2.51 we have

$$\begin{aligned}{}_1r(\mathfrak{X}_{w_0'}) &= {}_1r(\mathfrak{X}_1)\mathfrak{X}^{[54321]}\\ &= Q(1)C_1E_2\mathfrak{X}_1\mathfrak{X}^{[54321]}\\ &= Q(1)C_1E_2\mathfrak{X}_{w_0'}.\end{aligned}$$

We want to show that

$${}_2r(\mathfrak{X}_{w_0'}) = Q(3)C_2E_2\mathfrak{X}_{w_0'}.$$

For each $i = 1, \dots, 6$ let $\mathfrak{X}_{[i]} = K_2\mathfrak{X}^{[i]}K_2^{-1}$. By Property (2.78) of the skew derivative we have

$$\begin{aligned}{}_2r(\mathfrak{X}_{w_0'}) = &\,\mathfrak{X}_{[6]}\,{}_2r(\mathfrak{X}^{[5]})\mathfrak{X}^{[4321]} + \mathfrak{X}_{[65]}\,{}_2r(\mathfrak{X}^{[4]})\mathfrak{X}^{[321]} + \mathfrak{X}_{[654]}\,{}_2r(\mathfrak{X}^{[3]})\mathfrak{X}^{[21]}\\ &+ \mathfrak{X}_{[6543]}\,{}_2r(\mathfrak{X}^{[2]})\mathfrak{X}^{[1]} + \mathfrak{X}_{[65432]}\,{}_2r(\mathfrak{X}^{[1]}).\end{aligned} \tag{6.112}$$

We use Lemma 6.29 to calculate ${}_2r(\mathfrak{X}^{[i]})$ for $i = 1, \dots, 5$. We obtain

$${}_2r(\mathfrak{X}^{[1]}) \overset{(6.107)}{=} Q(3)C_2E_2\mathfrak{X}^{[1]}, \tag{6.113}$$

$${}_2r(\mathfrak{X}^{[5]}) \overset{(6.110)}{=} q^{-6}Q(1)^3Q(3)C_1^3C_2E_1^3T_{2121}(E_2)\mathfrak{X}^{[5]}. \tag{6.114}$$

We now consider ${}_2r(\mathfrak{X}^{[2]})$ in some detail using Equation (6.108). We have

$$\begin{aligned}{}_2r(\mathfrak{X}^{[2]}) = &\, q^{-3}Q(1)Q(3)C_1C_2E_1T_2(E_1)\mathfrak{X}^{[2]}\\ &+ q^{-1}C_1C_2Q(1)Q(3)T_{212}(E_1)\sum_{n\geq 0}\frac{Q(1)^n}{\{2n\}_1!!}C_1^nC_2^n\{2n+1\}_1T_2(E_1)^{2n}\\ &+ Q(1)^2Q(3)C_1^2C_2^2T_{21}(E_2)T_2(E_1)\sum_{n\geq 0}\frac{Q(1)^n}{\{2n\}_1!!}C_1^nC_2^n\{2n+3\}_1T_2(E_1)^{2n}.\end{aligned} \tag{6.115}$$

Recall that for $n \geq 1$ we have $\{2n+1\}_1 = 1 + q^2\{2n\}_1$. Additionally we have

$$\{2n+3\}_1 = (1 + q^2 + q^4) + q^6\{2n\}_1$$

for $n \geq 1$. Using this we get

$$\sum_{n\geq 0}\frac{Q(1)^n}{\{2n\}_1!!}C_1^nC_2^n\{2n+1\}_1T_2(E_1)^{2n} = \mathfrak{X}^{[2]} + q^2Q(1)C_1C_2\mathfrak{X}^{[2]}T_2(E_1)^2$$

and

$$\sum_{n\geq 0}\frac{Q(1)^n}{\{2n\}_1!!}C_1^nC_2^n\{2n+3\}_1T_2(E_1)^{2n}=(1+q^2+q^4)\mathfrak{X}^{[2]}+q^6Q(1)C_1C_2\mathfrak{X}^{[2]}T_2(E_1)^2.$$

Substituting both of these identities into (6.115) and using $(1+q^2+q^4)Q(1)=q^2Q(3)$ we obtain

$$\begin{aligned}{}_2r(\mathfrak{X}^{[2]}) &= q^{-3}Q(1)Q(3)C_1C_2E_1T_2(E_1)\mathfrak{X}^{[2]}+q^{-1}C_1C_2Q(1)Q(3)T_{212}(E_1)\mathfrak{X}^{[2]}\\ &\quad+qC_1^2C_2^2Q(1)^2Q(3)T_{212}(E_1)T_2(E_1)^2\mathfrak{X}^{[2]}\\ &\quad+q^2Q(1)Q(3)^2C_1^2C_2^2T_{21}(E_2)T_2(E_1)\mathfrak{X}^{[2]}\\ &\quad+q^6Q(1)^3Q(3)C_1^3C_2^3T_{21}(E_2)T_2(E_1)^3\mathfrak{X}^{[2]}.\end{aligned}\tag{6.116}$$

Similarly we obtain the following expressions for ${}_2r(\mathfrak{X}^{[3]})$ and ${}_2r(\mathfrak{X}^{[4]})$ using Equations (6.111) and (6.109), respectively.

$$\begin{aligned}{}_2r(\mathfrak{X}^{[3]}) &= q^{-2}Q(1)Q(3)^2C_1^3C_2^2E_1T_{212}(E_1)T_{21}(E_2)\mathfrak{X}^{[3]}\\ &\quad+q^{-2}Q(3)^2(q^2-1-q^{-2})C_1^3C_2^2T_{2121}(E_2)T_{21}(E_2)\mathfrak{X}^{[3]}\\ &\quad+Q(1)^2Q(3)C_1^3C_2^2T_{212}(E_1)^3\mathfrak{X}^{[3]}\\ &\quad+q^6Q(1)^3Q(3)^2C_1^6C_2^4T_{212}(E_1)^3T_{21}(E_2)^2\mathfrak{X}^{[3]},\end{aligned}\tag{6.117}$$

$$\begin{aligned}{}_2r(\mathfrak{X}^{[4]}) &= q^{-5}Q(1)^2Q(3)C_1^2C_2E_1^2T_{212}(E_1)\mathfrak{X}^{[4]}+q^{-4}Q(1)Q(3)^2C_1^2C_2E_1T_{2121}(E_2)\mathfrak{X}^{[4]}\\ &\quad+q^{-2}Q(1)^2Q(3)^2C_1^4C_2^2E_1T_{2121}(E_2)T_{212}(E_1)^2\mathfrak{X}^{[4]}\\ &\quad+q^{-4}Q(1)Q(3)^3C_1^4C_2^2T_{2121}(E_2)^2T_{212}(E_1)\mathfrak{X}^{[4]}\\ &\quad+Q(1)^3Q(3)^2C_1^6C_2^3T_{2121}(E_2)^2T_{212}(E_1)^3\mathfrak{X}^{[4]}.\end{aligned}\tag{6.118}$$

When calculating ${}_2r(\mathfrak{X}_{w_0'})$ we obtain a summand of the form

$$\mathfrak{X}_{[65432]}\,{}_2r(\mathfrak{X}^{[1]})=Q(3)C_2\mathfrak{X}_{[65432]}E_2\mathfrak{X}^{[1]}.$$

We bring the E_2 term to the front using Lemma 6.29. We have

$$\mathfrak{X}_{[2]}E_2\overset{(6.97)}{=}E_2\mathfrak{X}^{[2]},\tag{6.119}$$

$$\mathfrak{X}_{[3]}E_2\overset{(6.98)}{=}E_2\mathfrak{X}^{[3]}-q^6Q(1)^3C_1^3C_2^2\mathfrak{X}_{[3]}T_{21}(E_2)T_2(E_1)^3,\tag{6.120}$$

$$\begin{aligned}\mathfrak{X}_{[5]}E_2 &\overset{(6.100)}{=} E_2\mathfrak{X}^{[5]}-q^{-2}Q(1)Q(3)C_1^3C_2\mathfrak{X}_{[5]}T_{2121}(E_2)T_{212}(E_1)T_2(E_1)\\ &\quad-q^{-6}(q^6-q^4-q^2)Q(3)C_1^3C_2\mathfrak{X}_{[5]}T_{2121}(E_2)T_{21}(E_2)-Q(1)^3C_1^3C_2\mathfrak{X}_{[5]}T_{212}(E_1)^3\\ &\quad-Q(1)^3Q(3)C_1^6C_2^2\mathfrak{X}_{[5]}T_{2121}(E_2)^2T_{212}(E_1)^3.\end{aligned}\tag{6.121}$$

Additionally, using $\{2n+3\}_i=(1+q_i^2+q_i^4)+q_i^6\{2n\}_i$ for $i=1,2$ we obtain

$$\begin{aligned}\mathfrak{X}_{[4]}E_2 &\overset{(6.99)}{=} E_2\mathfrak{X}^{[4]}-qQ(1)^2C_1^2C_2\mathfrak{X}_{[4]}T_{212}(E_1)T_2(E_1)^2\\ &\quad-q^2Q(1)Q(3)C_1^2C_2\mathfrak{X}_{[4]}T_{21}(E_2)T_2(E_1)\\ &\quad-q^4Q(1)^2Q(3)C_1^4C_2^2\mathfrak{X}_{[4]}T_{212}(E_1)^2T_{21}(E_2)T_2(E_1)\end{aligned}$$

$$
\begin{aligned}
&- q^2Q(1)Q(3)^2C_1^4C_2^2\mathfrak{X}_{[4]}T_{212}(E_1)T_{21}(E_2)^2\\
&\quad - q^6Q(1)^3Q(3)C_1^6C_2^2\mathfrak{X}_{[4]}T_{212}(E_1)^3T_{21}(E_2)^2, \qquad (6.122)\\
\overset{(6.101)}{=}\ & E_2\mathfrak{X}^{[6]} - q^{-3}Q(1)C_1\mathfrak{X}_{[6]}E_1T_2(E_1) - q^{-1}Q(1)C_1\mathfrak{X}_{[6]}T_{212}(E_1)\\
&- q^{-5}Q(1)^2C_1^2\mathfrak{X}_{[6]}E_1^2T_{212}(E_1) - q^{-4}Q(1)Q(3)C_1^2\mathfrak{X}_{[6]}E_1T_{2121}(E_2)\\
&- q^{-6}Q(1)^3C_1^3\mathfrak{X}_{[6]}E_1^3T_{2121}(E_2). \qquad (6.123)
\end{aligned}
$$

We substitute the expressions (6.113), (6.114) and (6.116)–(6.118) into (6.112). Then we use Equations (6.119)–(6.123) to expand the term $\mathfrak{X}_{[65432]}\,{}_2r(\mathfrak{X}^{[1]})$. We do not write the full expression but many of the terms cancel by Theorem 2.59. For instance, there is an expression

$$
qQ(1)^2Q(3)C_1^2C_3^2\big(\mathfrak{X}_{[6543]}T_{212}(E_1)T_2(E_1)^2\mathfrak{X}^{[21]} - \mathfrak{X}_{[654]}T_{212}(E_1)T_2(E_1)^2\mathfrak{X}^{[321]}\big)
$$

which appears. We note that

$$
\begin{aligned}
\big[T_{212}(E_1), T_{21}(E_2)\big]_{q^{-3}} &= 0\\
\big[T_{21}(E_2), T_{212}(E_1)\big]_{q^{3}} &= 0
\end{aligned}
$$

by Theorem 2.59 and hence $T_{212}(E_1)T_2(E_1)^2\mathfrak{X}^{[3]} = \mathfrak{X}_{[3]}T_{212}(E_1)T_2(E_1)^2$ which implies that the expression above is zero. We are left with the following expression.

$$
\begin{aligned}
{}_2r(\mathfrak{X}_{w_0'}) = {}& Q(3)C_2E_2\mathfrak{X}_{w_0'} + q^{-2}Q(1)Q(3)^2C_1^3C_2^2\mathfrak{X}_{[654]}E_1T_{212}(E_1)T_{21}(E_2)\mathfrak{X}^{[321]}\\
&+ q^{-2}Q(1)^2Q(3)^2C_1^4C_2^2\mathfrak{X}_{[65]}E_1T_{2121}(E_2)T_{212}(E_1)^2\mathfrak{X}^{[4321]}\\
&+ q^{-4}Q(1)Q(3)^3C_1^4C_2^2\mathfrak{X}_{[65]}T_{2121}(E_2)^2T_{212}(E_1)\mathfrak{X}^{[4321]}\\
&- q^{-2}Q(1)Q(3)^2C_1^3C_2^2\mathfrak{X}_{[65]}T_{2121}(E_2)T_{212}(E_1)T_2(E_1)\mathfrak{X}^{[4321]}\\
&- q^{4}Q(1)^2Q(3)^2C_1^4C_2^3\mathfrak{X}_{[654]}T_{212}(E_1)^2T_{21}(E_2)T_2(E_1)\mathfrak{X}^{[321]}\\
&- q^{2}Q(1)Q(3)^3C_1^4C_2^3\mathfrak{X}_{[654]}T_{212}(E_1)T_{21}(E_2)^2\mathfrak{X}^{[321]}\\
&+ q^{-1}C_1C_2Q(1)Q(3)\big(\mathfrak{X}_{[6543]}T_{212}(E_1)\mathfrak{X}^{[21]} - \mathfrak{X}_{[6]}T_{212}(E_1)\mathfrak{X}^{[54321]}\big)\\
&+ q^{-3}Q(1)Q(3)C_1C_2\big(\mathfrak{X}_{[6543]}E_1T_2(E_1)\mathfrak{X}^{[21]} - \mathfrak{X}_{[6]}E_1T_2(E_1)\mathfrak{X}^{[54321]}\big).
\end{aligned} \tag{6.124}
$$

We consider the expression $E_1T_2(E_1)\mathfrak{X}^{[321]}$ in more detail using Lemma 6.30and the facts

$$
\begin{aligned}
T_{2121}(E_2)E_1 &= q^3E_1T_{2121}(E_2),\\
T_2(E_1)T_{212}(E_1) &= qT_{212}(E_1)T_2(E_1) + [3]_1T_{21}(E_2),\\
T_{212}(E_1)E_1 &= qE_1T_{212}(E_1) + [3]_1T_{2121}(E_2).
\end{aligned}
$$

We have

$$
\begin{aligned}
E_1T_2(E_1)\mathfrak{X}^{[5]} &\overset{(6.102)}{=} \mathfrak{X}_{[5]}E_1T_2(E_1) + qQ(1)Q(3)C_1^3C_2\mathfrak{X}_{[5]}E_1T_{2121}(E_2)T_{212}(E_1)^2, \qquad (6.125)\\
E_1T_2(E_1)\mathfrak{X}^{[4]} &\overset{(6.103)}{=} \mathfrak{X}_{[4]}E_1T_2(E_1) - qQ(3)C_1^2C_2T_{2121}(E_2)T_{212}(E_1)T_2(E_1)\mathfrak{X}^{[4]}
\end{aligned}
$$

$$+ qQ(3)C_1^2C_2\mathfrak{X}_{[4]}E_1T_{212}(E_1)T_{21}(E_2), \tag{6.126}$$

$$E_1T_2(E_1)\mathfrak{X}^{[3]} \overset{(6.104)}{=} \mathfrak{X}_{[3]}E_1T_2(E_1) - q^4Q(1)Q(3)C_1^3C_2^2T_{212}(E_1)^2T_2(E_1)T_{21}(E_2)\mathfrak{X}^{[3]}. \tag{6.127}$$

Substituting Equations (6.125)–(6.127) into (6.124) we obtain

$$\begin{aligned} {}_2r(\mathfrak{X}_{w_0'}) = {}& Q(3)C_2E_2\mathfrak{X}_{w_0'} + q^{-4}Q(1)Q(3)^3C_1^4C_2^2\mathfrak{X}_{[65]}T_{2121}(E_2)^2T_{212}(E_1)\mathfrak{X}^{[4321]} \\ & - q^2Q(1)Q(3)^3C_1^4C_2^3\mathfrak{X}_{[654]}T_{212}(E_1)T_{21}(E_2)^2\mathfrak{X}^{[321]} \\ & + q^{-1}C_1C_2Q(1)Q(3)\big(\mathfrak{X}_{[6543]}T_{212}(E_1)\mathfrak{X}^{[21]} - \mathfrak{X}_{[6]}T_{212}(E_1)\mathfrak{X}^{[54321]}\big). \end{aligned} \tag{6.128}$$

(6.129)

We use Equations (6.105) and (6.106) to simplify this expression further. We have

$$\begin{aligned} & T_{212}(E_1)\mathfrak{X}^{[5]} \\ & = \sum_{n\geq 0} \frac{Q(3)^n}{\{2n\}_2!!}C_1^{3n}C_2^nT_{212}(E_1)T_{2121}(E_2)^{2n} \\ & \overset{(6.105)}{=} T_{212}(E_1) + \sum_{n\geq 1} \frac{Q(3)^n}{\{2n\}_2!!}C_1^{3n}C_2^n\big(q^{-6n} + q^{-6n+3}Q(3)\{2n\}_2\big)T_{2121}(E_2)^{2n}T_{212}(E_1) \\ & = \left(1 + \sum_{n\geq 1} \frac{Q(3)^n}{\{2n\}_2!!}C_1^{3n}C_2^nq^{-6n}T_{2121}(E_2)^{2n}\right)T_{212}(E_1) \\ & \quad + q^3Q(3)\sum_{n\geq 1} \frac{Q(3)^n}{\{2n\}_2!!}C_1^{3n}C_2^nq^{-6n}\{2n\}_2T_{2121}(E_2)^{2n}T_{212}(E_1) \\ & = \mathfrak{X}_{[5]}T_{212}(E_1) + q^{-3}Q(3)^2C_1^3C_2\mathfrak{X}_{[5]}T_{2121}(E_2)^2T_{212}(E_1). \end{aligned}$$

This implies

$$q^{-3}Q(3)^2C_1^3C_2\mathfrak{X}_{[5]}T_{2121}(E_2)^2T_{212}(E_1) = T_{212}(E_1)\mathfrak{X}^{[5]} - \mathfrak{X}_{[5]}T_{212}(E_1).$$

In the same way, Equation (6.106) gives

$$-q^3Q(3)^2C_1^3C_2^2T_{212}(E_1)T_{21}(E_2)^2\mathfrak{X}^{[3]} = T_{212}(E_1)\mathfrak{X}^{[3]} - \mathfrak{X}_{[3]}T_{212}(E_1).$$

Substituting both of these into Equation (6.128) we obtain

$$\begin{aligned} {}_2r(\mathfrak{X}_{w_0'}) = {}& Q(3)C_2E_2\mathfrak{X}_{w_0'} + q^{-1}Q(1)Q(3)C_1C_2\mathfrak{X}_{[6]}\big(T_{212}(E_1)\mathfrak{X}^{[5]} - \mathfrak{X}_{[5]}T_{212}(E_1)\big)\mathfrak{X}^{[4321]} \\ & + q^{-1}Q(1)Q(3)C_1C_2\mathfrak{X}_{[654]}\big(T_{212}(E_1)\mathfrak{X}^{[3]} - \mathfrak{X}_{[3]}T_{212}(E_1)\big)\mathfrak{X}^{[21]} \\ & + q^{-1}C_1C_2Q(1)Q(3)\big(\mathfrak{X}_{[6543]}T_{212}(E_1)\mathfrak{X}^{[21]} - \mathfrak{X}_{[6]}T_{212}(E_1)\mathfrak{X}^{[54321]}\big). \\ = {}& Q(3)C_2\mathfrak{X}_{w_0'} \end{aligned}$$

as required. □

Chapter 7

Braid group actions for quantum symmetric pairs

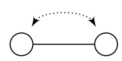
In the first part of this chapter, we return to the classical setting and establish a braid group action on the fixed Lie subalgebra $\mathfrak{k}$. This action depends on the subgroups $Br(W_X)$ and $Br(\widetilde{W})$ of $Br(\mathfrak{g})$. In Section 7.2 we explicitly construct an action of $Br(\widetilde{W})$ on $\mathfrak{k}$ by Lie algebra automorphisms.

We extend this theory to the quantum symmetric pair setting. In particular we show that there is an action of $Br(W_X)$ on $B_{\mathbf{c},\mathbf{s}}$ in Section 7.3. This is in full generality. In many cases one can exhibit an action of $Br(\widetilde{W})$ on $B_{\mathbf{c},\mathbf{s}}$, see [40]. In Section 7.4 we construct such an action when the Satake diagram is of type AIII. Our initial constructions require the use of the computer algebra program `GAP` but our arguments thereafter are general. We round off this chapter by combining the results of Sections 7.3 and 7.4 by showing that the two actions commute in type AIII.

7.1 The braid group action on $\mathfrak{k}$

Recall from Theorem 2.37 that we established a braid group action of $Br(\mathfrak{g})$ on $\mathfrak{g}$ by Lie algebra automorphisms. We aim to construct an analogous braid group action on $\mathfrak{k} = \{x \in \mathfrak{g} \mid \theta(x) = x\}$ by Lie algebra automorphisms. Generally we have $\mathrm{Ad}(b)(\mathfrak{k}) \neq \mathfrak{k}$ for $b \in Br(\mathfrak{g})$.

Example 7.1. Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and consider the Satake diagram $(\emptyset, (12))$, which is of type AIII. Graphically, we have the following diagram.



Here, the involution associated to this Satake diagram is given by $\theta = \tau \circ \omega$. By Lemma

3.12, the Lie subalgebra $\mathfrak{k}$ is generated by the elements

$$h_1 - h_2, \quad f_1 - e_2, \quad f_2 - e_1.$$

Then $\mathrm{Ad}(\varsigma_1)(f_1 - e_2) = -e_1 - [e_1, e_2] \notin \mathfrak{k}$ and therefore $Br(\mathfrak{g})$ does not act on $\mathfrak{k}$ by Lie algebra automorphisms.

In order to obtain a braid group action on $\mathfrak{k}$ we instead consider a suitable subgroup of $Br(\mathfrak{g})$. Since $\mathfrak{k}$ depends on θ, we expect that the subgroup we take should also depend on θ. We construct a subgroup $Br(\mathfrak{g})^\Theta$ analogously to W^Θ from Section 4.2. Recall from Section 3.1 that $m_X \in Br(\mathfrak{g})$ denotes the braid group element obtained from the longest element $w_X \in W_X$. Define a group automorphism

$$\Theta_B : Br(\mathfrak{g}) \to Br(\mathfrak{g}), \quad b \mapsto m_X \tau(b) m_X^{-1}. \tag{7.1}$$

Let $Br(\mathfrak{g})^\Theta = \{b \in Br(\mathfrak{g}) \mid \Theta_B(b) = b\}$ denote the subgroup of elements fixed by Θ_B. Recall from Section 4.12 that $\widetilde{W}$ is a Coxeter group generated by $\{\widetilde{\sigma}_i \mid i \in I \setminus X\}$. Hence $\widetilde{W}$ has a braid group associated to it. Let $Br(W_X)$ and $Br(\widetilde{W})$ denote the associated braid groups corresponding to the Weyl subgroups W_X and $\widetilde{W}$, respectively. Observe that both $Br(W_X)$ and $Br(\widetilde{W})$ are subgroups of $Br(\mathfrak{g})^\Theta$.

Remark 7.2. In Theorem 4.8 we showed that $W^\Theta = W_X \rtimes \widetilde{W}$. Proving a similar result for the structure of $Br(\mathfrak{g})^\Theta$ is a hard problem not considered in this thesis. Instead, we will only consider an action of $Br(W_X) \rtimes Br(\widetilde{W})$ on $\mathfrak{k}$ by Lie algebra automorphisms.

The following lemma shows that the action Ad restricted to $Br(W_X) \times Br(\widetilde{W})$ almost commutes with θ. Recall from Equation (3.3) the Chevalley involution $\omega : \mathfrak{g} \to \mathfrak{g}$.

Lemma 7.3. *For any $b \in Br(W_X) \rtimes Br(\widetilde{W})$, the relation*

$$\mathrm{Ad}(b) \circ \mathrm{Ad}(m_X) \circ \tau \circ \omega = \mathrm{Ad}(m_X) \circ \tau \circ \omega \circ \mathrm{Ad}(b) \tag{7.2}$$

holds.

Proof. By Equations (2.53) and (3.4), we observe that

$$\mathrm{Ad}(\varsigma_i) \circ \tau \circ \omega = \tau \circ \omega \circ \mathrm{Ad}(\varsigma_{\tau(i)})$$

for all $i \in I$. Hence for any $b \in Br(W_X) \times Br(\widetilde{W})$ and $x \in \mathfrak{g}$ we have

$$\begin{aligned} \mathrm{Ad}(m_X) \circ \tau \circ \omega \circ \mathrm{Ad}(b)(x) &= \mathrm{Ad}(m_X) \circ \mathrm{Ad}(\tau(b)) \circ \tau \circ \omega \\ &= \mathrm{Ad}(m_X \tau(b) m_X m_X^{-1}) \circ \tau \circ \omega(x) \\ &= \mathrm{Ad}(b) \circ \mathrm{Ad}(m_X) \circ \tau \circ \omega(x) \end{aligned}$$

since $b \in Br(W_X) \rtimes Br(\widetilde{W}) \subseteq Br(\mathfrak{g})^\Theta$. □

Generally, $\mathrm{Ad}(s)$ as defined in (3.12) does not commute with $\mathrm{Ad}(b)$ for $b \in Br(W_X) \times Br(\widetilde{W})$. However, we can find explicit maps $s' : I \to \mathbb{C}^\times$ such that $\theta' = \mathrm{Ad}(s') \circ \mathrm{Ad}(m_X) \circ$

$\tau \circ \omega$ commutes with $\mathrm{Ad}(b)$. The idea after finding such an s' is to modify the group homomorphism Ad to a new group homomorphism Ad' such that the $Br(W_X) \times Br(\widetilde{W})$ maps $\mathfrak{k}$ to itself under the action Ad'.

We fix a total order $>$ on the set I. With this total order, we let $s' : I \to \mathbb{C}$ be the function defined by

$$s'(\alpha_j) = \begin{cases} 1 & \text{if } j \in X \text{ or } \tau(j) = j, \\ i^{\alpha_j(2\rho_X^\vee)} & \text{if } j \notin X \text{ or } \tau(j) > j, \\ (-i)^{\alpha_j(2\rho_X^\vee)} & \text{if } j \notin X \text{ or } j > \tau(j). \end{cases} \tag{7.3}$$

for $j \in I$, where $i \in \mathbb{C}$ denotes the square root of -1. This is the same map as [38, Equation (2.7)]. If we choose a different total order on I, then the map s' may change by a factor -1 on the roots α_j and $\alpha_{\tau(j)}$ where $j \neq \tau(j)$ and $j, \tau(j) \notin X$. The following proposition implies that there are limited choices for s such that $\mathrm{Ad}(s)$ commutes with $\mathrm{Ad}(b)$.

Proposition 7.4. *Suppose that* $\mathrm{Ad}(s) \circ \mathrm{Ad}(b) = \mathrm{Ad}(b) \circ \mathrm{Ad}(s)$ *for all* $b \in Br(W_X) \times Br(\widetilde{W})$*. Then there exists a total order* $>$ *on* I *such that* $s(i) = s'(i)$ *for all* $i \in I$*.*

Proof. Since $\mathrm{Ad}(s)$ and $\mathrm{Ad}(b)$ are automorphisms of $\mathfrak{g}$, we only check on the Chevalley generators e_i, f_i and h_i for $i \in I$. Without loss of generality, we only consider the generators e_i, since $\mathrm{Ad}(s)|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}$ and the calculations for f_i are the same up to a change of sign. Let $\pi : Br(\mathfrak{g}) \to W$ denote the group homomorphism that associates an element $b \in Br(\mathfrak{g})$ to an element $w \in W$ by including the relation $\varsigma_i^2 = 1$ for all $i \in I$. Given $b \in Br(W_X) \times Br(\widetilde{W})$, we have $w = \pi(b) \in W^\Theta$. Then

$$\mathrm{Ad}(s) \circ \mathrm{Ad}(b)(e_i) = s_Q(w(\alpha_i))\mathrm{Ad}(b)(e_i),$$
$$\mathrm{Ad}(b) \circ \mathrm{Ad}(s)(e_i) = s_Q(\alpha_i)\mathrm{Ad}(b)(e_i).$$

Hence $\mathrm{Ad}(s) \circ \mathrm{Ad}(b) = \mathrm{Ad}(b) \circ \mathrm{Ad}(s)$ if and only if $s_Q(w(\alpha_i)) = s_Q(\alpha_i)$. If $w \in W_X$, then there is nothing to check by Condition (3.10). So suppose $w \in \widetilde{W}$. As $s_Q : Q \to \mathbb{C}^\times$ is a group homomorphism, we may assume that $w = \widetilde{\sigma}_j$ for some $j \in I \setminus X$. We make the following assumptions on i and j:

- $j \neq \tau(j)$, otherwise (3.10) implies that $s(k) = 1$ for all $k \in \{j, \tau(j)\} \cup X$ and hence $s_Q(\widetilde{\sigma}_j(\alpha_i)) = s_Q(\alpha_i)$. This further implies that the Satake diagram (I, X, τ) is of type ADE. Without loss of generality, we assume $\tau(j) > j$ with respect to the total order $>$.
- $i \in I \backslash X$, otherwise $w_{\{j,\tau(j)\} \cup X}(\alpha_i) = -\alpha_{\tau_{0,j}(i)} \in \Phi_X$ and thus $s_Q(\widetilde{\sigma}_j(\alpha_i)) = s_Q(\alpha_i) = 1$.
- If $X = \emptyset$, then $\widetilde{\sigma}_j(\alpha_i) = \alpha_i + n_j\alpha_j + n_{\tau(j)}\alpha_{\tau(j)}$ where $n_j, n_{\tau(j)} \in \{0, 1\}$. It follows that $s_Q(\widetilde{\sigma}_j(\alpha_i)) = s_Q(\alpha_i)$ if and only if $s_Q(n_j\alpha_j + n_{\tau(j)}\alpha_{\tau(j)}) = 1$. By Condition (3.11)

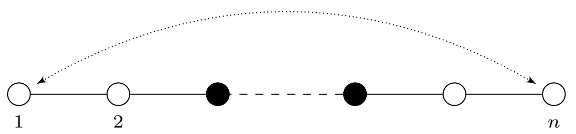
we have $s(j) = s(\tau(j))$ so we can choose $s(j) = \pm 1$. Hence we can assume $X \neq \emptyset$. Similarly we also assume that $\alpha_j(2\rho_X^\vee) \neq 0$. This means that the τ-orbit $\{j, \tau(j)\}$ is adjacent to a connected component of black nodes.

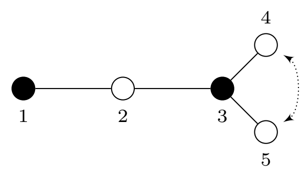
Let $i = j$. Then $w_{\{j,\tau(j)\}\cup X}(\alpha_j) = -\alpha_{\tau(j)}$ since $\tau \neq \mathrm{id}$ and $j \neq \tau(j)$. Then

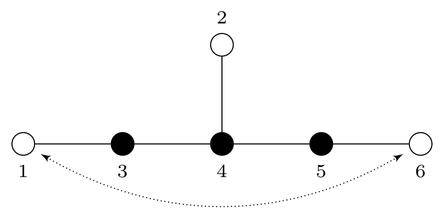
$$s_Q(\alpha_i) = s_Q(w_{\{j,\tau(j)\}\cup X}(\alpha_j)) = s_Q(-\alpha_{\tau(j)})$$

which implies $s(j)s(\tau(j)) = 1$. It follows from this and condition (3.11) that $s(j)^2 = (-1)^{\alpha_j(2\rho_X^\vee)}$. We hence obtain $s(j) = i^{\alpha_j(2\rho_X^\vee)}$ and $s(\tau(j)) = (-i)^{\alpha_j(2\rho_X^\vee)}$.

We now check the cases that arise when $i \neq j, \tau(j)$. We may assume that $X \neq \emptyset$ and $\alpha_j(2\rho_X^\vee) \neq 0$, since otherwise there is nothing to show. Further, we consider cases where $w_{\{j,\tau(j)\}\cup X}(\alpha_i) \neq w_X(\alpha_i)$. Graphically this means that the node i is connected to the τ-orbit $\{j, \tau(j)\}$ where any path only goes through nodes belonging to X. In other words, we need only check rank two cases. By our previous assumptions, there are only three cases to check.







In each case, one checks that $s_Q(w_{\{j,\tau(j)\}\cup X}(\alpha_i)) = s_Q(\alpha_i)$ for $i \neq j, \tau(j)$, as required. $\square$

Remark 7.5. The function s' has the additional advantage that $\mathrm{Ad}(s')$ commutes with the involutive automorphism θ. This is so since $\mathrm{Ad}(s')$ commutes with $\mathrm{Ad}(m_X)$ and $\tau \circ \omega$, see [38, Theorem 2.5]. The latter follows since $s'(\alpha_j) = s'(-\alpha_{\tau(j)})$.

Denote the involutive automorphism corresponding to s' and the Satake diagram (I, X, τ) by

$$\theta' = \mathrm{Ad}(s') \circ \mathrm{Ad}(m_X) \circ \tau \circ \omega \tag{7.4}$$

Let $\mathfrak{k}'$ be the associated fixed Lie subalgebra. Then Lemma 7.3 and Remark 7.5 imply that $Br(W_X) \rtimes Br(\widetilde{W})$ maps $\mathfrak{k}'$ to itself under the action Ad. We observe that given any involutive automorphism $\theta = \mathrm{Ad}(s) \circ \mathrm{Ad}(m_X) \circ \tau \circ \omega$ associated to (I, X, τ), we can find a Lie algebra automorphism ψ such that

$$\theta = \psi \circ \theta' \circ \psi^{-1}. \tag{7.5}$$

Since θ and θ' only differ by a scalar on each root space, such a Lie algebra automorphism will only act as a rescaling. In particular, we take $\psi = \mathrm{Ad}(\overline{s})$ where we choose $\overline{s} : I \to \mathbb{C}^\times$ such that

$$\overline{s}(i) = 1 \qquad \text{for } i \in X \text{ or } i = \tau(i), \tag{7.6}$$

$$\overline{s}(i) = (-1)^{\alpha_i(2\rho_X^\vee)}\overline{s}(\tau(i)) \qquad \text{for } i \notin X \text{ and } i \neq \tau(i). \tag{7.7}$$

Then for $x \in \mathfrak{g}_{\alpha_i}$ we have

$$\begin{aligned}\psi \circ \theta' \circ \psi^{-1}(x) &= \frac{s'(i)}{\overline{s}(i)\overline{s}(\tau(i))} \cdot \mathrm{Ad}(m_X) \circ \omega \circ \tau(x) \\ &= \frac{s'(i)s(\tau(i))}{(-1)^{\alpha_i(2\rho_X^\vee)}\overline{s}(i)^2} \cdot \theta(x).\end{aligned}$$

This implies that $\psi \circ \theta' \circ \psi^{-1} = \theta$ for $i \in X$ or $i = \tau(i)$. If $i \notin X$ and $i \neq \tau(i)$ then $\overline{s}(i)^2 = s'(i)s(i)$ and hence we choose to take $\overline{s}(i) = (s'(i)s(i))^{1/2}$.

Lemma 7.6. *Under the action $\psi \circ \mathrm{Ad} \circ \psi^{-1}$, the subgroup $Br(W_X) \rtimes Br(\widetilde{W})$ maps $\mathfrak{k}$ to itself.*

Proof. Let θ' be as in Equation (7.4). Then for any $b \in Br(W_X) \rtimes Br(\widetilde{W})$, we have

$$\theta' \circ \mathrm{Ad}(b) = \mathrm{Ad}(b) \circ \theta'$$

by Lemma 7.3 and Remark 7.5. Using Equation (7.5), we see that

$$\psi^{-1} \circ \theta \circ \psi \circ \mathrm{Ad}(b) = \mathrm{Ad}(b) \circ \psi^{-1} \circ \theta \circ \psi.$$

Applying ψ on the left and ψ^{-1} on the right gives

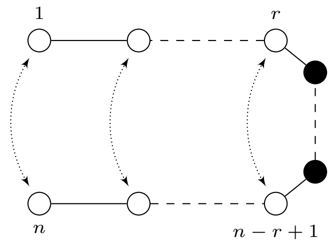
$$\theta \circ \psi \circ \mathrm{Ad}(b) \circ \psi^{-1} = \psi \circ \mathrm{Ad}(b) \circ \psi^{-1} \circ \theta.$$

This implies that $\psi \circ \mathrm{Ad}(b) \circ \psi^{-1}(\mathfrak{k}) = \mathfrak{k}$ as required. ☐

Remark 7.7. In [40, Lemma 2.1], the Lie algebra automorphism ψ does not appear. The reason for this is that Kolb and Pellegrini take $s(i) = 1$ for all $i \in I$ since they only consider cases where either $X = \emptyset$ or $\tau = \mathrm{id}$.

7.2 An explicit example: AIII

We construct the action of $Br(\widetilde{W})$ on $\mathfrak{k}$ explicitly when the Satake diagram (X, τ) is of type AIII.



In Section 7.4 we construct a quantum analogue of this action on the coideal subalgebra $B_{\mathbf{c},\mathbf{s}}$. Using Lemma 7.6 we calculate the action of $Br(\widetilde{W})$ on $\mathfrak{k}$. Recall that the generators of $\widetilde{W}$ are invariant under τ, i.e. $\widetilde{\sigma}_{\tau(i)} = \widetilde{\sigma}_i$ for all $i \in I \setminus X$. Hence in this setting $Br(\widetilde{W})$ is generated by elements $\widetilde{\varsigma}_i$ for $1 \leq i \leq r$ where

$$\widetilde{\varsigma}_i = m_{\{i,\tau(i)\}\cup X} m_X^{-1} = \begin{cases} \varsigma_i \varsigma_{\tau(i)} & \text{if } 1 \leq i < r, \\ \varsigma_r \varsigma_{r+1} \cdots \varsigma_{n-r+1} \cdots \varsigma_{r+1}\varsigma_r & \text{if } i = r. \end{cases} \tag{7.8}$$

Remark 7.8. In this case the subgroups $Br(W_X)$ and $Br(\widetilde{W})$ commute hence by Lemma 7.6 we have an action of $Br(W_X) \times Br(\widetilde{W})$ on $\mathfrak{k}$ by Lie algebra automorphisms. Indeed for $j \in X$ we have

$$\begin{aligned} \widetilde{\varsigma}_r \varsigma_j &= \varsigma_r \varsigma_{r+1} \cdots \varsigma_{n-r+1} \cdots \varsigma_j \varsigma_{j-1} \cdots \varsigma_{r+1} \varsigma_r \varsigma_j \\ &= \varsigma_r \varsigma_{r+1} \cdots \varsigma_{n-r+1} \cdots \varsigma_{j-1} \varsigma_j \varsigma_{j-1} \cdots \varsigma_{r+1} \varsigma_r \\ &= \varsigma_r \varsigma_{r+1} \cdots \varsigma_{j-1} \varsigma_j \varsigma_{j-1} \cdots \varsigma_{n-r+1} \cdots\cdots \varsigma_{r+1} \varsigma_r \\ &= \varsigma_j \widetilde{\varsigma}_r. \end{aligned}$$

Recall from Lemma 3.12 that the subalgebra $\mathfrak{k}$ is generated by elements

$$\begin{aligned} &e_i, f_i \quad \text{for } i \in X, \\ &h_i - w_X(h_{\tau(i)}) \quad \text{for } i \in I \setminus X, \\ &b_i := f_i + \theta(f_i) \quad \text{for } i \in I \setminus X. \end{aligned}$$

In this setting, let

$$e_X^+ := \big[e_{r+1}, [e_{r+2}, \dots, [e_{\tau(r+2)}, e_{\tau(r+1)}] \dots]\big], \tag{7.9}$$

$$e_X^- := \big[e_{\tau(r+1)}, [e_{\tau(r+2)}, \dots, [e_{r+2}, e_{r+1}] \dots]\big]. \tag{7.10}$$

Then the elements b_i are given explicitly by

$$b_i = \begin{cases} f_i - s(\tau(i)) e_{\tau(i)} & \text{if } i \neq r, \tau(r), \\ f_r - s(\tau(r))[e_X^+, e_{\tau(r)}] & \text{if } i = r, \\ f_{\tau(r)} - s(r)[e_X^-, e_r] & \text{if } i = \tau(r). \end{cases} \tag{7.11}$$

We only calculate the action of $Br(\widetilde{W})$ on the elements b_i for $i \in I \setminus X$. By Lemma 2.34 and Lemma 3.5, we effectively know how $Br(\widetilde{W})$ acts on the remaining elements of $\mathfrak{k}$.

To shorten notation, let $\mathrm{Ad}_i = \psi \circ \mathrm{Ad}(\widetilde{\varsigma}_i) \circ \psi^{-1}$ where $\psi = \mathrm{Ad}(\overline{s})$ as at the end of Section 7.1. In the current setting we have $\overline{s}(i) = s(i)^{1/2}$ if $i \neq r, \tau(r)$.

We recall how $\mathrm{Ad}(\varsigma_i)$ acts on e_j, f_j for $i, j \in I$. We have

$$\mathrm{Ad}(\varsigma_i)(e_j) = \begin{cases} e_j & \text{if } a_{ij} = 0, \\ [e_i, e_j] & \text{if } a_{ij} = -1, \\ -f_j & \text{if } a_{ij} = 2, \end{cases} \quad \mathrm{Ad}(\varsigma_i)(f_j) = \begin{cases} f_j & \text{if } a_{ij} = 0, \\ [f_j, f_i] & \text{if } a_{ij} = -1, \\ -e_j & \text{if } a_{ij} = 2. \end{cases} \tag{7.12}$$

There are many different cases to check in order to compute $\mathrm{Ad}_i(b_j)$ for $i, j \in I \setminus X$. We only look in more detail at the more involved calculations. The following is a consequence of (7.12).

Lemma 7.9. *For $i < r$ and $j \in I \setminus X$ we have*

$$\mathrm{Ad}_i(b_j) = \begin{cases} b_j & \textit{if } a_{ij} = 0 \textit{ and } a_{i\tau(j)} = 0, \\ b_{\tau(j)} & \textit{if } a_{ij} = 2 \textit{ or } a_{i\tau(j)} = 2, \\ s(i)^{-1/2}[b_j, b_i] & \textit{if } a_{ij} = -1, \\ s(\tau(i))^{-1/2}[b_j, b_{\tau(i)}] & \textit{if } a_{i\tau(j)} = -1. \end{cases} \tag{7.13}$$

Proof. We only calculate $\mathrm{Ad}_{r-1}(b_r)$ since $\mathrm{Ad}_{r-1}(b_{\tau(r)})$ is a similar calculation and the others are straightforward checks. We have

$$\begin{aligned} \mathrm{Ad}_{r-1}(b_r) &= \mathrm{Ad}(\overline{s}) \circ \mathrm{Ad}(\varsigma_{r-1}\varsigma_{\tau(r-1)}) \circ \mathrm{Ad}(\overline{s})^{-1}\Big(f_r - s(\tau(r))[e_X^+, e_{\tau(r)}]\Big) \\ &= \mathrm{Ad}(\overline{s})\Big(\overline{s}(r)\mathrm{Ad}(\varsigma_{r-1})(f_r) - s(\tau(r))\overline{s}(\tau(r))^{-1}\mathrm{Ad}(\varsigma_{\tau(r-1)})([e_X^+, e_{\tau(r)}])\Big) \\ &= \overline{s}(r-1)[f_r, f_{r-1}] - s(\tau(r))\overline{s}(\tau(r-1))\big[e_X^+, [e_{\tau(r-1)}, e_{\tau(r)}]\big]. \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} [b_r, b_{r-1}] &= \big[f_r - s(\tau(r))[e_X^+, e_{\tau(r)}], f_{r-1} - s(\tau(r-1))e_{\tau(r-1)}\big] \\ &= [f_r, f_{r-1}] - s(\tau(r-1))[f_r, e_{\tau(r-1)}] - s(\tau(r))\big[[e_X^+, e_{\tau(r)}], f_{r-1}\big] \\ &\quad + s(\tau(r-1))s(\tau(r))\big[[e_X^+, e_{\tau(r)}], e_{\tau(r)}\big] \\ &= [f_r, f_{r-1}] - s(\tau(r-1))s(\tau(r))\big[e_X^+, [e_{\tau(r-1)}, e_{\tau(r)}]\big] \end{aligned}$$

Since $s(r-1)^{-1/2}s(\tau(r-1)) = s(\tau(r-1))^{1/2}$ by Condition 3.11, it follows that

$$\mathrm{Ad}_{r-1}(b_r) = s(r-1)^{-1/2}[b_r, b_{r-1}]$$

as required. □

Now we only need to compute $\mathrm{Ad}_r(b_i)$ for $i \in I \setminus X$. The two key cases are when $i = r-1$ or $i = r$.

Lemma 7.10. *We have*

$$\mathrm{Ad}_r(b_r) = s'(\tau(r))b_r, \tag{7.14}$$

$$\mathrm{Ad}_r(b_{\tau(r)}) = s'(r)b_{\tau(r)}. \tag{7.15}$$

Proof. We check $\mathrm{Ad}_r(b_r)$; the computation for $\mathrm{Ad}_r(b_{\tau(r)})$ is similar. We make two observations. First of all,

$$\begin{aligned}\mathrm{Ad}(\widetilde{\varsigma}_r)(f_r) &= \mathrm{Ad}(\varsigma_r\varsigma_{r+1}\cdots\varsigma_{\tau(r)}\cdots\varsigma_{r+1}\varsigma_r)(f_r)\\ &= \mathrm{Ad}(\varsigma_{\tau(r)}\cdots\varsigma_{r+1}\varsigma_r\varsigma_{r+1})(f_r)\\ &= \mathrm{Ad}(\varsigma_{\tau(r)}\cdots\varsigma_{r+1})(f_{r+1})\\ &= \mathrm{Ad}(\varsigma_{\tau(r)}\cdots\varsigma_{r+2})(-e_{r+1})\\ &= (-1)^{|X|+1}[e_X^+, e_{\tau(r)}].\end{aligned}$$

Additionally, by Lemma 3.5 we have

$$\begin{aligned}\mathrm{Ad}(\widetilde{\varsigma}_r)([e_X^+, e_{\tau(r)}]) &= \mathrm{Ad}(w_{\{r,\tau(r)\}\cup X}w_X^{-1})\mathrm{Ad}(w_X)(e_{\tau(r)}) = \mathrm{Ad}(w_{\{r,\tau(r)\}\cup X})(e_{\tau(r)})\\ &= -f_r.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\mathrm{Ad}_r(b_r) &= \mathrm{Ad}(\overline{s})\circ\mathrm{Ad}(\widetilde{\varsigma}_r)\circ\mathrm{Ad}(\overline{s})^{-1}\big(f_r - s(\tau(r))[e_X^+, e_{\tau(r)}]\big)\\ &= \mathrm{Ad}(\overline{s})\Big(\overline{s}(r)\mathrm{Ad}(\widetilde{\varsigma}_r)(f_r) - \frac{s(\tau(r))}{\overline{s}(\tau(r))}\mathrm{Ad}(\widetilde{\varsigma}_r)([e_X^+, e_{\tau(r)}])\Big)\\ &= (-1)^{|X|+1}\overline{s}(r)\overline{s}(\tau(r))[e_X^+, e_{\tau(r)}] + \frac{s(\tau(r))}{\overline{s}(r)\overline{s}(\tau(r))}f_r.\end{aligned}$$

By the definition of $\overline{s}$ and s' we have

$$\begin{aligned}\frac{s(\tau(r))}{\overline{s}(r)\overline{s}(\tau(r))} &= \frac{(-1)^{\alpha_r(2\rho_X^\vee)}s(\tau(r))}{\overline{s}(r)^2}\\ &= \frac{(-1)^{\alpha_r(2\rho_X^\vee)}s(\tau(r))}{s(r)s'(r)}\\ &= \frac{1}{s'(r)} = s'(\tau(r))\end{aligned}$$

where the last equality follows since $s'(r)s'(\tau(r)) = (-i^2)^{\alpha_j(2\rho_X^\vee)} = 1$. We therefore have

$$\begin{aligned}\mathrm{Ad}_r(b_r) &= s'(\tau(r))f_r - (-1)^{|X|}\frac{s(\tau(r))}{s'(\tau(r))}[e_X^+, e_{\tau(r)}]\\ &= s'(\tau(r))\big(f_r - s(\tau(r))[e_X^+, e_{\tau(r)}]\big)\\ &= s'(\tau(r))b_r\end{aligned}$$

as required. ☐

We define elements f_X^+ and f_X^- similarly to the definitions of e_X^+ and e_X^- from (7.9) and

(7.10) by

$$f_X^+ := \big[f_{r+1}, [f_{r+2}, \dots, [f_{\tau(r+2)}, f_{\tau(r+1)}] \dots]\big], \tag{7.16}$$

$$f_X^- := \big[f_{\tau(r+1)}, [f_{\tau(r+2)}, \dots, [f_{r+2}, f_{r+1}] \dots]\big]. \tag{7.17}$$

Lemma 7.11. *We have*

$$\mathrm{Ad}_r(b_{r-1}) = \frac{s'(r)}{s(r)}\big[b_{r-1}, [b_r, [f_X^+, b_{\tau(r)}]]\big] + s'(r)b_{r-1}, \tag{7.18}$$

$$\mathrm{Ad}_r(b_{\tau(r-1)}) = \frac{s'(\tau(r))}{s(\tau(r))}\big[b_{\tau(r-1)}, [b_{\tau(r)}, [f_X^-, b_r]]\big] + s'(\tau(r))b_{\tau(r-1)}. \tag{7.19}$$

Proof. Since both alculations are similar, we only consider $\mathrm{Ad}_r(b_{r-1})$. We have

$$\begin{aligned}\mathrm{Ad}_r(b_{r-1}) &= \mathrm{Ad}(\overline{s}) \circ \mathrm{Ad}(\widetilde{\varsigma}_r) \circ \mathrm{Ad}(s)^{-1}\big(f_{r-1} - s(\tau(r-1))e_{\tau(r-1)}\big)\\ &= \mathrm{Ad}(\overline{s})\left(\overline{s}(r-1)\mathrm{Ad}(\widetilde{\varsigma}_r)(f_{r-1}) - \frac{s(\tau(r-1))}{\overline{s}(r-1)}\mathrm{Ad}(\widetilde{\varsigma}_r)(e_{\tau(r-1)})\right).\end{aligned}$$

Repeatedly using Equation (7.12) we obtain

$$\begin{aligned}\mathrm{Ad}(\widetilde{\varsigma}_r)(f_{r-1}) = \mathrm{Ad}(\varsigma_{\tau(r)} \cdots \varsigma_r)(f_{r-1}) &= \big[f_{r-1}, [f_r, \cdots, [f_{\tau(r+1)}, f_{\tau(r)}] \cdots]\big]\\ &= \big[f_{r-1}, [f_r, [f_X^+, f_{\tau(r)}]]\big],\\ \mathrm{Ad}(\widetilde{\varsigma}_r)(e_{\tau(r-1)}) = \mathrm{Ad}(\varsigma_r \cdots \varsigma_{\tau(r)})(e_{\tau(r-1)}) &= \big[e_r, [e_X^+, [e_{\tau(r)}, e_{\tau(r-1)}]]\big]\\ &= -\big[e_{\tau(r-1)}, [e_{\tau(r)}, [e_X^+, e_r]]\big].\end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{Ad}_r(b_{r-1}) = &\frac{1}{\overline{s}(r)\overline{s}(\tau(r))}\big[f_{r-1}, [f_r, [f_X^+, f_{\tau(r)}]]\big]\\ &- \overline{s}(r)\overline{s}(\tau(r))\big[-s(\tau(r-1))e_{\tau(r-1)}, [e_{\tau(r)}, [e_X^+, e_r]]\big].\end{aligned}$$

As $b_{r-1} = f_{r-1} - s(\tau(r-1))e_{\tau(r-1)}$ and $[e_i, f_j] = \delta_{ij}h_i$ for $i, j \in I$ we have

$$\begin{aligned}\mathrm{Ad}_r(b_{r-1}) = &\frac{1}{\overline{s}(r)\overline{s}(\tau(r))}\big[b_{r-1} + s(\tau(r-1))e_{\tau(r-1)}, [f_r, [f_X^+, f_{\tau(r)}]]\big]\\ &- \overline{s}(r)\overline{s}(\tau(r))\big[b_{r-1} - f_{r-1}, [e_{\tau(r)}, [e_X^+, e_r]]\big]\\ = &\frac{s'(r)}{s(r)}\Big[b_{r-1}, [f_r, [f_X^+, f_{\tau(r)}]] - \frac{s(r)^2}{s'(r)^2}[e_{\tau(r)}, [e_X^+, e_r]]\Big]\end{aligned}$$

where we use the fact that $\overline{s}(r)\overline{s}(\tau(r)) = \frac{s(r)}{s'(r)}$. Since $b_r = f_r - s(\tau(r))[e_X^+, e_{\tau(r)}]$ we have

$$\begin{aligned}\mathrm{Ad}_r(b_{r-1}) &= \frac{s'(r)}{s(r)}\Big[b_{r-1}, [b_r + s(\tau(r))[e_X^+, e_{\tau(r)}], [f_X^+, f_{\tau(r)}]] + \frac{s(r)^2}{s'(r)^2}[[e_X^+, e_{\tau(r)}], e_r]\Big]\\ &= \frac{s'(r)}{s(r)}\Big[b_{r-1}, [b_r, [f_X^+, f_{\tau(r)}]] + s(\tau(r))\big[[e_X^+, e_{\tau(r)}], [f_X^+, f_{\tau(r)}]\big]\\ &\quad + \frac{s(r)^2}{s'(r)^2}\big[[e_X^+, e_{\tau(r)}], e_r\big]\Big].\end{aligned}$$

Since $[e_X^+, e_{\tau(r)}] = \mathrm{Ad}(w_X)(e_{\tau(r)})$ and $[f_X^+, f_{\tau(r)}] = (-1)^{|X|}\mathrm{Ad}(w_X)(f_{\tau(r)})$ we rewrite

$$\begin{aligned}\big[[e_X^+, e_{\tau(r)}], [f_X^+, f_{\tau(r)}]\big] &= (-1)^{|X|}[\mathrm{Ad}(w_X)(e_{\tau(r)}, \mathrm{Ad}(w_X)(f_{\tau(r)})] \\ &= (-1)^{|X|}\mathrm{Ad}(w_X)(h_{\tau(r)}) \\ &= (-1)^{|X|}\sum_{i=r+1}^{\tau(r)} h_i.\end{aligned}$$

Hence

$$\mathrm{Ad}_r(b_{r-1}) = \frac{s'(r)}{s(r)}\Big[b_{r-1}, \big[b_r, [f_X^+, f_{\tau(r)}]\big] + s(\tau(r))(-1)^{|X|}\sum_{i=r+1}^{\tau(r)} h_i + \frac{s(r)^2}{s'(r)^2}\big[[e_X^+, e_{\tau(r)}], e_r\big]\Big].$$

Finally, we use $b_{\tau(r)} = f_{\tau(r)} - s(r)[e_X^-, e_r]$ and the fact that

$$\Big[b_{r-1}, s(\tau(r))(-1)^{|X|}\sum_{i=r+1}^{\tau(r)} h_i\Big] = [b_{r-1}, s(r)h_{\tau(r)}]$$

to obtain

$$\mathrm{Ad}_r(b_{r-1}) = \frac{s'(r)}{s(r)}\Big[b_{r-1}, \big[b_r, [f_X^+, b_{\tau(r)} + s(r)[e_X^-, e_r]]\big] + s(r)h_{\tau(r)} + \frac{s(r)^2}{s'(r)^2}\big[[e_X^+, e_{\tau(r)}], e_r\big]\Big].$$

We have $f_X^+ = \mathrm{Ad}(\varsigma_{\tau(r+1)}\cdots\varsigma_{r+2})(f_{r+1})$ and $e_X^- = \mathrm{Ad}(\varsigma_{\tau(r+1)}\cdots\varsigma_{r+2})(e_{r+1})$ which implies

$$\begin{aligned}\big[f_X^+, [e_X^-, e_r]\big] &= \big[[f_X^+, e_X^-], e_r\big] && (7.20)\\ &= [-\mathrm{Ad}(\varsigma_{\tau(r+1)}\cdots\varsigma_{r+2})(h_{r+1}), e_r] && (7.21)\\ &= -\Big[\sum_{i=r+1}^{\tau(r+1)} h_i, e_r\Big] = e_r. && (7.22)\end{aligned}$$

Substituting this into the expression for $\mathrm{Ad}_r(b_{r-1})$ and using the fact that

$$\frac{s(r)^2}{s'(r)^2} = (-1)^{|X|}s(r)^2 = s(r)s(\tau(r))$$

we hence have

$$\begin{aligned}\mathrm{Ad}_r(b_{r-1}) &= \frac{s'(r)}{s(r)}\Big[b_{r-1}, \big[b_r, [f_X^+, b_{\tau(r)}]\big] + s(r)h_{\tau(r)} + s(r)[b_r, e_r] + \frac{s(r)^2}{s'(r)^2}\big[[e_X^+, e_{\tau(r)}], e_r\big]\Big] \\ &= \frac{s'(r)}{s(r)}\Big[b_{r-1}, \big[b_r, [f_X^+, b_{\tau(r)}]\big] + s(r)h_{\tau(r)} + s(r)[f_r, e_r]\Big] \\ &= \frac{s'(r)}{s(r)}\big[b_{r-1}, [b_r, [f_X^+, b_{\tau(r)}]]\big] + s'(r)[b_{r-1}, h_r + h_{\tau(r)}] \\ &= \frac{s'(r)}{s(r)}\big[b_{r-1}, [b_r, [f_X^+, b_{\tau(r)}]]\big] + s'(r)b_{r-1}\end{aligned}$$

as required. □

7.3 Braid group action of $Br(W_X)$ on $B_{\mathbf{c},\mathbf{s}}$

By Lemma 7.6 the construction of the braid group action on $\mathfrak{k}$ by Lie algebra automorphisms is guided by the braid group action of $Br(\mathfrak{g})$ on $\mathfrak{g}$. We expect a similar connection in the setting of quantum symmetric pairs. In particular, we use the Lusztig automorphisms T_i as a guide for constructing a braid group action on $B_{\mathbf{c},\mathbf{s}}$ by algebra automorphisms. Recall that T_i for $i \in X$ give rise to a representation of $Br(W_X)$ on $U_q(\mathfrak{g})$. We show that the subalgebra $B_{\mathbf{c},\mathbf{s}}$ of $U_q(\mathfrak{g})$ is invariant under T_i for $i \in X$. This implies that there is a representation of $Br(W_X)$ on $B_{\mathbf{c},\mathbf{s}}$.

We note that although similar, the results of this section were found independently from [6, Section 4.1].

By Equations (2.66)–(2.72) it follows that $T_i(\mathcal{M}_X U^0_\Theta) = \mathcal{M}_X U^0_\Theta$. Hence we only need to compute $T_i(B_j)$ for $j \in I \setminus X$. The following lemma provides the key step.

Lemma 7.12. *The relation*

$$T_i T_{w_X}(E_{\tau(j)}) = \sum_{r=0}^{-a_{ij}} (-1)^r q_i^r F_i^{(r)} T_{w_X}(E_{\tau(j)}) K_j^{-1} F_i^{(-a_{ij}-r)} K_{\sigma_i(\alpha_j)} \tag{7.23}$$

holds for any $i \in X, j \in I \setminus X$.

Proof. First observe that for any $i \in X$ we have $T_i T_{w_X} = T_{w_X} T_{\tau(i)}$ since $\sigma_i w_X = w_X \sigma_{\tau(i)}$. Recall from (3.26) that $T_{w_X}(E_{\tau(i)}) = -F_i K_i$. Using this and Equation (2.69) we have

$$\begin{aligned} T_i T_{w_X}(E_{\tau(j)}) &= T_{w_X} T_{\tau(i)}(E_{\tau(j)}) \\ &= T_{w_X}\left(\sum_{s=0}^{-a_{ij}} (-1)^s q_i^{-s} E_{\tau(i)}^{(-a_{ij}-s)} E_{\tau(j)} E_{\tau(i)}^{(s)} \right) \\ &= \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{-s} \big(F_i K_i\big)^{(-a_{ij}-s)} T_{w_X}(E_{\tau(j)}) \big(F_i K_i\big)^{(s)}. \end{aligned}$$

By Relation (Q3) of Definition 2.26 we have $K_i F_i = q_i^{-2} F_i K_i$ and hence it follows that

$$\big(F_i K_i\big)^{(s)} = q_i^{-s(s-1)} F_i^{(s)} K_i^s$$

for all $i \in I$. Substituting this into the expression fr $T_i T_{w_X}(E_{\tau(j)})$ we obtain

$$T_i T_{w_X}(E_{\tau(j)}) = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{-a_{ij}^2 - 2sa_{ij} - 2s^2 - a_{ij} - s} F_i^{(-a_{ij}-s)} K_i^{-a_{ij}-s} T_{w_X}(E_{\tau(j)}) F_i^{(s)} K_i^s.$$

By Relation (Q2) of Definition 2.26 we have

$$K_i T_{w_X}(E_{\tau(j)}) = q^{(\alpha_i, w_X(\alpha_{\tau(j)}))} T_{w_X}(E_{\tau(j)}) K_i.$$

Since the inner product $(-,-)$ is invariant under w_X and τ we have $q^{(\alpha_i, w_X(\alpha_{\tau(j)}))} = q_i^{-a_{ij}}$.

Hence

$$T_iT_{w_X}(E_{\tau(j)}) = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{-sa_{ij}-2s^2-a_{ij}-s} F_i^{(-a_{ij}-s)} T_{w_X}(E_{\tau(j)}) K_i^{-a_{ij}-s} F_i^{(s)} K_i^s.$$

As $K_i^{-a_{ij}} = K_{\sigma_i(\alpha_j)} K_j^{-1}$ and $K_{\sigma_i(\alpha_j)} F_i = q_i^{a_{ij}} F_i K_{\sigma_i(\alpha_j)}$ we have

$$T_iT_{w_X}(E_{\tau(j)}) = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{-a_{ij}-s} F_i^{(-a_{ij}-s)} T_{w_X}(E_{\tau(j)}) K_j^{-1} F_i^{(s)} K_{\sigma_i(\alpha_j)}.$$

The result follows from a change of index by setting $r = -a_{ij} - s$ above. □

Proposition 7.13. *For any $i \in X$ we have $T_i(B_{\mathbf{c},\mathbf{s}}) = B_{\mathbf{c},\mathbf{s}}$.*

Proof. By Equation (2.71) and the previous lemma we have

$$T_i(B_j) = \sum_{r=0}^{-a_{ij}} (-1)^r q_i^r F_i^{(r)} \big(B_j - s_j K_j^{-1}\big) F_i^{(-a_{ij}-r)} + s_j K_{-\sigma_i(\alpha_j)}$$

for $j \in I \setminus X$. If $s_j \neq 0$ then $j \in I_{ns}$ and hence $a_{ij} = 0$. This implies

$$T_i(B_j) = B_j. \tag{7.24}$$

On the other hand, if $s_j = 0$, then

$$T_i(B_j) = \sum_{r=0}^{-a_{ij}} (-1)^r q_i^r F_i^{(r)} B_j F_i^{(-a_{ij}-r)} \in B_{\mathbf{c},\mathbf{s}}. \tag{7.25}$$

Therefore $T_i(B_{\mathbf{c},\mathbf{s}}) \subseteq B_{\mathbf{c},\mathbf{s}}$. Using the relation $T_i^{-1} = \sigma \circ T_i \circ \sigma$ one shows that $T_i^{-1}(B_{\mathbf{c},\mathbf{s}}) \subseteq B_{\mathbf{c},\mathbf{s}}$ and hence $B_{\mathbf{c},\mathbf{s}} \subseteq T_i(B_{\mathbf{c},\mathbf{s}})$. This implies $T_i(B_{\mathbf{c},\mathbf{s}}) = B_{\mathbf{c},\mathbf{s}}$ as required. □

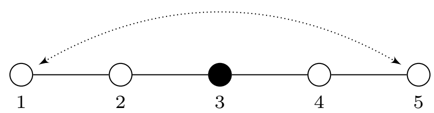
Since the algebra automorphisms T_i satisfy braid relations, the following corollary follows immediately.

Corollary 7.14. *There exists an action of $Br(W_X)$ on $B_{\mathbf{c},\mathbf{s}}$ by algebra automorphisms given by T_i for $i \in X$.*

7.4 Braid group action of $Br(\widetilde{W})$ on $B_{\mathbf{c},\mathbf{s}}$ in type AIII

Recall from Chapter 5 that the algebra automorphisms $\widetilde{T}_i = T_{\widetilde{\sigma}_i}$ for $i \in I \setminus X$ give rise to a representation of $Br(\widetilde{W})$ on $U_q(\mathfrak{g})$. The problem we encounter is that $\widetilde{T}_i$ does not restrict to an algebra automorphism of $B_{\mathbf{c},\mathbf{s}}$.

Example 7.15. Let $\mathfrak{g} = \mathfrak{sl}_6(\mathbb{C})$ and consider the Satake diagram



Then $B_1 = F_1 - c_1 s(5) E_5 K_1^{-1} + s_1 K_1^{-1}$. However, we have

$$\widetilde{T}_1(B_1) = T_1 T_5(B_1) = c_1 s(5) F_5 K_5 K_1 - K_1^{-1} E_1 + s_1 K_1$$

which is not an element of $B_{\mathbf{c},\mathbf{s}}$.

Recall from Chapter 5 that there is an algebra isomorphism $\varphi_{\mathbf{s},\mathbf{s}'} : B_{\mathbf{c},\mathbf{s}} \to B_{\mathbf{c},\mathbf{s}'}$ such that $\varphi_{\mathbf{s},\mathbf{s}'}(B_i^{\mathbf{c},\mathbf{s}}) = B_i^{\mathbf{c},\mathbf{s}'}$ and $\varphi_{\mathbf{s},\mathbf{s}'}|_{\mathcal{M}_X U_\Theta^0} = \mathrm{id}|_{\mathcal{M}_X U_\Theta^0}$. This implies that we can assume $\mathbf{s} = \mathbf{0}$. In order to give a corresponding algebra isomorphism for the parameters $\mathbf{c} \in \mathcal{C}$ we make an additional assumption. More specifically we assume

$$c_i \in \pm q^{\mathbb{Z}} \quad \text{for all } i \in I \setminus X. \tag{7.26}$$

It follows from this and Equation (5.4) that

$$c_i c_{\tau(i)} = q^{(\alpha_i, \Theta(\alpha_i) - 2\rho_X)} \quad \text{for all } i \in I \setminus X. \tag{7.27}$$

The following proposition is a general result that holds for all Satake diagrams. Recall that the subalgebra U_Θ^0 is generated by the elements $\{K_i K_{\tau(i)}^{-1} \mid i \in I \setminus X\}$ and $\{K_j \mid j \in X\} \subset \mathcal{M}_X$.

Proposition 7.16. *Let (X, τ) be any Satake diagram and suppose $\mathbf{c}, \mathbf{c}' \in \mathcal{C}$ satisfy Equation (5.4) and Condition (7.26). Then the map $\mathcal{A}_{\mathbf{c},\mathbf{c}'} : B_{\mathbf{c},\mathbf{s}} \to B_{\mathbf{c}',\mathbf{s}}$ defined by*

$$\mathcal{A}_{\mathbf{c},\mathbf{c}'}(B_i^{\mathbf{c},\mathbf{s}}) = B_i^{\mathbf{c}',\mathbf{s}} \qquad \textit{for all } i \in I \setminus X, \tag{7.28}$$

$$\mathcal{A}_{\mathbf{c},\mathbf{c}'}(K_i K_{\tau(i)}^{-1}) = \frac{c'_{\tau(i)}}{c_{\tau(i)}} K_i K_{\tau(i)}^{-1} \qquad \textit{for all } i \in I \setminus X, i \neq \tau(i), \tag{7.29}$$

and $\mathcal{A}_{\mathbf{c},\mathbf{c}'}|_{\mathcal{M}_X} = \mathrm{id}|_{\mathcal{M}_X}$ is an algebra isomorphism.

Proof. To show that $\mathcal{A}_{\mathbf{c},\mathbf{c}'}$ is an algebra homomorphism, we only need to check that all relations of $B_{\mathbf{c},\mathbf{s}}$ are preserved. By (7.27) we have

$$\mathcal{A}_{\mathbf{c},\mathbf{c}'}(K_i K_{\tau(i)}^{-1}) \mathcal{A}_{\mathbf{c},\mathbf{c}'}(K_{\tau(i)} K_i^{-1}) = \frac{c'_{\tau(i)}}{c_{\tau(i)}} \frac{c'_i}{c_i} = 1 = \mathcal{A}_{\mathbf{c},\mathbf{c}'}(K_{\tau(i)} K_i^{-1}) \mathcal{A}_{\mathbf{c},\mathbf{c}'}(K_i K_{\tau(i)}).$$

Since $\mathcal{A}_{\mathbf{c},\mathbf{c}'}$ rescales elements of U_Θ^0 and $\mathcal{A}_{\mathbf{c},\mathbf{c}'}|_{\mathcal{M}_X} = \mathrm{id}|_{\mathcal{M}_X}$ we only need to check relation (3.38). In particular $\mathcal{A}_{\mathbf{c},\mathbf{c}'}$ preserves (3.38) if

$$\mathcal{A}_{\mathbf{c},\mathbf{c}'}(C_{ij}(\mathbf{c})) = C_{ij}(\mathbf{c}')$$

for all $i, j \in I \setminus X$. This is immediate by Theorems 3.27 and 3.30 by noting that

$$\mathcal{A}_{\mathbf{c},\mathbf{c}'}(\mathcal{Z}_i) = \frac{c'_i}{c_i} \mathcal{Z}_i \quad \text{for all } i \in I \setminus X$$

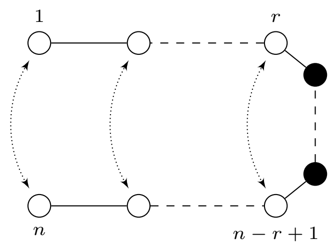
and

$$\mathcal{A}_{\mathbf{c},\mathbf{c}'}(\mathcal{W}_{ij}) = \frac{c'_i}{c_i} \mathcal{W}_{ij} \quad \text{for all } i \in I \setminus X, j \in X.$$

The checks for $a_{ij} = -2$ or $a_{ij} = -3$ are done using [3, Theorem 3.7/3.8]. □

7.4.1 Generators and relations in type AIII

For the remainder of this chapter, we consider quantum symmetric pairs arising from the Satake diagram of type AIII



where we assume $X \neq \emptyset$. This is the type A_n example containing black nodes and a non-trivial diagram automorphism.

Recall from (7.8) that $Br(\widetilde{W})$ is generated by the elements $\widetilde{\varsigma}_i$ for $1 \leq i \leq r$ since $\widetilde{\varsigma}_i = \widetilde{\varsigma}_{\tau(i)}$ for any $i \in I \setminus X$. In the type AIII setting the generators $\widetilde{\varsigma}_i$ are given explicitly by

$$\widetilde{\varsigma}_i = \begin{cases} \varsigma_i \varsigma_{\tau(i)} & \text{if } 1 \leq i < r, \\ \varsigma_r \varsigma_{r+1} \cdots \varsigma_{\tau(r)} \cdots \varsigma_{r+1} \varsigma_r & \text{if } i = r \end{cases}$$

subject to the relations

$$\widetilde{\varsigma}_i \widetilde{\varsigma}_j = \widetilde{\varsigma}_j \widetilde{\varsigma}_i \qquad \text{if } a_{ij} = 0 \text{ and } 1 \leq i, j \leq r, \tag{7.30}$$

$$\widetilde{\varsigma}_i \widetilde{\varsigma}_j \widetilde{\varsigma}_i = \widetilde{\varsigma}_j \widetilde{\varsigma}_i \widetilde{\varsigma}_j \qquad \text{if } a_{ij} = -1 \text{ and } 1 \leq i, j < r, \tag{7.31}$$

$$\widetilde{\varsigma}_i \widetilde{\varsigma}_j \widetilde{\varsigma}_i \widetilde{\varsigma}_j = \widetilde{\varsigma}_j \widetilde{\varsigma}_i \widetilde{\varsigma}_j \widetilde{\varsigma}_i \qquad \text{if } a_{ij} = -1 \text{ and } i = r, j = r-1. \tag{7.32}$$

Hence $Br(\widetilde{W})$ is isomorphic to the braid group of type B in r generators, denoted by $Br(\mathfrak{b}_r)$.

By Conditions (3.10) and (3.11) of $s : I \to \mathbb{C}^\times$ we have

$$s(i) = s(\tau(i)) \quad \text{if } i \in I \setminus (X \cup \{r, \tau(r)\}),$$
$$s(r) = (-1)^{|X|} s(\tau(r)).$$

We are free to choose s subject to these conditions so we let

$$s(i) = \begin{cases} (-1)^{|X|} & \text{if } i = r, \\ 1 & \text{otherwise.} \end{cases} \tag{7.33}$$

By the definition (3.33) and (7.27) we have $c_i \in \{\pm 1\}$ for all $i \in I \setminus (X \cup \{r, \tau(r)\})$ and $c_r c_{\tau(r)} = q^{|X|+1}$. In view of Proposition 7.16 we fix

$$c_i = \begin{cases} 1 & \text{if } i \neq r, \tau(r), \\ q^{|X|} & \text{if } i = r, \\ q & \text{if } i = \tau(r). \end{cases} \tag{7.34}$$

For technical reasons also observed in [40] we require the field $\mathbb{K}(q)$ to contain the square roots of q and -1.

Recall from Equation (5.21) the notation T_{i--j} and T_{j--i} for $1 \leq i \leq j \leq n$. In this setting, $B_{\mathbf{c}} = B_{\mathbf{c},\mathbf{0}}$ is the subalgebra of $U_q(\mathfrak{sl}_{n+1}(\mathbb{C}))$ generated by

$$B_i = \begin{cases} F_i - E_{\tau(i)}K_i^{-1} & \text{if } i \in I \setminus (X \cup \{r, \tau(r)\}), \\ F_r - (-q)^{|X|}T_{r+1--\tau(r+1)}(E_{\tau(r)})K_r^{-1} & \text{if } i = r, \\ F_{\tau(r)} - qT_{\tau(r+1)--r+1}(E_r)K_{\tau(r)}^{-1} & \text{if } i = \tau(r) \end{cases} \tag{7.35}$$

and the elements

$$\begin{aligned} &E_i, F_i, K_i^{\pm 1} \quad \text{for } i \in X, \\ &K_i K_{\tau(i)}^{-1} \quad \text{for } i \in I \setminus X. \end{aligned}$$

By (3.41) and (2.77) one finds that the elements $\mathcal{Z}_i$ for $i \in I \setminus X$ are given by

$$\mathcal{Z}_i = \begin{cases} -(1-q^{-2})T_{r+1--\tau(r+2)}(E_{\tau(r+1)})K_{\tau(r)}K_r^{-1} & \text{if } i = r, \\ -(-1)^{|X|}(1-q^{-2})T_{\tau(r+1)--r+2}(E_{r+1})K_rK_{\tau(r)}^{-1} & \text{if } i = \tau(r), \\ -K_{\tau(i)}K_i^{-1} & \text{otherwise.} \end{cases} \tag{7.36}$$

By Theorems 3.27 and 3.30 the algebra $B_{\mathbf{c}}$ is generated over $\mathcal{M}_X U_\Theta^0$ by the elements B_i for $i \in I \setminus X$ subject to the relations

$$B_iK_jK_{\tau(j)}^{-1} = q^{(\alpha_j - \alpha_{\tau(j)}, \alpha_i)}K_jK_{\tau(j)}^{-1}B_i \qquad \text{for } i, j \in I \setminus X, \tag{7.37}$$

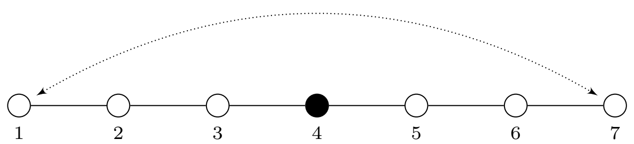
$$B_iE_j - E_jB_i = 0 \qquad \text{for } i \in I \setminus X, j \in X, \tag{7.38}$$

$$F_{ij}(B_i, B_j) = \delta_{i,\tau(j)}(q - q^{-1})^{-1}(c_i\mathcal{Z}_i - c_j\mathcal{Z}_j) \qquad \text{for } i, j \in I \setminus X,\ a_{ij} = 0, \tag{7.39}$$

$$F_{ij}(B_i, B_j) = 0 \qquad \text{otherwise.} \tag{7.40}$$

7.4.2 The case $|X| = 1$

In order to construct an action of $Br(\widetilde{W})$ on $B_{\mathbf{c}}$, we first complete the constructions in small rank cases with $|X| = 1$ and $|X| = 2$. We explain this procedure in detail through the following example.



In Theorem 7.22 we show that it is enough to consider this case to obtain a braid group action in general for $|X| = 1$. We first note that $\widetilde{T}_i(U_\Theta^0) = U_\Theta^0$ and $\widetilde{T}_i|_{\mathcal{M}_X} = \mathrm{id}|_{\mathcal{M}_X}$. This allows us to define

$$\mathcal{T}_i|_{U_\Theta^0} = \widetilde{T}_i|_{U_\Theta^0}, \quad \mathcal{T}_i|_{\mathcal{M}_X} = \mathrm{id}|_{\mathcal{M}_X} \tag{7.41}$$

for all $i = 1, \dots, r$. The generators B_i of $B_{\mathbf{c}}$ are given by

$$B_1 = F_1 - E_7 K_1^{-1}, \qquad B_5 = F_5 - q[E_4, E_3]_{q^{-1}} K_5^{-1},$$
$$B_2 = F_2 - E_6 K_2^{-1}, \qquad B_6 = F_6 - E_2 K_6^{-1},$$
$$B_3 = F_3 + q[E_4, E_5]_{q^{-1}} K_3^{-1}, \qquad B_7 = F_7 - E_1 K_7^{-1}$$

and the elements $\mathcal{Z}_i$ are given by

$$\mathcal{Z}_1 = -K_7 K_1^{-1}, \qquad \mathcal{Z}_5 = -(1-q^{-2}) E_4 K_3 K_5^{-1},$$
$$\mathcal{Z}_2 = -K_6 K_2^{-1}, \qquad \mathcal{Z}_6 = -K_2 K_6^{-1},$$
$$\mathcal{Z}_3 = (1-q^{-2}) E_4 K_5 K_3^{-1}, \qquad \mathcal{Z}_7 = -K_1 K_7^{-1}.$$

By evaluating $\widetilde{T}_1 = T_1 T_7$ on each generator B_i, we obtain an ansatz for $\mathcal{T}_1$ by calculating the summand with the highest order. We use the notation $\mathcal{T}_1'$ to denote this ansatz, and any updates to this are denoted by $\mathcal{T}_1''$, $\mathcal{T}_1'''$ and so on. We have

$$T_1 T_7(B_1) = -K_1^{-1} E_1 + F_7 K_7 K_1$$

with highest order summand $F_7 K_7 K_1$. We hence define

$$\mathcal{T}_1'(B_1) = B_7.$$

Calculating similarly we have

$$\begin{aligned}
T_1 T_7(B_2) &= [F_2, F_1]_q - [E_7, E_6]_{q^{-1}} K_1^{-1} K_2^{-1},\\
T_1 T_7(B_3) &= B_3,\\
T_1 T_7(B_5) &= B_5,\\
T_1 T_7(B_6) &= [F_6, F_7]_q - [E_1, E_2]_{q^{-1}} K_6^{-1} K_7^{-1},\\
T_1 T_7(B_7) &= -K_7^{-1} E_7 + F_1 K_1 K_7
\end{aligned}$$

and thus the ansatz for $\mathcal{T}_1$ on the generators B_i is

$$\mathcal{T}_1'(B_1) = B_7, \qquad \mathcal{T}_1'(B_5) = B_5,$$
$$\mathcal{T}_1'(B_2) = [B_2, B_1]_q, \qquad \mathcal{T}_1'(B_6) = [B_6, B_7]_q,$$
$$\mathcal{T}_1'(B_3) = B_3, \qquad \mathcal{T}_1'(B_7) = B_1.$$

Using the relations of $B_{\mathbf{c}}$ we modify this ansatz. For example, the relation

$$B_1 B_6 - B_6 B_1 = 0$$

holds in $B_{\mathbf{c}}$ but we see that

$$\mathcal{T}_1'(B_1)\mathcal{T}_1'(B_6) - q\mathcal{T}_1'(B_6)\mathcal{T}_1'(B_1) = 0.$$

In order to correct this we observe that

$$\mathcal{T}_1'(B_6) K_7 K_1^{-1} = q^{-1} K_7 K_1^{-1} \mathcal{T}_1'(B_6)$$

holds by Relation (7.37). This implies that we may correct $\mathcal{T}_1'(B_1)$ by setting

$$\mathcal{T}_1''(B_1) = B_7 K_7 K_1^{-1}. \tag{7.42}$$

By symmetry, we also let

$$\mathcal{T}_1''(B_7) = B_1 K_1 K_7^{-1}.$$

With this, we now have the relation

$$\mathcal{T}_1''(B_1)\mathcal{T}_1''(B_7) - \mathcal{T}_1''(B_7)\mathcal{T}_1''(B_1) = q^2(B_7B_1 - B_1B_7).$$

for symmetry reasons we give both $\mathcal{T}_1''(B_1)$ and $\mathcal{T}''(B_7)$ a factor q^{-1}. Similarly, in view of the relation

$$B_2B_6 - B_6B_2 = (q-q^{-1})^{-1}(\mathcal{Z}_2 - \mathcal{Z}_6)$$

one finds that

$$\mathcal{T}_1'(B_2)\mathcal{T}_1'(B_6) - \mathcal{T}_1'(B_6)\mathcal{T}_1'(B_2) = q(q-q^{-1})^{-1}(T_1T_7(\mathcal{Z}_2) - T_1T_7(\mathcal{Z}_6))$$

holds in $B_{\mathbf{c}}$ and hence we choose to give $\mathcal{T}_1'(B_2)$ and $\mathcal{T}_1'(B_6)$ a factor $q^{-1/2}$ each. Putting this together we define

$$\mathcal{T}_1(B_i) = \begin{cases} q^{-1}B_{\tau(i)}K_{\tau(i)}K_i^{-1} & \text{if } i = 1,7, \\ q^{-1/2}[B_2, B_1]_q & \text{if } i = 2, \\ q^{-1/2}[B_6, B_7]_q & \text{if } i = 6, \\ B_i & \text{if } i = 3,5. \end{cases} \tag{7.43}$$

Proposition 7.17. *Let $r = 3$ and $X = \{4\}$.*

(1) *There exists a unique algebra automorphism $\mathcal{T}_1$ of $B_{\mathbf{c}}$ such that $\mathcal{T}_1(B_i)$ is given by (7.43) for $i \in I \setminus X$ and $\mathcal{T}_1|_{\mathcal{M}_X U_\Theta^0} = T_1T_7|_{\mathcal{M}_X U_\Theta^0}$.*

(2) *The inverse automorphism $\mathcal{T}_1^{-1}$ of $B_{\mathbf{c}}$ is defined by*

$$\mathcal{T}_1^{-1}(B_i) = \begin{cases} qB_{\tau(i)}K_iK_{\tau(i)}^{-1} & \text{if } i = 1,7, \\ q^{-1/2}[B_1, B_2]_q & \text{if } i = 2, \\ q^{-1/2}[B_7, B_6]_q & \text{if } i = 6, \\ B_i & \text{if } i = 3,5. \end{cases} \tag{7.44}$$

with $\mathcal{T}_1^{-1}|_{\mathcal{M}_X U_\Theta^0} = T_1^{-1}T_7^{-1}|_{\mathcal{M}_X U_\Theta^0}$.

Sketch of proof. The proof is given by direct calculation using the package `QUAGROUP` under `GAP` using the file `A7_oneblacknode.txt` contained in [24]. First we define the generators of $B_{\mathbf{c}}$ and check that the defining relations are satisfied. We then check that the images $\mathcal{T}_1(B_j)$ and $\mathcal{T}_1^{-1}(B_j)$ also satisfy the relations which implies that $\mathcal{T}_1$ and $\mathcal{T}_1^{-1}$ are well-defined algebra endomorphisms of $B_{\mathbf{c}}$. Finally, we confirm that $\mathcal{T}_1$ and $\mathcal{T}_1^{-1}$ are mutual inverses to one another by showing $\mathcal{T}_1\mathcal{T}_1^{-1}(B_j) = \mathcal{T}_1^{-1}\mathcal{T}_1(B_j)$ for all $j \in I \setminus X$. □

Remark 7.18. The algebra automorphism $\mathcal{T}_1$ has already been observed in [40, Theorems 4.3 and 4.6] when $|X| = 0$. The main difference is [40, Equation (4.6)] which considers the case $a_{ij} = -1$ and $a_{\tau(i)j} = -1$. This condition does not appear in the current setting since $|X| \neq \emptyset$. It is hence reasonable to expect that the algebra automorphisms $\mathcal{T}_i$ for $i < r$ have the same form. This is so since the automorphisms $\widetilde{T}_i$ that we use to guide our constructions do not depend on X for $i < r$.

Taking the above remark into account we define

$$\mathcal{T}_2(B_i) = \begin{cases} q^{-1}B_{\tau(i)}K_{\tau(i)}K_i^{-1} & \text{if } i = 2,6, \\ q^{-1/2}[B_2, B_i]_q & \text{if } i = 1,3, \\ q^{-1/2}[B_6, B_i]_q & \text{if } i = 5,7. \end{cases} \tag{7.45}$$

The following proposition also requires the use of GAP, as in the sketch proof of Proposition 7.17. This is also contained in the file A7_oneblacknode.txt in [24].

Proposition 7.19. *Let $r = 3$ and $X = \{4\}$.*

(1) *There exists a unique algebra automorphism $\mathcal{T}_2$ of $B_\mathbf{c}$ such that $\mathcal{T}_2(B_i)$ is given by (7.45) for $i \in I \setminus X$ and $\mathcal{T}_2|_{\mathcal{M}_X U_\Theta^0} = T_2T_6|_{\mathcal{M}_X U_\Theta^0}$.*

(2) *The inverse automorphism $\mathcal{T}_2^{-1}$ of $B_\mathbf{c}$ is given by*

$$\mathcal{T}_2^{-1}(B_i) = \begin{cases} qB_{\tau(i)}K_iK_{\tau(i)}^{-1} & \text{if } i = 2,6, \\ q^{-1/2}[B_i, B_2]_q & \text{if } i = 1,3, \\ q^{-1/2}[B_i, B_6]_q & \text{if } i = 5,7. \end{cases} \tag{7.46}$$

and $\mathcal{T}_2^{-1}|_{\mathcal{M}_X U_\Theta^0} = T_2^{-1}T_6^{-1}|_{\mathcal{M}_X U_\Theta^0}$.

We now construct the algebra automorphism $\mathcal{T}_3$, using the Lusztig automorphism $T_{34543} = T_3T_4T_5T_4T_3$ as a starting point. Since

$$T_{34543}(F_2) = \big[F_2, [F_3, [F_4, F_5]_q]_q\big]_q$$

and

$$T_{34543}(F_6) = \big[F_6, [F_5, [F_4, F_3]_q]_q\big]_q$$

we obtain the ansatz

$$\mathcal{T}_3'(B_1) = B_1, \qquad \mathcal{T}_3'(B_5) = B_5,$$
$$\mathcal{T}_3'(B_2) = \big[B_2, [B_3, [B_4, B_5]_q]_q\big]_q, \qquad \mathcal{T}_3'(B_6) = \big[B_6, [B_5, [B_4, B_3]_q]_q\big]_q,$$
$$\mathcal{T}_3'(B_3) = B_3, \qquad \mathcal{T}_3'(B_7) = B_7.$$

The automorphism T_{34543} acts as the identity on B_1 and B_7 so we expect that no corrections need to be made to $\mathcal{T}_3'(B_1)$ and $\mathcal{T}_3'(B_7)$. In order to improve the ansatz for $\mathcal{T}_3'$, `GAP` is used directly to check relations. Such checks are not shown in our files. In view of the relation $B_2E_4 - E_4B_2$ we find that

$$\mathcal{T}_3'(B_2)E_4 - E_4\mathcal{T}_3'(B_2) = s(3)c_5q\big(E_4B_2K_3K_5^{-1}K_4^{-1} - B_2K_3K_5^{-1}K_4^{-1}E_4\big).$$

Here we keep note of c_i and $s(i)$ in order to make clear the dependence on the parameters. Rearranging the above equality we have

$$\big(\mathcal{T}_3'(B_2) + s(3)c_5qB_2K_3K_5^{-1}K_4^{-1}\big)E_4 = E_4\big(\mathcal{T}_3'(B_2) + s(3)c_5qB_2K_3K_5^{-1}K_4^{-1}\big).$$

Following this, we update the ansatz by letting

$$\mathcal{T}_3''(B_2) = \mathcal{T}_3'(B_2) + qs(3)c_5B_2K_3K_5^{-1}K_4^{-1}.$$

Similarly we let

$$\mathcal{T}_3''(B_6) = \mathcal{T}_3'(B_6) + qs(5)c_3B_6K_5K_3^{-1}K_4^{-1}.$$

We now consider the relation $B_2B_5 - B_5B_2 = 0$. In view of this we have

$$q\mathcal{T}_3''(B_2)\mathcal{T}_3'(B_5) - \mathcal{T}_3'(B_5)\mathcal{T}_3''(B_2) = 0.$$

Similar to the reasoning used to define $\mathcal{T}_1''(B_1)$ in Equation (7.42) we let

$$\mathcal{T}_3''(B_5) = B_5K_5K_3^{-1}$$

and symmetrically, we define

$$\mathcal{T}_3''(B_3) = B_3K_3K_5^{-1}$$

Finally, comparing with the relations

$$B_3B_5 - B_5B_3 = (q-q^{-1})^{-1}(q\mathcal{Z}_3 - q\mathcal{Z}_5),$$
$$B_2B_6 - B_6B_2 = (q-q^{-1})^{-1}(\mathcal{Z}_2 - \mathcal{Z}_6)$$

we see that

$$\mathcal{T}_3''(B_3)\mathcal{T}_3''(B_5) - \mathcal{T}_3''(B_5)\mathcal{T}_3''(B_3) = q^{-2}(q-q^{-1})^{-1}(q\widetilde{T}_3(\mathcal{Z}_3) - q\widetilde{T}_3(\mathcal{Z}_5)),$$
$$\mathcal{T}_3''(B_2)\mathcal{T}_3''(B_6) - \mathcal{T}_3''(B_6)\mathcal{T}_3''(B_2) = -q^{-3}(q-q^{-1})^{-1}(\widetilde{T}_3(\mathcal{Z}_2) - \widetilde{T}_3(\mathcal{Z}_6)).$$

Hence we give $\mathcal{T}_3''(B_3)$ and $\mathcal{T}_3''(B_5)$ a factor q^{-1} each, whilst both $\mathcal{T}_3''(B_2)$ and $\mathcal{T}_3''(B_6)$ are given a factor $\sqrt{-1}\,q^{-3/2}$.

Proposition 7.20. *Let $r = 3$ and $X = \{4\}$.*

(1) *There exists a unique algebra automorphism* $\mathcal{T}_3$ *of* $B_{\mathbf{c}}$ *such that*

$$\mathcal{T}_3(B_i) = \begin{cases} B_i & \text{if } i = 1,7, \\ \sqrt{-1}\, q^{-3/2}\big(\big[B_2, [B_3, [B_4, B_5]_q]_q\big]_q + s(3)c_5 q B_2 K_3 K_5^{-1} K_4^{-1}\big) & \text{if } i = 2, \\ \sqrt{-1}\, q^{-3/2}\big(\big[B_6, [B_5, [B_4, B_3]_q]_q\big]_q + s(5)c_3 q B_6 K_5 K_3^{-1} K_4^{-1}\big) & \text{if } i = 6, \\ q^{-1} B_i K_i K_{\tau(i)}^{-1} & \text{if } i = 3,5 \end{cases} \tag{7.47}$$

with $\mathcal{T}_3|_{\mathcal{M}_X U^0_\Theta} = \widetilde{T}_3|_{\mathcal{M}_X U^0_\Theta}$.

(2) *The inverse automorphism* $\mathcal{T}_3^{-1}$ *of* $B_{\mathbf{c}}$ *is given by*

$$\mathcal{T}_3^{-1}(B_i) = \begin{cases} B_i & \text{if } i = 1,7, \\ \sqrt{-1}\, q^{-3/2}\big(\big[B_5, [B_4, [B_3, B_2]_q]_q\big]_q + s(5)c_3 B_2 K_5 K_3^{-1} K_4^{-1}\big) & \text{if } i = 2, \\ \sqrt{-1}\, q^{-3/2}\big(\big[B_3, [B_4, [B_5, B_6]_q]_q\big]_q + s(3)c_5 B_6 K_3 K_5^{-1} K_4^{-1}\big) & \text{if } i = 6, \\ q B_i K_{\tau(i)} K_i^{-1} & \text{if } i = 3,5 \end{cases} \tag{7.48}$$

with $\mathcal{T}_3^{-1}|_{\mathcal{M}_X U^0_\Theta} = \widetilde{T}_3^{-1}|_{\mathcal{M}_X U^0_\Theta}$.

(3) *The algebra automorphisms* $\mathcal{T}_i$ *satisfy the braid relations* (7.30), (7.31) *and* (7.32).

Sketch of proof. Parts (1) and (2) proceed in the same way as in the proof of Proposition 7.17 with the disclaimer that computations that involve the terms $\mathcal{T}_3(B_2), \mathcal{T}_3(B_6), \mathcal{T}_3^{-1}(B_2)$ or $\mathcal{T}_3^{-1}(B_6)$ tend to take a few days to complete. For this reason, these checks are included at the end of the file `A7_oneblacknode.txt`. To prove part (3), we verify the braid relations on each generator B_i. Since the element $q^{1/2}$ can not be defined in `QUAGROUP` we track where half powers appear in our constructions. This has the effect of adding in extra powers of q. In order to cut the computation time down, we make the observation that $\mathcal{T}_2\mathcal{T}_3(B_i) = \mathcal{T}_3^{-1}(B_{\tau(i)})$ for $i = 2, 6$. □

We now consider the case of general r with $|X| = 1$. For $1 \le i \le r-1$ define

$$\mathcal{T}_i(B_j) = \begin{cases} q^{-1} B_{\tau(j)} K_{\tau(j)} K_j^{-1} & \text{if } j = i \text{ or } j = \tau(i), \\ q^{-1/2}[B_j, B_i]_q & \text{if } a_{ij} = -1, \\ q^{-1/2}[B_j, B_{\tau(i)}]_q & \text{if } a_{\tau(i)j} = -1, \\ B_j & \text{if } a_{ij} = 0 \text{ and } a_{\tau(i)j} = 0. \end{cases} \tag{7.49}$$

Theorem 7.21. *Let* $1 \le i \le r-1$ *and* $X = \{r+1\}$.

(1) *There exists a unique algebra automorphism* $\mathcal{T}_i$ *of* $B_{\mathbf{c}}$ *such that* $\mathcal{T}_i(B_j)$ *is given by* (7.49) *for* $j \in I \setminus X$ *and* $\mathcal{T}_i|_{\mathcal{M}_X U^0_\Theta} = T_i T_{\tau(i)}|_{\mathcal{M}_X U^0_\Theta}$.

(2) *The inverse automorphism $\mathcal{T}_1^{-1}$ is given by*

$$\mathcal{T}_i^{-1}(B_j) = \begin{cases} qB_{\tau(j)}K_jK_{\tau(j)}^{-1} & \textit{if } j = i \textit{ or } j = \tau(i), \\ q^{-1/2}[B_i, B_j]_q & \textit{if } a_{ij} = -1, \\ q^{-1/2}[B_{\tau(i)}, B_j]_q & \textit{if } a_{\tau(i)j} = -1, \\ B_j & \textit{if } a_{ij} = 0 \textit{ and } a_{\tau(i)j} = 0. \end{cases} \tag{7.50}$$

and $\mathcal{T}_i^{-1}|_{\mathcal{M}_X U_\Theta^0} = T_i^{-1}T_{\tau(i)}^{-1}|_{\mathcal{M}_X U_\Theta^0}$.

(3) *The relation $\mathcal{T}_i\mathcal{T}_{i+1}\mathcal{T}_i = \mathcal{T}_{i+1}\mathcal{T}_i\mathcal{T}_{i+1}$ holds for $1 \le i \le r-2$. Further, the relation $\mathcal{T}_i\mathcal{T}_j = \mathcal{T}_j\mathcal{T}_i$ holds if $a_{ij} = 0$.*

It remains to construct the algebra automorphism $\mathcal{T}_r$. For $j \in I \setminus X$ define

$$\mathcal{T}_r(B_j) = \begin{cases} B_j & \text{if } a_{rj} = 0 \text{ and } a_{\tau(r)j} = 0, \\ \sqrt{-1}\, q^{-3/2}\big(\big[B_j, [B_r, [B_{r+1}, B_{r+2}]_q]_q\big]_q & \\ \quad + s(r)c_{r+2}qB_jK_rK_{r+2}^{-1}K_{r+1}^{-1}\big) & \text{if } a_{rj} = -1, \\ \sqrt{-1}\, q^{-3/2}\big(\big[B_{\tau(j)}, [B_{r+2}, [B_{r+1}, B_r]_q]_q\big]_q & \\ \quad + s(r{+}2)c_rqB_{\tau(j)}K_{r+2}K_r^{-1}K_{r+1}^{-1}\big) & \text{if } a_{\tau(r)j} = -1, \\ q^{-1}B_jK_jK_{\tau(j)}^{-1} & \text{if } j = r \text{ or } j = \tau(r). \end{cases} \tag{7.51}$$

Theorem 7.22. *Let $X = \{r+1\}$.*

(1) *There exists a unique algebra automorphism $\mathcal{T}_r$ of $B_{\mathbf{c}}$ such that $\mathcal{T}_r(B_j)$ is given by* (7.51) *for $j \in I \setminus X$ and $\mathcal{T}_r|_{\mathcal{M}_X U_\Theta^0} = \widetilde{T}_r|_{\mathcal{M}_X U_\Theta^0}$.*

(2) *The inverse automorphism $\mathcal{T}_r^{-1}$ is given by*

$$\mathcal{T}_r^{-1}(B_j) = \begin{cases} B_j & \textit{if } a_{rj} = 0 \textit{ and } a_{\tau(r)j} = 0, \\ \sqrt{-1}\, q^{-3/2}\big(\big[B_{r+2}, [B_{r+1}, [B_r, B_j]_q]_q\big]_q & \\ \quad + s(r{+}2)c_rB_jK_{r+2}K_r^{-1}K_{r+1}^{-1}\big) & \textit{if } a_{rj} = -1, \\ \sqrt{-1}\, q^{-3/2}\big(\big[B_r, [B_{r+1}, [B_{r+2}, B_{\tau(j)}]_q]_q\big]_q & \\ \quad + s(r)c_{r+2}B_{\tau(j)}K_rK_{r+2}^{-1}K_{r+1}^{-1}\big) & \textit{if } a_{\tau(r)j} = -1, \\ qB_jK_{\tau(j)}K_j^{-1} & \textit{if } j = r \textit{ or } j = \tau(r). \end{cases} \tag{7.52}$$

(3) *The relation $\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1} = \mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r$ holds. Additionally, the relations $\mathcal{T}_r\mathcal{T}_i = \mathcal{T}_i\mathcal{T}_r$ hold for $a_{ri} = 0$ with $1 \le i < r$..*

Sketch of proof. We explain why it suffices to consider the case $r = 3$. In this case by Proposition 7.17 and Proposition 7.19, $\mathcal{T}_1$ and $\mathcal{T}_2$ are algebra automorphisms with inverses $\mathcal{T}_1^{-1}$ and $\mathcal{T}_2^{-1}$, respectively. This implies that $\mathcal{T}_i$ and $\mathcal{T}_i^{-1}$ are mutually inverse algebra automorphisms for general r and $1 \leq i \leq r-1$ through an appropriate relabelling of indices. Similarly, when $r = 3$ Proposition 7.20 implies that $\mathcal{T}_3$ and $\mathcal{T}_3^{-1}$ are mutually inverse algebra automorphisms. It follows that $\mathcal{T}_r$ and $\mathcal{T}_r^{-1}$ are mutually inverse algebra automorphisms in the general case. The braid relations for $r = 3$ imply the braid relations for general r. The result for $r = 1$ or $r = 2$ follows by noting that these cases embed into the case $r = 3$. □

7.4.3 The case $|X| = 2$

In the case $|X| \geq 2$ it is necessary to use a larger Hopf algebra than $U_q(\mathfrak{g})$ which we construct be enlarging the group algebra U^0. For $i \in I$ let $\varpi_i \in \mathfrak{h}^*$ denote the i^{th} fundamental weight. Recall that the fundamental weights have the property that $\varpi_i(h_j) = \delta_{ij}$ for all $i, j \in I$. Let $P = \sum_{i \in I} \mathbb{Z}\varpi_i$ denote the weight lattice. Recall from Section 2.2.3 the completion $\mathscr{U}$ of $U_q(\mathfrak{g})$. For any $\lambda \in P$ define $K_\lambda \in \mathscr{U}$ to be the element such that $K_\lambda \cdot v_\mu = q^{(\lambda,\mu)} v_\mu$ for all weight vectors v_μ of weight μ. With respect to the left adjoint action we have

$$K_{\varpi_i} E_j = q^{\delta_{ij}} E_j K_{\varpi_i},$$
$$K_{\varpi_i} F_j = q^{-\delta_{ij}} F_j K_{\varpi_i}$$

for all $i, j \in I$. Let $\check{U}^0$ denote the subalgebra of $\mathscr{U}$ generated by $\{K_\lambda \mid \lambda \in P\}$. As in [46, Section 1] we define $\check{U}_q(\mathfrak{g})$ to be the Hopf subalgebra of $\mathscr{U}$ generated by $U_q(\mathfrak{g})$ and $\check{U}^0$. Let $P^\Theta = \{\lambda \in P \mid \Theta(\lambda) = \lambda\}$. Denote by $\check{U}^0_\Theta$ the subalgebra of $\check{U}^0$ generated by the elements $\{K_\lambda \mid \lambda \in P^\Theta\}$. We extend the right coideal subalgebra to a larger subalgebra by including the elements $K_\lambda \in \check{U}^0_\Theta$. In the current setting, we can define such elements explicitly.

Lemma 7.23. *Let $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$. For any $i \in I$ we have*

$$\Theta(\varpi_i - \varpi_{\tau(i)}) = \varpi_i - \varpi_{\tau(i)}. \tag{7.53}$$

Proof. The proof is similar to that of 3.11 by noting that $\sigma_j(\varpi_i) = \varpi_i - \delta_{ij}(\alpha_j)$ for all $i, j \in I$ and hence $w_X(\varpi_i) - \varpi_i \in Q_X$ for any $i \in I$. □

We define $\check{B}_{\mathbf{c}}$ to be the subalgebra of $\check{U}_q(\mathfrak{g})$ generated by $\mathcal{M}_X$, $\check{U}^0_\Theta$ and the elements B_i for $i \in I$. This is also a right coideal subalgebra of $U_q(\mathfrak{g})$ and contains $B_{\mathbf{c}}$ as a subalgebra.

If $\lambda \in P^\Theta$ then for any $i \in I \setminus X$ we have

$$\begin{aligned}\Theta(\widetilde{\sigma}_i(\lambda)) &= -w_X \circ \tau \circ w_{X\cup\{i,\tau(i)\}} w_X(\lambda)\\ &= \widetilde{\sigma}_i \circ -\tau \circ w_X(\lambda) = \widetilde{\sigma}_i \circ \Theta(\lambda)\\ &= \widetilde{\sigma}_i(\lambda).\end{aligned}$$

It hence follows that $\widetilde{T}_i(\check{U}^0_\Theta) = \check{U}^0_\Theta$. We now repeat our constructions from the previous section in the case $|X| = 2$. For $1 \leq i \leq r-1$ we define algebra automorphisms $\widetilde{T}_i$ of $\check{B}_{\mathbf{c}}$ following the construction of Theorem 7.21. All that remains is to construct the algebra automorphism $\mathcal{T}_r$. Here, the elements $K_{\varpi_i - \varpi_{\tau(i)}} \in \check{U}^0_\Theta$ play a crucial role in our constructions. To shorten notation we write

$$\varpi'_i = \varpi_i - \varpi_{\tau(i)} \tag{7.54}$$

for $i \in I$. We define

$$\mathcal{T}_r(B_j) = \begin{cases} q^{-1}B_r K_r K_{r+3}^{-1} K_{\varpi'_{r+1}} & \text{if } j = r, \\ q^{-1}B_{r+3}K_{r+3}K_r^{-1}K_{\varpi'_{r+2}} & \text{if } j = r+3, \\ q^{-2}\big(\big[B_{r-1}, [B_r, [[F_{r+1}, F_{r+2}]_q, B_{r+3}]_q]_q\big]_q & \\ \quad + s(r)c_{r+3}qB_{r-1}K_rK_{r+3}^{-1}K_{r+1}^{-1}K_{r+2}^{-1}\big) & \text{if } j = r-1, \\ q^{-2}\big(\big[B_{r+4}, [B_{r+3}, [[F_{r+2}, F_{r+1}]_q, B_r]_q]_q\big]_q & \\ \quad + s(r+3)c_r qB_{r+4}K_{r+3}K_r^{-1}K_{r+1}^{-1}K_{r+2}^{-1}\big) & \text{if } j = r+4, \\ B_j & \text{if } a_{jr} = 0 \text{ and } a_{j(r+3)} = 0. \end{cases} \tag{7.55}$$

Theorem 7.24. (1) *There exists a unique algebra automorphism $\mathcal{T}_r$ of $\check{B}_{\mathbf{c}}$ such that $\mathcal{T}_r(B_i)$ is given by Equation (7.55) and $\mathcal{T}_r|_{\mathcal{M}_X\check{U}^0_\Theta} = \widetilde{T}_r|_{\mathcal{M}_X\check{U}^0_\Theta}$.*

(2) *The inverse automorphism $\mathcal{T}_r^{-1}$ is given by*

$$\mathcal{T}_r^{-1}(B_j) = \begin{cases} qB_rK_{r+3}K_r^{-1}K_{\varpi'_{r+2}} & \text{if } j = r, \\ qB_{r+3}K_rK_{r+3}^{-1}K_{\varpi'_{r+1}} & \text{if } j = r+3, \\ \big(\big[B_{r+3}, [[F_{r+2}, F_{r+1}]_q, [B_r, B_{r-1}]_q]_q\big]_q & \\ \quad + s(r)c_{r+3}B_{r-1}K_{r+3}K_r^{-1}K_{r+1}^{-1}K_{r+2}^{-1}\big) & \text{if } j = r-1, \\ \big(\big[B_r, [[F_{r+1}, F_{r+2}]_q, [B_{r+3}, B_{r+4}]_q]_q\big]_q & \\ \quad + s(r+3)c_rB_{r+4}K_rK_{r+3}^{-1}K_{r+1}^{-1}K_{r+2}^{-1}\big) & \text{if } j = r+4, \\ B_j & \text{if } a_{jr} = 0 \text{ and } a_{j(r+3)} = 0 \end{cases} \tag{7.56}$$

and $\mathcal{T}_r^{-1}|_{\mathcal{M}_X\check{U}^0_\Theta} = \widetilde{T}_r^{-1}|_{\mathcal{M}_X\check{U}^0_\Theta}$.

(3) *The relation $\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1} = \mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r$ holds. Additionally the relations $\mathcal{T}_r\mathcal{T}_i = \mathcal{T}_i\mathcal{T}_r$ hold for $a_{ir} = 0$.*

The proof is the same as that for Theorem 7.22 where we now consider the case $n = 8$ and $r = 3$. This is performed in the `GAP` file `A8_twoblacknodes.txt` contained in [24]. As in the $n = 7$ case of the previous section, we make the disclaimer that the time it takes to compute with $\mathcal{T}_3$ or $\mathcal{T}_3^{-1}$ is in the order of days.

Observe that we require the elements $K_{\varpi'_{r+1}}$ in order that the relations between $\mathcal{T}_r(B_j)$ and elements of $\mathcal{M}_X$ hold for $j = r, r+3$. For instance, defining $\mathcal{T}_r(B_r) = q^{-1}B_rK_rK^{-1}_{\tau(r)}$ is incorrect here, since then $\mathcal{T}_r(B_r)$ does not commute with E_{r+1}.

An important point to mention is that the `GAP` file does not include the elements $K_{\varpi'_i}$. The reason for this is that we require fractional powers in order to define $K_{\varpi'_i}$ as a product of K_j's for $j \in I$, which we can not do using `QUAGROUP`. However, in view of the fact that $K_{\varpi'_{r+1}}$ commutes with $\mathcal{T}_r(B_i)$ for all $i \in I \setminus X$ it is not necessary to include these elements in `GAP`. As a result of this, we no longer check the relations between $\mathcal{T}_r(B_j)$ and elements of $\mathcal{M}_X$ for $j = r, r+3$.

Remark 7.25. Introducing the additional elements $K_{\varpi'_i}$ does not lead to any consistency issues in the case where $X = \emptyset$ with n odd [40] and the case $|X| = 1$ from Theorem 7.22. In both cases we have $K_{\varpi'_{r+1}} = 1$ and hence we should not expect to see these elements appear.

7.4.4 The general case

The major difference between the results of [40] and the current setting is that `GAP` can not be used in order to construct a braid group action of $Br(\mathfrak{b}_r)$ on $B_{\mathbf{c}}$ in general. The reason for this is that `GAP` begins to encounter memory problems when $n \geq 9$. Based on the completion times of the files `A7_oneblacknode.txt` and `A8_twoblacknodes.txt`, this is to be expected. As a result `GAP` can, at best, only provide a braid group action for $|X| \leq 2$. For $1 \leq i \leq r-1$ define

$$\mathcal{T}_i(B_j) = \begin{cases} q^{-1}B_{\tau(j)}K_{\tau(j)}K_j^{-1} & \text{if } j = i \text{ or } j = \tau(i), \\ q^{-1/2}[B_j, B_i]_q & \text{if } a_{ij} = -1, \\ q^{-1/2}[B_j, B_{\tau(i)}]_q & \text{if } a_{\tau(i)j} = -1, \\ B_j & \text{if } a_{ij} = 0 \text{ and } a_{\tau(i)j} = 0. \end{cases} \tag{7.57}$$

The construction of $\mathcal{T}_i$ from 7.21 does not depend on X and hence implies the following theorem, also seen in [40, Theorem 4.6].

Theorem 7.26. *Let $1 \leq i \leq r-1$ and $X = \{r+1\}$.*

(1) *There exists a unique algebra automorphism $\mathcal{T}_i$ of $B_{\mathbf{c}}$ such that $\mathcal{T}_i(B_j)$ is given by (7.49) for $j \in I \setminus X$ and $\mathcal{T}_i|_{\mathcal{M}_X U^0_\Theta} = T_iT_{\tau(i)}|_{\mathcal{M}_X U^0_\Theta}$.*

(2) *The inverse automorphism* $\mathcal{T}_i^{-1}$ *is given by*

$$\mathcal{T}_i^{-1}(B_j) = \begin{cases} qB_{\tau(j)}K_jK_{\tau(j)}^{-1} & \text{if } j = i \text{ or } j = \tau(i), \\ q^{-1/2}[B_i, B_j]_q & \text{if } a_{ij} = -1, \\ q^{-1/2}[B_{\tau(i)}, B_j]_q & \text{if } a_{\tau(i)j} = -1, \\ B_j & \text{if } a_{ij} = 0 \text{ and } a_{\tau(i)j} = 0. \end{cases} \tag{7.58}$$

and $\mathcal{T}_i^{-1}|_{\mathcal{M}_X U_\Theta^0} = T_i^{-1}T_{\tau(i)}^{-1}|_{\mathcal{M}_X U_\Theta^0}$.

(3) *The relation* $\mathcal{T}_i\mathcal{T}_{i+1}\mathcal{T}_i = \mathcal{T}_{i+1}\mathcal{T}_i\mathcal{T}_{i+1}$ *holds for* $1 \le i \le r-2$. *Further, the relation* $\mathcal{T}_i\mathcal{T}_j = \mathcal{T}_j\mathcal{T}_i$ *holds if* $a_{ij} = 0$.

In order to define $\mathcal{T}_r$, we introduce the following notation in the spirit of (7.16) and (7.17). For a subset $J = \{i, i+1, \ldots, j\}$ of I with $i \le j$ define

$$F_J^+ = \big[F_i, [F_{i+1}, \ldots [F_{j-1}, F_j]_q \ldots]_q\big]_q, \tag{7.59}$$

$$F_J^- = \big[F_j, [F_{j-1}, \ldots [F_{i+1}, F_i]_q \ldots]_q\big]_q. \tag{7.60}$$

Additionally, let

$$K_J = K_iK_{i+1}\cdots K_{j-1}K_j. \tag{7.61}$$

Using Theorem 7.22 and 7.24 as a guide we define

$$\mathcal{T}_r(B_j) = \begin{cases} q^{-1}B_rK_rK_{\tau(r)}^{-1}K_{\varpi'_{r+1}} & \text{if } j = r, \\ q^{-1}B_{\tau(r)}K_{\tau(r)}K_r^{-1}K_{\varpi'_{\tau(r+1)}} & \text{if } j = \tau(r), \\ C\big(\big[B_{r-1}, [B_r, [F_X^+, B_{\tau(r)}]_q]_q\big]_q & \\ \quad + s(r)c_{\tau(r)}qB_{r-1}K_rK_{\tau(r)}^{-1}K_X^{-1}\big) & \text{if } j = r-1, \\ C\big(\big[B_{\tau(r-1)}, [B_{\tau(r)}, [F_X^-, B_r]_q]_q\big]_q & \\ \quad + s(\tau(r))c_rqB_{\tau(r-1)}K_{\tau(r)}K_r^{-1}K_X^{-1}\big) & \text{if } j = \tau(r-1), \\ B_j & \text{if } a_{jr} = 0 \text{ and } a_{j\tau(r)} = 0. \end{cases} \tag{7.62}$$

where

$$C = \begin{cases} iq^{-3/2} & \text{if } |X| \text{ odd}, \\ q^{-2} & \text{if } |X| \text{ even}. \end{cases} \tag{7.63}$$

In view of the relations

$$B_{r-1}B_{\tau(r-1)} - B_{\tau(r-1)}B_{r-1} = (q-q^{-1})^{-1}(c_{r-1}\mathcal{Z}_{r-1} - c_{\tau(r-1)}\mathcal{Z}_{\tau(r-1)}),$$
$$B_{r-1}^2B_r - (q+q^{-1})B_{r-1}B_rB_{r-1} + B_rB_{r-1}^2 = 0$$

the following is presented as a conjecture.

Conjecture 7.27. *Suppose* (X, τ) *is a Satake diagram of type AIII with* $X = \{r+1, \ldots, \tau(r+1)\}$ *and* $r \le \lceil \frac{n}{2} \rceil - 1$.

(1) *There is a unique algebra automorphism* $\mathcal{T}_r$ *of* $\check{B}_{\mathbf{c}}$ *such that* $\mathcal{T}_r(B_j)$ *is defined by* (7.62) *and* $\mathcal{T}_r|_{\mathcal{M}_X\check{U}^0_\Theta} = \widetilde{T}_r|_{\mathcal{M}_X\check{U}^0_\Theta}$.

(2) *The inverse automorphism* $\mathcal{T}_r^{-1}$ *is given by*

$$\mathcal{T}_r^{-1}(B_j) = \begin{cases} qB_rK_{\tau(r)}K_r^{-1}K_{\varpi'_{\tau(r+1)}} & \text{if } j = r, \\ qB_{\tau(r)}K_rK_{\tau(r)}^{-1}K_{\varpi'_{r+1}} & \text{if } j = \tau(r), \\ C\big(\big[B_{\tau(r)}, [F_X^-, [B_r, B_{r+1}]_q]_q\big]_q & \\ \quad + s(\tau(r))c_rB_{r-1}K_{\tau(r)}K_r^{-1}K_X^{-1}\big) & \text{if } j = r-1, \\ C\big(\big[B_r, [F_X^+, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q\big]_q & \\ \quad + s(r)c_{\tau(r)}B_{\tau(r-1)}K_rK_{\tau(r)}^{-1}K_X^{-1} & \text{if } j = \tau(r-1), \\ B_j & \text{if } a_{jr} = 0 \text{ and } a_{j\tau(r)} = 0 \end{cases} \tag{7.64}$$

and $\mathcal{T}_r^{-1}|_{\mathcal{M}_X\check{U}^0_\Theta} = \widetilde{T}_r^{-1}|_{\mathcal{M}_X\check{U}^0_\Theta}$.

Assuming that Conjecture 7.27 holds we obtain the following theorem which is the generalisation of part (3) of Theorem 7.22 and Theorem 7.24.

Theorem 7.28. *Let* (X, τ) *be a Satake diagram of type AIII with* $X \neq \emptyset$. *If Conjecture 7.27 is satisfied then the relation* $\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1} = \mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r$ *holds. Further, the relations* $\mathcal{T}_r\mathcal{T}_i = \mathcal{T}_i\mathcal{T}_r$ *hold for any* $i < r-1$.

The proof of Theorem 7.28 requires many calculations so it is given in Section 7.4.5.

Corollary 7.29. *Suppose* (X, τ) *is a Satake diagram of type AIII with* $X = \{r+1, r+2, \ldots, \tau(r+1)\}$ *and* $r \leq \lceil \frac{n}{2} \rceil - 1$. *If Conjecture 7.27 then there is a braid group action of* $Br(\widetilde{W})$ *on* $\check{B}_{\mathbf{c}}$ *by algebra automorphisms given by* $\mathcal{T}_i$ *for* $i \in I \setminus X$.

7.4.5 Proof of Theorem 7.28

Since $\mathcal{T}_j|_{\mathcal{M}_X\check{U}^0_\Theta} = \widetilde{T}_j|_{\mathcal{M}_X\check{U}^0_\Theta}$ for all $j \in I \setminus X$ it follows that the braid relations of Theorem 7.28 hold on elements of $\mathcal{M}_X\check{U}^0_\Theta$. It hence suffices to check the relations on the elements B_i for $i \in I \setminus X$. We first check that the relation $\mathcal{T}_r\mathcal{T}_i = \mathcal{T}_i\mathcal{T}_r$ holds for all $1 \leq i \leq r-2$.

Proposition 7.30. *For* $1 \leq i \leq r-2$ *and* $j \in I \setminus X$ *the relation*

$$\mathcal{T}_r\mathcal{T}_i(B_j) = \mathcal{T}_i\mathcal{T}_r(B_j) \tag{7.65}$$

holds.

Proof. By symmetry, we only check (7.65) for $1 \leq j \leq r$. We do this by a case-by-case analysis.

Case 3. $a_{ij} = 0, a_{jr} = 2$.
In this case we have $j = r$ and $\mathcal{T}_i(B_r) = B_r$. This implies

$$\mathcal{T}_r\mathcal{T}_i(B_r) = \mathcal{T}_r(B_r) = q^{-1}B_rK_rK_{\tau(r)}^{-1}K_{\varpi'_{r+1}} = \mathcal{T}_i\mathcal{T}_r(B_r)$$

as required.

Case 4. $a_{ij} = 0, a_{jr} = -1$.
Then $j = r-1$ and we have

$$\mathcal{T}_r\mathcal{T}_i(B_{r-1}) = \mathcal{T}_r(B_{r-1}) = \mathcal{T}_i\mathcal{T}_r(B_{r-1}).$$

Case 5. $a_{ij} = 0, a_{jr} = 0$.
Then $\mathcal{T}_r(B_j) = B_j$ and $\mathcal{T}_i(B_j) = B_j$ so the statement of the proposition holds in this case.

Case 6. $a_{ij} = -1, a_{jr} = 0$.
Then $\mathcal{T}_r(B_j) = B_j$ and $\mathcal{T}_i(B_j) = q^{-1/2}[B_j, B_i]_q$. Hence

$$\begin{aligned}\mathcal{T}_r\mathcal{T}_i(B_j) &= q^{-1/2}[\mathcal{T}_r(B_j), \mathcal{T}_r(B_i)]_q\\ &= q^{-1/2}[B_j, B_i]_q = \mathcal{T}_i\mathcal{T}_r(B_j).\end{aligned}$$

Case 7. $a_{ij} = -1, a_{jr} = -1$.
This case only occurs if $i = r-2$ and $j = r-1$. We have

$$\begin{aligned}\mathcal{T}_{r-2}\mathcal{T}_r(B_{r-1}) &= C\mathcal{T}_{r-2}\Big(\big[B_{r-1}, [B_r, [F_X^+, B_{\tau(r)}]_q]_q\big]_q + s(r)c_{\tau(r)}qB_{r-1}K_rK_{\tau(r)}^{-1}K_X^{-1}\Big)\\ &= q^{-1/2}C\Big(\big[[B_{r-1}, B_{r-2}]_q, [B_r, [F_X^+, B_{\tau(r)}]_q]_q\big]_q\\ &\qquad + s(r)c_{\tau(r)}q[B_{r-1}, B_{r-2}]_qK_rK_{\tau(r)}^{-1}K_X^{-1}\Big)\\ &= q^{-1/2}C\Big(\big[[B_{r-1}, [B_r, [F_X^+, B_{\tau(r)}]_q]_q]_q, B_{r-2}\big]_q\\ &\qquad + s(r)c_{\tau(r)}q[B_{r-1}K_rK_{\tau(r)}^{-1}K_X^{-1}, B_{r-2}]_q\Big)\\ &= q^{-1/2}[\mathcal{T}_r(B_{r-1}), \mathcal{T}_r(B_{r-2})]_q\\ &= \mathcal{T}_r\mathcal{T}_{r-2}(B_{r-1}).\end{aligned}$$

Case 8. $a_{ij} = 2$.
Then $j = i$. We have $\mathcal{T}_r(B_i) = B_i$ and $\mathcal{T}_r(\mathcal{T}_i(B_i)) = \mathcal{T}_i(B_i)$ which implies

$$\mathcal{T}_r\mathcal{T}_i(B_i) = \mathcal{T}_i\mathcal{T}_r(B_i)$$

as required.

□

We now check that the relations

$$\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}(B_j) = \mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r(B_j)$$

hold for all $j \in I \setminus X$. Again by symmetry, we need only consider $1 \le j \le r$. Many of the remaining claims in this section require the use of relations that are proved in Appendix A. The following lemma is immediate since $\mathcal{T}_r(B_j) = B_j$ and $\mathcal{T}_{r-1}(B_j) = B_j$ for $1 \le j < r-2$.

Lemma 7.31. *For $1 \le j < r-2$ the relation*

$$\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}(B_j) = \mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}(B_j) \tag{7.66}$$

holds.

Hence we need only consider the cases when $j \in \{r-2, r-1, r\}$. By (A.9) we have

$$\mathcal{T}_{r-1}\mathcal{T}_r(B_{r-1}) = \mathcal{T}_r^{-1}(B_{\tau(r-1)}).$$

Proposition 7.32. *The relation*

$$\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}(B_{r-1}) = \mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r(B_{r-1}) \tag{7.67}$$

holds.

Proof. Using Equation (A.9) we have

$$\begin{aligned}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}(B_{r-1}) &= \mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r(q^{-1}B_{\tau(r-1)}K_{\tau(r-1)}K_{r-1}^{-1})\\ &= q^{-1}B_{r-1}K_{r-1}K_{\tau(r-1)}^{-1}\\ &= \mathcal{T}_{r-1}(B_{\tau(r-1)})\end{aligned}$$

and hence (7.67) holds. □

We now consider the case $j = r-2$. By Lemma A.5 the element $[B_{r-1}, \mathcal{T}_r^{-1}(B_{\tau(r-1)})]_q$ is invariant under $\mathcal{T}_r$.

Proposition 7.33. *The relation*

$$\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}(B_{r-2}) = \mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r(B_{r-2}) \tag{7.68}$$

holds.

Proof. On one hand, we have

$$\begin{aligned}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}(B_{r-2}) &= \mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r\big(q^{-1/2}[B_{r-2}, B_{r-1}]_q\big)\\ &= \mathcal{T}_r\mathcal{T}_{r-1}\big(q^{-1/2}[B_{r-2}, \mathcal{T}_r(B_{r-1})]_q\big)\\ &= \mathcal{T}_r\big(q^{-1}[[B_{r-2}, B_{r-1}]_q, \mathcal{T}_{r-1}\mathcal{T}_r(B_{\tau(r-1)})]_q\big).\end{aligned}$$

Again by Equation (A.9) it follows that

$$\begin{aligned}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}(B_{r-2}) &= q^{-1}[[B_{r-2}, \mathcal{T}_r(B_{r-1})]_q, B_{\tau(r-1)}]_q\\ &= q^{-1}[B_{r-2}, [\mathcal{T}_r(B_{r-1}), B_{\tau(r-1)}]_q]_q\end{aligned}$$

where the last equality follows since B_{r-2} commutes with $B_{\tau(r-1)}$. On the other hand we obtain

$$\begin{aligned}\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r(B_{r-2}) &= \mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}(B_{r-2})\\ &= q^{-1}[[B_{r-2}, B_{r-1}]_q, \mathcal{T}_r^{-1}(B_{\tau(r-1)})]_q.\end{aligned}$$

Since $B_{\tau(r-1)}$ commutes with $\mathcal{T}_r(B_{r-2})$ it follows that

$$B_{r-2}\mathcal{T}_r^{-1}(B_{\tau(r-1)}) = \mathcal{T}_r^{-1}(B_{\tau(r-1)})B_{r-2}.$$

This and Lemma A.5 imply

$$\begin{aligned}\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r(B_{r-2}) &= q^{-1}[B_{r-2}, [B_{r-1}, \mathcal{T}_r^{-1}(B_{\tau(r-1)})]_q]_q\\ &= q^{-1}[B_{r-2}, [\mathcal{T}_r(B_{r-1}), B_{\tau(r-1)}]_q]_q\\ &= \mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}(B_{r-2})\end{aligned}$$

which proves the claim of the proposition. □

All that remains now is to consider the element B_r.

Proposition 7.34. *The relation*

$$\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}(B_r) = \mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r(B_r) \tag{7.69}$$

holds.

Proof. We consider the right hand side of (7.69) first. We have

$$\begin{aligned}&\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r(B_r)\\ &\quad= q^{-1}\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}(B_rK_rK_{\tau(r)}^{-1}K_{\varpi'_{r+1}})\\ &\quad= q^{-3/2}\mathcal{T}_{r-1}\mathcal{T}_r([B_r, B_{r-1}]_q)K_rK_{\tau(r)}^{-1}K_{\varpi'_{r+1}}\\ &\quad= q^{-5/2}\mathcal{T}_{r-1}([B_rK_rK_{\tau(r)}^{-1}K_{\varpi'_{r+1}}, \mathcal{T}_r(B_{r-1})]_q)K_rK_{\tau(r)}^{-1}K_{\varpi'_{r+1}}\\ &\quad= q^{-3}\big[[B_r, B_{r-1}]_qK_rK_{\tau(r)}^{-1}K_{r-1}K_{\tau(r-1)}^{-1}K_{\varpi'_{r+1}}, \mathcal{T}_r^{-1}(B_{\tau(r-1)})\big]_qK_rK_{\tau(r)}K_{\varpi'_{r+1}}\end{aligned}$$

Using Relation 7.37 and noting that $K_{\varpi'_{r+1}}$ commutes with $T_r^{-1}(B_{\tau(r-1)})$ we have

$$\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r(B_r) = q^{-2}\big[[B_r, B_{r-1}]_q, \mathcal{T}_r^{-1}(B_{\tau(r-1)})\big]_qK_r^2K_{\tau(r)}^{-2}K_{r-1}K_{\tau(r-1)}^{-1}K_{\varpi'_{r+1}}^2.$$

Using (A.11) and the fact that $[B_r, \mathcal{T}_r^{-1}(B_{\tau(r-1)})]_{q^{-1}} = 0$ by we obtain

$$\begin{aligned}\big[[B_r, B_{r-1}]_q, \mathcal{T}_r^{-1}(B_{\tau(r-1)})\big]_q &= \big[B_r, [B_{r-1}, \mathcal{T}_r^{-1}(B_{\tau(r-1)})]_q\big]\\ &= \big[B_r, [\mathcal{T}_r(B_{r-1}), B_{\tau(r-1)}]_q\big].\end{aligned}$$

It hence follows that

$$\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r(B_r) = q^{-2}\big[B_r, [\mathcal{T}_r(B_{r-1}), B_{\tau(r-1)}]_q\big]K_r^2K_{\tau(r)}^{-2}K_{r-1}K_{\tau(r-1)}^{-1}K_{\varpi'_{r+1}}^2.$$

Considering now the left hand side of (7.69) we obtain

$$
\begin{aligned}
&\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}(B_r)\\
&\qquad = q^{-1/2}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r([B_r, B_{r-1}]_q)\\
&\qquad = q^{-3/2}\mathcal{T}_r\mathcal{T}_{r-1}([B_rK_rK_{\tau(r)}^{-1}K_{\varpi'_{r+1}}.\mathcal{T}_r(B_{r-1})]_q)\\
&\qquad = q^{-2}\mathcal{T}_r\big(\big[[B_r, B_{r-1}]_qK_rK_{\tau(r)}^{-1}K_{r-1}K_{\tau(r-1)}^{-1}K_{\varpi'_{r+1}}, \mathcal{T}_r^{-1}(B_{\tau(r-1)})\big]_q\big)\\
&\qquad = q^{-2}\big[[B_rK_rK_{\tau(r)}^{-1}K_{\varpi'_{r+1}}, \mathcal{T}_r(B_{r-1})]_q, B_{\tau(r-1)}\big]K_rK_{\tau(r)}^{-1}K_{r-1}K_{\tau(r-1)}^{-1}K_{\varpi'_{r+1}}\\
&\qquad = q^{-2}\big[B_r, [\mathcal{T}_r(B_{r-1}), B_{\tau(r-1)}]_q\big]K_r^2K_{\tau(r)}^2K_{r-1}K_{\tau(r-1)}^{-1}K_{\varpi'_{r+1}}^2\\
&\qquad = \mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r(B_r)
\end{aligned}
$$

as required. □

The results of this section imply Theorem 7.28.

7.5 The action of $Br(W_X) \times Br(\widetilde{W})$ on $\check{B}_{\mathbf{c}}$ in type AIII

We now combine the results of Section 7.3 and Section 7.4 to give a quantum analogue of the action of $Br(W_X) \times Br(\widetilde{W})$ on $\mathfrak{k}$ by Lie algebra automorphisms established in Lemma 7.6 when we consider Satake diagrams of type AIII.

Theorem 7.35. *Let $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ and (X, τ) a Satake diagram of type AIII such that $|X| = 1$ or $|X| = 2$. Then there exists an action of $Br(W_X) \times B_r(\widetilde{W})$ on $\check{B}_{\mathbf{c}}$ by algebra automorphisms. The action of $Br(W_X)$ on $\check{B}_{\mathbf{c}}$ is given by the Lusztig automorphisms T_i for $i \in X$ and the corresponding action of $Br(\widetilde{W})$ is given by the algebra automorphisms $\mathcal{T}_i$ for $1 \le i \le r$ given by Theorem 7.26 and the formulas (7.62).*

Conjecture 7.36. *The statement of Theorem 7.35 above holds for all $|X| \geq 1$.*

In order to prove Theorem 7.35 it suffices by Corollary 7.14 and Corollary 7.29 to show that the actions of $Br(W_X)$ and $Br(\widetilde{W})$ on $\check{B}_{\mathbf{c}}$ commute. The remainder of this section shows this by casework on the elements of $\check{B}_{\mathbf{c}}$. We work in the general setting with $|X| \geq 1$. As a result of this it follows that if Conjecture 7.27 holds then also Conjecture 7.36 holds.

Lemma 7.37. *If $x \in \mathcal{M}_X\check{U}^0_\Theta$ then $\mathcal{T}_iT_j(x) = T_j\mathcal{T}_i(x)$ for all $j \in X$ and $1 \le i \le r$.*

Proof. Recall that $\mathcal{T}_i|_{\mathcal{M}_X\check{U}^0_\Theta} = \widetilde{T}_i|_{\mathcal{M}_X\check{U}^0_\Theta}$ for all $1 \le i \le r$. If $i \le r-1$ then $\mathcal{T}_i|_{\mathcal{M}_X\check{U}^0_\Theta} = T_iT_{\tau(i)}|_{\mathcal{M}_X\check{U}^0_\Theta}$ commutes with T_j for any $j \in X$. If $i = r$ then $\widetilde{T}_r|_{\mathcal{M}_X\check{U}^0_\Theta} = \mathrm{id}|_{\mathcal{M}_X\check{U}^0_\Theta}$ and hence there is nothing to show in this case. □

By symmetry it is enough to show

$$\mathcal{T}_iT_j(B_k) = T_j\mathcal{T}_i(B_k)$$

for $1 \leq i \leq r$, $j \in X$ and $1 \leq k \leq r$.

Lemma 7.38. *If $1 \leq i \leq r-1$ and $j \in X \setminus \{r+1\}$ then the relation*

$$\mathcal{T}_i T_j(B_k) = T_j \mathcal{T}_i(B_k)$$

holds for all $1 \leq k \leq r$.

Proof. We have $T_j(B_k) = B_k$ for all $j \in X \setminus \{r+1\}$ and $1 \leq k \leq r$. By the definition (7.57) of $\mathcal{T}_i$ the statement of the lemma follows. □

Lemma 7.39. *If $1 \leq i \leq r-1$ then*

$$\mathcal{T}_i T_{r+1}(B_k) = T_{r+1} \mathcal{T}_i(B_k)$$

holds for all $1 \leq k \leq r$.

Proof. By Equation (7.25) we have

$$T_{r+1}(B_k) = \begin{cases} B_k & \text{if } 1 \leq k \leq r-1, \\ [B_r, F_{r+1}]_q & \text{if } k = r. \end{cases}$$

There are three cases to consider, depending on the value of a_{ik}. If $a_{ik} = 0$, then $\mathcal{T}_i(B_k) = B_k$ for all $1 \leq k \leq r$ and hence the claim follows. If $a_{ik} = -1$ then $\mathcal{T}_i(B_k) = q^{-1/2}[B_k, B_i]_q$ which implies that we need only check the claim when $k = r$ and $i = r-1$. We obtain

$$\begin{aligned} T_{r+1}\mathcal{T}_{r-1}(B_r) &= q^{-1/2} T_{r+1}([B_r, B_{r-1}]_q) \\ &= q^{-1/2}[[B_r, F_{r+1}]_q, B_{r-1}]_q \\ &= q^{-1/2}[[B_r, B_{r-1}]_q, F_{r+1}]_q = \mathcal{T}_{r-1} T_{r+1}(B_r) \end{aligned}$$

since B_{r-1} commutes with F_{r+1}.

Finally, if $a_{ik} = 2$ the result follows immediately since $\mathcal{T}_i(B_i) = q^{-1} B_{\tau(i)} K_{\tau(i)} K_i^{-1}$ is invariant under T_{r+1}. □

Lemma 7.38 and Lemma 7.39 imply that

$$\mathcal{T}_i T_j(B_k) = T_j \mathcal{T}_i(B_k)$$

for all $1 \leq i \leq r-1$, $j \in X$ and $1 \leq k \leq r$. All that remains is the case $i = r$.

Lemma 7.40. *For all $j \in X$ and $1 \leq k \leq r$ with $k \neq r-1$, the relation*

$$\mathcal{T}_r T_j(B_k) = T_j \mathcal{T}_r(B_k)$$

holds.

Proof. For $1 \leq k \leq r-2$ both T_j and $\mathcal{T}_r$ act as the identity on B_k so there is nothing to show. Suppose that $k = r$. Then

$$\mathcal{T}_r(B_r) = q^{-1} B_r K_r K_{\tau(r)}^{-1} K_{\varpi_{r+1} - \varpi_{\tau(r+1)}}.$$

Let $\lambda = \alpha_r - \alpha_{\tau(r)} + \varpi_{r+1} - \varpi_{\tau(r+1)}$. Since $\alpha_r = -\varpi_{r+1} + 2\varpi_r - \varpi_{r-1}$ it follows $(\alpha_j, \lambda) = 0$ for all $j \in X$. This implies $\sigma_j(\lambda) = \lambda$ for all $j \in X$ and hence $T_j(K_\lambda) = K_\lambda$.

If $j \neq r+1$ then $T_j(B_r) = B_r$ and the result follows. Otherwise we have

$$\begin{aligned} T_{r+1}\mathcal{T}_r(B_r) &= q^{-1}T_{r+1}(B_r)K_\lambda \\ &= q^{-1}[B_r, F_{r+1}]_q K_\lambda = \mathcal{T}_r T_{r+1}(B_r) \end{aligned}$$

where we use the fact that F_{r+1} commutes with K_λ. This completes the proof. □

Lemma 7.41. *For all $j \in X \setminus \{r+1\}$ the relation*

$$\mathcal{T}_r T_j(B_{r-1}) = T_j \mathcal{T}_r(B_{r-1})$$

holds.

Proof. By Lemma A.6 the result is clear for $j \neq \tau(r+1)$ since T_j acts as the identity on the elements $F_X^+, K_r K_{\tau(r)}^{-1}, K_X^{-1}$ and B_k for $k \in I \setminus X$. On the other hand if $j = \tau(r+1)$ then

$$\begin{aligned} T_{\tau(r+1)}([F_X^+, B_{\tau(r)}]_q) &= [F^+_{X\setminus\{\tau(r+1)\}}, [F_{\tau(r+1)}, B_{\tau(r)}]_q]_q \\ &= [F_X, B_{\tau(r)}]_q \end{aligned}$$

by noting that $[F_{\tau(r+1)}, B_{\tau(r)}]_q = T^{-1}_{\tau(r+1)}(B_{\tau(r)})$. The result hence follows in this case also. □

Lemma 7.42. *The relation*

$$\mathcal{T}_r T_{r+1}(B_{r-1}) = T_{r+1}\mathcal{T}_r(B_{r-1})$$

holds.

Proof. Recall from (7.62) that

$$\mathcal{T}_r(B_{r-1}) = C\big[B_{r-1}, [B_r, [F_X^+, B_{\tau(r)}]_q]_q\big]_q + Cs(r)c_{\tau(r)}qB_{r-1}K_rK^{-1}_{\tau(r)}K_X^{-1}.$$

We are done if we show that $\mathcal{T}_r(B_{r-1})$ is invariant under T_{r+1}. We use Lemmas A.7 and A.8 to do this, depending on whether $|X| = 1$ or $|X| \geq 2$. If $|X| = 1$ then Lemma A.7 implies

$$\begin{aligned} T_{r+1}\big(\big[B_{r-1}, [B_r, [F_{r+1}, B_{r+2}]_q]_q\big]_q\big) &= \big[B_{r-1}, [B_r, [F_{r+1}, B_{r+2}]_q]_q\big]_q \\ &\quad + s(r)c_{r+2}qB_{r-1}K_rK^{-1}_{r+2}(K^{-1}_{r+1} - K_{r+1}). \end{aligned}$$

Further, we have

$$T_{r+1}(B_{r-1}K_rK^{-1}_{r+2}K^{-1}_{r+1}) = B_{r-1}K_rK^{-1}_{r+2}K_{r+1}.$$

Combining these we see that

$$\begin{aligned}T_{r+1}\mathcal{T}_r(B_{r-1}) &= C\big[B_{r-1},[B_r,[F_{r+1},B_{r+2}]_q]_q\big]_q + Cs(r)c_{r+2}qB_{r-1}K_rK_{r+2}^{-1}(K_{r+1}^{-1}-K_{r+1})\\ &\quad + Cs(r)c_{\tau(r+2)}qB_{r-1}K_rK_{r+2}^{-1}K_{r+1}\\ &= \mathcal{T}_r(B_{r-1}).\end{aligned}$$

On the other hand if $|X| = 2$ then Lemma A.8 implies that

$$\begin{aligned}T_{r+1}\big(\big[B_{r-1},[B_r,[F_X^+,B_{\tau(r)}]_q]_q\big]_q\big) &= \big[B_{r-1},[B_r,[F_X^+,B_{\tau(r)}]_q]_q\big]_q\\ &\quad + s(r)c_{\tau(r)}qB_{r-1}K_rK_{\tau(r)}^{-1}(K_{r+1}^{-1}-K_{r+1})K_{X\backslash\{r+1\}}^{-1}.\end{aligned}$$

We have

$$T_{r+1}(B_{r-1}K_rK_{\tau(r)}^{-1}K_X^{-1}) = B_{r-1}K_rK_{\tau(r)}^{-1}K_{r+1}K_{X\backslash\{r+1\}}^{-1}.$$

Calculating similarly, one obtains

$$T_{r+1}\mathcal{T}_r(B_{r-1}) = \mathcal{T}_r(B_{r-1})$$

also in this case. □

Appendix A

Useful relations in $B_{\mathbf{c}}$

In order to prove Theorem 7.28 and Theorem 7.35 we require the use of many relations, which are collected here for the reader's convenience. We recall that in Section 7.4 we only considered Satake diagrams of type AIII. This will be the setting for this appendix.

Further recall the notation F_J^+, F_J^- and K_J from (7.59), (7.60) and (7.61) where $J \subset I$ is of the form $J = \{i, i+1, \ldots, j-1, j\}$ for $i \leq j$. We similarly define elements

$$E_J^+ = \big[E_i, [E_{i+1}, \ldots, [E_{j-1}, E_j]_{q^{-1}} \cdots]_{q^{-1}}\big]_{q^{-1}} = T_i T_{i+1} \cdots T_{j-1}(E_j), \tag{A.1}$$

$$E_J^- = \big[E_j, [E_{j-1}, \ldots, [E_{i+1}, E_i]_{q^{-1}} \cdots]_{q^{-1}}\big]_{q^{-1}} = T_j T_{j-1} \cdots T_{i+1}(E_i), \tag{A.2}$$

By definition of F_J^+, F_J^-, K_J, E_J^+ and E_J^-1 the relation

$$E_J^+ F_J^- - F_J^- E_J^+ = \frac{K_J - K_J^{-1}}{q - q^{-1}} = E_J^- F_J^+ - F_J^+ E_J^- \tag{A.3}$$

holds in $U_q(\mathfrak{g})$. Additionally the q-commutator satisfies

$$\big[[x,y]_q, z\big]_q - \big[x, [y,z]_q\big]_q = q\big[[x,z], y\big] \tag{A.4}$$

for all $x, y, z \in U_q(\mathfrak{g})$. Recall from (7.57) the algebra automorphisms $\mathcal{T}_i$ for $1 \leq i \leq r-1$.

Lemma A.1. *The relation*

$$\big[\mathcal{T}_{r-1}(B_{r-1}), [F_X^+, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q\big] = 0 \tag{A.5}$$

holds in $B_{\mathbf{c}}$.

Proof. Since $\mathcal{T}_{r-1}(B_{r-1}) = q^{-1} B_{\tau(r-1)} K_{\tau(r-1)} K_{r-1}^{-1}$ it follows that $\mathcal{T}_{r-1}(B_{r-1})$ commutes with F_j for $j \in X$. Further the relation

$$B_{\tau(r-1)}^2 B_{\tau(r)} - (q + q^{-1}) B_{\tau(r-1)} B_{\tau(r)} B_{\tau(r-1)} + B_{\tau(r)} B_{\tau(r-1)}^2 = 0$$

implies that

$$B_{\tau(r-1)} [B_{\tau(r)}, B_{\tau(r-1)}]_q = q [B_{\tau(r)}, B_{\tau(r-1)}]_q B_{\tau(r-1)}$$

and hence $\mathcal{T}_{r-1}(B_{r-1})$ commutes with $[B_{\tau(r)}, B_{\tau(r-1)}]_q$. The result follows from this. □

Lemma A.2. *For any $i \in I \setminus (X \cup \{r, \tau(r)\})$ the relations*

$$\big[B_{\tau(i)}K_{\tau(i)}K_i^{-1}, [B_{i\pm1}, B_i]_q\big]_q = q^2 B_{i\pm1}, \tag{A.6}$$

$$\big[[B_i, B_{i\pm1}]_q, B_{\tau(i)}K_{\tau(i)}^{-1}K_i\big]_q = B_{i\pm1}. \tag{A.7}$$

hold in $B_{\mathbf{c}}$.

Proof. The relations follow immediately by applying the automorphisms $\mathcal{T}_i$ and $\mathcal{T}_i^{-1}$ to

$$\mathcal{T}_i^{-1}(B_{i\pm1}) = q^{-1/2}[B_i, B_{i\pm1}]_q,$$
$$\mathcal{T}_i(B_{i\pm1}) = q^{-1/2}[B_{i\pm1}, B_i]_q,$$

respectively. □

Lemma A.3. *The relation*

$$\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r(B_{r-1}) = \mathcal{T}_{r-1}(B_{\tau(r-1)}) \tag{A.8}$$

holds.

Proof. We first calculate $\mathcal{T}_{r-1}\mathcal{T}_r(B_{r-1})$. We have

$$\begin{aligned}
&\mathcal{T}_{r-1}\mathcal{T}_r(B_{r-1})\\
&\quad= C_n\Big(\big[\mathcal{T}_{r-1}(B_{r-1}), [\mathcal{T}_{r-1}(B_r), [F_X^+, \mathcal{T}_{r-1}(B_{\tau(r)})]_q]_q\big]_q\\
&\qquad + s(r)c_{\tau(r)}q\mathcal{T}_{r-1}(B_{r-1})\mathcal{T}_{r-1}(K_rK_{\tau(r)}^{-1})K_X^{-1}\Big)\\
&\quad= C_n\Big(q^{-2}\big[B_{\tau(r-1)}K_{\tau(r-1)}K_r^{-1}, \big[[B_r, B_{r-1}]_q, [F_X^+, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q\big]_q\big]_q\\
&\qquad + s(r)c_{\tau(r)}B_{\tau(r-1)}K_rK_{\tau(r-1)}^{-1}K_X^{-1}\Big)\\
&\quad= \mathcal{T}_r^{-1}(B_{\tau(r-1)}).
\end{aligned} \tag{A.9}$$

where the last equality holds by Equations (A.5) and (A.6). It follows from this that

$$\mathcal{T}_{r-1}\mathcal{T}_r\mathcal{T}_{r-1}\mathcal{T}_r(B_{r-1}) = \mathcal{T}_{r-1}(B_{\tau(r-1)})$$

as required. □

Lemma A.4. *The relation*

$$\big[B_{r-1}, [B_r, [F_X^+, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q]_q\big]_q = \big[[B_{r-1}, [B_r, [F_X^+, B_{\tau(r)}]_q]_q]_q, B_{\tau(r-1)}\big]_q \tag{A.10}$$

holds in $B_{\mathbf{c}}$.

Proof. First observe that since $B_{\tau(r-1)}$ commutes with B_r and F_X^+ we have

$$\big[B_r, [F_X^+, [B_{\tau(r)}, B_{\tau(r-1)}]_q]_q\big]_q = \big[[B_r, [F_X^+, B_{\tau(r)}]_q]_q, B_{\tau(r-1)}\big]_q.$$

To shorten notation let $Y = [B_r, [F_X^+, B_{\tau(r)}]_q]_q$. Recall the relation

$$B_{r-1}B_{\tau(r-1)} - B_{\tau(r-1)}B_{r-1} = (q - q^{-1})^{-1}\big(c_{r-1}\mathcal{Z}_{r-1} - c_{\tau(r-1)}\mathcal{Z}_{\tau(r-1)}\big)$$

where $\mathcal{Z}_{r-1} = -K_{\tau(r-1)}K_{r-1}^{-1}$ and $\mathcal{Z}_{\tau(r-1)} = -K_{r-1}K_{\tau(r-1)}^{-1}$, see (7.36). Further note that Y commutes with both $\mathcal{Z}_{r-1}$ and $\mathcal{Z}_{\tau(r-1)}$ by (7.37). Hence

$$\begin{aligned}
&\big[B_{r-1},[Y,B_{\tau(r-1)}]_q\big]_q\\
&\quad= B_{r-1}YB_{\tau(r-1)} - qB_{r-1}B_{\tau(r-1)}Y - qYB_{\tau(r-1)}B_{r-1} + q^2B_{\tau(r-1)}YB_{r-1}\\
&\quad= B_{r-1}YB_{\tau(r-1)} - q\big(B_{\tau(r-1)}B_{r-1} + (q-q^{-1})^{-1}(c_{r-1}\mathcal{Z}_{r-1} - c_{\tau(r-1)}\mathcal{Z}_{\tau(r-1)})\big)Y\\
&\qquad - qY\big(B_{r-1}B_{\tau(r-1)} - (q-q^{-1})^{-1}(c_{r-1}\mathcal{Z}_{r-1} - c_{\tau(r-1)}\mathcal{Z}_{\tau(r-1)})\big) + q^2B_{\tau(r-1)}YB_{r-1}\\
&\quad= \big[[B_{r-1},Y]_q,B_{\tau(r-1)}\big]_q - q(q-q^{-1})^{-1}\big[c_{r-1}\mathcal{Z}_{r-1} - c_{\tau(r-1)}\mathcal{Z}_{\tau(r-1)},Y\big]\\
&\quad= \big[[B_{r-1},Y]_q,B_{\tau(r-1)}\big]_q
\end{aligned}$$

as required. □

Lemma A.5. *The element* $[B_{r-1},\mathcal{T}_r^{-1}(B_{\tau(r-1)})]_q$ *is* $\mathcal{T}_r$*-invariant i.e.*

$$[B_{r-1},\mathcal{T}_r^{-1}(B_{\tau(r-1)})]_q = [\mathcal{T}_r(B_{r-1}),B_{\tau(r-1)}]_q. \tag{A.11}$$

Proof. We have

$$\begin{aligned}
[B_{r-1},\mathcal{T}_r^{-1}(B_{\tau(r-1)})]_q &= \big[B_{r-1},C_n[B_r,[F_X^+,[B_{\tau(r)},B_{\tau(r-1)}]_q]_q]_q\big]_q\\
&\quad + C_n s(r)c_{\tau(r)}\big[B_{r-1},B_{\tau(r-1)}K_rK_{\tau(r)}^{-1}K_X^{-1}\big]_q\\
&\overset{(A.10)}{=} \big[C_n[B_{r-1},[B_r,[F_X^+,B_{\tau(r)}]_q]_q]_q,B_{\tau(r-1)}\big]_q\\
&\quad + \big[C_n s(r)c_{\tau(r)}qB_{r-1}K_rK_{\tau(r)}^{-1}K_X^{-1},B_{\tau(r-1)}\big]_q\\
&= [\mathcal{T}_r(B_{r-1}),B_{\tau(r-1)}]_q
\end{aligned}$$

as required. □

Lemma A.6. *For any* $j \in X \setminus \{r+1,\tau(r+1)\}$ *the relation*

$$T_j(F_X^+) = F_X^+ \tag{A.12}$$

holds.

Proof. Recall from (7.59) that

$$F_X^+ = \big[F_{r+1},[F_{r+2},\dots,[F_{\tau(r+2)},F_{\tau(r+1)}]_q\cdots]_q\big]_q.$$

Observe that the automorphism T_j only acts non-trivially on the F_{j-1},F_j and F_{j+1}. The result follows from this, since

$$\begin{aligned}
T_j\big(\big[F_{j-1},[F_j,F_{j+1}]_q\big]_q\big) &= T_j([F_{j-1},T_j^{-1}(F_{j+1})]_q)\\
&= \big[[F_{j-1},F_j]_q,F_{j+1}\big]_q\\
&= \big[F_{j-1},[F_j,F_{j+1}]_q\big]_q.
\end{aligned}$$

□

Lemma A.7. *If $|X| = 1$ then the relation*

$$T_{r+1}\big([B_{r-1},[B_r,[F_{r+1},B_{r+2}]_q]_q]_q\big) = \big[B_{r-1},[B_r,[F_{r+1},B_{r+2}]_q]_q\big]_q + s(r)c_{r+2}qB_{r-1}K_rK_{r+2}^{-1}(K_{r+1}^{-1}-K_{r+1}) \tag{A.13}$$

holds.

Proof. We have

$$T_{r+1}\big([B_{r-1},[B_r,[F_{r+1},B_{r+2}]_q]_q]_q\big) = \big[B_{r-1},[[B_r,F_{r+1}]_q,B_{r+2}]_q\big]_q$$

by noting that $[F_{r+1},B_{r+2}]_q = T_{r+1}^{-1}(B_{r+2})$. Using the relation

$$[B_r,B_{r+2}] = (q-q^{-1})^{-1}(c_r\mathcal{Z}_r - c_{r+2}\mathcal{Z}_{r+2})$$

it follows that

$$\big[[B_r,F_{r+1}]_q,B_{r+2}\big]_q = \big[B_r,[F_{r+1},B_{r+2}]_q\big]_q - q(q-q^{-1})^{-1}[F_{r+1},c_r\mathcal{Z}_r - c_{r+2}\mathcal{Z}_{r+2}].$$

Recall that in the current setting we have

$$\mathcal{Z}_r = -(1-q^{-2})s(r+2)E_{r+1}K_{r+2}K_r^{-1},$$
$$\mathcal{Z}_{r+2} = -(1-q^{-2})s(r)E_{r+1}K_rK_{r+2}^{-1}.$$

Hence

$$[F_{r+1},c_r\mathcal{Z}_r - c_{r+2}\mathcal{Z}_{r+2}] = q^{-1}(K_{r+1}-K_{r+1}^{-1})(s(r+2)c_rK_{r+2}K_r^{-1} - s(r)c_{r+2}K_rK_{r+2}^{-1}).$$

This and the fact that $[B_{r-1},K_{r+2}K_r^{-1}]_q = 0$ imply

$$\begin{aligned}T_{r+1}\big([B_{r-1},[B_r,[F_{r+1},B_{r+2}]_q]_q]_q\big) &- \big[B_{r-1},[B_r,[F_{r+1},B_{r+2}]_q]_q\big]_q\\ &= (q-q^{-1})^{-1}[B_{r-1},(K_{r+1}-K_{r+1}^{-1})s(r)c_{r+2}K_rK_{r+2}^{-1})]_q\\ &= s(r)c_{r+2}qB_{r-1}K_rK_{r+2}^{-1}(K_{r+1}^{-1}-K_{r+1})\end{aligned}$$

as required. □

Lemma A.8. *If $|X| \geq 2$ then the relation*

$$T_{r+1}\big([B_{r-1},[B_r,[F_X^+,B_{\tau(r)}]_q]_q]_q\big) = \big[B_{r-1},[B_r,[F_X^+,B_{\tau(r)}]_q]_q + s(r)c_{\tau(r)}qB_{r-1}K_rK_{\tau(r)}^{-1}(K_{r+1}^{-1}-K_{r+1})K_{X\setminus\{r+1\}}^{-1} \tag{A.14}$$

holds.

Proof. Let $Y = X\setminus\{r+1\}$. Observing that $[F_X^+,B_{\tau(r)}]_q = T_{r+1}^{-1}([F_Y^+,B_{\tau(r)}]_q)$ we have

$$T_{r+1}\big([B_{r-1},[B_r,[F_X^+,B_{\tau(r)}]_q]_q]_q\big) = \big[B_{r-1},\big[[B_r,F_{r+1}]_q,[F_Y^+,B_{\tau(r)}]_q\big]_q\big]_q.$$

By (A.4) we obtain

$$\big[[B_r,F_{r+1}]_q,[F_Y^+,B_{\tau(r)}]_q\big]_q - \big[B_r,[F_X^+,B_{\tau(r)}]_q\big]_q = q\big[[B_r,[F_Y^+,B_{\tau(r)}]_q],F_{r+1}\big]$$

$$= q\big[[F_Y^+, [B_r, B_{\tau(r)}]]_q, F_{r+1}\big]. \tag{A.15}$$

Recall that

$$[B_r, B_{\tau(r)}] = (q - q^{-1})^{-1}(c_r \mathcal{Z}_r - c_{\tau(r)}\mathcal{Z}_{\tau(r)})$$

where

$$\mathcal{Z}_r = -s(\tau(r))(1 - q^{-2})E_X^+ K_{\tau(r)} K_r^{-1},$$
$$\mathcal{Z}_{\tau(r)} = -s(r)(1 - q^{-2})E_X^- K_r K_{\tau(r)}^{-1}.$$

Using Equation (A.3) it follows that

$$\begin{aligned}[F_Y, E_X^-] &= \big[[F_Y^+, E_Y^-], E_{r+1}]_{q^{-1}} \\ &= (q - q^{-1})^{-1}[K_Y^{-1} - K_Y, E_{r+1}]_{q^{-1}} \\ &= q^{-1}K_Y^{-1}E_{r+1}.\end{aligned}$$

This implies that

$$\begin{aligned}\big[[F_Y^+, E_X^- K_r K_{\tau(r)}^{-1}]_q, F_{r+1}\big] &= q\big[[F_Y^+, E_X^-], F_{r+1}\big]_{q^{-1}} K_r K_{\tau(r)}^{-1} \\ &= [K_Y^{-1}E_{r+1}, F_{r+1}]_{q^{-1}} K_r K_{\tau(r)}^{-1} \\ &= (q - q^{-1})^{-1} K_r K_{\tau(r)}^{-1}(K_{r+1} - K_{r+1}^{-1})K_Y^{-1}. \end{aligned} \tag{A.16}$$

Further, since $[B_{r-1}, K_{\tau(r)}K_r^{-1}]_q = 0$ it follows that

$$\big[B_{r-1}, \big[[F_Y^+, \mathcal{Z}_r]_q, F_{r+1}\big]\big]_q = 0. \tag{A.17}$$

By (A.16) and (A.17) we obtain

$$\begin{aligned}T_{r+1}&\big(\big[B_{r-1},\big[[B_r, F_{r+1}]_q, [F_Y^+, B_{\tau(r)}]_q\big]_q\big]_q\big) - \big[B_{r-1}, \big[[B_r, F_{r+1}]_q, [F_Y^+, B_{\tau(r)}]_q\big]_q\big]_q \\ &= -q(q - q^{-1})^{-1}c_{\tau(r)}\big[B_{r-1}, \big[[F_Y^+, \mathcal{Z}_{\tau(r)}]_q, F_{r+1}\big]\big]_q \\ &= s(r)c_{\tau(r)}(q - q^{-1})^{-1}\big[B_{r-1}, K_r K_{\tau(r)}^{-1}(K_{r+1} - K_{r+1}^{-1})K_Y^{-1}\big]_q \\ &= s(r)c_{\tau(r)}qB_{r-1}K_r K_{\tau(r)}^{-1}(K_{r+1}^{-1} - K_{r+1})K_Y^{-1}\end{aligned}$$

as required. □

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